Physics Research Survival Manual

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Cosmology TF

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CONTENTS

This is a notebook about physics. It's not done yet. (Maybe I'll never finish it because there are too many topics in physics.)

Use the source. Keep the source open.

CHAPTER

ONE

INTRODUCTION

Some notes continued from the full theoretical physics notes are here.

CHAPTER TWO

PREFACE

2.1 Whatever

I alway believe that making things open to everyone is one of the most powerful things that drive the world forward. So I DO think open source and open data, even open education are reforming the world.

This was a draft handbook for myself when I was studying in Fudan University. At that time, I learned how to use LaTeX and I was so exciting. So I thought I should start writing something using LaTeX since it's so beautiful. Well, the bad thing is, I just randomly wrote down my notes on some specific topics.

I was so greedy back then. I was trying to build up my own framework of physics by writing notes here. It never did the work by the way. Then I realized a framework should be somethng organized much better than this one. (I should draw a map of physics.)

Though these notes didn't help me building up my framework of phyiscs, I learn an important lesson. A physicist should build up his/her own style: the way to think, the way to solve problems, the way to check answers, the way to write, etc. (I just want to create a new word for his/her, hir or hes.)

Anyway, I got frastrated and gave up the effort to utilize it as a framework-building thing. However I won't just dump these notes. As I have more and more to add, I think I'll just let it be my notebook, which, of course, is open source and accessible to everyone.

Yes. Use the source. Keep the source open.

CHAPTER THREE

VOCABULARY

3.1 Basic

3.1.1 Dimension

How to find the relationship between two quantities? For example, what is the dimensional relationship between length and mass.

* Plank constant: $\hbar \sim [\text{Energy}] \cdot [\text{Time}] \sim [\text{Mass}] \cdot [\text{Length}]^2 \cdot [\text{Time}]^{-1}$
* Speed of light in vacuum: $\text{c} \sim [\text{Length}] \cdot [\text{Time}]^{-1}$
* Gravitational constant: $\text{G} \sim [\text{Length}]^3 \cdot [\text{Mass}]^{-1} \cdot [\text{Time}]^{-2}$

Then it is easy to find that a combination of $c/\hbar$ cancels the dimension of mass and leaves the inverse of length. That is

$$[L]^2 = \left[\frac{\hbar G}{c^3}\right]$$

$$[M]^2 = \left[\frac{\hbar c}{G}\right]$$

$$[T]^2 = \left[\frac{\hbar G}{c^5}\right]$$

As we can see, it is possible to use $c = 1, \hbar = 1, G = 1$ because we can always restore the units in a deterministic way. $c, \hbar, G$ are function of mass, length, time, and with $c = \hbar = G = 1$ give us only one solution of mass, length and time: three equations + three variables.

Planck Scales

As we have seen, the three constant can make up a length scale, a mass scale, a time scale. Then what are they?

Planck length:

$$l_P = \sqrt{\frac{\hbar G}{c^3}}$$

Planck mass:

$$m_P = \sqrt{\frac{\hbar c}{G}}$$

Planck time:

$$t_P = \sqrt{\frac{\hbar G}{c^5}}$$

Equations and Dimensions

Before solving equations, it is good to reform them in to dimensionless ones.

To make the equation dimensionless doesn't mean we can just divide arbitary terms on both sides. We need to find out the characteristic quantity of the system. For example, we can divide by $\hbar\omega$ on both sides of Schrodinger equation for Harmonic Oscillators. This is a good step because $\hbar\omega$ is the characteristic energy scale of system. At the same time, we can make the length terms dimensionless using the characteristic length. DO NOT use an arbitary length!

Most Wonderful Equations That Should Never Be Forgotten

Electrodynamics

$$\nabla \times \vec{E} = -\partial_t \vec{B}$$

$$\nabla \times \vec{H} = \vec{J} + \partial_t \vec{D}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

For linear meterials,

$$\vec{D} = \epsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

$$\vec{J} = \sigma \vec{E}$$

Dynamics

Hamilton conanical equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Thermodynamics and Statistical Physics

Liouville's Law

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} \equiv \frac{\partial \rho}{\partial t} + \sum_i \left[\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right] = 0$$

CHAPTER

FOUR

MATHEMATICS

4.1 Mathematics

4.1.1 Linear Algebra

Tensor Product Space

$|\phi\rangle_1$ and $|\phi\rangle_2$ are elements of Hilbert space H_1 and H_2. **Tensor Product** of $|\phi\rangle_1$ and $|\phi\rangle_2$ is denoted as $|\phi\rangle_1 \otimes |\phi\rangle_2$. This operation is linear and distributive.

Tensor product space $H_1 \otimes H_2$ is composed of all the linear combinations of all possible tensor products of elements in H_1 and H_2.

Inner Product

Inner product of two tensor products

$$(\langle\phi|_1 \otimes \langle\phi|_2)(|\psi\rangle_1 \otimes |\psi\rangle_2) = ({}_1\langle\phi \mid \psi\rangle_1)({}_2\langle\phi \mid \psi\rangle_2)$$

Operators Applied to Tensor Product

Two operators $\hat{O}_1$ and $\hat{O}_2$ works on H_1 and H_2 respectively applied to tensor product

$$(\hat{O}_1 \otimes \hat{O}_2)(|\phi\rangle_1 \otimes |\phi\rangle_2) = (\hat{O}_1 |\phi\rangle_1) \otimes (\hat{O}_2 |\phi\rangle_2)$$

4.1.2 Differential Geometry

Metric

Definitions

Denote the basis in use as $\hat{e}_\mu$, then the metric can be written as

$$g_{\mu\nu} = \hat{e}_\mu \hat{\cdot} e_\nu$$

if the basis satisfies

Inversed metric

$$g_{\mu\lambda} g^{\lambda\nu} = \delta_\mu^\nu = g_\mu^\nu$$

How to calculate the metric

Let's check the definition of metric again.

If we choose a basis $\hat{e}_\mu$, then a vector (at one certain point) in this coordinate system is

$$x^a = x^\mu \hat{e}_\mu$$

Then we can construct the expression of metric of this point under this coordinate system,

$$g_{\mu\nu} = \hat{e}_\mu \cdot \hat{e}_\nu$$

For example, in spherical coordinate system,

$$\vec{x} = r \sin\theta \cos\phi \hat{e}_x + r \sin\theta \sin\phi \hat{e}_y + r \cos\theta \hat{e}_z \tag{4.1}$$

Now we have to find the basis under spherical coordinate system. Assume the basis is $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$. Choose some scale factors $h_r = 1, h_\theta = r, h_\phi = r \sin\theta$. Then the basis is

$$\hat{e}_r = \frac{\partial \vec{x}}{h_r \partial r} = \hat{e}_x \sin\theta \cos\phi + \hat{e}_y \sin\theta \sin\phi + \hat{e}_z \cos\theta,$$

etc. Then collect the terms in formula (4.1) is we get $\vec{x} = r\hat{e}_r$, this is incomplete. So we check the derivative.

$$\begin{gathered}
\mathrm{d}\vec{x} = \hat{e}_x(\mathrm{d}r \sin\theta\cos\phi + r\cos\theta\cos\phi\mathrm{d}\theta - r\sin\theta\sin\phi\mathrm{d}\phi) \\
\hat{e}_y(\mathrm{d}r\sin\theta\sin\phi + r\cos\theta\sin\phi\mathrm{d}\theta + r\sin\theta\cos\phi\mathrm{d}\phi) \\
\hat{e}_z(\mathrm{d}r\cos\theta - r\sin\theta\mathrm{d}\theta) \\
= \mathrm{d}r(\hat{e}_x\sin\theta\cos\phi + \hat{e}_y\sin\theta\sin\phi - \hat{e}_z\cos\theta) \\
\mathrm{d}\theta(\hat{e}_x\cos\theta\cos\phi + \hat{e}_y\cos\theta\sin\phi - \hat{e}_z\sin\theta)r \\
\mathrm{d}\phi(-\hat{e}_x\sin\phi + \hat{e}_y\cos\phi)r\sin\theta \\
= \hat{e}_r\mathrm{d}r + \hat{e}_\theta r\mathrm{d}\theta + \hat{e}_\phi r\sin\theta\mathrm{d}\phi
\end{gathered}$$

