

# CONTENTS

1. Classical Hydrodynamical approach to the matter dominated Perturbation. ① - ⑥
2. ~~Black~~ Primordial Perturbations ⑨ - ⑩
3. Growth factor first trial ⑪ - ⑫
4. CPL Parameterization ⑬ - ⑭
5. Interacting DM & DE ⑯
6. Nonlinear effects ⑮ - ⑯
7. Gauge ⑰ - ⑱
8. Diffusion & Free-streaming ⑲ - ⑳
9. Including DE in regards Eq ㉑
10. Gauge invariant ㉒
11. Gauge (Synchronous) ㉓ - ㉔
12. Invariant quantities outside of horizon ㉕ - ㉖
13. Power spectrum. ㉗ - ㉘

ANSWER

Useful expressions and constants etc :

$$P_c = \frac{3H^2}{8\pi G}, M_{Pl} = \sqrt{\frac{8\pi G}{3}} \text{ or } \sqrt{\frac{8\pi G}{3}}$$

$$H^2 = \frac{1}{3M_{Pl}^2} (P_r + P_c + P_b + P_\phi)$$

# Project:

Construct other model's power spectra using a fiducial model.

## 1. Growth factor:

Scales  $l \ll cH^{-1}$  and velocities  $v \ll c$ . : Newtonian Linear perturbation theory.

For pressure gradient negligible gradient case, i.e.  $\vec{\nabla}p/p \ll \vec{\nabla}\phi$

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \vec{\nabla} \cdot \vec{v} = 0, \quad \frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a} \vec{v} + \frac{1}{a} \vec{\nabla} \phi = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 \delta}{\partial t^2} + 2 \frac{\dot{a}}{a} \frac{\partial \delta}{\partial t} = 4\pi G \rho_b \delta}$$

with a solution:  $\delta(x, t) = A(x) D_1(t) + B(x) D_2(t)$

$D_1(t)$  is the growth part;  $D_2(t)$  is the decaying mode

Decays don't care about this?

$$\rightarrow \boxed{D_1'' + 2H(z)D_1 - \frac{3}{2}\Omega_{m,0} H_0^2 (1+z)^3 D_1 = 0}$$

Solutions {

Approximation:  $d \ln D_1 / d \ln a = \Sigma_m^{0.6}$ , when  $\Delta \approx 1$ ,  $D_1 \propto a$ .

Fourier Mode:

$$\delta(x, t) = \int d^3 k \cdot \delta_k e^{i \vec{k} \cdot \vec{x}}$$

$$\delta_k = \int d^3 x \cdot \delta(x) e^{-i \vec{k} \cdot \vec{x}}$$

Gaussian if  $\delta_k$  is not correlated  $\downarrow$  complex number

$$\delta_k = A_k e^{i \theta_k}$$

$$\text{then } P(\delta) = (2\pi\sigma^2)^{-1/2} e^{-\delta^2/2\sigma^2}, \text{ is the 1-point distribution}$$

(in which  $\sigma$  is the variance and it reads  $\sigma^2 = \langle \delta^2 \rangle = \int_0^\infty 4\pi k^2 P(k) dk$ )

$$\text{Thus } P(k) = \langle A_k A_k^* \rangle$$

## In linear theory 2. Transfer function

① If the perturbation enters the Horizon at matter domination.

$\delta_H = \text{const}$ .  $\delta_H$  denotes the fluctuation reenter

$$4\pi k^3 P(k, t) = \delta_H^2 = \text{const} : \text{Scale invariant}$$

Since  $a(t_0) \sim ct_0$  and  $a \propto t^{\frac{2}{3}}$   
we have  $a^{\frac{1}{2}}(t_0) \propto \frac{1}{k}$

We already know  $P(k, t) \propto D^2(t) \propto a^2$  in MD.

$$P(kt) \underset{(1)}{=} P(k, t_0) \frac{a(t)}{a(t_0)} = \frac{\delta_H^2}{4\pi k^3} \cdot \frac{a^2(t)}{a^2(t_0)} \propto \frac{D^2(t)}{a^2(t)}$$

② If the fluctuation reenters at far RD, i.e.

$$\lambda_H \ll \lambda_{\text{eq}} \sim ct_{\text{eq}}(1+z_{\text{eq}})$$

In RD, fluctuations do not change for very small scale.

$$4\pi k^3 P(k, t) \underset{(1)}{=} 4\pi k^3 P(k, t_0) \underset{(2)}{=} \text{const} \Rightarrow$$

$$\Rightarrow P(k, t) \propto k^{-3} \quad (2)$$

During eras between ① and ②, there is a slow transition. That means

$$P(k, t) \propto \mathcal{T}(k) D^2(t) / P(k, t_0) \propto k^2 T(k) / \mathcal{T}(t_0)$$

in which  $\mathcal{T}(k)$  transits from  $k^{-3}$  to  $k^0$  during the transition from RD to MD.

(a). What does  $T(k)$  mean?

$T(k)$  stands for the effects of ~~EOS~~ EOS (or just the pressure and gravity effect)

(b). From ① & ② we know,  $T^2(k) = \text{const.}$  on large scale and  $T^2(k) \sim k^{-4}$  for small scale.

(k) The transition time has an important point : equality .  
it is said this is the  $P(k)$  turning-point

$$\lambda \approx \sim c t_{eq} (1+z_{eq}) = 50 \left( \frac{S_m \cdot h^2}{0.15} \right)^{-1} \text{Mpc}$$
$$= 36 \cdot \left( \frac{S_m h}{0.21} \right)^{-1} h^{-1} \cdot \text{Mpc}$$

may be modified by existence of baryon gravity .

### 3. 11 More about Power spectrum.

Generally .  $P(k) \propto k^n T^2(k) D^2(t)$

$n = 1$  : scale invariant

$n > 1$  : inflation

$n < 1$  : tilted spectrum : typical  $n = 0.9 - 1$   
but  $n > 1$  is possible

$n(k)$  : called "running of the spectral index".  
this effect should be small in typical  
inflation models .

With a sharp  $\lambda$ : "broken" scale invariance  
A feature in inflation potential  
can produce this

### 4. 91 Why CDM ?

Hot DM is relativistic at  $t_{eq}$  and suppressed  
the ~~fluctuations~~ fluctuations , changing  $T(k)$  dramatically

Warm DM is possible but have to be fine-tuned  
to fit the right scale .

CDM is natural choice . Heavy properties  
keeps it from erasing fluctuations .

Another point of view at linear perturbation theory.

In hydrodynamics MHD.

I give some explanations of Navier Stokes equations.

With a substitution from  $\rho_m$  to  $\delta_m$  we get the exact equations for matter

$$\left\{ \begin{array}{l} \frac{\partial \delta_m}{\partial t} + \alpha^1 \vec{v}_m \cdot \nabla \delta_m = -\alpha^{-1} (1 + f_m) \nabla \cdot \vec{v}_m \\ \frac{\partial \vec{v}_m}{\partial t} + \alpha^1 (\vec{v}_m \cdot \nabla) \vec{v}_m = -\frac{\nabla \Phi}{\alpha} - H \vec{v}_m \\ \alpha^{-2} \nabla^2 \Phi = 4\pi G \bar{\rho}_m \delta_m \end{array} \right.$$

Under linear approximation (no <sup>second or higher order</sup> of  $\delta_m$ ,  $\vec{v}_m$ ,  $\Phi$ , because we already use those as <sup>small</sup> perturbations)

Thus

$$\left\{ \begin{array}{l} \frac{\partial \delta_m}{\partial t} + \cancel{\alpha^{-1}} = -\alpha^1 \nabla \cdot \vec{v}_m \quad \text{①} \\ \frac{\partial \vec{v}_m}{\partial t} = -\frac{\nabla \Phi}{\alpha} - H \vec{v}_m \quad \text{②} \\ \alpha^{-2} \nabla^2 \Phi = 4\pi G \bar{\rho}_m \delta_m \quad \text{③} \end{array} \right.$$

Zero and First order perturbation equations for matter \*  
(P=0 actually)

1. Define  $\theta = \alpha^1 \vec{v} \cdot \vec{v}$  (as is the divergence of  $\vec{v}$ )

① & ② become

$$\ddot{\delta}_m + \cancel{QH\dot{\delta}_m} 2H \dot{\delta}_m - 4\pi G \bar{\rho}_m \delta_m = 0$$

(This is a trick. We always turn two correlated first order derivative eqns into a second order one,  $\ddot{\delta}_m$  in order to see the evolution of the quantities we are interested in)

2. a. Growth function:  $\delta_m \propto G(a)$

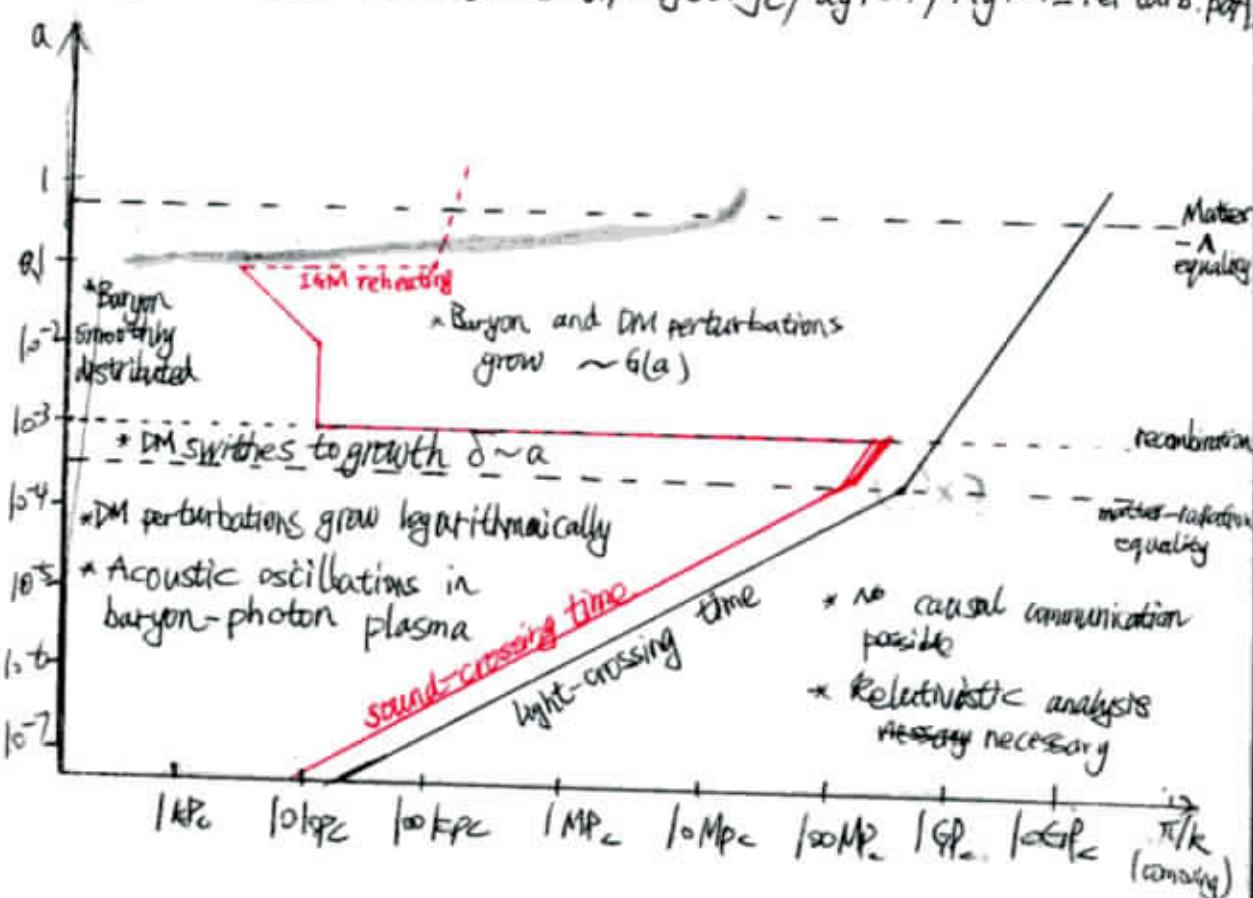
This is defined for growing mode, for decaying mode faded away at late times. For example, in Einstein-de Sitter model,  $\delta_m = C_1 t^{-3} + C_2 t^{-1}$ , when ~~t is~~  $t$  is large, ~~only~~ growing mode dominates.

b. Define Growth Rate as  $\frac{d \ln G(a)}{da}$

$$\text{Reform it } \frac{d \ln G(a)/dt}{d \ln a/dt} = \frac{d \ln(C \cdot \delta_m)/dt}{d \ln a/dt}$$

$$= \frac{\frac{1}{\delta_m} \frac{d \ln \delta_m}{dt}}{\frac{1}{a} \frac{d \ln a}{dt}} = \frac{\dot{\delta}_m}{H \delta_m} = \frac{-\nabla \cdot \vec{v}}{a H \delta_m}$$

3. George ~~gave~~ gives a description of perturbation with pressure  
[www.astro.caltech.edu/~george/ay127/Ay127-Perturb.pdf](http://www.astro.caltech.edu/~george/ay127/Ay127-Perturb.pdf)



Explanations on the next page ..

