Research Survival Handbook
(**Unfinished**)

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0.1 Differential Geometry

0.1.1 Metric

Definitions

Denote the basis in use as $\hat{e}_\mu$, then the metric can be written as

$$g_{\mu\nu} = \hat{e}_\mu \hat{\cdot} e_\nu \tag{1}$$

if the basis satisfies

Inversed metric

$$g_{\mu\lambda} g^{\lambda\nu} = \delta_\mu^\nu = g_\mu^\nu \tag{2}$$

How to calculate the metric

Let's check the definition of metric again.

If we choose a basis $\hat{e}_\mu$, then a vector (at one certain point) in this coordinate system is

$$x^a = x^\mu \hat{e}_\mu \tag{3}$$

Then we can construct the expression of metric of this point under this coordinate system,

$$g_{\mu\nu} = \hat{e}_\mu \cdot \hat{e}_\nu \tag{4}$$

For example, in spherical coordinate system,

$$\vec{x} = r \sin\theta \cos\phi \hat{e}_x + r \sin\theta \sin\phi \hat{e}_y + r \cos\theta \hat{e}_z \tag{5}$$

Now we have to find the basis under spherical coordinate system. Assume the basis is $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$. Choose some scale factors $h_r = 1, h_\theta = r, h_\phi = r \sin\theta$. Then the basis is $\hat{e}_r = \frac{\partial \vec{x}}{h_r \partial r} = \hat{e}_x \sin\theta \cos\phi + \hat{e}_y \sin\theta \sin\phi + \hat{e}_z \cos\theta$, etc. Then collect the terms in formula 5 is we get $\vec{x} = r\hat{e}_r$, this is incomplete. So we check the derivative.

$$\begin{aligned}
\mathrm{d}\vec{x} &= \hat{e}_x(\mathrm{d}r \sin\theta \cos\phi + r \cos\theta \cos\phi \mathrm{d}\theta - r \sin\theta \sin\phi \mathrm{d}\phi) && (6)\\
&\quad \hat{e}_y(\mathrm{d}r \sin\theta \sin\phi + r \cos\theta \sin\phi \mathrm{d}\theta + r \sin\theta \cos\phi \mathrm{d}\phi) && (7)\\
&\quad \hat{e}_z(\mathrm{d}r \cos\theta - r \sin\theta \mathrm{d}\theta) && (8)\\
&= \mathrm{d}r(\hat{e}_x \sin\theta \cos\phi + \hat{e}_y \sin\theta \sin\phi - \hat{e}_z \cos\theta) && (9)\\
&\quad \mathrm{d}\theta(\hat{e}_x \cos\theta \cos\phi + \hat{e}_y \cos\theta \sin\phi - \hat{e}_z \sin\theta) r && (10)\\
&\quad \mathrm{d}\phi(-\hat{e}_x \sin\phi + \hat{e}_y \cos\phi) r \sin\theta && (11)\\
&= \hat{e}_r \mathrm{d}r + \hat{e}_\theta r \mathrm{d}\theta + \hat{e}_\phi r \sin\theta \mathrm{d}\phi && (12)
\end{aligned}$$

Once we reach here, the component (e_r, e_θ, e_ϕ) of the point under the spherical coordinates system basis $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$ at this point are clear, i.e.,

$$
\begin{aligned}
\mathrm{d}\vec{x} &= \hat{e}_r \mathrm{d}r + \hat{e}_\theta r \mathrm{d}\theta + \hat{e}_\phi r \sin\theta \mathrm{d}\phi \qquad (13) \\
&= e_r \mathrm{d}r + e_\theta \mathrm{d}\theta + e_\phi \mathrm{d}\phi \qquad (14)
\end{aligned}
$$

In this way, the metric tensor for spherical coordinates is

$$
g_{\mu\nu} = (e_\mu \cdot e_\nu) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \qquad (15)
$$

0.1.2 Connection

First class connection can be calculated

$$
\Gamma^\mu{}_{\nu\lambda} = \hat{e}^\mu \cdot \hat{e}_{\mu,\lambda} \qquad (16)
$$

Second class connection is[1]

$$
[\mu\nu, \iota] = g_{\iota\mu} \Gamma^\mu{}_{\nu\lambda} \qquad (17)
$$

0.1.3 Gradient, Curl, Divergence, etc

Gradient

$$
T^b{}_{c;a} = \nabla_a T^b{}_c = T^b{}_{c,a} + \Gamma^b_{ad} T^d{}_c - \Gamma^d_{ac} T^b{}_d \qquad (18)
$$

Curl For an anti-symmetric tensor, $a_{\mu\nu} = -a_{\nu\mu}$

$$
\begin{aligned}
\mathrm{Curl}_{\mu\nu\tau}(a_{\mu\nu}) &\equiv a_{\mu\nu;\tau} + a_{\nu\tau;\mu} + a_{\tau\mu;\nu} \qquad (19) \\
&= a_{\mu\nu,\tau} + a_{\nu\tau,\mu} + a_{\tau\mu,\nu} \qquad (20)
\end{aligned}
$$

Divergence

$$
\begin{aligned}
\mathrm{div}_\nu(a^{\mu\nu}) &\equiv a^{\mu\nu}{}_{;\nu} = \frac{\partial a^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\nu\tau} a^{\tau\nu} + \Gamma^\nu_{\nu\tau} a^{\mu\tau} \qquad (21) \\
&= \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} a^{\mu\nu}) + \Gamma^\mu_{\nu\lambda} a^{\nu\lambda} \qquad (22)
\end{aligned}
$$

For an anti-symmetric tensor

$$
\mathrm{div}(a^{\mu\nu}) = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} a^{\mu\nu}) \qquad (23)
$$

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Annotation Using the relation $g = g_{\mu\nu}A_{\mu\nu}$, $A_{\mu\nu}$ is the algebraic complement, we can prove the following two equalities.

$$\Gamma^{\mu}_{\mu\nu} = \partial_{\nu} \ln \sqrt{-g} \tag{24}$$

$$V^{\mu}_{\ ;\mu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{-g}V^{\mu}) \tag{25}$$

In some simple case, all the three kind of operation can be demonstrated by different applications of the del operator, which $\nabla \equiv \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z$.
Gradient, ∇f, in which f is a scalar.
Divergence, $\nabla \cdot \vec{v}$
Curl, $\nabla \times \vec{v}$ Laplacian, $\Delta \equiv \nabla \cdot \nabla \equiv \nabla^2$

0.2 Linear Algebra

0.2.1 Basic Concepts

Trace Trace should be calculated using the metric. An example is the trace of Ricci tensor,

$$R = g^{ab}R_{ab} \tag{26}$$

Einstein equation is

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G T_{ab} \tag{27}$$

The trace is

$$\begin{aligned} g^{ab}R_{ab} - \frac{1}{2}g^{ab}g_{ab}R &= 8\pi G g^{ab}T_{ab} && (28)\\ \Rightarrow R - \frac{1}{2}4R &= 8\pi G T && (29)\\ \Rightarrow -R &= 8\pi G T && (30) \end{aligned}$$

0.2.2 Technique

Inverse of a matrix Many methods to get the inverse of a matrix. Check wikipedia for Invertible matrix.

Adjugate matrix method for example is here.

$$A^{-1} = \frac{A^*}{|A|} \tag{31}$$

in which, A^* is the adjugate matrix of A.

0.3 Differential Equations

0.3.1 Standard Procedure

0.3.2 Tricky

WKB Approximation When the highest derivative is multiplied by a small parameter, try this.