Once we reach here, the component (e_r, e_θ, e_ϕ) of the point under the spherical coordinates system basis $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$ at this point are clear, i.e.,

$$\begin{aligned}
\mathrm{d}\vec{x} &= \hat{e}_r\mathrm{d}r + \hat{e}_\theta r\mathrm{d}\theta + \hat{e}_\phi r\sin\theta\mathrm{d}\phi \\
&= e_r\mathrm{d}r + e_\theta\mathrm{d}\theta + e_\phi\mathrm{d}\phi
\end{aligned}$$

In this way, the metric tensor for spherical coordinates is

$$g_{\mu\nu} = (e_\mu \cdot e_\nu) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

Connection

First class connection can be calculated

$$\Gamma^{\mu}{}_{\nu\lambda} = \hat{e}^{\mu} \cdot \hat{e}_{\mu,\lambda}$$

Second class connection isfootnote{Kevin E. Cahill}

$$[\mu\nu, \iota] = g_{\iota\mu}\Gamma^{\mu}{}_{\nu\lambda}$$

Gradient, Curl, Divergence, etc

Gradient

$$T^{b}{}_{c;a} = \nabla_a T^{b}{}_{c} = T^{b}{}_{c,a} + \Gamma^{b}_{ad} T^{d}{}_{c} - \Gamma^{d}_{ac} T^{b}{}_{d}$$

Curl

For an anti-symmetric tensor, $a_{\mu\nu} = -a_{\nu\mu}$

$$\begin{aligned}\mathrm{Curl}_{\mu\nu\tau}(a_{\mu\nu}) &\equiv a_{\mu\nu;\tau} + a_{\nu\tau;\mu} + a_{\tau\mu;\nu} \\ &= a_{\mu\nu,\tau} + a_{\nu\tau,\mu} + a_{\tau\mu,\nu}\end{aligned}$$

Divergence

$$\begin{aligned}\mathrm{div}_{\nu}(a^{\mu\nu}) &\equiv a^{\mu\nu}{}_{;\nu} \\ &= \frac{\partial a^{\mu\nu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\nu\tau} a^{\tau\nu} + \Gamma^{\nu}_{\nu\tau} a^{\mu\tau} \\ &= \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\nu}} (\sqrt{-g} a^{\mu\nu}) + \Gamma^{\mu}_{\nu\lambda} a^{\nu\lambda}\end{aligned}$$

For an anti-symmetric tensor

$$\mathrm{div}(a^{\mu\nu}) = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\nu}} (\sqrt{-g} a^{\mu\nu})$$

Annotation Using the relation $g = g_{\mu\nu} A_{\mu\nu}$, $A_{\mu\nu}$ is the algebraic complement, we can prove the following two equalities.

$$\Gamma^{\mu}_{\mu\nu} = \partial_{\nu} \ln \sqrt{-g}$$

$$V^\mu{}_{;\mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} V^\mu)$$

In some simple case, all the three kind of operation can be demonstrated by different applications of the del operator, which $\nabla \equiv \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z$.

- Gradient, ∇f, in which f is a scalar.
- Divergence, $\nabla \cdot \vec{v}$
- Curl, $\nabla \times \vec{v}$
- Laplacian, $\Delta \equiv \nabla \cdot \nabla \equiv \nabla^2$

4.1.3 Linear Algebra

Basic Concepts

Trace

Trace should be calculated using the metric. An example is the trace of Ricci tensor,

$$R = g^{ab} R_{ab}$$

Einstein equation is

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab}$$

The trace is

$$\begin{aligned} g^{ab} R_{ab} - \frac{1}{2} g^{ab} g_{ab} R &= 8\pi G g^{ab} T_{ab} \\ \Rightarrow R - \frac{1}{2} 4R &= 8\pi G T \\ \Rightarrow -R &= 8\pi G T \end{aligned}$$

Technique

Inverse of a matrix

Many methods to get the inverse of a matrix. Check wikipedia for Invertible matrix.

Adjugate matrix method for example is here.

$$A^{-1} = \frac{A^*}{|A|}$$

in which, A^* is the adjugate matrix of A.

4.1.4 Differential Equations

Standard Procedure

Tricky

WKB Approximation

When the highest derivative is multiplied by a small parameter, try this.

4.2 Statistics

4.2.1 Famous Distributions

- Binomial distribution
- Poisson Distribution
- Chi-squared Distribution

CHAPTER

FIVE

FUNDAMENTAL PHYSICS

5.1 Classical Mechanics

5.1.1 Oscillators

In general, the Lagragian for a system with n general coordinates can be

$$L = \frac{1}{2} m_{jk} \dot q_j \dot q_k - V(q_1, \cdots, q_n)$$

To write down equation of motion, we need the following terms,

$$\frac{\partial L}{\partial \dot q_j} = m_{jk} \dot q_k \frac{\partial L}{\partial q_j} = \frac{1}{2} \frac{\partial m_{kl}}{\partial q_j} \dot q_k \dot q_l - \frac{\partial V}{\partial q_j}$$

Then equation of motion is

$$m_{jk} \ddot q_k + \frac{\partial m_{jk}}{\partial q_l} \dot q_k \dot q_l - \frac{1}{2} \frac{\partial m_{kl}}{\partial q_j} \dot q_k \dot q_l = - \frac{\partial V}{\partial q_j}$$

Generally, we can't solve this system. But there is an interesting limit. The system may have equilibrium points. We can study systems oscillating around equilibrium points.

At equilibrium, the system can stay steady, i.e., $\dot q_j^0 = 0$. This gives us

$$\frac{\partial V}{\partial q_j} = 0,$$

for all j.

Now for small deviations, we can expand the system around equilibrium points.

$$q_j = q_j^0 + \eta_j$$

Then

$$T = \frac{1}{2} m_{jk}|_0 \dot\eta_j \dot\eta_k \equiv \frac{1}{2} T_{jk} \dot\eta_j \dot\eta_k$$

$$V = V|_0 + \frac{\partial V}{\partial q_j}|_0 \eta_j + \frac{1}{2}\frac{\partial^2 V}{\partial q_j \partial q_k}|_0 \eta_j \eta_k + \cdots \equiv \frac{1}{2} V_{jk} \eta_j \eta_k$$

So we have the Lagrangian for small oscillations,

$$L = \frac{1}{2} T_{jk} \dot{\eta}_j \dot{\eta}_k - \frac{1}{2} V_{jk} \eta_j \eta_k$$

Typing indices using LaTeX is so annoying. So we'll use matrix notations and Lagragian becomes

$$L = \frac{1}{2}\dot{\tilde{\eta}} T \dot{\eta} - \frac{1}{2}\tilde{\eta} V \eta,$$

in which T and V matrices are n by n real and symmetric.

(We need to diagonalize T and V. First question comes to us is:

** Is is possible to diagonalize both T and V at the same time? **

We can have a look at the surface $\tilde{p} T p = C$, which is a elliptical surface with coordinates p.)

