Research Survival Handbook (**Unfinished**)

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0.1 Dimension

How to find the relationship between two quantities? For example, what is the dimensional relationship between length and mass.

Plank constant: $\hbar \sim [Energy] \cdot [Time] \sim [Mass] \cdot [Length]^2 \cdot [Time]^{-1}$
Speed of light in vacuum: $c \sim [Length] \cdot [Time]^{-1}$
Gravitational constant: $G \sim [Length]^3 \cdot [Mass]^{-1} \cdot [Time]^{-2}$

Then it is easy to find that a combination of $c/\hbar$ cancels the dimension of mass and leaves the inverse of length. That is

$$[Length]^2 = \frac{\hbar G}{c^3} \tag{1}$$

0.2 Most Wonderful Equations That Should Never Be Forgotten

0.2.1 Electrodynamics

Maxwell Equations

$$\begin{aligned} \nabla \times \vec{E} &= -\partial_t \vec{B} && (2)\\ \nabla \times \vec{H} &= \vec{J} + \partial_t \vec{D} && (3)\\ \nabla \cdot \vec{D} &= \rho && (4)\\ \nabla \cdot \vec{B} &= 0 && (5) \end{aligned}$$

For linear meterials,

$$\begin{aligned} \vec{D} &= \epsilon \vec{E} && (6)\\ \vec{B} &= \mu \vec{H} && (7)\\ \vec{J} &= \sigma \vec{E} && (8) \end{aligned}$$

Hamilton conanical equations

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} && (9)\\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} && (10) \end{aligned}$$

Liouville's Law

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} \equiv \frac{\partial \rho}{\partial t} + \sum_i \left[\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right] = 0 \tag{11}$$

Chapter 1

Classical Mechanics

- A worked example on velocity and acceleration in a curved path in a a plane: (the idea is to skillfully use $d(AB) = AdB + BdA$. This applies to change of momentum as well.)

  $$\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta, \qquad \hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$

  $$\hat{v} = \frac{d(R\hat{r})}{dt} = \frac{dR}{dt}\hat{r} + R\frac{d\hat{r}}{dt} = \dot{R}\hat{r} + R\omega\hat{\theta}$$

  Similarly,

  $$\vec{a} = (\ddot{R} - R\omega^2)\hat{r} + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{\theta}$$

- Firing rocket

  $$(v_g - v)dM + d(MV) = 0$$

  M is rocket mass, v is speed, v_g is relative speed of the waste fired out.

- Bernoulli's equation

  $$P + \frac{1}{2}\rho v^2 + \rho g y = \text{const}$$

  (conservation of energy)

- Torricelli's Theorem: The outlet speed is the free-fall speed. For a barrel with water depth d, an outlet at base has horizontal flow speed $v = \sqrt{2gd}$.

- Stoke's law: viscous drag is $6\pi\eta r_s\nu$.

- Poiseille's Law:

  $$\Delta P = \frac{8\mu LQ}{\pi r^4}$$

  where L is length of tube, Q is volume rate. This describes viscous incompressible flow through a constant circular cross-section.

- Kepler's laws.
  - An orbiting body travels in an ellipse

    $$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}$$

  - ``A line joining a planet and the Sun sweeps out equal areas during equal intervals of time."

    $$\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0$$

    or

    $$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \text{constant}$$

  - ``The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit."

    $$P = \frac{A}{dA/dt} = 2\sqrt{\frac{\mu}{R}}R^{3/2} \qquad P^2 \propto R^3$$

    or

    $$\frac{P^2}{a^3} = \frac{4\pi^2}{MG}$$

- Coriolis force:

  $$\vec{F} = -2m(\vec{\omega}\times\vec{v})$$

- Diffusion: Fick's law. The diffusion flux is given by

  $$\vec{J}_r = -D\nabla_n\phi$$

- Frequency of a pendulum of arbitrary shape:

  $$\omega = \sqrt{\frac{mgL}{I}} \qquad T = 2\pi\sqrt{\frac{I}{mgL}}$$

  where L is the distance between the axis of rotation and the center of mass.

