Research Survival Handbook (**Unfinished**)

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Contents

Preface

I have a bad memory, very bad. So bad that I can hardly remember anything.

I tried many many ways of pushing myself to the frontier of physics. That bad memory really pissed me off. So I decided to borrow the power of paper and computer.

This is only a draft handbook for myself in principal. However, I believe everyone need a handbook of his/her area and my version of handbook might be helpful for some people working on similiar things with mine.

Part I

Fundamental Physics

Chapter 1

Basic

1.1 Dimension

How to find the relationship between two quantities? For example, what is the dimensional relationship between length and mass.

Plank constant: $\hbar \sim [Energy] \cdot [Time] \sim [Mass] \cdot [Length]^2 \cdot [Time]^{-1}$
Speed of light in vacuum: $c \sim [Length] \cdot [Time]^{-1}$
Gravitational constant: $G \sim [Length]^3 \cdot [Mass]^{-1} \cdot [Time]^{-2}$

Then it is easy to find that a combination of $c/\hbar$ cancels the dimension of mass and leaves the inverse of length. That is

$$[Length]^2 = \frac{\hbar G}{c^3} \tag{1.1}$$

1.2 Equations That Should Never Be Forgotten

1.2.1 Electrodynamics

Maxwell Equations

$$\begin{aligned}
\nabla \times \vec{E} &= -\partial_t \vec{B} && (1.2)\\
\nabla \times \vec{H} &= \vec{J} + \partial_t \vec{D} && (1.3)\\
\nabla \cdot \vec{D} &= \rho && (1.4)\\
\nabla \cdot \vec{B} &= 0 && (1.5)
\end{aligned}$$

For linear meterials,

$$\begin{aligned}
\vec{D} &= \epsilon \vec{E} && (1.6)\\
\vec{B} &= \mu \vec{H} && (1.7)\\
\vec{J} &= \sigma \vec{E} && (1.8)
\end{aligned}$$

Hamilton conanical equations

$$\begin{aligned}
\dot{q}_i &= \frac{\partial H}{\partial p_i} && (1.9)\\
\dot{p}_i &= -\frac{\partial H}{\partial q_i} && (1.10)
\end{aligned}$$

Liouville's Law

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} \equiv \frac{\partial \rho}{\partial t} + \sum_i \left[\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i \right] = 0 \tag{1.11}$$

Part II

Advanced Physics

Chapter 2

Relativy

Part III

Tools

Chapter 3

Mathematics

3.1 Differential Geometry

3.1.1 Metric

Definations

Denote the basis in use as $\hat{e}_\mu$, then the metric can be written as

$$g_{\mu\nu} = \hat{e}_\mu \hat{\cdot} e_\nu \tag{3.1}$$

if the basis satisfies

Inversed metric

$$g_{\mu\lambda} g^{\lambda\nu} = \delta_\mu^\nu = g_\mu^\nu \tag{3.2}$$

How to calculate the metric

Let's check the definition of metric again.

If we choose a basis $\hat{e}_\mu$, then a vector (at one certain point) in this coordinate system is

$$x^a = x^\mu \hat{e}_\mu \tag{3.3}$$

Then we can construct the expression of metric of this point under this coordinate system,

$$g_{\mu\nu} = \hat{e}_\mu \cdot \hat{e}_\nu \tag{3.4}$$

For example, in spherical coordinate system,

$$\vec{x} = r \sin\theta \cos\phi \hat{e}_x + r \sin\theta \sin\phi \hat{e}_y + r \cos\theta \hat{e}_z \tag{3.5}$$

Now we have to find the basis under spherical coordinate system. Assume the basis is $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$. Choose some scale factors $h_r = 1, h_\theta = r, h_\phi = r\sin\theta$. Then the basis is $\hat{e}_r = \frac{\partial \vec{x}}{h_r \partial r} = \hat{e}_x \sin\theta\cos\phi + \hat{e}_y \sin\theta \sin\phi + \hat{e}_z \cos\theta$, etc. Then collect the terms in formula 3.5 is we get $\vec{x} = r\hat{e}_r$, this is incomplete. So we check the derivative.

