In sync. gauge. considering both DM & DE perturbations

(1)

$$\begin{cases} \delta_c' = -k v_c + \dfrac{\rho_c'}{\rho_c \mathcal{H}}\left(H_L + \frac{1}{3}H_T\right) & \boxed{1} \\ v_c' = -\mathcal{H} v_c & \boxed{2} \\ \delta_d' = -3\mathcal{H}\left(\dfrac{\delta P_d}{\delta \rho_d} - w\right)\delta_d - (1+w) k v_d - 3(1+w) H_L' & \boxed{3} \\ v_d' = \dfrac{k}{1+w}\dfrac{\delta P_d}{\delta \rho_d}\delta_d - \mathcal{H}(1-3w) v_d - \dfrac{w'}{1+w} v_d & \boxed{4} \end{cases}$$

Conservation eqns.

考虑DE扰动的情况

(2) Choose

Here	HE's
A	ψ
B	B
$2H_T$	E
H_L	ϕ

$H = \frac{\dot a}{a} = \left(\frac{a'}{a^2}\right) = \frac{1}{a}\mathcal{H}$

$\mathcal{H} = \frac{a'}{a} = \dot a = aH$

$\mathcal{R} = H_L + \frac{1}{3}H_T$

$$\mathcal{A} \equiv A - \frac{1}{a}\left[\frac{a^2}{a'}\left(H_L + \frac{1}{3}H_T\right)\right]'$$

$$= -\frac{1}{a}\left[\frac{a^2}{a'}\left(H_L + \frac{1}{3}H_T\right)\right]'$$

$$= -\frac{1}{a}\left[\left(a\left(\frac{a}{a'}\right)' + a'\frac{a}{a'}\right)\mathcal{R} + \frac{a^2}{a'}\mathcal{R}'\right]$$

$$= -\frac{a}{a'}\mathcal{R}' + \left[-\left(\frac{a'}{a}\right)^{-1\prime} - 1\right]\mathcal{R}$$

$$= -\frac{a}{a'}\mathcal{R}' + \left(\frac{a'}{a}\right)^{-2}\left[\left(\frac{a'}{a}\right)' - \left(\frac{a'}{a}\right)^2\right]\mathcal{R} \qquad \boxed{5}$$

$$\mathcal{B} \equiv B + \left(\frac{a'}{a}\right)^{-1} k\left(H_L + \frac{1}{3}H_T\right) - \frac{1}{k}H_T'$$

$$= k\left(\frac{a'}{a}\right)^{-1}\mathcal{R} - \frac{1}{k}H_T' \qquad \boxed{6}$$

(3)

And we have $V = v - \frac{1}{k}H_T'$ in Sync. gauge

$\delta G^0{}_j = \delta(\chi^2 T^0{}_j)$

$\Rightarrow k\frac{a'}{a}\mathcal{A} + \chi^2\frac{1}{2}a^2\sum(\rho_t + P_t)\mathcal{B} = \chi^2\frac{1}{2}a^2(\rho_t + P_t)V_t \; \sum_\lambda \rho_\lambda V_\lambda$ total

$\Rightarrow -k\mathcal{R}' + k\frac{a}{a'}\left[\mathcal{H}' - \mathcal{H}^2 + \frac{1}{2}\chi^2 a^2(\rho_t + P_t)\right]\mathcal{R}$

$-k\frac{a'}{a}\frac{a}{a'}\mathcal{R}' + k\frac{a'}{a}\left(\frac{a'}{a}\right)^{-2}\left[\left(\frac{a'}{a}\right)' - \left(\frac{a'}{a}\right)^2\right]\mathcal{R} + \frac{1}{2}\chi^2 a^2\sum(\rho_t + P_t)\frac{a}{a'}k\mathcal{R}$

$-\frac{1}{2}\chi^2 a^2(\rho_t + P_t)\frac{1}{k}H_T' = \frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda\left(v_\lambda - \frac{1}{k}H_T'\right)$

$\Rightarrow -k\mathcal{R}' + k\frac{a}{a'}\left[\left(\frac{a'}{a}\right)' - \left(\frac{a'}{a}\right)^2\right]\mathcal{R} + \frac{1}{2}\chi^2 a^2 \frac{a}{a'}k\mathcal{R}\sum(\rho_t + P_t) = \frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda v_\lambda$

$\Rightarrow -k\mathcal{R}' + k\frac{a}{a'}\left[\mathcal{H}' - \mathcal{H}^2 + \frac{1}{2}\chi^2 a^2\sum(\rho_t + P_t)\right]\mathcal{R} = \frac{1}{2}\chi^2 a^2 (\rho_t + P_t) v_t \quad \frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda v_\lambda$

$$\chi^2 \sum_\lambda \rho_\lambda \left(\delta_\lambda + 3\frac{\mathcal{H}}{k} v_\lambda (1+w_\lambda)\right) - \sum_\lambda \rho_\lambda \delta_\lambda - \sum_\lambda \rho_\lambda 3 \frac{\mathcal{H}}{k}(1+w_\lambda)\left(v_\lambda - \frac{1}{k}H_T'\right)$$

$$+ \frac{1}{k^2}\sum_\lambda \rho_\lambda \left(\delta_\lambda + 3\frac{\mathcal{H}}{k} v_\lambda (1+w_\lambda)\right) \left(-\frac{1}{2}\right)\chi^2 a^2 \sum_\lambda \rho_\lambda' / \mathcal{H} = -\sum_\lambda \rho_\lambda \frac{\rho_\lambda'}{\rho_\lambda \mathcal{H}}\left(H_L + \frac{1}{3}H_T\right)$$

$$\Rightarrow -\sum_\lambda \rho_\lambda 3 \frac{\mathcal{H}}{k}(1+w_\lambda)\frac{1}{k} H_T' + \frac{1}{k^2}\sum_\lambda \rho_\lambda \delta_\lambda \cdot \left(-\frac{1}{2}\right)\chi^2 a^2 \sum_\lambda \frac{\rho_\lambda'}{\mathcal{H}} = -\sum_\lambda \frac{\rho_\lambda'}{\mathcal{H}}\left(H_L + \frac{1}{3}H_T\right)$$

