

Power Spectrum & Its Evolution

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1 Harrison Zeldovich Prescription

All perturbations that come into the horizon have the same amplitude.

$$\Delta(k_i, t_i) = \Delta(k_e, t_e) = \text{Const.} \quad (1)$$

Attention $k = 2\pi/L$ is the comoving wavenumber.

1.1 What does that mean? & Evolution of This Prescription.

One kind of understanding of the assumption is

1. The perturbations are generated to be the same value (comoving value).
2. When the perturbations are outside of the horizon, their comoving measurement do not evolve.

However, this might be wrong (and it is wrong ¹). This only gives us a intuitive inspiration. The only thing we know is 1.

Am I wrong?

The assumption firstly used is $\delta_0 = \beta_E N^{-n}$. Or equivalently, $\delta_0 \propto L^{-3n}$. Since Harrison proved that there is a threshold epoch where the initial perturbations are located, all the perturbations generated at that epoch can be discribed as $\delta_0 \sim L^{-3n} \sim k^{3n}$ and $n = \frac{2}{3}$. i.e., primordial perturbations are $\delta_0 \sim L_0^{-2} \sim a(t_H)^2 / L_{H-physical}^2$. [t_0 denotes the time when the perturbation are generated. t_H is the time of horizon crossing, i.e., $2ct_H = \lambda_H$.]

During RD, $a(t)^2 = 2\sqrt{C}t$, then $\delta_H \sim a(t_H)^2 / t_H^2 \sim a(t_H)^2 / a(t_H)^4 \sim 1/t_H$. The amplitude of the perturbations are time dependent because the moment of horizon crossing t_H are different for different perturbations. So what are the mistakes here?

If $\delta_H \sim k_H^2$ remains the same for all perturbations and $P(k_H) \sim k_H$, from the defination of another measurement of perturbations

$$\Delta(k) = \frac{k^3}{2\pi^2} P(k), \quad (2)$$

we have $\Delta^2 = \delta_H^2$ when the perturbation comes into the horizon ($k = \lambda_H$), which indicates 1.

Generally speaking, one would choose the initial power spectrum to be power law form, i.e., $P(k) \propto k^n$, or even more simple, scale free Zeldovich spectrum, $P(k) \propto k$. ²

1.1.1 Primitive Spectrum

It is expected that Δ is scale free. The primitive spectrum given by inflation (or just a dimension analysis) is

$$P_\Phi(k) = \frac{50\pi^2}{9k^3} \left(\frac{k}{H_0} \right)^{\hat{n}-1} \delta_H^2 \left(\frac{\Omega_m}{D_1(a=1)} \right)^2 \quad (3)$$

When $\hat{n} = 1$, that is exact a scale invariant perturbation, i.e., $P(k) \sim k^{-3}$

THE PROBLEM IS, How do the orignal $\mu_0 \sim L_0^{-3n} \sim k_0^2$ evolve to $P(k_0) \sim k_0$?

$$P(k_0) \sim \mu_0^2 / k_0^3 \sim k_0. \quad (4)$$

¹Even the largest scale potential perturbations damp to 9/10 of their original value.

²http://fisica.usac.edu.gt/public/curccaf_proc/borganihtml/node5.html

2 Their Ideas

2.1 Some Notations

- 1. t_0 stands for current moment.
- 2. t_i is just the moment of horizon crossing.
- 3. In this section, k_e is the comoving wavenumber of the perturbation which cross the horizon at equality.
- Amplitude of perturbation

$$\delta^2(k, t) = \left(\frac{D_+(t)}{D_+(t_0)} \right)^2 \frac{k^3}{2\pi^2} P(k) \equiv \Delta(k, t) \quad (5)$$

where $P(k)$ is the initial power spectrum.

- Growth factor;

Defination

$$D_+(a) = \frac{5\Omega_{m0}}{2} H(a) H_0^2 \int_0^a \frac{1}{a'^3 H(a')^3} da' \quad (6)$$

Properties Since we are only going to use the ratio of growth factors, the properties of the ratio should be examined.

Before this, we have to calculate the Hubble function.

Hubble Function

$$H(a) = H_0 \sqrt{\Omega_{m0} a^{-3} + \Omega_{r0} a^{-4} + \Omega_{DE0} a^{-3(1+w)}} \quad (7)$$

$$D_+(t) = \begin{cases} D_+(t_i)(t/t_i)^{2/3} & t_e < t_i < t_{e2} \\ D_+(t_i) = \text{Const.} & t_i < t_e \end{cases} \quad (8)$$

- Crossing the particle horizon,

$$\lambda(t_i) a(t_i) = d_H(t_i) \quad (9)$$

Hubble distance is $d_H(t) \sim t$.³ During matter domination, we have $a(t) \sim t^{2/3}$. Thus,

$$\lambda = d_H(t_i)/a(t_i) \sim t^{1/3}. \quad (11)$$

Finally, since $\lambda(t) = 2\pi/k(t)$, we get

$$k_i^3 t_i = \text{Const.} = k_e^3 t_e. \quad (12)$$

³Hubble distance is

$$d_H(t) = c \frac{a}{\dot{a}} \begin{cases} c\sqrt{t}/\sqrt{t} & \text{radiation domination} \\ ct^{2/3}/t^{2/3} & \text{matter domination} \end{cases} \quad (10)$$

I have done the cal. under any w , but I do not remember the results.

2.2 Determine when the perturbations came into horizon.

Out of the prescription, we can determine when does the perturbations come into the horizon.

1. $t_i \gg t_e$, i.e., come into horizon at matter dominated era.

$$\Delta(k_e, t_0) = D_+^2(t_e, t_0)\Delta(k_e, t_e) = D_+^2(t_e, t_0)\Delta(k_i, t_i) = \frac{D_+^2(t_e, t_0)}{D_+^2(t_i, t_0)}\Delta(k_i, t_0) \quad (13)$$

$$= \left[\frac{a(t_0)/a(t_e)}{a(t_0)/a(t_i)} \right]^2 \Delta(k_i, t_0) = \left[\frac{a(t_i)}{a(t_e)} \right]^2 \Delta(k_i, t_0) = \left[\frac{t_i^{2/3}}{t_e^{2/3}} \right]^2 \Delta(k_i, t_0) \quad (14)$$

$$= \frac{k_e^4}{k_i^4} \Delta(k_i, t_0) \quad (15)$$

\Rightarrow

$$P(k_i, t_0) = (k_i/k_e)P(k_e, t_0) \quad (16)$$

2. $t_i \ll t_e$

$$P(k_i, t_0) = (k_i/k_e)^{-3}P(k_e, t_0) \quad (17)$$

2.3 Two different models diverge at later ages

Two models

1. Λ CDM
2. DE

Assumptions,

- DE do not cluster [?]
- During RD, perturbations evolve like a Λ CDM universe that has the same redshift of MR equality.
- model 1 and model 2 has the same initial perturbation amplitude.

Though two models has the same initial perturbations, the growth of them are different.

Since we have HZ, the power spectrum only diverge after equality, even more later when the transfer function becomes 1.

2.4 More

- [P] The HZP is not so accurate. It is calculated that the potential perturbation drops to about 9/10 (this may change in other models) of its value.
- [S] We do not need a too accurate calculation.
- [P] The factor Q is not only determined by the growth factor. Actually, this should be

$$Q^2(k_{in}, t_0, t_{in}^X, t_{in}^\Lambda) = \frac{D_X^2(t_{in}^X, t_0)}{D_\Lambda^2(t_{in}^\Lambda, t_0)} \cdot \frac{T_X^2(k_{in})}{T_\Lambda^2(k_{in})} = \frac{P_X(k_{in}, t_0)}{P_\Lambda(k_{in}, t_0)} \quad (18)$$

The subscript Λ mean these terms stand for Λ CDM model.

In their method, $P(k) = Ak^n T^2(k)$, is the k dependent part of the whole power sepctrum. So when calculating the fiducial Λ CDM model, two parts have to be calculated seperately, the whole power spectrum $P(k, t)$ and the growth factor $D_+(t, t_0)$ or identically, the k dependent part $P(k)$ and the growth factor.

Then for other models, we have only to calculate the growth factor to determine the growth for a particular mode and the transfer function for the variation of different modes, which employs the $P_X(k) = Q^2 P_{\Lambda CDM}(k)$ to generate the power spectrum for the new models.

[S] The two models have the same evolution at about radiation era and the early time of matter domination. Then the transfer function must be about the same.

To see this clearly, take a look at the Hubble function,

$$H^2 = H_0^2(\Omega_{M0}a^{-3} + \Omega_{R0}a^{-4} + \Omega_{DE0}a^{-3(1+w)} + \Omega_{K0}a^{-2}) \quad (19)$$

Since w is about the value of -1 and different DE models do not change the values of Ω_{M0} and Ω_{R0} very much, the Hubble function do not change very much from different models at about equality which is located at about $a \sim 10^{-3}$.

This is an example of the evolution of the background. Generally, the perturbation growth does not vary very much from different models.(?)⁴

- [P] One important thing before calculating the power spectrum, is to set the background of the two models to be the same.

There are several things to concern.

1. The background Hubble function which intends to pass the SN redshift exp. and samiliar receding exp.
2. The CMB temperature today. Since the total energy density is determined by the temperature if we assume the radiation today is a black body one, same temperature means the density of radiation is the same in the two models. From the first item, the Hubble function is the same. This doesn't mean the scale factor a is the same. But there is no way to write the density of the radiation to Hubble function rather than the scale factor (see Friedmann equation for an example). So this is really annoying.⁵

⁴Need proof. I just think, no proof referred.

⁵There is one special circumstance that the the scale factors and the first derivative of them are the same in two models. In this case, the radiation density at LSS should be the same since we in this special case the scale factors a are the same in the two models to insure they have the same background.

If we think about things happended before decoupling, the Hubble function is mainly dominated by radiation, or even just photons. Then we can set all thing to be the same during radiation era. This is roughly the assumption used by Ivan, Fernando and Diego (the evolution of the perturbations in X model is the same with a corresponding Λ CDM model). Growth factor is related to the ratio of matter energy density (might be related to the constituents of the universe) and the scale factor and the Hubble function. Then the growth of the perturbations diverges in different models because of the late effect of the different ratio of matter etc.

3 Even More

GOAL Our task is to check if the method can be applied to Λ CDM, sCDM, $f(R)$, $\omega(z)$, DGP, CPL and interacting model⁶, using the DE model as an fiducial model.

The ultimate purpose is to get the matter power spectrum today, or to find constrain on the parameters of different models or even to rule out models.

And other purposes?

3.1 FAQ

1. Why we use both growth factor and transfer function to discribe the evolution of the power sepctrum?

If we try to inspect the potentials that stays outside of the horizon, we can find that those potentials changes the amplitude by 10% during equality though they stay unchange both in radiation and matter epochs. This tells us that something special happened during equality. Thus we have to find a function to discribe this dropdown during equality, which is called transfer fuction.

(An alternate view is given below.) If the perturbations reenter at MD, the power spectrum $P(k, t) \propto C(t)k$. But it would be $P(k, t) \propto C(t)k^{-3}$ if the perturbations reenter at RD. These differences are caused by the different evolution of scale factor $a(t)$ during MD and RD. However, the universe dosen't transit from RD to MD suddenly. So if we want to write a unified equation of the power spectrum, it should be $P(k, t) \propto kT^2(k)D^2(t)$. The factor $T(k)$ (it is the transfer func. actually) should transit from k^{-3}/k in RD to k/k in MD gradually. We can see that the transfer function is of unit during MD.

More info. (including a Sokes-Navier view of the perturbation theory) is given in the *Cosmology Projects notebook*.

2. When did the equality happen?

This can be calculated with the Friedmann equations if only we use the total density. Anyway, the answer to this is

$$\tau_{eq} = \frac{2(\sqrt{2}-1)c}{H_0} \sqrt{\frac{a_{eq}}{\Omega_m}} \quad (20)$$

Conformal time is used here. **This function is useless because a_{eq} is still unknown.**

3. What about the transfer fuction?

There are many forms of transfer functions, for example, the BBKS (Bardeen, Bond, Kaiser, and Szalay, 1986) one reads

$$T^2(k) = \frac{\ln(1+2.34q)}{2.34q} \left[1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4 \right]^{-1/4} \quad (21)$$

in which, $q = k/\Gamma$ and $\Gamma = \Omega_M h$. There are also more accurate formulae. (Daniel J. Eisenstein and Wayne Hu, 1998, *Baryonic Features in The Matter Transfer Function*.)

How is it dealt with in their paper? Check the part about factor Q in 2.4.

4. Growth Factor? What is growth factor by Scott Dodelson? Does it change the spectrum of early time?

The growth factor is defined as the collective growth (k independent growth) of the perturbations in sub-horizon during MD (large a/a_{eq} limit or large a limit of extended Meszaros equation).

⁶This might be only possible for $Q = \text{simp}_{de}$

The equation for the perturbation of matter is ⁷

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m = 0 \quad (25)$$

It is a secondary ODE. So two linear indepent special solutions can be found. One mode stands for the growing of the perturbations and it is called the growth func., i.e., $\delta_m = C_1 D_1(t) + C_2 D_2(t)$ and $D_2(t) \rightarrow 0$ at $t \rightarrow \infty$.

Sometimes growth rate ($f(a) = d \ln G(a) / d \ln a$) is used (because it is convinient to be shown in a logarithmic figure, I think).