2. a. Growth function:  $\delta_m \propto g(a)$

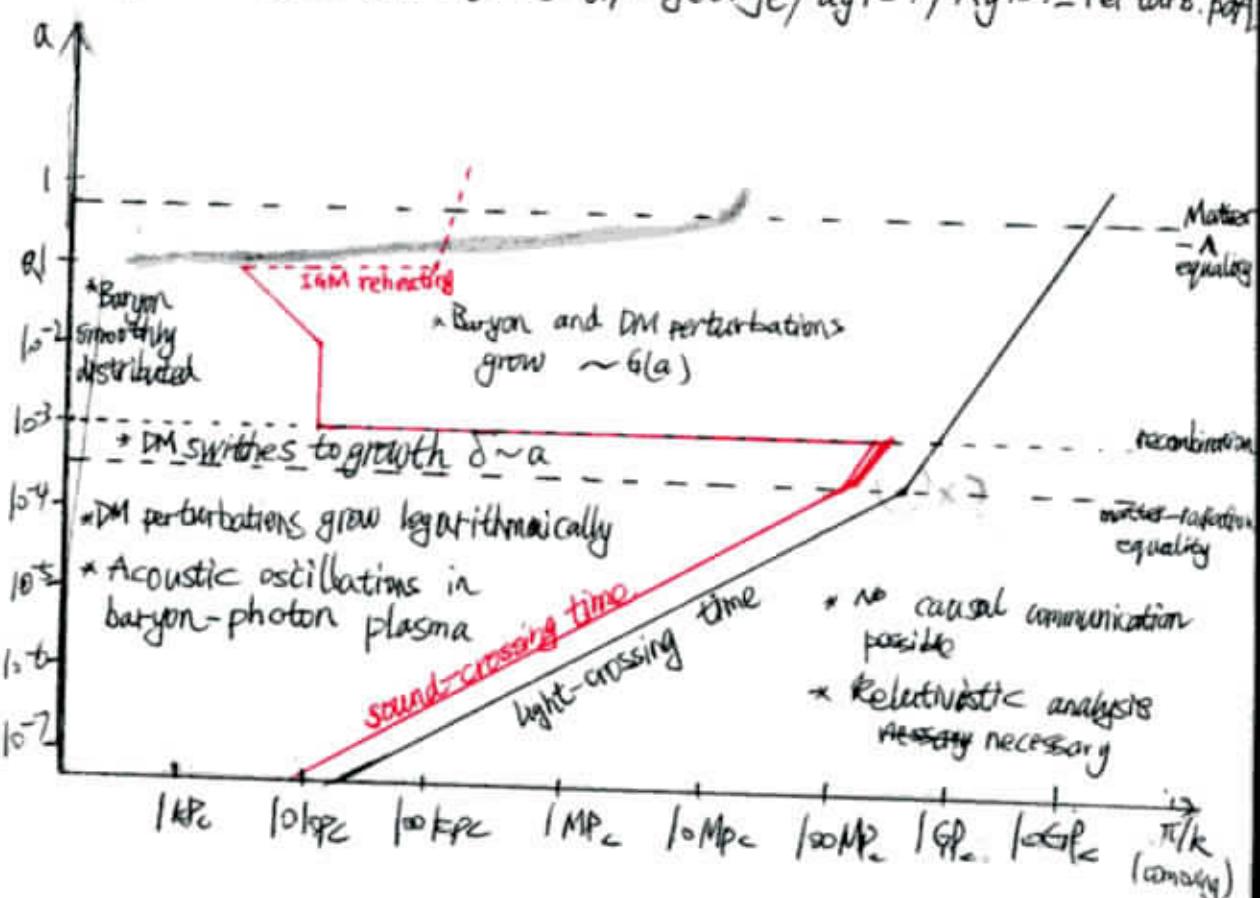
This is defined for growing mode, for decaying mode faded away at late times. For example, in Einstein-de Sitter model,  $\delta_m = C t^3 + G t^{-1}$ , when ~~this~~  $t$  is large, ~~only~~ growing mode dominates.

b. Define Growth Rate as  $\frac{d \ln G(a)}{da}$

$$\text{Reform it } \frac{d \ln G(a)/dt}{d \ln a/dt} = \frac{d \ln(C \cdot \delta_m)/dt}{d \ln a/dt}$$

$$= \frac{\frac{1}{\delta_m} \frac{d \ln \delta_m}{dt}}{\frac{1}{a} \frac{d \ln a}{dt}} = \frac{\dot{\delta}_m}{H \delta_m} = \frac{-\nabla \cdot \vec{v}}{a H \delta_m}$$

3. George gives a description of perturbation with pressure  
[www.astro.caltech.edu/~george/ay127/Ay127-Perturb.pdf](http://www.astro.caltech.edu/~george/ay127/Ay127-Perturb.pdf)



Explanations on the next page ..

# EXPLANATION:

## Borders:

- Matter Radiation equality
- Recombination: (of electrons and protons)  $e^- + p \rightarrow H + \gamma$   
when energy/temperature drops far below  $E_0 = 15.6 \text{ eV}$ ,  
 $\Rightarrow$  hydrogen produced heavily (appreciable recombination)
- Matter -  $\Lambda$  equality
- Light-crossing time: the time light needs to cross a certain distance.
  - ① Somehow ~~it is~~ it is the (event) horizon here??
  - ② This should not be a polygonal line, I think because the universe didn't transit from RD to MD suddenly.
- Sound-crossing time: (Sound ~~spreads~~  $\rightarrow$  horizon?)  
In these linearized equations  $\delta P = C_s^2 \frac{\delta P}{\text{sound speed}}$
- Sound ~~spreads~~ spreads quickly at first because the universe is a bath of photon baryon soup. It drops after matter radiation equality because matter turns out to be less relativistic and the support of photon pressure becomes less important.

① At recombination, it drops dramatically. Why?

Because recombination cut the trace of sound into pieces?

② Also the line should not change suddenly at matter-radiation equality.

③ Jeans instability? (<sup>When</sup> free-falling ~~is~~ faster than gaseous sound speed, the system collapse.)

### \* Areas with explanations

\* DM perturbations grow logarithmically:

$$\text{In radiation era } \delta_m = C_1 + C_2 \ln t$$

\* Acoustic Oscillations: In RD, baryon-photon fluid:

$$\frac{\partial^2 \delta_i}{\partial t^2} - \frac{C_s^2}{a^2} \nabla^2 \delta_i - 4\pi G a \bar{\rho}_i \delta_i + 2 \frac{\dot{a}}{a} H \frac{\partial \delta_i}{\partial t}. \quad \text{It's a damped wave func.}$$

\* Relativistic analysis is needed ~~on~~ on large scales in RD

~~classical~~ Jeans length:  $\lambda_J = \frac{2\pi}{k_J} = \sqrt{\frac{32}{3}} \frac{\pi}{C_s T}$

$$(k_J = \frac{a}{C_s} \sqrt{4\pi G P} \stackrel{RD}{=} \sqrt{\frac{3}{G}} \frac{aH}{C_s} = \sqrt{\frac{3}{G}} \frac{a}{C_s T})$$

For RD plasma,  $C_s = \frac{c}{\sqrt{E}}$ . Then  $\lambda_J$  is larger than light can travel. So we need to use  $G_P$  in order to get the right term. □

\* Baryon smoothly distributed.

In MD, potential dominated by dark matter.

Evolution eqn. for baryons ~~are~~ is

$$\ddot{\delta}_b + 2H\dot{\delta}_b - \frac{c_s^2 k^2}{a^2} \delta_b - \frac{R''}{a^2} = 0$$

$$\Rightarrow \ddot{\delta}_b + 2H\dot{\delta}_b - \frac{c_s^2 k^2}{a^2} \left( \delta_b + \frac{R''}{c_s^2} \right) = 0$$

- a. For the scales  $k > a/c_s t$ , the sound wave quickly affects the system. (shakes out the perturbation) the term  $\frac{c_s^2 k^2}{a^2} (\delta_b + \frac{R''}{c_s^2})$  dominates. Thus the system reaches an ~~equilibrium~~ <sup>equilibrium</sup>:  $\delta_b = -\frac{R''}{c_s^2}$ . (The oscillation terms do not affect the system because they are relatively small.)

- b. for scales  $k \ll a/c_s t$ , the sound waves hardly affects the whole system because it's too large. The terms  $\ddot{\delta}_b + 2H\dot{\delta}_b$  dominates and leads to ~~oscillations~~ the logarithmic growth of baryon. (Baryon grows with DM)

P.S. One more concern: will the two limits converge at  $k = a/c_s t$ ?

For  $k > a/c_s t$  solution POISSON EQU.

$$\delta_b = -\frac{R''}{c_s^2} = -\frac{1}{c_s^2} \cdot \frac{4\pi G a^2 \rho_{dm} \delta_{dm}}{k^2} = -\frac{2}{3} \left( \frac{a}{k c_s t} \right)^2 \delta_{dm}$$

When  $k = \frac{a}{c_s t}$ , it goes as

$$\delta_b = -\frac{2}{3} \delta_{dm}$$

That means the perturbations of baryons follow the perturbation of dark matter.

## About primordial Perturbations

1. A lumpy universe :  $ds^2 = dt^2 - \frac{a^2(t)}{c^2} \left[ 1 + 2\zeta(\vec{x}) \right] \sum_{i=1}^3 (dx^i)^2 + 2 \sum_{ij} h_{ij} \vec{e}^i \vec{e}^j$

$$ds^2 = dt^2 - \frac{a^2(t)}{c^2} \left\{ [1 + 2\zeta(\vec{x})] \sum_{i=1}^3 (dx^i)^2 + 2 \sum_{ij} h_{ij} \vec{e}^i \vec{e}^j \right\}$$

$h_{ij}(\vec{x})$  is a traceless symmetric tensor. ( $\zeta$  variables)

With a divergence-free condition on  $h_{ij}$ ,

$$\partial_i h_{ij} = 0 \quad (\text{three equations})$$

There are only 1 scalar and 1 tensor with 2 independent variables.

represents the primordial gravitational waves.

More explicitly,

a.  $\zeta(\vec{x})$  is a scalar represents the curvature perturbation.

b.  $h_{ij}(\vec{x})$  is a tensor with 2 free quantities. It represents the primordial gravitational waves.

2. Not all perturbations are represented in the geometry of universe. The distribution of different types of matter can do so. These are called curvature perturbations

e.g.

a.  $\frac{\text{Baryon}}{\text{photon}} = \eta(\vec{x})$  : Baryon iso-curvature

b.  $\frac{\text{Dark Matter}}{\text{photon}}$  is spatially variable : DM iso-cur...

yne the satt Classification

### Adiabatic

- \* Scalar / curvature perturbations,  $\delta\phi$
- \* Tensor / gravitational waves,  $\delta g_{ij}$

### Iso曲率

- \* Baryon
- \* DM

Way/Hu says,

The Density fluctuations could be represent initially. This represents the adiabatic mode.

Alternately, they can be generate from stresses in matter which causally push matter around. This represents the iso曲率 mode.

RS. Iso曲率 curvature here is spatial curvature.

"Iso" means initially there is no ~~curvature~~ perturbation.

If we set the spatial variations in the density of two types of material are initially of equal and opposite magnitude.  $\rightarrow$  the total density remains constant though different places while each species separately has ~~different~~ spatial density variations.

Adiabatic, All ~~spatia~~ species share the same perturbations.

Then adiabatic is completed reflected from the ~~examine~~ metric.

However, iso曲率 should be examined with ~~the~~ other information  $\Leftrightarrow$ . That's why the figure is drawn like that in the ~~top~~ of this page.

# 44 Problems to be explained.

## Exact equations for the electric matter

$$DM: \begin{cases} \delta' + \frac{ikv}{\alpha H y} = -3\Phi \\ v' + \frac{v}{y} = \frac{ik\Phi}{\alpha H y} \\ k^2 \Phi = \frac{3y}{2(y+1)} \alpha^2 H^2 \delta \end{cases}$$

① Poisson equation without radiation:

$$\frac{\partial^2 \Phi}{\partial r^2} = \frac{3Q_{dm}H^2}{2\pi G \alpha^2 H^2 \delta}$$

$$4\pi G \rho_{dm} = \frac{3}{2} H^2 J_2 m \alpha^{-3}$$

Solutions ① :

$$\begin{aligned} & \delta'' + ik \cancel{\left( v' \alpha H y - (\alpha H y)' v \right)} = -3\Phi'', \text{ initial } \Phi = \frac{P_0}{\alpha y} \\ & \delta'' + ikv \left( \frac{d(\alpha H y)}{dy} - \frac{1}{\alpha H y^2} \right) = \frac{3Q_{dm}H^2}{2y^3 \alpha^2 H^2 \delta} \cancel{\Phi''} \\ \Rightarrow & \cancel{ik \frac{d\delta}{da^2}} + \left( \frac{d(\ln(H))}{da} + \frac{3}{a} \right) \frac{d\delta}{da} - \frac{3Q_{dm}H^2}{2a^5 H^2 \delta} \cancel{\Phi''} = 0 \end{aligned}$$

↓

use  $v' + \frac{v}{y} = \frac{ik\Phi}{\alpha H y} \approx 0$   
(because in MD  
situation,  $\frac{ik}{\alpha H}$  is  
small)

## Meszaros equation (extended)

To solve this kind of equations, we have a general method.  
After getting a special solution  $\delta_1 \neq 0$ , the

Write the equation as

$$\frac{d^2 \delta}{da^2} + P(a) \frac{d\delta}{da} + q(a) \delta = 0$$

Substitute  $\delta = \delta_1 \int y da$ ,

$$\cancel{\delta_1} y' + 2\delta_1' y + p(a) \cancel{\delta_1} y = 0$$

The second order equation reduces to first order one.

$$y = C \cdot \frac{1}{\delta_1^2} e^{-\int p(a) da}$$

$$y = C \cdot \frac{1}{\delta_1^2} e^{-\int p(a) da}$$

Then

$$\cancel{\delta} = \delta_1 [C_1 + C_2 \cdot \int \frac{1}{\delta_1^2} e^{-\int p(a) da} da]$$

To verify that it's the general solution,  
take  $C_1 = 0$  and  $C_2 = 1$ , we get a special solution and  
it is linear independent to  $\delta_1$ .

Series expansion method is also possible after setting the initial conditions.

Finally, Scott gives ~~the~~ the solutions

$$\text{Since } \delta \propto H$$

$$\therefore \delta \propto H(a) \int_0^a \frac{da'}{(\hat{a}H(\hat{a}))^3}$$

⊕ -

These are quantities in Fourier space. Since we would like to set in real space  $\delta(k, t) \propto k^n T(k) D_f(a)$   $n$  is the spectral index

The ~~in~~ for a certain  $k$  mode.

$$\delta(k) \propto D_f(a)$$

Thus from ⊕ we know

$$D_f(a) \propto H(a) \int_0^a \frac{da}{(\hat{a}H(\hat{a}))^3}$$

$$D_f(a) = \frac{5P_m}{2} \cdot \frac{H(a)}{H_0} \int_0^a \frac{da}{(a'H(a')/H_0)^3}$$

P.T.E. Peebles and Bharat Ratra • The cosmological constant and dark energy Page 564 • Rev. Mod. Phys. Vol. 75, No. 2 April 2003

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 E(z)^2$$

$$= H_0^2 [\Omega_{M0} a^{-3} + \Omega_{R0}]$$

$$\text{That is } H_0^2 [\Omega_{M0} (Hz)^2 + \Omega_{R0} (Hz)^4 + \Omega_{\Lambda0} + \Omega_{k0} (Hz)^2]$$

$$H(a)^2 = H_0^2 [\Omega_{M0} a^{-3} + \Omega_{R0} a^{-4} + \Omega_{\Lambda0} + \Omega_{k0} a^{-2}]$$

For DE Model

$$H(a)^2 = H_0^2 [\Omega_{M0} a^{-3} + \Omega_{R0} a^{-4} + \Omega_{\Lambda0} a^{-1} + \Omega_{DE0} a^{3(H_w)}]$$

$$= H_0^2 \cdot \cancel{ff(a)} \cancel{ff^2(a)}$$

$$\text{That is } \frac{H(a)}{H_0} = \cancel{ff(a)}$$

Then growth function is

$$D_f(a) = \frac{5}{2} \Omega_M \cdot f^2(a) \int_0^a \frac{d\tilde{a}}{(\tilde{a} f^2(\tilde{a}))^{\frac{1}{2}}}$$

$$f^2(a) = \Omega_{m0} a^{-3} + \Omega_{k0} a^{-4} + \Omega_{r0} a^{-1} + \Omega_{DE0} a^{-3(1+w)}$$

(0 subscripts stand for now.)

$$\Omega_{m0}(t) = 8\pi G \rho_{m0} / (3H_0^2)$$

: P.J.E. Peebles & Bharat Ratra  
RMP, vol 75, No. 2 April 2003

This is scaled based on:  
early times when matter  
dominates,  $D_f \approx a$ .