<1> Background Einstein

Friedmann:
$$3\mathcal{H}^2 = \chi^2 a^2 \sum_\lambda \rho_\lambda$$
$$2\mathcal{H}' + \mathcal{H}^2 = -\chi^2 a^2 \sum_\lambda P_\lambda$$

: $h_\lambda = \rho_\lambda + P_\lambda$ for short

<2> Einstein Eqn (perturbed)

$$\delta G^0{}_0 = \chi^2 \sum_\lambda \delta T^0{}_{0(\lambda)}$$
$$\delta G^0{}_j = \chi^2 \sum_\lambda \delta T^0{}_{i(\lambda)}$$
$$\delta G^i{}_j = \chi^2 \sum_\lambda \delta T^i{}_{j(\lambda)}$$

Traceless part $\mathcal{A} + \frac{1}{k}\frac{1}{a^2}(a^2 \mathcal{B})' = -\chi^2 \frac{a^2}{k^2}\sum_\lambda \rho_\lambda \Pi_\lambda$ so chose a traceless T

<3> Gauge invariant Quantities

$$\mathcal{A} \equiv A - a^{-1}\left[\frac{a^2}{a'}\left(H_L + \frac{1}{3}H_T\right)\right]' = A - \frac{1}{a}\left[\frac{a^2}{a'}\mathcal{R}\right]'$$
$$\mathcal{B} \equiv B + \frac{k}{\mathcal{H}}\left(H_L + \frac{1}{3}H_T\right) - \frac{1}{k}H_T' = B + \frac{k}{\mathcal{H}}\mathcal{R} - \frac{1}{k}H_T'$$
$$\mathcal{R} = H_L + \frac{1}{3}H_T$$

$$\Delta = \Delta_g - 3(1+w_\lambda)\Phi + 3(1+w_\lambda)\mathcal{H}\frac{1}{k}V$$
$$\Phi = H_L + \frac{1}{3}H_T + \frac{1}{k}\mathcal{H}\left(B - \frac{1}{k}H_T'\right)$$
$$= \mathcal{R} + \frac{1}{k}\mathcal{H}\left(B - \frac{1}{k}H_T'\right)$$
$$= \frac{\mathcal{H}}{k}\left(\frac{k}{\mathcal{H}}\mathcal{R} + B - \frac{1}{k}H_T'\right) = \frac{\mathcal{H}}{k}\mathcal{B}$$
$$\Psi = \mathcal{A} + (ka)^{-1}(a\mathcal{B})' = A + \frac{1}{k}\mathcal{H}\left(B - \frac{1}{k}H_T'\right) + \frac{1}{k}\left(B' - \frac{1}{k}H_T''\right)$$

4) Perturbed Einstein Eqn $\Rightarrow$

(11) $\begin{cases} 3\mathcal{H}^2 A - \mathcal{H} k B = -\chi^2 a^2 \frac{1}{2} \sum_\lambda \rho_\lambda \Delta_{g\lambda} \\ k\mathcal{H} A + \chi^2 a^2 \frac{1}{2} \sum_\lambda h_\lambda B = \chi^2 a^2 \frac{1}{2} \sum_\lambda h_\lambda V_\lambda \end{cases}$

$\Rightarrow \begin{cases} 3k\mathcal{H}^2 A - \mathcal{H} k^2 B = -\chi^2 \frac{1}{2} a^2 \sum_\lambda \rho_\lambda \Delta_{g\lambda}\, k \\ 3k\mathcal{H}^2 A + \frac{1}{2} \chi^2 a^2 \sum_\lambda h_\lambda\, 3\mathcal{H} B = \frac{1}{2} \chi^2 a^2 \sum_\lambda h_\lambda V_\lambda \cdot 3\mathcal{H} \end{cases}$

$\Rightarrow \frac{3}{2}\mathcal{H} \chi^2 a^2 \sum_\lambda h_\lambda B + k^2 \mathcal{H} B = \frac{3}{2}\mathcal{H} \chi^2 a^2 \sum_\lambda h_\lambda V_\lambda + \frac{k}{2} \chi^2 a^2 \sum_\lambda \rho_\lambda \Delta_{g\lambda}$

$\Rightarrow \frac{3}{2}\mathcal{H} \chi^2 a^2 \sum_\lambda h_\lambda B + \mathcal{H} k^2 B = \frac{3}{2}\mathcal{H} \chi^2 a^2 \sum_\lambda h_\lambda V_\lambda + \frac{k}{2} \chi^2 a^2 \sum_\lambda \rho_\lambda \left(\Delta_\lambda + 3(1+w_\lambda)\Phi - 3(1+w_\lambda)\frac{a'}{a}\frac{1}{k} V_\lambda\right)$

$\frac{3}{2}\mathcal{H} \chi^2 a^2 \sum_\lambda h_\lambda B + k^2 \mathcal{H} B = \frac{3}{2}\mathcal{H} \chi^2 a^2 \sum_\lambda h_\lambda V_\lambda + \frac{k}{2} \chi^2 a^2 \sum_\lambda \rho_\lambda \Delta_\lambda + \frac{3k}{2} \chi^2 a^2 \sum_\lambda \rho_\lambda (1+w_\lambda)\Phi$
$- \frac{3}{2} k \chi^2 a^2 \sum_\lambda \rho_\lambda (1+w_\lambda) \mathcal{H} \frac{1}{k} V_\lambda$

$\Rightarrow \frac{1}{2}\chi^2 a^2 \sum_\lambda h_\lambda \cdot 3\mathcal{H} B + k^2 \mathcal{H} B = \frac{k}{2} \chi^2 a^2 \sum_\lambda \rho_\lambda \Delta_\lambda + \frac{3k}{2} \chi^2 a^2 \sum_\lambda h_\lambda \Phi$

$\Rightarrow \frac{3}{2} \chi^2 a^2 \sum_\lambda h_\lambda\, 3\mathcal{H} \frac{k}{\mathcal{H}} \Phi + k^2 \mathcal{H} \frac{k}{\mathcal{H}} \Phi = \frac{1}{2} k \chi^2 a^2 \sum_\lambda \rho_\lambda \Delta_\lambda + \frac{3}{2} k \chi^2 a^2 \sum_\lambda h_\lambda \Phi$

$$\Rightarrow \quad k^2 \Phi = \frac{1}{2} \chi^2 a^2 \sum_\lambda \rho_\lambda \Delta_\lambda \qquad \text{Poisson Eqn}$$