Since we would like to find the solution to δ_m to a , Scott rewrite the Meszaros equation

$$\frac{d^2 \delta}{da^2} + \left(\frac{d \ln H}{da} + \frac{3}{a} \right) \frac{d \delta}{da} - \frac{3\Omega_m H_0^2}{2a^5 H^2} \delta = 0. \quad (26)$$

We have a routine to solve such second order equations. (*Cosmology Projects* notebook)

The growing mode is

$$\delta \propto H(a) \int^a \frac{d\tilde{a}}{(\tilde{a}H(\tilde{a}))^3} \quad (27)$$

Equivalently,

$$D_+(a) = \frac{5\Omega_m}{2} \frac{H(a)}{H_0} \int_0^a \frac{d\tilde{a}}{(\tilde{a}H(\tilde{a})/H_0)^3}. \quad (28)$$

5. What is the Hubble function?

For dark energy model From the Friedmann equation,

$$\ddot{a}/a = -4/3\pi G(\rho + 3p) \quad (29)$$

$$\dot{\rho} = -3\dot{a}/a(\rho + p) \quad (30)$$

for any state equation $w = p/\rho$ (in which w is time independent), we have

$$\rho \cdot a^{3(1+w)} = \text{Const} = \rho_0 a_0^{3(1+w)} = C_1. \quad (31)$$

(Though it is easy to move on, we need nothing more.) ⁸

⁷A different method from Scott and Ivan etc given by George at CalTec is to use the Stoke-Navier equaiton for continuous matter. Define δ_m as the perturbation of ρ_m and drop the second and higher orders,

$$\frac{\partial \delta_m}{\partial t} = -a^{-1} \nabla \cdot \vec{v}_m \quad (22)$$

$$\frac{\partial \vec{v}}{\partial t} = -\frac{\nabla \Phi}{a} - H \vec{v}_m \quad (23)$$

$$a^{-2} \nabla^2 \Phi = 4\pi G \bar{\rho}_m \delta_m. \quad (24)$$

Meanings of these eqns. are given in *Cosmology Projects* notebook.

⁸Deform the second derivaive of a ,

$$\ddot{a} = -\frac{4}{3}\pi G \rho a(1+w) \quad (32)$$

$$= -\frac{4}{3}\pi G \rho a^{3(1+w)} \frac{a}{a^{3(1+w)}} (1+3w) \quad (33)$$

$$= -\frac{4}{3}\pi G (1+3w) C_1 a^{1-3(1+w)} \quad (34)$$

$$= C_2 a^{1-3(1+w)} \quad (35)$$

To solve that equation, we have to denote $y = \dot{a}$, then the second derivative of scale factor a goes $\ddot{a} = y \frac{dy}{da}$, the former equation becomes

$$y \frac{dy}{da} = C_2 a^{1-3(1+w)} \quad (36)$$

$$\rightarrow d\left(\frac{1}{2}y^2\right) = C_2 \frac{1}{2-3(1+w)} da^{2-3(1+w)} \quad (37)$$

$$\rightarrow \frac{1}{2}y^2 = C_3 a^{2-3(1+w)} + C_4 \quad (38)$$

$$\rightarrow y = \sqrt{2C_3 a^{2-3(1+w)} + 2C_4} \quad (39)$$

The hardest task is to determine how does scale factor a evolve, i.e., solving the equation of y . Actually this problem is a sole task of finding the initial states.

Then Hubble function is

$$H^2 = H_0^2[\Omega_{M0}(1+z)^3 + \Omega_{R0}(1+z)^4 + \Omega_{K0}(1+z)^2 + \Omega_{DE0}(1+z)^{3(1+w)}], \quad (40)$$

in which, $\Omega_{x0} = 8\pi G\rho_{x0}/(3H_0^2)$ and 0 subscript stands for the present value.

(For Λ CDM, substitute w with -1.)

Consequently, we can write down the growth factor analytically,⁹

$$D_+(a) = \frac{5}{2}\Omega_m\mathcal{H}(a)\int_0^a \frac{1}{(\tilde{a}\mathcal{H}(\tilde{a}))^3}d\tilde{a}, \quad (41)$$

in which, $\mathcal{H}(a)^2 = \Omega_{M0}a^{-3} + \Omega_{R0}a^{-4} + \Omega_{K0}a^{-2} + \Omega_{DE0}a^{-3(1+w)}$.

6. Why is there a σ_8 ?

Observations on present abundance of rich clusters of galaxies can only give constraints on $\sigma_8\Omega$.¹⁰ σ_8 stands for the normalization of the power spectrum on $8h^{-1}\text{Mpc}$ scale.

3.2 Models

3.2.1 sCDM Model

In this model, our universe is dominated by matter (including baryons and dark matter) today¹¹. Typically, when the time is the power spectrum is given by

$$P(k, a) = 2\pi^2\delta_H^2 \frac{k^n}{H_0^{n+3}} T^2(k) \left(\frac{D_1(a)}{D_1(a=1)} \right)^2 \quad (42)$$

- δ_H is the primordial horizon scale perturbation and we can set this to be the same in different models.
- H_0 is the Hubble constant today.
- We can just set $n = 1$ for Zeldovich spectrum since we are not requiring a very accurate calculation.

If we want to calculate the accurate power of the matter perturbations, we have to get T and D_1 .

However, we will just calculate the growth factor now. In **flat sCDM model, i.e., EdS model**, $\Omega_M(a) = \Omega_{M0} = 1$, $\Omega_{K0} = \Omega_{R0} = \Omega_{DE0} = 0$ ($\mathcal{H} = a^{-3}$ by applying 40). Then the growth factor is

$$D_+(a) = \frac{5}{2} \cdot 1 \cdot a^{-3/2} \cdot \int_0^a \tilde{a}^{-3/2} d\tilde{a} \quad (43)$$

$$= a \quad (44)$$

3.2.2 Λ CDM

Assume we have $\Omega_{M0} = n$, then $\Omega_{\Lambda0} = 1 - n$ since other components disappear far after equality in a flat Λ CDM model.

Take $\Omega_{\Lambda0} = 0.7$ as an example.¹²

Two examples are calculated to show the difference.¹³¹⁴

a. Red line is the growth factor of sCDM model while blue line is that of Λ CDM model.

⁹There are things to be clarified. The first thing is that this is scaled based on $D_+ = a$ during matter domination. The second one is a little complicated. This growth func. is derived with a $a \gg a_{eq}$. But here the integration starts from 0. The actual integration should be divided into several parts, i.e., RD, MD etc. Here it is OK to integrate from 0 because we have set the evolution in RD to be the same and lasts much shorter than MD and the perturbation changes little (compared to the changes in MD) during RD, if the perturbations come into horizon during RD. But for those come into horizon during deep MD, the integration should start from the moment they enter. **Check what have been done in their paper. Also, it is important to check how the approximation for those entered in RD changes the final result.**

¹⁰There are ways to break the degeneracy. Read astro-ph/9706018 for an example.

¹¹At late time of sCDM model, it becomes an Einstein-de Sitter model. This part (EdS) describes the real universe from $z \sim 1000$ to $z \sim 1$ very well.

¹²Check the mathematica file to find out more.

¹³They are normalised to the same value at $a = 0.1$.

¹⁴**These figures for growth factors are useless for our goal. I put them here to show they can be calculated. However, these calculations are just toy models because I used a lot approximation here. Formal calculations can be done by using NIntegrate in mathematica.**

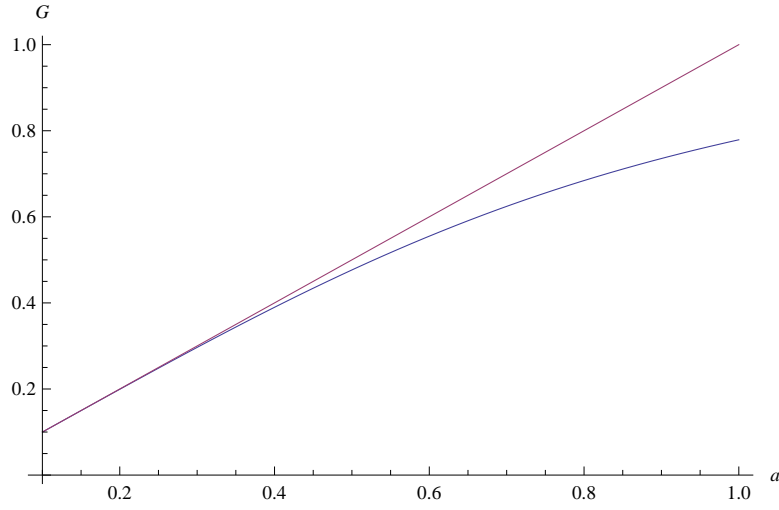


Figure 1: Growth factors of sCDM and Λ CDM

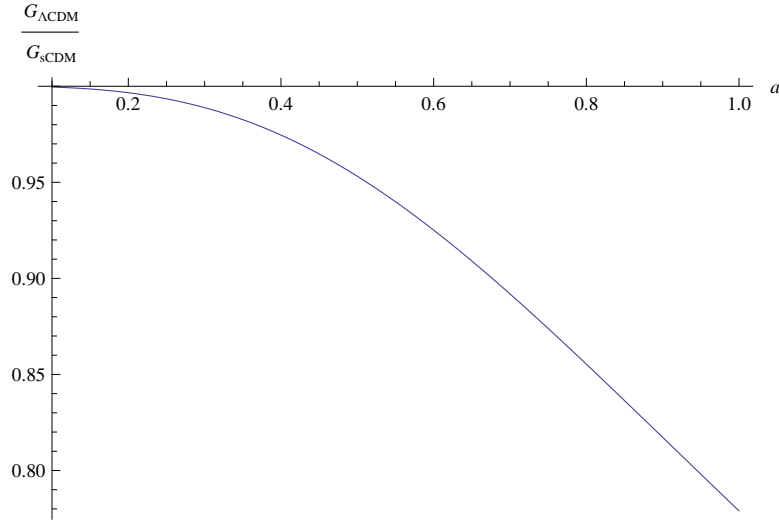


Figure 2: Ratio of the growth factors. (Λ CDM to sCDM.)

3.2.3 DE

Similar calculations generate the growth factor for dark energy model (figure 3) and the difference between sCDM and dark energy model (figure 4).¹⁵

3.2.4 f(R)

3.2.5 DGP

3.2.6 CPL

3.2.7 Interacting?

3.3 Power Spectrum

Calculate the power spectrum of a fiducial model using CMBEASY. I did a little calculation, but it seemed the data never went right. I will check this later.

¹⁵These are calculated using Mathematica. The mathematica notebook file is located in the same folder of the L^AT_EX file. Brief notations are given in that file.

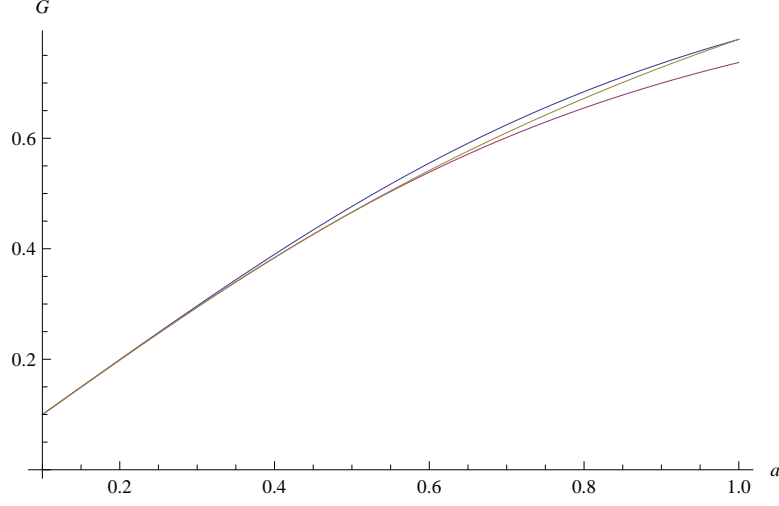


Figure 3: Growth factors of dark energy models with different EoS. The blue line, red line and green line stand for $w = -0.9$, $w = -1.0$ (it reduces to Λ CDM model) and $w = -1.1$ respectively.

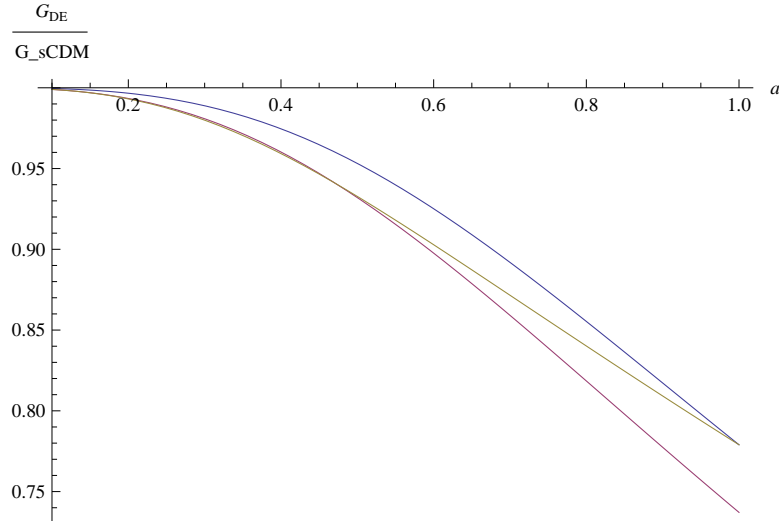


Figure 4: The ratio of growth factors of dark energy model to that of sCDM model. The blue line, red line and green line stand for $w = -0.9$, $w = -1.0$ (it reduces to Λ CDM model) and $w = -1.1$ respectively.

4 Work

4.1 Repeat Fernando's work

4.1.1 Preparing

To get the figures in their paper, a figure that describes the relation between growth and the wavenumber k (Growth VS k) should be calculated.

- Any growth factor can be calculated using the method in previous section.
- The next work is to calculate the Hubble distance (because this determines when do the perturbations come into the horizon) of scale factor.

Hubble distance is ¹⁶

$$d_H = \int_0^a \frac{da'}{a'^2 H(a')} \quad (45)$$

Then we can use the current $d_H(a(t_0))$ and MD approximation of $H(a(t))$ (since we will only cal. the MD evolution of the perturbations.) ¹⁷

$H(a(t))$ can be simplified into

$$H(a) = H_0(\Omega_{M0}a^{-3} + \Omega_{DE0}a^{-3(1+w)}) \quad (46)$$

according the fact that current observation shows the universe is filled with 73% dark energy and 27% matter (mostly dark matter).

