Research Survival Handbook
(**Unfinished**)

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Contents

Preface

I have a bad memory that I can hardly remember anything.

To get rid of it, I tried many ways of help myself memorizing things and pushing myself to the frontier of physics. Finally, I decided to collect some of my notes together and established this project on github.http://cosmologytaskforce.github.com/CosmologyResearchSurvivingManual

This is only a draft handbook for myself in principal. However, I believe everyone need a handbook of his/her area and my version of handbook might be helpful for some people working on similiar things with mine.

This book can never not be delivered formally because I borrowed many resources in this book. Here I list the most important of them here.

- In the part of physics, I just take the source file of a note on GRE Physics subject test written by Lin Cong and typeset by Duncan Watts and rearranged the sections.http://www.hcs.harvard.edu/~physics/?q=node/13;http://www.hcs.harvard.edu/~physics/files/GRE\%20notes.tex Then I did modifications based on it. He said in his notes everyone is welcome to typeset and improve his notes but if you are going to use this document commercially you need to contact him first.

Part I

Fundamental Physics

Chapter 1

Basic

1.1 Dimension

How to find the relationship between two quantities? For example, what is the dimensional relationship between length and mass.

Plank constant: $\hbar \sim [Energy] \cdot [Time] \sim [Mass] \cdot [Length]^2 \cdot [Time]^{-1}$
Speed of light in vacuum: $c \sim [Length] \cdot [Time]^{-1}$
Gravitational constant: $G \sim [Length]^3 \cdot [Mass]^{-1} \cdot [Time]^{-2}$

Then it is easy to find that a combination of $c/\hbar$ cancels the dimension of mass and leaves the inverse of length. That is

$$[Length]^2 = \frac{\hbar G}{c^3} \tag{1.1}$$

1.2 Most Wonderful Equations That Should Never Be Forgotten

1.2.1 Electrodynamics

Maxwell Equations

$$\begin{aligned}
\nabla \times \vec{E} &= -\partial_t \vec{B} && (1.2)\\
\nabla \times \vec{H} &= \vec{J} + \partial_t \vec{D} && (1.3)\\
\nabla \cdot \vec{D} &= \rho && (1.4)\\
\nabla \cdot \vec{B} &= 0 && (1.5)
\end{aligned}$$

For linear meterials,

$$\begin{aligned}
\vec{D} &= \epsilon \vec{E} && (1.6)\\
\vec{B} &= \mu \vec{H} && (1.7)\\
\vec{J} &= \sigma \vec{E} && (1.8)
\end{aligned}$$

Hamilton conanical equations

$$\begin{aligned}
\dot{q}_i &= \frac{\partial H}{\partial p_i} && (1.9)\\
\dot{p}_i &= -\frac{\partial H}{\partial q_i} && (1.10)
\end{aligned}$$

Liouville's Law

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} \equiv \frac{\partial \rho}{\partial t} + \sum_i \left[\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i\right] = 0 \tag{1.11}$$

Chapter 2

Classical Mechanics

- A worked example on velocity and acceleration in a curved path in a a plane: (the idea is to skillfully use $d(AB) = AdB + BdA$. This applies to change of momentum as well.)

  $$\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta, \qquad \hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$

  $$\hat{v} = \frac{d(R\hat{r})}{dt} = \frac{dR}{dt}\hat{r} + R\frac{d\hat{r}}{dt} = \dot{R}\hat{r} + R\omega\hat{\theta}$$

  Similarly,

  $$\vec{a} = (\ddot{R} - R\omega^2)\hat{r} + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{\theta}$$

- Firing rocket

  $$(v_g - v)dM + d(MV) = 0$$

  M is rocket mass, v is speed, v_g is relative speed of the waste fired out.

- Bernoulli's equation

  $$P + \frac{1}{2}\rho v^2 + \rho g y = \text{const}$$

  (conservation of energy)

- Torricelli's Theorem: The outlet speed is the free-fall speed. For a barrel with water depth d, an outlet at base has horizontal flow speed $v = \sqrt{2gd}$.

- Stoke's law: viscous drag is $6\pi\eta r_s \nu$.

- Poiseille's Law:

  $$\Delta P = \frac{8\mu L Q}{\pi r^4}$$

  where L is length of tube, Q is volume rate. This describes viscous incompressible flow through a constant circular cross-section.

- Kepler's laws.
  - An orbiting body travels in an ellipse

    $$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}$$

  - ``A line joining a planet and the Sun sweeps out equal areas during equal intervals of time."

    $$\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0$$

    or

    $$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \text{constant}$$

  - ``The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit."

    $$P = \frac{A}{dA/dt} = 2\sqrt{\frac{\mu}{R}}R^{3/2} \qquad P^2 \propto R^3$$

    or

    $$\frac{P^2}{a^3} = \frac{4\pi^2}{MG}$$

- Coriolis force:

  $$\vec{F} = -2m(\vec{\omega}\times\vec{v})$$

- Diffusion: Fick's law. The diffusion flux is given by

  $$\vec{J}_r = -D\nabla_n\phi$$

- Frequency of a pendulum of arbitrary shape:

  $$\omega = \sqrt{\frac{mgL}{I}} \qquad T = 2\pi\sqrt{\frac{I}{mgL}}$$

  where L is the distance between the axis of rotation and the center of mass.

- Hamiltonian formulation:

  $$\mathcal{H} = \sum_i p_i\dot{q}_i - \mathcal{L}, \qquad \dot{p} = -\frac{\partial\mathcal{H}}{\partial q}, \qquad \dot{q} = \frac{\partial\mathcal{H}}{\partial p}$$

- Circular orbits exist for almost all potentials. Stable non-circular orbits can occur for the simple harmonic potential and the inverse square law.

- Orbit questions:

$$V_{\text{eff}}(r) = V(r) + \frac{L^2}{2mr^2}$$

For a gravitational potential, $V(r) \propto \frac{1}{r}$. The total energy of an object

$$E = \frac{1}{2}mv^2 + V_{\text{eff}}$$

$E < V_{\text{min}}$ gives a spiral orbit, $E = V_{\text{min}}$ gives a circular orbit,, $V_{\text{min}} < E < 0$ gives an ellipse, $E = 0$ is a parabolic orbit, and $E > 0$ has a hyperbolic orbit.

- If we want to approximate the equation of motion as a small oscillation about a point of equilibrium $V'(x_0) = 0$ we can Taylor expand to get

$$V(x) = V(x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2$$

and then get the force

$$F = -\frac{dV}{dx} = -V''(x_0)(x - x_0)$$

so that we can approximate small oscillations has harmonic oscillations with $k = V''(x_0)$ and

$$\omega = \sqrt{\frac{V''(x_0)}{m}}.$$

Chapter 3

Electromagnetism

- Resistance is defined in terms of resistivity as

$$R = \frac{\rho L}{A}$$

- Faraday's laws of electrolysis
  - The mass liberated $\propto$ charge passed through
  - Mass of different elements liberated $\propto$ atomic weight/valence

$$m = \frac{QA}{Fv}$$

  where v is valence, A is atomic weight in kg/kmol, $F = 9.65 \times 10^7$C/kmol (Faraday's constant)

- Parallel plate capacitor $C = \epsilon_0 A/d$ or $\epsilon A/d$ for a dielectric. For a spherical capacitor,

$$C = \frac{4\pi\epsilon_0 ab}{a - b}$$

- In charging a capacitor,

$$q = q_0(1 - e^{-t/RC})$$

  discharging

$$q = q_0 e^{-t/RC}$$

- Cyclotron/magnetic bending

$$r = \frac{mv}{qB}$$

- Torque experienced by a planar coil of N loops, with current I in each loop.

  $$\tau = NIAB \sin\theta$$

  where θ is the angle between B and line perpendicular to coil plane:

  $$\vec{\tau} = \vec{\mu} \times \vec{B}$$

- B-field of a long wire

  $$B = \frac{\mu_0 I}{2\pi r}$$

  Center of a ring wire

  $$B = \frac{\mu_0 I}{2r}$$

  Long solenoid

  $$B = \mu_0 n I$$

  where n is the turn density.

- Ampere's Law:

  $$\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enc}}$$

- Conductors do not transmit EM wave, thus $\vec{E}$ vector is reversed upon reflecting, B vector is increased by a factor of 2 (by solving propogation of EM wave).

- Magnetic fields in matter:

  $$B = \mu H = \mu_0(H + M) = \mu_0(H + \chi_m H)$$

  Diamagnetic$\leftrightarrow$ χ_m very small and negative. Paramagnetic, $\leftrightarrow$ χ_m small and positive, inversely proportional to the absolute temperature. Ferromagnetic $\leftrightarrow$ χ_m positive, can be greater than 1. M is no longer proportional to H.

- For solenoid and toroid, $H = nI$, n is the number density.