2) Traceless part of Einstein Eqn:

$$A + \frac{1}{k}\frac{1}{a^2}(a^2 B)' = 0$$

$$\Rightarrow A + \frac{1}{k}\frac{1}{a^2}(2aa'B + a^2 B') = 0$$

$$\Rightarrow A + \frac{1}{k} B' + \frac{2}{k}\mathcal{H} B = 0$$

From the definition of Ψ.

$$\Psi = A + \frac{1}{ka} a' B + \frac{1}{ka} a B' = A + \frac{1}{k}\mathcal{H} B + \frac{1}{k} B'$$

$\Rightarrow \Psi + \frac{1}{k}\mathcal{H} B = 0 \quad \Psi = -\frac{1}{k}\mathcal{H}\frac{k}{\mathcal{H}}\Phi \Rightarrow \Psi = -\Phi$

(5) Conservation Eqns $\quad (D_{g\lambda} \equiv \Delta_{g\lambda} \quad . \; D_c = D_{gc} \; , \; D_d = D_{gd})$

(from 1112.1128) No interaction

$$\begin{cases} D_c' = -kV_c & (5\sim1) \\ V_c' = -\mathcal{H}V_c + k\Psi \cancel{+a(1+w)} & (5\sim2) \\ D_d' = -k(1+w)V_d - 3\mathcal{H}(c_e^2 - w)D_d - 9\mathcal{H}^2(c_e^2 - c_a^2)(1+w)\frac{V_d}{k} \\ \quad + \{3w' + 9\mathcal{H}(c_e^2 - w)(1+w)\}\Phi & (5\sim3) \\ V_d' = -\frac{w'}{1+w}V_d - \mathcal{H}(1-3w)V_d + kc_e^2\frac{D_d}{1+w} \\ \quad + 3\mathcal{H}(c_e^2 - c_a^2)V_d - 3kc_e^2\Phi + k\Psi & (5\sim4) \end{cases}$$

(5~1) & (5~2) $\qquad D_c'' = -kV_c' \; . \; \mathcal{H}D_c' = -k\mathcal{H}V_c$

$D_c'' + \mathcal{H}D_c' = -k^2\Psi \qquad . \; \Psi = -\Phi$

$D_c'' + \mathcal{H}D_c' = +k^2\Phi$

Switch to Δ_c and Δ_d :

$$\Delta_c' = -kV_c + \frac{3}{k}(\mathcal{H}V_c)' - 3\Phi'$$

$$V_c' = -\mathcal{H}V_c + k\Psi = -\mathcal{H}V_c - k\Phi$$

$$\Delta_d' = -k(1+w)V_d - 3\mathcal{H}(c_e^2 - w)(\Delta_d + 3(1+w)\Phi - 3(1+w)\tfrac{1}{k}\mathcal{H}V_d)$$

Choose $c_a^2 = w$

For scalar field $c_e^2 = 1$

So for simplicity choose $c_e^2 = 1$

$$- 9\mathcal{H}^2(c_e^2 - c_a^2)(1+w)\tfrac{1}{k}V_d + \{3w' + 9\mathcal{H}(c_e^2 - w)(1+w)\}\Phi \; -3[(1+w)\Phi]' + 3[(1+w)\mathcal{H}\tfrac{1}{k}V_d]'$$

$$= \Big[-k(1+w) + 3\mathcal{H}(c_e^2 - w)\cdot 3(1+w)\tfrac{1}{k}\mathcal{H} \cancel{- 9\mathcal{H}^2(c_e^2 - c_a^2)}$$

$$- 9\mathcal{H}^2(c_e^2 - c_a^2)(1+w)\tfrac{1}{k}\Big]V_d - 3\mathcal{H}(c_e^2 - w)\Delta_d$$

$$+ \Big[-3\mathcal{H}(c_e^2 - w)\cdot 3(1+w) + 3w' + 9\mathcal{H}(c_e^2 - w)(1+w)\Big]\Phi \; -3[(1+w)\Phi]' + 3[(1+w)\mathcal{H}\tfrac{1}{k}V_d]'$$

$\boxed{\text{use } c_a^2 = w}$

$$= \{-k(1+w)V_d - 3\mathcal{H}(c_e^2 - w)\Delta_d + 3w'\Phi - 3[(1+w)\Phi]' + 3[(1+w)\mathcal{H}\tfrac{1}{k}V_d]'$$

$\boxed{\text{use } c_e^2 = 1}$

$$= -k(1+w)V_d - 3\mathcal{H}(1-w)\Delta_d + 3w'\Phi - 3[(1+w)\Phi]' + 3[(1+w)\mathcal{H}\tfrac{1}{k}V_d]'$$

$$= -k(1+w)V_d - 3\mathcal{H}(1-w)\Delta_d - 3(1+w)\Phi' + 3[(1+w)\mathcal{H}\tfrac{1}{k}V_d]'$$

$$V_d' = -\frac{w'}{1+w}V_d - \mathcal{H}(1-3w)V_d + k\,\frac{\Delta_d + 3(1+w)\Phi - 3(1+w)\frac{\mathcal{H}}{k}V_d}{1+w}c_e^2$$
$$+ 3\mathcal{H}(c_e^2 - c_a^2)V_d - 3kc_e^2\Phi - k\Phi$$

$\boxed{\text{Use } c_e^2=1,\ c_a^2=w}$

$$= -\frac{w'}{1+w}V_d - \mathcal{H}(1-3w)V_d + k\frac{1}{1+w}\Delta_d + 3k\Phi - 3\mathcal{H}V_d$$
$$+ 3\mathcal{H}(1-w)V_d - 3k\Phi - k\Phi$$
$$= \left[-\frac{w'}{1+w} - \mathcal{H}(1-3w) - 3\mathcal{H} + 3\mathcal{H}(1-w)\right]V_d + k\frac{1}{1+w}\Delta_d - k\Phi$$
$$= -\left(\frac{w'}{1+w} + \mathcal{H}\right)V_d + k\frac{1}{1+w}\Delta_d - k\Phi$$