- Hamiltonian formulation:

  $$\mathcal{H} = \sum_i p_i\dot{q}_i - \mathcal{L}, \qquad \dot{p} = -\frac{\partial\mathcal{H}}{\partial q}, \qquad \dot{q} = \frac{\partial\mathcal{H}}{\partial p}$$

- Circular orbits exist for almost all potentials. Stable non-circular orbits can occur for the simple harmonic potential and the inverse square law.

- Orbit questions:

$$V_{\text{eff}}(r) = V(r) + \frac{L^2}{2mr^2}$$

  For a gravitational potential, $V(r) \propto \frac{1}{r}$. The total energy of an object

$$E = \frac{1}{2}mv^2 + V_{\text{eff}}$$

  $E < V_{\text{min}}$ gives a spiral orbit, $E = V_{\text{min}}$ gives a circular orbit,, $V_{\text{min}} < E < 0$ gives an ellipse, $E = 0$ is a parabolic orbit, and $E > 0$ has a hyperbolic orbit.

- If we want to approximate the equation of motion as a small oscillation about a point of equilibrium $V'(x_0) = 0$ we can Taylor expand to get

$$V(x) = V(x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2$$

  and then get the force

$$F = -\frac{dV}{dx} = -V''(x_0)(x - x_0)$$

  so that we can approximate small oscillations has harmonic oscillations with $k = V''(x_0)$ and

$$\omega = \sqrt{\frac{V''(x_0)}{m}}.$$

Chapter 2

Electromagnetism

- Resistance is defined in terms of resistivity as

$$R = \frac{\rho L}{A}$$

- Faraday's laws of electrolysis
  - The mass liberated $\propto$ charge passed through
  - Mass of different elements liberated $\propto$ atomic weight/valence

  $$m = \frac{QA}{Fv}$$

  where v is valence, A is atomic weight in kg/kmol, $F = 9.65 \times 10^7$C/kmol (Faraday's constant)

- Parallel plate capacitor $C = \epsilon_0 A/d$ or $\epsilon A/d$ for a dielectric. For a spherical capacitor,

$$C = \frac{4\pi\epsilon_0 ab}{a - b}$$

- In charging a capacitor,

$$q = q_0(1 - e^{-t/RC})$$

discharging

$$q = q_0 e^{-t/RC}$$

- Cyclotron/magnetic bending

$$r = \frac{mv}{qB}$$

- Torque experienced by a planar coil of N loops, with current I in each loop.

  $$\tau = NIAB\sin\theta$$

  where θ is the angle between B and line perpendicular to coil plane:

  $$\vec{\tau} = \vec{\mu} \times \vec{B}$$

- B-field of a long wire

  $$B = \frac{\mu_0 I}{2\pi r}$$

  Center of a ring wire

  $$B = \frac{\mu_0 I}{2r}$$

  Long solenoid

  $$B = \mu_0 n I$$

  where n is the turn density.

- Ampere's Law:

  $$\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enc}}$$

- Conductors do not transmit EM wave, thus $\vec{E}$ vector is reversed upon reflecting, B vector is increased by a factor of 2 (by solving propogation of EM wave).

- Magnetic fields in matter:

  $$B = \mu H = \mu_0(H + M) = \mu_0(H + \chi_m H)$$

  Diamagnetic$\leftrightarrow$ χ_m very small and negative. Paramagnetic, $\leftrightarrow$ χ_m small and positive, inversely proportional to the absolute temperature. Ferromagnetic $\leftrightarrow$ χ_m positive, can be greater than 1. M is no longer proportional to H.

- For solenoid and toroid, $H = nI$, n is the number density.

- Self inductance:

  $$\mathcal{E} = -L\frac{di}{dt}$$

  L is in henries, $1H = 1V \cdot S/A = 1J/A^2 = 1$ web$/A$

  $$N\Phi = LI$$

  is the flux linkage. Inductance of solenoid:

  $$L = \frac{\mu N^2 A}{c}$$

- Induced e.m.f

$$|\mathcal{E}_s| = N \left| \frac{d\Phi_B}{dt} \right|$$

- Time constant for $R - L$ circuit $t = L/R$. For an $R - C$ constant $t = RC$. For an $L - C$ circuit, $\omega_0 = 1/\sqrt{LC}$.

