Physics Research Surviving Manual Documentation

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ONE

WHATEVER

I alway believe that making things open to everyone is one of the most powerful things that drive the world forward. So I DO think open source and open data, even open education are reforming the world.

This was a draft handbook for myself when I was studying in Fudan University. At that time, I learned how to use LaTeX and I was so exciting. So I thought I should start writing something using LaTeX since it's so beautiful. Well, the bad thing is, I just randomly wrote down my notes on some specific topics.

I was so greedy back then. I was trying to build up my own framework of physics by writing notes here. It never did the work by the way. Then I realized a framework should be somethng organized much better than this one. (I should draw a map of physics.)

Though these notes didn't help me building up my framework of phyiscs, I learn an important lesson. A physicist should build up his/her own style: the way to think, the way to solve problems, the way to check answers, the way to write, etc. (I just want to create a new word for his/her, hir or hes.)

Anyway, I got frastrated and gave up the effort to utilize it as a framework-building thing. However I won't just dump these notes. As I have more and more to add, I think I'll just let it be my notebook, which, of course, is open source and accessible to everyone.

Yes. Use the source. Keep the source open.

CHAPTER

TWO

BASIC

2.1 Dimension

How to find the relationship between two quantities? For example, what is the dimensional relationship between length and mass.

* Plank constant: $\hbar \sim [\text{Energy}] \cdot [\text{Time}] \sim [\text{Mass}] \cdot [\text{Length}]^2 \cdot [\text{Time}]^{-1}$
* Speed of light in vacuum: $c \sim [\text{Length}] \cdot [\text{Time}]^{-1}$
* Gravitational constant: $G \sim [\text{Length}]^3 \cdot [\text{Mass}]^{-1} \cdot [\text{Time}]^{-2}$

Then it is easy to find that a combination of $c/\hbar$ cancels the dimension of mass and leaves the inverse of length. That is

$$[L]^2 = \left[\frac{\hbar G}{c^3}\right]$$

$$[M]^2 = \left[\frac{\hbar c}{G}\right]$$

$$[T]^2 = \left[\frac{\hbar G}{c^5}\right]$$

As we can see, it is possible to use $c = 1, \hbar = 1, G = 1$ because we can always restore the units in a deterministic way. $c, \hbar, G$ are function of mass, length, time, and with $c = \hbar = G = 1$ give us only one solution of mass, length and time: three equations + three variables.

2.1.1 Planck Scales

As we have seen, the three constant can make up a length scale, a mass scale, a time scale. Then what are they?

Planck length:

$$l_P = \sqrt{\frac{\hbar G}{c^3}}$$

Planck mass:

$$m_P = \sqrt{\frac{\hbar c}{G}}$$

Planck time:

$$t_P = \sqrt{\frac{\hbar G}{c^5}}$$

2.1.2 Equations and Dimensions

Before solving equations, it is good to reform them in to dimensionless ones.

To make the equation dimensionless doesn't mean we can just divide arbitary terms on both sides. We need to find out the characteristic quantity of the system. For example, we can divide by $\hbar\omega$ on both sides of Schrodinger equation for Harmonic Oscillators. This is a good step because $\hbar\omega$ is the characteristic energy scale of system. At the same time, we can make the length terms dimensionless using the characteristic length. DO NOT use an arbitary length!

Most Wonderful Equations That Should Never Be Forgotten

2.1.3 Electrodynamics

$$\nabla \times \vec{E} =$$
$$-\partial_t \vec{B}$$
$$\nabla \times \vec{H} =$$
$$\vec{J} + \partial_t \vec{D}$$
$$\nabla \cdot \vec{D} =$$
$$\rho$$
$$\nabla \cdot \vec{B} =$$
$$0$$

For linear meterials,

$$to$$

$$\vec{D} =$$
$$\epsilon \vec{E}$$
$$\vec{B} =$$
$$\mu \vec{H}$$
$$\vec{J} =$$
$$\sigma \vec{E} \quad (2.1)$$

=

=$\epsilon\vec{E}\vec{B}$

=$\mu\vec{H}\vec{J}$

$\sigma\vec{E}$

(2.1)

2.1.4 Dynamics

Hamilton conanical equations

$$to$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{2.2}$$

=

=$\frac{\partial H}{\partial p_i}\dot{p}_i$

$-\frac{\partial H}{\partial q_i}$

2.1.5 Thermodynamics and Statistical Physics

Liouville's Law

$$to$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} \equiv \frac{\partial \rho}{\partial t} + \sum_i \left[\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right] = 0(2.\text{-}1)$$

CHAPTER

THREE

MATHEMATICS

3.1 Differential Geometry

3.1.1 Metric

Definitions

Denote the basis in use as $\hat{e}_\mu$, then the metric can be written as

$$g_{\mu\nu} = \hat{e}_\mu \hat{\cdot} e_\nu$$

if the basis satisfies

Inversed metric

$$g_{\mu\lambda} g^{\lambda\nu} = \delta_\mu^\nu = g_\mu^\nu$$

How to calculate the metric

Let's check the definition of metric again.

