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Physics Research Survival Manual

Release 1.38

Cosmology TF

April 30, 2015

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This is a notebook about physics. It's not done yet. (Maybe I'll never finish it because there are too many topics in physics.)

Use the source. Keep the source open.

AT THE BEGINING OF THE UNIVERSE

I am not a good physicist so don't trust the notes written here.

PREFACE

2.1 Whatever

I always believe that making things open to everyone is one of the most powerful things that drive the world forward. So I DO think open source and open data, even open education are reforming the world.

This was a notebook for myself when I was studying at Fudan University. At that time, I learned how to use LaTeX and I was so excited. So I thought I should start writing something using LaTeX since it's so beautiful. Well, the bad thing is, I just randomly wrote down my notes on some specific topics.

I was so greedy back then. Then I tried to build up my own framework of physics by writing notes here. It never did the work by the way. Then I realized a framework should be something organized much better than this one. (I should draw a map of physics.)

Though these notes didn't help me building up my framework of physics, I learned an important lesson. A physicist should build up his/her own style: the way to think, the way to solve problems, the way to check answers, the way to write, etc.

Anyway, I got frustrated and gave up the effort to utilize it as a framework-building thing. However I won't just dump these notes. As I have more and more to add, I think I'll just let it be my notebook, which, of course, is open source and accessible to everyone.

Yes. Use the source. Keep the source open.

3.1 Fun

Here are some interesting physics related problems.

3.1.1 Quantum

Why Rutherford was wrong about the atom model?

Link to this: *rutherford-atom*

A Rutherford atom model is the combination of protons and electrons to give us a neutral nucleus.

Question 1: What is the energy of the electron if we confine it inside the nucleus? Consider only the kinetic energy due to uncertainty principle is enough to construct a contradiction.

Question 2: From the point of view of nuclear magnetic moment, the electron magnetic moment is way to large. Measurement tells us that nuclei usually have nuclear magnetic moment $-3\mu_N$ to $10\mu_N$ where $\mu_N = e\hbar/2m_p$ and m_p is the mass of proton.

Hint: The magnetic moment is of the magnitude $e\hbar/2m_x$.

3.1.2 Relativity

Blazers in astrophysics

Link to this: *Blazers*

Blazers has a very large apparant velocity, which is usually much larger than speed of light, eg, $34c$.

This problem is very nicely explained on this page: [Apparent Superluminal Velocity of Galaxies](#) .

3.1.3 Electrodynamics in 2+1 Spacetime

Maxwell's equations are mostly experiment determined, except for one term by Maxwell involves the induced current. The only hope to write down a real 2+1 electrodynamics formalism is to really understand the most fundamental properties of electrodynamics which I don't have at this moment.

So I turned to another approach. First of all we need to reach some basic agreement that which is not changed from our 3+1 theory to a 2+1 theory. As this being said, there could be a bunch of different versions of 2+1 theory.

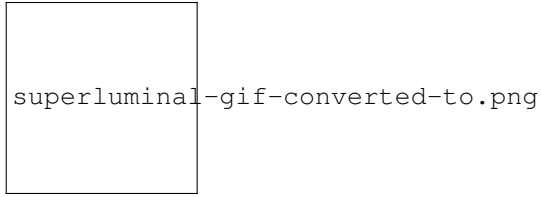


Fig. 3.1: This is taken from the link mentioned above. We measure the distance at 1 Jan then at 1 Feb. The apparent velocity would be the travelled distance divided by 1 month. However, the first measurement only measured the light from a place that is $1 + d/c$ further than the second spot due to the fact that light travels at a finite speed. So the distance we measured is larger than the actual distance at 1 Jan and 1 Feb. Thus leading to a apparent larger velocity and this velocity can exceed the limit of light speed.

To make sure we have a consistent theory, the following terms should be applied.

1. Something should be unchanged which will act as a connection between our 2+1 theory and 3+1 theory.

Maxwell's Theory

The equations could be written as

$$\partial_\mu F^{\mu\nu} = 0,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

However we would like to check the four laws independently since we really need to look into the meaning of the equations.

$$\begin{aligned}\partial_i E_i &= 4\pi\rho \\ \partial_i B_i &= 0 \\ \epsilon_{ijk} \partial_j E_k &= -\frac{1}{c} \partial_t B_i \\ \epsilon_{ijk} \partial_j B_k &= \frac{4\pi}{c} J_i + \frac{1}{c} \partial_t E_i.\end{aligned}$$

Here we write down the component form because the cross product doesn't have a clear meaning as we move to different dimensions.

The first naive version

Assumptions

1. The dimension of energy is not changed.
 2. The dimension of length and time are kept.
-

Gauss's Law

Gauss's law shows the source of the electric field, which should be in the form

$$\oint \vec{E} \cdot d\vec{l} = 2\pi \int \rho dS.$$

We have 2π instead of 4π is because we have only a integral of a closed loop not a closed surface.

Applying Stokes Theorem?

At first thought, we need to math the integral on the two sides thus Stokes theorem should be applied.

Surprisingly, we don't really get to the familiar Gauss's law of differential form. Instead, we have

$$\iint \nabla \times \vec{E} \cdot d\vec{S} = 2\pi \int \rho dS.$$

BUT think about this. Is this really true? We DO NOT have a third dimension! How could we define a curl? Back to the component form,

$$\nabla \times \vec{E} = \hat{e}_i \epsilon_{ijk} \partial_j E_k.$$

As a reminder,

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is } (1, 2, 3), (2, 3, 1) \text{ or } (3, 1, 2), \\ -1 & \text{if } (i, j, k) \text{ is } (3, 2, 1), (1, 3, 2) \text{ or } (2, 1, 3), \\ 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i \end{cases}$$

Now the problem is we have all the elements of this Levi-Civita symbol 0 because only 2 dimensions can be used in this theory.

That means we have no Gaussian theorem or no charge as a naive interpretation if we follow our idea that charge is source of static curl free electric field and followed up by using Stokes theorem.

This argument is WRONG. We need to reconsider the meaning of equations. This is 2D we don't have a third dimension to use Stokes theorem. We need divergence theorem.

Two match up the dimensions we do need to apply the divergence theorem, in 2D.

$$\oint \vec{E} \cdot d\vec{l} = \iint \partial_i E_i dS,$$

from which we are able to determine the differential form

$$\partial_i E_i = 2\pi\rho,$$

in which we have $i = 1, 2$.

Vectors in 2D

TBD

Faraday's Law

The change of magnetic flux generate electric field,

$$\oint \vec{E} \cdot d\vec{l} = -\frac{1}{c} \frac{d}{dt} \int \vec{B} d\vec{S}.$$

Wave

We still have a wave solution.

Refs

1. Electrodynamics in 2D by Kirk T. McDonald @ Princeton

VOCABULARY

4.1 Vocabulary

4.1.1 Functional Derivative

By definition,¹ functional derivative of a functional $G[f]$ with respect to f along the ‘direction’ of h is

$$\delta G[f][h] = \frac{d}{d\epsilon} G[f + \epsilon h]|_{\epsilon=0}.$$

As an example, the functional derivative of $G[f] = \int dx f^n(x)$ $\delta G[f][h]$ is

$$\begin{aligned} \delta G[f][h] &= \frac{d}{d\epsilon} G[f + \epsilon h]|_{\epsilon=0} \\ &= \int n f^{n-1}(x) h(x) dx. \end{aligned}$$

Now the problem appears. We have an unknown function h which makes sense because we haven’t specify a direction of the derivative yet.

For a physicist, the savior of integral is Dirac delta. So we use delta distribution as the direction in the functional derivative of action which is an integral,

$$\frac{\delta G[f]}{\delta f(y)} = \delta G[f][\delta_y].$$

It can be ambiguous to just write down δ_y without an example. Here is the previous example continued,

$$\begin{aligned} \frac{\delta G[f(y)]}{\delta f(y)} &= \int n f^{n-1}(x) h(x) dx|_{h(x)=\delta(x-y)} \\ &= \int n f^{n-1}(x) \delta(x-y) dx \\ &= n f^{n-1}(y). \end{aligned}$$

It seems that we can just think of f as a variable then take the ordinary derivative with respect to it. It is NOT true.

Consider such a functional $G[f] = \int (f'(x))^2 dx$ where ‘ \prime ’ means the derivative of $f(x)$.

$$\begin{aligned} \frac{\delta G[f]}{\delta f} &= - \int dx 2 f''(x) h(x)|_{h(x)=\delta(x-y)} \\ &= -2 f''(y), \end{aligned}$$

which is not that straightforward to understand from function derivatives.

¹ Chapter 15 of Physical Mathematics

4.1.2 Legendre Transformation

Legendre transformation is NOT just some algebra. Given $f(x)$ as a function of x , which is shown in blue, we could find the distance between a line $y = px_i$ and the function value $f(x_i)$.

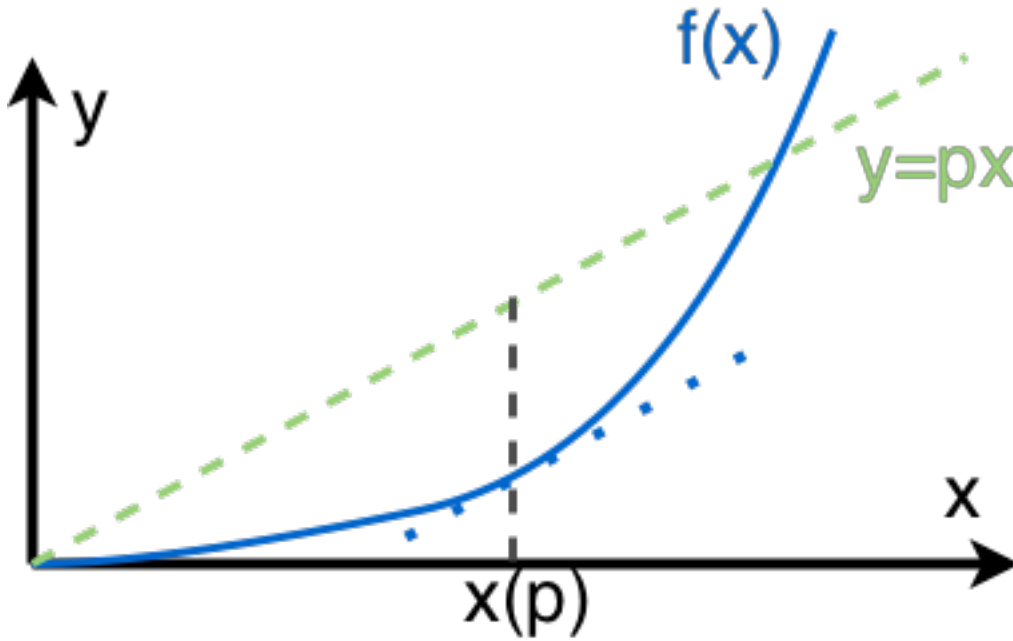


Fig. 4.1: Meaning of Legendre transformation

However, as we didn't fix x , this means that the distance

$$F(p, x) = px - f(x)$$

varies according to x . This is a transformation that maps a function $f(x)$ to some other function $F(p, x)$ which depends on the parameter p . A more pedagogical way of writing this is

$$px = F(x, p) + f(x).$$

To have a Legendre transformation, let's choose a relation between x and p . One choice is to make sure we have a maximum distance given p , which means the x we choose is the point that makes the slope of $f(x)$ the same as the line $y = px$. In the language of math, the condition we require is

$$0 = \frac{\partial F(p, x)}{\partial x} \equiv f'(x) - p,$$

which indeed shows that the slope of function and slope of the straight line match each other at the specified point. Thus we have a relation between x and p .

Substitute $x(p)$ back into $F(p, x)$, we will get the Legendre transformation $F(p, x(p))$ of $f(x)$.

Back to the math we learned in undergrad study. A Legendre transformation transforms a function of x to another function with variable $\frac{f(x)}{x}$. Using $f(x)$ and its Legendre transformation $F(p = px - f(x(p)))$ as an example, we can show that the slope of $F(p)$ is x ,

$$\frac{dF(p)}{dp} = x,$$

which is intriguing because the slope of $f(x)$ is p in our requirement. We removed the dependence of x in $F(p)$ because we have this extra constrain.

Let's Move to Another Level

We require the function $f(x)$ is convex (second order derivative is not negative). This is required because otherwise we would NOT have a one on one mapping of x and p .

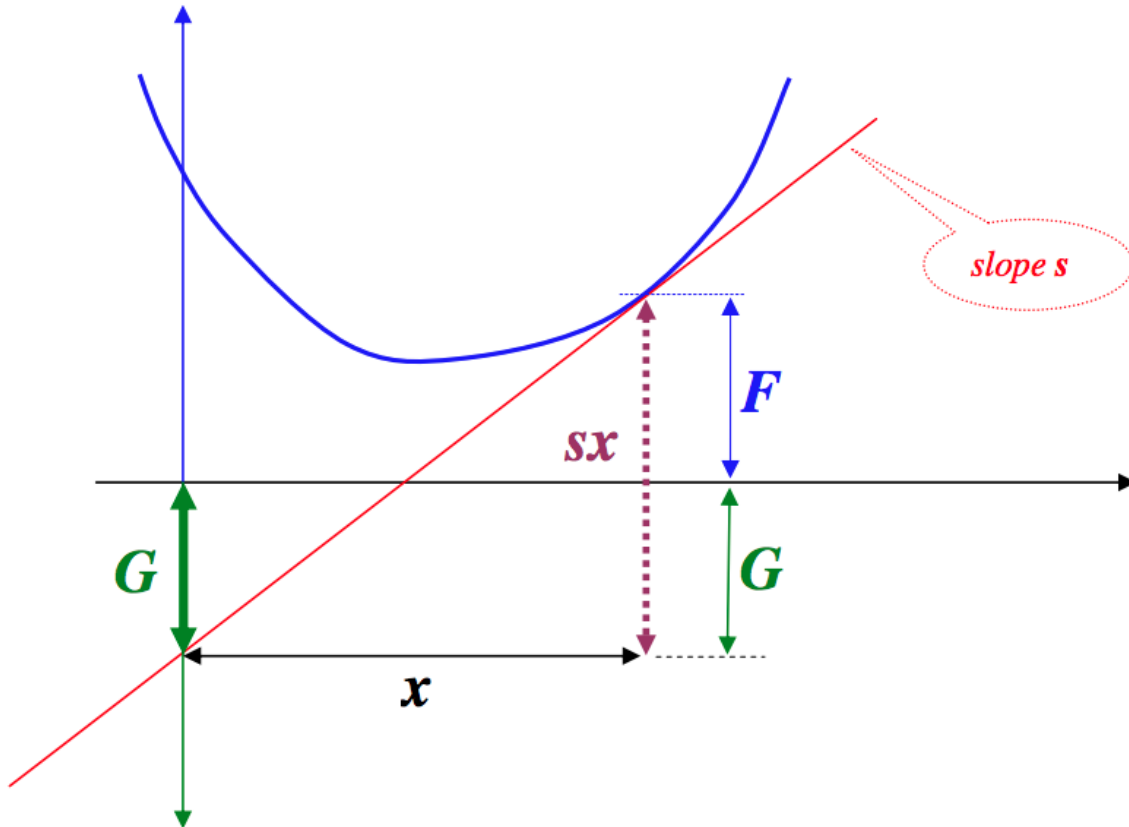


Fig. 4.2: This graph shows the Legendre transformation and triangles in which G is actually the F we used before and F in the graph corresponds to f .

One immediately notices the symmetry of Legendre transformation on interchanging of F and f .

This graph is taken from this paper [Making Sense of the Legendre Transform](#) .

This is the triangle that represents the Legendre transformation.

If we have a slope that vanishes, which means $f(x)$ is at minimum, then we have the relation

4.1.3 Vector Analysis

The ultimate trick is to use component form.

$$\begin{aligned}
 & \vec{a} \times (\vec{b} \times \vec{c}) \\
 &= \hat{e}_i \epsilon_{ijk} a_j (\epsilon_{kmn} b_m c_n) \\
 &= \hat{e}_i \epsilon_{kij} \epsilon_{kmn} a_j b_m c_n \\
 &= \hat{e}_i (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) a_j b_m c_n \\
 &= \hat{e}_i \delta_{im} \delta_{jn} a_j b_m c_n - \hat{e}_i \delta_{in} \delta_{jm} a_j b_m c_n \\
 &= \hat{e}_i a_j b_i c_j - \hat{e}_i a_j b_j c_i \\
 &= \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}).
 \end{aligned}$$

One should be able to find the component forms of gradient $\vec{\nabla}$, divergence $\vec{\nabla} \cdot$, Laplace operator, in **spherical coordinates, cylindrical coordinates and cartesian coordinates**.

4.1.4 Refs & Notes

4.2 Basic

4.2.1 Dimension

How to find the relationship between two quantities? For example, what is the dimensional relationship between length and mass.

* Plank constant: $\hbar \sim [\text{Energy}] \cdot [\text{Time}] \sim [\text{Mass}] \cdot [\text{Length}]^2 \cdot [\text{Time}]^{-1}$

* Speed of light in vacuum: $c \sim [\text{Length}] \cdot [\text{Time}]^{-1}$

* Gravitational constant: $G \sim [\text{Length}]^3 \cdot [\text{Mass}]^{-1} \cdot [\text{Time}]^{-2}$

Then it is easy to find that a combination of c/\hbar cancels the dimension of mass and leaves the inverse of length. That is

$$[L]^2 = \left[\frac{\hbar G}{c^3} \right]$$

$$[M]^2 = \left[\frac{\hbar c}{G} \right]$$

$$[T]^2 = \left[\frac{\hbar G}{c^5} \right]$$

As we can see, it is possible to use $c = 1, \hbar = 1, G = 1$ because we can always restore the units in a deterministic way. c, \hbar, G are function of mass, length, time, and with $c = \hbar = G = 1$ give us only one solution of mass, length and time: three equations + three variables.

Planck Scales

As we have seen, the three constant can make up a length scale, a mass scale, a time scale. Then what are they?

Planck length:

$$l_P = \sqrt{\frac{\hbar G}{c^3}}$$

Planck mass:

$$m_P = \sqrt{\frac{\hbar c}{G}}$$

Planck time:

$$t_P = \sqrt{\frac{\hbar G}{c^5}}$$

Equations and Dimensions

Before solving equations, it is good to reform them in to dimensionless ones.

To make the equation dimensionless doesn't mean we can just divide arbitrary terms on both sides. We need to find out the characteristic quantity of the system. For example, we can divide by $\hbar\omega$ on both sides of Schrodinger equation for Harmonic Oscillators. This is a good step because $\hbar\omega$ is the characteristic energy scale of system. At the same time, we can make the length terms dimensionless using the characteristic length. DO NOT use an arbitrary length!

Most Wonderful Equations That Should Never Be Forgotten

Electrodynamics

$$\nabla \times \vec{E} = -\partial_t \vec{B}$$

$$\nabla \times \vec{H} = \vec{J} + \partial_t \vec{D}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

For linear materials,

$$\vec{D} = \epsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

$$\vec{J} = \sigma \vec{E}$$

Dynamics

Hamilton conanical equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Thermodynamics and Statistical Physics

Liouville's Law

$$\frac{d\rho}{dt} \equiv \frac{\partial \rho}{\partial t} + \sum_i \left[\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right] = 0$$

MATHEMATICS

5.1 Mathematics

5.1.1 Kindergarten

Binominal theorem

$$(1+x)^n = \sum_{k=0}^n C_n^k x^k.$$

5.1.2 Complex Analysis

Some useful concepts: ¹

- Representation of a complex number and its conjugate
- Complex functions
- curves, closed curves, simple curves
- Infinity point
- Analytic functions: depends only on z not its complex conjugate
- Entire function: single-valued analytic all over \mathbb{C}
- Liouville theorem
- Pole
- Singularity, Essential Singularity
- Meromorphic function

For multi-valued functions,

- A branch of a function
- Analyticity of multi-valued function
- Branch point
- Cut

Operations

- Contour integral of a continuous function around some simple curve

¹ A handout note by Finly

- Cauchy's Integral Theorem

Cauchy-Riemann Equation

A function $f(z) = u(z) + iv(z)$ is a function of a complex variable $z = x + iy$.

$$\begin{aligned}\frac{\partial}{\partial x}u &= \frac{\partial}{\partial y}v \\ \frac{\partial}{\partial x}v &= -\frac{\partial}{\partial y}u\end{aligned}$$

Singularities

There are 3 common singularities,

1. Pole
2. Branch point
3. Essential singularity

Pole is very useful since it's related to the Residue Theorem. Thus one of the task in physics is to calculate the residue of a function.

The residue at a simple pole is given by

$$\text{Residue}(f(z_0)) = \lim_{z \rightarrow z_0} ((z - z_0)f(z)).$$

Meanwhile, the residue at a pole of nth order is

$$\text{Residue}(f(z_0)) = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} ((z - z_0)^n f(z)).$$

Branch points are points when we go around it in circles the values of our function would change. Examples of such points are $z = 0$ for $f(z) = \ln(z)$ and $z = 1$ for $f(z) = (z - 1)^{1/2}$.

5.1.3 Calculus

Differential of Functions

Integrals

Sometimes a integral on Real plane can be very hard, one of the techniques is to work on Complex plane and use contour integral.

1. Contours: use Ghost Contours so that we don't need to calculate these complicated integrals.
2. Branch Cut: cuts are needed if we have got branch points on the complex plane.
3. Residue Theorem: we can write down the integral by calculating the residue of the integrand,

$$\int_C f(z)dz = 2\pi i \sum_j \text{Residue}(f(z_j)),$$

where z_j are the poles.

Calculus on One Page

Differential	Integral
Fundamental Theorem of Calculus	
$\frac{d}{dx} \int_a^x f(t)dt = f(x)$	$\int_a^x \frac{d}{dt} f(t)dt = f(x) - f(a)$
Rules	
$[c_1 u(x) + c_2 v(x)]' = c_1 u'(x) + c_2 v'(x)$	$\int [c_1 f(x) + c_2 g(x)] dx = c_1 \int f(x)dx + c_2 \int g(x)dx$
$(uv)' = u'v + uv'$	$\int u'v dx = uv - \int uv' dx$
$\frac{df(u(x))}{dx} = \frac{df}{du} \frac{du}{dx}$	$\int \frac{df}{du} \frac{du}{dx} dx = f(u(x)) + c$
Mean Value Theorem	
$\frac{F(b)-F(a)}{b-a} = F'(\xi)$	$\int_a^b f(x)dx = (b-a)f(\xi)$
Useful Equations	
$(x^n)' = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c$
$(\sin x)' = \cos x$	$\int \cos x dx = \sin x + c$
$(\ln x)' = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln x + c$
$(e^x)' = e^x$	$\int e^x dx = e^x + c$
Multivariable Calculus (Fundamental Theorem)	
$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$	

Notes:

- One can derive other useful equation using just these in this table.
- The three equations of fundamental theorem for multivariable calculus can be derived using the exterior derivative.

Fig. 5.1: LaTeX source of this image is here .

5.1.4 Linear Algebra

Tensor Product Space

$|\phi\rangle_1$ and $|\phi\rangle_2$ are elements of Hilbert space H_1 and H_2 . **Tensor Product** of $|\phi\rangle_1$ and $|\phi\rangle_2$ is denoted as $|\phi\rangle_1 \otimes |\phi\rangle_2$. This operation is linear and distributive.

Tensor product space $H_1 \otimes H_2$ is composed of all the linear combinations of all possible tensor products of elements in H_1 and H_2 .

Inner Product

Inner product of two tensor products

$$(\langle\phi|_1 \otimes \langle\phi|_2)(|\psi\rangle_1 \otimes |\psi\rangle_2) = ({}_1\langle\phi| \psi\rangle_1)({}_2\langle\phi| \psi\rangle_2)$$

Operators Applied to Tensor Product

Two operators \hat{O}_1 and \hat{O}_2 works on H_1 and H_2 respectively applied to tensor product

$$(\hat{O}_1 \otimes \hat{O}_2)(|\phi\rangle_1 \otimes |\phi\rangle_2) = (\hat{O}_1 |\phi\rangle_1) \otimes (\hat{O}_2 |\phi\rangle_2)$$

Solving Linear Equations

First of all, write down the augmented matrix for the equation set.