- Self inductance:

  $$\mathcal{E} = -L\frac{di}{dt}$$

  L is in henries, $1H = 1V \cdot S/A = 1J/A^2 = 1$ web$/A$

  $$N\Phi = LI$$

  is the flux linkage. Inductance of solenoid:

  $$L = \frac{\mu N^2 A}{c}$$

- Induced e.m.f

$$|\mathcal{E}_s| = N \left| \frac{d\Phi_B}{dt} \right|$$

- Time constant for $R - L$ circuit $t = L/R$. For an $R - C$ constant $t = RC$. For an $L - C$ circuit, $\omega_0 = 1/\sqrt{LC}$.

- $X_L = 2\pi f L$ is the inductive reactance. $X_C = 1/2\pi f C$ is the capacitive reactance. The impedance is given by

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \qquad \text{series}$$

$$\frac{1}{Z} = \left[\left(\frac{1}{R} \right)^2 + \left(\frac{1}{X_C} - \frac{1}{X_L} \right)^2 \right]^{1/2} \qquad \text{parallel}$$

  Current is maximized at resonance $X_L = \omega L = X_C = 1/\omega C$ (there will be a lot of questions on this)

- Larmor formula for radiation

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \propto q^2 a^2$$

  where a is the acceleration. Energy per unit area decreases as distance increases (inverse square relation).

- Mean drift speed:

$$\vec{v} = \frac{\vec{J}}{ne}$$

  where n is the number of atoms per volume, J is current density I/A.

- Impedance of capacitor

$$Z = \frac{1}{i\omega C}$$

  Impedance of inductor

$$Z = i\omega L$$

- Magnetic field on axis of a circle of current

$$B = \frac{\mu_0 I}{2} \frac{r^2}{(r^2 + z^2)^{3/2}}$$

- Bremsstrahlung: electromagnetic radiation produced by the deceleration of a charged particle.

- For incident wave reflecting off a plane, just set up a boundary value problem.

  $$E_1^{\perp} - E_2^{\perp} = \sigma \qquad E_1^{\|} = E_2^{\|}$$

  and remember the Poynting vector

  $$\vec{S} \propto \vec{E} \times \vec{B}$$

  points in the direction of propagation.

  $$E_0 + E_0^{\text{reflected}} = E_0^{\text{transmitted}}$$

- Lenz's law: The idea is the system responds in a way to restore or at least attempt to restore to the original state.

- Impedance matching to maximize power transfer or to prevent terminal-end reflection.

  $$Z_{\text{rad}} = Z^*_{\text{source}}$$

  $$I(X_g) + I(X_L) = IR$$

  Generator impedance:

  $$R_g + jX_g$$

  Local impedance:

  $$R_L + jX_L$$

  $$Z = R + j(\omega L + 1/\omega C)$$

- Propagation vector $\vec{k}$

  $$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

  $$\vec{B}(\vec{r}, t) = \frac{1}{c}|\vec{E}(\vec{r}, t)|$$

  $$(\hat{k} \times \hat{n}) = \frac{1}{c}\hat{k} \times \hat{E}$$

- No electric field inside a constant potential enclosure implies constant V inside.

- Hall effect

  $$R_H = \frac{1}{(p - n)e}$$

  can be used to test the nature of charge carrier. p for positive, n for negative.

- Lorentz force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

- $\nabla \cdot (\nabla \times \vec{H}) = 0$, $\nabla \times (\nabla f) = 0$
- One usually has cycloid motion whenever the electric and magnetic fields are perpendicular.
- Faraday's law:

$$\mathcal{E} = \vec{E} \cdot d\vec{L} = -\frac{d\Phi}{dt}$$

- Visible spectrum in meters: Radio 10^3 (on the order of buildings); Microwave 10^{-2}; Infrared 10^{-5}; visible 700-900 nm (10^{-6}); UV 10^{-8}(molecules); X-ray 10^{-10}(atoms); gamma ray 10^{-12} (nuclei)
- Displacement field

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon_0\vec{E} + \epsilon_0\chi_e\vec{E} = \epsilon_0(1 + \chi_e)\vec{E} = \epsilon\vec{E}$$

  Dielectric constant

$$\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

$$\sigma_b = \vec{P} \cdot \vec{n}$$
$$\rho_b = -\vec{\nabla} \cdot \vec{P}$$

  These are the bound charge densities. Also note

$$\nabla \times \vec{D} = \nabla \times \vec{P}$$

  is not necessarily zero.

- We have

$$\vec{B} = \begin{cases} \mu_0 nI\hat{z} & \text{inside a solenoid} \\ 0 & \text{outside a solenoid} \end{cases}$$

  where n is density per length.

$$\vec{B} = \begin{cases} \frac{\mu_0 nI}{2\pi s}\hat{\phi} & \text{inside a toroid} \\ 0 & \text{outside} \end{cases}$$

- Force per unit length between two wires:

$$f = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}$$

- $B = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)$ looks like the magnetic field due to a segment of wire, where θ_i is the angle from the normal.

- Mutual inductance of two loops

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r_{ij}}$$

- Radiation pressure

$$P = \frac{I}{c} = \frac{\langle S \rangle}{c} \cos\theta$$

  It's twice that for a perfect reflector.

- $\nabla \cdot \vec{D} = \rho_f$ $\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$, $\nabla \cdot \vec{B} = 0$, $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$.

- Boundary conditions:

$$\epsilon_1 \vec{E}_1^\perp - \epsilon_2 \vec{E}_2^\perp = \sigma_f \qquad \vec{B}_1^\perp - \vec{B}_2^\perp = 0$$

$$\vec{E}_1^\parallel - \vec{E}_2^\parallel = 0, \qquad \mu_1 \vec{B}_1^\parallel - \mu_2 \vec{B}_2^\parallel = \vec{k}_f \times \hat{n}$$

- Biot-Savart law:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \vec{r}}{|\vec{r}^3|}$$

- B-field at a center of a ring

$$\vec{B} = \frac{\mu_0 I}{2r}$$

- $H = \frac{1}{\mu_0} B - M$, $J_b = \nabla \times \vec{M}$, $\vec{k}_b = \vec{M} \times \hat{n}$

$$\vec{B} = \mu \vec{H}, \qquad \mu = \mu_0(1 + \chi_m)$$

Chapter 4

Optics and Wave Phenomena

- Speed of propagation for waves
  - Transverse on string, $v = \sqrt{T/\rho}$
  - Longitudinal in liquid, $v = \sqrt{B/\rho}$, B is bulk modulus
  - Longitudinal in solid, $v = \sqrt{Y/\rho}$,Y is Young's modulus
  - Longitudinal in gases, $v = \sqrt{\gamma P/\rho}$
- For open pipe, fundamental frequency is $v/2L$ where v is the speed of sound. For a closed pipe it is $(2n-1)\lambda/4 = L$. The idea is $\lambda f = v$.
- Speed of sound in air is

  $$v = \sqrt{\frac{\gamma k T}{m}} = \sqrt{\frac{\gamma R T}{M}} \propto \sqrt{T}$$

  where m is the mass of a molecule, and M is the molar mass in kg/mole.
- Resonant frequency of a rectangular drum

  $$f_{mn} = \frac{\nu}{2}\sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2}$$

- Doppler effect

  $$f'' = \frac{v}{v + v_{\text{source}}} f$$

  v is the velocity in the medium, v_{source} is the source velocity w.r.t. medium. In general,

  $$\frac{f_{\text{listener}}}{v \pm v_{\text{lis}}} = \frac{f_{\text{source}}}{v \pm v_{\text{source}}}$$

  The $\pm$ can be determined by examining if the frequency received is higher or lower.

- Lens optics:

  $$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$

  Sign convention, real image has positive sign.

- Lens maker's equation:

  $$\frac{1}{f} \approx (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

  If R_1 is positive, it's convex, negative, concave. If R_2 is positive, it's concave, if it's negative, it's convex.

- Young's double slit:

  $$d \sin\theta = m\lambda \qquad \text{maxima}$$

  $$yd = mD\lambda \qquad d \ll D,\ \theta \text{ small}$$

  $$d \sin\theta = (m + \tfrac{1}{2})\lambda \qquad \text{minima}$$

- If we have a slab of material with thickness t and refractive index n_2, and the other medium is n_1.

  $$\frac{2n_2 t}{n_1 \lambda_1} = m + \frac{1}{2} \qquad \text{max}$$

  $$\frac{2n_2 t}{n_1 \lambda_1} = m + 1 \qquad \text{min}$$

- Conversely: if we have three layers of material, $n_1,\ n_t$, and n_2 (top to bottom), then we have a couple of different situations that would like to a maximum in intensity:

  $$d = \frac{m\lambda}{2n_t} \qquad n_1 > n_t > n_2, \qquad n_1 < n_t < n_2$$

  $$d = \frac{(m + \frac{1}{2})\lambda}{2n_t} \qquad n_1 < n_t > n_2, \qquad n_1 > n_t < n_2$$

  I think it's fair to assume that the minima occur when you replace $m + \frac{1}{2}$ with m and vice-versa.