Use the following transformation

$$\xi = T^{1/2} \eta$$

Then transpose

$$\tilde{\xi} = \tilde{\eta} T^{1/2}$$

$$\dot{\tilde{\xi}} \dot{\xi} = \dot{\tilde{\eta}} T \dot{\eta}$$

So we have the new Lagragian

$$L = \frac{1}{2}\dot{\tilde{\xi}}\dot{\xi} - \frac{1}{2}\tilde{\xi} T^{-1/2} V T^{-1/2} \xi$$

Define $T^{-1/2} V T^{-1/2} \equiv V'$.

Next we need to diagonalize V' by using its eigen vectors.

$$V' b = \lambda b$$

is equivalent to

$$V a = \lambda T a$$

with $b = T^{1/2} a$. So we have

$$\det(V' - \lambda \mathbf{I}) = 0$$

is same as

$$\det(V - \lambda T) = 0$$

in which λ is the eigen value of this function.

5.1.2 Hamiltonian Dynamics

Phase space

5.2 Quantum Mechanics

5.2.1 Quantum Mechanics Framework

What're the most important tricks in QM calculations?

- **Remember what basis we are working in**
- **Identity**

First Three Postulates

- Physical state is described by kets in a Hilbert space. We need to specify a complete basis $\{|i\rangle\}$ to do calculations.

$$|\psi\rangle = \sum_i |i\rangle \langle i| \, |\psi\rangle = \sum_i C_i \, |i\rangle$$

- Operators are given by Hermitian operators; A measurement of the variable $\hat{\Omega}$ will yield one of the eigenvalues ω with the probability

$$|\langle \omega \mid \psi \rangle|^2 .$$

  And the state of the system will change to $|\omega\rangle$.

- The state vector obeys the Schrödinger equation,

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = \hat{H} \, |\psi(t)\rangle ,$$

  where $\hat{H}$ is the Hamiltonian operator.

> The logic here is that we first find the way to describe a system, then think about how to find out the information we need from the state vector and also find the evolution of the state vector. Then we need the operator and Schrodinger equation. Finally, we would like to relate the theory to experiments, and it comes the measurement postulate.
>
> Later we will need the relation between position and momentum, which becomes the fourth postulate.

- How to solve the evolution of a system? We just define a magical operator, propagator

$$\hat{U}\left|\psi(t_0)\right\rangle = \left|\psi(t)\right\rangle.$$

This operator just gives us the evolution of state vector! Wait, can we write down the explicit expression of it?

Let's find out. The only thing we know about the evolution of a state vector is the third postulate up there.

$$\begin{aligned} i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi(t)\right\rangle &= \hat{H}\left|\psi(t)\right\rangle \\ i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}\left|\psi(t_0)\right\rangle &= \hat{H}\hat{U}\left|\psi(t_0)\right\rangle \\ i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{U} &= \hat{H}\hat{U} \end{aligned}$$

Looks familiar? This just gives us a exponential result, **if the Hamiltonian is time independent**.

$$\hat{U} = e^{-i\hat{H}(t-t_0)/\hbar}$$

We can prove that this operator is Unitary because $\hat{H}$ is Hermitian.

This is just the abstract representation, we work in some basis, and the most convenient basis is the eigenstates of Hamiltonian, { $|\epsilon_i\rangle$ },

$$\begin{aligned} \hat{U}\left|\phi\right\rangle &= e^{-i\hat{H}(t-t_0)/\hbar}\left|\psi\right\rangle \\ \hat{U}\left|\phi\right\rangle &= \sum_i e^{-i\hat{H}(t-t_0)/\hbar}\left|\epsilon_i\right\rangle\left\langle\epsilon_i\right|\left|\psi\right\rangle \\ \hat{U}\left|\phi\right\rangle &= \sum_i e^{-i\epsilon_i(t-t_0)/\hbar}\left|\epsilon_i\right\rangle\left\langle\epsilon_i\right|\left|\psi\right\rangle \end{aligned}$$

And we are going to use

$$\hat{U} = \sum_i e^{-i\epsilon_i(t-t_0)/\hbar}\left|\epsilon_i\right\rangle\left\langle\epsilon_i\right|$$

from now on. (Well, only on discrete eigenvalues ones.)

(See that? Identity does the work again.)

Position and Momentum Space

Summarize here.

- **Position**
  1. Define $\{|x\rangle\}$ basis.
  2. Define $\hat{x}$ operator.
  3. Find wave function in this basis.

4. Find measurement.

- **Evolution**
  1. Need propagator $\hat{U}$.
  2. Propagator needs the solution of Hamiltonian eigensystem.
  3. (Free particles) Hamiltonian needs the solution of momentum eigensystem.
- **Momentum**
  1. Before we define some arbitrary momentum space, we should check the relation between momentum and position. And it turns out to be related by a commutator.(Postulate IV)
  2. Use the postulate to momentum operator.
  3. Find eigenstates.
  4. (Calculate the propagator.)

Position Space

1. Define $|x\rangle$ basis.
   - Orthonormal:

   braket{x}{x'}=delta(x-x')

   - Complete:

   int braket{x'}{x'} d x' = mathbb{I}
2. Define position operator.

$$\hat{x}\,|x\rangle = x\,|x\rangle$$

   And in $\{|x\rangle\}$ basis, this operator becomes a function, which is

$$\begin{aligned}&\langle x|\,\hat{x}\,|x'\rangle\\ &= (\langle x|\,\hat{x})\,|x'\rangle\\ &= x\langle x \mid x'\rangle\\ &= x\delta(x-x')\end{aligned}$$

3. Find state vector in $\{|x\rangle\}$ basis.

$$\psi(t,x) = \langle x \mid \psi(t)\rangle$$

   - Normalized:

$$\int |\psi(t,x)|^2 \,\mathrm{d}x = 1.$$

And we are interpreting $|\psi(t,x)|^2$ as probability density.

4. Calculate probability of a measurement. Taking $\hat{x}$ as an example.

$$\begin{aligned}
&\langle\psi|\,\hat{x}\,|\psi\rangle \\
&= \iint \langle\psi \mid x\rangle \,\langle x|\,\hat{x}\,|x'\rangle\,\langle x' \mid \psi\rangle \mathrm{d}x\mathrm{d}x' \\
&= \iint \psi^*(t,x) x\delta(x-x')\psi(t,x')\mathrm{d}x\mathrm{d}x' \\
&= \int |\psi(t,x)|^2 \, x\mathrm{d}x
\end{aligned}$$

Momentum Space

To find the momentum operator, we need to check the relation between momentum and position before we just randomly define one. Truth is, we have a fourth postulate states the relation between them.

Postulate IV The commutator of $\hat{x}$, $\hat{p}$ is

$$[\hat{x},\hat{p}] = i\hbar$$

Two comments:

- Why i ? Eigenvalue of Anti-Hermitian operator.
- Why $\hbar$? Because people define the dimensions of position and momentum differently before they know this commutator. We would like to assign them the same dimension if we already know this relation.