4.1.2 Λ CDM

Parameters

$$w = -1; \Omega_{DE0} = 0.734; \Omega_{k0} = 0; \Omega_{m0} = 0.1334 / (0.71^2); \Omega_{r0} = 8.09 * 10^{-5};$$

Hubble distance:

growth factor:

Growth vs k

4.1.3 LCDM and Dark Energy

Parameters are listed below.

$$\Omega_{DE0} = 0.734; \Omega_{k0} = 0; \Omega_{m0} = 0.1334 / (0.71^2); \Omega_{r0} = 8.09 * 10^{-5}; \quad (47)$$

$$\Omega_{m0,s} = 1; \Omega_{r0,s} = 8.09 * 10^{-5}; \quad (48)$$

$$h = 0.71; H_0 = \frac{100h}{300000}; \quad (49)$$

Each color in the figures represents a model.

¹⁶For more details check the *Cosmology Projects* notebook

¹⁷There are many things to be considered.

- Different models might give different evolution of Hubble distance. And what is knotty is that the models might give different current value of Hubble distance when given a suitable set of parameters (because one might focus on other things to fit the parameters). (???)

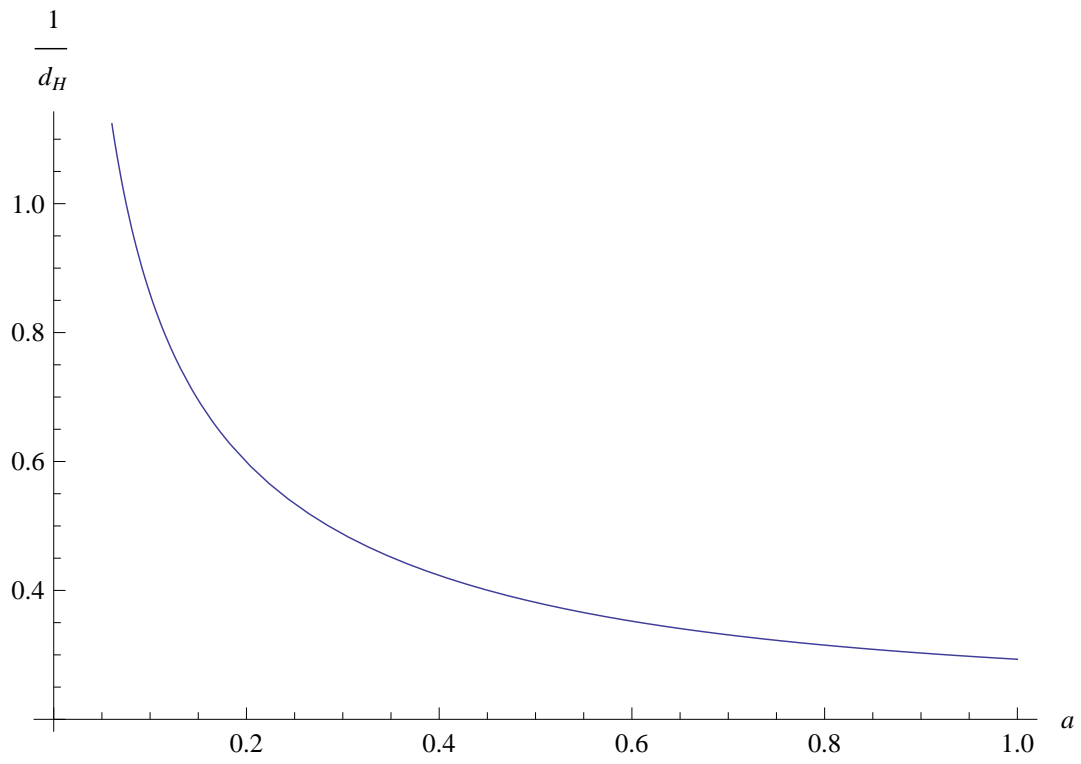


Figure 5: $\frac{1}{d_H}$ vs a

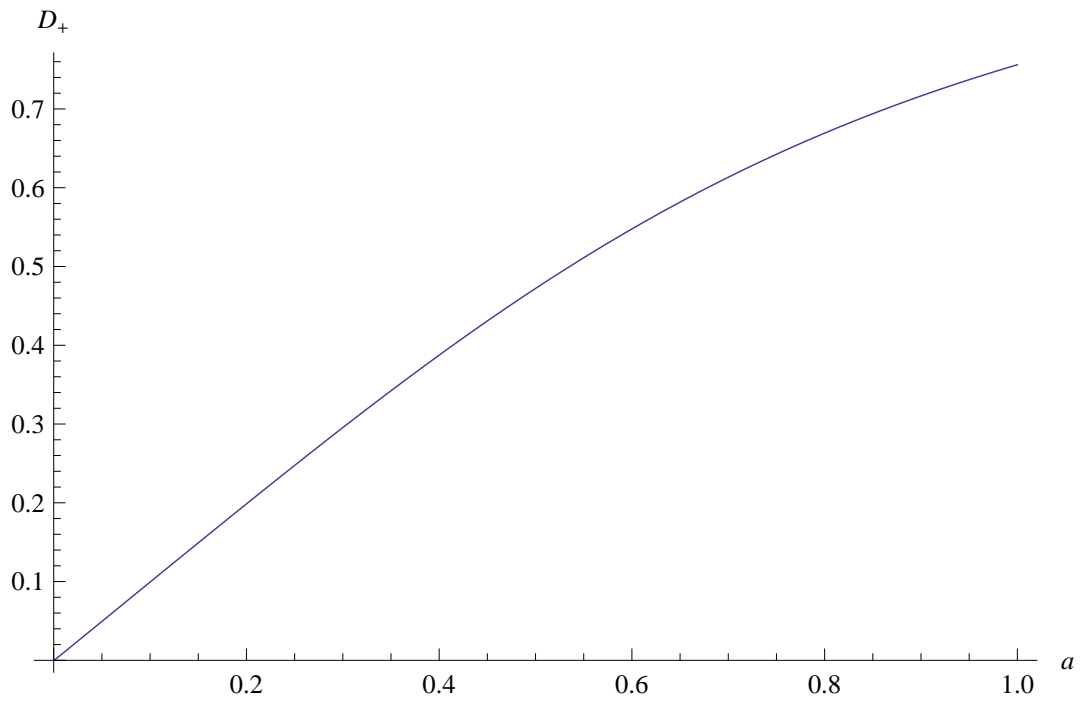


Figure 6: growth factor

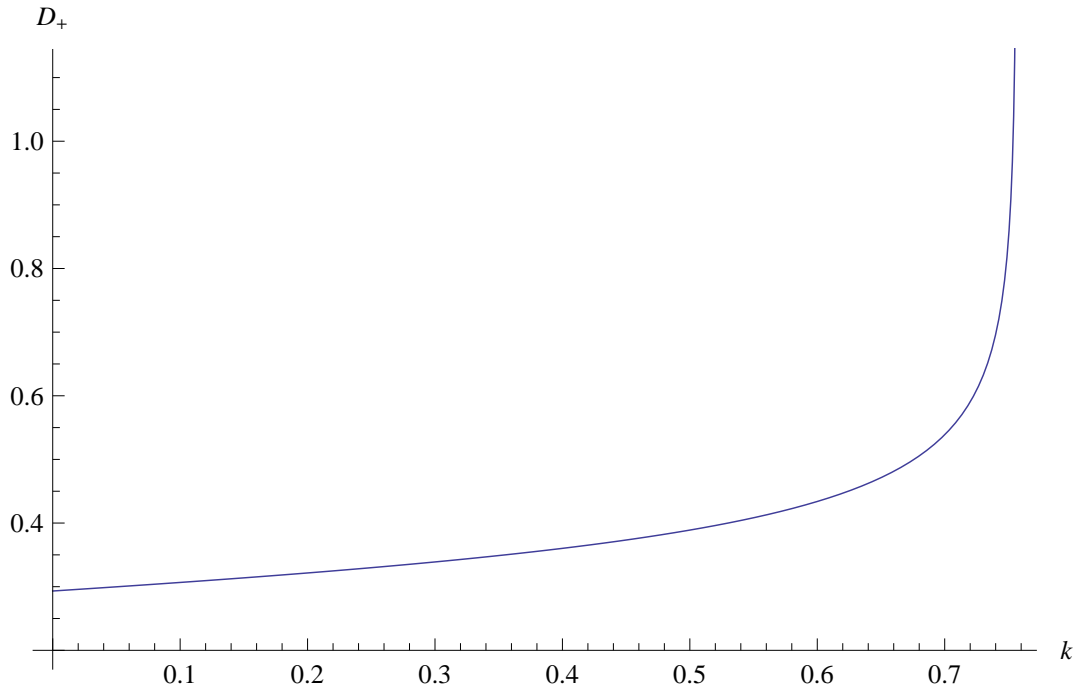


Figure 7: Growth

Color	Model
Red	sCDM
Orange	ΛCDM
Yellow	$w = -0.25$
Green	$w = -0.5$
Blue	$w = -0.75$

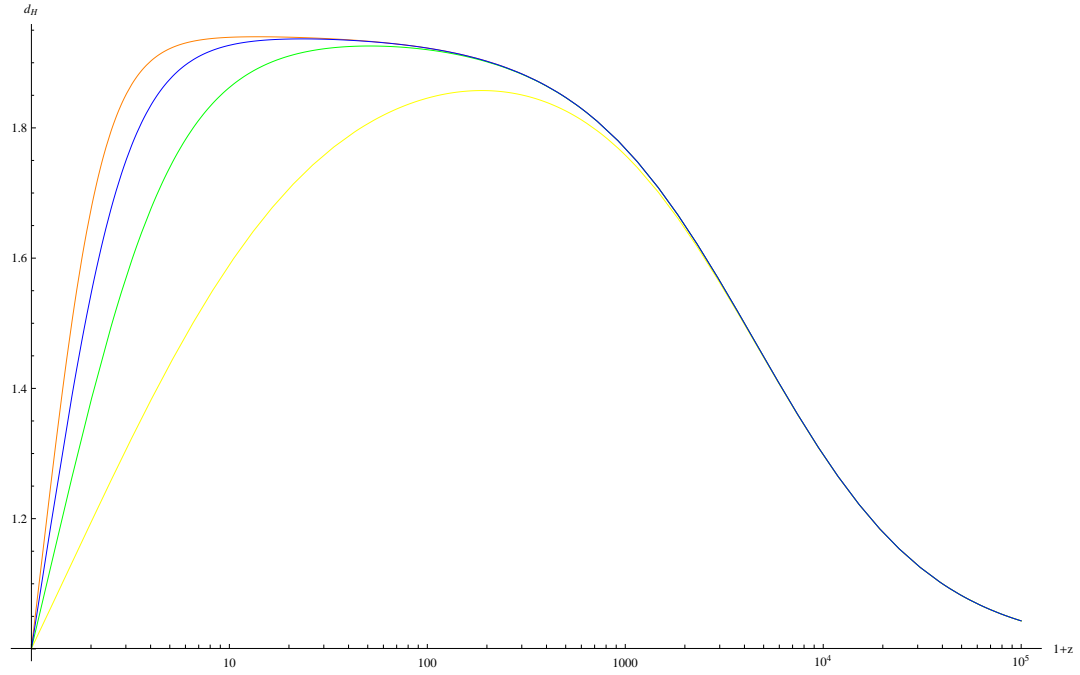


Figure 8: Hubble distances

Figure 8 shows the differences of the evolution of the Hubble distance. All the data are normalised with the

inverse of sCDM's Hubble distance. The shape of the lines can be explained by the fact that DE or Lambda only changes the background of the universe at late times after RD. The reason for the dropping down of the lines is that the Hubble functions should have the same value today ($1 + z = 1$). That is also the reason for the fact that they cross the same point at $1 + z = 1$. (Values of Hubble equations should converge at late times. So the part with $1 + z < 1$ is useless.)

Through figure 8 the three DE models fall between LCDM and sCDM which should be a straight line of value 1. Since the EoS of three DE models are exactly between 0 and -1, this result is quite reasonable. This figure also shows that the DE model with $w = -0.25$ obviously deviates from LCDM at an early age of $z \sim 1000$, while other models deviate after about $z \sim 50$.

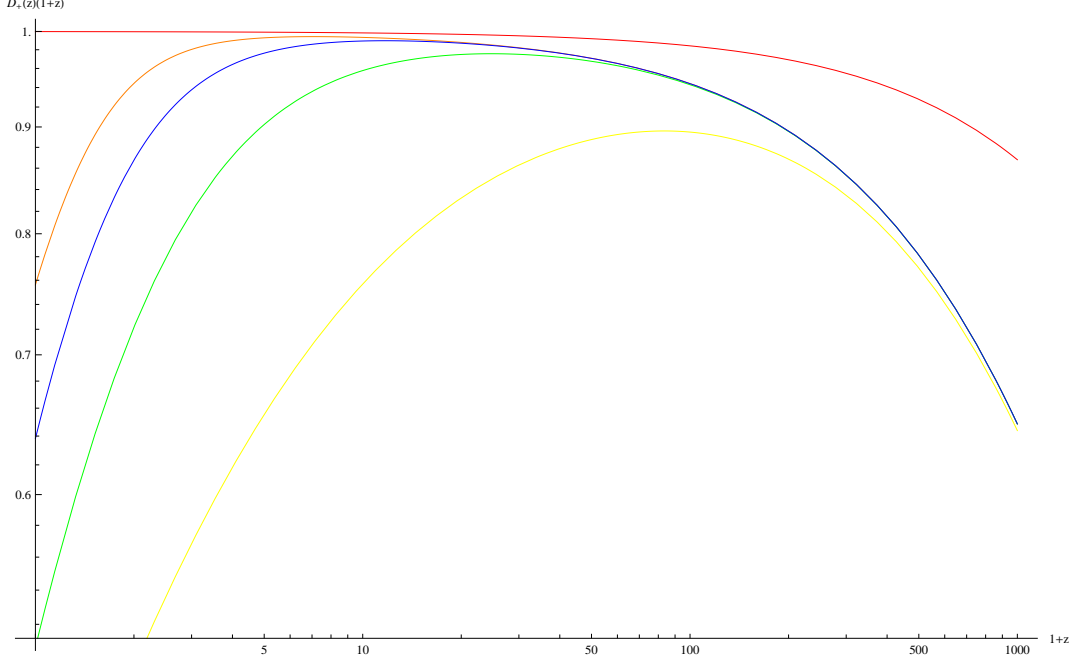


Figure 9: Growth factors vs $1 + z$ of DE and Λ CDM

Figure 9 are the growth factors of the models. The going down lines are due to the late age effect of dark energy which suppresses the evolution of perturbations.