- $X_L = 2\pi f L$ is the inductive reactance. $X_C = 1/2\pi f C$ is the capacitive reactance. The impedance is given by

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \qquad \text{series}$$

$$\frac{1}{Z} = \left[\left(\frac{1}{R} \right)^2 + \left(\frac{1}{X_C} - \frac{1}{X_L} \right)^2 \right]^{1/2} \qquad \text{parallel}$$

  Current is maximized at resonance $X_L = \omega L = X_C = 1/\omega C$ (there will be a lot of questions on this)

- Larmor formula for radiation

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \propto q^2 a^2$$

  where a is the acceleration. Energy per unit area decreases as distance increases (inverse square relation).

- Mean drift speed:

$$\vec{v} = \frac{\vec{J}}{ne}$$

  where n is the number of atoms per volume, J is current density I/A.

- Impedance of capacitor

$$Z = \frac{1}{i\omega C}$$

  Impedance of inductor

$$Z = i\omega L$$

- Magnetic field on axis of a circle of current

$$B = \frac{\mu_0 I}{2} \frac{r^2}{(r^2 + z^2)^{3/2}}$$

- Bremsstrahlung: electromagnetic radiation produced by the deceleration of a charged particle.

- For incident wave reflecting off a plane, just set up a boundary value problem.

  $$E_1^{\perp} - E_2^{\perp} = \sigma \qquad E_1^{\parallel} = E_2^{\parallel}$$

  and remember the Poynting vector

  $$\vec{S} \propto \vec{E} \times \vec{B}$$

  points in the direction of propagation.

  $$E_0 + E_0^{\text{reflected}} = E_0^{\text{transmitted}}$$

- Lenz's law: The idea is the system responds in a way to restore or at least attempt to restore to the original state.

- Impedance matching to maximize power transfer or to prevent terminal-end reflection.

  $$Z_{\text{rad}} = Z^*_{\text{source}}$$

  $$I(X_g) + I(X_L) = IR$$

  Generator impedance:

  $$R_g + jX_g$$

  Local impedance:

  $$R_L + jX_L$$

  $$Z = R + j(\omega L + 1/\omega C)$$

- Propagation vector $\vec{k}$

  $$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

  $$\vec{B}(\vec{r}, t) = \frac{1}{c}|\vec{E}(\vec{r}, t)|$$

  $$(\hat{k} \times \hat{n}) = \frac{1}{c}\hat{k} \times \hat{E}$$

- No electric field inside a constant potential enclosure implies constant V inside.

- Hall effect

  $$R_H = \frac{1}{(p-n)e}$$

  can be used to test the nature of charge carrier. p for positive, n for negative.

- Lorentz force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

- $\nabla \cdot (\nabla \times \vec{H}) = 0$, $\nabla \times (\nabla f) = 0$
- One usually has cycloid motion whenever the electric and magnetic fields are perpendicular.
- Faraday's law:

$$\mathcal{E} = \vec{E} \cdot d\vec{L} = -\frac{d\Phi}{dt}$$

- Visible spectrum in meters: Radio 10^3 (on the order of buildings); Microwave 10^{-2}; Infrared 10^{-5}; visible 700-900 nm (10^{-6}); UV 10^{-8}(molecules); X-ray 10^{-10}(atoms); gamma ray 10^{-12} (nuclei)
- Displacement field

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon_0 (1 + \chi_e) \vec{E} = \epsilon \vec{E}$$

Dielectric constant

$$\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

$$\sigma_b = \vec{P} \cdot \vec{n}$$

$$\rho_b = -\vec{\nabla} \cdot \vec{P}$$

These are the bound charge densities. Also note

$$\nabla \times \vec{D} = \nabla \times \vec{P}$$

is not necessarily zero.