Momentum Space

1. Find momentum operator in position basis $\{|x\rangle\}$.

$$\langle x|\,[\hat{x},\hat{p}]\,|x'\rangle = i\hbar\delta(x-x')$$

And write out the commutator and use the relation of delta function $x\delta'(x) = -\delta(x)$, we find out the momentum operator in $\{|x\rangle\}$ basis,

$$\langle x|\,\hat{p}\,|x'\rangle = -i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\delta(x-x')$$

Let's talk physics. What does that operator mean? We need to see what the result is when momentum operator is applied to a state. And remember we would work in $\{|x\rangle\}$ basis.

$$\begin{aligned}
&\langle x|\hat{p}|\psi\rangle \\
&=\iint \langle x \mid x'\rangle \langle x'|\hat{p}|x''\rangle \langle x'' \mid \psi\rangle \mathrm{d}x'\mathrm{d}x'' \\
&=\int \langle x|\hat{p}|x''\rangle \psi(t,x'')\mathrm{d}x'' \\
&=\int \left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x')\psi(t,x')\right)\mathrm{d}x' \\
&=\int \left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x'}\delta(x'-x)\psi(t,x')\right)\mathrm{d}x'
\end{aligned}$$

Integrate by parts, we will find the expression. (I am having a problem finding the right answer.)

$$\langle x|\hat{p}|\psi\rangle = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\psi(x).$$

2. Eigenfunction for momentum.

$$\hat{p}|p\rangle = p|p\rangle .$$

Again, we are going to project it on the $\{|x\rangle\}$ basis,

$$\langle x|\hat{p}|p\rangle = \langle x|p|p\rangle ,$$

where $\langle x \mid p\rangle$ is the eigenstates in $\{|x\rangle\}$ basis, we call it $\phi_p(x)$.

$$\begin{aligned}
\langle x|\hat{p}|p\rangle &= p\phi_p(x) \\
\int \langle x|\hat{p}|x'\rangle \langle x' \mid p\rangle\mathrm{d}x' &= p\phi_p(x) \\
-i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\phi_p(x) &= p\phi_p(x)
\end{aligned}$$

The solution is

$$\phi_p(x) = \mathrm{C}e^{ipx/\hbar}$$

This constant C is found by the normalization condition,

$$\langle p \mid p'\rangle = \int \phi_p^*(x)\phi_{p'}(x)\mathrm{d}x = \delta(p-p')$$

The final results should be

$$\phi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(ipx/\hbar\right)$$

3. Find the dynamics of free particles in quantum mechanics. **Find the propagator and everything solves.** The hamiltonian for a free particle is

$$\hat{H} = \frac{\hat{p}^2}{2m}.$$

We argue here that the eigenvectors of momentum are also the eigenvectors of this hamiltonian. And we can easily guess the eigenvalues are $p^2/2m$. So the propagator is

$$\hat{U} = \int e^{-ip^2 t/2m\hbar} \left|p\right\rangle \left\langle p\right| \mathrm{d}p$$

But that is too abstract to use, we can find the expression in $\{|x\rangle\}$ basis.

$$\begin{aligned}\langle x| \hat{U} |x\rangle &= \int e^{-ip^2 t/2m\hbar} \langle x \mid p\rangle\langle p \mid x\rangle \mathrm{d}p \\ &= \int e^{-ip^2 t/2m\hbar} \left|\phi_p\right|^2 \mathrm{d}p\end{aligned}$$

5.2.2 Quantum in 1D

General

Always start with the propagator for time independent Hamiltonian.

$$|\psi(t)\rangle = \hat{U} |\psi(0)\rangle$$

For cases that Hamiltonian with discrete eigenvalues,

$$|\psi(t)\rangle = \sum_n e^{-i\epsilon_n t/\hbar} |n\rangle \langle n \mid \psi(0)\rangle$$

If the initial state is just one of the eigenstates of Hamiltonian, say the mth one (normalized),

$$|\psi(t)\rangle = e^{-i\epsilon_m t/\hbar} |m\rangle$$

Well, that phase factor doesn't have any effect for the topic we discuss. So our time evolution will stay on the same state forever.

The same thing happens for continuous cases.

So our task is simplified to solve the eigensystem of Hamiltonian, which is

$$\hat{H} |\epsilon\rangle = \epsilon |\epsilon\rangle$$

Infinite Barriers

Math

Setup

- Potential in a box

$$
\begin{aligned}
V(x) = & 0, 0 < x < L \\
& \infty, \text{Other}
\end{aligned}
$$

Solve the Problem

- Hamiltonian

$$
\hat{H} = \frac{\hat{p}^2}{2m} + V(x)
$$

- Dynamic equation

$$
\hat{H} \left|\psi(t)\right\rangle = \epsilon \left|\psi(t)\right\rangle
$$

We are happy to work in $\{\left|x\right\rangle)\}$ basis,

$$
\left\langle x\right| \hat{H} \left|\psi(t)\right\rangle = \left\langle x\right| \epsilon \left|\psi(t)\right\rangle .
$$

Put the Hamiltonian in, and remember that in position basis

$$
\left\langle x\right| \hat{p} \left|\psi\right\rangle = -i\hbar \frac{\mathrm{d}}{\mathrm{d}x}\psi,
$$

the equation of motion becomes

$$
-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x,t) + V(x)\psi(x,t) = \epsilon\psi(x,t)
$$

- Boundary conditions

$$
\begin{aligned}
\psi_I(0,t) &= \psi_{II}(0,t) \\
\psi_{II}(L,t) &= \psi_{III}(L,t)
\end{aligned}
$$

- Guess the Solutions

$$
\psi_{II} = \psi = C\sin(kx) + D\cos(kx)
$$

- Find the wavenumber k, by putting the assumed solutions into equation of motion

$$
k = \pm\sqrt{\frac{2m\epsilon}{\hbar^2}}
$$

Since we can always merge the negative into the constants, it is fine to use

$$k = \sqrt{\frac{2m\epsilon}{\hbar^2}}$$

- **Use Boundary Condition**
  1. At x=0,

     $$\psi(0,t) = 0.$$

     This gives us $D = 0$.
  2. At $x = L$,

     $$\psi(L,t) = 0.$$

     This leads to

     $$kL = n\pi.$$

     Since $n = 0$ gives us a 0 wave function, we would just drop $n = 0$. For the same reason why we drop the negative values of k, we would drop all the negative values of n. This BC gives us the possible values of energy because wavenumber k is related to energy,

     $$\epsilon = \frac{\hbar^2}{2mL^2}(n\pi)^2,$$

     with

     $$n = 1, 2, 3, \cdots$$

- Normalization as the last constraint for the last undetermined parameter,

  $$C = \sqrt{\frac{2}{L}}$$

Physics

1. Estimation
   - Find the expression for energy using dimensional analysis.
   - Using uncertainty relation to estimate the expression for energy.
2. Comments
   - **Why is the solution quantized?**
     (a) Too many constraints. BCs + normalization.

- **Why do the n in the solution goes into the expression for energy?**
    (a) Have a look at the kinetic energy term, the derivative does it.
- **What's so weird?**
    (a) For $n = 2$, no particles found at $x = L/2$. And so on.

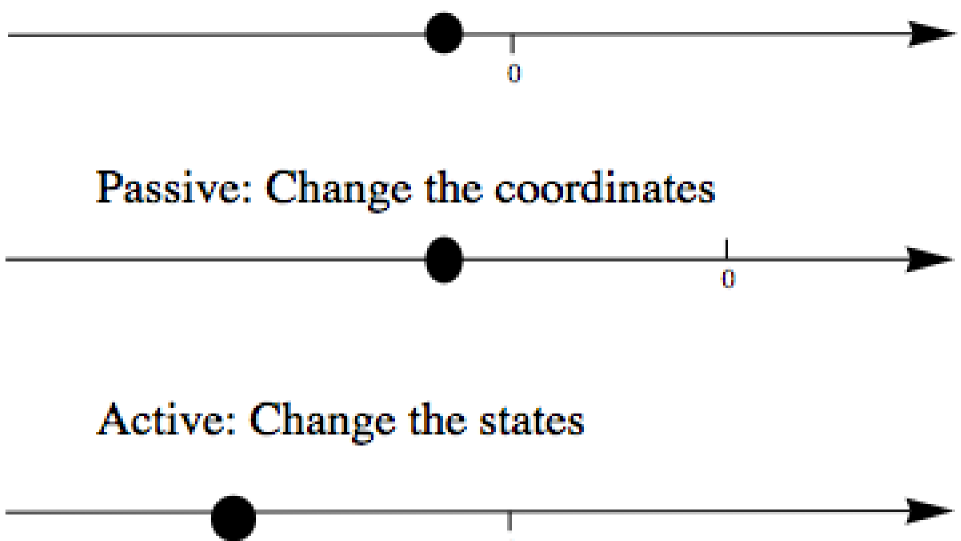
Some General Properties

1. 1D bound states have no degeneracy. Prove it by assume that there is a degeneracy state.
2. 1D bound states' wave function can be chosen to be real. (if potential V is real.)