[Why does the yellow line ($w = -0.25$) behave so strangely? Though we only use the part with $1 + z$ larger than 1, it is hard to imagine it crossing LCDM (while other lines crossing nothing).]

Figure 11 shows the power spectrums, which are generated by the standard LCDM model. The figure gives the lines the right trend when change the EoS, just as right as the Hubble distance.

One thing that is really annoying is that as the EoS becomes more and more close to LCDM, it is harder and harder to distinguish the DE models with LCDM models. The blue line is a good example for this statement.

These results supports the point that this method works well for these time independent EoS DE models and it can be show the differences of the models in between the time interval we are working on.

4.2 CPL

4.2.1 What Is CPL Parameterisation?

Notes in *Cosmology Project Notebook*.

The EoS here is $w = w_0 + w_a(1 - a)$.

4.2.2 Generating Power Sepctrum

The figures in this subsection follows the following rules unless exceptions are stated:

- Red for sCDM;orange for LCDM;yellow for

Parameters table (other parameters are exactly the same with the previous calculation):

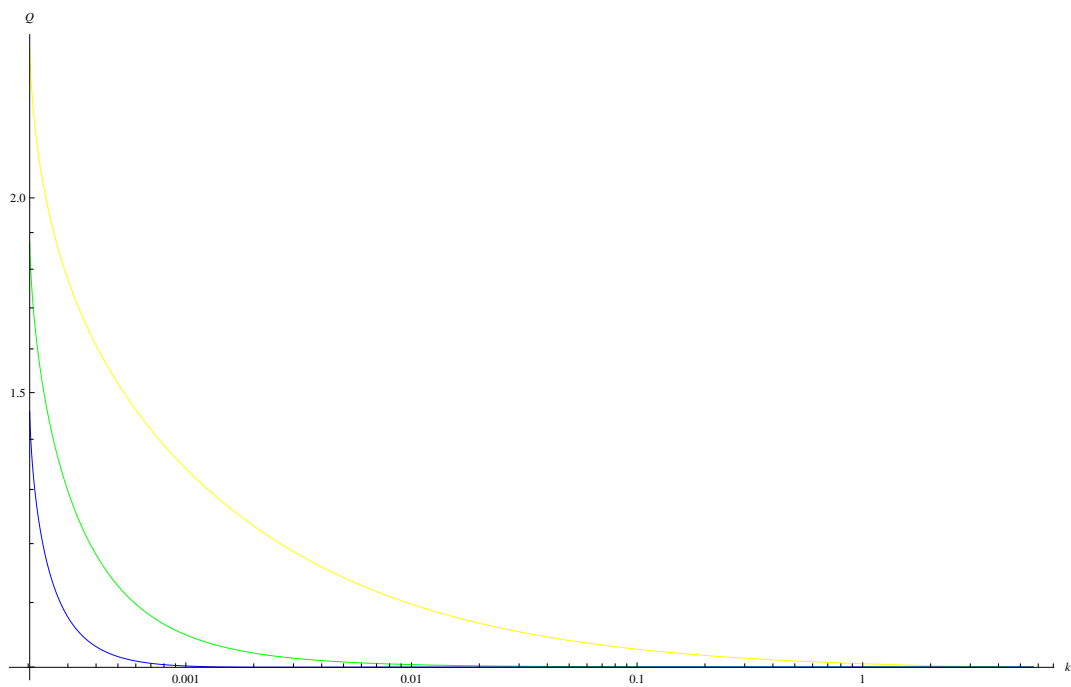


Figure 10: Q factors

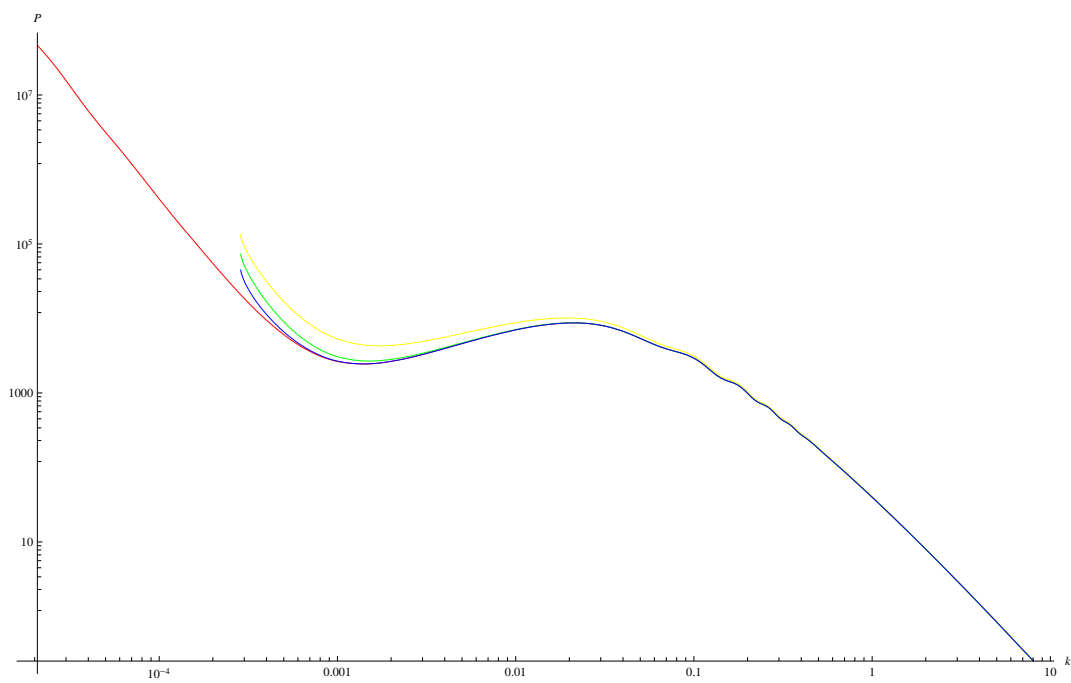


Figure 11: Power spectrum of LCDM and several DE models

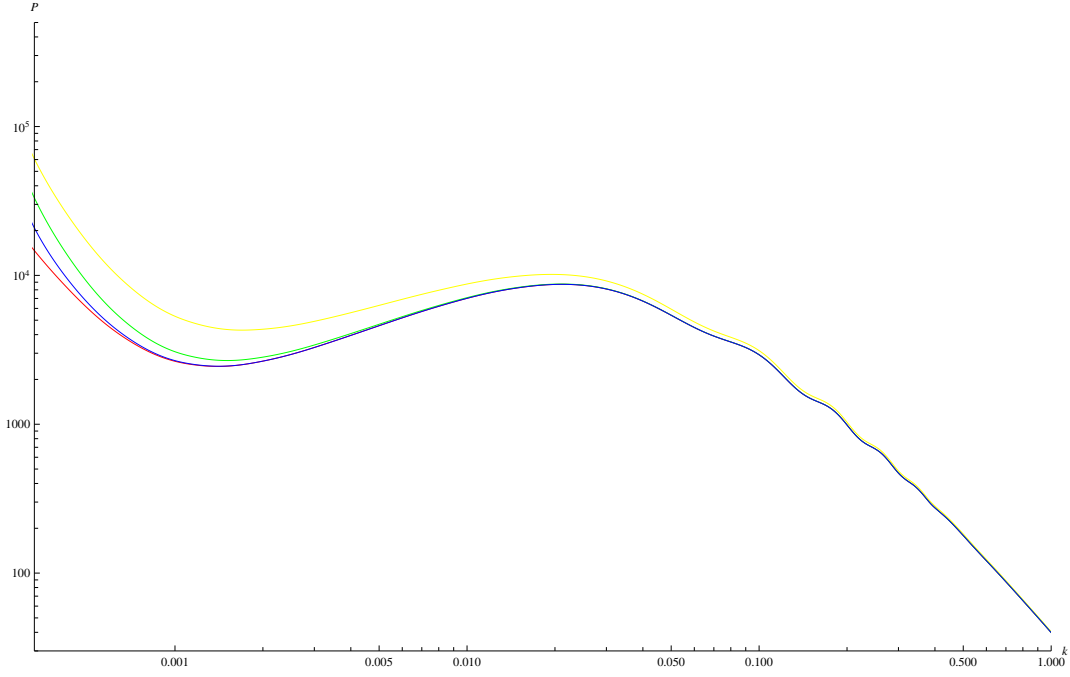


Figure 12: Power spectrum (within a range) of LCDM and several DE models

Color	Model
Red	sCDM
Orange	ΛCDM
Yellow	$w_a = -0.20$
Green	$w_a = -0.30$
Blue	$w_a = -0.32$
Cyan	$w_a = -0.34$
Purple	$w_a = -0.44$

In these CPL models, the parameters make sure that $w_0 + w_a = -1$ in order to generate a background analogous to ΛCDM.

It should be made clear that the range of validity is around $10^{-4} < k < 10$ or equivalently a range of $10^{-3} < a < 1$.

Figure 13 shows the Hubble distances. The $1 + z < 1$ part is also useless so I cut them off. The smaller the EoS is, the larger the Hubble distance is during late ages. The parameters chosen here are all fits to ΛCDM well. So the difference between them is small. However the differences are large enough to be noticed.

Figure 14 has peaks. These peaks come from the $1 + z < 1$ part of the growth factor. These should be cut off later.

The figures show the smaller the parameters w_a , the larger deviation from ΛCDM at late times.

Figure 18 shows that the power spectrum can hardly be recognised until today. (I am making too small changes in the parameters. More references needed here in order to make this clear.)

4.3 First Supplements

1. Figure 33 corresponds to Figure 1 a in Fernando's. I think $1 + z < 1$ is of nonsense because redshift z should be larger than 1 if we only care about the present and the past. So I only plotted the $1 + z > 1$. Here my figure is different from Fernando's at about $1 + z < 10$. All the lines converge at $z = 0$ in my plot because all the Hubble functions become the Hubble constant of today at that redshift. I have no idea why Fernando's plot does not. (And I have no idea why he plotted $z < 0$).
2. As said in the previous item, redshift less than zero is not so useful.
3. I have already checked the effect of Ω_{m0} and Ω_{de0} . The figure does not change very much.

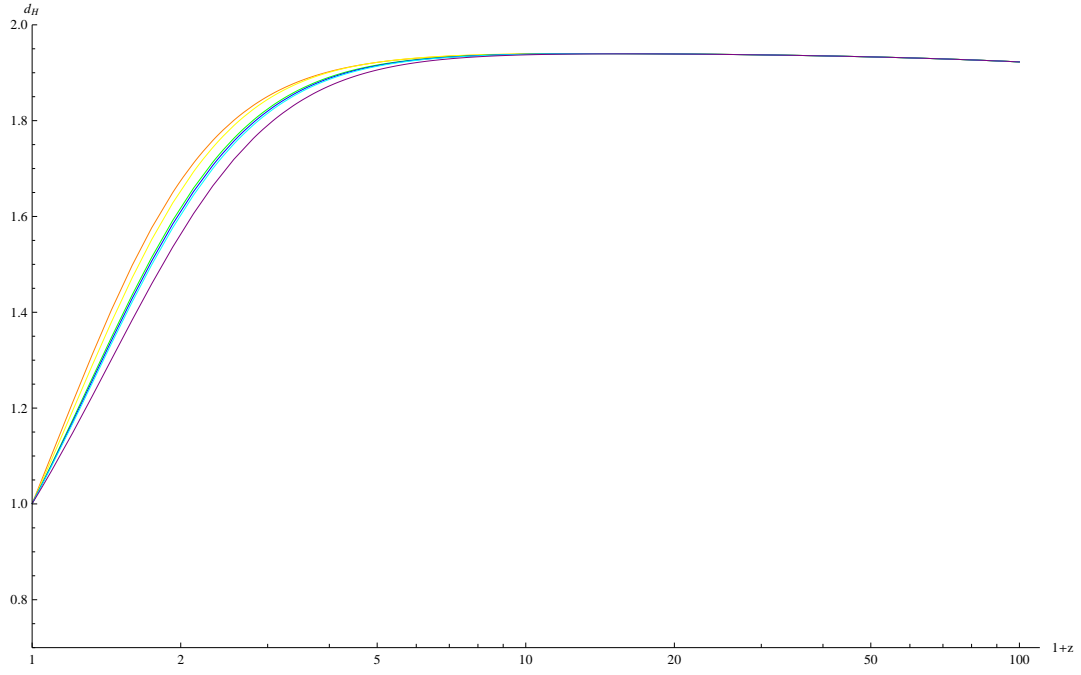


Figure 13: The Hubble distance of different models including Λ CDM, Λ CDM and five other CPL parameterised dark energy model

