

Research Survival Handbook (**Unfinished**)

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0.1 What's in the begining

In cosmology, the frame used most frequently is the cosmic microwave background radiation (CMB) stationary frame, not earth frame. We sometimes say it is earth frame because the speed of earth relative to CMB is rather small compared to the galaxies' movement observed by we earth beings.

0.2 Constants And Physical Quantities

- Deceleration parameter of today,

$$q = - \left(\frac{a}{\dot{a}} \ddot{a} \right) = - \frac{\ddot{a}}{\dot{a}} \frac{1}{H(a)}$$

Deceleration parameter is tightly related to Friedmann equation. ¹

0.2.1 Cosmographic Parameters

Co-Graphy-Par-1 Recession speed

$$v = H_0 \cdot d \tag{1}$$

Co-Graphy-Par-2 Hubble time

$$t_H = \frac{1}{H_0} \tag{2}$$

Co-Graphy-Par-3 Hubble distance

$$D_H = \frac{c}{H_0} = c \cdot t_H \tag{3}$$

Co-Graphy-Par-4 Dimensionless density parameters

Matter

$$\Omega_M = \frac{8\pi G \rho_M|_{a=1}}{3H_0^2} \tag{4}$$

Lambda

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} \tag{5}$$

Curvature can be viewed as some energy, too. From Friedmann equation we can sort out it. Or equivalently

$$\Omega_k = 1 - \Omega_M - \Omega_\Lambda \tag{6}$$

¹Friedmann equation can be written in terms of deceleration parameter,

$$q = 1/2(1 + 3w)(1 + k/(aH)^2)$$

Redshifts and Distances

Co-Dis-1 Redshift defined in observation

$$z = \frac{\nu_e}{\nu_o} - 1 = \frac{\lambda_o - \lambda_e}{\lambda_e} \quad (7)$$

Co-Dis-2 In linear range, given a radial velocity, redshift can be written as

$$z = \sqrt{\frac{1 + v/c}{1 - v/c}} - 1 \quad (8)$$

Co-Dis-3 Peculiar velocity

$$v_{pec} = c \frac{z_{obs} - z_{cos}}{1 + z}, \quad (9)$$

in which, z_{obs} is the observed redshift, while z_{cos} is the cosmological redshift. Cosmological redshift is the Hubble flow "due solely to the expansion of the universe". **This definition is valid when v_{pec} is much smaller than speed of light, i.e., $v_{pec} \ll c$.**

Co-Dis-4 Scale factor

$$a|_{t_0} = (1 + z)a|_{t_e}. \quad (10)$$

In cosmology, we often use another form which is derived from and setting $a|_{t_0} = 1$

$$a = \frac{1}{1 + z}. \quad (11)$$

This is the scale factor we used in line element.

Co-Dis-5 Relative redshift

$$z_{12} = \frac{a|_{t_1}}{a|_{t_2}} - 1 = \frac{1 + z_2}{1 + z_1} \quad (12)$$

Co-Dis-6 Line-of-sight comoving distance

$$D_c = D_H \int_0^z \frac{1}{E(z')} dz' \quad (13)$$

in which $E(z) \equiv \sqrt{\Omega_M(1 + z)^3 + \Omega_\Lambda + \Omega_k(1 + z)^2}$.

Co-Dis-7 Transverse comoving distance

$$D_M = \frac{D_H}{\sqrt{|\Omega_k|}} f(\sqrt{|\Omega_k|} \frac{D_C}{D_H}) \quad (14)$$

in which

$$f(x) = \begin{cases} \sinh(x), & \Omega_k > 0 \\ x, & \Omega_k = 0 \\ \sin(x), & \Omega_k < 0 \end{cases} \quad \begin{matrix} \text{hyperbola} \\ \text{flat} \\ \text{parabola} \end{matrix}. \quad (15)$$

Co-Dis-8 Angular diameter distance

$$D_A = \frac{1}{1+z} D_M \quad (16)$$

We just divide the transverse distances by $1+z$ The advantage of it is that it is not singular at $z \rightarrow \infty$

Co-Dis-9 Luminosity distance

$$D_L \equiv \sqrt{\frac{L}{4\pi S}} \quad (17)$$

in which L is bolometric luminosity², S is the bolometric flux.