- We have

$$\vec{B} = \begin{cases} \mu_0 n I \hat{z} & \text{inside a solenoid} \\ 0 & \text{outside a solenoid} \end{cases}$$

where n is density per length.

$$\vec{B} = \begin{cases} \frac{\mu_0 n I}{2\pi s} \hat{\phi} & \text{inside a toroid} \\ 0 & \text{outside} \end{cases}$$

- Force per unit length between two wires:

$$f = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d}$$

- $B = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)$ looks like the magnetic field due to a segment of wire, where θ_i is the angle from the normal.

- Mutual inductance of two loops

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{l_1} \cdot d\vec{l_2}}{r_{ij}}$$

- Radiation pressure

$$P = \frac{I}{c} = \frac{\langle S \rangle}{c} \cos\theta$$

  It's twice that for a perfect reflector.

- $\nabla \cdot \vec{D} = \rho_f$ $\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$, $\nabla \cdot \vec{B} = 0$, $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$.

- Boundary conditions:

$$\epsilon_1 \vec{E}_1^{\perp} - \epsilon_2 \vec{E}_2^{\perp} = \sigma_f \qquad \vec{B}_1^{\perp} - \vec{B}_2^{\perp} = 0$$

$$\vec{E}_1^{\|} - \vec{E}_2^{\|} = 0, \qquad \mu_1 \vec{B}_1^{\|} - \mu_2 \vec{B}_2^{\|} = \vec{k}_f \times \hat{n}$$

- Biot-Savart law:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \vec{r}}{|\vec{r}^3|}$$

- B-field at a center of a ring

$$\vec{B} = \frac{\mu_0 I}{2r}$$

- $H = \frac{1}{\mu_0} B - M$, $J_b = \nabla \times \vec{M}$, $\vec{k}_b = \vec{M} \times \hat{n}$

$$\vec{B} = \mu \vec{H}, \qquad \mu = \mu_0 (1 + \chi_m)$$

Chapter 3

Optics and Wave Phenomena

- Speed of propagation for waves
  - Transverse on string, $v = \sqrt{T/\rho}$
  - Longitudinal in liquid, $v = \sqrt{B/\rho}$, B is bulk modulus
  - Longitudinal in solid, $v = \sqrt{Y/\rho}$,Y is Young's modulus
  - Longitudinal in gases, $v = \sqrt{\gamma P/\rho}$
- For open pipe, fundamental frequency is $v/2L$ where v is the speed of sound. For a closed pipe it is $(2n-1)\lambda/4 = L$. The idea is $\lambda f = v$.
- Speed of sound in air is

  $$v = \sqrt{\frac{\gamma kT}{m}} = \sqrt{\frac{\gamma RT}{M}} \propto \sqrt{T}$$

  where m is the mass of a molecule, and M is the molar mass in kg/mole.
- Resonant frequency of a rectangular drum

  $$f_{mn} = \frac{\nu}{2}\sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2}$$

- Doppler effect

  $$f'' = \frac{v}{v + v_{\text{source}}} f$$

  v is the velocity in the medium, v_{source} is the source velocity w.r.t. medium. In general,

  $$\frac{f_{\text{listener}}}{v \pm v_{\text{lis}}} = \frac{f_{\text{source}}}{v \pm v_{\text{source}}}$$

  The $\pm$ can be determined by examining if the frequency received is higher or lower.

- Lens optics:

  $$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$

  Sign convention, real image has positive sign.

- Lens maker's equation:

  $$\frac{1}{f} \approx (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

  If R_1 is positive, it's convex, negative, concave. If R_2 is positive, it's concave, if it's negative, it's convex.