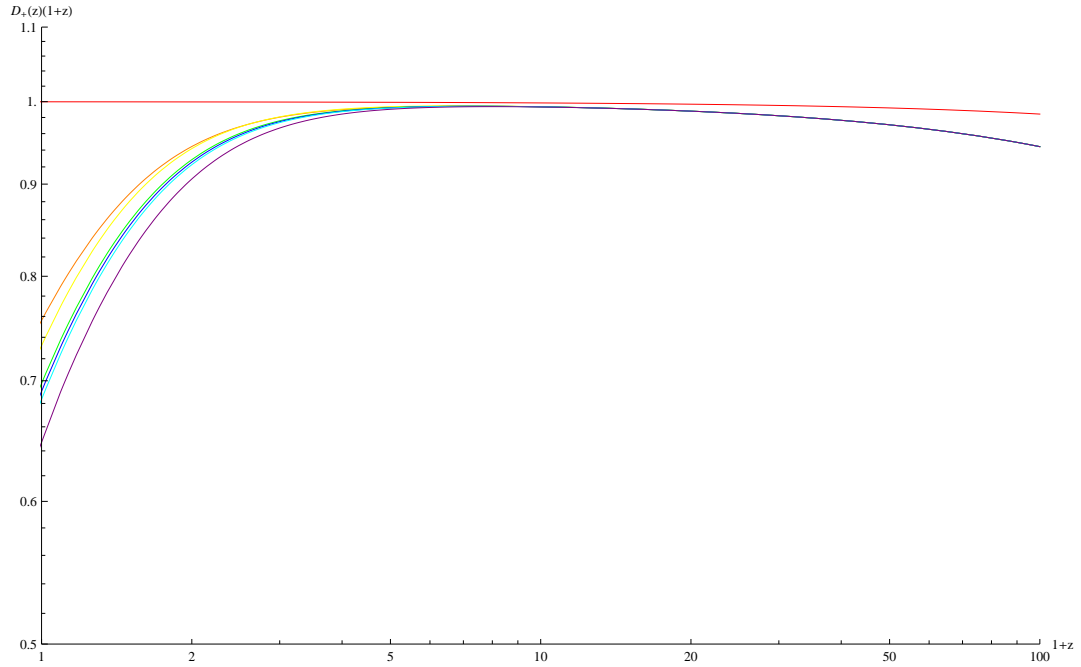


Figure 14: The growth factors of different models including Λ CDM, Λ CDM and five other CPL parameterised dark energy model

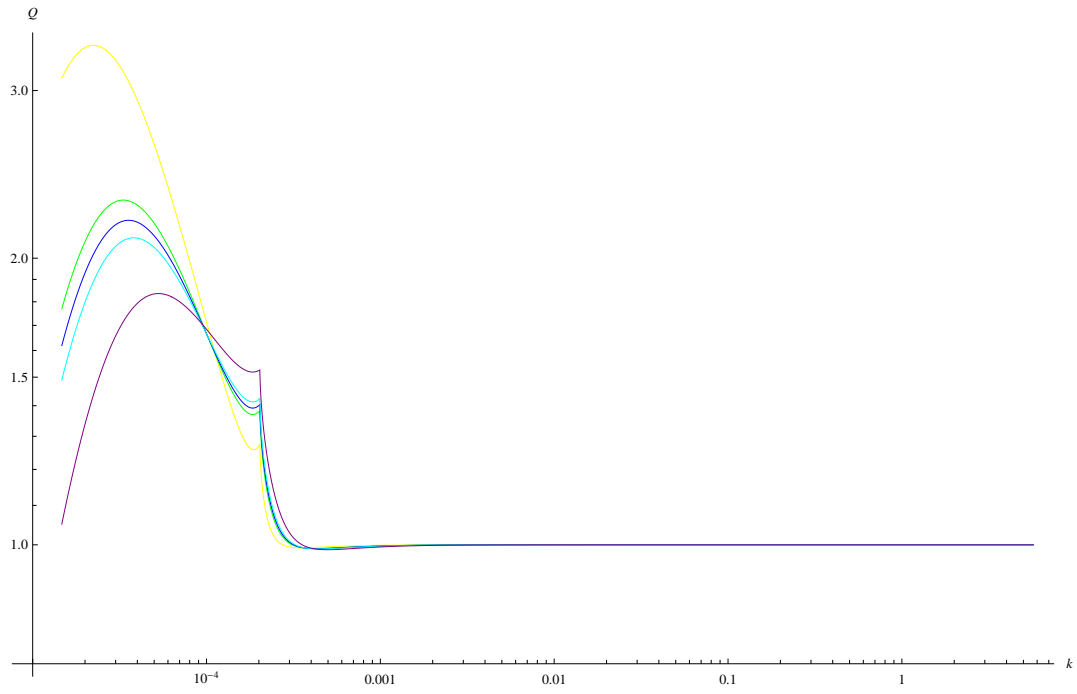


Figure 15: The Q factors of different models including Λ CDM, LCDM and five other CPL parameterised dark energy model

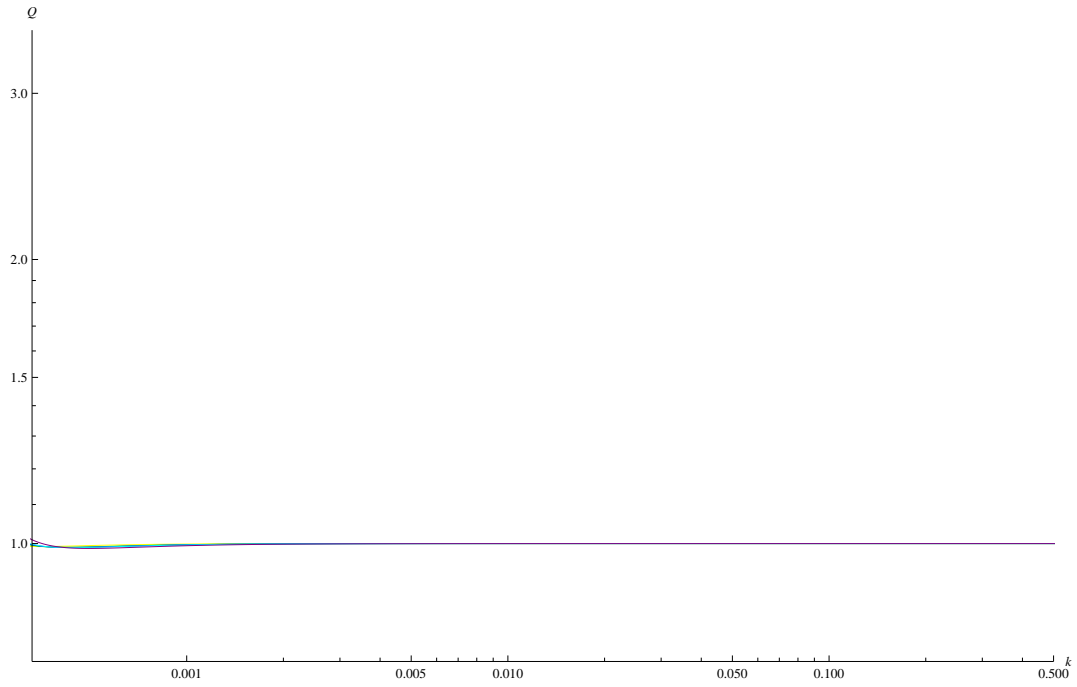


Figure 16: The Q factors (within a range) of different models including Λ CDM, LCDM and five other CPL parameterised dark energy model

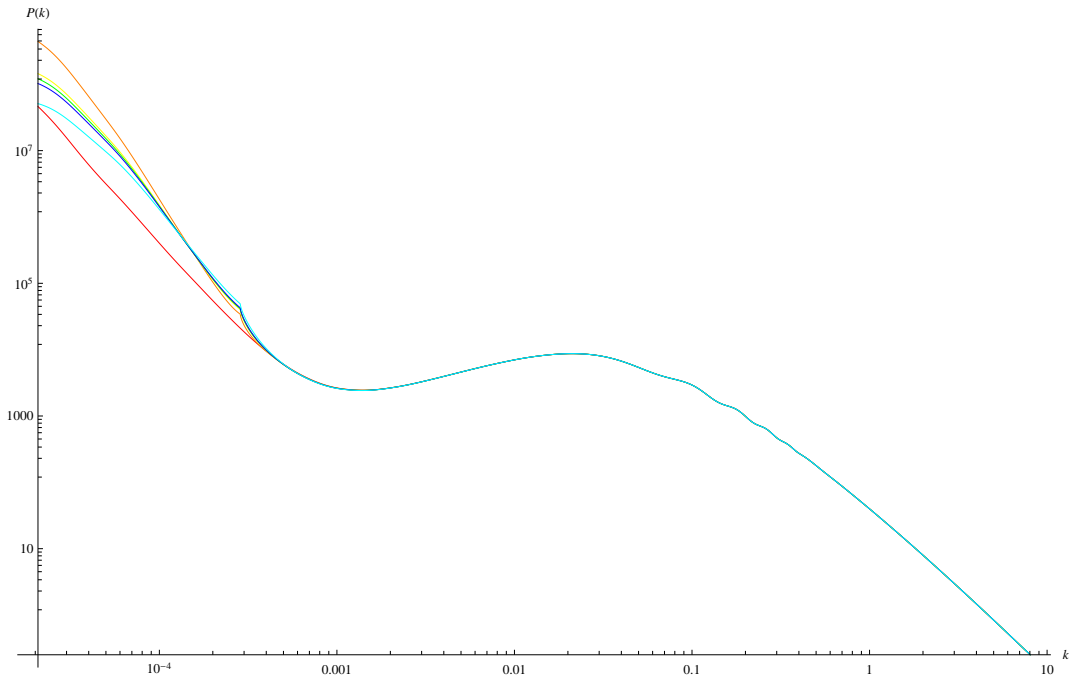


Figure 17: The power spectrums of different models including Λ CDM, LCDM and five other CPL parameterised dark energy model

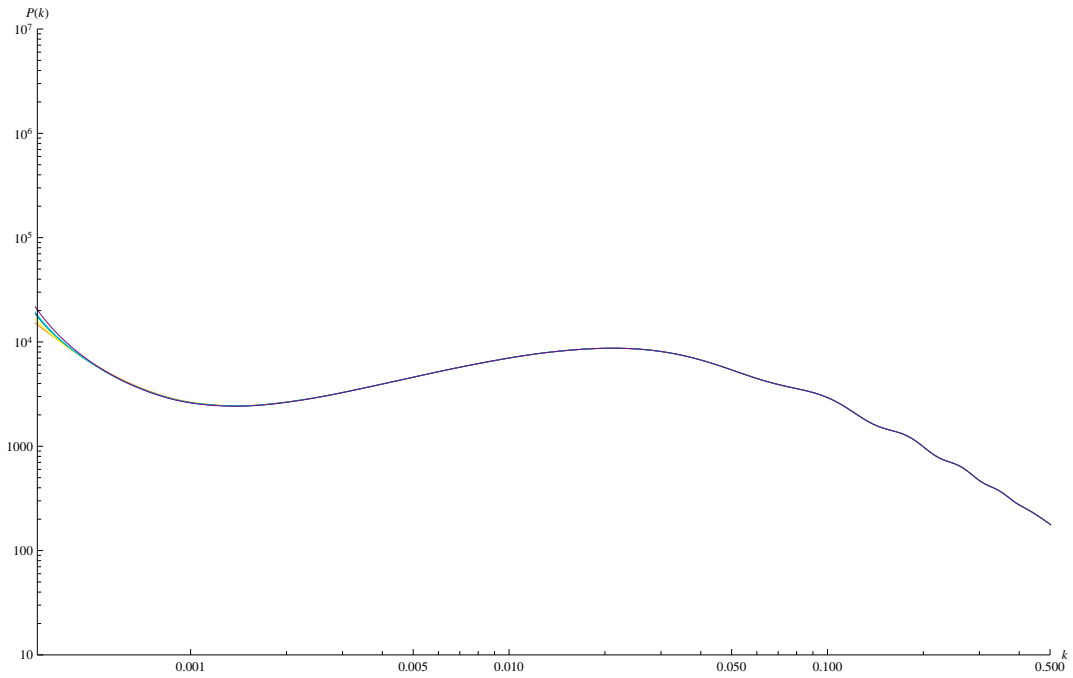


Figure 18: The power spectrums (within a range) of different models including Λ CDM, LCDM and five other CPL parameterised dark energy model

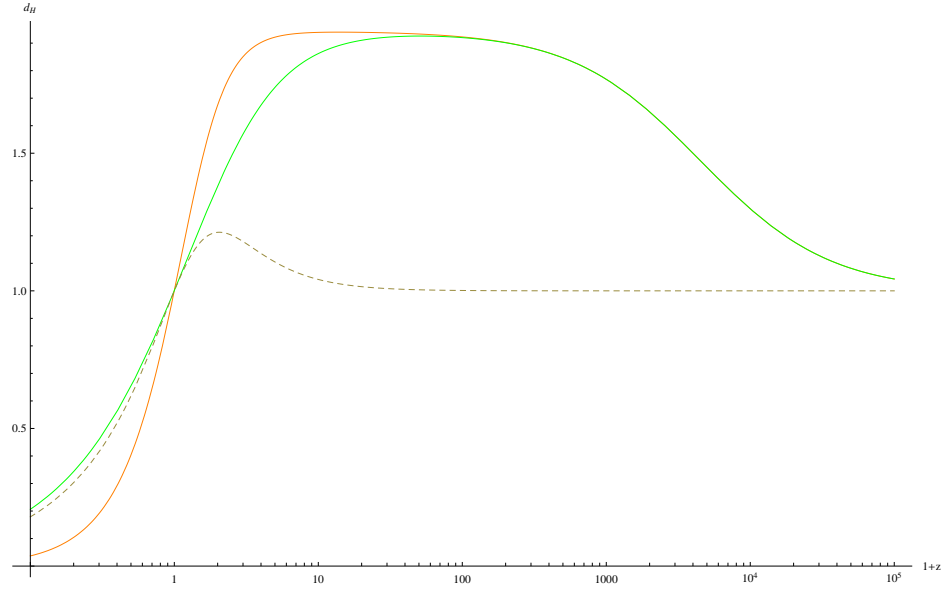


Figure 19: Hubble distance. Dashed line is the Hubble distance ratio of DE (with $w = -0.5$) and LCDM.