Elementary row operations are allowed on the augmented matrix. Operate on the matrix until one can read out the solutions.

5.1.5 Differential Geometry

Metric

Definitions

Denote the basis in use as \hat{e}_μ , then the metric can be written as

$$g_{\mu\nu} = \hat{e}_\mu \cdot \hat{e}_\nu$$

if the basis satisfies

Inversed metric

$$g_{\mu\lambda} g^{\lambda\nu} = \delta_\mu^\nu = g_\mu^\nu$$

How to calculate the metric

Let's check the definition of metric again.

If we choose a basis \hat{e}_μ , then a vector (at one certain point) in this coordinate system is

$$x^a = x^\mu \hat{e}_\mu$$

Then we can construct the expression of metric of this point under this coordinate system,

$$g_{\mu\nu} = \hat{e}_\mu \cdot \hat{e}_\nu$$

For example, in spherical coordinate system,

$$\vec{x} = r \sin \theta \cos \phi \hat{e}_x + r \sin \theta \sin \phi \hat{e}_y + r \cos \theta \hat{e}_z \quad (5.1)$$

Now we have to find the basis under spherical coordinate system. Assume the basis is $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$. Choose some scale factors $h_r = 1, h_\theta = r, h_\phi = r \sin \theta$. Then the basis is

$$\hat{e}_r = \frac{\partial \vec{x}}{\partial r} = \hat{e}_x \sin \theta \cos \phi + \hat{e}_y \sin \theta \sin \phi + \hat{e}_z \cos \theta,$$

etc. Then collect the terms in formula (4.1) is we get $\vec{x} = r \hat{e}_r$, this is incomplete. So we check the derivative.

$$\begin{aligned} d\vec{x} &= \hat{e}_x (dr \sin \theta \cos \phi + r \cos \theta \cos \phi d\theta - r \sin \theta \sin \phi d\phi) \\ &\quad + \hat{e}_y (dr \sin \theta \sin \phi + r \cos \theta \sin \phi d\theta + r \sin \theta \cos \phi d\phi) \\ &\quad + \hat{e}_z (dr \cos \theta - r \sin \theta d\theta) \\ &= dr (\hat{e}_x \sin \theta \cos \phi + \hat{e}_y \sin \theta \sin \phi + \hat{e}_z \cos \theta) \\ &\quad + d\theta (\hat{e}_x \cos \theta \cos \phi + \hat{e}_y \cos \theta \sin \phi - \hat{e}_z \sin \theta) r \\ &\quad + d\phi (-\hat{e}_x \sin \phi + \hat{e}_y \cos \phi) r \sin \theta \\ &= \hat{e}_r dr + \hat{e}_\theta r d\theta + \hat{e}_\phi r \sin \theta d\phi \end{aligned}$$

Once we reach here, the component (e_r, e_θ, e_ϕ) of the point under the spherical coordinates system basis $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$ at this point are clear, i.e.,

$$\begin{aligned} d\vec{x} &= \hat{e}_r dr + \hat{e}_\theta r d\theta + \hat{e}_\phi r \sin \theta d\phi \\ &= e_r dr + e_\theta d\theta + e_\phi d\phi \end{aligned}$$

In this way, the metric tensor for spherical coordinates is

$$g_{\mu\nu} = (e_\mu \cdot e_\nu) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$$

Connection

First class connection can be calculated

$$\Gamma^\mu_{\nu\lambda} = \hat{e}^\mu \cdot \hat{e}_{\mu,\lambda}$$

Second class connection is footnote{Kevin E. Cahill}

$$[\mu\nu, \iota] = g_{\iota\mu} \Gamma^\mu_{\nu\lambda}$$

Gradient, Curl, Divergence, etc

Gradient

$$T^b_{c;a} = \nabla_a T^b_c = T^b_{c,a} + \Gamma^b_{ad} T^d_c - \Gamma^d_{ac} T^b_d$$

Curl

For an anti-symmetric tensor, $a_{\mu\nu} = -a_{\nu\mu}$

$$\begin{aligned}\text{Curl}_{\mu\nu\tau}(a_{\mu\nu}) &\equiv a_{\mu\nu;\tau} + a_{\nu\tau;\mu} + a_{\tau\mu;\nu} \\ &= a_{\mu\nu,\tau} + a_{\nu\tau,\mu} + a_{\tau\mu,\nu}\end{aligned}$$

Divergence

$$\begin{aligned}\text{div}_\nu(a^{\mu\nu}) &\equiv a^{\mu\nu}{}_{;\nu} \\ &= \frac{\partial a^{\mu\nu}}{\partial x^\nu} + \Gamma_{\nu\tau}^\mu a^{\tau\nu} + \Gamma_{\nu\tau}^\nu a^{\mu\tau} \\ &= \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} a^{\mu\nu}) + \Gamma_{\nu\lambda}^\mu a^{\nu\lambda}\end{aligned}$$

For an anti-symmetric tensor

$$\text{div}(a^{\mu\nu}) = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} a^{\mu\nu})$$

Annotation Using the relation $g = g_{\mu\nu} A_{\mu\nu}$, $A_{\mu\nu}$ is the algebraic complement, we can prove the following two equalities.

$$\Gamma_{\mu\nu}^\mu = \partial_\nu \ln \sqrt{-g}$$

$$V^\mu{}_{;\mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} V^\mu)$$

In some simple case, all the three kind of operation can be demonstrated by different applications of the del operator, which $\nabla \equiv \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z$.

- Gradient, ∇f , in which f is a scalar.
- Divergence, $\nabla \cdot \vec{v}$
- Curl, $\nabla \times \vec{v}$
- Laplacian, $\Delta \equiv \nabla \cdot \nabla \equiv \nabla^2$

5.1.6 Linear Algebra

Basic Concepts

Trace

Trace should be calculated using the metric. An example is the trace of Ricci tensor,

$$R = g^{ab} R_{ab}$$

Einstein equation is

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab}$$

The trace is

$$\begin{aligned} g^{ab}R_{ab} - \frac{1}{2}g^{ab}g_{ab}R &= 8\pi Gg^{ab}T_{ab} \\ \Rightarrow R - \frac{1}{2}4R &= 8\pi GT \\ \Rightarrow -R &= 8\pi GT \end{aligned}$$

Technique

Inverse of a matrix

Many methods to get the inverse of a matrix. Check wikipedia for Invertible matrix.

Adjugate matrix method for example is here.

$$A^{-1} = \frac{A^*}{|A|}$$

in which, A^* is the adjugate matrix of A .

5.1.7 Refs & Notes

5.2 Special Functions

There are a lot of useful special function in physics. Some of them provides physics understanding of the problem, some of them helps us writing down a solution quickly.

Among them, Gamma functions, Legendre polynomials, Bessel functions, spherical harmonics, modified bessel functions, spherical bessel functions, and elliptical functions are the most used ones.

5.2.1 Gamma Functions

Gamma function satisfies the following relation,

$$\Gamma(z+1) = z\Gamma(z).$$

For some cases, it can also be written as

$$\Gamma(n) = \int_0^\infty dt t^{n-1} e^{-t}.$$

One can prove that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

5.2.2 Legendre Polynomials

Legendre polynomials are solutions to Legendre equation, which is

$$\left(\frac{d}{dx} \left[(1-x^2) \frac{d}{dx} \right] + n(n+1) \right) P_n(x) = 0.$$

Legendre polynomials has many different representations.

Integral

$$P_n(z) = \frac{1}{2\pi i} \oint (1 - 2tz + t^2)^{1/2} t^{-n-1} dt.$$

Rodrigues representation

$$P_n(z) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l.$$

It's generation function is

$$\frac{1}{\sqrt{1 + \eta^2 - 2\eta x}} = \sum_{k=0}^{\infty} \eta^k P_k(x).$$

Properties

Orthogonality

$$\int_{-1}^1 P_m(x) P_n(x) dx = \frac{2}{2n+1} \delta_{mn}.$$

They all have value 1 at $z = 1$.

The parity is alternating.

Examples

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1). \end{aligned}$$

Through these, we can solve out

$$\begin{aligned} x &= P_1(x) \\ x^2 &= \frac{1}{3}(P_0(x) + 2P_2(x)). \end{aligned}$$

Notice that they have physics meanings although it's better to understand it together with spherical harmonics.

5.2.3 Associated Legendre Polynomials

The associated Legendre equation is

$$\left(\frac{d}{dx} \left[(1 - x^2) \frac{d}{dx} \right] + n(n+1) - \frac{m^2}{1 - x^2} \right) P_n(x) = 0.$$

The solution to this equation is Associated Legendre polynomial, which can be represented by

$$P_n^\nu(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x).$$

5.2.4 Bessel Functions

Bessel functions are solutions to Bessel equation,

$$\left(x \frac{d}{dx} x \frac{d}{dx} + x^2 - \nu^2\right) J_\nu(x) = 0.$$

They all satisfy these recurrence relations,

$$\begin{aligned} Z_{n+1} + Z_{n-1} &= \frac{2n}{x} Z_n \\ Z_{n-1} - Z_{n+1} &= 2 \frac{d}{dx} Z_n. \end{aligned}$$

Bessel Function of the first kind

Use notation $J_n(x)$ for the first kind.

Generating function is

$$e^{\frac{z}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} i^n f(t) t^n J_n(z).$$

Integral representation

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n\tau - x \sin \tau)} d\tau.$$

It also has a **summation representation**,

$$J_\alpha(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \alpha + 1)} \left(\frac{z}{2}\right)^{2m + \alpha}.$$

At large $|x|$ limits, we have

$$\begin{aligned} \lim_{|x| \rightarrow \infty} J_l(x) &= \frac{\sin\left(z - l \frac{\pi}{2}\right)}{x} \\ \lim_{|x| \rightarrow \infty} J'_l(x) &= \frac{\cos\left(z - l \frac{\pi}{2}\right)}{x}. \end{aligned}$$

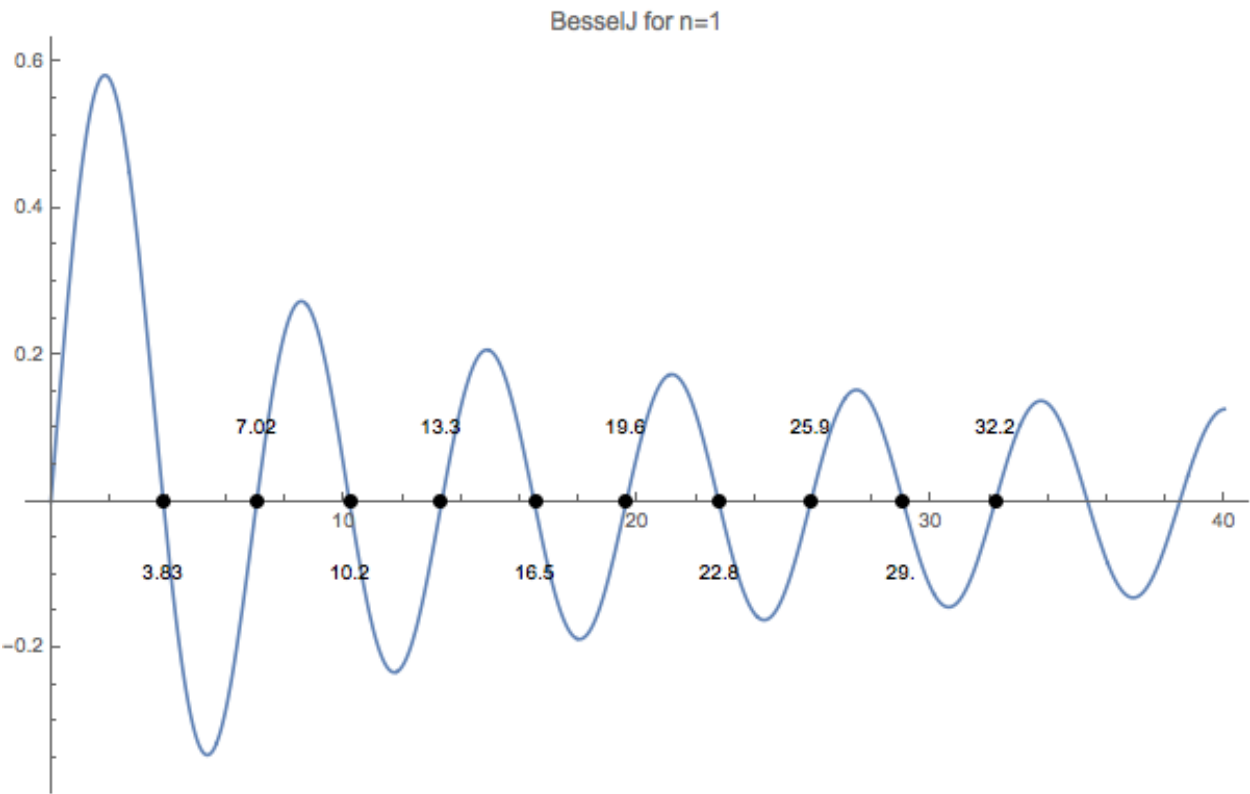
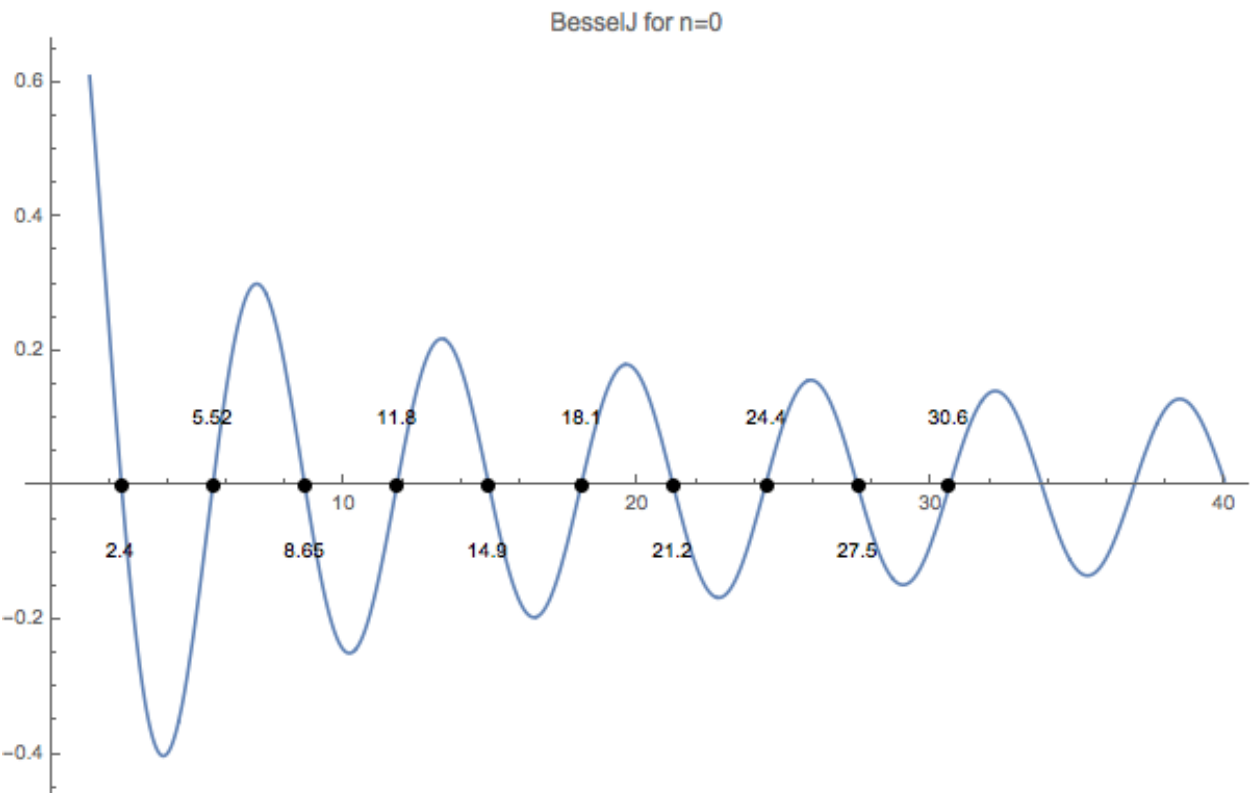
By playing with the recurrence relation,

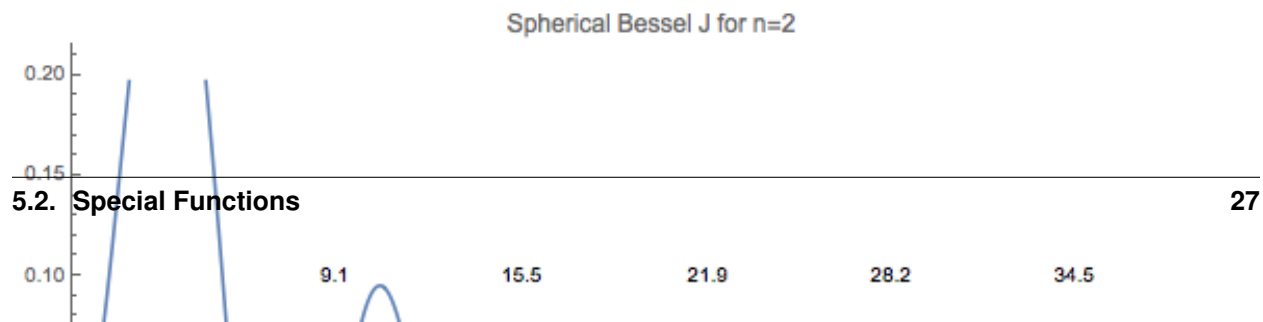
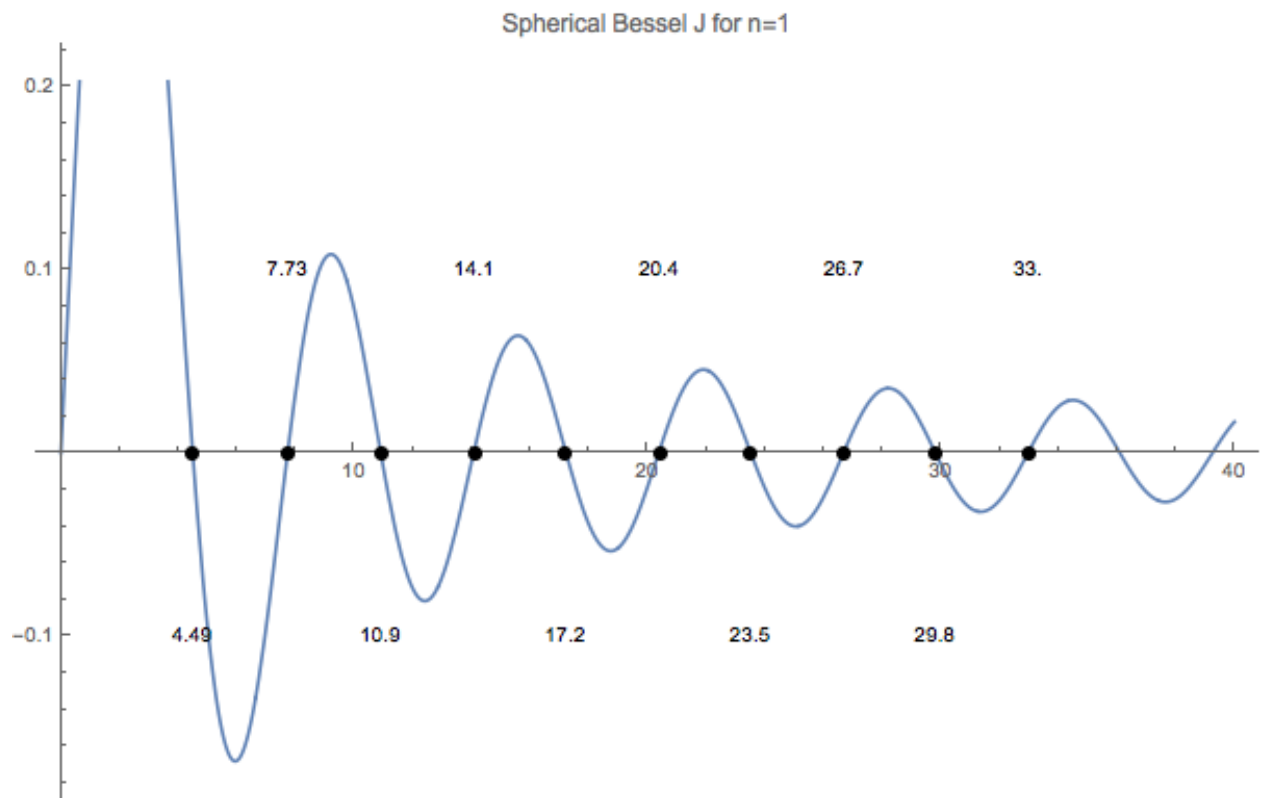
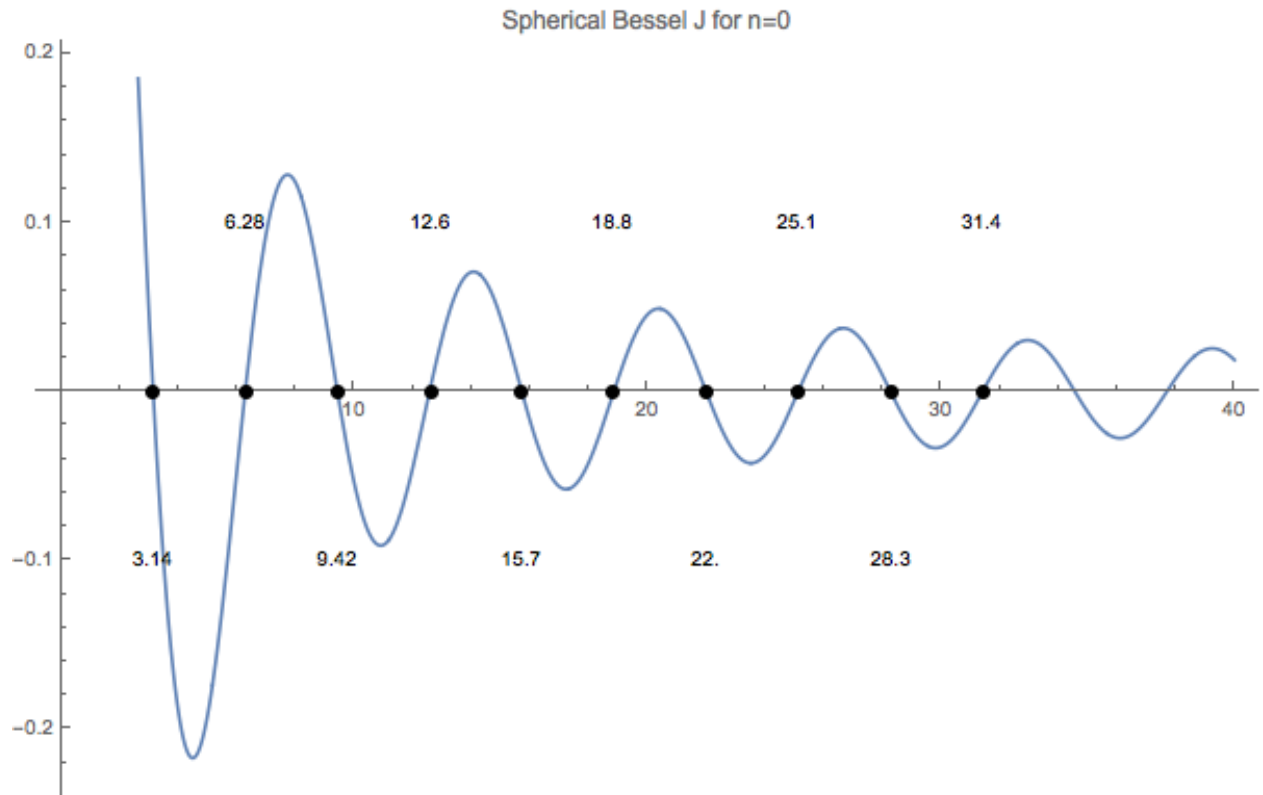
$$\begin{aligned} 2J'_n &= J_{n-1} - J_{n+1} \\ 2nJ_n &= J_{n+1} + J_{n-1}, \end{aligned}$$

we can get two more useful relations,

$$\begin{aligned} \frac{d}{dz}(z^n J_n) &= z^n J_{n-1} \\ \frac{d}{dz}(z^{-n} J_n) &= -z^{-n} J_{n+1}. \end{aligned}$$

They are very useful when integrating by part.