$$
\begin{aligned}
\mathrm{d}\vec{x} &= \hat{e}_x(\mathrm{d}r \sin\theta\cos\phi + r\cos\theta\cos\phi\mathrm{d}\theta - r\sin\theta\sin\phi\mathrm{d}\phi) && (3.6)\\
& \quad \hat{e}_y(\mathrm{d}r\sin\theta\sin\phi + r\cos\theta\sin\phi\mathrm{d}\theta + r\sin\theta\cos\phi\mathrm{d}\phi) && (3.7)\\
& \quad \hat{e}_z(\mathrm{d}r\cos\theta - r\sin\theta\mathrm{d}\theta) && (3.8)\\
&= \mathrm{d}r(\hat{e}_x\sin\theta\cos\phi + \hat{e}_y\sin\theta\sin\phi - \hat{e}_z\cos\theta) && (3.9)\\
& \quad \mathrm{d}\theta(\hat{e}_x\cos\theta\cos\phi + \hat{e}_y\cos\theta\sin\phi - \hat{e}_z\sin\theta)r && (3.10)\\
& \quad \mathrm{d}\phi(-\hat{e}_x\sin\phi + \hat{e}_y\cos\phi)r\sin\theta && (3.11)\\
&= \hat{e}_r\mathrm{d}r + \hat{e}_\theta r\mathrm{d}\theta + \hat{e}_\phi r\sin\theta\mathrm{d}\phi && (3.12)
\end{aligned}
$$

Once we reach here, the component (e_r, e_θ, e_ϕ) of the point under the spherical coordinates system basis ($\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$) at this point are clear, i.e.,

$$\begin{aligned} \mathrm{d}\vec{x} &= \hat{e}_r \mathrm{d}r + \hat{e}_\theta r \mathrm{d}\theta + \hat{e}_\phi r \sin\theta \mathrm{d}\phi && (3.13)\\ &= e_r \mathrm{d}r + e_\theta \mathrm{d}\theta + e_\phi \mathrm{d}\phi && (3.14)\end{aligned}$$

In this way, the metric tensor for spherical coordinates is

$$g_{\mu\nu} = (e_\mu \cdot e_\nu) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{3.15}$$

3.1.2 Connection

First class connection can be calcuated

$$\Gamma^\mu{}_{\nu\lambda} = \hat{e}^\mu \cdot \hat{e}_{\mu,\lambda} \tag{3.16}$$

Second class connection is[1]

$$[\mu\nu, \iota] = g_{\iota\mu}\Gamma^\mu{}_{\nu\lambda} \tag{3.17}$$

3.1.3 Gradient, Curl, Divergence, etc

Gradient

$$T^b{}_{c;a} = \nabla_a T^b{}_c = T^b{}_{c,a} + \Gamma^b_{ad} T^d{}_c - \Gamma^d_{ac} T^b{}_d \tag{3.18}$$

Curl For an anti-symmetric tensor, $a_{\mu\nu} = -a_{\nu\mu}$

$$\begin{aligned} \mathrm{Curl}_{\mu\nu\tau}(a_{\mu\nu}) &\equiv a_{\mu\nu;\tau} + a_{\nu\tau;\mu} + a_{\tau\mu;\nu} && (3.19)\\ &= a_{\mu\nu,\tau} + a_{\nu\tau,\mu} + a_{\tau\mu,\nu} && (3.20)\end{aligned}$$

Divergence

$$\begin{aligned} \mathrm{div}_\nu(a^{\mu\nu}) &\equiv a^{\mu\nu}{}_{;\nu} = \frac{\partial a^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\nu\tau} a^{\tau\nu} + \Gamma^\nu_{\nu\tau} a^{\mu\tau} && (3.21)\\ &= \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu}(\sqrt{-g} a^{\mu\nu}) + \Gamma^\mu_{\nu\lambda} a^{\nu\lambda} && (3.22)\end{aligned}$$