- Diffraction grating

  $$d \sin\theta = m\lambda$$

  If incident at angle θ_i

  $$d(\sin\theta_m + \sin\theta_i) = m\lambda$$

The overall result is an interference pattern modulated by single slit diffraction envelope. Intensity of interference

$$I = I_0 \frac{\sin^2(N\phi/2)}{\sin^2(\phi/2)} \qquad \phi = \frac{2\pi}{\lambda} d \sin\theta$$

Minima occurs at $N\phi/2 = \pi, \dots n\pi$ where $n/N \notin \mathbb{Z}$. Maxima occurs at $\phi/2 = 0, \pi, 2\pi, \dots$. Single-slit envelope,

$$I = I_0 \frac{\sin^2(\phi'/2)}{(\phi'/2)^2} \qquad \phi' = \frac{2\pi}{\lambda} w \sin\theta$$

where w is the width of the slit. Overall,

$$I = I_0 \frac{\sin^2(\phi'/2)\sin^2(N\phi/2)}{(\phi'/2)^2 \sin^2(\phi/2)}$$

- Bragg's law of reflection

  $$m\lambda = 2d\sin\theta$$

  Make sure that θ is a glancing angle, not angle of incidence (relative to the plane). This gives the angles for coherent and incoherent scattering from a crystal lattice.

- Index of refraction is defined as

  $$n = \frac{c}{v}$$

  Again,

  $$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

- Brewster's angle is the angle of incidence at which light with a particular polarization is perfectly transmitted, no reflection.

  $$\tan\theta = \frac{n_2}{n_1}$$

- Diffraction again (more background info). The light diffracted by a grating is found by summing the light diffracted from each of the elements, and is essentially a convolution of diffraction and interference pattern. Fresnel diffraction is near field, and fraunhofer diffraction is far field.

- Diffraction limited imaging

  $$d = 1.22\lambda N$$

  where N is the focal length/diameter. Angular resolution is

  $$\sin\theta = 1.22\frac{\lambda}{D}$$

  where D is the lens aperture.

- Thin-film theory. Say the film has higher refractive index. Then there's a phase change for reflection off front surface, no phase change for reflection off back surface. Constructive interference thickness t: $2t = (n + 1/2)\lambda$. Destructive interference $2t = n\lambda$.

- The key idea for many questions is to scrutinize path difference (optical)

- Some telescopes have two convex lenses, the objective and the eyepiece. For the telescope to work the lenses have to be at a distance equal to the sum of their focal lengths, i.e. $d = f_{\text{objective}} + f_{\text{eye}}$:

$$M = \left| \frac{f_{\text{objective}}}{f_{\text{eye}}} \right|$$

  Magnifying power = max angular magnification = image size with lens/image size without lens.

- Microscopy

$$\text{magnifying power} = \frac{\beta}{\alpha}$$

- In Michelson interferometer a change of distance $\lambda/2$ of the optical path between the mirrors generally results in a change of λ of optical path of light ray, thus potentially giving a cycle of bright$\rightarrow$dark$\rightarrow$bright fringes.

- Mirror with curvature $f \approx R/2$.

- Beats: the beat frequency is $f_1 - f_2$:

$$\sin(2\pi f_1 t) + \sin(2\pi f_2 t) = 2 \cos\left(2\pi \frac{f_1 - f_2}{2} t\right) \sin\left(2\pi \frac{f_1 + f_2}{2} t\right)$$

Part II

Advanced Physics

Chapter 5

Thermodynamics and Statistical Mechanics

- PV diagram plots change in pressure wrt to volume for some process. The work done by the gas is the area **under the curve.**
- If the cyclic process moves clockwise around the loop, then W will be positive, and it represents a heat engine. If it moves counterclockwise, then W will be negative, and it represents a heat pump.
- The most basic definition of entropy is

$$dS \geq \frac{dQ}{T}$$

- Heat transfer
  - Conduction: rate

$$H = \frac{\Delta Q}{\Delta t} = -kA\frac{T_2 - T_1}{L}, \qquad \frac{dQ}{dt} = -kA\frac{dT}{dx}$$

    where A is area, k is a constant.
  - Convection (probably not in GRE),

$$H = \frac{\Delta Q}{\Delta T} = hA(T_s - T_\infty)$$

    where T_s is the surface temperature, h =convective heat-transfer coefficient. There are both natural and forced convections.
- Radiation

$$\text{Power} = \epsilon\sigma AT^4$$

ϵ =emissivity, $\epsilon \in [0,1]$. Net loss= $\epsilon\sigma A(T^4_{\text{emission}} - T^4_{\text{absorption}})$

- Wien's displacement law: The absolute temperature of a blackbody and the peak wavelength of its radiation are inversely proportional:

$$\lambda_{\text{max}} T = 2.898 \times 10^{-3} \text{ m·K}$$

- Ideal gas law

$$PV = nRT = NkT$$

- Kinetic theory of gas

$$P = \frac{1}{3}\rho v_{\text{rms}}^2 \qquad v_{\text{rms}} = \sqrt{\frac{3kT}{m}}, \qquad \bar{v} = \sqrt{\frac{8kT}{\pi m}}, \qquad v_{\text{most probable}} = v_m = \sqrt{\frac{2kT}{m}}$$

- Maxwell-Boltzmann distribution (less likely to be in GRE), number of molecules with energy between E and $E + dE$

$$N(E)dE = \frac{2N}{\sqrt{\pi}(kT)^{3/2}}\sqrt{E}e^{-E/kT}dE$$

$$f(v)d^3v = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}d^3v$$

$$P(v) = \sqrt{\frac{2}{\pi}}\left(\frac{m}{kT}\right)^{3/2} v^2 e^{-mv^2/2kT}$$

(from which we can derive v_m)

- Mean free path of a gas molecule of radius b

$$l = \frac{1}{4\pi T b^2 (N/V)}$$

- Van der Waals equation of state

$$(P + an^2/V^2)(V - bn) = nRT$$

$$(P + aN^2/V^2)(V - Nb) = NkT$$

- Adiabatic process

$$PV^\gamma = \text{const}$$

For an ideal gas to expand adiabatically from $(P_1, V_1) \to (P_2, V_2)$, work done by the gas is

$$W = \frac{P_1V_1 - P_2V_2}{\gamma - 1}$$

derived from $W = \int_{V_1}^{V_2} PdV$.

- The greatest possible thermal efficiency of an engine operating between two heat reservoirs is that of a Carnot engine, one that operates in the Carnot cycle. Max efficiency is

  $$y^\star = 1 - \frac{T_{\text{cold}}}{T_{\text{hot}}}$$

  For the case of the refrigerator

  $$\kappa = \frac{Q_{\text{cold}}}{W} \qquad \kappa_{\text{Carnot}} = \left(\frac{T_{\text{hot}}}{T_{\text{cold}}} - 1\right)^{-1}$$

  Carnot=adiabatic+isothermal, $dS = 0$. Otto=adiabatic+isobaric

  $$y = 1 - \frac{T_d - T_a}{T_c - T_b}$$

- Dalton's Law

  $$P = P_1 + P_2 = (n_1 + n_2)\frac{RT}{V}$$

- The critical isotherm is the line that just touches the critical liquid-vapor region

  $$\left(\frac{dP}{dV}\right)_c = 0 \qquad \left(\frac{d^2P}{dV^2}\right)_c = 0$$

  with c the critical point. Equilibrium region is where pressure and chemical potential for the two states of matter equal, usually a pressure constant region in the $P - V$ diagram.

- In the Dulong-Petit law,

  $$C_V = \frac{dE}{dT} = 3R$$

- Laws of thermodynamics

  - 0th: If two thermodynamic systems are each in thermal equilibrium with a third, then they are in thermal equilibrium with each other.
  - 1st: $\Delta U = Q - W$ (conservation of energy)
  - 2nd: Entropy increases/heat flows from hot to cold/heat cannot be completely converted into work.
  - 3rd: As $T \to 0$, $S \to$constant minimum.

- Change in entropy for a system where specific heat and temperature are constant;

  $$\Delta S = Nk \ln \frac{V}{V_0}$$

- Change in energy for an ideal gas:

$$\Delta U = C_V \Delta T$$

- Work done by ideal gas:

$$W = \int P dV = \begin{cases} NkT \ln \frac{V_2}{V_1} & \text{Isothermal} \\ P\Delta V & \text{Ideal gas, constant Pressure} \end{cases}$$

- Partition function:

$$Z = \sum_i e^{-\beta E_i} = \int dE\, \Omega(E) e^{-\beta E} = \int dE\, e^{-\beta A(E)}$$

where $A(E)$ is the Helmholtz free energy and $\Omega(E)$ is the degeneracy.

$$P(E_i) = \frac{e^{-\beta E_i}}{Z}$$

$$S = k \ln \Omega = -k \sum_i P_i \ln P_i$$

- Equipartition Theorem: (1) Classical canonical and (2) quadratic dependence: each particle has energy $\frac{1}{2}kT$ for each quadratic canonical degree of freedom.