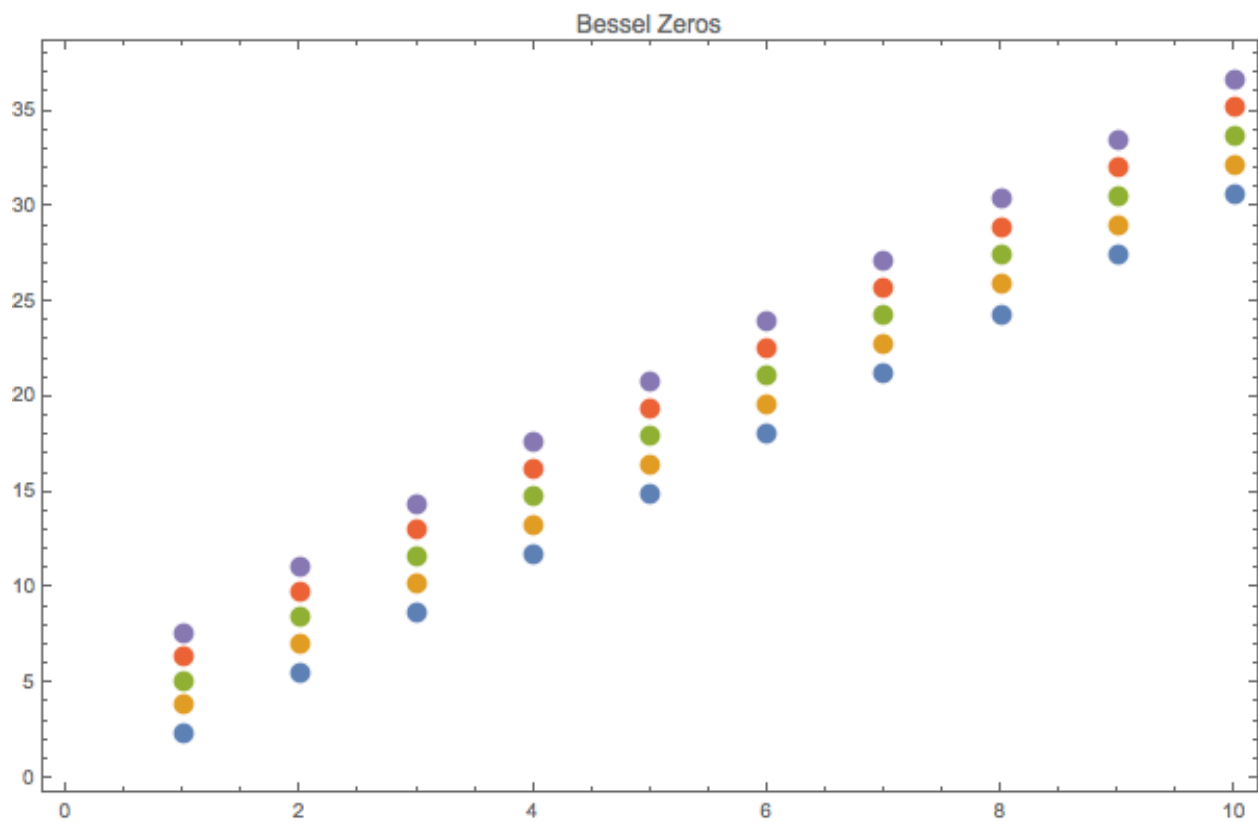


Fig. 5.4: Bessel function zeros in a list plot. Horizontal axis is nth zero point, while vertical axis is the value.

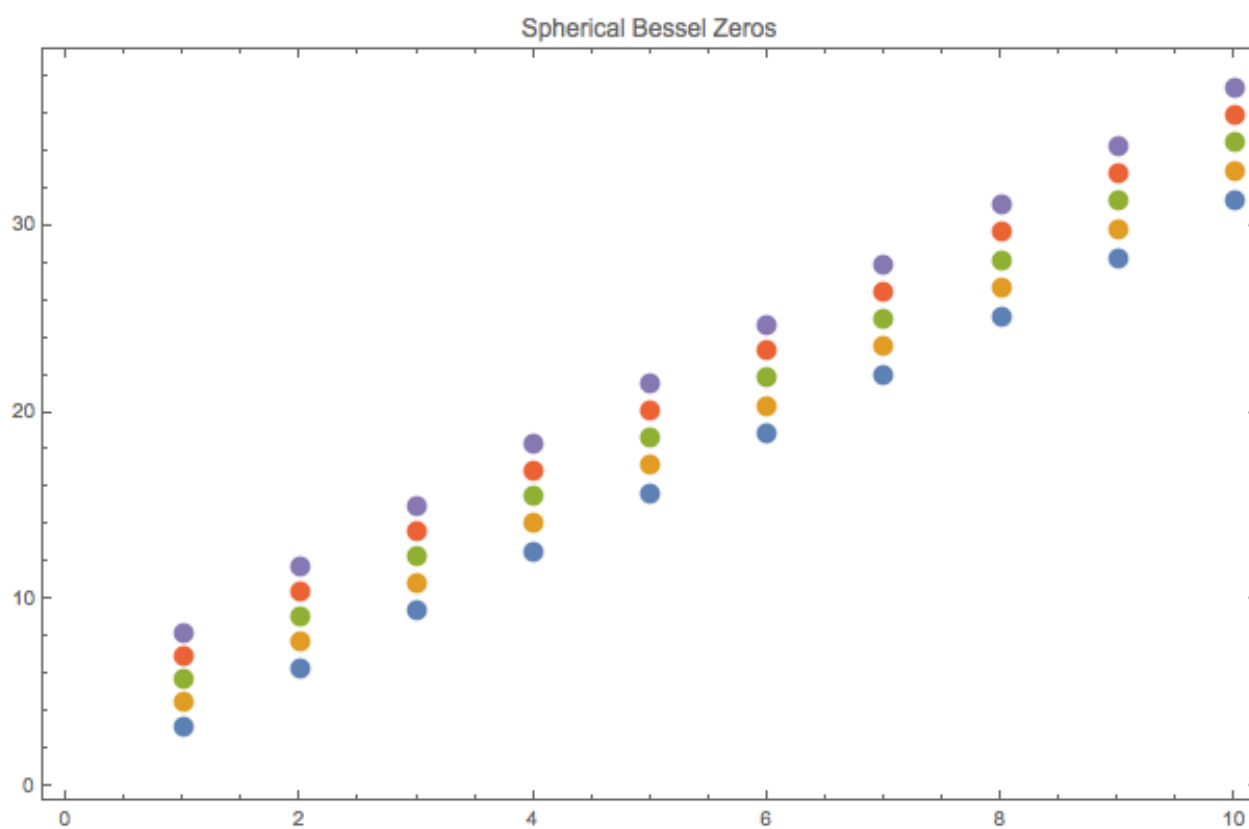


Fig. 5.5: Spherical Bessel function zeros.

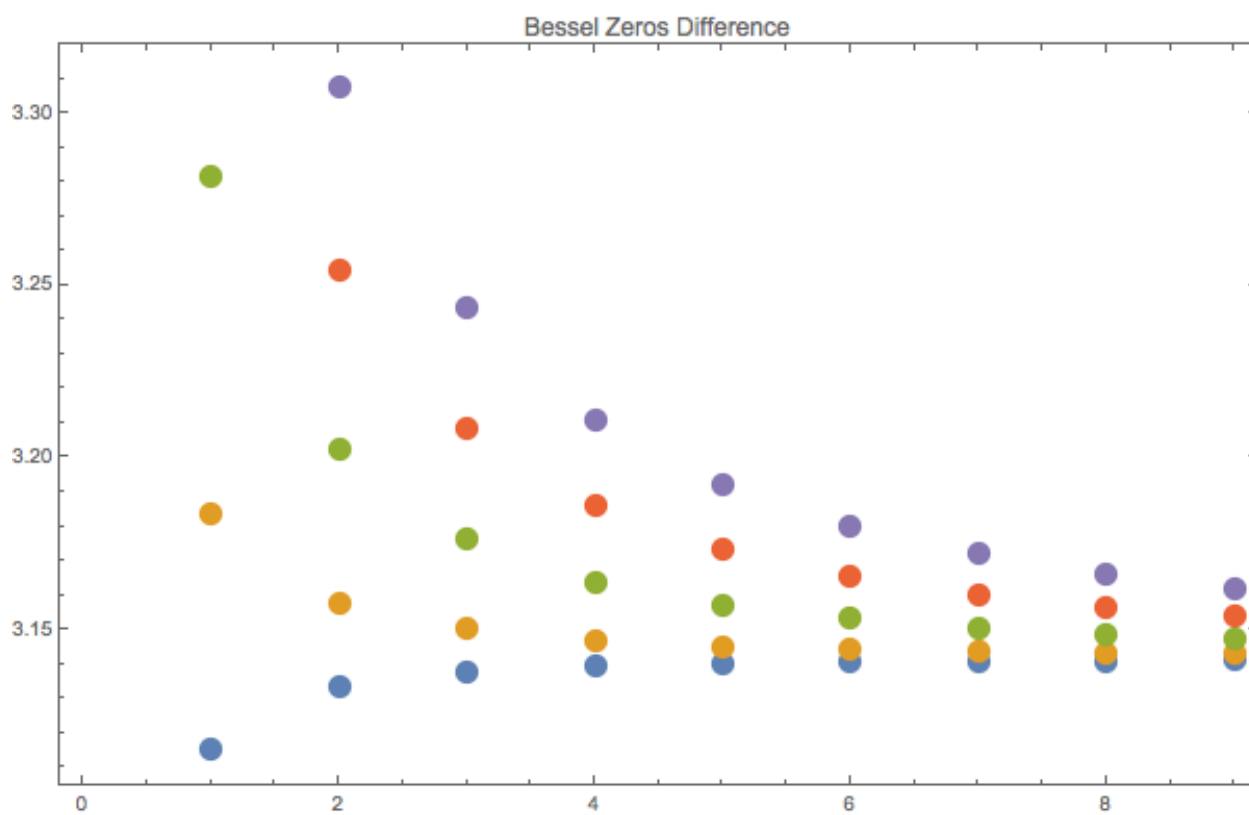


Fig. 5.6: The difference between zeros of Bessel functions. They are almost the same, which is around π .

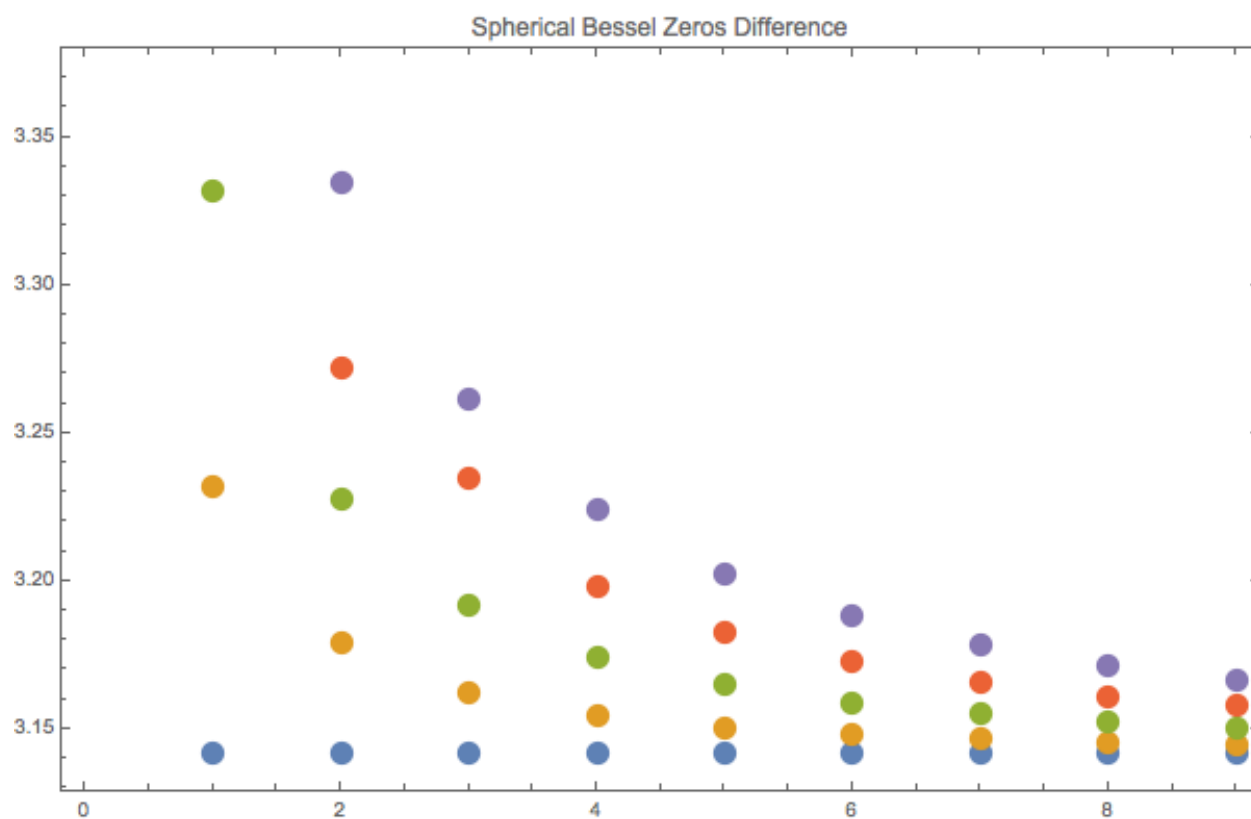


Fig. 5.7: Spherical Bessel function zeros differences.

Graphics and Properties

5.2.5 Refs & Notes

5.3 Equation Solving

There are so many methods and techniques to solve an equation. Here we will review only some of them.

5.3.1 Ordinary Differential Equations

There are many important equations in physics.

There are many methods to solve an ODE,

1. Green's function.
2. Series solution
3. Laplace transform
4. Fourier transform

Green's Function

Suppose we have a differential operator L_x , for example L_x can be $L_x \equiv \frac{d^2}{dx^2} + 1$. The definition of GF is

$$L_x G(x, z) = \delta(x - z).$$

with the constrain of boundary condition of the ODE.

In most cases, GF is a stepwisid function.

The application of GF to ODE follows the precEDURE,

1. Find the general form of GF for operator L_x ;
2. Apply BC to GF;
3. Continuity at $n - 2$ order of derivatives at point $x = z$, i.e., $G^{(n-2)}(x, z)|_{x < z} = G^{(n-2)}(x, z)|_{x > z}$ at $x = z$;
4. Discontinuity of the first order derivative at $x = z$, i.e., $G^{(n-1)}(x, z)|_{x > z} - G^{(n-1)}(x, z)|_{x < z} = 1$ at point $x = z$;
5. Solve the coefficients to get the GF;
6. The solution to an inhomogeneous ODE $L_x y(x) = f(x)$ is given immediately by

$$y(x) = \int_{Lower}^{Upper} G(x, z) f(z) dz$$

An Example

Solve equation

$$y'' + \frac{1}{4}y = f(x).$$

{bf Green's Function }

The operator is $\hat{L} = \partial^2 + 1/4$ with boundary condition $y(0) = y(\pi) = 0$.

Equation	Regular singularities	Essential singularities
Hypergeometric $z(1-z)y'' + [c - (a+b+1)z]y' - aby = 0$	$0, 1, \infty$	—
Legendre $(1-z^2)y'' - 2zy' + \ell(\ell+1)y = 0$	$-1, 1, \infty$	—
Associated Legendre $(1-z^2)y'' - 2zy' + \left[\ell(\ell+1) - \frac{m^2}{1-z^2}\right]y = 0$	$-1, 1, \infty$	—
Chebyshev $(1-z^2)y'' - zy' + v^2y = 0$	$-1, 1, \infty$	—
Confluent hypergeometric $zy'' + (c-z)y' - ay = 0$	0	∞
Bessel $z^2y'' + zy' + (z^2 - v^2)y = 0$	0	∞
Laguerre $zy'' + (1-z)y' + vy = 0$	0	∞
Associated Laguerre $zy'' + (m+1-z)y' + (v-m)y = 0$	0	∞
Hermite $y'' - 2zy' + 2vy = 0$	—	∞
Simple harmonic oscillator $y'' + \omega^2y = 0$	—	∞

Table 16.1 Important second-order linear ODEs in the physical sciences and engineering.

Fig. 5.8: Taken from Riley's book.

First step is to find the Green's function of this operator, which is defined as the solution to

$$\hat{L}G(x, x') = \delta(x - x'),$$

where \hat{L} only operates on x not x' .

The general solutions for $\hat{L}G(x, x') = 0$ is

$$G(x, x') = A \cos(x/2) + B \sin(x/2).$$

Applying the BC, we reach a step function expression for $G(x, x')$,

$$\begin{aligned} G(x, x') &= B \sin(x/2), \text{ if } 0 \leq x \leq x', \\ &= A \cos(x/2), \text{ if } x' \leq x \leq \pi. \end{aligned}$$

{bf Continuity and Discontinuity }

It is required by the equation for Green's function that,

$$G(x', x') = A \sin(x'/2) = B \cos(x'/2),$$

and

$$\frac{dG}{dx} \Big|_{x'+} - \frac{dG}{dx} \Big|_{x'-} = 1.$$

Put the expressions for Green's function in, we can solve the coefficients,

$$\begin{aligned} G(x, x') &= -2 \cos(x'/2) \sin(x/2), \text{ if } 0 \leq x \leq x', \\ &= -2 \sin(x'/2) \cos(x/2), \text{ if } x' \leq x \leq \pi. \end{aligned}$$

In one line this can be written as,

$$G(x, x') = -2 \sin(x_{<}/2) \cos(x_{>}/2).$$

The final step is to find the solution to original equation, which is straightforward.

Series Solution

A second order ODE,

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

Wronskian of this is

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix},$$

where y_1 and y_2 are linearly independent solutions, i.e., $c_1 y_1 + c_2 y_2 = 0$ is only satisfied when $c_1 = c_2 = 0$.
Wronskian is NOT zero if they are linearly independent.

Singularities of an ODE is defined when $p(x)$ or $q(x)$ or both of them have singular points. For example, Legendre equation

$$(1 - z^2)y'' - 2zy' + l(l + 1)y = 0$$

has three singular points which are $z = \pm 1, \infty$ while $z = 0$ is an ordinary point.

Solution at Ordinary Points

Series expansion of the solution can be as simple as

$$y(z) = \sum_{n=0}^{\infty} a_n z^n,$$

which converges in a radius R where R is the distance from $z = 0$ to the nearest singular point of our ODE.

Solution at Regular Singular Points

Frobenius series of the solution

$$y(z) = z^{\sigma} \sum_{n=0}^{\infty} a_n z^n.$$

The next task is to find the indicial equation.

If the roots are not differing by an integer, we just plug the two solutions to σ in and find two solutions independently.

If the roots differ by an integer, on the other side, we can only plug in the **larger** root and find one solution. As for the second solution, we need some other techniques, such as Wronskian method and derivative method.

Wronskian method requires two expression of Wronskian, which are

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix},$$

and

$$W(z) = C e^{-\int^z p(u) du}.$$

From the first expression, we have

$$y_2(z) = y_1(z) \int^z \frac{W(u)}{y_1(u)^2} du.$$

However, we don't know $W(z)$ at this point. We should apply the second expression of Wronskian,

$$y_2(z) = y_1(z) \int^z \frac{C e^{-\int^z p(u) du}}{y_1(u)^2} du,$$

where the constant C can be set to 1 as one wish.

TO DO

The **derivative method** is on my to do list.

Comparing With A General Form

For equation that take the following form,

$$y'' + \frac{1-2a}{x} y' + \left((bcx^{c-1})^2 + \frac{a^2 - p^2 c^2}{x^2} \right) y = 0,$$

where $y \equiv y(x)$, we can write down the solutions immediately,

$$y(x) = x^a \mathcal{Z}_p(bx^c),$$

in which \mathcal{Z}_p is the solution to Bessel equation, i.e., is one kind of Bessel function with index p .

A Pendulum With A Uniformly Changing String Length

As an example, let's consider the case of length changing pendulum,

$$\frac{d}{dt} (ml^2 \dot{\theta}) = -mgl \sin \theta \approx -mgl\theta.$$

Notice that l is a function of time and

$$l = l_0 + vt.$$

Then the equation can be rewritten as

$$\frac{d^2}{dl^2} \theta + \frac{2}{l} \frac{d}{dl} \theta + \frac{g/v^2}{l} \theta = 0.$$

Comparing with the general form, we have one of the possible solutions

$$\begin{aligned} a &= -1/2, \\ pc &= 1/2, \\ c &= 1/2, \\ p &= 1, \\ b &= 2\sqrt{g}/v. \end{aligned}$$

This solution should be

$$\begin{aligned} \theta &= l^a \mathcal{Z}_p(bl^c) \\ &= \frac{1}{\sqrt{l}} J_1\left(\frac{2\sqrt{g}}{v} \sqrt{l}\right). \end{aligned}$$

Airy Equation

Time-independent Schrödinger equation with a simple potential,

$$\ddot{\Psi} + \alpha x \Psi = 0.$$

Comparing it with general form, we should set

$$\begin{aligned} a &= 1/2, \\ |pc| &= 1/2, \\ c &= 3/2, \\ b^2 c^2 &= \alpha^2. \end{aligned}$$

So the two possible solutions are

$$\begin{aligned} \Psi_1(x) &= \sqrt{x} \mathcal{Z}_{1/3}(2/3\alpha x^{3/2}), \\ \Psi_2(x) &= \sqrt{x} \mathcal{Z}_{-1/3}(2/3\alpha x^{3/2}). \end{aligned}$$

The general solution is

$$\Psi(x) = a\Psi_1(x) + b\Psi_2(x).$$

Second Order Differential Equations and Gauss' Equation

Gauss' equation has the form

$$z(z-1)\frac{d^2}{dz^2}u(z) + [(a+b+1)z-c]\frac{d}{dz}u(z) + abu(z) = 0,$$

which has a solution of the hypergeometric function form

$$u(z) = {}_2F_1(a, b; c; z).$$

The interesting thing about this equation is that its Paperitz symbol is

$$\left(\begin{array}{ccc|c} 0 & 1 & \infty & \\ 0 & 0 & a & \\ 1-c & c-a-b & b & \end{array} \middle| z \right),$$

in which the first three columns are the singularities at points $0, 1, \infty$ while the last column just points out that the argument of this equation is z .

This means, in some sense, the solution to any equation with three singularities can be directly written down by comparing the equation with Gauss' equation. If you care, the actual steps are changing variables, rewriting the equation into Gauss' equation form, writing down the solutions.

5.3.2 Integral Equations

Neumann Series AKA WKB

For differential equation, whenever the highest derivative is multiplied by a small parameter, try this. But generally, the formalism is the following.

First of all, we use Hilbert space $\mathcal{L}[a, b; w]$ which means the space is defined on $[a, b]$ with a weight w , i.e.,

$$\langle f | g \rangle = \int_a^b dx w(x) f(x) g(x).$$

Quantum Mechanics Books

Notice that this is very different from the notation we used in most QM books.

What is the catch? Try to write down $\langle x | u \rangle$. It's not that different because one can always go back to the QM notation anyway.

With the help of Hilbert space, one can always write down the vector form of some operators. Suppose we have an equation

$$\hat{L}u(x) = f(x),$$

where $\hat{L} = \hat{I} + \hat{M}$. So the solution is simply

$$\begin{aligned} u(x) &= \hat{L}^{-1}f(x) \\ &= (\hat{I} + \hat{M})^{-1}f(x). \end{aligned}$$

However, it's not a solution until we find the inverse. A most general approach is the Neumann series method. We require that

$$\|\hat{M}u\| < \gamma\|u\|,$$

where $\gamma \in (0, 1)$ and should be independent of x .

As long as this is satisfied, the equation can be solved using Neumann series, which is an iteration method with

$$\begin{aligned} u(x) &= u_0(x) + \delta u_1(x) + \delta^2 u_2(x) + \dots \\ u_0(x) &= f(x). \end{aligned}$$

As an example, we can solve this equation

$$(\hat{I} + |t\rangle \langle \lambda|)u(t) = f(t).$$

We define $\hat{M} = |t\rangle \langle \lambda|$ and check the convergence condition for λ .