For an anti-symmetric tensor

$$\mathrm{div}(a^{\mu\nu}) = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu}(\sqrt{-g} a^{\mu\nu}) \tag{3.23}$$

[1] Kevin E. Cahill

Annotation Using the relation $g = g_{\mu\nu}A_{\mu\nu}$, $A_{\mu\nu}$ is the algebraic complement, we can prove the following two equalities.

$$\Gamma^{\mu}_{\mu\nu} = \partial_\nu \ln \sqrt{-g} \tag{3.24}$$

$$V^{\mu}{}_{;\mu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{-g}V^{\mu}) \tag{3.25}$$

In some simple case, all the three kind of operation can be demonstrated by different applications of the del operator, which $\nabla \equiv \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z$.
Gradient, ∇f, in which f is a scalar.
Divergence, $\nabla \cdot \vec{v}$
Curl, $\nabla \times \vec{v}$ Laplacian, $\Delta \equiv \nabla \cdot \nabla \equiv \nabla^2$

3.2 Linear Algebra

3.2.1 Basic Concepts

Trace Trace should be calculated using the metrc. An example is the trace of Ricci tensor,

$$R = g^{ab}R_{ab} \tag{3.26}$$

Einstein equation is

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G T_{ab} \tag{3.27}$$

The trace is

$$\begin{aligned} g^{ab}R_{ab} - \frac{1}{2}g^{ab}g_{ab}R &= 8\pi G g^{ab}T_{ab} && (3.28)\\ \Rightarrow R - \frac{1}{2}4R &= 8\pi G T && (3.29)\\ \Rightarrow -R &= 8\pi G T && (3.30) \end{aligned}$$

3.3 Differential Equations

3.3.1 Standard Procedure

3.3.2 Tricky

WKB Approximation When the highest derivative is multiplied by a small parameter, try this.

Chapter 4

Cosmology

4.1 What's in the begining

In cosmology, the frame used most frequently is the cosmic microwave background radiation (CMB) stationary frame, not earth frame. We sometimes say it is earth frame because the speed of earth relative to CMB is rather small compared to the galaxies' movement observed by we earth beings.

4.2 Constants And Physical Quantities

- Deceleration parameter of today,

  $$q = -\left(\frac{a}{\dot{a}}\ddot{a}\right) = -\frac{\ddot{a}}{\dot{a}}\frac{1}{H(a)}$$

  Deceleration parameter is tightly related to Friedmann equation. [1]

4.2.1 Cosmographic Parameters

Co-Graphy-Par-1 Recession speed

$$v = H_0 \cdot d \tag{4.1}$$

Co-Graphy-Par-2 Hubble time

$$t_H = \frac{1}{H_0} \tag{4.2}$$

Co-Graphy-Par-3 Hubble distance

$$D_H = \frac{c}{H_0} = c \cdot t_H \tag{4.3}$$

Co-Graphy-Par-4 Dimensionless density parameters

Matter

$$\Omega_M = \frac{8\pi G \rho_M|_{a=1}}{3H_0^2} \tag{4.4}$$

Lambda

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} \tag{4.5}$$

Curvature can be viewed as some energy, too. From Friedmann equation we can sort out it. Or equivalently

$$\Omega_k = 1 - \Omega_M - \Omega_\Lambda \tag{4.6}$$

[1] Firedmann equation can be written in terms of deceleration parameter,

$$q = 1/2(1+3w)(1+k/(aH)^2)$$

Redshifts and Distances

Co-Dis-1 Redshift defined in observation

$$z = \frac{\nu_e}{\nu_o} - 1 = \frac{\lambda_o - \lambda_e}{\lambda_e} \tag{4.7}$$