- Internal energy

$$dU = TdS - PdV$$

Enthalpy

$$H = U + PV \qquad dH = TdS + VdP \qquad \text{isobaric}$$

Helmholtz

$$F = U - TS, \qquad dF = -SdT - PdV \qquad \text{isothermal}$$

Gibbs free energy

$$G = U - TS + PV, \qquad dG = -SdT + VdP$$

- Heat capacities:

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V$$

$$C_P = \left(\frac{\partial U}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P = T\left(\frac{\partial S}{\partial T}\right)_P = \left(\frac{\partial H}{\partial T}\right)_P$$

- Fun stuff:

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z, \qquad F = -kT\ln Z$$

$$S = k\ln Z + \langle E\rangle/T, \qquad dS = \int\frac{dQ}{T}$$

Gibbs-Helmholtz equation.

$$U = F + TS = F - T\left(\frac{\partial F}{\partial T}\right)_V = -T^2\left(\frac{\partial}{\partial T}\right)_V\left(\frac{F}{T}\right)$$

- Availability of system

$$A = U + P_0V - T_0S$$

In natural change, A cannot increase.

- Diatomic gas

$$U = \frac{5}{2}kT$$

- Maxwell Relations

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V = \frac{\partial^2 U}{\partial S\partial V}$$

$$\left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial V}{\partial S}\right)_P = \frac{\partial^2 H}{\partial S\partial P}$$

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V = -\frac{\partial^2 A}{\partial T\partial V}$$

$$-\left(\frac{\partial S}{\partial P}\right)_T = \left(\frac{\partial V}{\partial T}\right)_P = \frac{\partial^2 G}{\partial T\partial P}$$

- For ideal gas in adiabatic process, $W = \Delta U = \frac{3}{2}Nk\Delta T$
- Clockwise enclosed area in a $P - V$ diagram is the work done by the gas in a cycle.
- Chemical potential

$$\mu(T, V, N) = \left(\frac{\partial F}{\partial N}\right)_{T,V}$$

At equilibrium μ is uniform, F achieves minimum.

- $P_{\text{boson}} \propto T^{5/2}$, $P_{\text{classical}} \propto T$, $P_{\text{fermion}} \propto T_F$ (very big). $T_{\text{classical}} \gg T_{\text{boson}}$
- A thermodynamic system in maximal probability state is stable.
- Both Debye and Einstein assume $3N$ independent Harmonic oscillators for lattice. Einstein took a constant frequency

Chapter 6

Quantum Mechanics

- Uncertainty principle

$$\Delta x \Delta p \geq \frac{\hbar}{2}, \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

- Schrodinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \Psi + V\Psi$$

- Commutator relation:

$$[AB, C] = ABC - CAB = ABC - ACB + ACB - CAB = A[B, C] + [A, C]B$$

- De Broglie

$$\lambda = \frac{h}{p} = \frac{hc}{E} = \frac{h}{\sqrt{2mkT}}$$

  (The last equality is thermal)

- A one-dimensional problem has no degenerate states.
- Heisenberg's uncertainty principle generalized:

$$\Delta A \Delta B \geq \frac{1}{2}|\langle [A, B] \rangle|$$

- Infinite square well

$$\psi_n = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}, \qquad E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}, \qquad n \geq 1$$

  Delta-function well $V = -\alpha\delta(x)$. Only one bound state, many scattering states.

$$\psi(x) = \sqrt{\frac{m\alpha}{\hbar}} e^{-m\alpha|x|/\pi^2}, \qquad E = -\frac{m\alpha^2}{2\hbar^2}$$

  Shallow, narrow well, there is always at least one bound state.

- Selection rule

  $$\Delta l = \pm 1, \qquad \Delta m_l = \pm 1 \text{ or } 0, \qquad \Delta j = \pm 1 \text{ or } 0$$

  Electric dipole radiation $\Leftrightarrow \Delta l = 0$. Magnetic dipole or electric quadrupole transitions are ``forbidden'' but do occur occasionally.

- Stimulated and spontaneous emission rate $\propto |p|^2$ where

  $$p \equiv q\langle\psi_b|z|\psi_a\rangle$$

  The lifetime of an excited state is $\tau = (\sum A_i)^{-1}$ where A_i are spontaneous emission rates.

- Time-independent first order perturbation

  $$E_n^1 = E_n^0 + \langle\psi_n^0|H'|\psi_n^0\rangle, \qquad \psi_n^1 = \psi_n^0 + \sum_{m\neq n} \frac{\langle\psi_m^0|H'|\psi_n^0\rangle}{E_n^0 - E_m^0}\psi_m^0$$

- Quantum approximation of rotational energy

  $$E_{\text{rot}} = \frac{\hbar^2 l(l+1)}{2I}$$

- Fermi energy

  $$E_F = kT_F \simeq \frac{1}{2}mv^2$$

- Differential cross-section

  $$\frac{d\sigma}{d\Omega} = \frac{\text{scattered flux/unit of solid angle}}{\text{incident flux/unit of surface}}$$

- Intrinsic magnetic moment

  $$\vec{\mu} = \gamma\vec{S}, \qquad \gamma = \frac{eg}{2m}$$

  where g is the Lande g-factor. If m points up, $\vec{\mu}$points down.

- Total cross section

  $$\sigma = \int D(\theta)d\Omega, \qquad D(\theta) = \frac{d\sigma}{d\Omega}$$

- Stark effect is the electrical analog to the Zeeman effect.

- Born-Oppenheimer approximation: the assumption that the electronic motion and the nuclear motion in molecules can be separated, i.e.

  $$\psi_{\text{molecule}} = \psi_e\psi_{\text{nuclei}}$$

- In Stern-Gerlach experiment, a beam of neutral silver atoms are sent through an inhomogeneous magnetic field. Classically, nothing happens as the atoms are neutral with Larmor precession, the beam would be deflected into a smear. But it actually deflects into $2s+1$ beams, thus corroborating with the fact electrons are at spin $\frac{1}{2}$

- Know the basic spherical harmonics

$$Y_0^0 = \sqrt{\frac{1}{4\pi}}, \qquad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}, \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta$$

- Probability density current

$$\vec{J} = \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*) = \Re\left(\psi^*\frac{\hbar}{im}\nabla\psi\right)$$

- Laser operates by going from lower state to high state (population inversion), then falls back on a metastable state in between (not all the way down due to selection rule).

- Neat identities:

$$\langle\mathcal{O}\rangle = \int \Psi^*\mathcal{O}\Psi dx, \qquad [f(x), p] = i\hbar\frac{\partial f}{\partial x}, \qquad p = -i\hbar\nabla$$

- Ehrenfest's Theorem: expectation values obey classical laws.

$$m\frac{d^2\langle x\rangle}{dt^2} = \frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

- If $V(x)$ is even, $\psi(x)$ can always be taken to be even or odd.

- More identities:

$$\langle H\rangle = \sum_{n=1}^{\infty}|c_n|^2 E_n, \qquad \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk$$

- Tunneling shows exponential decay.

- The ground state of even potential is even and has no nodes.

- In stationary states, all expectation values are independent of t.

- Harmonic oscillators:

$$H = \hbar\omega(a_-a_+ - \tfrac{1}{2}) = \hbar\omega(a_+a_- + \tfrac{1}{2}), \qquad a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x \mp ip)$$

$$[a_-a_+] = 1, \qquad N \equiv a_+a_-, \qquad N\psi_n = n\psi_n$$

$$a_+\psi_n = \sqrt{n+1}\psi_{n+1}, \qquad a_-\psi_n = \sqrt{n}\psi_{n-1}$$

$$\psi_n = \frac{1}{\sqrt{n!}}(a_+)^n\psi_0, \qquad x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-), \qquad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$$

- Fourier transforms:

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-px/\hbar}\Psi(x,t)dx$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar}\Phi(p,t)dp$$

- Operators changing in time:

$$\frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle[H,Q]\rangle + \left\langle \frac{\partial Q}{\partial t}\right\rangle$$

- Virial theorem, in stationary state

$$2\langle T\rangle = \left\langle x\frac{dV}{dx}\right\rangle$$

- Hydrogen atom revisited:

$$\begin{aligned} E_n &\propto \text{reduced mass} \\ &\propto Z^2 \\ &\propto 1/n^2 \\ &= -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{2\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} = \frac{E_1}{n^2} \\ E_n(Z) &= Z^2 E_n \\ a(Z) &= \frac{a}{Z} \\ R(Z) &= Z^2 R \end{aligned}$$

Bohr radius $a = 4\pi\epsilon_0\hbar^2/me^2 = 0.528 \times 10^{-10}$ meters.