Step one is always checking condition of convergence.

Step two is to write down the series and zeroth order. Then we reach the key point. The iteration relation is

$$u_n(t) + \int_0^1 ds s u_{n-1}(s) = 0.$$

One can write down u_1 immediately

$$u_1(t) = - \int_0^1 ds s u_0(s).$$

Keep on going.

Using Dyads in Vector Space

For the same example,

$$\hat{L}u(x) = f(x),$$

where $\hat{L} = \hat{I} + \hat{M}$, we can solve it using vector space because if operator is linear.

Suppose we have a $\hat{M} = |a\rangle \langle b|$, the equation, in some Hilbert space, is

$$|u\rangle + |a\rangle \langle b | u \rangle = |f\rangle.$$

Multiplying through by $\langle b|$, we have

$$\langle b | u \rangle + \langle b | a \rangle \langle b | u \rangle = \langle b | f \rangle,$$

which reduces to a linear equation. We only need to solve out $\langle b | u \rangle$ then plug it back into the original equation.

FUNDAMENTAL PHYSICS

6.1 Classical Mechanics

6.1.1 Oscillators

In general, the Lagrangian for a system with n general coordinates can be

$$L = \frac{1}{2} m_{jk} \dot{q}_j \dot{q}_k - V(q_1, \dots, q_n)$$

To write down equation of motion, we need the following terms,

$$\frac{\partial L}{\partial \dot{q}_j} = m_{jk} \dot{q}_k \frac{\partial L}{\partial q_j} = \frac{1}{2} \frac{\partial m_{kl}}{\partial q_j} \dot{q}_k \dot{q}_l - \frac{\partial V}{\partial q_j}$$

Then equation of motion is

$$m_{jk} \ddot{q}_k + \frac{\partial m_{jk}}{\partial q_l} \dot{q}_k \dot{q}_l - \frac{1}{2} \frac{\partial m_{kl}}{\partial q_j} \dot{q}_k \dot{q}_l = - \frac{\partial V}{\partial q_j}$$

Generally, we can't solve this system. But there is an interesting limit. The system may have equilibrium points. We can study systems oscillating around equilibrium points.

At equilibrium, the system can stay steady, i.e., $\dot{q}_j^0 = 0$. This gives us

$$\frac{\partial V}{\partial q_j} = 0,$$

for all j .

Now for small deviations, we can expand the system around equilibrium points.

$$q_j = q_j^0 + \eta_j$$

Then

$$T = \frac{1}{2} m_{jk} |_0 \dot{\eta}_j \dot{\eta}_k \equiv \frac{1}{2} T_{jk} \dot{\eta}_j \dot{\eta}_k$$

$$V = V|_0 + \frac{\partial V}{\partial q_j} |_0 \eta_j + \frac{1}{2} \frac{\partial^2 V}{\partial q_j \partial q_k} |_0 \eta_j \eta_k + \dots \equiv \frac{1}{2} V_{jk} \eta_j \eta_k$$

So we have the Lagrangian for small oscillations,

$$L = \frac{1}{2} T_{jk} \dot{\eta}_j \dot{\eta}_k - \frac{1}{2} V_{jk} \eta_j \eta_k$$

Typing indices using LaTeX is so annoying. So we'll use matrix notations and Lagrangian becomes

$$L = \frac{1}{2} \dot{\eta}^T T \dot{\eta} - \frac{1}{2} \eta^T V \eta,$$

in which T and V matrices are n by n real and symmetric.

(We need to diagonalize T and V . First question comes to us is:

**** Is it possible to diagonalize both T and V at the same time? ****

We can have a look at the surface $\tilde{p}^T T p = C$, which is an elliptical surface with coordinates p .)

Use the following transformation

$$\xi = T^{1/2} \eta$$

Then transpose

$$\tilde{\xi} = \tilde{\eta} T^{1/2}$$

$$\dot{\xi}^T \dot{\xi} = \dot{\eta}^T T \dot{\eta}$$

So we have the new Lagrangian

$$L = \frac{1}{2} \dot{\xi}^T \dot{\xi} - \frac{1}{2} \tilde{\xi} T^{-1/2} V T^{-1/2} \xi$$

Define $T^{-1/2} V T^{-1/2} \equiv V'$.

Next we need to diagonalize V' by using its eigen vectors.

$$V' b = \lambda b$$

is equivalent to

$$V a = \lambda T a$$

with $b = T^{1/2} a$. So we have

$$\det(V' - \lambda \mathbf{I}) = 0$$

is same as

$$\det(V - \lambda T) = 0$$

in which λ is the eigen value of this function.

6.1.2 Hamiltonian Dynamics

Phase space

6.2 Quantum Mechanics

6.2.1 Quantum Mechanics Framework

What're the most important tricks in QM calculations?

- Remember what basis we are working in
- Identity

First Three Postulates

- Physical state is described by kets in a Hilbert space. We need to specify a complete basis $\{|i\rangle\}$ to do calculations.

$$|\psi\rangle = \sum_i |i\rangle \langle i|\psi\rangle = \sum_i C_i |i\rangle$$

- Operators are given by Hermitian operators; A measurement of the variable $\hat{\Omega}$ will yield one of the eigenvalues ω with the probability

$$|\langle\omega|\psi\rangle|^2.$$

And the state of the system will change to $|\omega\rangle$.

- The state vector obeys the Schrödinger equation,

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle,$$

where \hat{H} is the Hamiltonian operator.

The logic here is that we first find the way to describe a system, then think about how to find out the information we need from the state vector and also find the evolution of the state vector. Then we need the operator and Schrodinger equation. Finally, we would like to relate the theory to experiments, and it comes the measurement postulate.

Later we will need the relation between position and momentum, which becomes the fourth postulate.

- How to solve the evolution of a system? We just define a magical operator, propagator

$$\hat{U} |\psi(t_0)\rangle = |\psi(t)\rangle.$$

This operator just gives us the evolution of state vector! Wait, can we write down the explicit expression of it?

Let's find out. The only thing we know about the evolution of a state vector is the third postulate up there.

$$\begin{aligned} i\hbar \frac{d}{dt} |\psi(t)\rangle &= \hat{H} |\psi(t)\rangle \\ i\hbar \frac{d}{dt} \hat{U} |\psi(t_0)\rangle &= \hat{H} \hat{U} |\psi(t_0)\rangle \\ i\hbar \frac{d}{dt} \hat{U} &= \hat{H} \hat{U} \end{aligned}$$

Looks familiar? This just gives us an exponential result, **if the Hamiltonian is time independent.**

$$\hat{U} = e^{-i\hat{H}(t-t_0)/\hbar}$$

We can prove that this operator is Unitary because \hat{H} is Hermitian.

This is just the abstract representation, we work in some basis, and the most convenient basis is the eigenstates of Hamiltonian, $\{|\epsilon_i\rangle\}$,

$$\begin{aligned} \hat{U} |\phi\rangle &= e^{-i\hat{H}(t-t_0)/\hbar} |\psi\rangle \\ \hat{U} |\phi\rangle &= \sum_i e^{-i\hat{H}(t-t_0)/\hbar} |\epsilon_i\rangle \langle\epsilon_i|\psi\rangle \\ \hat{U} |\phi\rangle &= \sum_i e^{-i\epsilon_i(t-t_0)/\hbar} |\epsilon_i\rangle \langle\epsilon_i|\psi\rangle \end{aligned}$$

And we are going to use

$$\hat{U} = \sum_i e^{-i\epsilon_i(t-t_0)/\hbar} |\epsilon_i\rangle \langle \epsilon_i|$$

from now on. (Well, only on discrete eigenvalues ones.)

(See that? **Identity does the work again.**)

Position and Momentum Space

Summarize here.

- **Position**

1. Define $\{|x\rangle\}$ basis.
2. Define \hat{x} operator.
3. Find wave function in this basis.
4. Find measurement.

- **Evolution**

1. Need propagator \hat{U} .
2. Propagator needs the solution of Hamiltonian eigensystem.
3. (Free particles) Hamiltonian needs the solution of momentum eigensystem.

- **Momentum**

1. Before we define some arbitrary momentum space, we should check the relation between momentum and position. And it turns out to be related by a commutator.(Postulate IV)
 2. Use the postulate to momentum operator.
 3. Find eigenstates.
 4. (Calculate the propagator.)
-

Position Space

1. Define $|x\rangle$ basis.

- Orthonormal:

$$\langle x | x' \rangle = \delta(x - x')$$

- Complete:

$$\int \langle x' | x' \rangle dx' = \mathbb{I}$$

2. Define position operator.

$$\hat{x} |x\rangle = x |x\rangle$$

And in $\{|x\rangle\}$ basis, this operator becomes a function, which is

$$\begin{aligned}\langle x | \hat{x} | x' \rangle &= (\langle x | \hat{x} | x' \rangle) \\ &= x \langle x | x' \rangle \\ &= x \delta(x - x')\end{aligned}$$

3. Find state vector in $\{|x\rangle\}$ basis.

$$\psi(t, x) = \langle x | \psi(t) \rangle$$

- Normalized:

$$\int |\psi(t, x)|^2 dx = 1.$$

And we are interpreting $|\psi(t, x)|^2$ as probability density.

4. Calculate probability of a measurement. Taking \hat{x} as an example.

$$\begin{aligned}\langle \psi | \hat{x} | \psi \rangle &= \iint \langle \psi | x \rangle \langle x | \hat{x} | x' \rangle \langle x' | \psi \rangle dx dx' \\ &= \iint \psi^*(t, x) x \delta(x - x') \psi(t, x') dx dx' \\ &= \int |\psi(t, x)|^2 x dx\end{aligned}$$

Momentum Space

To find the momentum operator, we need to check the relation between momentum and position before we just randomly define one. Truth is, we have a fourth postulate states the relation between them.

Postulate IV The commutator of \hat{x} , \hat{p} is

$$[\hat{x}, \hat{p}] = i\hbar$$

Two comments:

- Why i ? Eigenvalue of Anti-Hermitian operator.
- Why \hbar ? Because people define the dimensions of position and momentum differently before they know this commutator. We would like to assign them the same dimension if we already know this relation.

Momentum Space

1. Find momentum operator in position basis $\{|x\rangle\}$.

$$\langle x | [\hat{x}, \hat{p}] | x' \rangle = i\hbar \delta(x - x')$$

And write out the commutator and use the relation of delta function $x\delta'(x) = -\delta(x)$, we find out the momentum operator in $\{|x\rangle\}$ basis,

$$\langle x | \hat{p} | x' \rangle = -i\hbar \frac{d}{dx} \delta(x - x')$$

Let's talk physics. What does that operator mean? We need to see what the result is when momentum operator is applied to a state. And remember we would work in $\{|x\rangle\}$ basis.

$$\begin{aligned}
 \langle x | \hat{p} | \psi \rangle &= \iint \langle x | x' \rangle \langle x' | \hat{p} | x'' \rangle \langle x'' | \psi \rangle dx' dx'' \\
 &= \int \langle x | \hat{p} | x'' \rangle \psi(x'') dx'' \\
 &= \int \left(-i\hbar \frac{d}{dx} \delta(x - x'') \psi(x'') \right) dx'' \\
 &= \int \left(-i\hbar \frac{d}{dx'} \delta(x' - x) \psi(x') \right) dx'
 \end{aligned}$$

Integrate by parts, we will find the expression. (I am having a problem finding the right answer.)

$$\langle x | \hat{p} | \psi \rangle = -i\hbar \frac{d}{dx} \psi(x).$$

2. Eigenfunction for momentum.

$$\hat{p} | p \rangle = p | p \rangle.$$

Again, we are going to project it on the $\{|x\rangle\}$ basis,

$$\langle x | \hat{p} | p \rangle = \langle x | p | p \rangle,$$

where $\langle x | p \rangle$ is the eigenstates in $\{|x\rangle\}$ basis, we call it $\phi_p(x)$.

$$\begin{aligned}
 \langle x | \hat{p} | p \rangle &= p \phi_p(x) \\
 \int \langle x | \hat{p} | x' \rangle \langle x' | p \rangle dx' &= p \phi_p(x) \\
 -i\hbar \frac{d}{dx} \phi_p(x) &= p \phi_p(x)
 \end{aligned}$$

The solution is

$$\phi_p(x) = C e^{ipx/\hbar}$$

This constant C is found by the normalization condition,

$$\langle p | p' \rangle = \int \phi_p^*(x) \phi_{p'}(x) dx = \delta(p - p')$$

The final results should be

$$\phi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp(ipx/\hbar)$$

3. Find the dynamics of free particles in quantum mechanics. **Find the propagator and everything solves.** The hamiltonian for a free particle is

$$\hat{H} = \frac{\hat{p}^2}{2m}.$$

We argue here that the eigenvectors of momentum are also the eigenvectors of this hamiltonian. And we can easily guess the eigenvalues are $p^2/2m$. So the propagator is

$$\hat{U} = \int e^{-ip^2 t/2m\hbar} |p\rangle \langle p| dp$$

But that is too abstract to use, we can find the expression in $\{|x\rangle\}$ basis.

$$\begin{aligned}\langle x|\hat{U}|x\rangle &= \int e^{-ip^2t/2m\hbar} \langle x|p\rangle \langle p|x\rangle dp \\ &= \int e^{-ip^2t/2m\hbar} |\phi_p|^2 dp\end{aligned}$$

6.2.2 Quantum in 1D

General

Always start with the propagator for time independent Hamiltonian.

$$|\psi(t)\rangle = \hat{U} |\psi(0)\rangle$$

For cases that Hamiltonian with discrete eigenvalues,

$$|\psi(t)\rangle = \sum_n e^{-i\epsilon_n t/\hbar} |n\rangle \langle n|\psi(0)\rangle$$

If the initial state is just one of the eigenstates of Hamiltonian, say the m th one (normalized),

$$|\psi(t)\rangle = e^{-i\epsilon_m t/\hbar} |m\rangle$$

Well, that phase factor doesn't have any effect for the topic we discuss. So our time evolution will stay on the same state forever.

The same thing happens for continuous cases.

So our task is simplified to solve the eigensystem of Hamiltonian, which is

$$\hat{H} |\epsilon\rangle = \epsilon |\epsilon\rangle$$

Infinite Barriers

Math

Setup

- Potential in a box

$$V(x) = \begin{cases} 0, & 0 < x < L \\ \infty, & \text{Other} \end{cases}$$

Solve the Problem

- Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$$

- Dynamic equation

$$\hat{H} |\psi(t)\rangle = \epsilon |\psi(t)\rangle$$

We are happy to work in $\{|x\rangle\}$ basis,

$$\langle x | \hat{H} | \psi(t) \rangle = \langle x | \epsilon | \psi(t) \rangle.$$

Put the Hamiltonian in, and remember that in position basis

$$\langle x | \hat{p} | \psi \rangle = -i\hbar \frac{d}{dx} \psi,$$

the equation of motion becomes

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x, t) + V(x) \psi(x, t) = \epsilon \psi(x, t)$$

- Boundary conditions

$$\psi_I(0, t) = \psi_{II}(0, t)$$

$$\psi_{II}(L, t) = \psi_{III}(L, t)$$

- Guess the Solutions

$$\psi_{II} = \psi = C \sin(kx) + D \cos(kx)$$

- Find the wavenumber k, by putting the assumed solutions into equation of motion

$$k = \pm \sqrt{\frac{2m\epsilon}{\hbar^2}}$$

Since we can always merge the negative into the constants, it is fine to use

$$k = \sqrt{\frac{2m\epsilon}{\hbar^2}}$$

- Use Boundary Condition

1. At $x=0$,

$$\psi(0, t) = 0.$$

This gives us $D = 0$.

2. At $x = L$,

$$\psi(L, t) = 0.$$

This leads to

$$kL = n\pi.$$

Since $n = 0$ gives us a 0 wave function, we would just drop $n = 0$. For the same reason why we drop the negative values of k, we would drop all the negative values of n. This BC gives us the possible values of energy because wavenumber k is related to energy,

$$\epsilon = \frac{\hbar^2}{2mL^2} (n\pi)^2,$$

with

$$n = 1, 2, 3, \dots$$

- Normalization as the last constraint for the last undetermined parameter,

$$C = \sqrt{\frac{2}{L}}$$

Physics

1. Estimation

- Find the expression for energy using dimensional analysis.
- Using uncertainty relation to estimate the expression for energy.

2. Comments

- **Why is the solution quantized?**
 - (a) Too many constraints. BCs + normalization.
- **Why do the n in the solution goes into the expression for energy?**
 - (a) Have a look at the kinetic energy term, the derivative does it.
- **What's so weird?**
 - (a) For $n = 2$, no particles found at $x = L/2$. And so on.

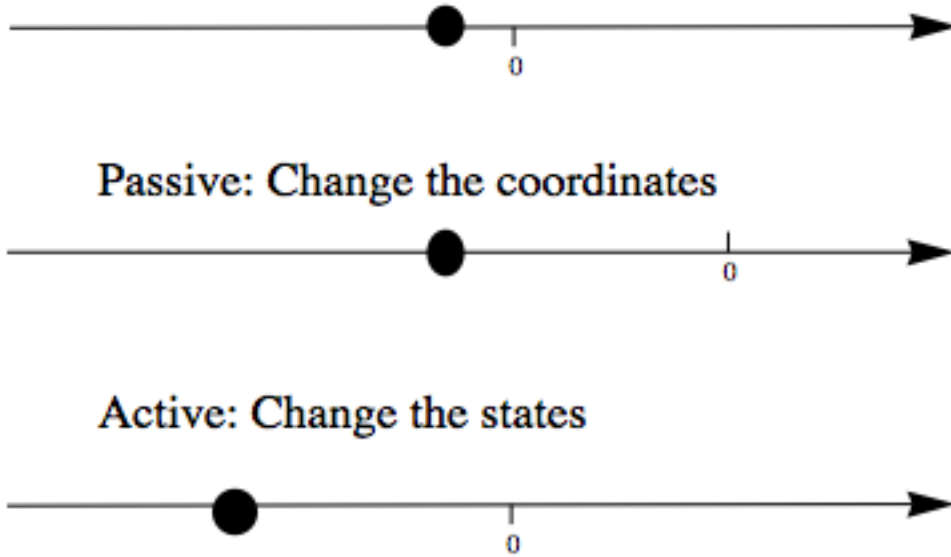
Some General Properties

1. 1D bound states have no degeneracy. Prove it by assume that there is a degeneracy state.
2. 1D bound states' wave function can be chosen to be real. (if potential V is real.)

6.2.3 Parity

Passive and Active Transformations

Generally, there are two ways of interpreting a transformation.



Here in QM, passive means transform the operator $\hat{\Omega}$, while active means change the state $|\psi\rangle$. Suppose we have a system $|\psi\rangle$, an operator $\hat{\Omega}$, a transformation \hat{U} .

Transformation $\hat{U}|\psi\rangle$ is identical to $\hat{U}^\dagger\hat{\Omega}\hat{U}$ because they give the same observation results. The first one is called active, while the second one is called passive.

Parity

Definition

$$\hat{\Pi}|x\rangle = |-x\rangle$$

Properties

1. Act on momentum eigenvectors,

$$\hat{\Pi}|p\rangle = |-p\rangle.$$

- Physics: Parity changes the coordinate, so the direction of momentum is also changed.

- Math:

$$\hat{\Pi} |p\rangle = \int \hat{\Pi} |x\rangle \langle x | p\rangle dx = \int |-x\rangle \langle x | p\rangle dx$$

Change coordinate from x to -x,

$$\hat{\Pi} |p\rangle = \int |x\rangle \langle -x | p\rangle dx = \int |x\rangle \langle x | -p\rangle dx = |-p\rangle$$

2. Hermitian,

$$\langle x | \hat{\Pi} | x'\rangle = \delta(x + x') (\langle x' | \hat{\Pi} | x\rangle)^\dagger = \langle x | \hat{\Pi}^\dagger | x'\rangle = \delta(x + x')$$

3. Unitary

$$\langle x | \hat{\Pi}^\dagger \hat{\Pi} | x'\rangle = \langle -x | -x'\rangle = \delta(-x + x') = \delta(x - x') = \langle x | x'\rangle$$

4. Inverse of parity

$$\hat{\Pi} \hat{\Pi} = \hat{\Pi} \hat{\Pi}^\dagger = \hat{I}$$

5. Eigensystem of parity.

$$\hat{\Pi} |\pi\rangle = \pi |\pi\rangle$$

Apply another operator

$$\hat{\Pi}^2 |\pi\rangle = \pi^2 |\pi\rangle$$

So, * Eigenvalues: 1, -1; * Eigenvectors: Even function, Odd function

6. Parity applied to operators a. Apply to position operator,

$$\hat{\Pi}^\dagger \hat{X} \hat{\Pi} = -\hat{X}$$

Proof:

$$\langle x | \hat{\Pi}^\dagger \hat{X} \hat{\Pi} | x'\rangle = \langle -x | \hat{X} | -x'\rangle = -x' \delta(x - x') = \langle x | (-\hat{X}) | x'\rangle$$

- (a) Apply to momentum operator,

$$\hat{\Pi}^\dagger \hat{p} \hat{\Pi} = -\hat{p}$$

Proof: Similar to the previous one, just change x basis to momentum basis.

2. Symmetry related to Hamiltonian.

$$[\hat{\Pi}, \hat{H}] = 0$$

When this happens, parity of Hamiltonian won't change the wave function. Or the wave function should have an specific parity for 1D problem.

6.2.4 Classical Limit of QM

Ehrenfest's Theorem

Schrödinger equation and its adjoint

$$\begin{aligned} i\hbar \frac{d}{dt} |\psi(t)\rangle &= \hat{H} |\psi(t)\rangle \\ -i\hbar \frac{d}{dt} \langle\psi(t)| &= \langle\psi(t)| \hat{H} \end{aligned}$$

For any observable $\hat{\Omega}$,

$$\begin{aligned} \frac{d}{dt} \langle\hat{\Omega}\rangle &= \left(\frac{d}{dt} \langle\psi(t)| \right) \hat{\Omega} |\psi(t)\rangle + \langle\psi(t)| \dot{\hat{\Omega}} |\psi(t)\rangle + \langle\psi(t)| \hat{\Omega} \left(\frac{d}{dt} |\psi(t)\rangle \right) \\ &= \frac{1}{i\hbar} \left(-\langle\psi(t)| \hat{H} \hat{\Omega} |\psi(t)\rangle + \langle\psi(t)| \hat{\Omega} \hat{H} |\psi(t)\rangle \right) + \langle\psi(t)| \dot{\hat{\Omega}} |\psi(t)\rangle \\ &= \frac{1}{i\hbar} \langle\psi(t)| [\hat{\Omega}, \hat{H}] |\psi(t)\rangle + \langle\psi(t)| \dot{\hat{\Omega}} |\psi(t)\rangle \end{aligned}$$

This is called Ehrenfest's Theorem.

Simple Example of Ehrenfest's Theorem

Suppose we have a system with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

We need to figure some commutators first.

$$2m [\hat{x}, \hat{H}] = [\hat{x}, \hat{p}^2] = \hat{x}\hat{p}\hat{p} - \hat{p}\hat{p}\hat{x} = \hat{x}\hat{p}\hat{p} - \hat{p}\hat{x}\hat{p} + \hat{p}\hat{x}\hat{p} - \hat{p}\hat{p}\hat{x} = [\hat{x}, \hat{p}] \hat{p} + \hat{p} [\hat{x}, \hat{p}] = 2i\hbar\hat{p}$$

$$[\hat{p}, \hat{H}] = [\hat{p}, V(\hat{x})] = \left[\hat{p}, \sum_0^{\infty} \frac{V^{(n)}}{n!} \hat{x}^n \right] = \dots = -i\hbar V'(\hat{x})$$

1. Position average

$$\begin{aligned} \frac{d}{dt} \langle\hat{x}\rangle &= \frac{1}{i\hbar} \langle\psi(t)| [\hat{x}, \hat{H}] |\psi(t)\rangle \\ &= \frac{\langle\hat{p}\rangle}{m} \end{aligned}$$

We are familiar with this in classical mechanics.