In sync. gauge: $A = B = 0$

$$\begin{cases} \Delta_c = \delta_c + 3\frac{\mathcal{H}}{k}v_c \\ V_c = v_c - \frac{1}{k}H_T' \\ \Delta_d = \delta_d + 3(1+w)\mathcal{H}\frac{1}{k}v_d \\ V_d = v_d - \frac{1}{k}H_T' \end{cases}$$

Here	HE's
A	ψ
B	B
$2H_T$	E
H_L	ϕ

$\mathcal{H} = \frac{a'}{a} = \dot{a} = aH$

Then the conservation eqns in Sync. gauge can be

$$\delta_c' + 3\frac{1}{k}(\mathcal{H}v_c)' = -k\left(v_c - \frac{1}{k}H_T'\right) + \frac{3}{k}(\mathcal{H}v_c)' - 3\Phi'$$
$$v_c' - \frac{1}{k}H_T'' = -\mathcal{H}\left(v_c - \frac{1}{k}H_T'\right) - k\Phi$$
$$\delta_d' + \frac{3}{k}\left[(1+w)\mathcal{H}v_d\right]' = -k(1+w)\left(v_d - \frac{1}{k}H_T'\right) - 3\mathcal{H}(1-w)\left(\delta_d + \frac{3}{k}(1+w)\mathcal{H}v_d\right)$$
$$-3(1+w)\Phi' \cancel{+ 3w'\Phi - 3[(1+w)\Phi]'} + 3\left[(1+w)\mathcal{H}\frac{1}{k}\left(v_d - \frac{1}{k}H_T'\right)\right]'$$
$$v_d' - \frac{1}{k}H_T'' = -\left(\frac{w'}{1+w} + \mathcal{H}\right)\left(v_d - \frac{1}{k}H_T'\right) + \frac{k}{1+w}\left(\delta_d + \frac{3}{k}(1+w)\mathcal{H}v_d\right) - k\Phi$$

$\delta_c' = -kv_c - 3\Phi' + H_T'$ $\quad v_c' = -\mathcal{H} v_c + \frac{1}{k}\mathcal{H} H_T' - k\Phi + \frac{1}{k}H_T''$

$\delta_c'' = -kv_c' - 3\Phi'' + H_T''$

$\delta_c'' + \mathcal{H}\delta_c' = -k(v_c' + \mathcal{H}v_c) - 3\Phi'' + H_T'' - 3\mathcal{H}\Phi' + \mathcal{H}H_T'$ $\cancel{-3H_L'' - 3\mathcal{H}H_L' - H_T''}$

$\Rightarrow \delta_c'' + \mathcal{H}\delta_c' = -k\left[\frac{1}{k}\mathcal{H}H_T' - k\Phi + \frac{1}{k}H_T''\right] - 3\Phi'' + H_T'' - 3\mathcal{H}\Phi' + \mathcal{H}H_T'$ $\cancel{-3H_L'' - 3\mathcal{H}H_L' - H_T''}$

$\Rightarrow \delta_c'' + \mathcal{H}\delta_c' = k^2\Phi - \mathcal{H}H_T' - H_T'' - 3\Phi'' - 3\mathcal{H}\Phi' + H_T'' + \mathcal{H}H_T'$ $\cancel{-3H_L'' - 3\mathcal{H}H_L' - H_T''}$ $\cancel{k^2\Phi - 3\mathcal{H}}$

$\Rightarrow \boxed{\delta_c'' + \mathcal{H}\delta_c' = k^2\Phi \cancel{- 3\mathcal{H}(H_L + \frac{1}{3}H_T)' - 3H_L'' - H_T''} - 3\Phi'' - 3\mathcal{H}\Phi'}$

$\cancel{\Rightarrow \delta_c'' + \mathcal{H}\delta_c' = -3\Phi'' - 3\mathcal{H}\Phi}$

$\cancel{\Rightarrow \delta_c'' + \mathcal{H}\delta_c' = k\frac{\mathcal{H}}{k} k^2(H_L + \frac{1}{3}H_T) \Rightarrow k\frac{1}{k}\mathcal{H}H_T' - 3\mathcal{H}(H_L + \frac{1}{3}H_T)' - 3H_L''}$ $\delta\rho_d$

Need to find some expressions about $H_L + \frac{1}{3}H_T$, H_T', H_L'', or

List all known eqns

$$\begin{cases} -3\mathcal{H}^2\frac{1}{a}\left[\frac{a^2}{a'}\mathcal{R}\right]' - \mathcal{H}k\left[\frac{k}{\mathcal{H}}\mathcal{R} - \frac{1}{k}H_T'\right] = -\frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda(\Delta_\lambda + 3(1+w_\lambda)\Phi - 3(1+w_\lambda)\mathcal{H}\frac{v_\lambda}{k}) \\ -k\mathcal{H}\frac{1}{a}\left[\frac{a}{\mathcal{H}}\mathcal{R}\right]' + \frac{1}{2}\chi^2 a^2 \sum_\lambda h_\lambda\left[\frac{k}{\mathcal{H}}\mathcal{R} - \frac{1}{k}H_T'\right] = \frac{1}{2}\chi^2 a^2 \sum_\lambda h_\lambda V_\lambda \end{cases}$$

The first one

$\Rightarrow -3\mathcal{H}^2\frac{1}{a}\left(\frac{a}{\mathcal{H}}\right)'(H_L + \frac{1}{3}H_T) - 3\mathcal{H}^2\frac{1}{a}\frac{a}{\mathcal{H}}(H_L' + \frac{1}{3}H_T') - k^2(H_L + \frac{1}{3}H_T) + \mathcal{H}H_T'$

$= -\frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda(\delta_\lambda + 3(1+w_\lambda)\mathcal{H}\frac{1}{k}v_\lambda) - \frac{3}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda(1+w_\lambda)\Phi$

$+ \frac{3}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda(1+w_\lambda)\frac{\mathcal{H}}{k}(v_\lambda - \frac{1}{k}H_T')$ $\qquad \chi^2 a^2 \sum_\lambda h_\lambda = \mathcal{H}^2 - \mathcal{H}'$

$\Rightarrow -3\mathcal{H}^2\frac{1}{a}\left(\frac{a}{\mathcal{H}}\right)'\mathcal{R} - 3\mathcal{H}\mathcal{R}' - k^2\mathcal{R} + \mathcal{H}H_T'$