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}$$

- Angular momentum

$$[L_i, L_j] = i\hbar L_k \epsilon_{ijk}$$

where $\epsilon_{ijk} = 1$ for even permutations, -1 for odd permutations, zero otherwise.

$$L_\pm = L_x \pm iL_y, \qquad [L^2, L_i] = 0$$

$$L^2 f_l^m = \hbar^2 l(l+1), \qquad L_z f_l^m = \hbar m f_l^m$$

$$L_\pm f_l^m = \hbar\sqrt{(l \mp m)(l \pm m + 1)}f_l^m = \hbar\sqrt{l(l+1) - m(m \pm 1)}f_l^m$$

$$[L_z, x] = i\hbar y, \qquad [L_z, p_x] = i\hbar p_y, \qquad [L_z, y] = -i\hbar x, \qquad [L_z, p_y] = -i\hbar p_x$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

- Spin,

$$S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

$$\chi_+^{(x)} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \qquad \chi_-^{(x)} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \frac{\hbar^2}{4}$$

- Clebsch-Gorden coefficients

$$|sm\rangle = \sum_{m_1+m_2=m} C^{s_1 s_2 s}_{m_1 m_2 m} |s_1 m_1\rangle |s_2 m_2\rangle$$

$$|s_1 m_1\rangle |s_2 m_2\rangle = \sum_s C^{s_1 s_2 s}_{m_1 m_2 m} |sm\rangle$$

- Continuity equation

$$\nabla \cdot \vec{J} = -\frac{\partial}{\partial t}|\psi|^2$$

$$\int_S \vec{J} \cdot d\vec{a} = -\frac{d}{dt}\int_V |\psi|^2 d^3\vec{r}$$

- Representation of angular momentum.

$${}^{2s+1}\mathcal{L}_J$$

  where $s =$spin, $\mathcal{L} =$orbital, $J =$total. Hund's rule: (1) State with highest spin will have lowest energy given Pauli principle satisfied; (2) For given spin and anti-symmetrization highest $\mathcal{L}$ have lowest energy; (3) Lowest level has $J = |L - S|$, if more than half-filled $J = L + S$.

- Fermi gas

$$k_F = (3\rho\pi^2)^{1/3}, \qquad \rho = Nq/V, \qquad v_F = \sqrt{2E_F/m}$$

  Degeneracy pressure

$$P \propto \rho^{5/3} m_e^{-1} m_p^{-5/3}$$

- Particle distributions

$$n(\epsilon) = \begin{cases} e^{-\beta(\epsilon-\mu)} & \text{Maxwell-Boltzmann} \\ (e^{\beta(\epsilon-\mu)}+1)^{-1} & \text{Fermi-Dirac} \\ (e^{\beta(\epsilon-\mu)}-1)^{-1} & \text{Bose-Einstein} \end{cases}$$

  Blackbody density

$$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3 (e^{\hbar\omega/kT}-1)}$$

- Fine structure$\rightarrow$spin-orbit coupling. Relativistic correction $\alpha = 1/137.056$. Then Lamb shift is from the electric field, then Hyperfine structure due to magnetic interaction between electrons and protons, then spin-spin coupling (21 cm line)

- Fine structure breaks degeneracy in l but still have j

- Fermi's golden rule is a way to calculate the transition rate (probability of transition per unit time) from one energy eigenstate of a quantum system into a continuum of energy eigenstates, due to a perturbation.

- Full shell and close to a full shell configuration are more difficult to ionize.

- Larmor precession:

$$\vec{\Gamma} = \vec{\mu} \times \vec{B} = \gamma \vec{J} \times \vec{B}$$

  and we get $\omega = \gamma B$, where Γ is the torque, μ is the magnetic moment, and J is total angular momentum.

Chapter 7

Atomic Physics

- $\Delta E = hf = \hbar\omega = hc/\lambda$. $hc = 12.4$ keV·Å $= 1240$ eV·nm, de Broglie wavelength $\lambda = h/p$.
- Emission due to transition from level n to level m

  $$\frac{1}{\lambda} = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

  $m = 1$ Lyman series, $m = 2$ Balmer series.

  $$R = 1.097 \times 10^7\text{m}^{-1}, \qquad E_n = -\frac{13.6 \text{ eV}}{n^2}$$

- Hydrogen model extended, $Z =$number of protons, quantities scale as

  $$E \sim Z^2, \qquad \lambda \sim \frac{1}{Z^2}$$

  Reduced-mass correction to emission formula is

  $$\frac{1}{\lambda} = \frac{RZ^2}{1 + m/M}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

  where m is the mass of electron, M is the mass of the proton, $m/M = 1/1836$.

- Bohr postulate $L = mvr = n\hbar$
- Zeeman effect: splitting of a spectral line into several components in the presence of a static magnetic field.
- k series refers to the innermost shell (K, L, M, N) so transition to innermost shell.

  $$E = -13.6(Z-1)^2\left(1 - \frac{1}{n_i^2}\right) \text{ eV}$$

  where the $(Z-1)^2$ is a shielding approximation.

- Frank-Hertz Experiment: Electrons of a certain energy range can be scattered inelastically, and the energy lost by electrons is discrete.

- Spectroscopic notation is a standard way to write down the angular momentum quantum number of a state,

  $$^{2s+1}L_j$$

  where s is the total spin quantum number, $2s+1$ is the number of spin states, L refers to the orbital angular momentum quantum number ℓ but is written as $S, P, D, F, \ldots$ for $\ell = 0, 1, 2, 3, \ldots$ and j is the total angular momentum quantum number. So for hydrogen we could have things like

  $$^{2}P_{\frac{3}{2}}, ^{2}P_{\frac{1}{2}}$$

  (since $s = 1/2$ and $\ell = 1$, spin up versus spin down).

Chapter 8

Special Relativity

- Energy:

$$E^2 = (pc)^2 + (mc^2)^2$$

For massless particles, $E = pc = h\nu$

- Relativistic Doppler Effect

$$\lambda = \sqrt{\frac{1 \pm \beta}{1 \mp \beta}}\lambda_0$$

$\beta = v/c$. Sign is determined by whether source is moving away or closer.

- Space-time interval

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

- Lorentz transformation

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

- Relativistic addition of velocities

$$u'_x = \frac{u_x + v}{1 + u_x v/c^2}, \qquad u'_y = \frac{u_y}{\gamma(1 + u_x v/c^2)}, \qquad u'_z = \frac{u_z}{\gamma(1 + u_x v/c^2)}, \qquad \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

- Lorentz-Transformation of EM, parallel and perpendicular to direction o motion.

$$\vec{E}'_\parallel = \vec{E}_\parallel, \qquad \vec{E}'_\perp = \gamma(\vec{E}_\perp + \vec{v} \times \vec{B}_\perp)$$

$$\vec{B}'_\parallel = \vec{B}_\parallel, \qquad \vec{B}'_\perp = \gamma(\vec{B}_\perp - \vec{v} \times \vec{E}_\perp/c^2)$$

- Relativistic energy/momentum

$$E = \gamma mc^2, \qquad p = \gamma mv$$

- In every closed system, the total relativistic energy and momentum are conserved.
- Spacelike separation means two events can happen at the same time, which requires

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 < 0$$

- Transverse Doppler shift:

$$f = \frac{f'}{\sqrt{1-\beta^2}} \text{ or } f = f'\sqrt{1-\beta^2}$$

- Four-vectors can be useful. We can define

$$\mathbf{P} = \left(\frac{E}{c}, \mathbf{p}\right)$$

and the dot product

$$\mathbf{P}^2 = \frac{E^2}{c^2} - p^2 = m^2c^2$$

to get

$$E^2 = m^2 + p^2.$$

Remember, this mass is invariant, so we can equate the **P** vector at different times.

Chapter 9

Laboratory Methods

- If measurements are independent (or intervals in a Poisson process are independent) both expected value and variance increase linearly with time, so longer time can improve uncertainty, which is usually defined as

$$\frac{\sigma}{R} \propto \frac{1}{\sqrt{t}}$$

- In Poisson distribution, $\sigma = \sqrt{\bar{x}}$.

- Error analysis, estimating uncertainties. If you are sure the value is closer to 26 than to 25 or 27, then record best estimate 26 ± 0.5.

- Propagation of uncertainties for sum of random and independent variables

$$\delta x = \sqrt{\sum_i (\delta x_i)^2}$$

If multiplication or divisions are involved, use fractional uncertainty:

$$\frac{\delta q}{|q|} = \sqrt{\sum_i \left(\frac{\delta x_i}{x_i}\right)^2}$$

- Experimental uncertainties can be revealed by repeating the measurements are called random errors; those that cannot be revealed in this way are called systematic errors.

- If the the uncertainties are different for different measurements, we have

$$\bar{x} = \frac{\sum (x_i/\sigma_i^2)}{\sum_i (1/\sigma_i)^2} \qquad \sigma_{\bar{x}}^2 = \frac{1}{\sum_i (1/\sigma_i^2)}$$

Chapter 10

Specialized Topics

- Photoelectric effect.

  $$E_{\text{photon}} = \phi + K_{\text{max}}$$

  (or the sum of the work function and the kinetic energy).

- Compton scattering:

  $$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$$

  where m_e is the mass of the atom: $h/m_e c$ is the Compton wavelength of the electron, and λ' is the new wavelength.