2. Momentum average

$$\begin{aligned} \frac{d}{dt} \langle\hat{p}\rangle &= \frac{1}{i\hbar} \langle\psi(t)| [\hat{p}, \hat{H}] |\psi(t)\rangle \\ &= \frac{1}{i\hbar} \langle\psi(t)| (-i\hbar V'(\hat{x})) |\psi(t)\rangle \\ &= -\langle V'(\hat{x}) \rangle \end{aligned}$$

In classical mechanics, the derivative of potential is force. And the result is just like Newton's 2n Law except the right hand side is not exactly like a force which should be $-\frac{d}{dx} \langle V(\hat{x}) \rangle$.

What does $\langle V'(\hat{x}) \rangle$ mean

Suppose the potential area is fairly small and distributed around some coordinate $x_0 = \langle \hat{x} \rangle$, we can do Taylor expansion around x_0 .

$$\begin{aligned} \langle V(\hat{x}) \rangle &= V(x_0) + V'(x_0) \langle (x - x_0) \rangle + V''(x_0) \langle (x - x_0)^2 \rangle / 2 + \dots \\ &= V(x_0) + 0 + V''(x_0) (\Delta x)^2 / 2 + \dots \end{aligned}$$

If the uncertainty is small enough, every term except the first one becomes small. So to the lowest order, average of potential is approximately the potential at x_0 .

Similarly, the average of first derivative of potential $\langle V'(\hat{x}) \rangle$ is approximately $V'(x_0)$.

These gives us a hint for the previous result we got for the time evolution of average momentum. The result reduces to classical mechanics one as long as we keep the lowest order of Taylor expansion. Those higher order terms show the quantum effect.

Picture

We can see deeper into Ehrenfest's Theorem through Heisenberg Picture of quantum mechanics.

Schrödinger & Heisenberg Pictures

Pictures are the ways we look at the evolution of systems.

Schrödinger Picture In Schrödinger picture the states are evolving with time.

$$i\hbar \frac{d}{dt} |\psi\rangle_S = \hat{H} |\psi\rangle_S$$

And for time independent Hamiltonian,

$$|\psi\rangle_S = U^\dagger |\psi_0\rangle_S$$

Heisenberg Picture In Heisenberg Picture, the states do not change with time.

$$|\psi\rangle_H = |\psi_0\rangle_H,$$

and of course the initial is the same with Schrödinger Picture,

$$|\psi_0\rangle_H = |\psi_0\rangle_S.$$

How do we relate to Heisenberg Picture to Schrödinger Picture? Through investigation of observables. We should have the same observation results in both Pictures.

$$\begin{aligned} {}_H \langle \psi | \hat{\Omega}_H | \psi \rangle_H &= {}_S \langle \psi | \hat{\Omega}_S | \psi \rangle_S \\ {}_H \langle \psi | \hat{\Omega}_H | \psi \rangle_H &= {}_S \langle \psi_0 | \hat{U}^\dagger \hat{\Omega}_S \hat{U} | \psi_0 \rangle_S \\ \hat{\Omega}_H &= \hat{U}^\dagger \hat{\Omega}_S \hat{U} \end{aligned}$$

So the operators change with time in Heisenberg Picture.

Ehrenfest's Theorem in Heisenberg Picture

$$\frac{d}{dt}\hat{\Omega}_H = \frac{1}{i\hbar} [\hat{\Omega}_H, \hat{H}] + \hat{U}^\dagger \frac{\partial}{\partial t} \hat{\Omega}_H \hat{U}$$

This can be easily proved by throwing every definition need in to it. We also need the following equations.

$$\frac{d}{dt}\hat{U} = \frac{d}{dt}e^{-i\hat{H}t/\hbar} = \frac{\hat{H}}{i\hbar}\hat{U}$$

And REMEMBER that propagator commute with time independent Hamiltonian, so

$$\hat{H} = \hat{U}^\dagger \hat{U} \hat{H} = \hat{U}^\dagger \hat{U} \hat{H} \equiv \hat{H}_H$$

So this Ehrenfest's Theorem can also be written as

$$\frac{d}{dt}\hat{\Omega}_H = \frac{1}{i\hbar} [\hat{\Omega}_H, \hat{H}_H] + \hat{U}^\dagger \frac{\partial}{\partial t} \hat{\Omega}_H \hat{U}$$

We can **define**

$$\frac{\partial}{\partial t}\hat{\Omega}_H \equiv \hat{U}^\dagger \frac{\partial}{\partial t} \hat{\Omega}_S \hat{U},$$

which is the time derivative of operator in Heisenberg Picture.

Reminder: The time derivative of an observable (average) depends not only the time derivative of itself, but also the commutator of the observable and Hamiltonian.

Example of Ehrenfest's Theorem in Heisenberg Picture We will show why it is better to work in Heisenberg Picture to show the meanings of Ehrenfest's Theorem.

Suppose we have a Hamiltonian in Heisenberg Picture,

$$\hat{H}_H = \frac{\hat{p}_H^2}{2m} + V(\hat{x}_H).$$

Time derivative of position operator

$$\frac{d}{dt}\hat{x}_H = \frac{1}{i\hbar} [\hat{x}_H, \hat{H}_H] = \frac{\hat{p}_H}{m}$$

Time derivative of momentum operator

$$\frac{d}{dt}\hat{p}_H = \frac{1}{i\hbar} [\hat{p}_H, \hat{H}_H] = -V'(\hat{x}_H)$$

So the operator in Heisenberg Picture just have a sense of the physical quantities in classical mechanics. That's why we like it.

Conservation

We say a observable is conserved if the corresponding operator commutes with Hamiltonian,

$$[\hat{\Omega}, \hat{H}] = 0$$

1. Energy Hamiltonian always commutes with itself.

$$\frac{d}{dt}\langle \epsilon \rangle = \langle \psi | \left(\frac{\partial}{\partial t} \hat{H} \right) | \psi \rangle$$

If Hamiltonian is time independent, then energy is conserved. (If Hamiltonian is time dependent, energy is not conserved. This is kind of obvious in classical mechanics.)

What is the nature of time dependence

We can see this by looking at a simple example.

Assume we have a system with energy eigenstates $|\epsilon_n\rangle$, and initially,

$$|\psi_0\rangle = \sum_n C_n |\epsilon_n\rangle.$$

So

$$|\psi(t)\rangle = \sum_n C_n e^{-i\epsilon_n t/\hbar} |\epsilon_n\rangle.$$

We can calculate the expectation value of some operator $\hat{\Omega}$,

$$\langle\omega(t)\rangle = \sum_{n,m} \left(C_n^* e^{i\epsilon_n t/\hbar} \langle\epsilon_n| \right) \hat{\Omega} \left(C_m e^{-i\epsilon_m t/\hbar} |\epsilon_m\rangle \right) = \sum_{n,m} C_n^* C_m e^{-i(\epsilon_m - \epsilon_n)t/\hbar} \langle\epsilon_n| \hat{\Omega} |\epsilon_m\rangle$$

If $|\epsilon_n\rangle$ are also the eigenvectors of $\hat{\Omega}$, then

$$\langle\epsilon_n| \hat{\Omega} |\epsilon_m\rangle = \omega_m \delta_{n,m}$$

And the expectation value

$$\langle\omega(t)\rangle = \sum_n C_n^* C_n \omega_n$$

The important thing is that the time dependence of this expectation value actually arise from this term

$$e^{-i(\epsilon_m - \epsilon_n)t/\hbar}.$$

As it is so important, we call

$$(\epsilon_m - \epsilon_n)/\hbar$$

Bohr frequency.

6.2.5 Harmonic Oscillators

Why Harmonic Oscillators

Many systems can reduce to it. Use Taylor expansion for the potential and redefine parameters we will find harmonic oscillators in the potential.

Hamiltonian for 1D is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} k \hat{x}^2$$

Standard Solution

We can use polynomial expansion for part of the solution.

Dimension Schrodinger Equation

First step is always finding out the characteristic length scale and characteristic energy scale. Assume we have an characteristic length η and characteristic energy scale ϵ_0 . Through uncertainty principle we know only for dimensional analysis

$$[\hat{p}] = \frac{\hbar}{\eta}$$

Kinetic energy and potential energy have the same dimension

$$\frac{\hbar^2}{\eta^2 m} = k\eta^2,$$

so we have

$$\eta = \sqrt{\frac{\hbar}{m\omega}}$$

with $\omega^2 = k/m$. A dimensional analysis shows that $\epsilon_0 = \hbar\omega$.

Now we can define dimensionless variables,

$$z = x/\eta, e = \epsilon/\epsilon_0$$

The time independent Schrodinger equation in position basis is

$$-\hbar^2 \frac{d^2}{dx^2} \psi''/m + kx^2 = 2\epsilon\psi.$$

Using those characteristic scales, we can rewrite this equation into a dimensionless one, which is

$$\psi'' + (2e - z^2)\psi = 0$$

in which $\psi' = \frac{d}{dz}\psi$.

Take Limits

We need to look at the behavior of the solutions before we can guess a proper general solution.

$z \rightarrow \infty$, we have $\psi'' - z^2\psi = 0$. Solution to this equation is $\psi(z) e^{-z^2/2}$.

The solution of the the equation should be in the form

$$\psi(z) = u(z)e^{-z^2/2}.$$

Insert this to time independent Schrodinger equation, we can get the equation of $u(z)$.

$$u'' - 2zu' + (2e - 1)u = 0$$

Polynomial Method

The simplest form of $u(z)$ is polynomial,

$$u(z) = \sum_{n=0}^{\infty} u_n z^n.$$

Put this back to equation of u, we can get the recursion relation,

$$(n+2)(n+1)u_{n+2} = [2n - (2e-1)]u_n.$$

If u_0 and u_1 are given, we can get the whole polynomial.

Notice that we have definite parity here. So u_1 branch vanish because they are even.

u_0 is set by the normalization condition.

Terminate The Series

The series blow up if it doesn't terminate. So we need to terminate the series using the following relation,

$$2e - 1 = 2n.$$

Then we have the energy levels, which is $e = n + 1/2$.

Complete Series

By picking proper normalization factor, we can write down the energy levels and corresponding wave functions. In fact, this polynomial can be found in mathematical physics books.

$$H_{n+1} = 2zH_n - nH_{n-1}$$

Tricky Solution

Find out the characteristic length and energy

$$\begin{aligned}\eta &= \sqrt{\frac{\hbar}{m\omega}} \\ \epsilon &= \hbar\omega \\ \omega &= \sqrt{\frac{k}{m}}\end{aligned}$$

One way to get the intrinsic length without writing down the dimensions of each quantity is to use the following relation

$$\begin{aligned}[E] &= [m\omega^2\hat{x}^2] \\ \hbar\omega &= m\omega^2\eta^2 \\ \eta &= \sqrt{\frac{\hbar}{m\omega}}\end{aligned}$$

Or if we are given the Hamiltonian in terms of k ,

$$\begin{aligned}\left[\frac{\hat{p}^2}{2m}\right] &= [k\hat{x}^2] \\ \frac{\hbar^2/\eta^2}{m} &= k\eta^2 \\ \eta &= \sqrt{\hbar}\sqrt{mk} = \sqrt{\hbar m\omega}\end{aligned}$$

Rewrite the Hamiltonian

$$\begin{aligned}
 \hat{H} &= \frac{1}{2m} \left[\left(\frac{\hat{p}}{\hbar/\eta} \right)^2 \left(\frac{\hbar}{\eta} \right)^2 + \frac{1}{2} m \omega^2 \left(\frac{\hat{x}}{\eta} \right)^2 \right] \\
 &= \frac{1}{2} \hbar \omega \left[\left(\frac{\hat{p}}{\hbar/\eta} \right)^2 + \left(\frac{\hat{x}}{\eta} \right)^2 \right] \\
 &= \frac{1}{2} \hbar \omega \left(\frac{\hat{x}}{\eta} - i \frac{\hat{p}}{\hbar/\eta} \right) \left(\frac{\hat{x}}{\eta} + i \frac{\hat{p}}{\hbar/\eta} \right) - \frac{i}{\hbar} [\hat{x}, \hat{p}] \\
 &= \frac{1}{2} \hbar \omega (\sqrt{2} \hat{a}^\dagger \sqrt{2} \hat{a} + 1) \\
 &= \hbar \omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)
 \end{aligned}$$

Now we can define $\hat{a}^\dagger \hat{a} = \hat{N}$, which is just like an operator for (energy) quanta numbers.

An important relation is

$$\begin{aligned}
 [\hat{a}, \hat{a}^\dagger] &= 1 \\
 [\hat{a}, \hat{N}] &= \hat{a}
 \end{aligned}$$

The eigen equation for this weird energy quanta number operator is

$$\hat{N} |n\rangle = n |n\rangle$$

To find out the eigen state of \hat{a} and \hat{a}^\dagger , we try this,

$$\begin{aligned}
 \hat{N}(\hat{a} |n\rangle) &= (n-1)(\hat{a} |n\rangle) \\
 \hat{N}(\hat{a}^\dagger |n\rangle) &= (n+1)(\hat{a}^\dagger |n\rangle)
 \end{aligned}$$

This means $\hat{a} |n\rangle$ and $\hat{a}^\dagger |n\rangle$ are also eigen states of \hat{N} .

The next step is very crucial. Since $\hat{a} |n\rangle$ and $\hat{a}^\dagger |n\rangle$ are eigen states of \hat{N} , we know that

$$\begin{aligned}
 \hat{a} |n\rangle &= C1 |n\rangle \\
 \hat{a}^\dagger |n\rangle &= C2 |n\rangle
 \end{aligned}$$

Then our next step is to find out what are $C1$ and $C2$ exactly.

The way of finding them is to use invariant quantities, such as the inner product. Here we use average of \hat{N} operator.

$$\begin{aligned}
 \hat{a} |n\rangle &= \sqrt{n} |n-1\rangle \\
 \hat{a}^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle
 \end{aligned}$$

Final step is to constrain on n , which should be integrals. This is true because we need a cut off for the eigen equation of \hat{N} , whose average is n and it should be non negative.

$$\langle n | \hat{N} | n \rangle \geq 0$$

leads to $n \geq 0$. To get this proper cut off, n should be integer because if it's not, according to

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle$$

n can go to negative numbers. If n is positive integer,

$$\begin{aligned}
 \hat{a} |1\rangle &= |0\rangle \\
 \hat{a} |0\rangle &= 0 |0\rangle
 \end{aligned}$$

show an cut off at 0.

We can even find out the wave functions of these $|n\rangle$ by finding the ground state first and apply \hat{a}^\dagger to the ground state.

Ground state in $|x\rangle$ basis can be found by solving the differential equation,

$$\langle x | \hat{a} | 0 \rangle = 0$$

Very important:

- The Hermitian conjugate of $\hat{a} |n\rangle$ is $\langle n | \hat{a}^\dagger$.
- Hermitian conjugate of $\hat{a}\hat{a}^\dagger$ is $\hat{a}\hat{a}^\dagger$ *aggr*. This can be a trap. Hermitian conjugate is the complex conjugate AND TRANSPOSE!

Semiclassical

Classical

In phase space, the trajectory of phase space points ($\{x/\eta$ and $p/(\hbar/\eta)\}$) is on a circle of radius x_{max}/η .

Quantum semiclassical

Key points:

1. What is the trajectory of $\langle \hat{x}/\eta \rangle$ and $\langle \hat{p}/(\hbar/\eta) \rangle$
2. Can we make the trajectory just like the classical case by choosing some special conditions?
3. What do these special cases mean?
 - Expectation value of creation and annihilation operators

Apply Ehrenfest theorem to annihilation operator,

$$i\hbar \frac{d}{dt} \langle \hat{a}(t) \rangle = \langle \psi | [\hat{a}(t), \hat{H}] | \psi \rangle = \hbar\omega \langle \hat{a}(t) \rangle$$

Excellent. Now we can solve out $\langle \hat{a}(t) \rangle$, which is

$$\langle \hat{a}(t) \rangle = \alpha_0 \exp(-i\omega t)$$

Take the hermitian conjugate,

$$\langle \hat{a}^\dagger(t) \rangle = \alpha_0^* \exp(i\omega t)$$

- Expectation value of position and momentum

With these two operators, we can find out the average of \hat{x} and \hat{p} because

$$\begin{aligned} \hat{x} &= \eta \frac{1}{\sqrt{2}} (\hat{a}^\dagger + \hat{a}) \\ \hat{p} &= \frac{\hbar}{\eta} i \frac{1}{\sqrt{2}} (\hat{a}^\dagger - \hat{a}), \end{aligned}$$

we have

$$\begin{aligned} \langle \hat{x}(t) \rangle &= \eta \frac{1}{\sqrt{2}} (\langle \hat{a}^\dagger(t) \rangle + \langle \hat{a}(t) \rangle) \\ \langle \hat{p}(t) \rangle &= \frac{\hbar}{\eta} i \frac{1}{\sqrt{2}} (\langle \hat{a}^\dagger(t) \rangle - \langle \hat{a}(t) \rangle) \end{aligned}$$

We can have a look at these two averages,

$$\frac{\langle \hat{x}(t) \rangle}{\eta} = \frac{1}{\sqrt{2}} [(\alpha_0 + \alpha_0^*) \cos(\omega t) + i(\alpha_0^* - \alpha_0) \sin(\omega t)]$$

$$\frac{\langle \hat{p}(t) \rangle}{\hbar/\eta} = \frac{1}{\sqrt{2}} [(\alpha_0 + \alpha_0^*) \sin(\omega t) + i(\alpha_0 - \alpha_0^*) \cos(\omega t)]$$

It is obvious that the average reduces to classical case if $\alpha_0 = \alpha_0^*$. **But this is too strong for a semiclassical limit.**

- Coherent state

Coherent state is the eigenstate of creation operator. Its wave package has the smallest spread allowed by quantum mechanics.

The most special part about coherent state is that the system stays on coherent state if it start with coherent state.

$$\hat{a} |\alpha(t)\rangle = \alpha(t) |\alpha(t)\rangle$$

Take the hermitian conjugate,

$$\langle \alpha(t) | \hat{a}^\dagger = \langle \alpha(t) | \alpha(t)^*$$

At $t = 0$, we have

$$\langle \psi(0) | N | \psi(0) \rangle = |\alpha_0|^2$$

That is to say, energy should be

$$\langle \psi(0) | \hat{H} | \psi(0) \rangle = \hbar\omega \left(|\alpha_0|^2 + \frac{1}{2} \right)$$

Initially, we also have

$$\langle \psi(0) | (\hat{a} - \alpha_0)^\dagger (\hat{a} - \alpha_0) | \psi(0) \rangle = 0$$

This means

$$\hat{a} |\psi(0)\rangle = \alpha_0 |\psi(0)\rangle$$

- Coherent state expanded using energy eigenstates

(This result)

(To Be Finished...)

6.3 Quantum Mechanics 2

6.3.1 Tensor Product Space

This part has been moved to *Tensor Product Space*

6.3.2 Density Matrix

6.3.3 Angular Momentum

Angular Momentum

For an new operator, we would like to know

1. Commutation relation: with their own components, with other operators;
2. Eigenvalues and their properties;
3. Eigenstates and their properties;
4. Expectation and classical limit.

Definition of Angular Momentum

In classical mechanics, angular momentum is defined as

$$\vec{L} = \vec{X} \times \vec{P}.$$

One way of defining operator is to change position and momentum into operators and check if the operator is working properly in QM. So we just define

$$\hat{L} = \hat{X} \times \hat{P}.$$

It is Hermitian. So it can be an operator. We also find

$$\hat{L} \times \hat{L} = i\hbar \hat{L}$$

$$[\hat{L}_i, \hat{L}_j] = \sum_k i\epsilon_{ijk} \hat{L}_k.$$

More generally, we can define angular momentum as

$$[\hat{J}_i, \hat{J}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{J}_k$$

We can prove that

$$[\hat{J}^2, \hat{J}_z] = 0.$$

So they can have the same eigenstates

$$\hat{J}_z |\lambda m\rangle = m\hbar |\lambda m\rangle$$

$$\hat{J}^2 |\lambda m\rangle = \lambda^2 \hbar^2 |\lambda m\rangle$$

To find the constraints on these eigenvalues, we can use positive definite condition of certain inner products, such as,

$$\langle \psi | \hat{J}_+ \hat{J}_- | \psi \rangle \geq 0$$

$$\langle \psi | \hat{J}_- \hat{J}_+ | \psi \rangle \geq 0$$

where

$$\hat{J}_{\pm} = \hat{J}_x \pm i\hat{J}_y$$

and we have

$$[\hat{J}_+, \hat{J}_-] = 2\hbar \hat{J}_z$$

$$[\hat{J}_z, \hat{J}_\pm] = \pm \hbar \hat{J}_\pm.$$

It's easy to find out that

$$\hat{J}_z(\hat{J}_\pm |\lambda m\rangle) = (m \pm 1)\hbar(\hat{J}_\pm |\lambda m\rangle)$$

i.e., $\hat{J}_\pm |\lambda m\rangle$ is eigenstate of \hat{J}_z .

Follow the plan of finding out the bounds through these positive inner products, we can prove that

$$\hat{J}^2 |jm\rangle = j(j+1)\hbar^2 |jm\rangle$$

$$\hat{J}_\pm |jm\rangle = \sqrt{j(j+1) - m(m \pm 1)}\hbar |j, m \pm 1\rangle$$

Eigenstates of Angular Momentum

As we have proposed, the eigenstates of both \hat{J}_z and \hat{J}^2 are $|j, m\rangle$, where $j = 0, 1, 2, \dots$ and $m = -j, -j+1, \dots, j-1, j$.

We can also find out the wave function in $|\theta, \phi\rangle$ basis. Before we do that, the definition of this basis should be made clear. This basis spans the surface of a 3D sphere in Euclidean space and satisfies the following orthonormal and complete condition.

$$\int d\Omega \langle \theta', \phi' | \theta, \phi \rangle = \delta(\cos \theta' - \cos \theta, \phi' - \phi) \int d\Omega |\theta', \phi'\rangle \langle \theta, \phi| = 1$$

Now we have an arbitrary state $|\psi\rangle$,

$$\begin{aligned} |\psi\rangle &= \sum_{l,m} \psi_{lm} |l, m\rangle \\ &= \sum_{l,m} \int d\Omega |\theta', \phi'\rangle \langle \theta, \phi | \psi_{lm} |l, m\rangle \\ &= \sum_{l,m} \int d\Omega |\theta', \phi'\rangle (\langle \theta, \phi | l, m \rangle) \psi_{lm} \end{aligned}$$

Then we define

$$\langle \theta, \phi | l, m \rangle = Y_l^m(\theta, \phi)$$

which is the spherical harmonic function.

Then

$$|\psi\rangle = \sum_{l,m} \psi_{lm} \int d\Omega Y_l^m(\theta, \phi) |\theta', \phi'\rangle$$

So as long as we find out what ψ_{lm} is, any problem is done.

6.4 Quantum Mechanics 3

6.4.1 Topics

Quantum Vocabulary

Vocabulary of physics, the fountain of research ideas.

0. Fine Structure Constant

$$\alpha = \frac{k_e e^2}{\hbar c} = \frac{1}{(4\pi\epsilon_0)} \frac{e^2}{\hbar c} = \frac{e^2 c \mu_0}{2\hbar}$$

In electrostatic cgs units, $\alpha = \frac{e^2}{\hbar c}$.

In natural units, $\alpha = \frac{e^2}{4\pi}$.