If we choose a basis $\hat{e}_\mu$, then a vector (at one certain point) in this coordinate system is

$$x^a = x^\mu \hat{e}_\mu$$

Then we can construct the expression of metric of this point under this coordinate system,

$$g_{\mu\nu} = \hat{e}_\mu \cdot \hat{e}_\nu$$

For example, in spherical coordinate system,

$$\vec{x} = r \sin\theta \cos\phi \hat{e}_x + r \sin\theta \sin\phi \hat{e}_y + r \cos\theta \hat{e}_z$$

Now we have to find the basis under spherical coordinate system. Assume the basis is $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$. Choose some scale factors $h_r = 1, h_\theta = r, h_\phi = r\sin\theta$. Then the basis is

$$\hat{e}_r = \frac{\partial \vec{x}}{h_r \partial r} = \hat{e}_x \sin\theta \cos\phi + \hat{e}_y \sin\theta \sin\phi + \hat{e}_z \cos\theta,$$

etc. Then collect the terms in formula (3.1.1) is we get $\vec{x} = r\hat{e}_r$, this is incomplete. So we check the derivative.

$$to$$

$$\mathrm{d}\vec{x} =$$
$$\hat{e}_x(\mathrm{d}r\sin\theta\cos\phi + r\cos\theta\cos\phi\mathrm{d}\theta - r\sin\theta\sin\phi\mathrm{d}\phi)$$
$$\hat{e}_y(\mathrm{d}r\sin\theta\sin\phi + r\cos\theta\sin\phi\mathrm{d}\theta + r\sin\theta\cos\phi\mathrm{d}\phi)$$
$$\hat{e}_z(\mathrm{d}r\cos\theta - r\sin\theta\mathrm{d}\theta)$$
$$=$$
$$\mathrm{d}r(\hat{e}_x\sin\theta\cos\phi + \hat{e}_y\sin\theta\sin\phi - \hat{e}_z\cos\theta)$$
$$\mathrm{d}\theta(\hat{e}_x\cos\theta\cos\phi + \hat{e}_y\cos\theta\sin\phi - \hat{e}_z\sin\theta)r$$
$$\mathrm{d}\phi(-\hat{e}_x\sin\phi + \hat{e}_y\cos\phi)r\sin\theta$$
$$=$$
$$\hat{e}_r\mathrm{d}r + \hat{e}_\theta r\mathrm{d}\theta + \hat{e}_\phi r\sin\theta\mathrm{d}\phi \quad (3.-10)$$

=
$\hat{e}_x(\mathrm{d}r\sin\theta\cos\phi + r\cos\theta\cos\phi\mathrm{d}\theta - r\sin\theta\sin\phi\mathrm{d}\phi)$
$\hat{e}_y(\mathrm{d}r\sin\theta\sin\phi + r\cos\theta\sin\phi\mathrm{d}\theta + r\sin\theta\cos\phi\mathrm{d}\phi)$
=$\hat{e}_z(\mathrm{d}r\cos\theta - r\sin\theta\mathrm{d}\theta)$
$\mathrm{d}r(\hat{e}_x\sin\theta\cos\phi + \hat{e}_y\sin\theta\sin\phi - \hat{e}_z\cos\theta)$
$\mathrm{d}\theta(\hat{e}_x\cos\theta\cos\phi + \hat{e}_y\cos\theta\sin\phi - \hat{e}_z\sin\theta)r$
=$\mathrm{d}\phi(-\hat{e}_x\sin\phi + \hat{e}_y\cos\phi)r\sin\theta$
$\hat{e}_r\mathrm{d}r + \hat{e}_\theta r\mathrm{d}\theta + \hat{e}_\phi r\sin\theta\mathrm{d}\phi$

Once we reach here, the component (e_r, e_θ, e_ϕ) of the point under the spherical coordinates system basis $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$

at this point are clear, i.e.,

to

$\mathrm{d}\vec{x} =$
$\hat{e}_r \mathrm{d}r + \hat{e}_\theta r \mathrm{d}\theta + \hat{e}_\phi r \sin\theta \mathrm{d}\phi$
$=$
$e_r \mathrm{d}r + e_\theta \mathrm{d}\theta + e_\phi \mathrm{d}\phi$(3.-13)