$= -\frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda\delta_\lambda - \cancel{\frac{3}{2}\chi^2 a^2 \sum_\lambda h_\lambda \Phi}\ \frac{3}{2}(\mathcal{H}^2 - \mathcal{H}')\Phi$

$- \frac{3}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda(1+w_\lambda)\frac{\mathcal{H}}{k^2}H_T'$

$\Rightarrow -3\mathcal{H}^2\frac{1}{a}\left(\frac{a}{\mathcal{H}}\right)'\mathcal{R} - 3\mathcal{H}\mathcal{R}' - k^2\mathcal{R} + \mathcal{H}H_T' = -\frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda\delta_\lambda$

$- \frac{3}{2}(\mathcal{H}^2 - \mathcal{H}')\Phi - \frac{3}{2}(\mathcal{H}^2 - \mathcal{H}')\frac{\mathcal{H}}{k^2}H_T'$

$\Rightarrow -3\mathcal{H}^2\frac{1}{a}\left(\frac{a}{\mathcal{H}}\right)'\mathcal{R} - 3\mathcal{H}\mathcal{R}' - k^2\mathcal{R} + \mathcal{H}H_T' = -\frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda\delta_\lambda$

$- \frac{3}{2}(\mathcal{H}^2 - \mathcal{H}')(\mathcal{R} - \frac{1}{k^2}\mathcal{H}H_T') - \frac{3}{2}(\mathcal{H}^2 - \mathcal{H}')\frac{\mathcal{H}}{k^2}H_T'$

$$\Rightarrow -3\mathcal{H}^2\frac{1}{a}\frac{a\mathcal{H}^2}{\mathcal{H}^2}R + 3\mathcal{H}^2\frac{1}{a}\frac{a\mathcal{H}'}{\mathcal{H}^2}R - 3\mathcal{H}R' - k^2R + \mathcal{H}H_T'$$
$$\doteq -\frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda\delta_\lambda - \frac{3}{2}(\mathcal{H}^2 - \mathcal{H}')R$$
$$\Rightarrow -3\mathcal{H}^2R + 3\mathcal{H}'R - 3\mathcal{H}R' - k^2R + \mathcal{H}H_T'$$
$$= -\frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda\delta_\lambda - \frac{3}{2}(\mathcal{H}^2 - \mathcal{H}')R$$
$$\Rightarrow -3(\mathcal{H}^2 - \mathcal{H}')R - 3\mathcal{H}R' - k^2R + \mathcal{H}H_T' = -\frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda\delta_\lambda - \frac{3}{2}(\mathcal{H}^2 - \mathcal{H}')R$$
$$\Rightarrow -\frac{3}{2}(\mathcal{H}^2 - \mathcal{H}')R - 3\mathcal{H}R' - k^2R + \mathcal{H}H_T' = -\frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda\delta_\lambda$$

~~The second one~~ Poisson eqn:

$$k^2\Phi = \frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda \Delta_\lambda$$
$$\Rightarrow k^2\Phi = \frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda\left(\delta_\lambda + 3(1+w_\lambda)\mathcal{H}\frac{1}{k}v_\lambda\right)$$
$$\Rightarrow k^2\left(R - \frac{\mathcal{H}}{k^2}H_T'\right) = \frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda\delta_\lambda + \frac{3}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda(1+w_\lambda)\frac{\mathcal{H}}{k}v_\lambda$$
$$\Rightarrow k^2R - \mathcal{H}H_T' = \frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda\delta_\lambda \cancel{+\frac{3}{2}(\mathcal{H}^2-\mathcal{H}')\frac{\mathcal{H}}{k}v} \frac{3}{2}\chi^2 a^2 \frac{\mathcal{H}}{k}\sum_\lambda h_\lambda v_\lambda$$
$$k^2\Phi = \frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda\delta_\lambda + \frac{3}{2}\chi^2 a^2 \frac{\mathcal{H}}{k}\sum_\lambda h_\lambda v_\lambda$$

Solve $-3\Phi'' - 3\mathcal{H}\Phi' + k^2\Phi$

Subhorizon: $\frac{\mathcal{H}}{k} \ll 1$, $\Phi \sim \frac{1}{k^2}$

$$-3\Phi'' - 3\mathcal{H}\Phi' + k^2\Phi \simeq k^2\Phi = \frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda\delta_\lambda + \underbrace{\frac{3}{2}\chi^2 a^2 \frac{\mathcal{H}}{k}\sum_\lambda h_\lambda v_\lambda}_{\sim\frac{\mathcal{H}}{k}}$$
$$\doteq \frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda\delta_\lambda$$

~~Perturb~~ Dark energy perturbation

$$\delta' + \frac{3}{k}[(1+w)\mathcal{H}v_d]' = -k(1+w)v_d + (1+w)H_T' - 3\mathcal{H}(1-w)$$
$$-3\mathcal{H}(1-w)\frac{3}{k}(1+w)\mathcal{H}v_d - 3(1+w)\Phi' + 3\left[(1+w)\mathcal{H}\frac{1}{k}\left(v_d - \frac{1}{k}H_T'\right)\right]'$$

Drop $\left(\frac{\mathcal{H}}{k}\right)^n$ terms, $\left(\Phi \sim \frac{1}{k^2}\right)$

$$\delta_d' = -k(1+w)v_d + (1+w)H_T' - 3\mathcal{H}(1-w)\delta_d$$

dark energy velocity ~~perturbation~~

$$v_d' - \frac{1}{k}H_T'' = -\left(\frac{w'}{1+w}+\mathcal{H}\right)v_d + \left(\frac{w'}{1+w}+\mathcal{H}\right)\frac{1}{k}H_T'$$
$$+ \frac{k}{1+w}\delta_d + \frac{k}{1+w}\cdot\frac{3}{k}(1+w)\mathcal{H}v_d - k\Phi$$

drop $\left(\frac{\mathcal{H}}{k}\right)^2$ terms if any ~~need~~ (remember $\Phi \sim \frac{1}{k^2}$, $v \sim \frac{1}{k}$)