1. Hydrogen Atom

$$\text{Potential } V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}.$$

$$\text{Energy levels: } E_n = -\left(\frac{Z^2 \mu e^4}{32\pi^2 \epsilon_0^2 \hbar^2}\right) \frac{1}{n^2} = -\left(\frac{Z^2 \hbar^2}{2\mu a_\mu^2}\right) \frac{1}{n^2} = \frac{\mu c^2 Z^2 \alpha^2}{2n^2}.$$

$$\text{Ground state of hydrogen atom } \psi_{100}(r) = \frac{1}{\sqrt{\pi}} \frac{1}{a^{3/2}} e^{-Zr/a}.$$

Quantum Questions

Wedge Product, Cross Product & Commutation relation

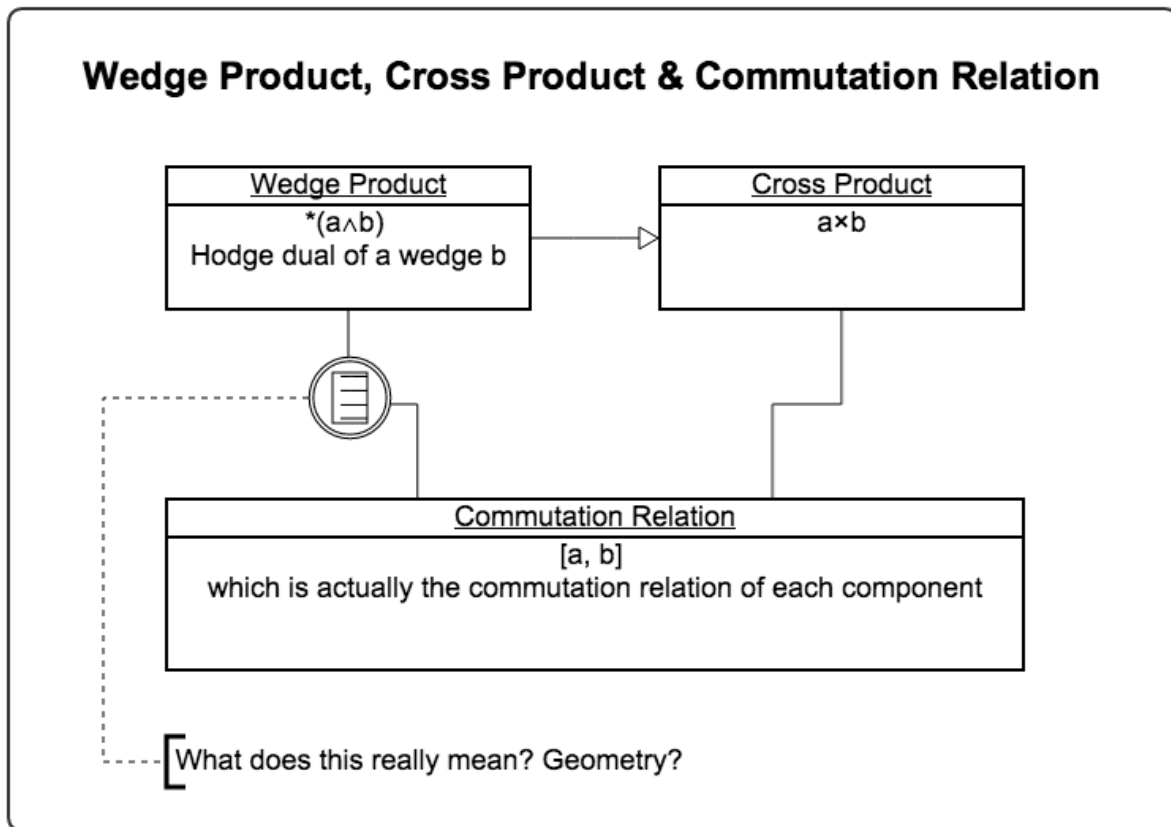


Fig. 6.1: Geometry language here?

Quantum Approximation Methods

Variational Method

Trial functions Some of the calculable trial functions:

1. $\psi(x) = \cos \alpha x$, for $|\alpha x| < \pi/2$, otherwise 0.
2. $\psi(x) = \alpha^2 - x^2$, for $|x| < \alpha$, otherwise 0.
3. $\psi(x) = C \exp(-\alpha x^2/2)$.
4. $\psi(x) = C(\alpha - |x|)$, for $|x| < \alpha$, otherwise 0.
5. $\psi(x) = C \sin \alpha x$, for $|\alpha x| < \pi$, otherwise 0.

Procedure

1. Pick a trial function.

Note: How to pick a trial function? For ground state energy, we should pick a function that has the same property as the real ground state. This requires some understanding of the problem we are dealing with.

Things to consider:

- (a) The new problem is just a modification of a known solved problem. Then we can easily find out what really is different and interpret the new problem in terms of the old one.
 - (b) If the Hamiltonian have definite parity, the ground state wave function should pick up some parity which is usually even to make it the lowest energy.
 - (c) Continuous function? A C^∞ Hamiltonian can only have continuous functions as solutions for a finite system.
 - (d) Nodes determines the kinetic energy so check the nodes for ground state wave function.
 - (e) Check the behavior of the wave function at different limits. In most cases, the Schrödinger equation can be reduced to something solvable at some limits.
 - (f) **One more thing, the trial function should make the problem calculable.**
-

Why Not General Variational Method Why don't we just use a most general variational method to find out the ground state? Because we will eventually come back to the time-independent Schrödinger equation.

Suppose we have a functional form

$$E(\psi^*, \psi, \lambda) = \int dx \psi^* H \psi - \lambda \left(\int dx \psi^* \psi - 1 \right)$$

The reason we have this Lagrange multiplier method is that the wave function should be normalized and this multiplier provides the degree of freedom. We would only get a wrong result if we don't include this DoF.

Variation of ψ^* ,

$$\delta E = \int dx \delta \psi^* H \psi - \int dx \delta \psi^* \psi = 0$$

Now what?

$$H \psi - \lambda \psi = 0$$

Not helpful.

Variational Method and Virial Theorem

For a potential $V(x) = bx^n$, we can prove that virial theorem is valid for ground state if we use Gaussian trial function $e^{-\alpha x^2/2}$.

A MMA proof is here.

Virial theorem is pretty interesting. It shares the same math with equipartition theorem.

WKB

This is a semi-classical method. It is semi classical because we will use the classical momentum

$$\hbar k(x) = \sqrt{2m(E - V(x))}$$

The following points are important for this method.

0. WKB start from a classical estimation of wave number at a certain energy E which is later quantified by the Bohr-Sommerfeld quantization rule.

1. Conservation law:

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \vec{j} = 0$$

where $\rho = \psi^* \psi$, $\vec{j} = -\frac{\hbar}{2mi} (\psi \nabla \psi^* - \psi^* \nabla \psi)$. This can be derived from Shrödinger equation easily.

2. Phase: Wave function is generally $A(x) \exp(\phi(x))$. However, $\phi(x)$ should be the area of the phase function starting from some initial point. For example in WKB, $k(x) = \phi'(x)$ and $\phi(x) = \int \phi'(x') dx' = \int k(x') dx'$.

Using this general wave function and conservation law we find out that $A(x) \propto \frac{1}{\sqrt{k(x)}}$. Then we can apply the two boundary conditions. However we will find two different wave functions given by two boundary conditions. Now we should connect them because $\psi(a) = \psi(b)$ exactly. By comparing the two wave functions we can find something like Bohr-Sommerfeld quantization rule.

3. Correction at boudlary: However, this method requires that the potential varies slowly or equivalently the wave number varies slowly. Basicly we are just using the following approximation:

$$A'(x) = 0, k'(x) = 0$$

For example when taking the derivative of wave function,

$$\psi'(x) = A'(x)e^{i \int \dots} + A(x)k(x)e^{i \int \dots} \approx A(x)k(x)e^{i \int \dots}$$

where we drop the term with $A'(x)$. That is to say

$$|A'| \ll |Ak| \Rightarrow |k'| \ll k^2$$

But at boundary where $E = V$, this is obviously not valid because $k = 0$. So we need to fix this problem.

The solution is to use first order of the potential in a Taylor expansion. Then solve the problem exactly. Finally we connect regions that is far out from the boundary, need the boundary and between the boundary.

If we can have a good boundary condition, then the energy spectrum given by WKB can be very good. Even we don't have a good boundary condtion, the excited states given by this method are always close to the exact ones.

How does it work

Symmetries in QM

Time and Space Translation

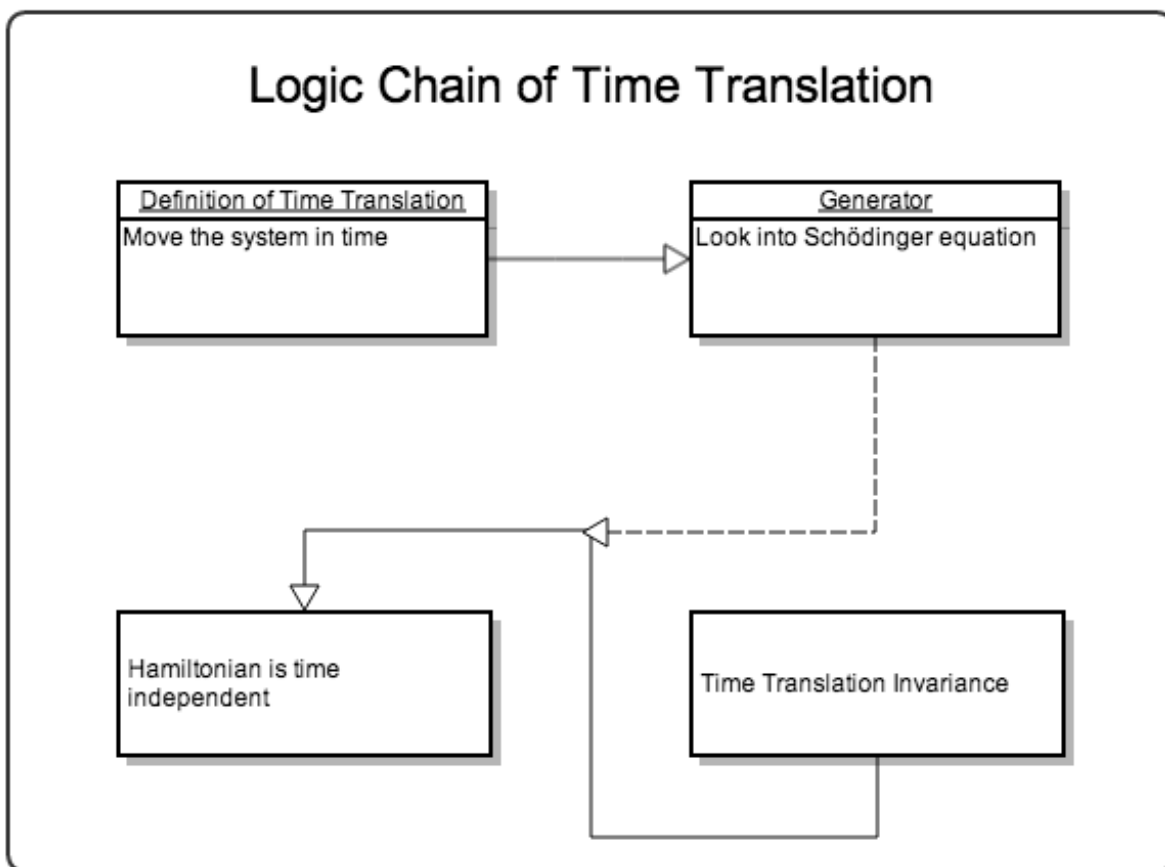
First of all I want to know what is not changed or what is the invariant quantity in a transformation.

There are three kind of common transformations.

1. Time translation: move the system in time. In this sense time translation is just the time evolution operator or propagator.
2. Space translation: move the system in space.
3. Gauge transformation

The invariance of them corresponds to:

1. Time translation invariance (T.T.I.) means the evolution of the system is not changing under time translations.
Hamiltonian is invariant.
2. Space translation invariance (S.T.I.) means that the



Time Translation Symmetry Time translation Gliffy Source

Definition of Time Translation

Move the system in time.

Generator of Time Translation

T.T.I. is generated by Hamiltonian which can be easily understood by looking into Schrödinger equation.

Hint: Starting from Schrödinger equation,

$$i\hbar \frac{|\psi(t + \Delta t)\rangle - |\psi(t)\rangle}{\Delta t} = H(t)\psi(t)$$

Then we get the state after a evolution of time Δt ,

$$|\psi(t + \Delta t)\rangle = \left(\hat{I} - i \frac{\Delta t \hat{H}(t)}{\hbar} \right) |\psi(t)\rangle$$

Time translation symmetry means the state evolution in the same time interval Δt no matter when to start the evolution. Mathematically,

$$|\psi(t_1 + \Delta t)\rangle = \left(\hat{I} - i \frac{\Delta t \hat{H}(t_1)}{\hbar} \right) |\psi(t_1)\rangle$$

should get the same final state if we start from some other time t_2 ,

$$|\psi(t_2 + \Delta t)\rangle = \left(\hat{I} - i \frac{\Delta t \hat{H}(t_2)}{\hbar} \right) |\psi(t_2)\rangle$$

That means the two Hamiltonian should be the same. Now we reach the conclusion that Hamiltonian is time independent.

The logic is to prove that Hamiltonian is time independent by using infinitesimal time translation approach. Given that Hamiltonian is time independent, we immediately know that time translation operator is just the propagator with the form

$$\hat{T}_{\Delta t} \equiv \hat{U}(\Delta t) = e^{-i\hat{H}\Delta t/\hbar}$$

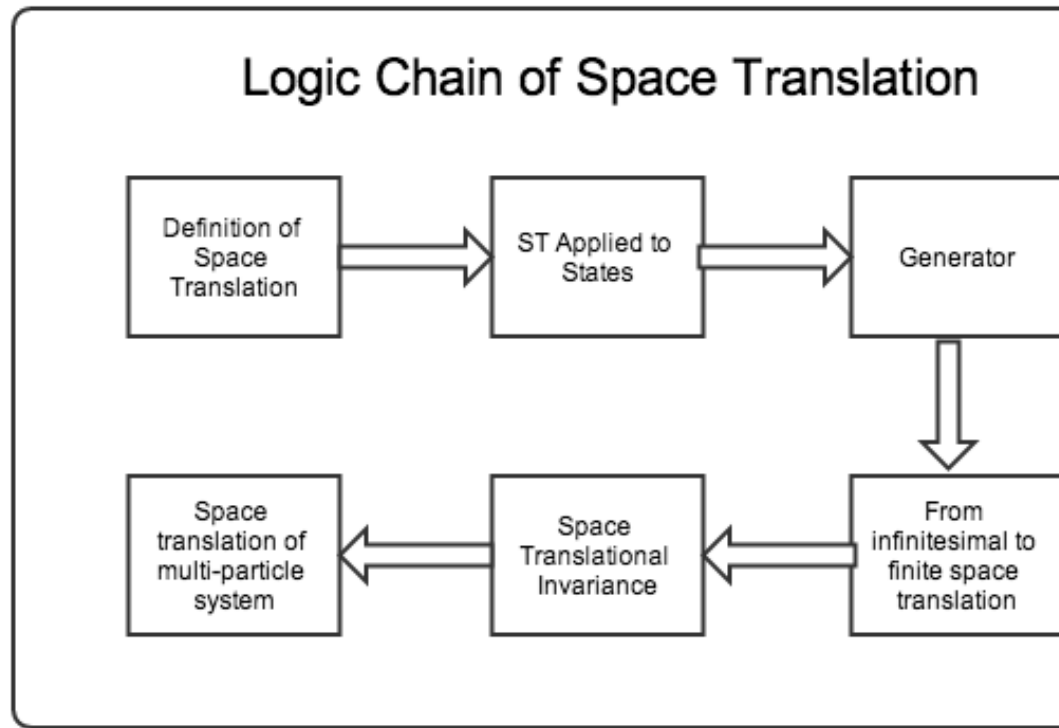
All other conclusions come from the fact that Hamiltonian is a constant of motion.

Hint: Ehrenfest theorem tells us that time independent Hamiltonian is a constant of motion.

$$\frac{d}{dt} \langle H \rangle = \frac{1}{i\hbar} \langle [\hat{H}, \hat{H}] \rangle + \left\langle \frac{\partial}{\partial t} H \right\rangle = 0$$

Important: For an isolated system, T.T.I. should always be satisfied because there is nothing more else to change the system but to leave the system with energy conserved.

My concern is if we don't have an Hamiltonian for TdS , we can't actually say this because of what the second law of thermodynamics tells us.



Space Translation Symmetry

Space Translation Gliffy Source

S.T.I. is generated by canonical momentum. This is not so obvious as time translation. To prove this we need to understand what space translation really means.

Definition of Space Translation

Space translation means we change the position of the system by some spatial distance a . In math this means a transformation from $|x\rangle$ to $|x + a\rangle$ where the plus sign is by definition. We invent this space translation operator,

$$\hat{T}_a |x\rangle = |x + a\rangle.$$

Space Translation Applied to States

Next we can obtain the result of space translation operator applied to state in position basis

$$\langle x | \hat{T}_a | \psi \rangle = (\langle x | \hat{T}_a^\dagger) | \psi \rangle = \langle x - a | \psi \rangle = \psi(x - a)$$

where we used the relation

$$(\langle x | \hat{T}_a) = (\hat{T}_a^\dagger | x \rangle)^\dagger = (\hat{T}_{-a} | x \rangle)^\dagger = (| x - a \rangle)^\dagger = \langle x - a |$$

which of course is because the normalization of coordinate basis tells us that space translation operator is unitary,

$$\langle x + a | x + a \rangle = \langle x | \hat{T}_a^\dagger \hat{T}_a | x \rangle$$

Generator of Space Translation

Similarly to time translation, we can find out the generator out of this definition. For infinitesimal translation,

$$-i\hbar \frac{|\psi(x)\rangle - |\psi(x - \Delta)\rangle}{\Delta} = \hat{p} |\psi(x)\rangle$$

i.e.,

$$|\psi(x - \Delta)\rangle = |\psi(x)\rangle - \frac{i\Delta}{\hbar} |\psi(x)\rangle$$

which shows that the generator of space translation is momentum operator.

From Infinitesimal to Finite Space Translation

$$\hat{T}_a = \lim_{N \rightarrow \infty} \hat{T}_{a/N}^N = \lim_{N \rightarrow \infty} \left(1 - \frac{i\hat{p}}{\hbar} \frac{a}{N} \right)^N = \exp\left(-\frac{i\hat{p}a}{\hbar}\right)$$

Now we have the explicit expression for space translation operators.

Space Translation on Operators

1. Use the invariant scalar – inner product.
2. Passive vs Active

Space Translational Invariance

Space translational invariance of arbitrary operator is

$$\hat{\Omega} = \hat{T}_a^\dagger \hat{\Omega} \hat{T}_a$$

is equivalent to

$$\hat{T}_a \hat{\Omega} = \hat{\Omega} \hat{T}_a \Rightarrow [\hat{T}_a, \hat{\Omega}] = 0$$

We say some system has space translational invariance we mean the Hamiltonian is space translational invariant,

$$[\hat{H}, \hat{T}_a] = 0.$$

Such a system has space translational invariance.

Hint: I once thought Hamiltonian is space/time translational invariant is not enough for the statement that the whole system is invariant under space or time translation for all observables. Of course I was wrong. Once the Hamiltonian and initial condition is given the whole system can be determined completely in principle.

Gauge Symmetry

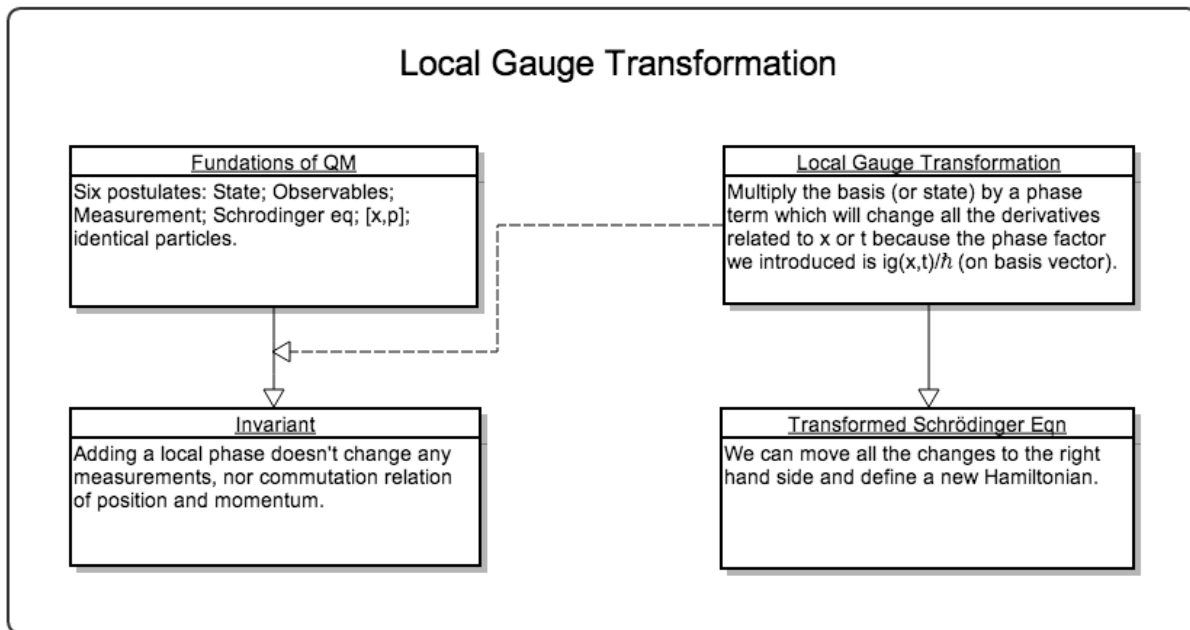
Global Gauge Transformation

$$|\psi\rangle \rightarrow e^{ig\hat{I}} |\psi\rangle$$

All quantum states are invariant under such transformation. This is not a nonsense transformation because the two states are different in some sense if we put them in a phase space where the phase factor assigns a position for the state vector in the phase space and we can see the difference directly in this image.

The invariant thing is the probability density which is obvious.

Hint: This is global because the phase factor doesn't depend on position and time.



Local Gauge Transformation Local Gauge Transformation Gliffy Source

What if we have a local phase factor: $g(x, t)$?

One way of implemment this phase factor is to transform the basis, for example:

$$|x\rangle \rightarrow e^{ig(x,t)/\hbar} |x\rangle$$

By changing the basis, we can transform anything on position basis. Since the first principle of QM is Schrödinger equation, we would like to check what happens to that.

It turns out that both space derivative and time derivative of the wave function changed. For both of them,

$$\frac{d}{dw}(\exp(-ig/\hbar)\phi) = \exp(-ig/\hbar)\frac{d}{dw}\phi - i/\hbar\left(\frac{d}{dw}g\right)\phi$$

equivalently, we can just change all the derivatives to

$$\frac{d}{dw} \rightarrow \exp(-ig/\hbar)\frac{d}{dw} - i/\hbar\frac{d}{dw}g$$

where w can be x or t .

Parity

Logic The only thing we need is the definition:

$$\hat{\Pi} |\vec{x}\rangle = |-\vec{x}\rangle$$

Starting from that, we can derive properties.

1. Hermition? **The way to find out something is Hermition or not is to take the Hermitian conjugate of the inner product sandwiched by the operator.**

We know

$$\langle x | \hat{\Pi} | x \rangle = \delta(x + x')$$

Take the Hermitian conjugate of the whole expression,

$$(\langle x' | \hat{\Pi} | x \rangle)^\dagger = \delta(x + x')$$

We know the LHS is $\langle x | \hat{\Pi}^\dagger | x' \rangle$. So we have

$$\langle x | \hat{\Pi}^\dagger | x' \rangle = \langle x | \hat{\Pi} | x' \rangle$$

Then we get that parity operator is Hermitian.

2. Inversion? Parity operator is Unitary.

$$\hat{\Pi}\hat{\Pi}|\pi\rangle = \hat{\Pi}\pi|\pi\rangle = \pi^2|\pi\rangle$$

By physics we know that parity twice gets back to the original state. So $\pi^2 = 1$. Then we can find inverse parity operator. What's important is that it's unitary.

3. Acts on states? From definition, we need to go to position basis.

$$\langle x | \hat{\Pi} | \psi \rangle = \langle -x | \psi \rangle.$$

We can also find the results on momentum eigenbasis, which is

$$\langle x | \hat{\Pi} | p \rangle = \langle -x | p \rangle.$$

We already know momentum eigen state in position is some kind of plane wave and it's easily proved that $\langle -x | p \rangle = \langle x | -p \rangle$.

4. Commutators with any observables? Just sandwich $\hat{\Pi}^\dagger \hat{O} \hat{\Pi}$ then act on arbitrary state and put it into position basis.