5.2.3 Parity

Passive and Active Transformations

Generally, there are two ways of interpreting a transformation.



Here in QM, passive means transform the operator $\hat{\Omega}$, while active means change the state $|\psi\rangle$. Suppose we have a system $|\psi\rangle$, an operator $\hat{\Omega}$, a transformation $\hat{U}$.

Transformation $\hat{U}|\psi\rangle$ is identical to $\hat{U}^\dagger\hat{\Omega}\hat{U}$ because they give the same observation results. The first one is called active, while the second one is called passive.

Parity

Definition

$$\hat{\Pi}\left|x\right\rangle = \left|-x\right\rangle$$

Properties

1. Act on momentum eigenvectors,

$$\hat{\Pi}\left|p\right\rangle = \left|-p\right\rangle.$$

- Physics: Parity changes the coordinate, so the direction of momentum is also changed.
- Math:

$$\hat{\Pi}\left|p\right\rangle = \int \hat{\Pi}\left|x\right\rangle \left\langle x \mid p\right\rangle \mathrm{d}x = \int \left|-x\right\rangle \left\langle x \mid p\right\rangle \mathrm{d}x$$

Change coordinate from x to -x,

$$\hat{\Pi}\left|p\right\rangle = \int \left|x\right\rangle \left\langle -x \mid p\right\rangle \mathrm{d}x = \int \left|x\right\rangle \left\langle x \mid -p\right\rangle \mathrm{d}x = \left|-p\right\rangle$$

2. Hermitian,

$$\left\langle x\right|\hat{\Pi}\left|x'\right\rangle = \delta(x+x')(\left\langle x'\right|\hat{\Pi}\left|x\right\rangle)^{\dagger} = \left\langle x\right|\hat{\Pi}^{\dagger}\left|x'\right\rangle = \delta(x+x')$$

3. Unitary

$$\left\langle x\right|\hat{\Pi}^{\dagger}\hat{\Pi}\left|x'\right\rangle = \left\langle -x \mid -x'\right\rangle = \delta(-x+x') = \delta(x-x') = \left\langle x \mid x'\right\rangle$$

4. Inverse of parity

$$\hat{\Pi}\hat{\Pi} = \hat{\Pi}\hat{\Pi}^{\dagger} = \hat{I}$$

5. Eigensystem of parity.

$$\hat{\Pi}\left|\pi\right\rangle = \pi\left|\pi\right\rangle$$

Apply another operator

$$\hat{\Pi}^2\left|\pi\right\rangle = \pi^2\left|\pi\right\rangle$$

So, * Eigenvalues: 1, -1; * Eigenvactors: Even function, Odd function

6. Parity applied to operators a. Apply to position operator,

$$\hat{\Pi}^\dagger \hat{X} \hat{\Pi} = -\hat{X}$$

Proof:

$$\langle x| \hat{\Pi}^\dagger \hat{X} \hat{\Pi} |x'\rangle = \langle -x| \hat{X} |-x'\rangle = -x'\delta(x-x') = \langle x| (-\hat{X}) |x'\rangle$$

(a) Apply to momentum operator,

$$\hat{\Pi}^\dagger \hat{p} \hat{\Pi} = -\hat{p}$$

Proof: Similar to the previous one, just change x basis to momentum basis.

2. Symmetry related to Hamiltonian.

$$\left[\hat{\Pi}, \hat{H}\right] = 0$$

When this happens, parity of Hamiltonian won't change the wave function. Or the wave function should have an specific parity for 1D problem.

5.2.4 Classical Limit of QM

Ehrenfest's Theorem

Schrödinger equation and its adjoint

$$\begin{aligned} i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle &= \hat{H} |\psi(t)\rangle \\ -i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \langle\psi(t)| &= \langle\psi(t)| \hat{H} \end{aligned}$$

For any observable $\hat{\Omega}$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left\langle \hat{\Omega} \right\rangle &= \left(\frac{\mathrm{d}}{\mathrm{d}t} \langle\psi(t)|\right) \hat{\Omega} |\psi(t)\rangle + \langle\psi(t)| \dot{\hat{\Omega}} |\psi(t)\rangle + \langle\psi(t)| \hat{\Omega} \left(\frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle\right) \\ &= \frac{1}{i\hbar} \left(- \langle\psi(t)| \hat{H}\hat{\Omega} |\psi(t)\rangle + \langle\psi(t)| \hat{\Omega}\hat{H} |\psi(t)\rangle\right) + \langle\psi(t)| \dot{\hat{\Omega}} |\psi(t)\rangle \\ &= \frac{1}{i\hbar} \langle\psi(t)| \left[\hat{\Omega}, \hat{H}\right] |\psi(t)\rangle + \langle\psi(t)| \dot{\hat{\Omega}} |\psi(t)\rangle \end{aligned}$$

This is called Ehrenfest's Theorem.

Simple Example of Ehrenfest's Theorem

Suppose we have a system with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

We need to figure some commutators first.

$$2m\left[\hat{x},\hat{H}\right]=\left[\hat{x},\hat{p}^2\right]=\hat{x}\hat{p}\hat{p}-\hat{p}\hat{p}\hat{x}=\hat{x}\hat{p}\hat{p}-\hat{p}\hat{x}\hat{p}+\hat{p}\hat{x}\hat{p}-\hat{p}\hat{p}\hat{x}=\left[\hat{x},\hat{p}\right]\hat{p}+\hat{p}\left[\hat{x},\hat{p}\right]=2i\hbar\hat{p}$$

$$\left[\hat{p},\hat{H}\right]=\left[\hat{p},V(\hat{x})\right]=\left[\hat{p},\sum_0^\infty\frac{V^{(n)}}{n!}\hat{x}^n\right]=\cdots=-i\hbar V'(\hat{x})$$

1. Position average

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{x}\rangle&=\frac{1}{i\hbar}\langle\psi(t)|\left[\hat{x},\hat{H}\right]|\psi(t)\rangle\\&=\frac{\langle\hat{p}\rangle}{m}\end{aligned}$$

We are familiar with this in classical mechanics.

2. Momentum average

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{p}\rangle&=\frac{1}{i\hbar}\langle\psi(t)|\left[\hat{p},\hat{H}\right]|\psi(t)\rangle\\&=\frac{1}{i\hbar}\langle\psi(t)|\left(-i\hbar V'(\hat{x})\right)|\psi(t)\rangle\\&=-\langle V'(\hat{x})\rangle\end{aligned}$$

In classical mechanics, the derivative of potential is force. And the result is just like Newton's 2n Law except the right hand side is not exactly like a force which should be $-\frac{\mathrm{d}}{\mathrm{d}x}\langle V(\hat{x})\rangle$.