### Models

1.  $\Lambda CDM$ :  $f^2(a) = a^{-\frac{3}{2}}$ ,  $\Omega_M(a) = 1$  (at late times)

$$\begin{aligned} D_f(a) &= \frac{5}{2} \cdot \tilde{a}^{\frac{1}{2}} \cdot \int_0^a \frac{d\tilde{a}}{(\tilde{a} \tilde{a}^{-\frac{3}{2}})^{\frac{1}{2}}} \\ &= \frac{5}{2} \tilde{a}^{-\frac{1}{2}} \int_0^a \tilde{a}^{\frac{3}{2}} d\tilde{a} \\ &= a \end{aligned}$$

2.  $\Lambda CDM$ : ~~Mathematica~~  $\Omega_M = n$ ,  $\Omega_{k0} = 0$

$$\begin{aligned} D_f(a) &= \frac{5}{2} \cdot n \cdot \left( n a^{-3} + \frac{(1-n)}{3} \right)^{\frac{1}{2}} \cdot \int_0^a \frac{dx}{x \cdot \left[ x \cdot (x^{-3} + \frac{(1-n)}{3})^{\frac{1}{2}} \right]^3} \\ &= \frac{5}{2} \cdot n \cdot [na^{-3} + 1-n]^{\frac{1}{2}} \cdot \int_0^a \dots \end{aligned}$$

3. DE:

$$D_f(a) = \frac{5}{2} \cdot n \left( n a^{-3} + (1-n) a^{-3(1+w)} \right)^{\frac{1}{2}} \int_0^a \dots$$

$w = -0.5$

$$a_i = a_i^{0.75}$$



# A Review on Inflation & Digr.

Problem: Hubble distance.

Basically, the Hubble distance ~~is~~ (comoving) is

$$d_H = \int_0^{a(t)} \frac{dt'}{a(t')} = \int_0^{a(t)} \frac{dt'}{\frac{da(t')}{a(t')} a(t')} = \int_0^{a(t)} \frac{da(t')}{a(t')^2 H(a(t'))}$$

Since we mostly care about MD, & the Hubble distance

$$d_H = \int_0^{a(t_0)} \frac{da(t')}{a(t')^2 H(a(t'))} + \int_{a(t_0)}^{a(t)} \frac{da(t')}{a(t')^2 H(a(t'))}$$
$$= d_H(t_0) + \int_{a(t_0)}^{a(t)} \frac{da(t')}{a(t')^2 H(a(t'))}$$
$$= H(t_0) \int_{a(t_0)}^{a(t)} \frac{da(t')}{a(t')^2 H(a(t'))}$$

Should we use Hubble distance or sound horizon?

At first, when ~~out~~. Sound horizon is used to describe the ~~as~~ acoustic oscillations in baryon-photon plasma. The ~~evolution~~ evolution of the perturbation is driven by gravitation and pressure which is travelling with the speed of light.

Here the comoving distance ~~is~~ actually ~~refered~~ refers to the comoving ~~gen~~ distance grow after inflation, approximately standed by stand by a integration from 0 ~~is the~~ of  $\frac{dt}{a(t')}$  in a model without inflation.

Examine this integration carefully:

Equality :  $a \sim 10^{-4} \Rightarrow 10^2 \gtrsim 10^2$

Since  $\Omega_{\text{rad}} \sim 10^{-4} \sim 10^{-2}$ , the effect of  $\Omega_{\text{rad}} a^{-4}$  is comparable with the effect of matter  $\Omega_{\text{mat}} a^{-3}$

$$\Omega_{\text{rad}} a^{-3} = 10^{-9}, (10^{-3})^4 = 10^{-12}$$
$$(10^{-4})^3 = 10^{-12}, (10^{-4})^4 = 10^{-16}$$

$\Omega_{\text{rad}} / \Omega_{\text{mat}} \approx 10^{-3}$  In order to eliminate radiation term,  $a$  should be ~~too large~~ larger than  $10^{-3}$ .

### Eqn set of Late time

$$\left\{ \begin{array}{l} \dot{\delta}_{\text{dm}} + ikv_{\text{cdm}} = -3\dot{\varphi} \\ v_{\text{cdm}} + \frac{\dot{a}}{a} v_{\text{cdm}} = -ik\dot{\varphi} \\ \dot{\delta}_b + ikv_b = -3\dot{\varphi} \\ v_b + \frac{\dot{a}}{a} v_b = -ik\dot{\varphi} \end{array} \right.$$

with metric

$$g_{\mu\nu}^{\text{FLRW}} = -1 - 2\Phi(\vec{x}, t)$$

$$g_{ij}(\vec{x}, t) = D$$

$$g_{ij}(\vec{x}, t) = a^2 \delta_{ij} (1 + 2\Phi(\vec{x}, t))$$

Late times

$$+ \frac{i}{R} [D_b + 3iB_b]$$

interaction between photon and electron and other  
at late times, ~~if~~  $I \ll 1, i \ll 1$

$$\Rightarrow \left\{ \begin{array}{l} \dot{\delta}_{\text{dm}} + ikv_{\text{cdm}} = -3\dot{\varphi} \\ v_{\text{cdm}} + \frac{\dot{a}}{a} v_{\text{cdm}} = +ik\dot{\varphi} \\ \dot{\delta}_b + ikv_b = -3\dot{\varphi} \\ v_b + \frac{\dot{a}}{a} v_b = ik\dot{\varphi} \end{array} \right.$$

Since,  $P_m - \bar{P}_m = \bar{P}_{\text{cdm}} \delta_{\text{cdm}} + \bar{P}_b \delta_b$

$$\delta_m = \frac{P_m - \bar{P}_m}{P_m} = \frac{\bar{P}_{\text{cdm}} \delta_{\text{cdm}} + \bar{P}_b \delta_b}{P_m}$$

?

$$\ddot{\delta_m} + 2H\dot{\delta_n} - 4\pi G \bar{P}_m \cdot \delta_m = 0$$

• stands for derivative  
to time +  $\ddot{x}$

Change to derivative with respect to  $a$ .

$$\frac{d^2\delta_m}{da^2} + \left( \frac{d\ln(H)}{da} + \frac{3}{a} \right) \frac{d\delta_m}{da} - \frac{3\Omega_{m0} H^2}{2a^5 H^2} \delta_m = 0$$

$$A(a, H) \cancel{\delta_m} / A(a, \dot{a}, \ddot{a})$$

$$B(a, H) / B(a, \dot{a}, \ddot{a})$$

$$\ddot{\delta_m} + A(a, H) \dot{\delta_m} + B(a, H) \delta_m = 0$$

可有精确解：(先待解再接着面方程解)

$$\delta_m = \frac{5}{2} \Omega_m \frac{H(a)}{H_0} \int_a^{a_{in}} \frac{da'}{a' (a' H(a')/H_0)^3} + C$$

设定初始条件： $\Rightarrow a_{in}$  时刻， $\delta_m = \delta_{in}$

猜得一个特解为  $H$ 。（用 Mathematica 把  $H$  代入  $\ddot{\delta_m}$  求解）

$$\text{Then } \delta_m = H \left[ C + C \left( \frac{1}{H^2} e^{-\int A(a, H) da} \right) \right]$$

$$= H \cdot G + C \int_{a_{in}}^a \frac{1}{a^3 H^3} da$$

$\Rightarrow$  在 MD 遵守时

初始条件：①  $a = a_{in}$ ,  $\Rightarrow \delta_m = \delta_{in}$

②  $a$  是 in MD,  $\Rightarrow H = H_0 \sqrt{\Omega_m a^2}$ ,  $\delta_m \propto a$

~~①:  $\delta_m(a_{in}) = C_1 H = \delta_{in} \Rightarrow C_1 = \frac{\delta_{in}}{H(a_{in})}$~~

~~②:  $\delta_{in}(a) = \delta_{in} + \frac{2}{5} C_1 \cdot a^{-\frac{3}{2}} \frac{1}{H_0^2 \Omega_{m0}} \left. \left\{ a^2 \right\} \right|_{a_{in}}$~~

~~$= \delta_{in} + \frac{2}{5} C_1 \frac{1}{H_0^2 \Omega_{m0}} a - \frac{2}{5} C_1 \frac{1}{H_0^2 \Omega_{m0}} a_{in}$~~ 
 ~~$\frac{H \delta_{in} - \frac{3}{5} C_1}{H_0^2 \Omega_{m0}} a_{in} = 0 \Rightarrow C_1 = \frac{5}{2} \frac{\sin H_0^2 \Omega_{m0}}{H(a_{in})} H(a_{in})$~~

~~$\delta_m = \frac{\delta_{in} H_0^2}{H(a_{in})} + \frac{5}{2} \frac{\sin H_0^2 \Omega_{m0}}{H(a_{in}) H(a)} \int_{a_{in}}^a \frac{1}{a^3 H^3} da$~~

~~$= \delta_{in} \left( \frac{H_0^2}{H(a_{in})} + \frac{5}{2} \frac{H_0^2 \Omega_{m0}}{H(a_{in}) H(a)} \int_{a_{in}}^a \frac{1}{a^3 H^3} da \right) = C_2(a)$~~

However,  $H$  is in a decaying mode, as time goes, it will disappear. So actually most of the time (when  $a$  is much larger than  $a_{in}$ ) it can be dropped. This is the solution of the ~~fact~~ function.

As it is related to  $a_{in}$  in the lower limit of the integral, we would like to find another kind of solution.

Here we have two special solutions:

$$\tilde{\delta}_{m,1} \approx H, \quad \tilde{\delta}_{m,2} = H(a) \int_0^a \frac{1}{a^3 H^2} da$$

Then  $\tilde{\delta}_m = C_1 H + C_2 H(a) \int_0^a \frac{1}{a^3 H^2} da$   
as  $H$  is decaying.  $\tilde{\delta}_m = C_2 H(a) \int_0^a \frac{1}{a^2 H^2} da$   
the lower limit  $a$  can be freely chosen.  
actually,

Then we know  $\tilde{\delta}_m$  (when  $a$  is in MP).  $\tilde{\delta}_m \approx a$  (proportional, perturbation.)

$$\begin{aligned} \tilde{\delta}_m(a) &= a \Rightarrow C_2 H(a) \int_0^a \frac{1}{a^2 (H(a) \sqrt{R_{mo} a^3})^2} da = a \\ \Rightarrow C_2 H(a) \int_0^a \frac{1}{H_0^2 R_{mo}^2 a^{\frac{3}{2}}} da &= a \Rightarrow C_2 H(a) \frac{1}{H_0^2 R_{mo}^2} \frac{2}{5} a^{\frac{5}{2}} = a \\ \Rightarrow C_2 &= \frac{2}{5} \cdot H_0^2 R_{mo} \\ \tilde{\delta}_m(a) &= \frac{2 R_{mo}}{5} \frac{H(a)}{H(a) H_0} \int_0^a \frac{1}{a^3 (H/H_0)^2} da \end{aligned}$$

Problem: (1) When we set  $\tilde{\delta}_{in}(a_{in}) = \tilde{\delta}_{in}$ , for different mode,  $\tilde{\delta}_{in}$  is different for  $a_{in}$  is different for each mode. Because there is no freedom to adjust.  $\frac{\tilde{\delta}(a)}{\tilde{\delta}_{in}} = \frac{\tilde{\delta}_m(a)}{\tilde{\delta}_m(a_{in})} = \frac{P_+(a)}{P_+(a_{in})}$  is not right for this expression is should only be valid for  $a \gg a_{in}$ .

④ Let's go back to  $\delta_m = C H + C H(a) \int_0^a \frac{1}{a'^3 H^3} da'$

Since  $H$  is decaying, drop  $C H$ , then  
 $\delta_m = C H(a) \int_0^a \frac{1}{a'^3 H^3} da'$

We have an initial condition.  $\delta_m(a_i) = \delta_{in}$

$$C H(a_{in}) \int_0^{a_{in}} \frac{1}{a'^3 H(a')^3} da' = \delta_{in}$$

$$C = \frac{\delta_{in}}{H(a_{in})} \frac{1}{\int_0^{a_{in}} \frac{1}{a'^3 H(a')^3} da'}$$

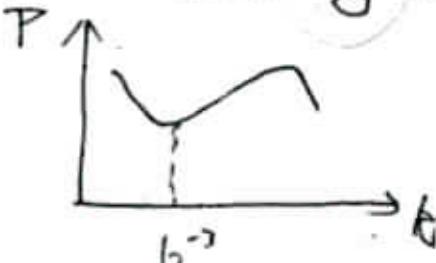
To simplify it, define  $D_+(a) = \frac{1}{H(a) H_0^2} \int_0^a \frac{1}{a'^2 H(a')^2} da'$ .  
 and hence we have

$$C = \delta_{in} \frac{\frac{1}{2} D_{in} H_0^2}{D_+(a_{in})}$$

Finally, the perturbation at any  $a$  is

$$\delta_m(a) = \delta_{in} \frac{1}{D_+(a)} D_+(a)$$

Why the power calculated by the CMBEASY is like this



$a \rightarrow t_1 -$

Since we ~~are~~ introduced scale ~~free~~ power (that is  $P(k) \propto k^{-3}$  is scale free, equivalently  $\delta$  is scale free).