As an example, find commutation relation with position operator.

$$\langle x | \hat{\Pi}^\dagger \hat{X} \hat{\Pi} | \psi \rangle = \langle -x | \hat{X} \hat{\Pi} | \psi \rangle = -x \langle -x | \hat{\Pi} | \psi \rangle = -x \langle x | \psi \rangle$$

which is $\langle x | (-\hat{X}) | \psi \rangle$. This proves the following equation.

$$\hat{\Pi}^\dagger \hat{X} \hat{\Pi} = -\hat{X}$$

which can also be interpreted as passive transformation.

Another example is the commutation relation with (canonical) momentum.

$$\langle x | \hat{\Pi}^\dagger \hat{P} \hat{\Pi} | \psi \rangle = \langle -x | \hat{P} \hat{\Pi} | \psi \rangle = \int \langle -x | \hat{P} | x' \rangle \langle x' | \hat{\Pi} | \psi \rangle dx'.$$

By carefully applying parity on position basis, we have

$$\int \langle -x | \hat{P} | x' \rangle \langle -x' | \psi \rangle dx' = \int \langle -x | \hat{P} | -x' \rangle \langle x' | \psi \rangle dx'$$

Because commutation relation tells us

$$\langle x' | [\hat{X}, \hat{P}] | x \rangle = \langle x' | \hat{X} \hat{P} | x \rangle - \langle x' | \hat{P} \hat{X} | x \rangle = (x' - x) \langle x' | \hat{P} | x \rangle = i\hbar \delta(x' - x)$$

Here comes the keypoint. Recall that

$$x\delta'(x) = -\delta$$

we know that

$$(x - x') \langle x | \hat{P} | x' \rangle = i\hbar \delta(x' - x)$$

gives us the expression of momentum in position basis,

$$\langle x' | \hat{P} | x \rangle = -i\hbar \partial_x \delta(x' - x)$$

So to continue our calculation of parity applied to momentum,

$$\int \langle -x | \hat{P} | -x' \rangle \langle x' | \psi \rangle dx' = \int \langle x | (-\hat{P}) | x' \rangle \langle x' | \psi \rangle dx'$$

So we can prove that momentum actually inverts when parity is applied to it.

Tensors and Groups in Quantum

A rank-k tensor \hat{T}_k^q is defined as

$$\begin{aligned} [\hat{J}_z, \hat{T}_k^1] &= q\hbar\hat{T}_k^q \\ [\hat{J}_\pm, \hat{T}_k^q] &= \sqrt{(k \mp q)(k \pm q + 1)}\hbar\hat{T}_k^{q\pm 1}. \end{aligned}$$

Wigner-Eckart Theorem

Wigner-Eckart theorem is

$$\langle n'j'm' | \hat{T}_k^q | njm \rangle = \langle n'j' | \hat{T}_k | nj \rangle \langle j'm'; kj | kq; jm \rangle,$$

where j, j' are the angular momentum quantum numbers and n, n' are quantum numbers which are not related to angular momentum.

It seems that tensor \hat{T}_k^q is a source of angular momentum. The maximum angular momentum it can provide is k .

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Perturbation Theory in Quantum Mechanics

6.5 Statistical Physics

This part has been moved to [here](#) .

6.6 Electrodynamics

At this moment, the update will be done here [Electrodynamics](#) as a subproject and will be merged to this to this project when it is done.

6.6.1 Basics

Coulomb Potential Energy for a point charge Q with the appearance of a test charge q at distance r

$$V(r) = k \frac{qQ}{r}.$$

The ability to keep storage of charge is called capacitance, which is straight forward to have such a definition as

$$C = \frac{q}{U},$$

where U is the electric potential (not the potential energy).

Maxwell's equations are

$$\begin{aligned} \mathbf{E} \cdot d\mathbf{S} &= \frac{1}{\varepsilon_0} \iiint_{\Omega} \rho dV \\ \mathbf{B} \cdot d\mathbf{S} &= 0 \\ \oint_{\partial\Sigma} \mathbf{E} \cdot d\boldsymbol{\ell} &= -\frac{d}{dt} \iint_{\Sigma} \mathbf{B} \cdot d\mathbf{S} \\ \oint_{\partial\Sigma} \mathbf{B} \cdot d\boldsymbol{\ell} &= \mu_0 \iint_{\Sigma} \mathbf{J} \cdot d\mathbf{S} + \mu_0 \varepsilon_0 \frac{d}{dt} \iint_{\Sigma} \mathbf{E} \cdot d\mathbf{S} \end{aligned}$$

or

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

6.6.2 Ref & Notes

ADVANCED PHYSICS

7.1 Special Relativity

7.1.1 Conventions

Metric in special relativity

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Quantities and Operations

d'Alembertian

d'Alembert operator, or wave operator, is the Lapace operator in Minkowski space. ¹

$$\square \equiv \partial_\mu \partial^\mu = \eta_{\mu\nu} \partial^\mu \partial^\nu$$

In the usual $\{t, x, y, z\}$ natural orthonormal basis,

$$\begin{aligned} \square &= -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 \\ &= -\partial_t^2 + \Delta^2 \\ &= -\partial_t^2 + \nabla^2 \end{aligned}$$

On wiki ², they give some applications to it.

- klein-Gordon equation $(\square + m^2)\phi = 0$
- wave equation for electromagnetic field in vacuum: For the electromagnetic four-potential $\square A^\mu = 0$ footnote{Gauge}
- wave equation for small vibrations $\square_c u(t, x) = 0 \rightarrow u_{tt} - c^2 u_{xx} = 0$

¹ Actually, there are more general definitions for Lapacian, which includes this d'Alembertian of course.

Footnotes

7.2 General Relativity

7.2.1 Description of Space-time Manifold

How to describe space-time manifold?

- Metric (with a set of local coordinates), connection (Christoffel symbols).
- Metric (in the form of tetrads), connection (Ricci rotation coefficients).
- 1+3 covariantly defined variables.

7.2.2 Description of Space-time Manifold - Coordinates

7.2.3 Description of Space-time Manifold - Tetrads

7.2.4 Description of Space-time Manifold - 1+3 Covariant Description

Physics in description is easier to understand.

Definitions

Definitions of some physical quantities and operators are listed below.

Here we have

1. **geometrical variables:** Volume
2. **Kinematical variables:** Velocity, Expansion rate, Shear rate
3. **Thermodynamical variables:** Energy density, Momentum density, Pressure, Equation of state

Volume

To calculate volume, the volume element should be defined first in order to integrate. Before that, orientation on manifolds is to be figured out.

On an oriented manifold with metric, the defined volume element (a n-form) should be compatible with the orientation and also determined by the metric. ³

Introducing those requirements, a compatible volume element is

$$\epsilon_{a_1 \dots a_n} = \pm \sqrt{|g|} (e^1)_{a_1} \wedge \dots \wedge (e^n)_{a_n}$$

Alternatively, this can be expressed in the way Ellis used in arXiv:gr-qc/9812046v5.

$$\eta_{abcd} = \eta_{[abcd]}, \quad \text{with } \eta_{0123} = \sqrt{|\det g_{ab}|}$$

Induced volume element $\hat{\epsilon}_{a_1 \dots a_{n-1}}$ is defined use the normal vector u^a of the hypersurface,

$$\hat{\epsilon}_{a_1 \dots a_{n-1}} = u^b \epsilon_{ba_1 \dots a_{n-1}}$$

³ For more information, check out Canbin Liang's book. Volume 1, page 115.

4-velocity

4-velocity of observed matter is

$$u^\alpha = \frac{dx^\alpha}{d\tau}$$

with $u^\alpha u_\alpha = -1$, τ is the proper time along the worldlines of investigated matter.

Projection Tensors

We can use 4-velocity to project variables to parts that is parallel to u^α and parts that is orthogonal to u^α .

$$U^a_b = -u^a u_b$$

$$h_{ab} = g_{ab} + u_a u_b, \quad \text{induced metric from } g_{ab}$$

Some properties of the two projections.

$$U^a_b U^b_c = U^a_c, U^a_a = 1, U_{ab} = g_{ac} U^c_b, U_{ab} u^b = -g_{ac} u^c u_b u^b = u_a$$

$$h^a_b = g^{ac} h_{cb} = \delta^a_b + u^a u_b = \delta^a_b - U^a_b$$

$$h^a_c h^c_b = (\delta^a_c - U^a_c)(\delta^c_b - U^c_b) = \delta^a_b - U^a_b = h^a_b$$

$$h^a_a = 4 - 1 = 3, h_{ab} u^b = 0$$

Covariant time derivative ($\dot{}$)

This is the derivative along the fundamental worldlines (projection on the worldlines),

$$\dot{T}^{ab}_{cd} = u^e \nabla_e T^{ab}_{cd}$$

Fully orthogonally projected covariant derivative ($\tilde{\nabla}$)

This derivative is the project orghogonal to the normal vector of the hyperspace or orthogonal to the observer's 4-velocity or along the tagent of the hyperspace.

$$\tilde{\nabla}_e T^{ab}_{cd} = h^a_f h^b_g h^p_c h^q_d h^r_e \nabla_r T^{fg}_{pq}$$

Orthogonal projections of vectors

Orthogonal projection of vectors

$$v^{<a>} = h^a_b v^b$$

And the orthogonally projected symmetric trace-free part of tensors

$$T^{<ab>} = [h^{(a}_c h^{b)}_d - \frac{1}{3} h^{ab} h_{cd}] T^{cd}$$

Othogonal projected covariant time derivatives along u^a

$$\dot{v}^{<a>} = h^a_b \dot{v}^b$$

$$\dot{T}^{<ab>} = [h^{(a}_b h^{b)}_d - \frac{1}{3} h^{ab} h_{cd}] \dot{T}^{cd}$$

Properties

- Projected time and space derivatives of U_{ab} , h_{ab} and η_{abc} vanish.

7.2.5 Fields and Particles

Energy-Momentum Tensor for Particles

$$S_p \equiv -mc \int \int ds d\tau \sqrt{-\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu} \delta^4(x^\mu - x^\mu(s)),$$

in which $x^\mu(s)$ is the trajectory of the particle. Then the energy density ρ corresponds to $m\delta^4(x^\mu - x^\mu(s))$.

The Lagrange density

$$\mathcal{L} = - \int ds mc \sqrt{-\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu} \delta^4(x^\mu - x^\mu(s))$$

Energy-momentum density is $\mathcal{T}^{\mu\nu} = \sqrt{-g}T^{\mu\nu}$ is

$$\mathcal{T}^{\mu\nu} = -2 \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}$$

Finally,

$$\begin{aligned} \mathcal{T}^{\mu\nu} &= \int ds \frac{mc \dot{x}^\mu \dot{x}^\nu}{\sqrt{-\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu}} \delta(t - t(s)) \delta^3(\vec{x} - \vec{x}(t)) \\ &= m \dot{x}^\mu \dot{x}^\nu \frac{ds}{dt} \delta^3(\vec{x} - \vec{x}(s(t))) \end{aligned}$$

7.2.6 Theorems

Killing Vector Related

ξ^a is Killing vector field, T^a is the tangent vector of geodesic line. Then $T^a \nabla_a (T^b \xi_b) = 0$, that is $T^b \xi_b$ is a constant on geodesics.

7.2.7 Specific Topics

Redshift

In geometrical optics limit, the angular frequency ω of a photon with a 4-vector K^a , measured by an observer with a 4-velocity Z^a , is $\omega = -K_a Z^a$.

Stationary vs Static

Stationary

“A stationary spacetime admits a timelike Killing vector field. That a stationary spacetime is one in which you can find a family of observers who observe no changes in the gravitational field (or sources such as matter or electromagnetic fields) over time.”

When we say a field is stationary, we only mean the field is time-independent.

Static

“A static spacetime is a stationary spacetime in which the timelike Killing vector field has vanishing vorticity, or equivalently (by the Frobenius theorem) is hypersurface orthogonal. A static spacetime is one which admits a slicing into spacelike hypersurfaces which are everywhere orthogonal to the world lines of our ‘bored observers’”

When we say a field is static, the field is both time-independent and symmetric in a time reversal process.

CUTTING EDGE PHYSICS

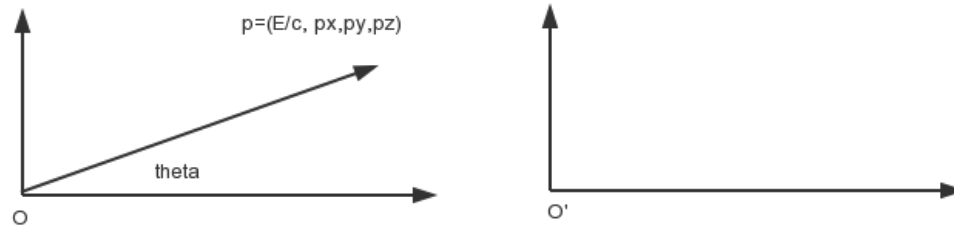
8.1 Astrophysics

8.1.1 Wave Bands

A table of wavebands in astronomy is something like this.

8.1.2 Doppler Shift

The special relativistic doppler shift can be derived using the fact that 4-momentum is a vector thus it transforms under Lorentz transformation.



The observer is fixed in O' frame and source is in O frame. Emitting angle in O frame is θ .

Since momentum is a vector, we have the Lorentz transformation which transform it in to O' frame,

$$\frac{E'}{c} = \gamma \left(\frac{E}{c} - \beta p_x \right),$$

where we also have

$$p_x = p \cos \theta, p = E/c.$$

Combining these equations, the energy of the photons in O' frame is

$$E' = E\gamma(1 - \beta \cos \theta).$$

In quantum mechanics, energy is related to angular frequency,

$$E = \hbar\omega.$$

The angular frequency in O' frame is

$$\omega' = \omega\gamma(1 - \beta \cos \theta).$$

Redshift is define as

$$\begin{aligned} z &= \frac{\nu_e - \nu_o}{\nu_o} \\ &= \frac{\omega_e - \omega_o}{\omega_o} \\ &= \frac{1/\gamma - 1 + \beta \cos \theta}{1 - \beta \cos \theta}. \end{aligned}$$

Non-relativistic Doppler Shift

To understand the effect of relativity, we would first recall the non-relativistic doppler shift.

$$\omega'_{non-rel} = \omega_{non-rel}(1 - v/c \cos \theta).$$

where no γ is relavent. It's obvious that we have only two kinds of shift, redshift due to the source is closing, or blueshift due to the fact that the source is moving away.

Here we have three different kinds, the additional one is the transverse redshift due to the γ factor or the contraction of time.

An gif from wikipedia shows this explicitly,

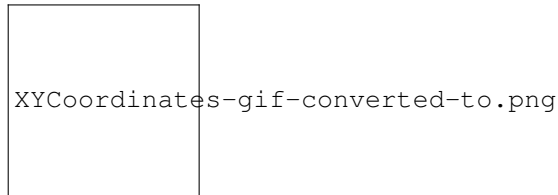


Fig. 8.1: Image Source: [File:XYCoordinates.gif](#)

8.1.3 Photon Production

There are two important processes that produces photons in astrophysical environment, namely thermal Bremsstrahlung and synchrotron radiation.

Bremsstrahlung Radiation

8.1.4 Compton Scattering

The quantities we are interested in are cross section and radiated power, both of which are Lorentz invariant.

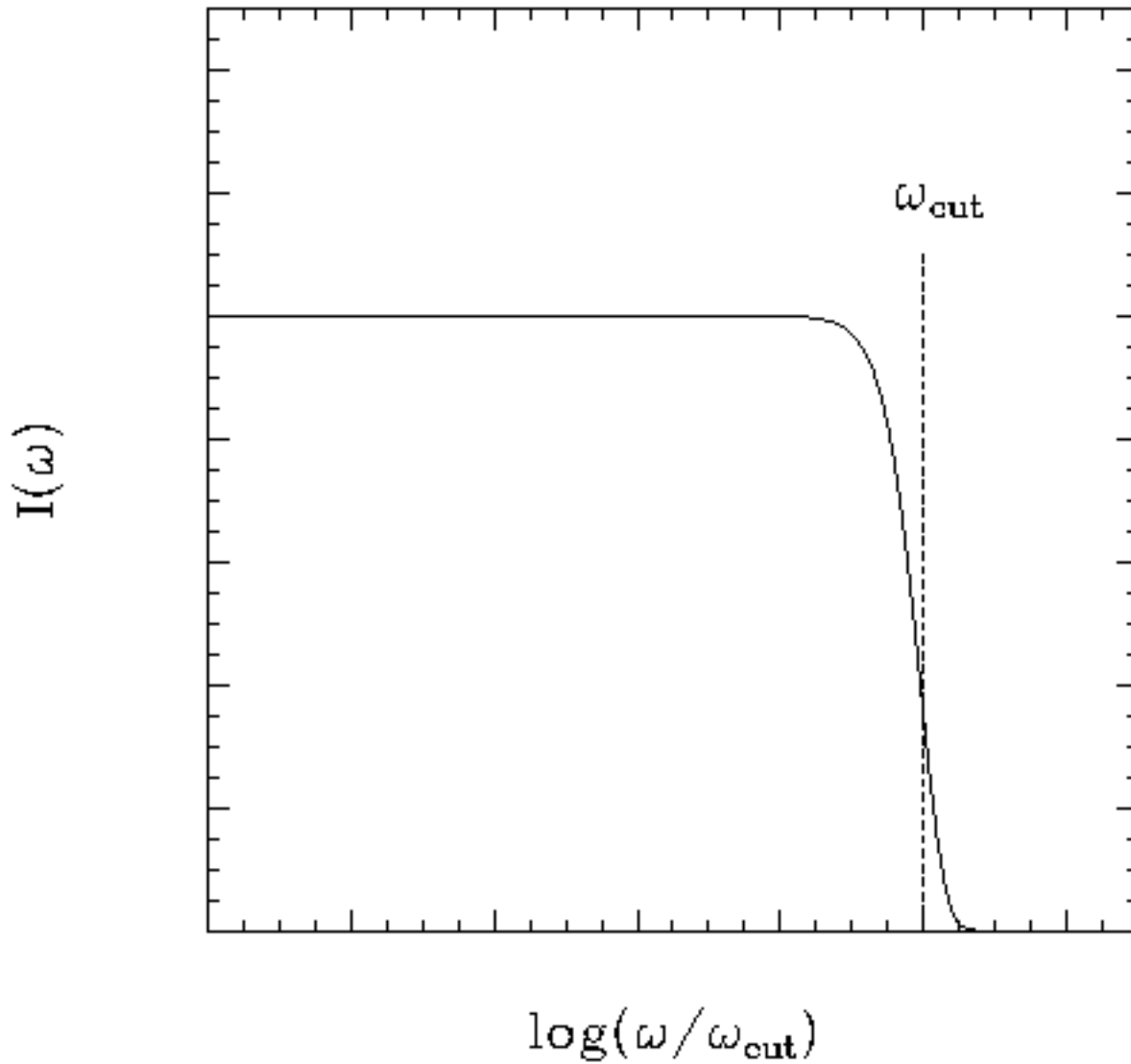


Fig. 8.2: This is the spectrum of frequency-dependent emissivity of the process which happens when a flux of non-relativistic regime with thermal distributon of temperature T is shot into a plasma of ions or protons.

Thomson Scattering

Total cross section of Thomson scattering can be obtained using only classical electrodynamics. The physics behind it is that the electric field exerts force on an electron then the electron emits photons to all possible direction as it oscillates.

The incoming power per unit area is

$$P_{inc} = cu_{rad},$$

as $u_{rad} = \frac{E^2}{4\pi}$ is the energy density of EM field.

The outgoing or scattered wave power (total) is

$$P_{scatt} = \frac{2}{3} \frac{e^4}{m_e^2 c^3} E^2.$$

The total cross section is ratio of the two quantity, i.e.,

$$\sigma_T \equiv \frac{P_{scatt}}{P_{inc}} = \frac{8\pi e^4}{3m_e^2 c^4} = \frac{8\pi(\alpha\hbar c)^2}{3m_e^2 c^4},$$

in which the fine structure constant is defined as $\alpha = \frac{e^2}{\hbar c}$.

Compton Scattering

The full quantum electrodynamics result is called Klein-Nishina formula, which describes the total cross section of colliding photon and electron,

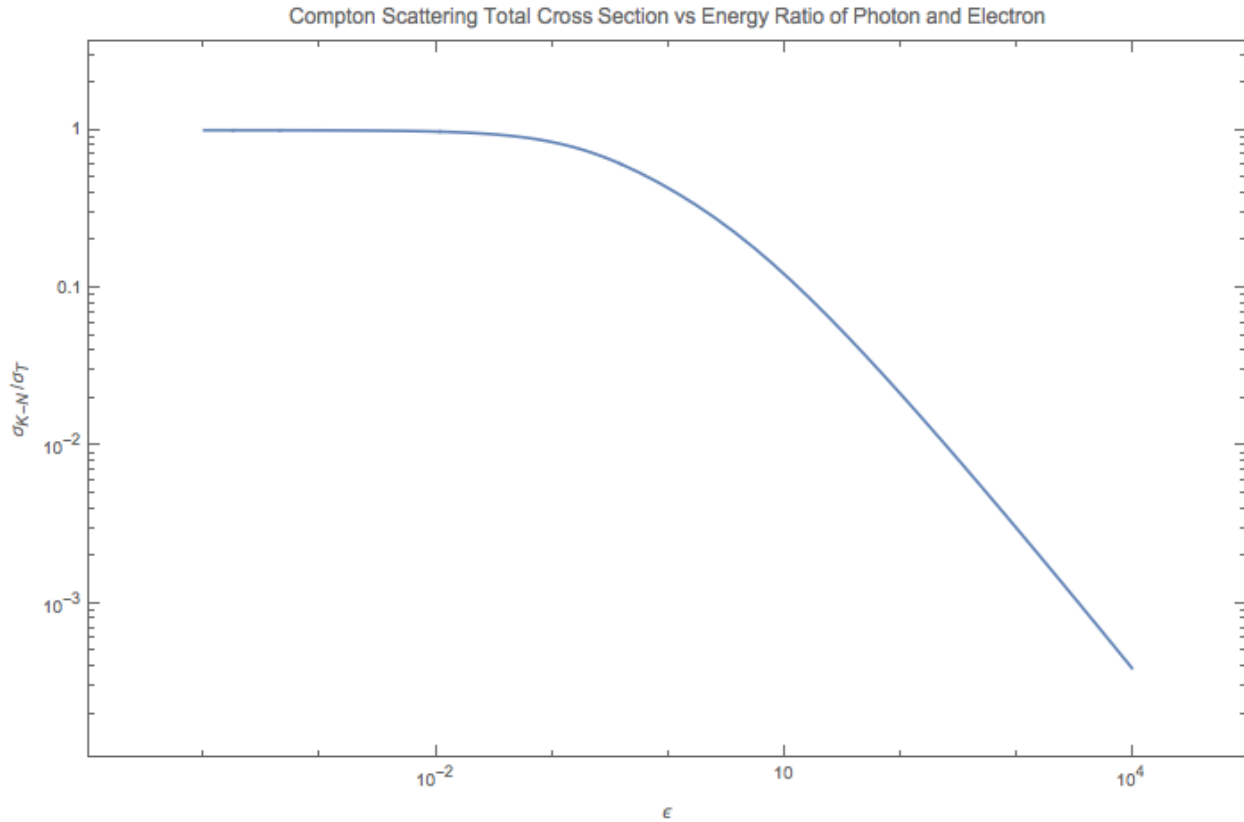
$$\sigma_{K-N} = \frac{\pi e^4}{m_e^2 c^4} \frac{1}{\epsilon} \left[\left(1 - \frac{2(\epsilon+1)}{\epsilon^2} \right) \ln(2\epsilon+1) + \frac{1}{2} + \frac{4}{\epsilon} - \frac{1}{2(2\epsilon+1)^2} \right],$$

where $\epsilon = \frac{E}{m_e c^2}$.

In the limit that energy of electron is much larger than photons, we have ϵ is much smaller than 1, we would come back to the Thomson limit, which is true for our equation,

$$\begin{aligned} \sigma_{K-N} &= \sigma_T \frac{3 \left(\left(1 - \frac{2(x+1)}{x^2} \right) \log(2x+1) + \frac{4}{x} - \frac{1}{2(2x+1)^2} + \frac{1}{2} \right)}{8x} \\ &= \sigma_T (1 - 2\epsilon + O(x^2)). \end{aligned}$$

To have more understanding on this formula, I plotted σ_{K-N} in terms of σ_T as the energy scale ϵ changes.