- Young's double slit:

  $$d \sin\theta = m\lambda \qquad \text{maxima}$$

  $$yd = mD\lambda \qquad d \ll D,\ \theta \text{ small}$$

  $$d \sin\theta = (m + \tfrac{1}{2})\lambda \qquad \text{minima}$$

- If we have a slab of material with thickness t and refractive index n_2, and the other medium is n_1.

  $$\frac{2n_2 t}{n_1 \lambda_1} = m + \frac{1}{2} \qquad \text{max}$$

  $$\frac{2n_2 t}{n_1 \lambda_1} = m + 1 \qquad \text{min}$$

- Conversely: if we have three layers of material, n_1, n_t, and n_2 (top to bottom), then we have a couple of different situations that would like to a maximum in intensity:

  $$d = \frac{m\lambda}{2n_t} \qquad n_1 > n_t > n_2, \qquad n_1 < n_t < n_2$$

  $$d = \frac{(m + \frac{1}{2})\lambda}{2n_t} \qquad n_1 < n_t > n_2, \qquad n_1 > n_t < n_2$$

  I think it's fair to assume that the minima occur when you replace $m + \frac{1}{2}$ with m and vice-versa.

- Diffraction grating

  $$d \sin\theta = m\lambda$$

  If incident at angle θ_i

  $$d(\sin\theta_m + \sin\theta_i) = m\lambda$$

The overall result is an interference pattern modulated by single slit diffraction envelope. Intensity of interference

$$I = I_0 \frac{\sin^2(N\phi/2)}{\sin^2(\phi/2)} \qquad \phi = \frac{2\pi}{\lambda} d \sin\theta$$

Minima occurs at $N\phi/2 = \pi, \ldots n\pi$ where $n/N \notin \mathbb{Z}$. Maxima occurs at $\phi/2 = 0, \pi, 2\pi, \ldots$. Single-slit envelope,

$$I = I_0 \frac{\sin^2(\phi'/2)}{(\phi'/2)^2} \qquad \phi' = \frac{2\pi}{\lambda} w \sin\theta$$

where w is the width of the slit. Overall,

$$I = I_0 \frac{\sin^2(\phi'/2)\sin^2(N\phi/2)}{(\phi'/2)^2 \sin^2(\phi/2)}$$

- Bragg's law of reflection

  $$m\lambda = 2d\sin\theta$$

  Make sure that θ is a glancing angle, not angle of incidence (relative to the plane). This gives the angles for coherent and incoherent scattering from a crystal lattice.

- Index of refraction is defined as

  $$n = \frac{c}{v}$$

  Again,

  $$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

- Brewster's angle is the angle of incidence at which light with a particular polarization is perfectly transmitted, no reflection.

  $$\tan\theta = \frac{n_2}{n_1}$$

- Diffraction again (more background info). The light diffracted by a grating is found by summing the light diffracted from each of the elements, and is essentially a convolution of diffraction and interference pattern. Fresnel diffraction is near field, and fraunhofer diffraction is far field.

- Diffraction limited imaging

  $$d = 1.22\lambda N$$

  where N is the focal length/diameter. Angular resolution is

  $$\sin\theta = 1.22\frac{\lambda}{D}$$

  where D is the lens aperture.

- Thin-film theory. Say the film has higher refractive index. Then there's a phase change for reflection off front surface, no phase change for reflection off back surface. Constructive interference thickness t: $2t = (n + 1/2)\lambda$. Destructive interference $2t = n\lambda$.

- The key idea for many questions is to scrutinize path difference (optical)

- Some telescopes have two convex lenses, the objective and the eyepiece. For the telescope to work the lenses have to be at a distance equal to the sum of their focal lengths, i.e. $d = f_{\text{objective}} + f_{\text{eye}}$:

  $$M = \left| \frac{f_{\text{objective}}}{f_{\text{eye}}} \right|$$

  Magnifying power = max angular magnification = image size with lens/image size without lens.

- Microscopy

  $$\text{magnifying power} = \frac{\beta}{\alpha}$$

- In Michelson interferometer a change of distance $\lambda/2$ of the optical path between the mirrors generally results in a change of λ of optical path of light ray, thus potentially giving a cycle of bright$\to$dark$\to$bright fringes.

- Mirror with curvature $f \approx R/2$.

- Beats: the beat frequency is $f_1 - f_2$:

  $$\sin(2\pi f_1 t) + \sin(2\pi f_2 t) = 2\cos\left(2\pi \frac{f_1 - f_2}{2} t\right) \sin\left(2\pi \frac{f_1 + f_2}{2} t\right)$$