=
=$\hat{e}_r \mathrm{d}r + \hat{e}_\theta r \mathrm{d}\theta + \hat{e}_\phi r \sin\theta \mathrm{d}\phi$
$e_r \mathrm{d}r + e_\theta \mathrm{d}\theta + e_\phi \mathrm{d}\phi$

In this way, the metric tensor for spherical coordinates is

$$g_{\mu\nu} = (e_\mu \cdot e_\nu) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \quad (3.-17)$$

00
00
$r^2 \sin^2\theta$

3.1.2 Connection

First class connection can be calculated

$$\Gamma^{\mu}{}_{\nu\lambda} = \hat{e}^{\mu} \cdot \hat{e}_{\mu,\lambda}$$

Second class connection isfootnote{Kevin E. Cahill}

$$[\mu\nu, \iota] = g_{\iota\mu} \Gamma^{\mu}{}_{\nu\lambda}$$

3.1.3 Gradient, Curl, Divergence, etc

Gradient

$$T^b{}_{c;a} = \nabla_a T^b{}_c = T^b{}_{c,a} + \Gamma^b_{ad} T^d{}_c - \Gamma^d_{ac} T^b{}_d$$

Curl

For an anti-symmetric tensor, $a_{\mu\nu} = -a_{\nu\mu}$

$$to$$

$$\mathrm{Curl}_{\mu\nu\tau}(a_{\mu\nu}) \equiv$$
$$a_{\mu\nu;\tau} + a_{\nu\tau;\mu} + a_{\tau\mu;\nu}$$
$$=$$
$$a_{\mu\nu,\tau} + a_{\nu\tau,\mu} + a_{\tau\mu,\nu}(3.\text{-}23)$$

$a_{\mu\nu;\tau} + a_{\nu\tau;\mu} + a_{\tau\mu;\nu}$
$a_{\mu\nu,\tau} + a_{\nu\tau,\mu} + a_{\tau\mu,\nu}$

Divergence

$$to$$

$$\mathrm{div}_\nu(a^{\mu\nu}) \equiv$$
$$a^{\mu\nu}{}_{;\nu} = \frac{\partial a^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\nu\tau} a^{\tau\nu} + \Gamma^\nu_{\nu\tau} a^{\mu\tau}$$
$$=$$
$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu}(\sqrt{-g} a^{\mu\nu}) + \Gamma^\mu_{\nu\lambda} a^{\nu\lambda}(3.\text{-}26)$$

$$a^{\mu\nu}{}_{;\nu} = \frac{\partial a^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\nu\tau} a^{\tau\nu} + \Gamma^\nu_{\nu\tau} a^{\mu\tau}$$

$$= \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} a^{\mu\nu}) + \Gamma^\mu_{\nu\lambda} a^{\nu\lambda}$$

For an anti-symmetric tensor

$$\mathrm{div}(a^{\mu\nu}) = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} a^{\mu\nu})$$

Annotation Using the relation $g = g_{\mu\nu} A_{\mu\nu}$, $A_{\mu\nu}$ is the algebraic complement, we can prove the following two equalities.

$$\Gamma^\mu_{\mu\nu} = \partial_\nu \ln \sqrt{-g}$$

$$V^\mu{}_{;\mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} V^\mu)$$

In some simple case, all the three kind of operation can be demonstrated by different applications of the del operator, which $\nabla \equiv \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z$.

- Gradient, ∇f, in which f is a scalar.
- Divergence, $\nabla \cdot \vec{v}$
- Curl, $\nabla \times \vec{v}$
- Laplacian, $\Delta \equiv \nabla \cdot \nabla \equiv \nabla^2$

3.2 Linear Algebra

3.2.1 Basic Concepts

Trace

Trace should be calculated using the metric. An example is the trace of Ricci tensor,

$$R = g^{ab} R_{ab}$$

Einstein equation is

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab}$$

The trace is

$$to$$

$$g^{ab}R_{ab} - \frac{1}{2}g^{ab}g_{ab}R =$$
$$8\pi G g^{ab}T_{ab}$$
$$\Rightarrow R - \frac{1}{2}4R =$$
$$8\pi G T$$
$$\Rightarrow -R =$$
$$8\pi G T(3.-39)$$

=

=$8\pi G g^{ab}T_{ab} \Rightarrow R - \frac{1}{2}4R$

=$8\pi G T \Rightarrow -R$

$8\pi G T$

3.2.2 Technique

Inverse of a matrix

Many methods to get the inverse of a matrix. Check wikipedia for Invertible matrix.