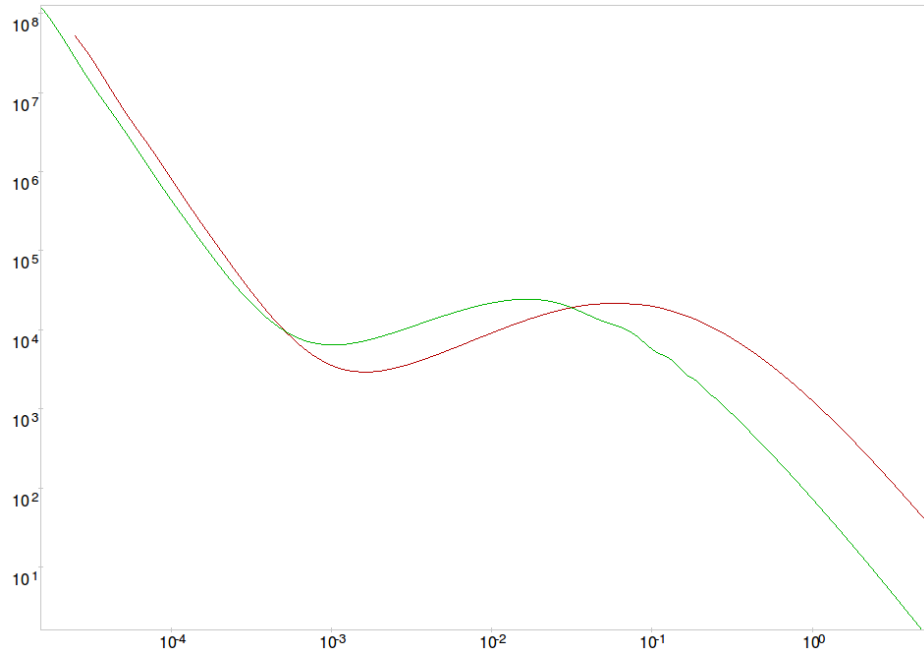


Figure 20: Power spectrum of dark energy model with $\Omega_{DE0} = 0$ and $\Omega_{DE0} = 0.7$. The red line is the Λ CDM model.

That's why I did not plot figure varying in Ω_{DE0} .

4. I think there is no need to plot figures on different DE and DM abundances. Is there anything to be expected from plotting these figures?
Small k corresponds to negative redshift.

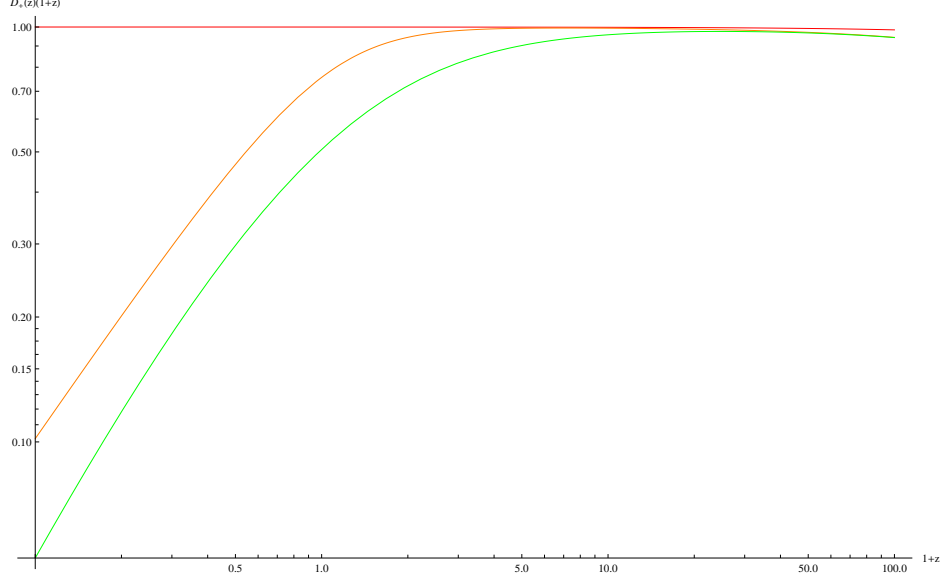


Figure 21: Growth factors are given in this figure. Dashed line is the Hubble distance ratio of DE (with $w = -0.5$) and LCDM.

5. The growth factors in figure 21 is plotted within $1+z \sim [0.1, 1000]$. As argued in 1, the data is useless when $z < 0$ unless one is going to check the future evolution of the universe. **That is why I give two figures for one quantity sometimes: one for a larger range and one for a suitable range.**
6. The CPL EoS used here is $w = w_0 + w_a(1 - a)$. The parameters chosen are listed in the table below.

Color	Model
Red	sCDM
Orange	LCDM
Yellow	$w_0 = -1$ & $w_a = 0.1$
Green	$w_0 = -1$ & $w_a = -0.1$
Blue	$w_0 = -0.9$ & $w_a = -0.1$
Cyan	$w_0 = -0.9$ & $w_a = 0.1$
Purple	$w_0 = -0.9$ & $w_a = -0.2$

The parameters chosen only have effects on small redshift. This is because the parameters all satisfy the condition that $w_0 \rightarrow -1$ and $w_a \rightarrow 0$.

By comparing yellow line ($w_0 = -1, w_a = 0.1$) and green line ($w_0 = -1, w_a = -0.1$), we can see positive w_a leads to a larger than LCDM Hubble distance when $w_0 = -1$. This is in accord with the series form of Hubble equation 85 since $w_0 = -1$ eliminates the first order term and the w_a determines whether the expression is larger than one. Other comparisons like green line and blue line, blue line and purple line in Figure 22 23 24 can also be interpreted by equation 85.

Growth factors are also following the expansion. Cyan line ($w_0 = -0.9, w_a = 0.1$) has smaller Hubble distances than others at late time, so the growth drops since growth factor is positively correlated to the size of the Hubble distance. The changes in growth factors when changing the parameters are exactly positively correlated to the size of Hubble distance changes. For

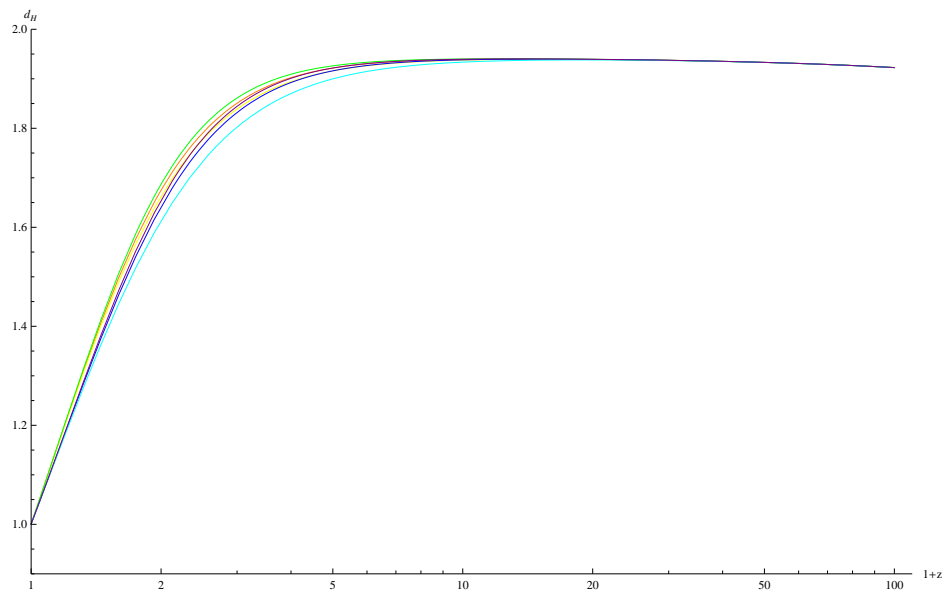


Figure 22: Hubble distance of CPL dark energy.

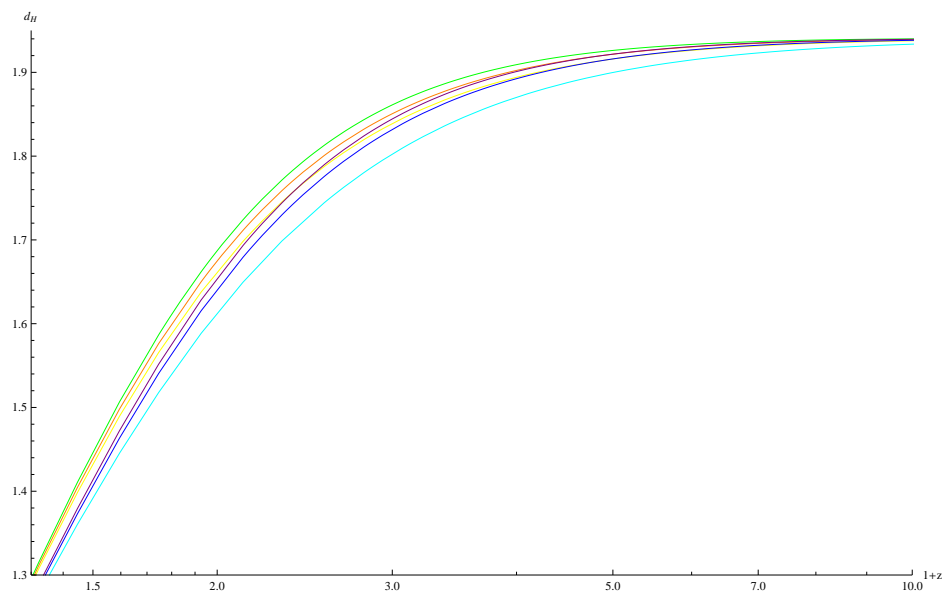


Figure 23: Hubble distance of CPL dark energy.

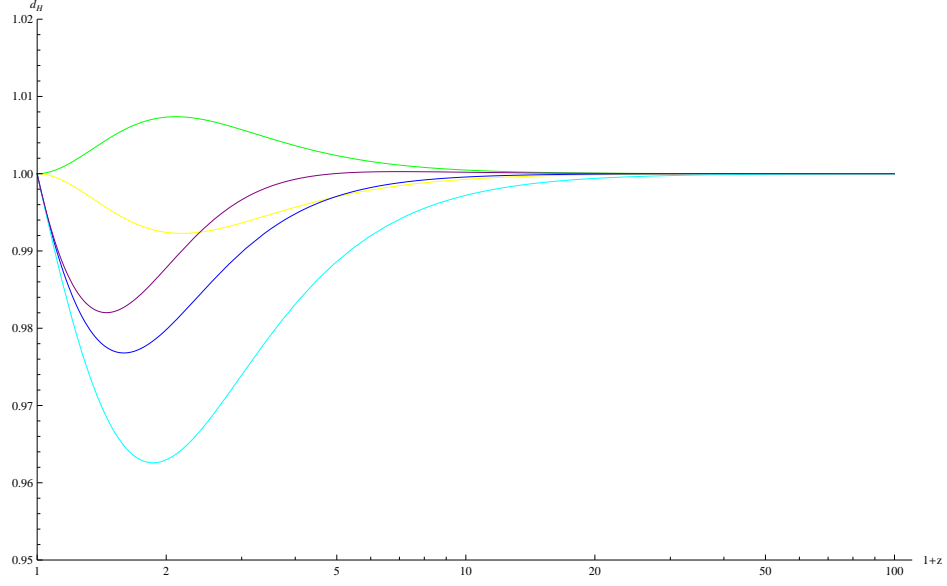


Figure 24: Hubble distance of CPL dark energy in units of LCDM Hubble distances at the corresponding value.

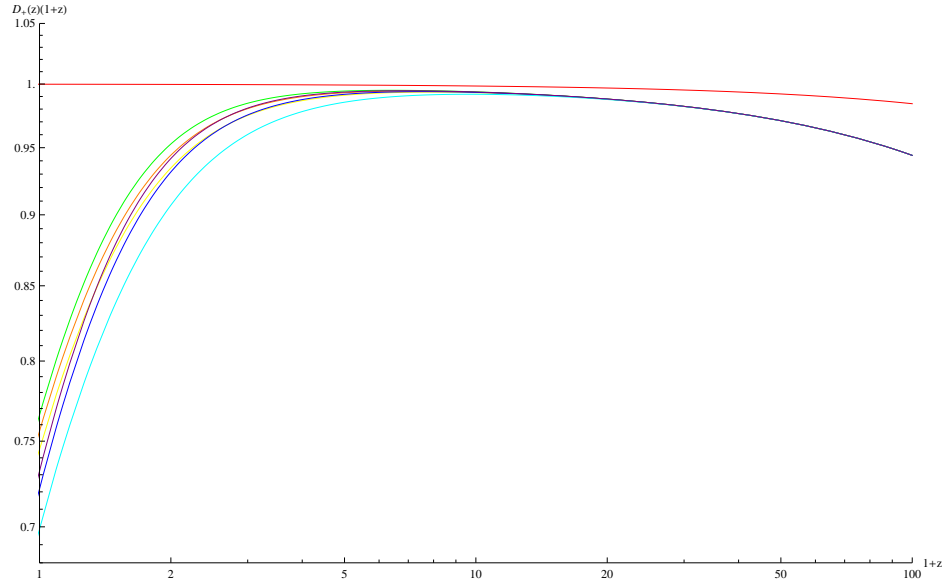


Figure 25: Growth factors of CPL dark energy and LCDM in units of sCDM.