What does $-\langle V'(\hat{x})\rangle$ mean

Suppose the potential area is fairly small and distributed around some coordinate $x_0=\langle\hat{x}\rangle$, we can do Taylor expansion around x_0.

$$\begin{aligned}<V(\hat{x})>&=V(x_0)+V'(x_0)<(x-x_0)>+V''(x_0)<(x-x_0)^2>/2+\cdots\\&=V(x_0)+0+V''(x_0)(\Delta x)^2+\cdots\end{aligned}$$

If the uncertainty is small enough, every term except the first one becomes small. So to the lowest order, average of potential is approximately the potential at x_0.

Similarly, the average of first derivative of potential $<V'(\hat{x})>$ is approximately $V'(x_0)$.

These gives us a hint for the previous result we got for the time evolution of average momentum. The result reduces to classical mechanics one as long as we keep the lowest order of Taylor expansion. Those higher order terms show the quantum effect.

Picture

We can see deeper into Ehrenfest's Theorem through Heisenberg Picture of quantum mechanics.

Schrödinger & Heisenberg Pictures

Pictures are the ways we look at the evolution of systems.

Schrödinger Picture In Schrödinger picture the states are evolving with time.

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \left|\psi\right\rangle_S = \hat{H} \left|\psi\right\rangle_S$$

And for time independent Hamiltonian,

$$\left|\psi\right\rangle_S = U^\dagger \left|\psi_0\right\rangle_S$$

Heisenberg Picture In Heisenberg Picture, the states do not change with time.

$$\left|\psi\right\rangle_H = \left|\psi_0\right\rangle_H ,$$

and of course the initial is the same with Schrödinger Picture,

$$\left|\psi_0\right\rangle_H = \left|\psi_0\right\rangle_S .$$

How do we relate to Heisenberg Picture to Schrödinger Picture? Through investigation of observables. We should have the same observation results in both Pictures.

$$\begin{gathered}
{}_H\left\langle\psi\right| \hat{\Omega}_H \left|\psi\right\rangle_H = {}_S\left\langle\psi\right| \hat{\Omega}_S \left|\psi\right\rangle_S \\
{}_H\left\langle\psi\right| \hat{\Omega}_H \left|\psi\right\rangle_H = {}_S\left\langle\psi_0\right| \hat{U}^\dagger \hat{\Omega}_S \hat{U} \left|\psi_0\right\rangle_S \\
\hat{\Omega}_H = \hat{U}^\dagger \hat{\Omega}_S \hat{U} \\
So the operators change with time in Heisenberg Picture.
\end{gathered}$$

Ehrenfest's Theorem in Heisenberg Picture

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Omega}_H = \frac{1}{i\hbar}\left[\hat{\Omega}_H, \hat{H}\right] + \hat{U}^\dagger \frac{\partial}{\partial t}\Omega_H \hat{U}$$

This can be easily proved by throwing every definition need in to it. We also need the following equations.

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{U} = \frac{\mathrm{d}}{\mathrm{d}t} e^{-i\hat{H}t/\hbar} = \frac{\hat{H}}{i\hbar}\hat{U}$$

And REMEMBER that propagator commute with time independent Hamiltonian, so

$$\hat{H} = \hat{U}^\dagger \hat{U} \hat{H} = \hat{U}^\dagger \hat{U} \hat{U} \equiv \hat{H}_H$$

So this Ehrenfest's Theorem can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\Omega}_H = \frac{1}{i\hbar}\left[\hat{\Omega}_H, \hat{H}_H\right] + \hat{U}^\dagger \frac{\partial}{\partial t}\Omega_H \hat{U}$$

We can **define**

$$\frac{\partial}{\partial t}\hat{\Omega}_H \equiv \hat{U}^\dagger \frac{\partial}{\partial t}\hat{\Omega}_S \hat{U},$$

which is the time derivative of operator in Heisenberg Picture.

Reminder: The time derivative of an observable (average) depends not only the time derivative of itself, but also the commutator of the observable and Hamiltonian.

Example of Ehrenfest's Theorem in Heisenberg Picture We will show why it is better to work in Heisenberg Picture to show the meanings of Ehrenfest's Theorem.

Suppose we have a Hamiltonian in Heisenberg Picture,

$$\hat{H}_H = \frac{\hat{p}_H^2}{2m} + V(\hat{x}_H).$$

Time derivative of position operator

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_H = \frac{1}{i\hbar}\left[\hat{x}_H, \hat{H}_H\right] = \frac{\hat{p}_H}{m}$$

Time derivative of momentum operator

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{p}_H = \frac{1}{i\hbar}\left[\hat{p}_H, \hat{H}\right] = -V'(\hat{x}_H)$$

So the operator in Heisenberg Picture just have a sense of the physical quantities in classical mechanics. That's why we like it.

Conservation

We say a observable is conserved if the corresponding operator commutes with Hamiltonian,

$$\left[\hat{\Omega}, \hat{H}\right] = 0$$

1. Energy Hamiltonian always commutes with itself.

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\epsilon\rangle = \langle\psi|\left(\frac{\partial}{\partial t}\hat{H}\right)|\psi\rangle$$

If Hamiltonian is time independent, then energy is conserved. (If Hamiltonian is tide dependent, energy is not conserved. This is kind of obvious in classical mechanics.)

What is the nature of time dependence

We can see this by looking at a simple example.

Assume we have a system with energy eigenstates $|\epsilon_n\rangle$, and initially,

$$|\psi_0\rangle = \sum_n C_n |\epsilon_n\rangle .$$

So

$$|\psi(t)\rangle = \sum_n C_n e^{-i\epsilon_n t/\hbar} |\epsilon_n\rangle .$$

We can calculate the expectation value of some operator $\hat{\Omega}$,

$$\langle\omega(t)\rangle = \sum_{n,m} \left(C_n^* e^{i\epsilon_n t/\hbar} \langle\epsilon_n|\right) \hat{\Omega} \left(C_m e^{-i\epsilon_m t/\hbar} |\epsilon_m\rangle\right) = \sum_{n,m} C_n^* C_m e^{-i(\epsilon_m - \epsilon_n)t/\hbar} \langle\epsilon_n| \hat{\Omega} |\epsilon_m\rangle$$

If $|\epsilon_n\rangle$ are also the eigenvectors of $\hat{\Omega}$, then

$$\langle\epsilon_n| \hat{\Omega} |\epsilon_m\rangle = \omega_m \delta_{n,m}$$

And the expectation value

$$\langle\omega(t)\rangle = \sum_n C_n^* C_n \omega_n$$

The important thing is that the time dependence of this expectation value actually arise from this term

$$e^{-i(\epsilon_m - \epsilon_n)t/\hbar}.$$

As it is so important, we call

$$(\epsilon_m - \epsilon_n)/\hbar$$

Bohr frequency.

5.2.5 Harmonic Oscillators

Why Harmonic Oscillators

Many systems can reduce to it. Use Taylor expansion for the potential and redefine parameters we will find harmonic oscillators in the potential.

Hamiltonian for 1D is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} k \hat{x}^2$$

Standard Solution

We can use polynomial expansion for part of the solution.

Dimension Schrodinger Equation

First step is always finding out the characteristic length scale and characteristic energy scale. Assume we have an characteristic length η and characteristic energy scale ϵ_0. Through uncertainty principle we know only for dimensional analysis

$$[\hat{p}] = \frac{\hbar}{\eta}$$

Kinetic energy and potential energy have the same dimension

$$\frac{\hbar^2}{\eta^2 m} = k\eta^2,$$

so we have

$$\eta = \sqrt{\frac{\hbar}{m\omega}}$$

with $\omega^2 = k/m$. A dimensional analysis shows that $\epsilon_0 = \hbar\omega$.