Co-Dis-2 In linear range, given a radial velocity, redshift can be written as

$$z = \sqrt{\frac{1 + v/c}{1 - v/c}} - 1 \tag{4.8}$$

Co-Dis-3 Peculiar velocity

$$v_{pec} = c\frac{z_{obs} - z_{cos}}{1 + z}, \tag{4.9}$$

in which,z_{obs} is the observed redshift, while z_{cos} is the cosmological redshift. Cosmological redshift is the Hubble flow "due solely to the expansion of the universe". **This definition is valid when v_{pec} is much smaller than speed of light, i.e., $v_{pec} \ll c$.**

Co-Dis-4 Scale factor

$$a|_{t_0} = (1 + z)a|_{t_e}. \tag{4.10}$$

In cosmology, we often use another form which is derived from and setting $a|_{t_0} = 1$

$$a = \frac{1}{1 + z}. \tag{4.11}$$

This is the scale factor we used in line element.

Co-Dis-5 Relative redshift

$$z_{12} = \frac{a|_{t_1}}{a|_{t_2}} - 1 = \frac{1 + z_2}{1 + z_1} \tag{4.12}$$

Co-Dis-6 Line-of-sight comoving distance

$$D_c = D_H \int_0^z \frac{1}{E(z')} \mathrm{d}z' \tag{4.13}$$

in which $E(z) \equiv \sqrt{\Omega_M(1 + z)^3 + \Omega_\Lambda + \Omega_k(1 + z)^2}$.

Co-Dis-7 Transverse comoving distance

$$D_M = \frac{D_H}{\sqrt{|\Omega_k|}} f(\sqrt{|\Omega_k|}\frac{D_C}{D_H}) \tag{4.14}$$

in which

$$f(x) = \begin{cases} \text{sinh}(x), & \Omega_k > 0 \quad \text{hyperbola} \\ x, & \Omega_k = 0 \quad \text{flat} \\ \text{sin}(x), & \Omega_k < 0 \quad \text{parabola} \end{cases} . \tag{4.15}$$

Co-Dis-8 Angular diameter distance

$$D_A = \frac{1}{1+z} D_M \tag{4.16}$$

We just divide the transverse distances by $1+z$ The advantage of it is that it is not singular at $z \to \infty$

Co-Dis-9 Luminosity distance

$$D_L \equiv \sqrt{\frac{L}{4\pi S}} \tag{4.17}$$

in which L is bolometric luminosity[2], S is the bolometric flux.

Related to other distances

$$D_L = (1+z) D_M = (1+z)^2 D_A \tag{4.18}$$

Co-Dis-10 Distance modules

$$DM \equiv 5 \log D_L / 10 \mathrm{p}c \tag{4.19}$$

In Mpc unit,

$$DM \equiv 5 \log D_L + 25 \tag{4.20}$$

Co-Dis-11 Comoving volume

$$\mathrm{d}V_C = D_H \frac{(1+z)^2 D_\Lambda^2}{E(z)} \mathrm{d}\Omega \mathrm{d}z \tag{4.21}$$

Co-Dis-12 Lookback time

$$t_L = t_H \int_0^z \frac{\mathrm{d}z'}{(1+z')E(z')} \tag{4.22}$$

Lookback time is "the difference between the age t_o of the Universe now (at observation) and the age t_e of the Universe at the time the photons were emmitted (according to the object). "

Co-Dis-13 Probability of intersecting objects

$$\mathrm{d}P = n(z)\sigma(z) D_H \frac{(1+z)^2}{E(z)} \mathrm{d}z \tag{4.23}$$

"Given a population of objects with comoving number density $n(z)$ (number per unit volume) and cross section $\sigma(z)$ (area), what is the incremental probability dP atha a line of sight will intersect one of the objects in redshift interval dz at redshift z?"[3]

[2]Bolometric luminosity: The total energy radiated by an object at all wavelengths, usually given in joules per second.