$$v_d' = -\left(\frac{w'}{1+w}+\mathcal{H}\right)v_d + \left(\frac{w'}{1+w}+\mathcal{H}\right)\frac{1}{k}H_T' + \frac{1}{k}H_T''$$
$$+ \frac{k}{1+w}\delta_d + 3\mathcal{H}v_d - k\Phi$$

$$\Rightarrow v_d' = -\left(\frac{w'}{1+w}-2\mathcal{H}\right)v_d + \frac{k}{1+w}\delta_d + \left(\frac{w'}{1+w}+\mathcal{H}\right)\frac{1}{k}H_T' + \frac{1}{k}H_T'' - k\Phi$$

$$\begin{cases} \delta_d'' = -kw'v_d - k(1+w)v_d' + w'H_T' + (1+w)H_T'' \\ \qquad + 3\mathcal{H}w'\delta_d - 3\mathcal{H}(1-w)\delta_d' \\ -2\mathcal{H}\delta_d' = 2k(1+w)\mathcal{H}v_d - 2(1+w)\mathcal{H}H_T' + 6\mathcal{H}^2(1-w)\delta_d \end{cases}$$

$$\Rightarrow \delta_d'' - 2\mathcal{H}\delta_d' = -kw'v_d - k(1+w)v_d' + 2k(1+w)\mathcal{H}v_d$$
$$+ w'H_T' + (1+w)H_T'' + 3\mathcal{H}w'\delta_d - 3\mathcal{H}(1-w)\delta_d' - 2(1+w)\mathcal{H}H_T'$$
$$+ 6\mathcal{H}^2(1-w)\delta_d$$

~~$\Rightarrow \delta_d'' + \mathcal{H}\delta_d' = -k(1+w)\left[\frac{w'}{1+w}v_d + v_d' + \mathcal{H}v_d\right] + \left[w' + (1+w)\mathcal{H}\right]H_T'$~~
~~$+ (1+w)H_T''$~~ $+ 3\mathcal{H}\left[w' - \mathcal{H}(1-w)\right]\delta_d - 3\mathcal{H}(1-w)\delta_d'$

$$\Rightarrow \delta_d'' - 2\mathcal{H}\delta_d' = -k(1+w)\left[\frac{w'}{1+w} + v_d' - 2\mathcal{H}v_d\right] + \left[w' - 2(1+w)\mathcal{H}\right]H_T'$$
$$+ (1+w)H_T'' + 3\mathcal{H}\left[w' + 2\mathcal{H}(1-w)\right]\delta_d - 3\mathcal{H}(1-w)\delta_d'$$

$$\delta_d'' + \mathcal{H}(1-3w)\delta_d' = -k(1+w)\left[\frac{k}{1+w}\delta_d + \left(\frac{w'}{1+w}+\mathcal{H}\right)\frac{1}{k}H_T' + \frac{1}{k}H_T'' - k\Phi\right]$$
$$+ [w' - 2(1+w)\mathcal{H}]H_T' + (1+w)H_T'' + 3\mathcal{H}[w' + 2\mathcal{H}(1-w)]\delta_d$$
$\Rightarrow$

$$\Rightarrow \delta_d'' + \mathcal{H}(1-3w)\delta_d' = -k^2\delta_d - (1+w)\left(\frac{w'}{1+w}+\mathcal{H}\right)H_T' - (1+w)H_T''$$
$$+ k^2(1+w)\Phi + \cancel{[w'-2(1+w)]}\,[w' - 2(1+w)\mathcal{H}]H_T' + (1+w)H_T''$$
$$+ 3\mathcal{H}[w' + 2\mathcal{H}(1-w)]\delta_d .$$

$$\Rightarrow \delta_d'' + \mathcal{H}(1-3w)\delta_d' - 3\mathcal{H}[w' + 2\mathcal{H}(1-w)]\delta_d + k^2\delta_d$$
$$= -(w' + \mathcal{H}(1+w))H_T' \cancel{-(1+w)H_T''} + k^2(1+w)\Phi + [w' - 2(1+w)\mathcal{H}]H_T'$$
$$\cancel{+ 3\mathcal{H}[w' + 2\mathcal{H}(1-w)]\delta_d}$$

$$\Rightarrow \delta_d'' + \mathcal{H}(1-3w)\delta_d' - 3\mathcal{H}[w' + 2\mathcal{H}(1-w)]\delta_d + k^2\delta_d$$
$$= [-w' - \mathcal{H}(1+w) + w' - 2(1+w)\mathcal{H}]H_T' + k^2(1+w)\Phi$$

$$\Rightarrow \delta_d'' + \mathcal{H}(1-3w)\delta_d' - 3\mathcal{H}[w' + 2\mathcal{H}(1-w)]\delta_d + k^2\delta_d$$
$$= -3(1+w)\mathcal{H}H_T' + k^2(1+w)\Phi$$

[Since $\Phi = -\Psi \Rightarrow \mathcal{R} - \frac{1}{k^2}\mathcal{H}H_T' = -\frac{1}{k^2}\mathcal{H}(H_T' - \frac{1}{k^2}H_T'' \Rightarrow \mathcal{R} = -\frac{1}{k^2}H_T''$

$\Phi = -\frac{1}{k^2}(H_T'' + \mathcal{H}H_T')$]

$$\Rightarrow \delta_d'' + \mathcal{H}(1-3w)\delta_d' - 3\mathcal{H}[w' + 2\mathcal{H}(1-w)]\delta_d + k^2\delta_d$$
$$= -3(1+w)\mathcal{H}H_T' - (1+w)(H_T'' + \mathcal{H}H_T')$$

$$\delta_c'' + \mathcal{H}\delta_c' = -H_T'' - \mathcal{H}H_T'$$

$$\rho_{cr} = \frac{3H_0^2}{\chi^2}$$

$$-H_T'' - \mathcal{H}H_T' = \frac{1}{2}\chi^2 a^2 \sum_\lambda \rho_\lambda\delta_\lambda \Rightarrow -H_T'' - \mathcal{H}H_T' = \frac{3}{2}H_0^2 \sum_\lambda \frac{\rho_\lambda}{\rho_{cr}}\delta_\lambda\, a^2$$