Now we can define dimensionless variables,

$$z = x/\eta, e = \epsilon/\epsilon_0$$

The time independent Schrodinger equation in position basis is

$$-\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi''/m + kx^2 = 2\epsilon\psi.$$

Using those characteristic scales, we can rewrite this equation into a dimensionless one, which is

$$\psi'' + (2e - z^2)\psi = 0$$

in which $\psi' = \frac{\mathrm{d}}{\mathrm{d}z}\psi$.

Take Limits

We need to look at the behavior of the solutions before we can guess a proper general solution.

$z \to \infty$, we have $\psi'' - z^2\psi = 0$. Solution to this equation is $\psi(z)\ e^{-z^2/2}$.

The solution of the the equation should be in the form

$$\psi(z) = u(z)e^{-z^2/2}.$$

Insert this to time independent Schrodinger equation, we can get the equation of $u(z)$.

$$u'' - 2zu' + (2e-1)u = 0$$

Polynomial Method

The simplest form of $u(z)$ is polynomial,

$$u(z) = \sum_{n=0}^{\infty} u_n z^n.$$

Put this back to equation of u, we can get the recursion relation,

$$(n+2)(n+1)u_{n+2} = [2n - (2e-1)]\, u_n.$$

If u_0 and u_1 are given, we can get the whole polynomial.

Notice that we have definite parity here. So u_1 branch vanish because they are even.

u_0 is set by the normalization condition.

Terminate The Series

The series blow up if it doesn't terminate. So we need to terminate the series using the following relation,

$$2e - 1 = 2n.$$

Then we have the energy levels, which is $e = n + 1/2$.

Complete Series

By picking proper normalization factor, we can write down the energy levels and corresponding wave functions. In fact, this polynomial can be found in mathematical phyisics books.

$$H_{n+1} = 2zH_n - nH_{n-1}$$

Tricky Solution

Find out the characteristic length and energy

$$\eta = \sqrt{\frac{\hbar}{m\omega}}$$
$$\epsilon = \hbar\omega$$
$$\omega = \sqrt{\frac{k}{m}}$$

One way to get the intrinsic length without writing down the dimensions of each quantity is to use the following relation

$$\begin{aligned}[E] &= \left[m\omega^2\hat{x}^2\right] \\ \hbar\omega &= m\omega^2\eta^2 \\ \eta &= \sqrt{\frac{\hbar}{m\omega}}\end{aligned}$$

Or if we are given the Hamiltonian in terms of k,

$$\begin{aligned}\left[\frac{\hat{p}^2}{2m}\right] &= \left[k\hat{x}^2\right] \\ \frac{\hbar^2/\eta^2}{m} &= k\eta^2 \\ \eta = \sqrt{\hbar}\sqrt{mk} &= \sqrt{\hbar}m\omega\end{aligned}$$

Rewrite the Hamiltonian

$$\begin{aligned}\hat{H} &= \frac{1}{2m}\left[\left(\frac{\hat{p}}{\hbar/\eta}\right)^2\left(\frac{\hbar}{\eta}\right)^2 + \frac{1}{2}m\omega^2\left(\frac{\hat{x}}{\eta}\right)^2\right] \\ &= \frac{1}{2}\hbar\omega\left[\left(\frac{\hat{p}}{\hbar/\eta}\right)^2 + \left(\frac{\hat{x}}{\eta}\right)^2\right] \\ &= \frac{1}{2}\hbar\omega\left(\frac{\hat{x}}{\eta} - i\frac{\hat{p}}{\hbar/\eta}\right)\left(\frac{\hat{x}}{\eta} + i\frac{\hat{p}}{\hbar/\eta}\right) - \frac{i}{\hbar}\left[\hat{x},\hat{p}\right] \\ &= \frac{1}{2}\hbar\omega(\sqrt{2}\hat{a}^\dagger\sqrt{2}\hat{a} + 1) \\ &= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\end{aligned}$$

Now we can define $\hat{a}^\dagger\hat{a} = \hat{N}$, which is just like an operator for (energy) quanta numbers.

An impoertan relation is

$$\begin{aligned}\left[\hat{a},\hat{a}^\dagger\right] &= 1 \\ \left[\hat{a},\hat{N}\right] &= \hat{a}\end{aligned}$$

The eigen equation for this weird energy quanta number operator is

$$\hat{N}\left|n\right\rangle = n\left|n\right\rangle$$

To find out the eigen state of $\hat{a}$ and $\hat{a}^\dagger$, we try this,

$$\begin{aligned}\hat{N}(\hat{a}\left|n\right\rangle) &= (n-1)(\hat{a}\left|n\right\rangle) \\ \hat{N}(\hat{a}^\dagger\left|n\right\rangle) &= (n+1)(\hat{a}^\dagger\left|n\right\rangle)\end{aligned}$$

This means $\hat{a}\,|n\rangle$ and $\hat{a}^\dagger\,|n\rangle$ are also eigen states of $hatN$.

The next step is very crucial. Since $\hat{a}\,|n\rangle$ and $\hat{a}^\dagger\,|n\rangle$ are eigen states of $hatN$, we know that

$$\hat{a}\,|n\rangle = C1\,|n\rangle$$
$$\hat{a}^\dagger\,|n\rangle = C2\,|n\rangle$$

Then our next step is to find out what are $C1$ and $C2$ exactly.

They way of finding them is to use invariant quantities, such as the inner product. Here we use average of $\hat{N}$ operator.

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$$
$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Final step is to constrain on n, which should be integrals. This is true because we need a cut off for the eigen equation of $\hat{N}$, whose avarage is n and it should be non negative.

$$\langle n|\,\hat{N}\,|n\rangle \geq 0$$

leads to $n \geq 0$. To get this proper cut off, n should be integer because if it's not, according to

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

n can go to negative numbers. If n is positive integer,

$$\hat{a}\,|1\rangle = |0\rangle$$
$$\hat{a}\,|0\rangle = 0\,|0\rangle$$

show an cut off at 0.

We can even find out the wave functions of these $|n\rangle$ by finding the ground state first and apply $\hat{a}^\dagger$ to the ground state.

Ground state in $|x\rangle$ basis can be found by solving the differential equation,

$$\langle x|\,\hat{a}\,|0\rangle = 0$$

Very important:

- The Hermitian conjugate of $\hat{a}\,|n\rangle$ is $\langle n|\,\hat{a}^\dagger$.
- Hermitian conjugate of $\hat{a}\hat{a}^\dagger$ is $\hat{a}\hat{a}^dagger$. This can be a trap. Hermitian conjugate is the complex conjugate AND TRANSPOSE!

Semiclassical

Classical

In phase space, the trajectory of phase space points ($\{x/\eta$ and $p/(\hbar/\eta)\}$) is on a circle of radius x_{max}/η.

Quantum semiclassical

Key points:

1. What is the trajectory of $\langle \hat{x}/\eta \rangle$ and $\langle \hat{p}/(\hbar/\eta) \rangle$
2. Can we make the trajectory just like the classical case by choosing some special conditions?
3. What do these special cases mean?