$a \sim 10^{-3} \rightarrow k \sim 1$  (FIR)  $\rightarrow$  Growth factor with amplitude  $\propto \delta$  by at least 100 times

$a \sim 1 \rightarrow k \sim 10^{-4} - 10^{-3}$  (since  $P \propto \frac{1}{k^3}$ , this will be much larger than  $a \sim 10^{-3}$  !)

There is a mistake here!  $\delta_{in}$  is not the real initial condition from end of inflation. Instead, this should be negative.

Related to the evolution after this solution begins to fluctuate

Or substitute an with  $a_0$  which is the current day scale factor

$$\delta_m(a) = \delta_0 \frac{D_+(a)}{D_+(a_0)}$$

$a_0$ : current day

$\delta_0$ : current perturbation

# ABOUT THE PRIMORDIAL PERTURBATION

1. { CMB fluctuations to probe the density perturbation  
 @ on scales larger than  $100 \text{ Mpc}$  (comoving)  
 Luminous galaxy distribution fluctuation generated  
 by the gravity field. on scales out to  $\sim 100 \text{ Mpc}$   
 (comoving)

2. Scale-invariance corresponds to  $n=1$  for adiabatic (constant entropy) and  $n=-3$  for isocurvature (constant potential) initial density perturbation.

$$P(k) = A k^n$$

A: square of a free normalization parameter

$\rightarrow n=-3$  means  $\delta$  is k independent, isocurvature  
 $n=1$  . . . .  $\delta \propto k^2$  adiabatic

$\delta \propto f(P(k))$

3. If These gravitational waves from inflation would have a significant impact only on large angular scales and are not traced by the large-scale structure observations

4. Primordial BH may behave as CDM.
5. Big Bang nucleosynthesis :  $0.01 < \Omega_{bh}^2 < 0.026$   
D. Tytler, X-M. Fan, S. Burles, Nature 361, 207 (1996)  
(This is old, check the newest result if this is cited)
6. Peacock & Rodds, Mon. Not. R. Astr. Soc. 267, 120 (1994) : model-dependent corrections for redshift distortions for each galaxy power spectrum  
Also (4), (45) Refs. **My Question: Why bias?**  
**Where does this freedom come from?**

The answer:  $\bar{\delta}_{in}$  is normalized

7.

# ABOUT CPL Parameterization!

$$W^{CPL}(z) = W_0 + W_a \frac{z}{1+z} = W_0 + W_a(1-a)$$

Friedmann Eqn  $\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0$

$$\frac{dp}{dt} = -3(1+w(a))\rho \frac{da}{adt}$$

$$\frac{dp}{\rho} = -3(1+w(a)) \frac{da}{a}$$

$$d\ln p = -3(1+W_0+W_a) \frac{da}{a} + 3W_a da$$

$$p = a^{-3(1+W_0+W_a)} + \int (3W_a a + \text{Const}) da$$

$$p = a^{-3(1+W_0+W_a)} e^{3W_a a + \text{Const}}$$

That is to say the CPL dark energy term in Hubble function is  $\Omega_{DE} \cdot a^{-3(1+W_0+W_a)} e^{3W_a a + \text{Const}}$

To find out the [Const], consider the fact that we want

\$\Omega\_{DE} a^{-3(1+W\_0+W\_a)} e^{3W\_a a + \text{Const}}\$ to be just  $\Omega_{DE}$  today. Then the whole term becomes  $\Omega_{DE} a^{-3W_a(1-a)} e^{-3W_a(1-a)}$

Check

When  $a \approx 1$ ,  $e^{-3W_a(1-a)} \sim 1 + 3W_a(1-a) + \frac{1}{2}(3W_a(1-a))^2 + \dots$

$$\text{Thus } \Omega_{DE} a^{-3(1+W_0+W_a)} e^{-3W_a(1-a)}$$

$$\sim \Omega_{DE0} a^{-3(1+W_0+W_a)} \left[ 1 + 3W_a(1-a) + \frac{1}{2}(3W_a(1-a))^2 + \dots \right]$$

$$\sim \Omega_{DE0} a^{-3(1+W_0+W_a)} \cdot (1 + 3W_a(1-a) + \frac{1}{2}(3W_a(1-a))^2 + \dots)$$

$$\sim \Omega_{DE0} a^{-3(1+W_0+W_a)} \left[ (1 - 3W_a) + \cancel{a^{-3(1+W_0+W_a)}} + \cancel{(a^{-1-2(1+W_0+W_a)} + \dots)} \right]$$

$$\text{Change to } z : \Omega_{DE0} \frac{e^{-3W_a(1-z)}}{1+z} \quad \text{expand at } z=0$$

$$\sim \Omega_{DE0} \left[ 1 + (1+2W_a+W_a^2)z + \frac{1}{2}(W_a + 4W_a^2 + W_0 + 4W_0W_a + W_a^2)z^2 + \dots \right]$$

# Introducing DM & DE interaction

0. In an expanding universe. Only matter, no interaction

$$\frac{\partial \rho}{\partial t} + a^{-1} \vec{v} \cdot \nabla \rho = -a^{-1} \nabla \cdot (\vec{v} + aH\vec{r})$$

$$\frac{\partial \vec{v}}{\partial t} + a^{-1}(\vec{v} \cdot \nabla) \vec{v} = -a^{-1} \nabla \Phi - a^{-1} \frac{\nabla \rho}{\rho} - H\vec{v}$$

$$a^{-2} \nabla^2 \Phi = 4\pi G \bar{\rho} P$$

Perturbation eqns.

$$\Rightarrow \frac{\partial \delta_m}{\partial t} + a^{-1} \vec{v}_m \cdot \nabla \delta_m = -a^{-1} (1 + \delta_m) \nabla \cdot \vec{v}_m$$

$$\frac{\partial \vec{v}_m}{\partial t} + a^{-1}(\vec{v}_m \cdot \nabla) \vec{v}_m = -\frac{\nabla \Phi}{a} - \frac{\nabla \bar{\rho}_m}{a \bar{\rho}_m} - H\vec{v}_m$$

$$a^{-2} \nabla^2 \Phi = 4\pi G \bar{\rho}_m \delta_m$$

1. Introduce \* interaction between DE and DM.  
this only affects the conservation equation of each component.

$$a^2 k^2 \dot{\phi} = 4\pi G \sum_i (\Delta_i + \frac{a^2 \bar{\rho}_i^0}{\bar{\rho}_i} \frac{V_i}{k}) \bar{P}_i$$

Subhorizon means  $\frac{aH}{k} \ll 1$ , then

$$\Rightarrow a^2 k^2 \dot{\phi} = 4\pi G \sum_i \Delta_i \bar{P}_i \quad \Delta_i = \bar{\Delta}_i - \bar{\rho}_i \frac{k^2 + R}{k}$$

$$\Rightarrow a^{-2} k^2 \dot{\phi} = 4\pi G \sum_i \bar{\Delta}_i \bar{P}_i$$

2. Approximate:

$$\begin{cases} \frac{\partial \delta_m}{\partial t} = -a^{-1} \nabla \cdot \vec{v}_m \\ \frac{\partial \vec{v}_m}{\partial t} = -\frac{\nabla \Phi}{a} - H\vec{v}_m \\ a^{-2} \nabla^2 \dot{\phi} = 4\pi G \bar{\rho}_1 \bar{\Delta}_1 \end{cases} \Rightarrow \begin{aligned} \ddot{\delta}_m + 2H\dot{\delta}_m - a^2 \nabla^2 \dot{\phi} &= 0 \\ \ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G \bar{\rho}_1 \bar{\Delta}_1 &= 0 \end{aligned}$$

Well, next, two things should be done next:

① express  $\delta_{de}$  in terms of  $\delta_m$

② find out how does interaction change Hubble eqn with the effective  $W$ .

②

$$\begin{cases} \dot{P}_c + 3H P_c = a \cdot 3H \cancel{\delta}(P_c + P_d) Q_c \\ \dot{P}_d + 3H(1+w) P_d = a Q_d \end{cases}$$

choose:  $Q_c \cancel{Q} = -Q_d = Q = 3\delta H P_d$

$$\Rightarrow \begin{cases} \dot{P}_c + 3H P_c = a \cdot 3\delta H P_d & (1) \\ \dot{P}_d = -a \cdot 3\delta H P_d & (2) \end{cases}$$

$$(2) \Rightarrow P_d = e^{-3\delta a + \text{const}} \quad \text{Apply } P_d(a=1) = \Omega_{DE0} \text{ physical}$$
$$\text{const} = 3g + \ln \Omega_{DE0} P \quad \text{Denote } p_{\text{initial}} = p_0$$
$$P_d = \Omega_{DE0} P^c \cdot e^{-3\delta(a-1)}$$

The dark energy term in Hubble equation

$$\Omega_{DE} e^{-3\delta(a-1)}$$

1. CPL参数化值  $w_0 + w_a(1-a)$

( $w_0$ ,  $w_a$ )

2. CPL - - -  $D_f$  上去 ~~看~~ 对  $P$  影响不大?

3.  $\Phi$  LCDM  $\Phi$   $\Omega_m$ ,  $\Omega_b$  和  $P(k)$  的影响.

## About Non linear effect.

Linear effect happens at about ~~at~~  $a = \text{a few}$  below  $10 \text{ Mpc}$ .

$$a = 0.1 \quad 1$$

$$k_{\text{fwhm}} = 0.005 \quad 0.003$$

$$\lambda > 10^2 \text{ Mpc} \quad > 10^3 \text{ Mpc}$$

Thus non linear effects won't effect this calculation.

Comments on the first period results:

1. The power spectrum goes up at small  $k$ . Reasons are given:

Those smaller ~~blocks~~ do not ~~fit~~ go in to horizon.

$$P \sim \frac{1}{k^2} \delta_{\text{lin}}, \delta_{\text{lin}} \text{ independent of } k.$$

And a comment for the equation (1) and (2) in P. Bhattacharya's

It's only right before matter enters the equality.

At DE domination,  $D_+ \sim a^{-2}$ ,  $H \sim \text{const}$

## Gauge

### \* Conformal Newtonian Gauge

$$ds^2 = \alpha^2(z) [-(1+2\psi)dt^2 + (1-2\psi)\delta_{ij}dx^i dx^j]$$

Also called longitudinal gauge.

- Slicing & threading : orthogonal
- No shear ( $E_{ij} = 0$ )

### \* Total matter gauge ~~& comoving gauge~~

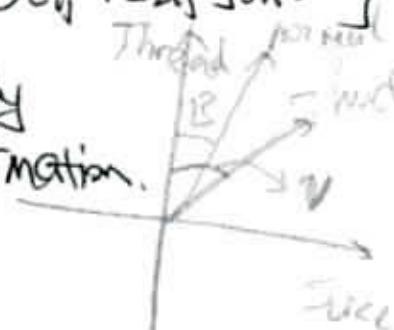
$$V = B \quad \tilde{V} = V - \frac{k}{a}$$

C I & LSS by Andrew etc. P43

$$\delta x = 0, \quad \delta t = \frac{V}{k}$$

$$ds^2 = \alpha^2(z) \{ -(1+2A)dt^2 - B^2 dx^i dx^i + [(1+2D)\delta_{ij} + 2E_{ij}] dx^i dx^j \}$$

\* Check Page 340 of CI ~~& LSS by~~ Andrew etc for the gauge transformation.



# Diffusion & Freestreaming at the Universe

\* Conformal Newtonian Gauge here.

\* Eqs:

$$\bullet \text{DM: } \begin{cases} \frac{\partial \delta_c^N}{\partial z} = -kV_c + 3\frac{\partial \Psi}{\partial z} \\ \frac{\partial V_c}{\partial z} = -\alpha HV_c + k\Psi \end{cases}$$

No coupling because no interaction/weak interaction

\* Particles (including ~~top~~ electrons)

$$\begin{cases} \frac{\partial \delta_e^N}{\partial z} = -kV_b + 3\frac{\partial \Psi}{\partial z} \\ \frac{\partial V_b}{\partial z} = -\alpha HV_b + G^2 k \delta_e^N + k\Psi + \frac{4P_F}{3\rho_0} \alpha v_{eT} (V_r - V_b) \end{cases}$$

\* Gas.

$$\left[ \frac{\delta T}{T} \right]_{SW} = \frac{1}{3} \Psi(\hat{e} \chi_w) + 2 \int_{z_0}^{z_0} \frac{\partial \Psi(x, z)}{\partial z} dz$$

Sachs-Wolfe effect of the freestreaming photons. All due to  ~~$\Psi(x, z)$~~   $\Psi$

\* Complete system of equations:

PS6E 15.6 Initial conditions and the transfer function

\* Transfer function  $T(k)$  in (5.4)

$$R_k^{(m)} = f(k) R_k$$

$$\delta_k(t) = \frac{3}{5} \left( \frac{k}{aH} \right)^2 T(k) R_k$$

Ref:

Cosmological Inflation & Large-Scale Structure by Andrew et al. - Chap: 15

- ①  $P_0 = n T_b$  in validation
  - equilibrium
  - Boltzmann
  - non-relativistic

②  $\Psi(z_0)$  doesn't help for the anisotropy

# A simple analysis

~~DE-matter equality~~:  $t_0$

Nowday  $t$ .

$$\begin{aligned}\Delta(k_e, t_0) &= \left(\frac{D_f(t_0)}{D_f(t_e)}\right)^2 \Delta(k_e, t_e) = \left(\frac{D_f(t_0)}{D_f(t_e)}\right)^2 \Delta(k, t) \\ &= \left(\frac{D_f(t_0)}{D_f(t_e)}\right)^2 \left(\frac{D_f(t)}{D_f(t_e)}\right)^2 \delta(k, t_0) \\ &= \left(\frac{D_f(t)}{D_f(t_e)}\right)^2 \Delta(k, t_0)\end{aligned}$$

Remember  $t$  is corresponds to the  $k$  mode, i.e.  $k$  mode enters horizon at  $t$ .  
 $t$  is not arbitrary

This relation gives us the information about different modes today.

- a. if  ~~$t \leq t_e, t_{eq} < t \leq t_c$~~  (during the after RM equality and before Dark energy-matter eq.)