Related to other distances

$$D_L = (1+z)D_M = (1+z)^2 D_A \quad (18)$$

Co-Dis-10 Distance modules

$$DM \equiv 5 \log D_L / 10 \text{pc} \quad (19)$$

In Mpc unit,

$$DM \equiv 5 \log D_L + 25 \quad (20)$$

Co-Dis-11 Comoving volume

$$dV_C = D_H \frac{(1+z)^2 D_A^2}{E(z)} d\Omega dz \quad (21)$$

Co-Dis-12 Lookback time

$$t_L = t_H \int_0^z \frac{dz'}{(1+z')E(z')} \quad (22)$$

Lookback time is "the difference between the age t_o of the Universe now (at observation) and the age t_e of the Universe at the time the photons were emitted (according to the object). "

Co-Dis-13 Probability of intersecting objects

$$dP = n(z)\sigma(z)D_H \frac{(1+z)^2}{E(z)} dz \quad (23)$$

"Given a population of objects with comoving number density $n(z)$ (number per unit volume) and cross section $\sigma(z)$ (area), what is the incremental probability dP that a line of sight will intersect one of the objects in redshift interval dz at redshift z ?"³

²Bolometric luminosity: The total energy radiated by an object at all wavelengths, usually given in joules per second.

³arXiv:astro-ph/9905116v4

This part is mostly taken from [arXiv:astro-ph/9905116v4](#)

Page 418 of Gravitation And Cosmology: Principles And Applications Of The General Theory Of Relativity written by Weinberg in 1972.

0.3 The Homogeneous and Isotropic Universe

It is always pointed out that most theories are based on the Cosmological Principle.

I have stated this principle of in the chapter telling the perturbation theory in cosmology.

Empedocles: 'God is an infinite sphere whose center is everywhere and circumference nowhere.'

Cosmological principle states that VIEWED on a sufficiently large scale, the properties of the Universe are the same for all observers. The key to understand this is that "same" means same physics principles and physical constants. Wikipedia gives three qualifications and two testable consequences. The two consequences are isotropic and homogeneous. Isotropic indicates no matter where we are looking at the universe is the same while homogeneous indicates no matter where we are located the universe is about the same (i.e., we are looking at a fair sample of the universe).

Problem is how to describe isotropy and homogeneity. [Liang, P360]

Actually, we have a rigorous form for the principle, as I mentioned in these paragraphs.

Homogeneous Space-time (M, g_{ab}) , sliced into Σ_t , induced metric of g_{ab} on Σ_t . (M, g_{ab}) is spatially homogeneous if we can always find Σ_t , ensuring the existence of a set of isometry on Σ_t itself.

0.4 Quantities

0.4.1 Energy-momentum Tensor

Energy-momentum tensor can be made clear using a standard procedure of field theory.

Check out the table below. [From Ohanian and Ruffini's GRAVITATION AND SPACETIME, P494.]

	Particle	Field
Quantities	$q_i(t)$	$\phi(\vec{x}, t)$
independent quantitie	$t, i[?]$	\vec{x}, t
Lagrange	$L = L(q_i, \dot{q}_i)$	$L = \int \mathcal{L}(\phi, \partial\phi/\partial x^\mu) d^3x$
Equation of motion	$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$	$\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$
Hamiltonian	$H = \sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$	$H = \int (\phi_{,0} \frac{\partial \mathcal{L}}{\partial \phi_{,0}} - \mathcal{L}) d^3x$

The integrand in H is the energy density of the complete system. We would like to choose the t_0^0 component of a canonical energy-momentum tensor energy-momentum tensor to describe energy density,

$$t_0^0 = \phi_{,0} \frac{\partial \mathcal{L}}{\partial \phi_{,0}} - \mathcal{L}. \quad (24)$$

It can be proved that the following equation is the only one that satisfies the constrain we proposed before.

$$t_\mu^\nu = \phi_{,\mu} \frac{\partial \mathcal{L}}{\partial \phi_{,\nu}} - \delta_\mu^\nu \mathcal{L} \quad (25)$$