[3]**arXiv:astro-ph/9905116v4**

This part is mostly taken from **arXiv:astro-ph/9905116v4**

Page 418 of Gravitation And Cosmology: Principles And Applications Of The General Theory Of Relativity written by Weinberg in 1972.

4.3 The Homogeneous and Isotropic Universe

It is always pointed out that most thoeries are based on the Cosmological Principle.

I have stated this principle of in the chapter telling the perturbation theory in cosmology.

> Empedocles: `God is an infinite sphere whose center is everywhere and circumference nowhere.'
>
> Cosmological principle states that VIEWED on a sufficiently large scale, the properties of the Universe are the same for all observers. The key to understand this is that "same" means same physics principles and physical constants. Wikipedia gives three qualifications and two testable consequences. The two consequences are isotropic and homogeneous. Isotropic indicates no matter where we are looking at the universe is the same while homogeneous indicates no matter where we are located the universe is about the same (i.e., we are looking at a fair sample of the universe).
>
> Problem is how to describe isotropy and homogeneity. [Liang, P360]

Actually, we have a rigorous form for the principle, as I mentioned in these paragraphs.

Homogeneous Space-time (M, g_{ab}), sliced into Σ_t, induced metric of g_{ab} on Σ_t. (M, g_{ab}) is spatially homogeneous if we can always find Σ_t, ensuring the existance of a set of isometry on Σ_t itself.

4.4 Quantities

4.4.1 Energy-momentum Tensor

Energy-momentum tensor can be made clear using a standard procedure of field theory.

Check out the table below. [From Ohanian and Ruffini's GRAVATATION AND SPACETIME, P494.]

	Particle	Field
Quantities	$q_i(t)$	$\phi(\vec{x},t)$
independent quantitie	t,i[?]	$\vec{x},t$
Lagrange	$L=L(q_i,\dot{q}_i)$	$L=\int\mathscr{L}(\phi,\partial\phi/\partial x^\mu)\mathrm{d}^3x$
Equation of motion	$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_i}-\frac{\partial L}{\partial q_i}=0$	$\frac{\partial}{\partial x^\mu}\frac{\partial\mathscr{L}}{\partial\phi_{,\mu}}-\frac{\partial\mathscr{L}}{\partial\phi}=0$
Hamiltonian	$H=\sum\dot{q}_i\frac{\partial L}{\partial\dot{q}}-L$	$H=\int(\phi_{,0}\frac{\partial\mathscr{L}}{\partial\phi_{,0}}-\mathscr{L})\mathrm{d}^3x$

The integrand in H is the energy density of the complete system. We would like to choose the ${t_0}^0$ component of a canonical energy-momentum tensor energy-momentum tensor to describe energy density,

$$ {t_0}^0=\phi_{,o}\frac{\partial\mathscr{L}}{\partial\phi_{,0}}-\mathscr{L}. \tag{4.24}$$

It can be proved that the following equation is the only one that satisfies the constrain we proposed before.

$$ {t_\mu}^\nu=\phi_{,\mu}\frac{\partial\mathscr{L}}{\partial\phi_{,\nu}}-\delta^\nu_\mu\mathscr{L} \tag{4.25}$$

We have the conservation law:

$$ \frac{\partial}{\partial x^\nu}{t_\mu}^\nu=0 \tag{4.26}$$

Start from this we can find the Hamiltonian is conserved.

$$ \frac{\mathrm{d}H}{\mathrm{d}t}=0 \tag{4.27}$$

The total momentum is conserved too.

$$ P_k=\int {t_k}^0\mathrm{d}^3x \tag{4.28}$$

This is all about a scalar field. If we switch to vector field (EM field) and tensor field (1-order gravitation field), the canonical energy-momentum tensors are

$$ {t_{(em)}}_\mu{}^\nu = A^\alpha_{,\mu}\frac{\partial\mathscr{L}_{(em)}}{\partial A^\alpha_{,\nu}}-\delta^\nu_\mu\mathscr{L}_{(em)} \tag{4.29}$$

$$ {t_{(g1)}}_\mu{}^\nu = h^{\alpha\beta}_{,\mu}\frac{\partial\mathscr{L}_{(g1)}}{\partial h^{\alpha\beta}_{,\nu}}-\delta^\nu_\mu\mathscr{L}_{(g1)} \tag{4.30}$$

In special relativity, the energy-momentum tensor for ideal fluid is

$$ T_{ab}=(\rho+p)U_aU_b+p\eta_{ab} \tag{4.31}$$

Each component has its physical meaning.