~~but~~

- b. if  $t \geq t_e$ , i.e. after DE-matter equality.

$$\frac{D_f(t)}{D_f(t_e)} \approx \left(\frac{a}{a_e}\right)^{\frac{3}{1+3w}} \quad \left(\frac{a}{a_e}\right)^{\cancel{\frac{3}{1+3w}}} \cancel{H^{\frac{3}{1+3w}}}$$

$$\cancel{H(a) \sim a^{-\frac{3}{1+3w}}} \rightarrow k \sim aH \sim a^{1-\frac{3}{1+3w}} \left(\frac{a_e}{a}\right)^{\frac{3(1+3w)}{1+3w}}$$

$$\frac{D_f(a)}{D_f(a_e)} \sim \left(\frac{k_e}{k}\right)^{\cancel{\frac{3}{1+3w}}} \left(\frac{a_e}{a}\right)^{\cancel{\frac{3}{1+3w}}} \left(\frac{k_e}{k}\right)^{\frac{3(1+3w)}{1+3w}}$$

This means if we are going to calculate the future power spectrum, then the small  $k$  goes down as ~~small  $k$~~  becomes smaller.

- c. if  ~~$t \leq t_e, t < t_m$~~

$$\frac{D_f(a)}{D_f(a_{eq})} \sim \left(\frac{a}{a_{eq}}\right)^{\frac{1}{1+w}}$$

$$H \sim a^{-\frac{3}{2}} \rightarrow k \sim aH \sim a^{-\frac{1}{2}}$$

$$\frac{D_f(a)}{D_f(a_{eq})} \sim \left(\frac{k_e}{k}\right)^{\frac{3}{2}}$$

$$\frac{P}{P_{eq}} \sim \left(\frac{k}{k_e}\right)$$

$$\frac{P}{P_{eq}} \sim \left(\frac{k_e}{k}\right)^{\cancel{\frac{1}{1+w}}} \cancel{H^{-1-w}}$$

$$\sim \left(\frac{k}{k_e}\right)^{\cancel{\frac{3}{2}}} \left(\frac{k}{k_e}\right)^{\frac{3}{1+w}}$$

$t_{eq}$ : Equality of matter and radiation

If  $\dot{y}_M$ : Still matter evolution!  
 $w=0$ !!  
 Wrong!

the problem is if we are going to calculate the monodromy power spectrum, all the growth factors will remain 1 because nothing is growing and the equation for

It should be noticed that ~~for~~ those modes that is outside of horizon now, ~~the difference~~ the gauge difference between gauges counts.

## INCLUDING THE DE in Heszardos $\delta$

Background:

$$(\bar{\rho}_d + \bar{\rho}_m) + 3(\bar{\rho}_d + \bar{\rho}_m + \bar{P}_d + \bar{P}_m)H = 0.$$

$$\frac{d}{dt}(\bar{\rho}_d + \bar{\rho}_m) = -3(H\dot{w})\bar{\rho}_d H - 3\bar{\rho}_m H$$

$$\Rightarrow \text{With } \delta = \frac{\rho_d + \rho_m - \bar{\rho}_d - \bar{\rho}_m}{\bar{\rho}_d + \bar{\rho}_m} - 1 \\ \dot{\delta} + \alpha^1 \vec{V} \cdot \nabla \delta = -\alpha^1(1+\delta) \vec{V} \cdot \nabla \vec{V} + (H\delta) \frac{3H\dot{w}\bar{\rho}_d}{\bar{\rho}_d + \bar{\rho}_m}$$

For any component with a EoS  $w$ ,

$$\dot{\delta}_x + \alpha^1 \vec{V}_x \cdot \nabla \delta_x = -\alpha^1 (\epsilon_x + 1) \vec{V}_x \cdot \nabla \vec{V}_x$$

$$\delta_x = \frac{\rho_x - \bar{\rho}_x}{\bar{\rho}_x}$$

A little time : ~~don't use~~  $\bar{\rho} = \bar{\rho}_d + \bar{\rho}_m - \bar{\rho}_d \bar{\rho}_m$

If  $\bar{\rho}_d a^{3(1+w)} = \text{const 1}$ .  $\bar{\rho}_m a^3 = \text{const 2}$   
and assume  $\bar{\rho}_d = \bar{\rho}_d$ .  $\bar{\rho}_d a^{3(1+w)} = \lambda \bar{\rho}_m a^3$   
 $\bar{\rho} = \frac{1}{1+\lambda a^{3w}} \bar{\rho}_m$ ,  $\bar{\rho}_d = \lambda \bar{\rho}_m a^{-3w}$

$$\dot{\delta}_m + \cancel{a^{-1} \vec{V} \cdot \nabla \delta_m} = -a^{-1} \nabla \vec{V} (1 + \delta_m + \lambda a^{3w}) \\ + \frac{2 + \lambda a^{3w} + \delta_m}{1 + \lambda a^{3w}} 3Hw \lambda a^{3w} + \cancel{\frac{\lambda a^{3w+1} \dot{a}}{1 + \lambda a^{3w}}}$$

$\lambda$  is constant., Since dark energy is only behave as a background, then  $\vec{V} = \vec{V}_m$ .

I don't think this equation works because

Take into account of nonlinear effect.

$\delta^2 \sim 1$  is the criteria.

Apply the approximation!

- ① Perturbations are small
- ② Ignore second order terms

$$\dot{\delta}_m = -(a^{-1} \nabla \vec{V}) (1 + \cancel{\delta_m}) + \frac{2 + \lambda a^{3w} + \delta_m}{1 + \lambda a^{3w}} 3Hw \lambda a^{3w} \quad ①$$

$$\frac{\partial \vec{V}_m}{\partial t} + a^{-1} (\vec{V}_m \cdot \nabla) \vec{V}_m = -\frac{\nabla \phi}{a} - \frac{T(\bar{\rho}_m + \bar{\rho}_d)}{a(\bar{\rho}_m + \bar{\rho}_d)} \vec{H} \vec{V}_m$$

With  $\bar{\rho}_d = \bar{\rho}_d = \lambda \bar{\rho}_m a^{3w}$  and  $\nabla \bar{\rho}_m = 0$   
Then

$$\frac{\partial \vec{V}_m}{\partial t} + a^{-1} (\vec{V}_m \cdot \nabla) \vec{V}_m = -\frac{\nabla \phi}{a} - H \vec{V}_m$$

Approximately,

$$\frac{\partial \vec{V}_m}{\partial t} = -\frac{\nabla \phi}{a} - H \vec{V}_m \quad ②$$

Of course the Poisson equation is

$$\alpha^{-2} \nabla^2 \Phi = 4\pi G \bar{\rho}_m \delta_m \quad (3)$$

(Only consider the  $\phi$   $\rho - \bar{\rho} = \rho_m - \bar{\rho}_m + \rho_d - \bar{\rho}_d$   
 $= \rho_m - \bar{\rho}_m = \bar{\rho}_m \delta_m$ , background is  
included in Friedmann eqn.)

$$\textcircled{1} \rightarrow \dot{\delta}_m = -\theta + \frac{2 + \lambda a^{3w} + \dot{\Delta}_m}{1 + \lambda a^{3w}} \approx H\lambda a^{3w}, \quad \theta = \alpha^{-1} \nabla^2 V$$
$$\textcircled{2} \rightarrow \dot{\theta} + 2H\theta + 4\pi G \bar{\rho}_m \delta_m = 0$$

Then

$$\ddot{\delta}_m + (2H - M_2) \dot{\delta}_m - (2HM_2 + 4\pi G \bar{\rho}_m + \dot{M}_1) \delta_m = 2H M_1 + \dot{M}_1$$

in which  $M = 3HW\lambda a^{3w}$

$$= 3HW\lambda a^{3w} \frac{2 + \lambda a^{3w}}{1 + \lambda a^{3w}} + \frac{3HW\lambda a^{3w}}{1 + \lambda a^{3w}} \delta_m$$
$$= M_1 + M_2 \delta_m$$

This is the  
equation  
for matter  
perturbation  
in de  
universe.

$$\ddot{\delta}_m + \frac{3\lambda w a^{3w-1} \dot{a} \dot{\delta}_m}{1 + \lambda a^{3w}} + a^{-1} \delta_m \nabla^2 \vec{V} - \frac{3HW\lambda a^{3w}}{1 + \lambda a^{3w}} \delta_m$$

$$+ a^{-1} \vec{V} \nabla \delta_m = -a^{-1} (1 + \lambda a^{-3w}) \nabla \vec{V} + \frac{3HW\lambda a^{3w}}{1 + \lambda a^{3w}}$$

$$+ \frac{3\lambda w H a^{3w}}{1 + \lambda a^{3w}} \delta_m - \frac{3HW\lambda a^{3w}}{1 + \lambda a^{3w}} \delta_m + a^{-1} \delta_m \nabla \vec{V}$$

$$+ a^{-1} \vec{V} \nabla \delta_m = -a^{-1} (1 + \lambda a^{-3w}) \nabla \vec{V} + 3HW\lambda a^{-3w}$$

$$\ddot{\delta}_m + a^{-1} \vec{V} \nabla \delta_m + a^{-1} \delta_m \nabla \vec{V} = -a^{-1} (1 + \lambda a^{-3w}) \nabla \vec{V} + 3HW\lambda a^{-3w}$$

Apply the approximation that  ~~$\vec{v}$~~  and  ~~$\vec{a}$~~  are both small, keep only the first order of them.

$$\dot{\delta}_m = -\alpha^{-1}(1+\lambda a^{-3w}) \nabla \vec{v} + 3Hw\lambda a^{-3w} \quad (2)$$

Peculiar velocity equation

$$\frac{d\vec{v}_m}{dt} + \alpha^{-1}(\vec{v}_m \cdot \nabla) \vec{v}_m = -\frac{\nabla \Phi}{a} - H\vec{v}_m \quad (3)$$

Approximately,

$$\frac{d\vec{v}_m}{dt} = -\frac{\nabla \Phi}{a} - H\vec{v}_m \quad (4)$$

Poisson equation is

$$a^2 \nabla^2 \Phi = 4\pi G \bar{\rho}_m \delta_m \quad (5)$$

Denote  $\Theta = \alpha^{-1} \nabla \vec{v}$ , the (2) becomes

$$\dot{\delta}_m = -(H\lambda a^{-3w}) \Theta + 3w\lambda H a^{-3w}$$

(4) becomes

$$\dot{\theta} + 2H\theta + 4\pi G \bar{\rho}_m \delta_m = 0$$

Finally,

$$\begin{aligned} \dot{\delta}_m &+ \cancel{\frac{3w\lambda a^{-3w} H + 2H + 2F\lambda a^{-3w}}{H\lambda a^{-3w}}} \cancel{\dot{\delta}_m H \left( \frac{3w\lambda a^{-3w}}{1+\lambda a^{-3w}} + 2 \right)} \\ &- (1+\lambda a^{-3w}) 4\pi G \bar{\rho}_m \delta_m = 3w\lambda a^{-3w} \left( 2 - \frac{3w}{1+\lambda a^{-3w}} \right) H^2 \\ &\quad + 3w\lambda H a^{-3w} \end{aligned}$$

Will this one be true?

# Gauge :

+ Ref 1. Physical Foundations of Cosmology  
By Mukhanov

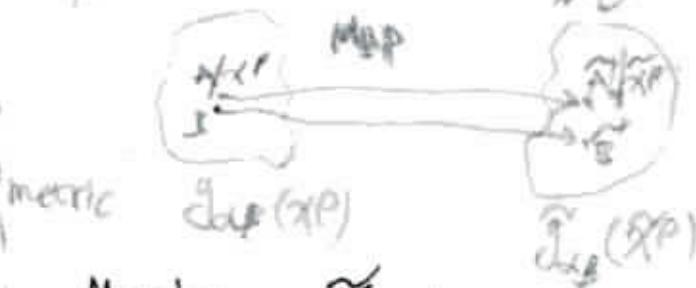
Step 1. Coordinate transformation.

$$x^\alpha \rightarrow \tilde{x}^\alpha = x^\alpha + \xi^\alpha$$

Step 2. Change in metric of a point  $\tilde{x}^P$

$$\tilde{g}_{\alpha\beta}(\tilde{x}^P) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\mu\nu}(x^P) \quad (1)$$

manifold  $M$



Metric on  $M'$  changed because this map may change the relative position of the ~~vicinity~~ adjacent points. So we use  $\tilde{g}_{\alpha\beta}(\tilde{x}^P)$  to stand for the metric at point  $\tilde{x}^P$  on the new manifold.

$$\begin{aligned} 0 \rightarrow \tilde{g}_{\alpha\beta}(\tilde{x}^P) &= g_{\alpha\beta}(x^P) + \cancel{\frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu}} (\delta_{\alpha\mu} - \xi^\gamma_{,\alpha})(\delta_{\beta\nu} - \xi^\delta_{,\beta}) g_{\mu\nu}(x^P) \\ &= g_{\alpha\beta}(x^P) + \xi^\mu_{,\alpha} \xi^\nu_{,\beta} g_{\mu\nu}(x^P) - \cancel{\xi^\mu_{,\mu} \xi^\nu_{,\nu}} g_{\mu\nu}(x^P) - \cancel{\xi^\mu_{,\alpha} \xi^\nu_{,\beta}} g_{\mu\nu}(x^P) \end{aligned}$$

~~Cancel off~~ Get rid of high orders of  $\xi^\alpha$ ,  $g_{\alpha\beta}$ . Then

$$\tilde{g}_{\alpha\beta}(\tilde{x}^P) \approx g_{\alpha\beta}(x^P) - \cancel{\xi^\mu_{,\alpha} \xi^\nu_{,\beta} g_{\mu\nu}(x^P)} - \cancel{\xi^\mu_{,\alpha} \xi^\nu_{,\beta} g_{\mu\nu}(x^P)}$$

$$\approx \underline{g_{\alpha\beta}(x^P) + \delta g_{\alpha\beta}} - \cancel{\frac{1}{2} g_{\mu\nu}(x^P) \xi^\mu_{,\alpha} \xi^\nu_{,\beta}}$$

order 1, no higher orders

(2)

~~take~~ Step 3: Look at this  $\tilde{g}_{\alpha\beta}(\tilde{x}^\mu)$

In perturbation theory  $\rightarrow$

$$\tilde{g}_{\alpha\beta}(\tilde{x}^\mu) \sim {}^{(0)}g_{\alpha\beta}(\tilde{x}^\mu) + \delta g_{\alpha\beta} \quad (3)$$

in which

- a.  ${}^{(0)}g_{\alpha\beta}(\tilde{x}^\mu)$  is the background. Background metric ~~shouldn't be changed~~ under such coordinate transformations (or this transformation is really bad and makes the calculation complex)

This is the reason for the notation  $g_{\alpha\beta}(\tilde{x}^\mu)$  without a tilde  $\tilde{\ }$

- b.  $\delta g_{\alpha\beta}$  is the perturbation of  $\tilde{g}_{\alpha\beta}$

Step 4. Insert a relation here:

$${}^{(0)}g_{\alpha\beta}(\tilde{x}^\mu) \approx {}^{(0)}g_{\alpha\beta}(\tilde{x}^\mu) - {}^{(1)}g_{\alpha\beta,r} \{^r \quad (4)$$

Step 5. Compute from  $\textcircled{1} \otimes \textcircled{4}$ ,

$$\delta \tilde{g}_{\alpha\beta} = \delta g_{\alpha\beta} - {}^{(1)}g_{\alpha\beta,r} \{^r - {}^{(1)}g_{\alpha\beta,\mu} \{^\mu - {}^{(1)}g_{\alpha\beta} \}^\mu, \mu$$

This method can be used to calculate the transformation for scalar and vector.