- Expectation value of creation and annihilation operators

Apply Ehrenfest theorem to annihilation operator,

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \langle \hat{a}(t) \rangle = \langle \psi | \left[\hat{a}(t), \hat{H} \right] | \psi \rangle = \hbar\omega \langle \hat{a}(t) \rangle$$

Excellent. Now we can solve out $\langle \hat{a}(t) \rangle$, which is

$$\langle \hat{a}(t) \rangle = \alpha_0 \exp(-i\omega t)$$

Take the hermitian conjugate,

$$\left\langle \hat{a}^\dagger(t) \right\rangle = \alpha_0^* \exp(i\omega t)$$

- Expectation value of position and momentum

With these two operators, we can find out the average of $\hat{x}$ and $\hat{p}$ because

$$\begin{aligned} \hat{x} &= \eta \frac{1}{\sqrt{2}} \left(\hat{a}^\dagger + \hat{a} \right) \\ \hat{p} &= \frac{\hbar}{\eta} i \frac{1}{\sqrt{2}} \left(\hat{a}^\dagger - \hat{a} \right), \end{aligned}$$

we have

$$\begin{aligned} \langle \hat{x}(t) \rangle &= \eta \frac{1}{\sqrt{2}} \left(\left\langle \hat{a}^\dagger(t) \right\rangle + \langle \hat{a}(t) \rangle \right) \\ \langle \hat{p}(t) \rangle &= \frac{\hbar}{\eta} i \frac{1}{\sqrt{2}} \left(\left\langle \hat{a}^\dagger(t) - \langle \hat{a}(t) \rangle \right\rangle \right) \end{aligned}$$

We can have a look at these two averages,

$$\begin{aligned} \frac{\langle \hat{x}(t) \rangle}{\eta} &= \frac{1}{\sqrt{2}} \left[(\alpha_0 + \alpha_0^*) \cos(\omega t) + i(\alpha_0^* - \alpha_0) \sin(\omega t) \right] \\ \frac{\langle \hat{p}(t) \rangle}{\hbar/\eta} &= \frac{1}{\sqrt{2}} \left[(\alpha_0 + \alpha_0^*) \sin(\omega t) + i(\alpha_0 - \alpha_0^*) \cos(\omega t) \right] \end{aligned}$$

It is obvious that the average reduces to classical case if $\alpha_0 = \alpha_0^*$. **But this is too strong for a semiclassical limit.**

- Coherent state

Coherent state is the eigenstate of creation operator. Its wave package has the smallest spread allowed by quantum mechanics.

The most special part about coherent state is that the system stays on coherent state if it start with coherent state.

$$\hat{a} \left|\alpha(t)\right\rangle = \alpha(t) \left|\alpha(t)\right\rangle$$

Take the hermitian conjugate,

$$\left\langle\alpha(t)\right| \hat{a}^\dagger = \left\langle\alpha(t)\right| \alpha(t)^*$$

At $t = 0$, we have

$$\left\langle\psi(0)\right| N \left|\psi(0)\right\rangle = |\alpha_0|^2$$

That is to say, energy should be

$$\left\langle\psi(0)\right| \hat{H} \left|\psi(0)\right\rangle = \hbar\omega \left(|\alpha_0|^2 + \frac{1}{2}\right)$$

Initially, we also have

$$\left\langle\psi(0)\right| (\hat{a} - \alpha_0)^\dagger (\hat{a} - \alpha_0) \left|\psi(0)\right\rangle = 0$$

This means

$$\hat{a} \left|\psi(0)\right\rangle = \alpha_0 \left|\psi(0)\right\rangle$$

- Coherent state expanded using energy eigenstates

(This result)

(To Be Finished...)

5.3 Quantum Mechanics 2

5.3.1 Tensor Product Space

This part has been moved to *Tensor Product Space*

5.3.2 Density Matrix

5.3.3 Angular Momentum

Angular Momentum

For an new operator, we would like to know

1. Commutation relation: with their own components, with other operators;
2. Eigenvalues and their properties;
3. Eigenstates and their properties;
4. Expectation and classical limit.

Definition of Angular Momentum

In classical mechanics, angular momentum is defined as

$$\vec{L} = \vec{X} \times \vec{P}.$$

One way of defining operator is to change position and momentum into operators and check if the operator is working properly in QM. So we just define

$$\hat{\vec{L}} = \hat{\vec{X}} \times \hat{\vec{P}}.$$

It is Hermitian. So it can be an operator. We also find

$$\hat{\vec{L}} \times \hat{\vec{L}} = i\hbar\hat{\vec{L}}$$

$$\left[\hat{L}_i, \hat{L}_j\right] = \sum_k i\epsilon_{ijk}\hat{L}_k.$$

More generally, we can define angular momentum as

$$\left[\hat{J}_i, \hat{J}_j\right] = i\hbar\sum_k \epsilon_{ijk}\hat{J}_k$$

We can prove that

$$\left[\hat{J}^2, \hat{J}_z\right] = 0.$$

So they can have the same eigenstates

$$\hat{J}_z \left|\lambda m\right\rangle = m\hbar \left|\lambda m\right\rangle$$

$$\hat{J}^2 \left|\lambda m\right\rangle = \lambda^2\hbar^2 \left|\lambda m\right\rangle$$

To find the constraints on these eigenvalues, we can use positive definite condition of certain inner porducts, such as,

$$\left\langle\psi\right| \hat{J}_+\hat{J}_- \left|\psi\right\rangle \geq 0$$

$$\langle\psi|\,\hat{J}_-\hat{J}_+\,|\psi\rangle \geq 0$$

where

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$$

and we have

$$\left[\hat{J}_+, \hat{J}_-\right] = 2\hbar\hat{J}_z$$

$$\left[\hat{J}_z, \hat{J}_\pm\right] = \pm\hbar\hat{J}_\pm.$$

It's easy to find out that

$$\hat{J}_z(\hat{J}_\pm\,|\lambda m\rangle) = (m\pm 1)\hbar(\hat{J}_\pm\,|\lambda m\rangle)$$

i.e., $\hat{J}_\pm\,|\lambda m\rangle$ is eigenstate of $\hat{J}_z$.

Follow the plan of finding out the bounds through these positive inner products, we can prove that

$$\hat{J}^2\,|jm\rangle = j(j+1)\hbar^2\,|jm\rangle$$

$$\hat{J}_\pm\,|jm\rangle = \sqrt{j(j+1)-m(m\pm 1)}\hbar\,|j, m\pm 1\rangle$$

Eigenstates of Angular Momentum

As we have proposed, the eigenstates of both $\hat{J}_z$ and $\hat{\hat{J}}^2$ are $|j,m\rangle$, where $j = 0, 1, 2, \cdots$ and $m = -j, -j+1, \cdots, j-1, j$.

We can also find out the wave function in $|\theta,\phi\rangle$ basis. Before we do that, the definition of this basis should be made clear. This basis spans the surface of a 3D sphere in Euclidean space and satisfies the following orthonormal and complete condition.

$$\int \mathrm{d}\Omega\langle\theta',\phi' \mid \theta,\phi\rangle = \delta(\cos\theta' - \cos\theta, \phi'-\phi)\int \mathrm{d}\Omega\,|\theta',\phi'\rangle\,\langle\theta,\phi| = 1$$

Now we have an arbitary state $|\psi\rangle$,

$$\begin{aligned}
|\psi\rangle &= \sum_{l,m}\psi_{lm}\,|l,m\rangle \\
&= \sum_{l,m}\int \mathrm{d}\Omega\,|\theta',\phi'\rangle\,\langle\theta,\phi|\,\psi_{lm}\,|l,m\rangle \\
&= \sum_{l,m}\int \mathrm{d}\Omega\,|\theta',\phi'\rangle\,(\langle\theta,\phi \mid l,m\rangle)\psi_{lm}
\end{aligned}$$

Then we define

$$\langle \theta, \phi \mid l, m \rangle = Y_l^m(\theta, \phi)$$

which is the spherical harmonic function.

Then

$$|\psi\rangle = \sum_{l,m} \psi_{lm} \int \mathrm{d}\Omega Y_l^m(\theta, \phi) \, |\theta', \phi'\rangle$$

So as long as we find out what ψ_{lm} is, any problem is done.

5.4 Quantum Mechanics 3

This part has been moved to http://emptymalei.github.io/quantum/ .

5.5 Statistical Physics

This part has been moved to here .

CHAPTER

SIX

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