- X-ray Bragg reflection

  $$n\lambda = 2d\sin\theta$$

  (compare to **diffraction grating** $n\lambda = d\sin\theta$)

- 1.602×10^{-19}J$= e(1\text{ V}) = 1$ eV.

- In solid-state physics, effective mass is

  $$m^* = \frac{\hbar^2}{d^2E/dk^2}$$

- Electronic filters: high pass means $\omega \to \infty$, $V_{\text{in}} = V_{\text{out}}$. Usually look at $I = V_{\text{in}}/Z$, $Z = R + i(X_L - X_C)$, $X_L = \omega L$, $X_C = 1/\omega C$.

- Band spectra is a term that refers to using EM waves to probe molecules.

- Solid state:

  $$\text{primitive cell} = \frac{\text{unit cell}}{\text{\# of lattice points in a Bravais lattice}}$$

  Simple cubic $\to$ 1 point, body-centered $\to$ 2 points, face-centered $\to$ 4 points.

- Resistivity of undoped semiconductor varies as $1/T$.
- Nuclear physics: binding energy is a form of potential energy, convention is to take it as positive. It's the energy needed to separate into different constituents. It is usually subtracted for other energy to tally total energy.
- Pair production refers to the creation of an elementary particle and its antiparticle. Usually need high energy (at least the total mass).
- At low energies, photoelectric-effect dominates Compton scattering.
- Radioactivity: Beta decay

  $$X_Z^A \to X'^A_{Z+1} + \beta^0_{-1} + \nu$$

  Alpha:

  $$X_Z^A \to X'^{A-4}_{Z-2} + \text{He}^4_2$$

  Gamma

  $$X_Z^A \to X_Z^A + \gamma$$

  Deuteron decay (not natural)

  $$X_Z^A \to X^{A-2}_{Z-1} + \text{H}^2_1$$

  Radioactivity usually follows Poisson distribution.
- Coaxial cable terminated at an end with characteristic impedance in order to avoid reflection of signals from the terminated end of cable.
- Human eyes can only see things in motion up to ~ 25 Hz.
- In magnetic field, e are more likely to be emitted in a direction opposite to the spin direction of the decaying atom.
- Op-amp (operational amplifiers): if you only have two days to prepare for the GRE, this is not worth the effort, maximum one question on this. Read ``The Art of Electronics'' to check this out.
- The specific heat of a superconductor jumps to a lower value at the critical temperature (resistivity jumps too)
- Elementary particles: review the quarks, leptons, force carriers, generations, hadrons.
  - Family number conserved
  - Lepton number conserved

  - Strangeness is conserved (except for weak interactions)
  - Baryon number is conserved

- Internal conversion is a radioactive decay where an excited nucleus interacts with an electron in one of the lower electron shells, causing the electron to be emitted from the atom. It is not beta decay.

Chapter 11

Relativity

11.1 How to survive the calculations of Special Relativity

11.1.1 Important Relations

Metric in use

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11.1}$$

11.2 Quantities and Operations

d'Alembertian d'Alembert operator, or wave operator, is the Lapace operator in Minkowski space. [1]

$$\Box \equiv \partial_\mu \partial^\nu = \eta_{\mu\nu} \partial^\mu \partial^\nu \tag{11.2}$$

In the usual t,x,y,z natural orthonormal basis,

$$\begin{aligned} \Box &= -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 && (11.3) \\ &= -\partial_t^2 + \Delta^2 && (11.4) \\ &= -\partial_t^2 + \nabla && (11.5) \end{aligned}$$

On wiki [2], they give some applications to it.

klein-Gordon equation $(\Box + m^2)\phi = 0$

wave equation for electromagnetic field in vacuum For the electromagnetic four-potential $\Box A^\mu = 0$[3]

wave equation for small vibrations $\Box_c u(t,x) = 0 \rightarrow u_{tt} - c^2 u_{xx} = 0$

[1] Actually, there are more general definations for Lapacian, which includes this d'Alembertian of course.

[2] wiki:D'Alembert_operator

[3] Gauge

11.3 Fields and Particles

11.3.1 Energy-Momentum Tensor for Particles

$$S_p \equiv -mc \int\int \mathrm{d}s\mathrm{d}\tau \sqrt{-\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu}\delta^4(x^\mu - x^\mu(s)), \tag{11.6}$$

in which $x^\mu(s)$ is the trajectory of the particle. Then the energy density ρ corresponds to $m\delta^4(x^\mu - x^\mu(s))$.

The Largrange density

$$\mathcal{L} = -\int \mathrm{d}smc\sqrt{-\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu}\delta^4(x^\mu - x^\mu(s)) \tag{11.7}$$

Energy-momentum density is $\mathcal{T}^{\mu\nu} = \sqrt{-g}T^{\mu\nu}$ is

$$\mathcal{T}^{\mu\nu} = -2\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} \tag{11.8}$$

Finally,

$$\mathcal{T}^{\mu\nu} = \int \mathrm{d}s\frac{mc\dot{x}^\mu\dot{x}^\nu}{\sqrt{-\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu}}\delta(t - t(s))\delta^3(\vec{x} - \vec{x}(t)) \tag{11.9}$$

$$= m\dot{x}^\mu\dot{x}^\nu\frac{\mathrm{d}s}{\mathrm{d}t}\delta^3(\vec{x} - \vec{x}(s(t))) \tag{11.10}$$

11.4 Theorems

11.4.1 Killing Vector Related

11.1. *ξ^a is Killing vector field, T^a is the tangent vector of geodesic line. Then $T^a\nabla_a(T^b\xi_b) = 0$, that is $T^b\xi_b$ is a constant on geodesics.*

11.5 Topics

11.5.1 Redshift

In geometrical optics limit, the angular frequency ω of a photon with a 4-vector K^a, measured by a observer with a 4-velocity Z^a, is $\omega = -K_aZ^a$.

Part III

Tools

Chapter 12

Mathematics

12.1 Differential Geometry

12.1.1 Metric

Definitions

Denote the basis in use as $\hat{e}_\mu$, then the metric can be written as

$$g_{\mu\nu} = \hat{e}_\mu \hat{\cdot} e_\nu \tag{12.1}$$

if the basis satisfies

Inversed metric

$$g_{\mu\lambda} g^{\lambda\nu} = \delta^\nu_\mu = g^\nu_\mu \tag{12.2}$$

How to calculate the metric

Let's check the definition of metric again.

If we choose a basis $\hat{e}_\mu$, then a vector (at one certain point) in this coordinate system is

$$x^a = x^\mu \hat{e}_\mu \tag{12.3}$$

Then we can construct the expression of metric of this point under this coordinate system,

$$g_{\mu\nu} = \hat{e}_\mu \cdot \hat{e}_\nu \tag{12.4}$$

For example, in spherical coordinate system,

$$\vec{x} = r \sin\theta \cos\phi \hat{e}_x + r \sin\theta \sin\phi \hat{e}_y + r \cos\theta \hat{e}_z \tag{12.5}$$

Now we have to find the basis under spherical coordinate system. Assume the basis is $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$. Choose some scale factors $h_r = 1, h_\theta = r, h_\phi = r \sin\theta$. Then the basis is $\hat{e}_r = \frac{\partial \vec{x}}{h_r \partial r} = \hat{e}_x \sin\theta \cos\phi + \hat{e}_y \sin\theta \sin\phi + \hat{e}_z \cos\theta$, etc. Then collect the terms in formula 12.5 is we get $\vec{x} = r\hat{e}_r$, this is incomplete. So we check the derivative.

$$
\begin{aligned}
\mathrm{d}\vec{x} &= \hat{e}_x(\mathrm{d}r \sin\theta \cos\phi + r \cos\theta \cos\phi \mathrm{d}\theta - r \sin\theta \sin\phi \mathrm{d}\phi) && (12.6)\\
& \quad \hat{e}_y(\mathrm{d}r \sin\theta \sin\phi + r \cos\theta \sin\phi \mathrm{d}\theta + r \sin\theta \cos\phi \mathrm{d}\phi) && (12.7)\\
& \quad \hat{e}_z(\mathrm{d}r \cos\theta - r \sin\theta \mathrm{d}\theta) && (12.8)\\
&= \mathrm{d}r(\hat{e}_x \sin\theta \cos\phi + \hat{e}_y \sin\theta \sin\phi - \hat{e}_z \cos\theta) && (12.9)\\
& \quad \mathrm{d}\theta(\hat{e}_x \cos\theta \cos\phi + \hat{e}_y \cos\theta \sin\phi - \hat{e}_z \sin\theta) r && (12.10)\\
& \quad \mathrm{d}\phi(-\hat{e}_x \sin\phi + \hat{e}_y \cos\phi) r \sin\theta && (12.11)\\
&= \hat{e}_r \mathrm{d}r + \hat{e}_\theta r \mathrm{d}\theta + \hat{e}_\phi r \sin\theta \mathrm{d}\phi && (12.12)
\end{aligned}
$$