1. T^{00} is the energy density;
2. $T^{0k} = T^{k0}$ is the k momentum density (energy flux density);
3. $T^{kl} = T^{lk}$ is the k momentum flux density in l direction.

Just change all the η_{ab} into g_{ab}.
For any system,

$$\begin{aligned}
\nabla_\mu T^{\mu\nu} &= U^\mu U^\nu \nabla_\mu \rho + U^\mu U^\nu \nabla_\mu p + (\rho+p) U^\mu \nabla_\mu U^\nu + (\rho+p) U^\nu \nabla_\mu U^\mu + g^{\mu\nu}\nabla_\mu p && (4.32)\\
&= U^\mu U^\nu \dot{\rho} + (g^{\mu\nu} + U^\mu U^\nu)\nabla_\mu p + (\rho+p)A^\nu + (\rho+p)U^\nu \nabla_\mu \Theta && (4.33)\\
&= U^\mu U^\nu \dot{\rho} + (\rho+p)U^\nu \nabla_\mu \Theta && (4.34)\\
&= Q^\nu && (4.35)
\end{aligned}$$

A^ν vanishes because we have chosen a comoving observer. $\nabla_\mu p$ vanishes because in our standard model ρ, p are coordinate free. Actually, since U^μ is timelike, we have $(g^{\mu\nu} + U^\mu U^\nu)\nabla_\mu p = h^{\mu\nu}\nabla_\mu p$.

4.4.2 Friedmann Universe

Cosmological Principle Empedocles: `God is an infinite sphere whose center is everywhere and circumference nowhere.'

Cosmological principle states that VIEWED on a sufficiently large scale, the properties of the Universe are the same for all observers. The key to understand this is that "same" means same physics principles and physical constants. Wikipedia gives three qualifications and two testable consequences. The two consequences are isotropic and homogeneous. Isotropic indicates no matter where we are looking at the universe is the same while homogeneous indicates no matter where we are located the universe is about the same (i.e., we are looking at a fair sample of the universe).

Problem is how to describe isotropy and homogeneity. [Liang, P360]

Robertson-Walker Metric The next problem is how to set up a simple and useful coordinate system which metric to use in cosmology. [Liang, P366]

Evolution of Scale Factor Friedmann equation

$$\begin{aligned}
3(\dot{a}^2 + k)/a^2 &= 8\pi p && (4.36)\\
2\ddot{a}/a + (\dot{a}^2 + k)/a^2 &= -8\pi p && (4.37)
\end{aligned}$$

4.36 is called Friedmann equation.
It is better to use another set of equations which are identical to 4.36 and 4.37

$$\begin{aligned}
\ddot{a} &= -4\pi a(\rho + 3p)/3 && (4.38)\\
0 &= \dot{\rho} + 3(\rho + p)\dot{a}/a && (4.39)
\end{aligned}$$

These equations can be solved in some circumstances. [Liang, P376]
It could be useful to reform these equation.

$$H^2 + \frac{k}{a^2} = \frac{8}{3}\pi\rho \tag{4.40}$$

$$2\dot{H} + 3H^2 + \frac{k}{a^2} = -8\pi p \tag{4.41}$$

Mixing

$$3\ddot{a} = -4\pi a(\rho + 3p) \tag{4.42}$$

$$0 = \dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} \tag{4.43}$$

$$\dot{H} = -4\pi G(\rho + p) + \frac{k}{a^2} \tag{4.44}$$