Step 5 get the form of transformation of the metric.

~~Further~~ Further more,  $\xi^d = (\xi^0, \xi^i)$  and  $\xi^i = \xi_L^i + \xi_{\perp}^i$  ( $\xi_L^i$  is the transverse part and  $\xi_{\perp}^i$  is the scalar part )  
 $\xi_{\perp i} = 0$ ;

And use these new variables, we get new expressions (7.16 in ref 1)

Finally, for scalar perturbation with ~~met~~

$$ds^2 = a^2(z) \left\{ -(1+2A)dt^2 - B_i dx^i dt + [(1+2D)\delta_i + 2E_{ij}] dx^i dx^j \right\}$$

We have

$$\tilde{A} = A - \frac{1}{2}a(\dot{a}\xi^0)$$

$$\tilde{B} = B + \dot{\xi} + k\xi^0$$

$$\tilde{D} = D - \frac{1}{3}k\dot{\xi} - \cancel{aH\xi^0}$$

$$\tilde{E} = E + k\xi$$

Some additional notes:

1. The perturbation eq calculated directly from Einstein eqn do not satisfy the condition that coordinate transformation changes the form of the equation
2. Large Scale Structure by Andrew

Gauges from Longitudinal to synchronous

Longitudinal :  $\Phi = \phi_c, \Psi = \psi_c,$   
 $B_i = E_i = 0, \delta \sim D_c$

Synchronous :  $\delta_c = \tilde{\delta} - \frac{\dot{P}}{\rho a} \int a \dot{\Phi} dt$   
 $\Phi = \tilde{\Phi} \cdot \frac{1}{a} (a \dot{E}_c)$

$$\dot{P}_m = -3(H+w)\bar{P}_m H, \bar{P}_m = \Omega_{mo} \bar{L}_c = \frac{\Omega_{mo} 3H_0^2}{8\pi G}$$

$$\delta_c = \tilde{\delta} + \frac{3H}{a} \frac{\Omega_{mo} 3H_0^2}{8\pi G} \int a \dot{\Phi} dt$$

$$\tilde{\delta} = \delta_c + \frac{3H}{a} \frac{9H_0^2 H S_{mo}}{8\pi G} \int \dot{\Phi} dt$$

Arxiv: astro-ph/9401007 :

Synchronous : time is ~~synchronised~~ synchronised  
for different observers.

$$\begin{cases} \dot{\delta} = -(1+w)(\theta + \frac{h}{2}) - 3\frac{\dot{a}}{a} \left( \frac{\delta P}{\delta \rho} - w \right) \delta \\ \dot{\theta} = -\frac{\dot{a}}{a} (1-3w)\theta - \frac{w}{1+w}\theta + \frac{\delta P / \delta \rho}{1+w} k^2 \delta - k^2 \theta \end{cases}$$
$$\Rightarrow \begin{cases} \dot{\delta} = -\theta - \frac{1}{2}h \\ \dot{\theta} = -H\theta - k^2 \theta \end{cases}$$
$$\begin{cases} k^2 \eta - \frac{1}{2}Hh = 4\pi G a^2 \delta \rho T \\ h + 2Hh - 2k^2 \eta = -8\pi G a^2 \delta T \end{cases}$$

set  $\Theta = \Omega = 0$ , for this because if we can choose it to define coordinates thus no peculiar velocity

$$\vec{\delta} = \vec{e} - \frac{1}{2}\vec{h}$$

from 20b & 20d  $\vec{h} = \vec{a} - \frac{1}{2}\vec{a}^2$

$$\vec{\delta} \sim \vec{k}^2 \vec{\eta} = 0$$

$$\vec{h} + \vec{h}'' + 2H(\vec{h} + \vec{h}'') - 2\vec{k}\vec{\eta} = 0$$

$$\Rightarrow \vec{h} + 2H\vec{h} - 2\vec{k}\vec{\eta} = 0 \Rightarrow \vec{h} + 2H\vec{h} - 2\vec{k}^2\vec{\eta} = 0$$

$$\therefore \vec{h} = 2\vec{k}^2 \int a^{-2} dt + a^{-2} \vec{C}$$

$$\vec{\delta} = \frac{1}{2}\vec{C}/a^2 dt - \frac{1}{2} \int \vec{k}^2 \vec{\eta} a^{-2} \int a^2 dt dt$$

## Progress of Theoretical Physics Supplement No. 7E, 1984

### Cosmological Perturbation Theory

Part A : Perturbed tensors

$$\tilde{g}_{\mu\nu} = -a^2 [1 + 2A] \gamma$$

$$\tilde{g}_{ij} = -a^2 B \gamma_j$$

$$\tilde{g}_{ij} = a^2 [K_{ij} + 2H_a Y_{ij} + 2H_T Y_{ij}]$$

Scalar type component of vectors

$$Y_i = -k^i Y_{12}$$

$$(D + k^2) \gamma = 0$$

Tensors:

$$Y_{ij} = k^{-2} (Y_{ij} - \Lambda^{ij} Y_{12} \Delta \gamma) = (k^2 Y_{12} + \Lambda^{ij} Y_{ij}) \gamma$$

$$Y_{ij} = -(2k)^{-1}(Y_{ij}^{(1)} + Y_{jk}^{(1)})$$

That's vector-type component

$$\begin{cases} (0k^2)Y_i^{(1)} = 0 \\ Y_{kk}^{(1)} = 0 \end{cases}$$

Tensor-type (divergenceless & traceless second rank symmetric tensors):  $Y_{ij}^{(2)}$

$$(\Delta + k^2)Y_{ij}^{(2)} = 0 \Rightarrow$$

$$Y_{ij}^{(2)} = 0$$

$$Y_{ij}^{(2)} = 0$$

Part B

Using Perturbations of the metric  $\tilde{g}_{ij}$ .  
Using the decompositions, the perturbed metric is

$$\tilde{g}_{00} = -\alpha^2 [1 + 2AY]$$

$$\tilde{g}_{ij} = -\alpha^2 BY_{ij}$$

$$\tilde{g}_{ij} = \alpha^2 [Y_{ij} + 2H_L Y_{ij} + 2H_T Y_{ij}]$$

$$\text{Inverse } g_{ij} \tilde{g}^{ij} = -\alpha^2 [1 + 2AY]$$

$$\tilde{g}^{ij} = -\alpha^2 BY^{ij}$$

$$g^{ij} = \alpha^{-2} [Y^{ij} - 2H_L Y^{ij} - 2H_T Y^{ij}]$$

- ① Solar: 2AY
- ②  $\tilde{g}^{ij}$ : Vector-like
- ③  $2H_L Y_{ij}$ : trace part scalar-like
- ④  $2H_T Y_{ij}$ : tensor-like / traceless

Some comments: Conventionally in canonical quantization of gravity, the metric is decomposed

$$\text{into } g_{\mu\nu} dx^\mu dx^\nu = (-N^2 + f_k f_k) dt^2 + \sum_k f_k dx^k$$

$f_k dx^k$ . In this decomposition,  $N$  is

called the lapse function (because this causes some dilation in time),  $f_k$  are the shift functions

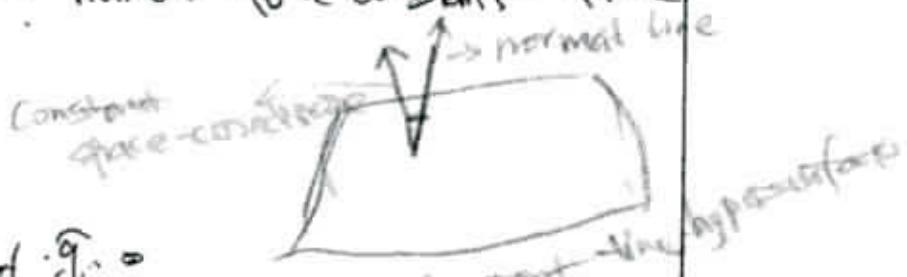
So  $A$  is the amplitude of the perturbation in the lapse function.

## Interpretation:

- 1: amplitude of perturbation to the lapse function
- 2: amplitude i.e., between two neighboring constant time hypersurface, ~~A~~ stand the ratio of proper time to coordinate time
- 3: amplitude of

1) perturbation in the shift function (vector)

i.e.: the rate of deviation of a constant space-coordinate line from a line normal to a constant time hypersurface.



$H_2$ : Since trace of  $\tilde{g}_{ij}$  =

$$\text{Tr}[\tilde{g}_{ij}] = -a^{-2}(1 - 2nH_2Y) \text{Tr}[g_{ij}]$$

n+1 dimension

Interpretation:

## Part C . Dealing with the Energy momentum tensor

1. Perturbed ~~matter~~ velocity ~~vector~~  $\tilde{u}^\mu$

$$\tilde{T}^\mu_{\nu}, \tilde{u}^\nu = -\tilde{\rho}\tilde{u}^\mu \quad (\text{eigen function, eigenvalue is perturbed proper density})$$

$$\tilde{u}_\mu \tilde{u}^\mu = -1 \quad (\text{normalization})$$

Using the perturbed metric

$$(\tilde{u}_\mu \tilde{u}_\nu \tilde{g}^{\mu\nu} = -1 \Rightarrow \tilde{u}_\mu \tilde{u}^\nu + 2\tilde{u}_\mu \tilde{u}^\lambda \tilde{g}^{\mu\lambda} + \tilde{u}_\mu \tilde{u}^\lambda \tilde{g}^{\lambda\mu} = -1)$$

$$\Rightarrow -a^{-2}(1 - 2AY)\tilde{u}_\mu \tilde{u}^\mu = -2a^{-2}B^2 \tilde{u}_\mu \tilde{u}^\mu + a^{-2}[Y^2(2H_2Y^2 - 2H_1Y^1)]\tilde{u}_\mu \tilde{u}^\mu = -1$$

Scalar part  $\tilde{u}^\mu$  shouldn't be related to vector or tensor parts. thus  $\tilde{u}_\mu \tilde{u}^\mu = -a^{-2}(1 - 2AY)^{-\frac{1}{2}} \Rightarrow \tilde{u}^\mu = a^{\frac{1}{2}}(1 - AY)^{-\frac{1}{2}}$

~~time-like~~

expand to 1st order.



Similarly, vector part:  $\tilde{U}_i = \alpha(v - B) Y_i$ , (to finish off)  
 in which  $v$  is defined as  $\tilde{U}^0 \tilde{U}^i = \tilde{U}^i / \tilde{U}^0 = v Y^i$   $\rightarrow$  decomposition

2. The ~~sp~~ normal to  $U^i$  part of the energy-momentum tensor:

$$C-④ \quad \tilde{T}_{\mu\nu} = \tilde{P}_\mu^\lambda \tilde{P}_\nu^\beta \tilde{T}_{\lambda\beta}, \quad \tilde{P}^\mu_\nu = \delta^\mu_\nu + \tilde{U}^\mu \tilde{U}_\nu$$

$$\tilde{U}^\mu \tilde{T}_{\mu\nu} = 0$$

$$\tilde{T}^0_0 = 0$$

$$\tilde{T}^0_j = \cancel{\rho} P(v - B) Y_j$$

$$\tilde{T}^j_0 = -\cancel{\rho} P v Y_j$$

$$\tilde{T}^i_j = \tilde{T}^i_j$$

Is the projection tensor.

$$\tilde{P}^0_0 = 0$$

$$\tilde{P}^0_j = (v - B) Y_j$$

$$\tilde{P}^j_0 = -v Y^j$$

$$\tilde{P}^i_j = \delta^i_j$$

$\Rightarrow$  in which  $P$  is some scalar.

$\tilde{T}^i_j$  can be decomposed

$$T^i_j = P[\delta^i_j + \Pi_L \delta^i_j + \Pi_T Y^i_j]$$

C-⑤

Interpretation:

\*  $\Pi_L$ : From C-⑤; we can directly understand it as ~~the~~  $\Pi_L$  is isotropic.

$\Pi_L$  is the isotropic pressure perturbation amplitude

The pressure perturbation

$$\tilde{P} \equiv \frac{1}{n} \tilde{T}^0_0 = P(1 + \Pi_L)$$

\*  $\Pi_T$ : Anisotropic pressure perturbation's amplitude

3. Finally, we can write down the Energy-momentum tensor:  $\tilde{T}^\mu_\nu = \tilde{P}^\mu_\nu \tilde{U}_0 + \tilde{T}^0_\nu$  (from C-④)

Only one quantity is left now:  $\tilde{P}$ .

In which  $\delta$  is the ~~the~~ amplitude of density perturbation,

Finally:  $\tilde{T}^0_0 = -P[1 + \delta Y]$  And: ~~only~~

$$\tilde{T}^i_j = (\rho + \tilde{P})(v - B) Y^i_j$$

$$\tilde{T}^i_j = -(\rho + P) v Y^i_j$$

$\delta, v, \Pi_L, \Pi_T$   
 are complete to describe  
 a certain matter's energy momentum.

39

Interpretation:

\* The previous derivation is only for scalar perturbation because only scalar-like decomposition are involved? that is,  $\Gamma$ ,  $Y_{ij} = -k^i Y_{ij}$ ,  $Y_{ij} = k^i k^j Y_{ij}$

Part D.