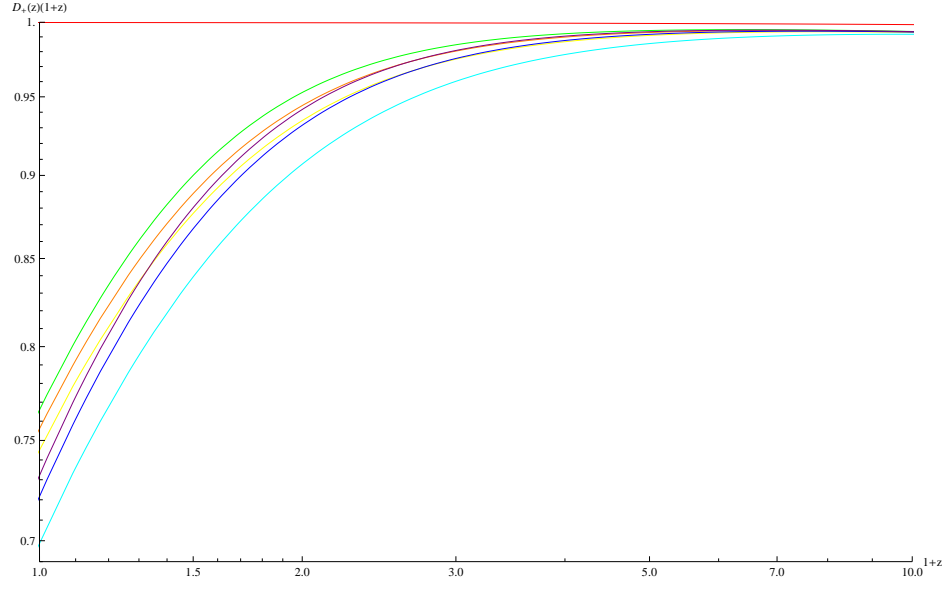


Figure 26: Growth factors of CPL dark energy and LCDM in units of sCDM.

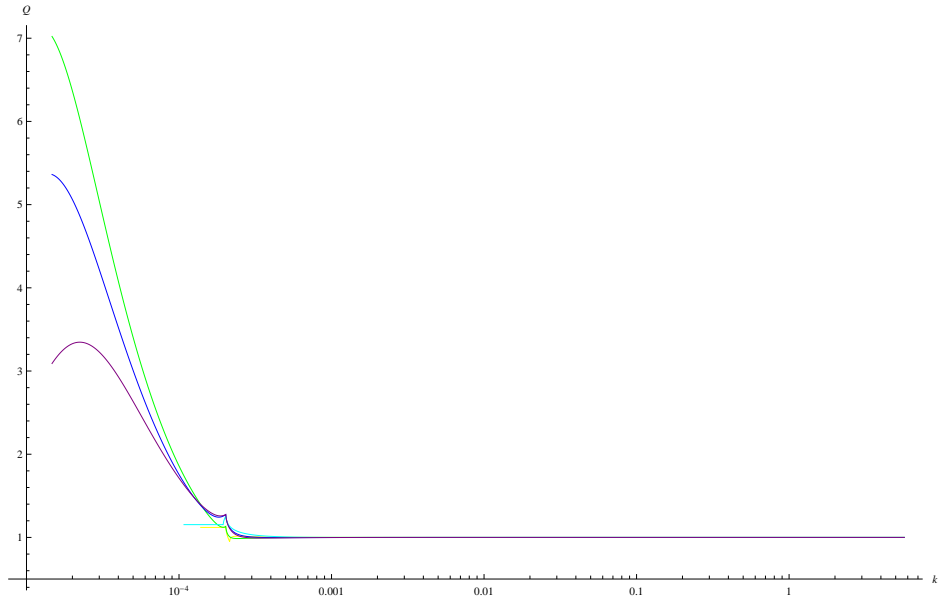


Figure 27: Q factors of CPL dark energy and LCDM in units of sCDM.

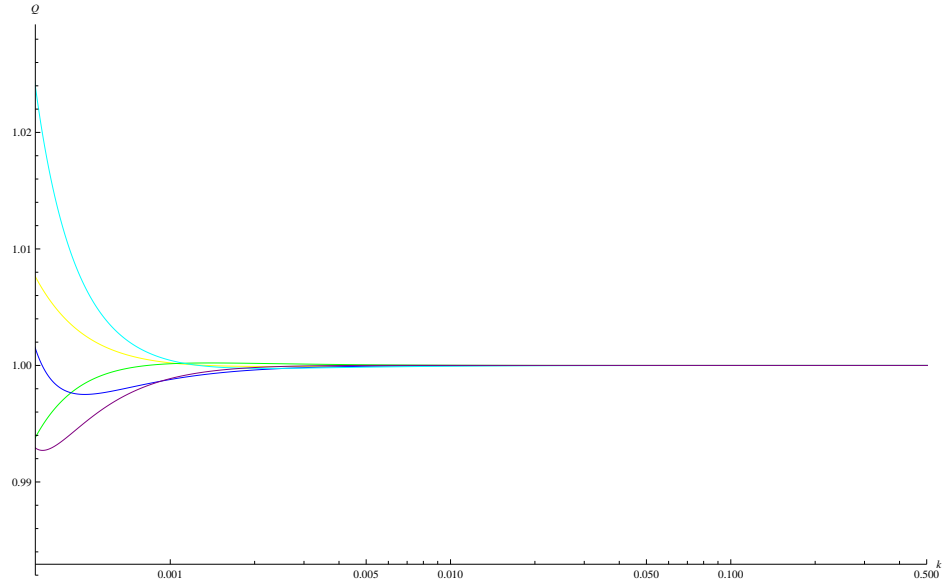


Figure 28: Q factors of CPL dark energy and LCDM in units of sCDM.

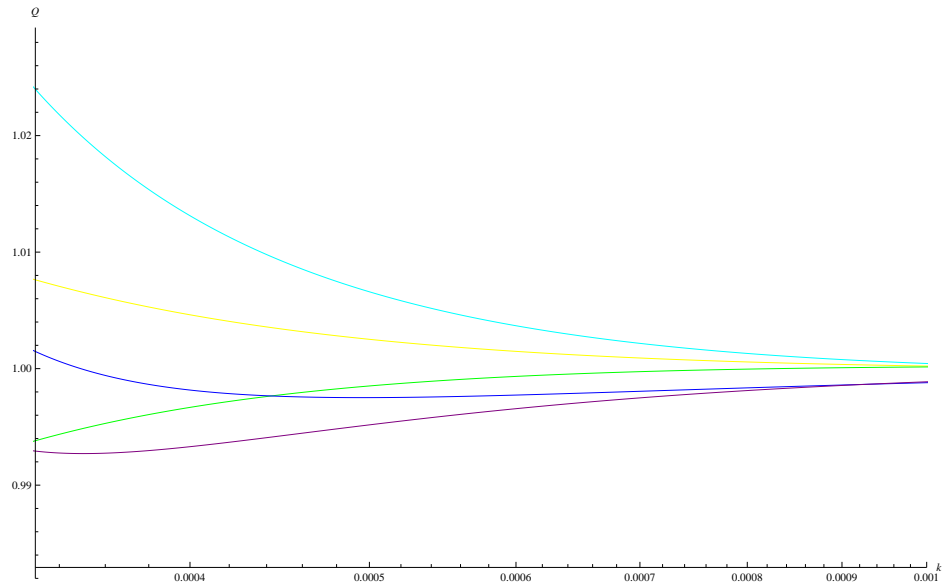


Figure 29: Q factors of CPL dark energy and LCDM in units of sCDM.

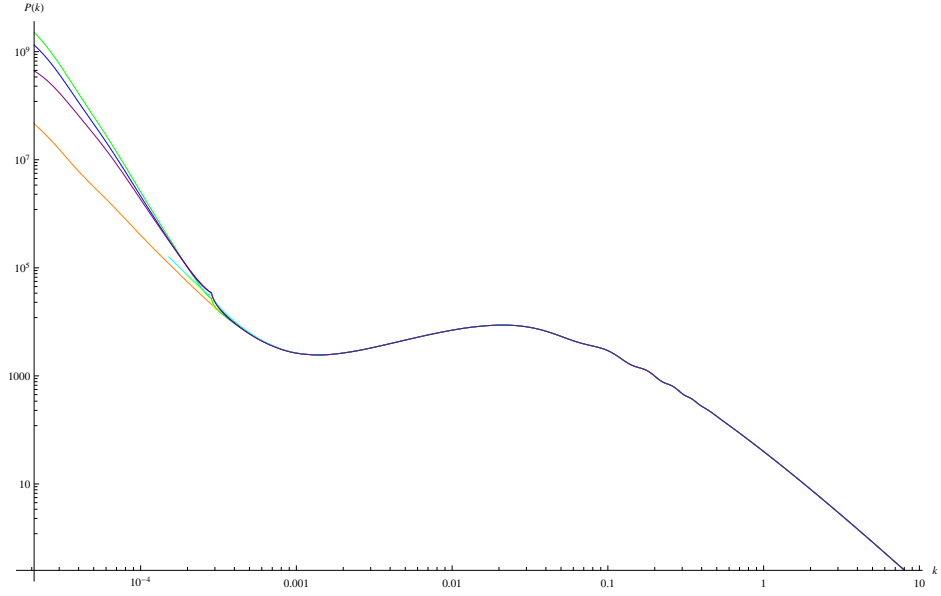


Figure 30: Power spectrum of CPL dark energy and LCDM in units of sCDM.

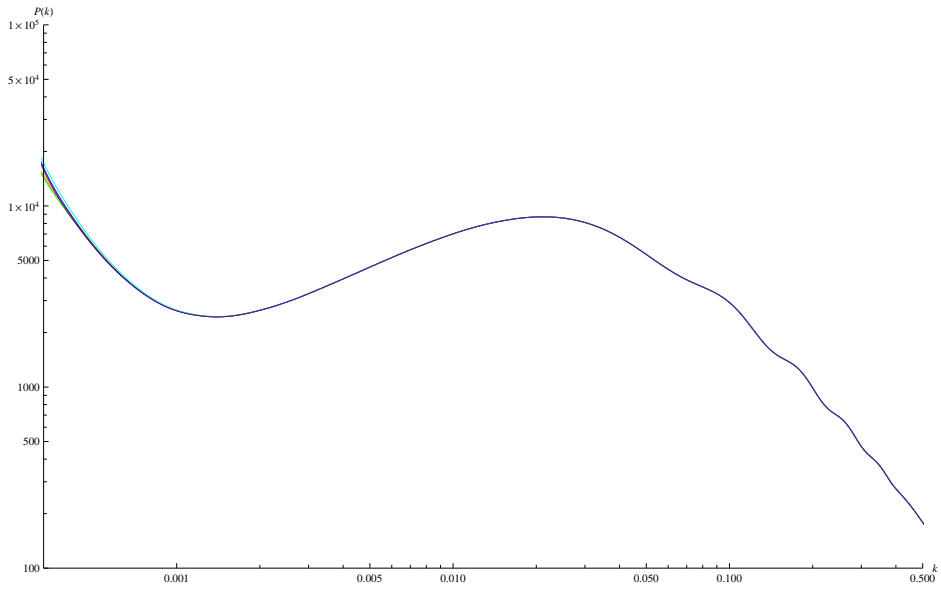


Figure 31: Power spectrum of CPL dark energy and LCDM in units of sCDM.

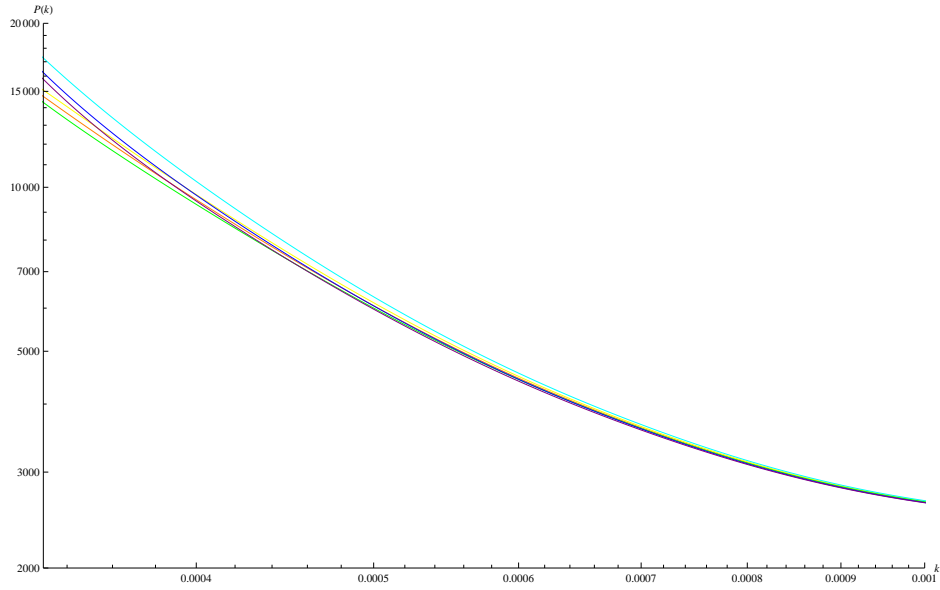


Figure 32: Power spectrum of CPL dark energy and LCDM in units of sCDM.

example, the yellow line ($w_0 = -1, w_a = 0.1$) crosses the purple line ($w_0 = -0.9, w_a = -0.2$) in figure 25 26 as they cross each other in the figure of Hubble distance.

Figure 27 has a region $k < 3.3 * 10^{-4}$ that is corresponding to scale factor $a > 1$. So I give another two plots figure 28 and 29. The same thing is done for power spectrum.

The power spectrum is consistent with the Hubble distance figure: larger Hubble distance generates weaker power.

7. Because we are using adiabatic initial conditions with a form $P \sim \frac{1}{k^3}$.

At $a \sim 10^{-3}$ and $a \sim 1$, we have $k \sim 1$ and $k \sim 10^{-3}$ correspondingly. The growth factor with amplifies the initial perturbation by about 10^3 to get the current value $P \sim 10^6$ for $k \sim 1$ and the current value is 10^9 for initial perturbation at $a \sim 1$ (or $k \sim 10^{-3}$) since it has not evolved. This simple calculation show the shape of power spectrum is definitely going up at small k.

item As stated previously, the inconsistent part is $z < 0$, so there is no need to worry about.