Once we reach here, the component (e_r, e_θ, e_ϕ) of the point under the spherical coordinates system basis $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$ at this point are clear, i.e.,

$$\begin{aligned} \mathrm{d}\vec{x} &= \hat{e}_r \mathrm{d}r + \hat{e}_\theta r \mathrm{d}\theta + \hat{e}_\phi r \sin\theta \mathrm{d}\phi && (12.13) \\ &= e_r \mathrm{d}r + e_\theta \mathrm{d}\theta + e_\phi \mathrm{d}\phi && (12.14) \end{aligned}$$

In this way, the metric tensor for spherical coordinates is

$$g_{\mu\nu} = (e_\mu \cdot e_\nu) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{12.15}$$

12.1.2 Connection

First class connection can be calculated

$$\Gamma^\mu{}_{\nu\lambda} = \hat{e}^\mu \cdot \hat{e}_{\mu,\lambda} \tag{12.16}$$

Second class connection is[1]

$$[\mu\nu, \iota] = g_{\iota\mu} \Gamma^\mu{}_{\nu\lambda} \tag{12.17}$$

12.1.3 Gradient, Curl, Divergence, etc

Gradient

$$T^b{}_{c;a} = \nabla_a T^b{}_c = T^b{}_{c,a} + \Gamma^b_{ad} T^d{}_c - \Gamma^d_{ac} T^b{}_d \tag{12.18}$$

Curl For an anti-symmetric tensor, $a_{\mu\nu} = -a_{\nu\mu}$

$$\begin{aligned} \mathrm{Curl}_{\mu\nu\tau}(a_{\mu\nu}) &\equiv a_{\mu\nu;\tau} + a_{\nu\tau;\mu} + a_{\tau\mu;\nu} && (12.19) \\ &= a_{\mu\nu,\tau} + a_{\nu\tau,\mu} + a_{\tau\mu,\nu} && (12.20) \end{aligned}$$

Divergence

$$\begin{aligned} \mathrm{div}_\nu(a^{\mu\nu}) &\equiv a^{\mu\nu}{}_{;\nu} = \frac{\partial a^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\nu\tau} a^{\tau\nu} + \Gamma^\nu_{\nu\tau} a^{\mu\tau} && (12.21) \\ &= \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} a^{\mu\nu}) + \Gamma^\mu_{\nu\lambda} a^{\nu\lambda} && (12.22) \end{aligned}$$

For an anti-symmetric tensor

$$\mathrm{div}(a^{\mu\nu}) = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} a^{\mu\nu}) \tag{12.23}$$

[1] Kevin E. Cahill

Annotation Using the relation $g = g_{\mu\nu}A_{\mu\nu}$, $A_{\mu\nu}$ is the algebraic complement, we can prove the following two equalities.

$$\Gamma^{\mu}_{\mu\nu} = \partial_\nu \ln \sqrt{-g} \tag{12.24}$$

$$V^{\mu}_{;\mu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{-g}V^{\mu}) \tag{12.25}$$

In some simple case, all the three kind of operation can be demonstrated by different applications of the del operator, which $\nabla \equiv \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z$.
Gradient, ∇f, in which f is a scalar.
Divergence, $\nabla \cdot \vec{v}$
Curl, $\nabla \times \vec{v}$ Laplacian, $\Delta \equiv \nabla \cdot \nabla \equiv \nabla^2$

12.2 Linear Algebra

12.2.1 Basic Concepts

Trace Trace should be calculated using the metric. An example is the trace of Ricci tensor,

$$R = g^{ab}R_{ab} \tag{12.26}$$

Einstein equation is

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G T_{ab} \tag{12.27}$$

The trace is

$$\begin{aligned} g^{ab}R_{ab} - \frac{1}{2}g^{ab}g_{ab}R &= 8\pi G g^{ab}T_{ab} && (12.28)\\ \Rightarrow R - \frac{1}{2}4R &= 8\pi G T && (12.29)\\ \Rightarrow -R &= 8\pi G T && (12.30)\end{aligned}$$

12.2.2 Technique

Inverse of a matrix Many methods to get the inverse of a matrix. Check wikipedia for Invertible matrix.

Adjugate matrix method for example is here.

$$A^{-1} = \frac{A^*}{|A|} \tag{12.31}$$

in which, A^* is the adjugate matrix of A.

12.3 Differential Equations

12.3.1 Standard Procedure

12.3.2 Tricky

WKB Approximation When the highest derivative is multiplied by a small parameter, try this.

Part IV

Cutting Edge

Chapter 13

Cosmology

13.1 What's in the begining

In cosmology, the frame used most frequently is the cosmic microwave background radiation (CMB) stationary frame, not earth frame. We sometimes say it is earth frame because the speed of earth relative to CMB is rather small compared to the galaxies' movement observed by we earth beings.

13.2 Constants And Physical Quantities

- Deceleration parameter of today,

  $$q = -\left(\frac{a}{\dot{a}}\ddot{a}\right) = -\frac{\ddot{a}}{\dot{a}}\frac{1}{H(a)}$$

  Deceleration parameter is tightly related to Friedmann equation. [1]

13.2.1 Cosmographic Parameters

Co-Graphy-Par-1 Recession speed

$$v = H_0 \cdot d \tag{13.1}$$

Co-Graphy-Par-2 Hubble time

$$t_H = \frac{1}{H_0} \tag{13.2}$$

Co-Graphy-Par-3 Hubble distance

$$D_H = \frac{c}{H_0} = c \cdot t_H \tag{13.3}$$

Co-Graphy-Par-4 Dimensionless density parameters

Matter

$$\Omega_M = \frac{8\pi G \rho_M|_{a=1}}{3H_0^2} \tag{13.4}$$

Lambda

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} \tag{13.5}$$

Curvature can be viewed as some energy, too. From Friedmann equation we can sort out it. Or equivalently

$$\Omega_k = 1 - \Omega_M - \Omega_\Lambda \tag{13.6}$$

---

[1] Firedmann equation can be written in terms of deceleration parameter,

$$q = 1/2(1+3w)(1+k/(aH)^2)$$

Redshifts and Distances

Co-Dis-1 Redshift defined in observation

$$z = \frac{\nu_e}{\nu_o} - 1 = \frac{\lambda_o - \lambda_e}{\lambda_e} \tag{13.7}$$

Co-Dis-2 In linear range, given a radial velocity, redshift can be written as

$$z = \sqrt{\frac{1 + v/c}{1 - v/c}} - 1 \tag{13.8}$$

Co-Dis-3 Peculiar velocity

$$v_{pec} = c\frac{z_{obs} - z_{cos}}{1+z}, \tag{13.9}$$

in which,z_{obs} is the observed redshift, while z_{cos} is the cosmological redshift. Cosmological redshift is the Hubble flow "due solely to the expansion of the universe". **This definition is valid when v_{pec} is much smaller than speed of light, i.e., $v_{pec} \ll c$.**

Co-Dis-4 Scale factor

$$a|_{t_0} = (1+z)a|_{t_e}. \tag{13.10}$$

In cosmology, we often use another form which is derived from and setting $a|_{t_0} = 1$

$$a = \frac{1}{1+z}. \tag{13.11}$$

This is the scale factor we used in line element.

Co-Dis-5 Relative redshift

$$z_{12} = \frac{a|_{t_1}}{a|_{t_2}} - 1 = \frac{1+z_2}{1+z_1} \tag{13.12}$$

Co-Dis-6 Line-of-sight comoving distance

$$D_c = D_H \int_0^z \frac{1}{E(z')} \mathrm{d}z' \tag{13.13}$$

in which $E(z) \equiv \sqrt{\Omega_M(1+z)^3 + \Omega_\Lambda + \Omega_k(1+z)^2}$.

Co-Dis-7 Transverse comoving distance

$$D_M = \frac{D_H}{\sqrt{|\Omega_k|}} f(\sqrt{|\Omega_k|}\frac{D_C}{D_H}) \tag{13.14}$$

in which

$$f(x) = \begin{cases} \sinh(x), & \Omega_k > 0 \quad \text{hyperbola} \\ x, & \Omega_k = 0 \quad \text{flat} \\ \sin(x), & \Omega_k < 0 \quad \text{parabola} \end{cases} . \tag{13.15}$$

Co-Dis-8 Angular diameter distance

$$D_A = \frac{1}{1+z} D_M \tag{13.16}$$

We just divide the transverse distances by $1+z$ The advantage of it is that it is not singular at $z \to \infty$

Co-Dis-9 Luminosity distance

$$D_L \equiv \sqrt{\frac{L}{4\pi S}} \tag{13.17}$$

in which L is bolometric luminosity[2], S is the bolometric flux.