8.1.5 Asteroseismology

The stars do shake, from inside out.

Long period variable such as Cepheids pulsate in the luminosity. This is because of the radial oscillation mode with a approximate period of

$$P_{dynamical} \approx \left(\frac{R^3}{GM} \right)^{1/2} \approx (G\bar{\rho})^{-1/2},$$

in which we have the radius of the star as R , mass of the star as M and mean density $\bar{\rho}$. **The good thing of this oscillation immediately shows us the mean density of the star, even without any further inspection.**

There are double mode Cepheids, whose modes provides information about mass and radius.

Our sun, up to now we have identified thousands of individual modes. And more modes as many as 10^6 modes can be determined accurately.[^helioosc]

Papers, Researches and More

1. Double mode Cepheids, J. Otzen Petersen, 1973, 1974, 1978.
2. An introduction of seismology applied to stars. <http://ap.smu.ca/~guenther/seismology/seismology.html>

8.1.6 Refs & Notes

8.2 General Relativity Revisited

This post lists the experiments which are used to test gravity theories carried out on the earth.

The test of gravity theories can be viewed as test of the foundations of gravity theories and the theories themselves, say test of equivalent principle and general relativity or $f(R)$ gravity theory. Thus we should break down general relativity theory into several stages. Here, we use the following table to do so.

- **Physical Foundations: Hyperthesis:**

Theory	Mach	WEP	EEP	SEP	GC	Notes
GR	Partial	Y	Y	Y	Y	

- **Mathematical Description:**

Theory	Topoplogy	Manifold	Connection	Metric
GR			No torsion	Non-metricity tensor vanishes

- **Theoretical Implications:**

Theory	Gravitational Waves	Newtonian Limit	GR Limit	Notes
GR				

Most items in mathematics are the same in different theories.

8.2.1 Hyperthesis

- **WEP:** weak equivalence principle
- **EEP:** Einstein equivalence principle
- **SEP:** strong equivalence principle
- **GC,** General Covariance
- **Mach Principle:** gravity coupled to matter

8.2.2 Experiments

Eotvos Torsion Balance

How

- Inertial mass m_I
- Gravitational mass m_G

In Newtonian system, the acceleration of an object will be \vec{a}

In a static and uniform gravitation field, the gravity force is $\vec{F}_G = -m_G \vec{g}$

Thus the acceleration in this case should be $\vec{a} = -\vec{g}$

When m_G/m_I is constant, the falling acceleration are the same for different objects with same mass. However, if m_G/m_I is not a constant, say $m_G \neq m_I$, different objects would fall at different acceleration.

Now if we put two ball with different mass on the Eotvos torsion balance, the balance would rotate and we can measure it.

Results

Detection of $R_{0l0}^k = (1/c^2)\partial^2\Phi/\partial x^k\partial x^l \sim 10^{-32}\text{cm}^{-2}$.

Hughes-Drevershiy Experiment, etc

Anisotropy of gravitation/electromagnetism is not proved in our galaxy.

Radio Signal

Similar to Eddington and Dyson’s bending light observation, radio signals serve as a more precise experiment to test Einstein’s theory. And these experiments are against scalar tensor theories because scalar tensor theories give a smaller bending angle (1.66 second of arc less than the observations).

8.2.3 Summary Table

Tables constructed according to arXiv:1106.2476v3.

Test of fundamental principles

	Experiment	Results	Note
WEP	Eotvos torsion balance	$\eta = (0.3 \pm 1.8) \times 10^{-13}$	More precise in space exp. [[^1a] [[^1b]] [[^1c]]
	Gravitational redshift of light		[[^2]]
EEP	Hughes-Drever Experiment		

8.3 Cosmology

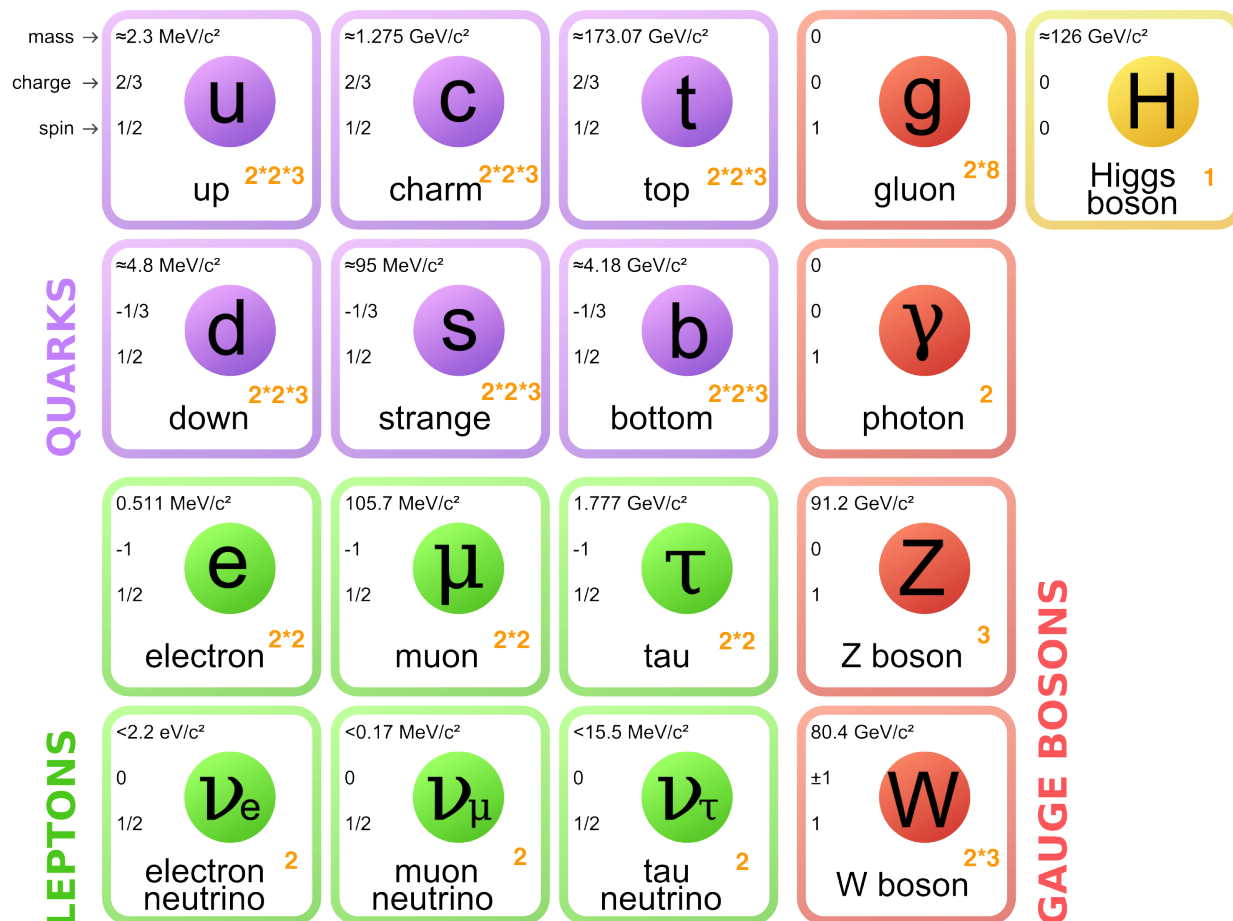
8.3.1 Thermal History of The Universe

Review of Standard Model for Particle Physics

SM of particle physics

- 1. describes elementary particles and their interactions.
- 2. is well test with experiments.

Degree of Freedom of Elementary Particles



IMG Source: https://en.wikipedia.org/wiki/File:Standard_Model_of_Elementary_Particles.svg

The orange numbers at the right bottom of each particle is the degrees of freedom it has. Here are some comments.

1. Photons have only two DoF because it is mass 0. Same reason can apply to gluon. But according to symmetry, there are 8 kinds of gluons.
2. W bosons carry charges. This is where the 2 come from.
3. Electrons and quarks have antiparticles. So there DoF will be doubled after counting the spin.
4. Each quark have 3 different colors and this gives us the 3 when calculating there DoF.

Finally, we can make this table.

Partilces	Higgs	Messengers	Quarks	Leptons
DoF	1	27	72	18

Expansion and Temperature

We can see that the heaviest particle is top quark with a mass of $m_t = 170\text{GeV}$.

Temperature Greater Than Mass of Top Quark

If temperature of the universe $T \gg m_t$, all particles should be in relativistic regime and the decay (annihilation) and inverse decay (inverse annihilation) are in equilibrium so all particles contribute to the thermal quantities in a relativistic way.

$$g_B = 28$$

$$g_F = 90$$

Then

$$g_* = g_B + \frac{7}{8}g_F = 106.75$$

For convinience, define the following reduced Planck mass

$$8\pi G = \frac{1}{M_p^2}$$

And it's good to know its value, which is $2.4 \times 10^{18} \text{GeV}$.

We would like to know the relation between expansion and temperature. We already know that the energy density is

$$\rho = g_* \frac{\pi^2}{30} T^4$$

So the expansion is

$$H^2 = \frac{8\pi G}{3} \rho = 106.75 \times \frac{\pi^2}{30} \frac{T^4}{3M_p^2}$$

So Hubble function is

$$H \approx 3 \frac{T^2}{M_p}$$

Temperature Down to Mass of A Particle

As temperature drops down, particle dacy (annihilation) will be greater than its inverse which is suppressed by Boltzmann factor $\exp(-m/T)$. The decay rate is so quick that the particle will almost dispear before the universe expand a lot.

So when the temperature drops below the mass of a particle, it won't contribute to the energy density. Their DoF will just disappear.

For example, if T MeV, Higgs and W and Z will decay and quarks are combined with gluons. So we only have **photons, electrons, neutrinos** as elementary particles, that is $g_* = 10.75$.

The Hubble function,

$$H \approx \frac{T^2}{M_p}$$

Decay Rate VS Expansion Rate

We can generally prove that decay rate is much faster than the expansion rate. To be added.

8.3.2 Two Parameters

Why is Cosmology Dedicated to Finding Two Parameters Before 90's

Basically, the cosmology before the 90's have only two tasks. The first one is to find out the Hubble constant, while the second one is looking for the deceleration parameter.

We don't rush to define what Hubble constant and deceleration parameter are, but have a look at what observations do at that time.

Observations

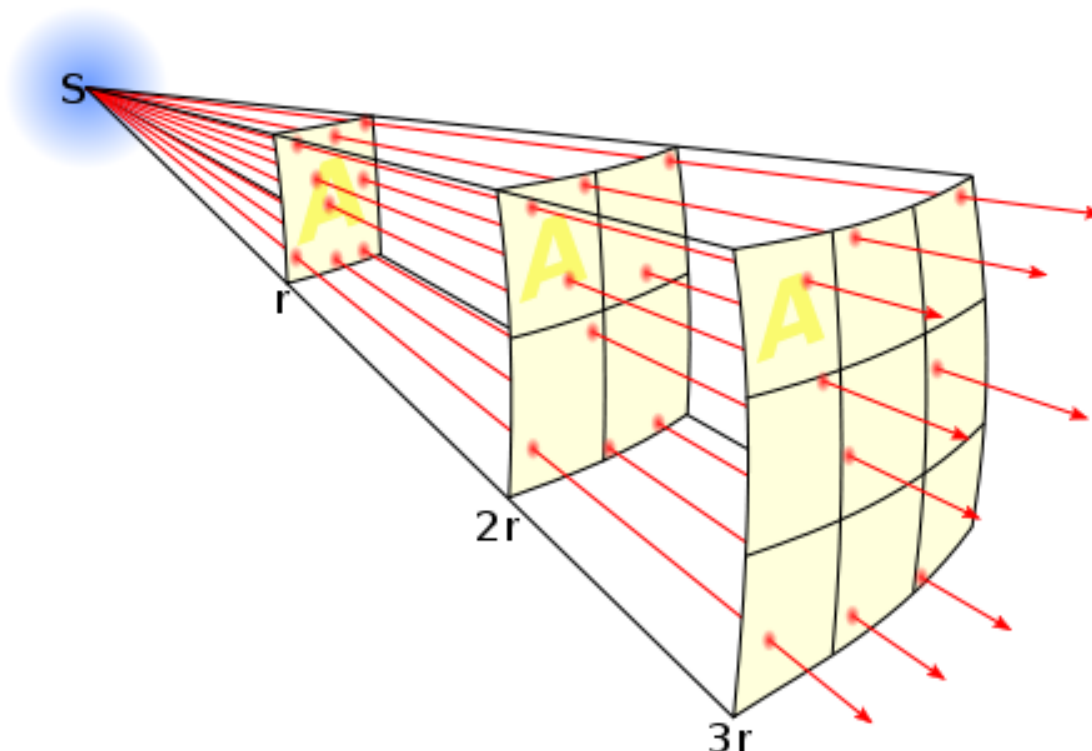
Astronomers are really good at measuring distances. They have infinite tricky ways to find out some distance.

Luminosity Distance

Luminosity Distance from Observation We can find out how bright a star is by observation. One way to represent the brightness is to use the energy crossed per unit area per unit time at the observer, because this is what our eyes do.

This quantity is related to how much energy was emitted at the star, how far we are from the star. The more energy the star emitted, the brighter it look like. The nearer the star is, the brighter it is. Just like what we feel like with a candle.

This schematic picture shows that energy spread out on a surface because the total energy is conserved. Isotropic energy flux through the same solid angle at different radius must be the same.



Through a very simple calculation, it is as simple as

$$L_0 = \frac{L}{4\pi r^2}.$$

We are dealing with Cosmology now. The space-time manifold should be a great concern. The luminosity turns out to be

$$L_0 = \frac{L_{\text{abs}}}{4\pi d^2} \frac{1}{1+z} \frac{1}{1+z}.$$

Here d is the physical distance between the star and the observer. L is the absolute luminosity of the star, which stands for the power of the star. z is the redshift of the star.

The first $\frac{1}{1+z}$ term comes from the fact that the energy of each photon decrease due to expansion of the universe, while the second is the result that the rate of photons arrived at the observer is less.

We are happy to define

$$d_L = d(1+z),$$

then the luminosity becomes simpler,

$$L_0 = \frac{L_{\text{abs}}}{4\pi d_L^2}.$$

Now we come back to have a look at this luminosity.

- We can measure how much energy is passing through a unit area at a unit time, which means **we can determine this luminosity directly from observations.**
- We can **predict the absolute luminosity** from a star evolution model.
- The $d_L = d(1+z)$ is only valid for a flat universe, with curvature term $K = 0$ in Friedmann equation.

Then we can find out this so called luminosity distance

$$d_L = \frac{L_{\text{abs}}}{4\pi L_0}$$

from some data.

Luminosity Distance from Theory We don't just do the observation for the luminosity distance itself. We observe to test theories.

What is this distance in theory?

$$d_L = d(1+z)$$

Wait, didn't we just mention that this is only valid for a flat universe? So we just do some extension.

$$d_L = R(d)(1+z)$$

where $R(d)$ is a function of d and can be determined through geometry,

- Spherical: $4\pi \sin^2 d$,
- Flat: $4\pi d$,
- Hyperbolic: $4\pi \sinh^2 d$.

Nearby Objects For nearby objects, we can always use flat geometry and use Taylor expansion at current time for $a(t)$.

Luminosity distance is

$$d_L = d(1+z) = ra(t)(1+z),$$

where r is the comoving distance and $a(t)$ is the scale factor at time t .

We know

$$r = \int_t^{t_0} \frac{1}{a(t')} dt'.$$

So we are happy to use Taylor expansion around t_0 for $a(t)$, and keep only up to the second order of time. And do some substitution with

$$H_0 = \dot{a}(t_0)/a(t_0)$$

$$q_0 = \ddot{a}(t_0)/a(t_0)$$

We then do the same thing on redshift

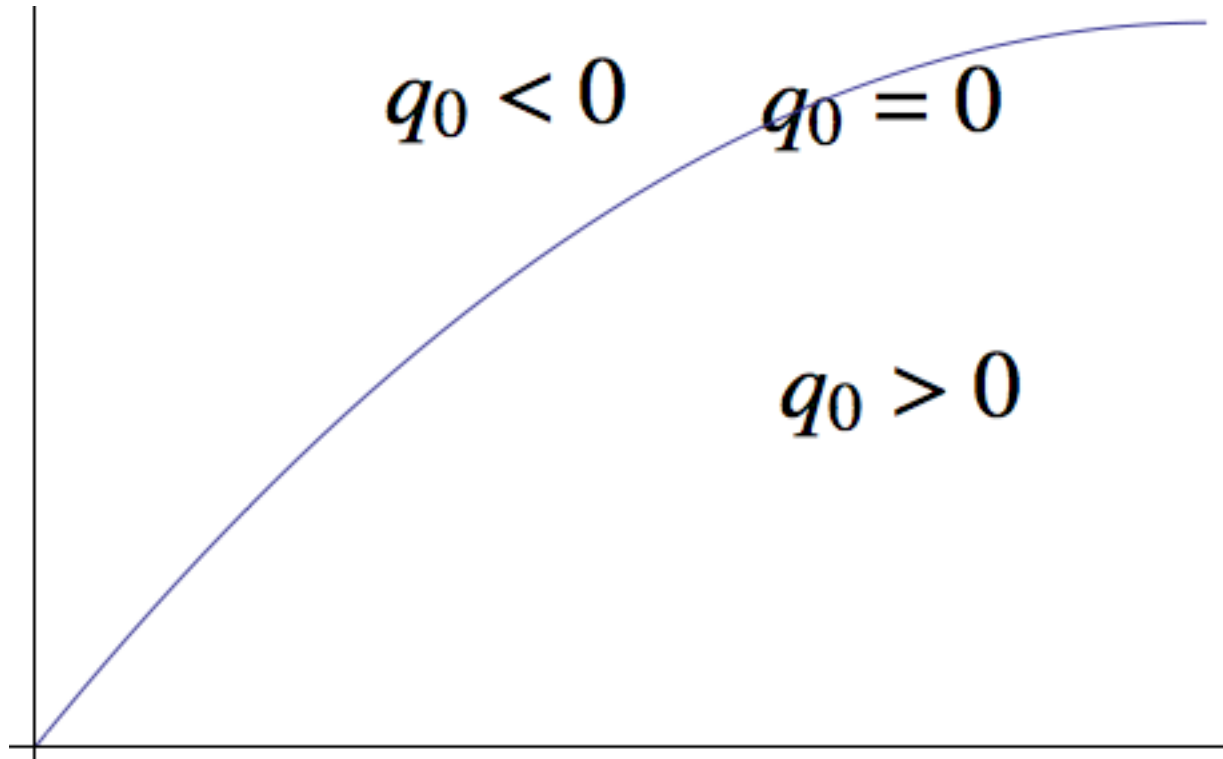
$$z = a(t_0)/a(t) - 1.$$

Finally, we can find out the relation $r(z)$, which leads us to the result we need, $d_L(z) = H_0^{-1}(z - \frac{1}{2}(1 + q_0)z^2)$.

- For very near objects (not as near as our sun of course),

$$d_L = H_0^{-1}z.$$

This is a model independent observation and derivation. We can draw a line to represent the case when deceleration parameter is zero, lines higher than this stands for a accelerating universe while lower region show a decelerating universe.

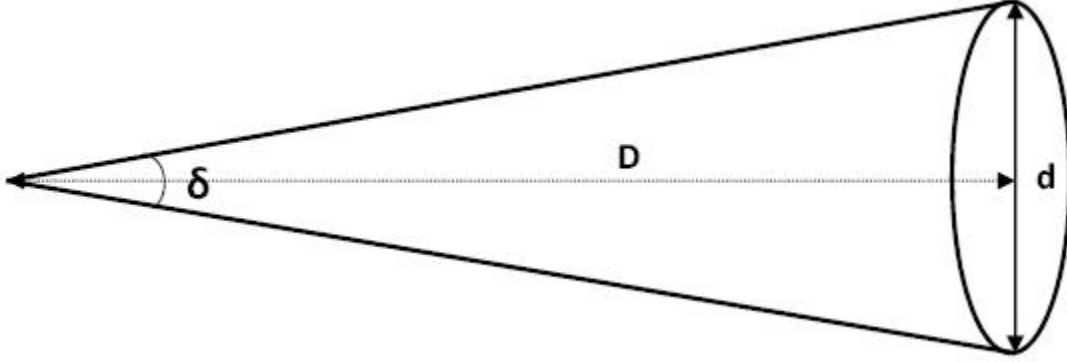


We can show that for a vacuum energy dominated universe, the line would go up and for a matter dominated universe, it would below the zero deceleration line.

Comment In this model independent method, the only two parameters occur are Hubble constant H_0 and deceleration parameter q_0 .

Angular Diameter Distance

Observation Angular diameter distance is really useful if we have some standard ruler. Now assume we have a ruler d , we can find out the angle between the two ends of the ruler, by some kind of measurement.



At the same time, we can use magic of math

$$\theta = d/D.$$

Now as we already find out what θ is by a measurement, and we said about the d is a standard ruler, which means we know the length of it very well. Then we can find out the distance D , which is the distance between us and the standard ruler.

Theory We can find out this kind of distance, which we will denote it as d_A from now on. What is it for?

A angular diameter distance is the physical distance between us and the standard ruler,

$$d_A = a(t)r.$$

We can use the same trick we used in luminosity distance calculations, and it is easy to find that

$$d_A = H_0^{-1} \left(z - \frac{1}{2}(3 + q_0)z^2 \right).$$

Again, the observation is related to only two parameters, Hubble constant H_0 and deceleration parameter q_0 .

Standard Rulers It is hard to imagine that we really have some standard rulers. In fact, we do. They are

- Baryon Acoustic Oscillation
- Sound Horizon at Recombination

Galaxy Number Count

Now we can see anything that is only (simply) related to physical or comoving distance can be determined by this trick. The result is that only two cosmological parameters would come in our equation as long as we keep only upper to order two of redshift.

Here another example is the galaxy number count.

$$\frac{dN_g}{dz d\Omega} = z^2 \frac{n_0}{H_0^3} (1 - 2(1 + q_0)z).$$

8.4 Quantum Optics

8.4.1 Quantum Optics Framework

From Professor [Ivan Deutsch](#).

1. Classical foundations
 - (a) Oscillators, interference, and coherence.
 - (b) Simple harmonic oscillators, quadratures, and Fourier analysis.
 - (c) Lorentz oscillator model.
2. Quantum foundations
 - (a) Density matrix and coherence.
 - (b) Two level systems – Pauli algebra, Bloch-sphere, magnetic resonance.
 - (c) Quantum simple harmonic oscillator.
3. Optical resonance for two level atoms
 - (a) Atom-photon interaction in electric dipole approximation.
 - (b) Pseudo-spin formulation, Rabi flopping.
 - (c) Density matrix formulation.
 - (d) Phenomenological damping – master equation and rate equations.
4. The electromagnetic vacuum
 - (a) Quantization of the electromagnetic field.
 - (b) Spontaneous emission and Wigner-Weisskopf theory.
 - (c) Resonance fluorescence – Mollow triplet.
 - (d) Jaynes-Cummings model – Dressed states.
5. Three level quantum coherence
 - (a) Raman resonance.
 - (b) Dark states and EIT.
 - (c) Slow light, fast light, and polaritons.
6. Quantum-Optical Coherence
 - (a) Photon counting statistics – Mandel’s formula.
 - (b) Theory of partial coherence - Classical statistical optics
 - (c) Coherent states as quasi-classical states.
 - (d) Glauber’s correlation functions.
 - (e) Hanbury-Brown and Twiss interferometry.
 - (f) Bunching, antibunching ,and photon statistics.
7. Nonclassical Light
 - (a) Phase space methods – Quasiprobability distributions, P-Glauber, Q-Husimi, W-Wigner functions.
 - (b) Nonlinear optics and nonclassical light.

- (c) Squeezed states.
- (d) Homodyne detection.
- (e) Correlated twin photons.
- (f) Photon interferometry.

8.5 Super-Symmetric Quantum Mechanics

Here is a note on this.

The idea of supersymmetric quantum mechanics is to introduce a hamiltonian related to supercharge, which is defined through

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CHAPTER
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This open source project is hosted on GitHub: [Physics Research Manual](#) .