Adjugate matrix method for example is here.

$$A^{-1} = \frac{A^*}{|A|}$$

in which, A^* is the adjugate matrix of A.

3.3 Differential Equations

3.3.1 Standard Procedure

3.3.2 Tricky

WKB Approximation

When the highest derivative is multiplied by a small parameter, try this.

CHAPTER

FOUR

CLASSICAL MECHANICS

CHAPTER

FIVE

QUANTUM MECHANICS

5.1 Quantum Mechanics Framework

5.1.1 What're the most important tricks in QM calculations?

- **Remember what basis we are working in**
- **Identity**

5.1.2 First Three Postulates

- Physical state is described by kets in a Hilbert space. We need to specify a complete basis $\{i\}$ to do calculations.

$$\psi = \sum_i i i \psi = \sum_i C_i i$$

- Operators are given by Hermitian operators; A measurement of the variable $\hat{\Omega}$ will yield one of the eigenvalues ω with the probability

$$|\omega\psi|^2 .$$

  And the state of the system will change to ω.

- The state vector obeys the Schrdinger equation,

$$i\hbar$$

where $\hat{H}$ is the Hamiltonian operator.

The logic here is that we first find the way to describe a system, then think about how to find out the information we need from the state vector and also find the evolution of the state vector. Then we need the operator and Schrodinger equation. Finally, we would like to relate the theory to experiments, and it comes the measurement postulate.

Later we will need the relation between position and momentum, which becomes the fourth postulate.

How to solve the evolution of a system? We just define a magical operator, propagator

$$\hat{U}\psi(t_0) = \psi(t).$$

This operator just gives us the evolution of state vector! Wait, can we write down the explicit expression of it?

Let's find out. The only thing we know about the evolution of a state vector is the third postulate up there.

$$\overline{\dot{t}\hat{U}\psi(t_0) = \hat{H}\hat{U}\psi(t_0)}$$

to

i $\overline{\dot{t}\hat{U}=\hat{H}\hat{U}}$(5.-7) $i\hbar$

=

$\hat{H}\psi(t)i\hbar$

.

$\dot{t}\hat{U}\psi(t_0)=\hat{H}\hat{U}\psi(t_0)$

Looks familiar? This just gives us a exponential result, **if the Hamiltonian is time independent**.

$$\hat{U} = e^{-i\hat{H}(t-t_0)/\hbar}$$

We can prove that this operator is Unitary because $\hat{H}$ is Hermitian.

This is just the abstract representation, we work in some basis, and the most convenient basis is the eigenstates of Hamiltonian, { ϵ_i },

to

$\hat{U}\phi =$

$e^{-i\hat{H}(t-t_0)/\hbar}\psi$

$\hat{U}\phi =$

$\sum_i e^{-i\hat{H}(t-t_0)/\hbar}\epsilon_i\epsilon_i\psi$

$\hat{U}\phi =$

$\sum_i e^{-i\epsilon_i(t-t_0)/\hbar}\epsilon_i\epsilon_i\psi$(5.-11)

=

$$e^{-i\hat{H}(t-t_0)/\hbar}\psi\hat{U}\phi = \sum_i e^{-i\hat{H}(t-t_0)/\hbar}\epsilon_i\epsilon_i\psi\hat{U}\phi = \sum_i e^{-i\epsilon_i(t-t_0)/\hbar}\epsilon_i\epsilon_i\psi$$

And we are going to use

$$\hat{U} = \sum_i e^{-i\epsilon_i(t-t_0)/\hbar}\epsilon_i\epsilon_i$$

from now on. (Well, only on discrete eigenvalues ones.)

(See that? Identity does the work again.)

5.1.3 Position and Momentum Space

Summarize here.

- **Position**
    1. Define $\{x\}$ basis.
    2. Define $\hat{x}$ operator.
    3. Find wave function in this basis.
    4. Find measurement.
- **Evolution**
    1. Need propagator $\hat{U}$.
    2. Propagator needs the solution of Hamiltonian eigensystem.
    3. (Free particles) Hamiltonian needs the solution of momentum eigensystem.
- **Momentum**
    1. Before we define some arbitrary momentum space, we should check the relation between momentum and position. And it turns out to be related by a commutator.(Postulate IV)
    2. Use the postulate to momentum operator.
    3. Find eigenstates.
    4. (Calculate the propagator.)