5 Appendix

5.1 Growth Factor Revisited

In this part, prime (') stands for the derivative with time t , over dot stands for the derivative with scale factor a .

Use Navier Stokes equations for matter (with pressure $p = 0$) as the basic equation set.¹⁸

Using $\delta_m = \frac{\rho_m - \bar{\rho}_m}{\bar{\rho}_m}$, the equations become

$$\frac{\partial \delta_m}{\partial t} + a^{-1} \vec{v}_m \cdot \delta_m = -a^{-1}(1 + \delta_m) \nabla \cdot \vec{v}_m \quad (50)$$

$$\frac{\partial \vec{v}_m}{\partial t} + a^{-1}(\vec{v}_m \cdot \nabla) \vec{v}_m = -\frac{\nabla \Phi}{a} - H \vec{v}_m \quad (51)$$

$$a^{-2} \nabla^2 \Phi = 4\pi G \bar{\rho}_m \delta_m \quad (52)$$

In these equations, Φ is the gravitational potential.

This is too complex to solve. So we only use the linear approximation, that is, no second or higher order of δ_m , \vec{v}_m , Φ appear, because we already use those as perturbations. Finally,

$$\frac{\partial \delta_m}{\partial t} = -a^{-1} \nabla \cdot \vec{v}_m \quad (53)$$

$$\frac{\partial \vec{v}_m}{\partial t} = -\frac{\nabla \Phi}{a} - H \vec{v}_m \quad (54)$$

$$a^{-2} \nabla^2 \Phi = 4\pi G \bar{\rho}_m \delta_m \quad (55)$$

Easily, one can use the well known trick for this kind of equations to find the second derivative equation for δ_m , or the basic function for this problem,

$$\delta_m'' + 2H\delta_m' - 4\pi G \bar{\rho}_m \delta_m = 0 \quad (56)$$

However, we usually use scale factor a as a measurement of time. Thus we would transform it the following form

$$\ddot{\delta}_m + \left(\frac{d \ln H}{da} + \frac{3}{a}\right) \dot{\delta}_m - \frac{3\Omega_{m0}H_0^2}{2a^5H^2} \delta_m = 0 \quad (57)$$

Almost each book of ODE will tell us how to solve such an equation.

After a little work, we will get two special solutions:

$$\delta_{m,1} = H \quad (58)$$

$$\delta_{m,2} = H \int_0^a \frac{1}{a'^3 H(a')^3} da' \quad (59)$$

General solution is

$$\delta_m = C_1 H + C H \int_0^a \frac{1}{a'^3 H(a')^3} da' \quad (60)$$

Since H is decaying, drop $C_1 H$, then

$$\delta_m = C H \int_0^a \frac{1}{a'^3 H(a')^3} da' \quad (61)$$

Apply initial condition $\delta_m(a_i) = \delta_i$

$$C = \frac{\delta_i}{H(a_i)} \frac{1}{\int_0^{a_i} \frac{1}{a'^3 H(a')^3} da'} \quad (62)$$

To simplify these expressions, we now define $D_+ = \frac{5}{2} \Omega_{m0} H(a) \int_0^a \frac{1}{a'^3 H(a')^3} da'$. (One might be confused with this definition at first. The constant multiplied the the original part is to ensure one condition: at matter domination, D_+ is a . This condition endow D_+ with more physics.) Then

$$C = \delta_i \frac{5\Omega_{m0}H_0^2}{2} \frac{1}{D_+(a_i)} \quad (63)$$

¹⁸ *Cosmology Project Notebook*

With this, we have

$$\delta_m(a) = \delta_i \frac{D_+(a)}{D_+(a_i)} \quad (64)$$

Actually, this $D_+(a)$ is a growing mode of the equation, and we all call it growth factor.

5.2 CPL

5.2.1 Equation of State

$$w(a) = w_0 + w_a(1 - a) = w_0 + w_a \frac{z}{1 + z} \quad (65)$$

5.2.2 Hubble Function

By solving the Friedmann equation using the CPL parameterisation, the contribution to the background of CPL dark energy is $\Omega_{de0} a^{-3(1+w_0+w_a)} e^{-3w_a(1-a)}$.

Thus the Hubble function should be

$$H(a) = H_0 \sqrt{\Omega_{m0} a^{-3} + \Omega_{r0} a^{-4} + \Omega_{de0} a^{-3(1+w_0+w_a)} e^{-3w_a(1-a)}} \quad (66)$$

The Taylor series of dark energy contribution is

$$a^{-3(1+w_0+w_a)} e^{-3w_a(1-a)} = 1 + 3(1+w_0)(a-1) + \frac{3}{2}(4+w_a+7w_0+3w_0^2)(1-a)^2 + \dots \quad (67)$$

To create a mimic of LCDM at low redshift, $w_0 \rightarrow 0$ should be satisfied. To make a more rigorous condition, w_a should be much smaller than 1 for this cancels the contribution of z^2 term once $w_0 = -1$.

In order to generate a background similar to LCDM, the parameters should carefully chosen according to the criteria that $w_0 \rightarrow 0$ and $w_a \rightarrow 0$.

5.3 Second Supplements

5.3.1 To Think More

1. A little dimensional analysis is shown here, in case of future mistakes.
 - (a) Hubble constant, H_0 carries the dimension of $[Time]^{-1}$
 - (b) Gravitational constant, G carries the dimension of $[Length]^3/[Mass][Time]^2$
 - (c) Speed of light in vacuum, c of course, has the dimension of $[Length]/[Time]$
2. Large k corresponds to the future. Besides, large k should have the same amplitude because they have not come inside the horizon.
3. What's more, the comoving horizon becomes constant in the far future.

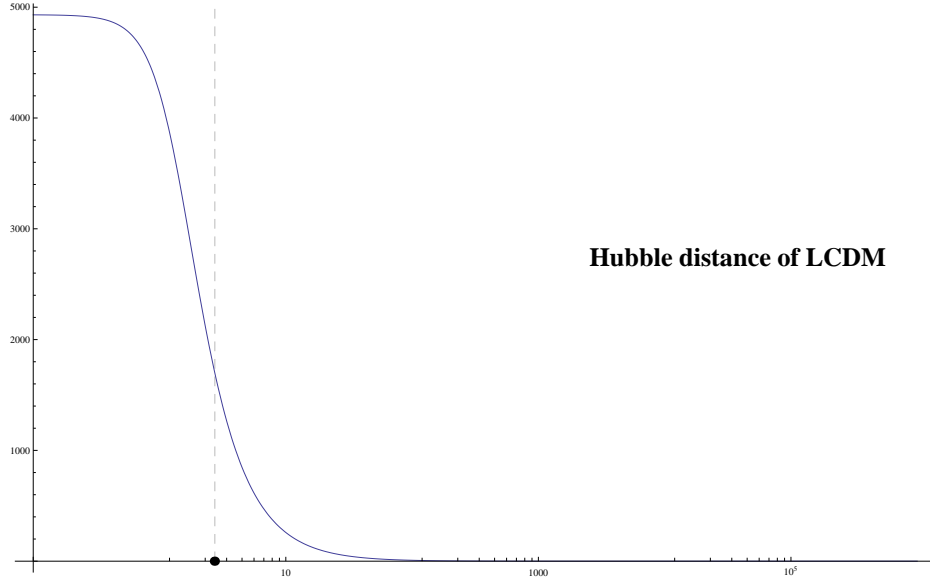


Figure 33: Hubble distance of LCDM model. The dashed vertical line is zero redshift.

This simply means there are perturbations that will never come into the horizon.

To calculate the power spectrum, one should cut the growth functions at $z = 0$ and set the matter perturbation the same then calculate on. Since these are outside of the horizon, today's power spectrum seems weird.

4. The power spectrum calculated by me is the power of today. However, observations certainly won't get the power spectrum of the present day. On large scales, the distribution of matter is the in the state of earlier ages.

In paper **Matter power spectrum from the Lyman-alpha forest: myth or reality?** (*Mon. Not. R. Astron. Soc.* 334, 107-116, 2002) calculated the distribution of matter at $z = 2.72$. This paper also assumed that the underlying baryon density follows that of dark matter (their reference is Croft et al, 2001, astro-ph/0012324).

5.4 Including DM-DE perturbation

6 Todo List

- Find all the growth factors if they are useful.
- Calculate the fiducial power spectrum.
- The spectrum is normalised to be the same at $a \sim 0$. So that the σ_8 of different models are the same.
- Switch from adiabatic initial condition to isocurvature initial condition will change the expression of growth factors (changing the power of H), thus leading to different results.

- Introduce interacting between dark energy and dark matter.
- Other quantities in cosmology? Can one generalise this method to more calculations?

7 Appendix

7.1 Growth Factor Revisited

In this part, prime (') stands for the derivative with time t , over dot stands for the derivative with scale factor a .

Use Navier Stokes equations for matter (with pressure $p = 0$) as the basic equation set.¹⁹

Using $\delta_m = \frac{\rho_m - \bar{\rho}_m}{\bar{\rho}_m}$, the equations become

$$\frac{\partial \delta_m}{\partial t} + a^{-1} \vec{v}_m \cdot \delta_m = -a^{-1} (1 + \delta_m) \nabla \cdot \vec{v}_m \quad (68)$$

$$\frac{\partial \vec{v}_m}{\partial t} + a^{-1} (\vec{v}_m \cdot \nabla) \vec{v}_m = -\frac{\nabla \Phi}{a} - H \vec{v}_m \quad (69)$$

$$a^{-2} \nabla^2 \Phi = 4\pi G \bar{\rho}_m \delta_m \quad (70)$$

In these equations, Φ is the gravitational potential.

This is too complex to solve. So we only use the linear approximation, that is, no second or higher order of δ_m , \vec{v}_m , Φ appear, because we already use those as perturbations. Finally,

$$\frac{\partial \delta_m}{\partial t} = -a^{-1} \nabla \cdot \vec{v}_m \quad (71)$$

$$\frac{\partial \vec{v}_m}{\partial t} = -\frac{\nabla \Phi}{a} - H \vec{v}_m \quad (72)$$

$$a^{-2} \nabla^2 \Phi = 4\pi G \bar{\rho}_m \delta_m \quad (73)$$

Easily, one can use the well known trick for this kind of equations to find the second derivative equation for δ_m , or the basic function for this problem,

$$\delta_m'' + 2H\delta_m' - 4\pi G \bar{\rho}_m \delta_m = 0 \quad (74)$$

However, we usually use scale factor a as a measurement of time. Thus we would transform it the following form

$$\ddot{\delta}_m + \left(\frac{d \ln H}{da} + \frac{3}{a} \right) \dot{\delta}_m - \frac{3\Omega_{m0}H_0^2}{2a^5H^2} \delta_m = 0 \quad (75)$$

Almost each book of ODE will tell us how to solve such an equation.

After a little work, we will get two special solutions:

$$\delta_{m,1} = H \quad (76)$$

$$\delta_{m,2} = H \int_0^a \frac{1}{a'^3 H(a')^3} da' \quad (77)$$

General solution is

$$\delta_m = C_1 H + C H \int_0^a \frac{1}{a'^3 H(a')^3} da' \quad (78)$$

Since H is decaying, drop $C_1 H$, then

$$\delta_m = C H \int_0^a \frac{1}{a'^3 H(a')^3} da' \quad (79)$$

Apply initial condition $\delta_m(a_i) = \delta_i$

$$C = \frac{\delta_i}{H(a_i)} \frac{1}{\int_0^{a_i} \frac{1}{a'^3 H(a')^3} da'} \quad (80)$$

To simplify these expressions, we now define $D_+ = \frac{5}{2} \Omega_{m0} H(a) \int_0^a \frac{1}{a'^3 H(a')^3} da'$. (One might be confused with this definition at first. The constant multiplied the the original part is to ensure one condition: at matter domination, D_+ is a . This condition endow D_+ with more physics.) Then

$$C = \delta_i \frac{5\Omega_{m0}H_0^2}{2} \frac{1}{D_+(a_i)} \quad (81)$$

¹⁹ *Cosmology Project Notebook*

With this, we have

$$\delta_m(a) = \delta_i \frac{D_+(a)}{D_+(a_i)} \quad (82)$$

Actually, this $D_+(a)$ is a growing mode of the equation, and we all call it growth factor.

7.2 CPL

7.2.1 Equation of State

$$w(a) = w_0 + w_a(1 - a) = w_0 + w_a \frac{z}{1 + z} \quad (83)$$

7.2.2 Hubble Function

By solving the Friedmann equation using the CPL parameterisation, the contribution to the background of CPL dark energy is $\Omega_{de0} a^{-3(1+w_0+w_a)} e^{-3w_a(1-a)}$.

Thus the Hubble function should be

$$H(a) = H_0 \sqrt{\Omega_{m0} a^{-3} + \Omega_{r0} a^{-4} + \Omega_{de0} a^{-3(1+w_0+w_a)} e^{-3w_a(1-a)}} \quad (84)$$

The Taylor series of dark energy contribution is

$$a^{-3(1+w_0+w_a)} e^{-3w_a(1-a)} = 1 + 3(1+w_0)(a-1) + \frac{3}{2}(4+w_a+7w_0+3w_0^2)(1-a)^2 + \dots \quad (85)$$

To create a mimic of LCDM at low redshift, $w_0 \rightarrow 0$ should be satisfied. To make a more rigorous condition, w_a should be much smaller than 1 for this cancels the contribution of z^2 term once $w_0 = -1$.

In order to generate a background similar to LCDM, the parameters should carefully chosen according to the criteria that $w_0 \rightarrow 0$ and $w_a \rightarrow 0$.