We have the conservation law:

$$\frac{\partial}{\partial x^\nu} t_\mu^\nu = 0 \quad (26)$$

Start from this we can find the Hamiltonian is conserved.

$$\frac{dH}{dt} = 0 \quad (27)$$

The total momentum is conserved too.

$$P_k = \int t_k^0 d^3x \quad (28)$$

This is all about a scalar field. If we switch to vector field (EM field) and tensor field (1-order gravitation field), the canonical energy-momentum tensors are

$$t_{(em)\mu}^\nu = A_{,\mu}^\alpha \frac{\partial \mathcal{L}_{(em)}}{\partial A_{,\nu}^\alpha} - \delta_\mu^\nu \mathcal{L}_{(em)} \quad (29)$$

$$t_{(g1)\mu}^\nu = h_{,\mu}^{\alpha\beta} \frac{\partial \mathcal{L}_{(g1)}}{\partial h_{,\nu}^{\alpha\beta}} - \delta_\mu^\nu \mathcal{L}_{(g1)} \quad (30)$$

In special relativity, the energy-momentum tensor for ideal fluid is

$$T_{ab} = (\rho + p)U_a U_b + p\eta_{ab} \quad (31)$$

Each component has its physical meaning.

1. T^{00} is the energy density;
2. $T^{0k} = T^{k0}$ is the k momentum density (energy flux density);
3. $T^{kl} = T^{lk}$ is the k momentum flux density in l direction.

Just change all the η_{ab} into g_{ab} .

For any system,

$$\nabla_\mu T^{\mu\nu} = U^\mu U^\nu \nabla_\mu \rho + U^\mu U^\nu \nabla_\mu p + (\rho + p) U^\mu \nabla_\mu U^\nu + (\rho + p) U^\nu \nabla_\mu U^\mu + g^{\mu\nu} \nabla_\mu p \quad (32)$$

$$= U^\mu U^\nu \dot{\rho} + (g^{\mu\nu} + U^\mu U^\nu) \nabla_\mu p + (\rho + p) A^\nu + (\rho + p) U^\nu \nabla_\mu \Theta \quad (33)$$

$$= U^\mu U^\nu \dot{\rho} + (\rho + p) U^\nu \nabla_\mu \Theta \quad (34)$$

$$= Q^\nu \quad (35)$$

A^ν vanishes because we have chosen a comoving observer. $\nabla_\mu p$ vanishes because in our standard model ρ, p are coordinate free. Actually, since U^μ is timelike, we have $(g^{\mu\nu} + U^\mu U^\nu) \nabla_\mu p = h^{\mu\nu} \nabla_\mu p$.

0.4.2 Friedmann Universe

Cosmological Principle Empedocles: 'God is an infinite sphere whose center is everywhere and circumference nowhere.'

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Problem is how to describe isotropy and homogeneity. [Liang, P360]

Robertson-Walker Metric The next problem is how to set up a simple and useful coordinate system which metric to use in cosmology. [Liang, P366]

Evolution of Scale Factor Friedmann equation

$$3(\dot{a}^2 + k)/a^2 = 8\pi p \quad (36)$$

$$2\ddot{a}/a + (\dot{a}^2 + k)/a^2 = -8\pi p \quad (37)$$

?? is called Friedmann equation.

It is better to use another set of equations which are identical to ?? and ??

$$\ddot{a} = -4\pi a(\rho + 3p)/3 \quad (38)$$

$$0 = \dot{\rho} + 3(\rho + p)\dot{a}/a \quad (39)$$

These equations can be solved in some circumstances. [Liang, P376]
 It could be useful to reform these equation.

$$H^2 + \frac{k}{a^2} = \frac{8}{3}\pi\rho \quad (40)$$

$$2\dot{H} + 3H^2 + \frac{k}{a^2} = -8\pi p \quad (41)$$

Mixing

$$3\ddot{a} = -4\pi a(\rho + 3p) \quad (42)$$

$$0 = \dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} \quad (43)$$

$$\dot{H} = -4\pi G(\rho + p) + \frac{k}{a^2} \quad (44)$$