Gauge invariant variables.

in MT  
space-time

$$\Delta = \Delta s + n(HW)(a'/a)k^i V \\ = \delta + n(HW)(a'/a)k^i (w - B)$$

$w = c_s$

$k \equiv \Delta Y_j = -[k^2 - (n-1)K] Y_j$

Prime is  
 $\frac{d}{dt}$

$K$  is the curvature term.  $(\frac{\partial}{\partial t})^2 \frac{\partial}{\partial x} = \frac{\partial^2}{\partial t^2}$

$$\Delta'' = \left\{ n(2w - c_s^2) - 1 \right\} \frac{a'}{a} \Delta' + n \left[ \left\{ \frac{n}{2} w^2 - (n-1)w \right\} \frac{a'}{a} \right. \\ \left. - \frac{n-2}{2} + n c_s^2 \right] \left( \frac{a'}{a} \right)^2 + \frac{nw^2 + 1 - 3n}{2} K + \frac{k^2 - nK}{n} c_s^2 \Delta = \Psi$$

$$\Psi = -(K^2 - nK)wT - (n-1) \left( 1 - \frac{nK}{K^2} \right) \frac{a'}{a} w T' + \left[ \left\{ n(w^2 + c_s^2) \right. \right. \\ \left. \left. - 2w \right\} \left( \frac{a'}{a} \right)^2 + w(nw + n-1)K + \frac{k^2 - nK}{n} c_s^2 \right] \left( 1 - \frac{nK}{K^2} \right) (n-1)T$$

D-0

$\Gamma = \Gamma_L - \frac{c_s^2}{w} \delta$ ,  $\Pi = \Pi_R$

$\bar{\Pi}_L = \Pi_L + \frac{c_s^2}{w} n(1+u) \frac{a'}{a} T$ ,  $\bar{\Pi}_T = \Pi_T$

\* Synchronous gauge:  $A = B = 0$  a special occasion of proper time slicing  
still leaves a freedom  $\circlearrowleft$   $\Gamma_{15}$   $A = 0$

Fundamental variables:

$$h_L, H_T, V, \delta, \Gamma, \Pi$$

Relations are

$$h_L = 2nH_L, A = B = 0$$

Ans

$$\text{Deflection: } V = v - k^{-1} H'_T \quad (D-1)$$

Put (D-1) into the following

$$V' + \frac{a'}{a} V = -k \left[ \frac{n-1}{n-1} \chi^2 \frac{a^2 p}{k^2 n k} - \frac{c_s^2}{H w} \right] \Delta + k \frac{w}{H w} \Gamma - k \left[ \frac{n-1}{n} \left( 1 - \frac{n k}{R^2} \right) \frac{1}{H w} + \chi^2 \frac{a^2 p}{k^2} \right] w \Pi$$

in which  $\chi^2$  is from  $\delta G^a v = \chi^2 \delta T^a v$

$$c_s^2 = \frac{\delta p}{\delta p}, \quad \Gamma = \Pi_L - \frac{c_s^2}{w} \delta, \quad \Pi = \Pi_T$$

$$\Pi_L = \Pi_{L0} + \frac{c_s^2}{w} n (1+w) \frac{a'}{a} \Gamma, \quad \Pi_T = \Pi_T$$

We get

$$v' + (1 - n c_s^2) \frac{a'}{a} v = k \frac{c_s^2}{H w} \delta + k \frac{w}{H w} \left[ \Gamma - \frac{n-1}{n} \left( 1 - \frac{n k}{R^2} \right) \Pi \right] \quad \text{Main Eq 1}$$

(D-1)

We have a deflection for

$$\begin{aligned} \Delta &= \delta_s + n(1+w) \left( \frac{a'}{a} \right) k^{-1} v \\ &= \delta + n(1+w) \frac{a'}{a} k^{-1} (v - B) \end{aligned} \quad (D-2)$$

Put (D-1) and (D-2) into

$$\delta - n w \frac{a'}{a} \Delta = - \left( 1 - \frac{n k}{R^2} \right) (H w) k v - (n-1) \left( 1 - \frac{n k}{R^2} \right) \frac{a'}{a} w \Pi$$

We get

$$\begin{aligned} \delta' + n(c_s^2 - w) \frac{a'}{a} \delta' &= (H w) \left[ -k^2 + \frac{n}{n-1} (1+w) \chi^2 p a^2 \right] \frac{1}{k} \\ &+ \left( 1 - \frac{n k}{R^2} \right) (H w) H'_T - n w \frac{a'}{a} \Gamma \end{aligned} \quad (D-3)$$

In this equation, there are terms including  $H'_T$ ,

to eliminate  $H'_T$ , put (D-2) and

$$A = - \left( \frac{a'}{a} \right)^{-1} R' + \left( \frac{a'}{a} \right)^{-2} \left\{ \left( \frac{a'}{a} \right)^2 - \left( \frac{a'}{a} \right)^{22} \right\} R$$

$$\text{into } k \frac{a'}{a} A - \left\{ k - \chi^2 \frac{a^2 h}{n-1} \right\} B = \chi^2 \frac{a^2 h}{n-1} v \quad (\text{this is } \delta q_i = \chi^2 \delta T_i)$$

with  $\mathcal{B} = k\left(\frac{a}{a'}\right)^2 R - \frac{1}{k} H_T^2$  and  $\mathcal{L} = \frac{1}{2n} h_L + \frac{1}{n} H_T$   
 we get:

$$\frac{1}{2} h_L' + \left(1 - \frac{nK}{R^2}\right) H_T' = -\frac{n}{n-1} \gamma^2 h a^2 \frac{v}{k} \quad (D-4)$$

Put this into (D-3) again,

$$\delta + n(s^2 - w)\frac{a'}{a}\delta = - (Hw) \cdot \left(kv - \frac{1}{2} h_L'\right) - nw \frac{a'}{a t} \quad \text{Main Eq 2}$$

We have to find another main equation.

$$(4.3) \quad \gamma^2 \rho \delta = (n-1) a^{-2} (k^2 - nK) \bar{\phi}$$

$$(4.4) \quad (n-2) \bar{\Psi} + \bar{\Psi} = - \gamma^2 a^2 K^2 P T \Pi \quad | \Rightarrow$$

$$(17) \quad \bar{\Psi} = - \frac{1}{K^2} (H_T'' + \frac{a'}{a} H_T')$$

$$H_T'' + \frac{a'}{a} H_T' = \frac{n-2}{n-1} \frac{\gamma^2 \rho a^2}{1 - nK/k} [\delta + n(Hw) \frac{a'}{a} \frac{v}{k}] + \gamma^2 \rho a^2 \Pi \quad (D-5)$$

Put (D-5) and (D-6) and

$$1.23 \quad \frac{1}{2} h_L' + \left(1 - \frac{nK}{R^2}\right) H_T' = - \frac{n}{n-1} \gamma^2 h a^2 \frac{v}{k}$$

into (D-3), we know

$$h_L'' + \frac{a'}{a} h_L' = - \frac{n-2+nK^2}{n-1} \gamma^2 \rho a^2 \delta - \frac{2n}{n-1} \gamma^2 \rho a^2 \Pi \quad \text{Main Eq 3}$$

Main Eq 1, 2, 3 are the complete set of fundamental perturbation equations in synchronous gauge.

Simplification under <sup>3+1</sup> flat universe  $K=0, n=3$   
 adiabatic perturbations  $\frac{\delta P}{P} = \frac{\delta \rho}{\rho}$   
 matter  $w=0, P=0$

$$\begin{cases} v' + \frac{a'}{a} v = 0 \\ \delta' = -(kv - \frac{1}{2} h_L') \\ h_L'' + \frac{a'}{a} h_L' = - \gamma^2 \rho a^2 \delta \end{cases}$$

At superhorizon limit.  $k\eta \ll 1$ , drop all  $k$  terms

$$\delta'' + \frac{a'}{a}\delta' - \frac{1}{2}\chi^2\rho a^2\delta = 0, \quad \chi^2 = 8\pi G$$

When  $a \rightarrow \infty, a \rightarrow t, \delta \rightarrow 0$

~~PT~~ explains the

P.S. DE:  $a \sim a(t_m) \exp[(t-t_m)\sqrt{8\pi G P_0/3}]$

See the  
solution of this equation  
on the next page

Gauge invariant variables and its evolution.

D-~~O~~ is reduced to a simple eqn within the following conditions:

$n=3$  (~~NP~~ 3+1 formalism)

$K=0$  (flat universe)

~~large scale~~ adiabatic perturbation:  $\frac{\partial p}{\partial \rho} = \frac{\dot{p}}{\dot{\rho}}$   
matter.  $w=0, p_{\text{co}}$

$$\Delta'' + \frac{a'}{a}\Delta' + 3\left(-\frac{1}{2}\frac{a'}{a}\right)^2\Delta = 0$$

$$\Delta'' + \frac{a'}{a}\Delta' - \frac{3}{2}\left(\frac{a'}{a}\right)^2\Delta = 0 \Rightarrow \ddot{\delta} + \frac{2\dot{a}}{a}\dot{\delta} - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2\delta = 0$$

Two solutions for this equation when  $t \rightarrow \infty$  and

$$a(t) = a(t_m) \exp[(t-t_m)\sqrt{8\pi G P_0/3}]$$

$$= \exp[Bt] \cdot M$$

$$B = \sqrt{8\pi G P_0/3}, \quad M = a(t_m) \exp\left(-t_m\sqrt{8\pi G P_0/3}\right)$$

$$\text{then } C = \frac{3}{2}B^2 > 0$$

General solution:

$$\Delta = C_1 e^{ct} + C_2 e^{-ct-2Bt}$$

$$C = \frac{-B \pm \sqrt{B^2 + 4C}}{2}, \quad C_1, C_2 \neq 0$$

$$\Delta = e^{\frac{1}{2}[-B - \sqrt{B^2 + 4C}]t} N[1] + e^{\frac{1}{2}[B + \sqrt{B^2 + 4C}]t} N[2]$$

~~N[1] and N[2] are the integration constants.~~

~~At large  $t$ ,~~

$$\Delta \rightarrow e^{[-B + \sqrt{B^2 + 4C}]t} N[2]$$

~~and this one goes up at large scale.~~

12

# PART E. Perturbation Evolution.

\* Synchronous Gauge

~~After~~ After horizon crossing.

$$\delta'' + \frac{a'}{a} \delta' + \frac{1}{2} \gamma^2 \rho a^2 \delta = 0 \quad (\text{Same as large scale})$$

$$\delta'' + H(\delta' - 4\pi G \rho a^2 \delta) = 0 \Rightarrow \ddot{\delta} + 2H\dot{\delta} - 4\pi G P \delta = 0$$

A particular solution is  $\text{Const. } H$ .

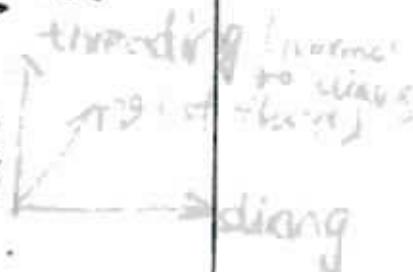
Similar to what we did under Newtonian gauge.

$$\bar{\delta}_m = c_1 H + c_2 H(a) \int_{a_0}^a \frac{1}{H(a')^2 a'^3} da'$$

@ Synchronous gauge

Time ~~is~~ has no dilation  $\pm$  and normal to

Synchronous ~~slicing~~ with background



\* ~~Notation:~~ Ruth define power spectrum of DM

$$P_D(k) = \langle |D_g^{(m)}(k, \eta_*)|^2 \rangle$$

in which  $D_g = \delta^{(\text{long})} - 3(H_W)\phi = \delta + B(H_N)(H_B + \frac{1}{3}H_D)$ ,

~~it corresponds to~~ is Bardeen potential

$$\begin{aligned} \delta &= H_B + N^{-1} H_D + k^{-1} \left( \frac{a'}{a} \right) \left( B - k^{-1} H_D \right) \\ \delta^{(\text{long})} &= D_S \end{aligned}$$

In Synchronous gauge,  $B=0$ .

$$H_T'' + \frac{a'}{a} H_T' = \frac{1}{2} \frac{\gamma^2 \rho a^2}{\gamma^2 \rho a^2} \left[ \delta + 3 \frac{a'}{a} \frac{\eta}{k} \right] + \gamma^2 \rho a^2 \Pi$$

make ~~to~~  $\Pi = \pi_T = 0$

$$\text{Solve } \ddot{\nu} + H\nu = 0 \Rightarrow \nu = \text{Const } e^{-\int H dt} = C e^{-t}$$

\* Review the power spectrum definition.

① correlation function:

$$\cancel{P(k)} \leftarrow \langle \delta(k), \delta(k') \rangle P(k) \delta(k) = \frac{\langle \tilde{\delta}(k) \tilde{\delta}(k') \rangle}{(2\pi)^3}$$

$\langle \cdot \rangle$  denotes ensemble average

② Observables?

What is observable? Matter perturbation?

③ Gauge invariant & density perturbation

$$D_g = \delta + \beta(1+w)(H_L + \frac{1}{3}H_T)$$

$$= \delta^{(\text{long})} - \beta(1+w)\beta$$

In which  $\delta$  is Bardeen potential

and  $\Phi = k \cancel{H} H_L + n' H_T + k^{-1} \left( \frac{a'}{a} \right) (B - k' H_T')$

$$= k^{-1} \left( \frac{a'}{a} \right) B$$

Prime stands  
for  $\frac{d}{d\eta} \rightarrow$  conformal  
time.

Use gauge invariant variable to  
define quantities, leave all the other  
work to astrophysicists.

LET'S CHECK WHAT DOES

Wittenberg say in his cosmology (2008?)

~~Variables~~:

$\eta$ : is the expandable parameter in Fourier Integral.