Related to other distances

$$D_L = (1+z)D_M = (1+z)^2 D_A \tag{13.18}$$

Co-Dis-10 Distance modules

$$DM \equiv 5\,\mathrm{log}\, D_L/10\mathrm{p}c \tag{13.19}$$

In Mpc unit,

$$DM \equiv 5\,\mathrm{log}\, D_L + 25 \tag{13.20}$$

Co-Dis-11 Comoving volume

$$\mathrm{d}V_C = D_H \frac{(1+z)^2 D_\Lambda^2}{E(z)} \mathrm{d}\Omega \mathrm{d}z \tag{13.21}$$

Co-Dis-12 Lookback time

$$t_L = t_H \int_0^z \frac{\mathrm{d}z'}{(1+z')E(z')} \tag{13.22}$$

Lookback time is "the difference between the age t_o of the Universe now (at observation) and the age t_e of the Universe at the time the photons were emmitted (according to the object). "

Co-Dis-13 Probability of intersecting objects

$$\mathrm{d}P = n(z)\sigma(z) D_H \frac{(1+z)^2}{E(z)} \mathrm{d}z \tag{13.23}$$

"Given a population of objects with comoving number density $n(z)$ (number per unit volume) and cross section $\sigma(z)$ (area), what is the incremental probability dP atha a line of sight will intersect one of the objects in redshift interval dz at redshift z?"[3]

[2]Bolometric luminosity: The total energy radiated by an object at all wavelengths, usually given in joules per second.

[3]**arXiv:astro-ph/9905116v4**

This part is mostly taken from **arXiv:astro-ph/9905116v4**

Page 418 of Gravitation And Cosmology: Principles And Applications Of The General Theory Of Relativity written by Weinberg in 1972.

13.3 The Homogeneous and Isotropic Universe

It is always pointed out that most thoeries are based on the Cosmological Principle.

I have stated this principle of in the chapter telling the perturbation theory in cosmology.

> Empedocles: `God is an infinite sphere whose center is everywhere and circumference nowhere.'
>
> Cosmological principle states that VIEWED on a sufficiently large scale, the properties of the Universe are the same for all observers. The key to understand this is that "same" means same physics principles and physical constants. Wikipedia gives three qualifications and two testable consequences. The two consequences are isotropic and homogeneous. Isotropic indicates no matter where we are looking at the universe is the same while homogeneous indicates no matter where we are located the universe is about the same (i.e., we are looking at a fair sample of the universe).
>
> Problem is how to describe isotropy and homogeneity. [Liang, P360]

Actually, we have a rigorous form for the principle, as I mentioned in these paragraphs.

Homogeneous Space-time (M, g_{ab}), sliced into Σ_t, induced metric of g_{ab} on Σ_t. (M, g_{ab}) is spatially homogeneous if we can always find Σ_t, ensuring the existance of a set of isometry on Σ_t itself.

13.4 Quantities

13.4.1 Energy-momentum Tensor

Energy-momentum tensor can be made clear using a standard procedure of field theory.

Check out the table below. [From Ohanian and Ruffini's GRAVATATION AND SPACETIME, P494.]

	Particle	Field
Quantities	$q_i(t)$	$\phi(\vec{x}, t)$
independent quantitie	t,i[?]	$\vec{x}, t$
Lagrange	$L = L(q_i, \dot{q}_i)$	$L = \int \mathscr{L}(\phi, \partial\phi/\partial x^\mu)\mathrm{d}^3 x$
Equation of motion	$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$	$\frac{\partial}{\partial x^\nu}\frac{\partial \mathscr{L}}{\partial \phi_{,\mu}} - \frac{\partial \mathscr{L}}{\partial \phi} = 0$
Hamiltonian	$H = \sum \dot{q}_i \frac{\partial L}{\partial \dot{q}} - L$	$H = \int (\phi_{,0}\frac{\partial \mathscr{L}}{\partial \phi_{,0}} - \mathscr{L})\mathrm{d}^3 x$

The integrand in H is the energy density of the complete system. We would like to choose the ${t_0}^0$ component of a canonical energy-momentum tensor energy-momentum tensor to describe energy density,

$$ {t_0}^0 = \phi_{,o}\frac{\partial \mathscr{L}}{\partial \phi_{,0}} - \mathscr{L}. \tag{13.24} $$

It can be proved that the following equation is the only one that satisfies the constrain we proposed before.

$$ {t_\mu}^\nu = \phi_{,\mu}\frac{\partial \mathscr{L}}{\partial \phi_{,\nu}} - \delta^\nu_\mu \mathscr{L} \tag{13.25} $$

We have the conservation law:

$$ \frac{\partial}{\partial x^\nu}{t_\mu}^\nu = 0 \tag{13.26} $$

Start from this we can find the Hamiltonian is conserved.

$$ \frac{\mathrm{d}H}{\mathrm{d}t} = 0 \tag{13.27} $$

The total momentum is conserved too.

$$ P_k = \int {t_k}^0 \mathrm{d}^3 x \tag{13.28} $$

This is all about a scalar field. If we switch to vector field (EM field) and tensor field (1-order gravitation field), the canonical energy-momentum tensors are

$$ {t_{(em)}}_\mu{}^\nu = A^\alpha_{,\mu}\frac{\partial \mathscr{L}_{(em)}}{\partial A^\alpha_{,\nu}} - \delta^\nu_\mu \mathscr{L}_{(em)} \tag{13.29} $$

$$ {t_{(g1)}}_\mu{}^\nu = h^{\alpha\beta}_{,\mu}\frac{\partial \mathscr{L}_{(g1)}}{\partial h^{\alpha\beta}_{,\nu}} - \delta^\nu_\mu \mathscr{L}_{(g1)} \tag{13.30} $$

In special relativity, the energy-momentum tensor for ideal fluid is

$$ T_{ab} = (\rho + p)U_a U_b + p\eta_{ab} \tag{13.31} $$

Each component has its physical meaning.

1. T^{00} is the energy density;
2. $T^{0k} = T^{k0}$ is the k momentum density (energy flux density);
3. $T^{kl} = T^{lk}$ is the k momentum flux density in l direction.

Just change all the η_{ab} into g_{ab}.
For any system,

$$\begin{aligned}
\nabla_\mu T^{\mu\nu} &= U^\mu U^\nu \nabla_\mu \rho + U^\mu U^\nu \nabla_\mu p + (\rho + p)U^\mu \nabla_\mu U^\nu + (\rho + p)U^\nu \nabla_\mu U^\mu + g^{\mu\nu}\nabla_\mu p && (13.32) \\
&= U^\mu U^\nu \dot{\rho} + (g^{\mu\nu} + U^\mu U^\nu)\nabla_\mu p + (\rho + p)A^\nu + (\rho + p)U^\nu \nabla_\mu \Theta && (13.33) \\
&= U^\mu U^\nu \dot{\rho} + (\rho + p)U^\nu \nabla_\mu \Theta && (13.34) \\
&= Q^\nu && (13.35)
\end{aligned}$$

A^ν vanishes because we have chosen a comoving observer. $\nabla_\mu p$ vanishes because in our standard model ρ, p are coordinate free. Actually, since U^μ is timelike, we have $(g^{\mu\nu} + U^\mu U^\nu)\nabla_\mu p = h^{\mu\nu}\nabla_\mu p$.

13.4.2 Friedmann Universe

Cosmological Principle Empedocles: `God is an infinite sphere whose center is everywhere and circumference nowhere.'

Cosmological principle states that VIEWED on a sufficiently large scale, the properties of the Universe are the same for all observers. The key to understand this is that "same" means same physics principles and physical constants. Wikipedia gives three qualifications and two testable consequences. The two consequences are isotropic and homogeneous. Isotropic indicates no matter where we are looking at the universe is the same while homogeneous indicates no matter where we are located the universe is about the same (i.e., we are looking at a fair sample of the universe).

Problem is how to describe isotropy and homogeneity. [Liang, P360]

Robertson-Walker Metric The next problem is how to set up a simple and useful coordinate system which metric to use in cosmology. [Liang, P366]

Evolution of Scale Factor Friedmann equation

$$\begin{aligned}
3(\dot{a}^2 + k)/a^2 &= 8\pi p && (13.36) \\
2\ddot{a}/a + (\dot{a}^2 + k)/a^2 &= -8\pi p && (13.37)
\end{aligned}$$

13.36 is called Friedmann equation.
It is better to use another set of equations which are identical to 13.36 and 13.37

$$\begin{aligned}
\ddot{a} &= -4\pi a(\rho + 3p)/3 && (13.38) \\
0 &= \dot{\rho} + 3(\rho + p)\dot{a}/a && (13.39)
\end{aligned}$$

These equations can be solved in some circumstances. [Liang, P376]
It could be useful to reform these equation.

$$H^2 + \frac{k}{a^2} = \frac{8}{3}\pi\rho \tag{13.40}$$
$$2\dot{H} + 3H^2 + \frac{k}{a^2} = -8\pi p \tag{13.41}$$

Mixing

$$3\ddot{a} = -4\pi a(\rho + 3p) \tag{13.42}$$
$$0 = \dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} \tag{13.43}$$

$$\dot{H} = -4\pi G(\rho + p) + \frac{k}{a^2} \tag{13.44}$$