No dark energy perturbation

Sync. Gauge: $A=B=0$

不考虑DE扰动的情况

A	ψ
B	B
$2H_T$	E
H_L	ϕ

<1> Dark matter perturbations

$$\begin{cases} v_c' + \frac{a'}{a} v_c = 0 & \boxed{1} \\ \delta_c' = (-k^2 + \frac{3}{2}\chi^2 \rho_c a^2)\frac{v_c}{k} + H_T' & \boxed{2} \end{cases}$$

<2>
$$\mathcal{A} = -\frac{1}{a}\left[\frac{a}{\mathcal{H}}(H_L + \frac{1}{3}H_T)\right]'$$
$$= -\frac{1}{\mathcal{H}}\mathcal{R}' + \mathcal{H}^{-2}[\mathcal{H}' - \mathcal{H}^2]\mathcal{R}$$
$$\mathcal{B} = k\frac{1}{\mathcal{H}}\mathcal{R} - \frac{1}{k}H_T' \qquad \boxed{3}$$

$\boxed{4}$

<3> $\delta G^0_{\ j} = \delta \chi^2 T^0_{\ j}$

with $\{ V_c = v_c - \frac{1}{k}H_T'$, $\{ h_c = \rho_c + P_c)$ Only matter perturbation considered.

$$\Rightarrow k\mathcal{H}\mathcal{A} + \chi^2 \frac{1}{2}a^2 h_c \mathcal{B} = \chi^2 \frac{a^2 h_c}{2} V_c \qquad \boxed{5}$$

$$\Rightarrow -k\mathcal{R}' + \frac{k}{\mathcal{H}}[\mathcal{H}' - \mathcal{H}^2 + \frac{1}{2}\chi^2 a^2 h_c]\mathcal{R} = \frac{1}{2}\chi^2 a^2 h_c v_c \qquad \boxed{6}$$

<4> $\begin{cases} \text{Friedmann Eqn:} \quad 3\mathcal{H}^2 = \chi^2 a^2 \sum_\lambda \rho_\lambda \\ \text{Another eqn of BG:} \quad 2\mathcal{H}' + \mathcal{H}^2 = -\chi^2 a^2 \sum_\lambda P_\lambda \end{cases}$

$$\Rightarrow 2\mathcal{H}^2 - 2\mathcal{H}' = \chi^2 a^2 \sum_\lambda h_\lambda$$

$$\Rightarrow \frac{1}{2}\chi^2 a^2 h_c = \mathcal{H}^2 - \mathcal{H}' - \frac{1}{2}\chi^2 a^2 h_d \qquad \boxed{7}$$

Put ⑦ back to ⑥

$$-k\mathcal{R}' - \frac{k}{\mathcal{H}}\frac{1}{2}\chi^2 a^2 h_d \mathcal{R} = \frac{1}{2}\chi^2 a^2 h_c v_c$$

⑤ Poisson Eqn: $\chi^2 \rho_c \Delta_c = \frac{2}{a^2} k^2 \Phi$

$$\Phi = -\Psi = \frac{1}{k^2}(H_T'' + \mathcal{H} H_T')$$

$$\chi^2 \rho_c(\delta_c + 3\mathcal{H}\frac{1}{k}v_c) = \frac{2}{a^2}k^2\Phi$$

$$\Rightarrow \chi^2\rho_c(\delta_c + 3\mathcal{H}\frac{1}{k}v_c) = \frac{2}{a^2}\cdot\frac{1}{k^2}k^2(H_T''+\mathcal{H}H_T')$$

$$\frac{1}{2}\chi^2a^2\rho_c(\delta_c+3\frac{\mathcal{H}}{k}v_c) = H_T''+\mathcal{H}H_T' \qquad \boxed{8}$$

Combine ① and ②.

$$① \Rightarrow \delta_c'' = (-k^2+\frac{1}{2}\chi^2a^2\rho_c)\frac{v_c'}{k} + \frac{1}{2}\chi^2(\rho_c a^2)'\frac{v_c}{k} + H_T''$$

$$\Rightarrow \delta_c'' = -kv_c' + \frac{1}{2}\chi^2\rho_c a^2\frac{1}{k} v_c' + \frac{1}{2}\chi^2(\rho_c a^2)'\frac{v_c}{k}+H_T''$$

$$\Rightarrow \delta_c'' = -kv_c' + H_T'' \qquad \boxed{9}$$

$$② \Rightarrow \mathcal{H}\delta_c' = -k\mathcal{H}v_c + \mathcal{H}H_T' \qquad \boxed{10}$$

$$⑨+⑩ \Rightarrow \delta_c''+\mathcal{H}\delta_c' = H_T''+\mathcal{H}H_T'$$

Apply ⑧,

$$\delta_c''+\mathcal{H}\delta_c' = \frac{1}{2}\chi^2a^2\rho_c(\delta_c+3\frac{\mathcal{H}}{k}v_c)$$

Since $v_c \sim \frac{1}{k}$, drop $(\frac{\mathcal{H}}{k})^n$ term / subhorizon approx.

$$\delta_c''+\mathcal{H}\delta_c' - \frac{1}{2}\chi^2a^2\rho_c\delta_c = 0 \qquad \boxed{11}$$

We need to change from $\frac{d}{d\tau}$ to $\frac{d}{da}$. $\hat{\ } = \frac{d}{da}$ $\hat{\hat{\ }} = \frac{d^2}{da^2}$

$$\delta_c' = \frac{d}{d\tau}\delta_c = \frac{d}{da}\delta_c \cdot \frac{da}{d\tau} = a'\hat{\delta}_c = a\mathcal{H}\hat{\delta}_c$$

$$\delta_c'' = \frac{d}{d\tau}(a\mathcal{H}\hat{\delta}_c) = a\mathcal{H}(a\mathcal{H}\hat{\delta}_c)^{\wedge}$$

$$= (a\mathcal{H})^2\hat{\hat{\delta}}_c + a\mathcal{H}(\mathcal{H} + a\hat{\mathcal{H}})\hat{\delta}_c$$

① ⇒

$$(a\mathcal{H})^2\hat{\hat{\delta}}_c + a\mathcal{H}(\mathcal{H} + a\hat{\mathcal{H}})\hat{\delta}_c + \mathcal{H}a\mathcal{H}\hat{\delta}_c - \frac{1}{2}\chi^2 a^2 \rho_c \delta_c = 0$$

$$\Rightarrow \hat{\hat{\delta}}_c + \left(\frac{1}{a} + \frac{\hat{\mathcal{H}}}{\mathcal{H}}\right)\hat{\delta}_c + \frac{1}{a}\hat{\delta}_c - \frac{1}{2}\chi^2\frac{1}{\mathcal{H}^2}\rho_c\delta_c = 0$$

$$\Rightarrow \hat{\hat{\delta}}_c + \left(\frac{2}{a} + \frac{\hat{\mathcal{H}}}{\mathcal{H}}\right)\hat{\delta}_c - \frac{1}{2}\chi^2\frac{1}{\mathcal{H}^2}\rho_c\delta_c = 0$$

~~$\Rightarrow \hat{\hat{\delta}}_c + \left(\frac{2}{a} + \frac{\hat{\mathcal{H}}}{\mathcal{H}}\right)\hat{\delta}_c - \frac{1}{2}\frac{1}{\mathcal{H}^2}\cdot\left(\frac{\chi}{3}3\mathcal{H}^2\right.$~~

$$2\mathcal{H}' + \mathcal{H}^2 = -\chi^2 a^2 \rho_d w$$ ← (Einstein Eqn Space part)

$$\Rightarrow 2a\mathcal{H}\hat{\mathcal{H}} = -\mathcal{H}^2 - \chi^2 a^2 w \rho_d$$

$$\Rightarrow \frac{\hat{\mathcal{H}}}{\mathcal{H}} = \frac{-\mathcal{H}^2}{2a\mathcal{H}^2} - \frac{\chi^2 a^2 w\rho_d}{2a\mathcal{H}^2}$$

$\rho_{critical} = \rho_{cr} = \frac{3H_0^2}{\chi^2}$

$$\Rightarrow \frac{\hat{\mathcal{H}}}{\mathcal{H}} = -\frac{1}{2a} - \frac{w\rho_d}{2a\mathcal{H}^2}\cdot\frac{3H_0^2}{\rho_{cr}}a^2$$

$$\Rightarrow \frac{\hat{\mathcal{H}}}{\mathcal{H}} = -\frac{1}{2a} - \cancel{\frac{3}{2a}w\frac{\rho_d}{\rho_{cr}}a^2}\ \frac{3}{2a}\frac{\rho_d}{\rho_{cr}}\frac{wH_0^2}{\mathcal{H}^2}a^2$$

~~$\Rightarrow \hat{\hat{\delta}}_c + \left(\frac{2}{a} - \frac{1}{2a} - \frac{3}{2a}w\frac{\rho_d}{\rho_{cr}}\right)\hat{\delta}_c - \frac{1}{2}\frac{1}{\mathcal{H}^2}\frac{1}{a^2}\frac{3\mathcal{H}^2}{\rho_{cr}}\rho_c\delta_c = 0$~~

~~$\Rightarrow \hat{\hat{\delta}}_c + \frac{3}{2a}\left(1 - w\frac{\rho_d}{\rho_{cr}}\right)\hat{\delta}_c - \frac{3}{2}\frac{1}{a^2}\frac{\rho_c}{\rho_{cr}}\delta_c = 0$~~

$$\Rightarrow \hat{\hat{\delta}}_c + \left(\frac{2}{a} - \frac{1}{2a} - \frac{3}{2a}a^2\frac{wH_0^2}{\mathcal{H}^2}\frac{\rho_d}{\rho_{cr}}\right)\hat{\delta}_c - \frac{1}{2}\frac{1}{\mathcal{H}^2}\delta_c\rho_c\cdot\frac{3H_0^2}{\rho_{cr}} = 0$$

$$\Rightarrow \hat{\hat{\delta}}_c + \frac{3}{2a}\left(1 - a^2\frac{wH_0^2}{\mathcal{H}^2}\frac{\rho_d}{\rho_{cr}}\right)\hat{\delta}_c - \frac{3}{2}\frac{H_0^2}{\mathcal{H}^2}\frac{\rho_c}{\rho_{cr}}\delta_c = 0$$

For CPL models with $w = w_0 + w_a(1-a)$

$$\frac{\rho_d}{\rho_{cr}} = \Omega_{DE0} \cdot a^{-3(1+w_0+w_a)} e^{-3w_a(1-a)}$$

$$\frac{\rho_c}{\rho_{cr}} = \Omega_{m0}\, a^{-3}$$

Thus

$$\hat{\delta}_c + \frac{3}{2a}\left(1 - w \cdot \frac{a^2 H_0^2}{\mathcal{H}^2} \Omega_{DE0} \cdot a^{-3(1+w_0+w_a)} e^{-3w_a(1-a)}\right)\hat{\delta}_c$$

$$- \frac{3}{2}\frac{H_0^2}{\mathcal{H}^2}\Omega_{m0}\frac{1}{a^3}\delta_c = 0$$

$$\Rightarrow \hat{\hat{\delta}}_c + \frac{3}{2a}\left(1 - w\left(\frac{H_0}{H}\right)^2 \Omega_{DE0}\, a^{-3(1+w_0+w_a)} e^{-3w_a(1-a)}\right)\hat{\delta}_c$$

$$- \frac{3}{2a^2}\frac{H_0^2}{H^2}\Omega_{m0}\frac{1}{a^3}\delta_c = 0$$

Define $\tilde{H} = \frac{H}{H_0}$

$$\Rightarrow \hat{\hat{\delta}}_c + \frac{3}{2a}\left(1 - w\frac{1}{\tilde{H}^2}\Omega_{DE0}\, a^{-3(1+w_0+w_a)} e^{-3w_a(1-a)}\right)\hat{\delta}_c$$

$$- \frac{3}{2a^2}\frac{1}{\tilde{H}^2}\Omega_{m0}\frac{1}{a^3}\delta_c = 0$$

When CPL $\rightarrow$ LCDM model

$w = -1$, ~~w_0+w_a~~ $\Omega_{DE}(a) = \Omega_{DE0}$

$$\sim \hat{\hat{\delta}}_c + \frac{3}{2a}\left(1 + \frac{1}{\tilde{H}^2}\Omega_{DE0}\right)\hat{\delta}_c - \frac{3}{2a^2}\frac{1}{\tilde{H}^2}\Omega_{m0}\frac{1}{a^3}\delta_c = 0$$