$\eta$  is actually the comoving number,  $\frac{\eta}{a}$  is the physically wave number. (In the way  $H$  is some kind of physical numbers)

Theoretically, as I derived previously, the perturbation eqn is

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G \bar{\rho}\delta$$

$$\Delta_{ij} = \cancel{\beta \delta_{ij}} + \beta \delta_{ij}(t_L) + (1-\beta)\delta_{ij}(t_0)$$

$$- t_0 \psi_q(t_L) + \beta t_0 \left( \frac{q^2}{a^2} \right) \Delta U_{qj}(t_L)$$

in which  $\beta = \frac{\Omega_B}{\Omega_M} \doteq \frac{1}{6}$ ,  $t_L$  is for last scattering

Switch to Weinberg's notations

Synchronous Gauge:

$$g_{00} = -1, g_{0i} = 0, g_{ij} = a^4 [(\Lambda A) \delta_{ij} + \frac{\partial B}{\partial x^i x^j}]$$

\*: Kodama's  $\rightarrow 2D \Rightarrow A, E \rightarrow B, E_{ij} \rightarrow \frac{\partial B}{\partial x^i \partial x^j}$

$$\Psi = \frac{1}{2} B \dot{A} + \nabla^2 B \int \frac{\partial (h_{ii})}{\partial t} \left( \frac{h_{ii}}{2a^2} \right) \quad \text{It's equation is}$$

$$\therefore \frac{\partial}{\partial t} (a^2 \Psi) = -4\pi G a^4 (\delta\rho + 3\delta P + \nabla^2 \Pi^S)$$

$$\Rightarrow \frac{\partial (a^2 \Psi)}{\partial t} = -4\pi G a^4 (\delta\rho_{12} + \delta P_{12} + 2\delta\rho_{22} + 2\delta P_{22})$$

Early times ~ Model in W's book.

$$\left\{ \begin{array}{l} \dot{\delta}_q = -\psi_q \\ \frac{d}{dt} \left( \frac{\delta U_{qj}}{3\pi} \right) = -\frac{1}{3\pi} \delta_q \\ \frac{d}{dt} (t \psi_q) = -\frac{1}{t} \delta_q \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \ddot{\delta}_q + \frac{1}{t} \dot{\delta}_q - \frac{1}{t^2} \delta_q = 0 \\ \dot{\psi}_q = -\dot{\delta}_q \end{array} \right.$$

Solutions are

$$\delta_q = C_{s1} t + C_{s2} \cos \frac{1}{t} \quad \text{Remove decaying mode}$$

And normalise  $\psi_q = \frac{-q^2 R_0}{a^2}$  this according to

the constrain that  $q^2 R_0^2 = -a^4 \psi_q + 4\pi G a^2 \delta_q + q^2 \delta_{qj}$   
Should be time independent outside of horizon

This part will be used to cal the initial for later universe.

$$\delta_{1g} = \delta_{Rg} = \delta_{2g} = \delta_{Dg} = \frac{q^2 t^2 R_g^0}{a^2} = \delta_g$$

$$\psi_g = -\frac{t q^2 R_g^0}{a^2}$$

$$\delta U_{1g} = \delta U_{2g} = -\frac{2 t^2 q^2 R_g^0}{9 a^2}$$

And Why should  $q^2 R_g^0$  be time-independent over horizon scale?

## UNDER NEWTONIAN GAUGE

constraint:  $\tilde{\rho} \delta p - 3H a^4 (\tilde{\rho} + \tilde{p}) \partial t - \left(\frac{a}{4\pi G}\right) \nabla^2 \tilde{\psi} = 0$

We construct two quantities:

\*:  $R_g = -\psi_g + H \delta U_g$

\*:  ~~$\tilde{\rho}$~~   $S_g = -\psi_g + \frac{\delta p}{3(\tilde{\rho} + \tilde{p})} = \psi_g - \frac{q^2 \psi_g}{12\pi G(\tilde{\rho} + \tilde{p}) a^2}$

①  $R_g$  is conserved "outside of the horizon in

② Adiabatic condition ensures that the field equations always have solutions ~~of~~ outside the

HORIZON, regardless of what constituents the universe ~~has~~ has.

That is to say, solutions exists which makes  $R_g$  is time-independent when  $q \ll aH$  (Also tensor mode  $D_g$ , check the book if useful)

Put  $R_g$  and  $S_g$  into other gauges.

$$R_g = \frac{1}{2} A_g + H \delta U_g$$

$$S_g = \frac{1}{2} A_g - H \frac{\delta p}{\tilde{p}}$$

calculate under the same gauge.

Or, under synchronous gauge:

$$q^2 S_g = -a^2 H \psi_g + 4\pi G a^2 \delta p_g - q^2 H \delta A_g / \tilde{p}$$

$$q^2 R_g = -a^2 H \psi_g + 4\pi G a^2 \delta p_g + q^2 H \delta U_g$$

General eqn

Cosmology,  
Weinberg, 2008

Outside of the horizon., Adiabatic Modes.

$$\begin{cases} \dot{\delta}_{Bq} - (q^2/a^2) \delta U_{Rq} = -4q \\ \dot{\delta}_{Bq} - (q^2/a^2) \delta U_{Rq} = -4q \end{cases} \Rightarrow \frac{d(\delta_{Rq} - \delta_{Bq})}{dt} = 0$$

If initially,  $\delta_{Rq} = \delta_{Bq}$ , then this property will beerved all the way down (thereafter)

② Discuss the modes that after M-R equality and still outside of horizon.

$$q \ll aH, \quad \delta_{Bq} = \delta_{Vq}.$$

$$\begin{cases} \frac{d}{dt}(a^2 \dot{\delta}_q) = -4\pi G a^2 \left[ \bar{P}_D \dot{\delta}_q + \left( \bar{P}_E + \frac{8}{3} \bar{P}_R \right) \delta_{Vq} + \frac{8}{3} \bar{P}_R \dot{\delta}_{Vq} \right] \\ \dot{\delta}_{Vq} = \dot{\delta}_{Uq} = \dot{\delta}_{Bq} = -4q \\ \frac{d}{dt} \left( \frac{(1+R) \delta_{Uq}}{a} \right) = -\frac{1}{3a} \dot{\delta}_{Vq} \\ \frac{d}{dt} \left( \frac{\delta_{Uq}}{a} \right) = -\frac{1}{3a} \dot{\delta}_{Vq} \\ R = \frac{3 \bar{P}_B}{4 \bar{P}_E} \rightarrow \text{Baron} \\ \rightarrow \text{Radiation} \quad \bar{P}_R = \bar{P}_P + \bar{P}_N, \quad \bar{P}_U = \bar{P}_E + \bar{P}_D \end{cases}$$

Substitution:  $y = \frac{a}{a_{eq}} = \frac{\bar{P}_U}{\bar{P}_R}$

$$\delta^{(1)} = y^{-2} (1 + 4y - 2y^2 + y^3)$$

$$\delta^{(2)} = y^2 \sqrt{1+y}$$

General solution should be a mixing of  $\delta^{(1)}, \delta^{(2)}$

$$\delta = C_1 \delta^{(1)} + C_2 \delta^{(2)} \quad \text{at } y \rightarrow 0 \quad (\text{ass, or } t \rightarrow 0)$$

~~radiation era~~ to fit with the mode 1 in W's book. (and should be normalized to be like mode 1, which is the general solution)

$$\Delta q = \frac{4q^2 R_q}{3H_{\infty}^2 a_{\infty}^2 y_2} \left( (6 + 8y - 2y^2 + y^3) / 6 \ln(y) \right)$$

" $R_q$ " is the time independent value of  $R_q$  outside of horizon.

## Power Spectra.

$$\Delta_{Mq}(t) = A(q) F(t)$$

$\Delta(q)$  is some kind of initial ~~contribution~~ value

$F(t)$  is the ~~time~~ evolution.

Thus  $\Delta(q)$  can be set to the LSS/recombination.

$$\Delta(q) = f \delta_{rel}(t_c) + (1-f) \Delta_{Dq}(t_c) + t_c \psi_q(t_c) + \beta t_c (q^2/a_c^2) \Delta_{Ly\alpha}(t_c)$$

$$\text{In which } \beta = \frac{\bar{P}_q}{P_m} = \frac{S_q}{S_m} \approx \frac{1}{6}$$

Leaving  $F(t)$  satisfies :

$$\frac{d}{dt} [a^2 \frac{d}{dt} F(t)] = 4\pi G a^2 \bar{P}_M F(t) \Rightarrow F(t) = \frac{3}{5} \frac{(a/t)^2}{(\ln(a/t))^3} T(x)$$

Defination:  $P(k) = (2\pi)^3 a_0^3 F^2(t_0) |\delta(a_k)|^2$   
present power spectral function  $P(k)$

$T(x)$  is correction factor with  $T(0)=1$

Since ① reduces to

$$\Delta(q) = \Delta_{Dq}(t_c) - t_c \psi_q(t_c)$$

when neglecting  $\beta$ . (which is about  $\frac{1}{6}$ )

$$\Delta(q) = \frac{3q^2 t_c^2 (R_q^0 T(x))}{2 a_c^2} = \frac{2q^2 R_q^0 T(x)}{3H_c^2 a_c^2}$$

$$\text{So } k = \sqrt{2} \frac{q}{a_c H_c}$$

$$P(k) = \frac{4(2\pi)^4 a_c^2 \bar{C}^2(x, a_m)}{25 S_m^2 H_c^4} \frac{R_q^0}{R_q^0} k^{3/2} T^2(k/k_m)$$

$$\bar{C}(x) = \frac{5}{6} x^{-5/6} \int_{1/x}^x \frac{du}{u^{5/6} (1+u)^{5/2}}$$

Q.E.D.

$$P_R \approx N \cdot q^{-\frac{3}{2}} (q/q_0)^{(n_s-1)/2} \xrightarrow{\text{spectral index } n_s \text{ and constant } N}$$

$$n_s = 1 \rightarrow$$

$$P(k) = \text{Const} \cdot k^2 (\bar{k}/k_{B0})$$

where  $\bar{T}^2(\bar{k}/k_{B0})$  is transfer func.

$$\bar{k} = \frac{q J_2}{a_0 H_{B0}}$$

Transfer function.

$T \rightarrow 1$  when  $\bar{k} \ll 1$

$$T(k) \rightarrow \frac{45}{2k^2} \left[ -\frac{1}{2} + \gamma + \ln\left(\frac{4k}{\pi}\right) \right] \quad \text{when } \bar{k} \gg 1,$$

$$T(k) \underset{20 \ll k}{=} \begin{cases} 1 & \bar{k} \ll 1 \\ \frac{45}{2k^2} \left[ -\frac{1}{2} + \gamma + \ln\left(\frac{4k}{\pi}\right) \right] & \bar{k} \gg 1 \end{cases}$$

$\gamma = 0.572 \dots$   
is the Euler Constant

Bogolyubov  
Cosmology

At large scales ( $q \ll 1 / k \ll 1 / \bar{k} \ll 1$ )

$$P(k) \sim k$$



Switch back to Kodama

in k space.

We need dimension free power spectrum: Then we

make gauge invariant variation of Einstein Eqs.: we

$$\Delta_0 = c_1 H + c_2 H \int_0^a \frac{1}{\alpha^3 H^2} d\tilde{a}$$

To normalise the solution and eliminate the integration constants, we need to make the time independent, so we

Box

have to calculate the early evolution of  $\Delta_R$  (for matter). Since we are using adiabatic condition,  $\Delta_R = \Delta_{mR} = \Delta_{BR} = \Delta_{GR} = \Delta_{2R} = \Delta_{DR}$   
 $R$  stands for invariant wave number.  $k = \frac{a}{\lambda}$  is physical.  
 Outside of the horizon,  $G_S = 0$  for matter. Then the evolution of  $\Delta_R$  (which is in space) is the same as the late time one.

$$\ddot{\Delta} + 2\frac{\dot{a}}{a}\dot{\Delta} - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 \Delta = 0$$

growing mode  $\Delta \propto \sqrt{2Bt}$ ,  $B = \sqrt{\frac{8\pi P_0}{3}}$ , then  $H = \frac{\dot{a}}{a} = \frac{1}{2t}$

$$\Delta = H \int_0^a \frac{1}{(xH)^3} dx \cdot \text{const}$$

$$= \frac{1}{2t} \int_0^a \frac{1}{x^3} dx$$

$$= \frac{1}{2t} \cdot \frac{1}{2B} \cdot 2Bt = \frac{1}{2} \cdot \text{const}$$

Next, solve the equation for  $\Delta U_i$  in Weinberg's here corresponds to what?

Gauge transformation in Weinberg (I use more general symbol)

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \Delta h_{\mu\nu}$$

$$\begin{cases} \tilde{g}_{tt} = g_{tt} - \frac{\partial \epsilon_0}{\partial t} \\ \tilde{g}_{ti} = g_{ti} - \frac{\partial \epsilon_i}{\partial t} - \frac{\partial \epsilon_0}{\partial x_i} + 2\frac{\dot{a}}{a}\epsilon_i \\ \tilde{g}_{ij} = g_{ij} - \frac{\partial \epsilon_i}{\partial x_j} - \frac{\partial \epsilon_j}{\partial x_i} + 2\dot{a}\dot{a}\delta_{ij} \epsilon_0 \end{cases}$$

Corresponding to what in Redam (in k space)

$$\tilde{g}_{00} = g_{00} - a^2 \cdot 2T \left( \Lambda - T' - \frac{a'}{a}T \right) \rightarrow \epsilon_0 = -aT$$

$$\tilde{g}_{0j} = -a^2 T' - a^2 \left( L' + kT \right) + \tilde{g}_{0j}$$

$$\tilde{g}_{ij} = a^2 g_{ij} + a^2 \left[ 2T_{ij} \left( -\frac{k}{h}L - \left( \frac{a'}{a} \right) T \right) + 2T_{ij}(kL) \right]$$

In order to understand which is which, I have to understand how does the transformation or decomposition works!

Anyway,  $\tilde{R}_g = \frac{1}{3} H^2 + \frac{1}{3} \tilde{\Theta} H_T$

Normalization condition satisfies

$\nabla^2 \tilde{R}_g$  is time independent.

Eqs for matter.

$$V' + \frac{\alpha'}{\alpha} V = \nabla^2 k \Psi +$$

$$V, \Psi$$

$$\Delta' = -kV$$

$$\Delta,$$

Andrew Liddle  
etc.

Cosmological  
Inflation  
and large-  
scale structure

I think I'll do this latter however, where I introduce a simple understanding.

In Newtonian gauge,

$$\tilde{R}_g = \frac{q^2 t^2 R_g}{a^2}$$

$$\gamma$$

$R_g$  is gauge invariant (Waldberg)

$$H_c \propto \frac{1}{t^{1/2}}$$

+ Dimensional analysis.

$$\Rightarrow \tilde{R}_g \sim \frac{q^2 t^2 R_g}{a^2}$$

$$\text{choose } \Delta = \bar{\delta} + 3(1+w)\frac{a'}{a}(B-\bar{B})$$

$$\text{For matter, } \Delta = \bar{\delta} + 3\frac{a'}{a}\frac{1}{k}(V-B)$$

Under any gauge, perturbations inside horizon

$$\Delta \approx \bar{\delta} \quad (\text{since } k \gg aH/k \text{ is large})$$

Eq for  $\Delta$  inside horizon is

$$\square \quad \ddot{\Delta} + 2\frac{a'}{a}\dot{\Delta} - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 \Delta = 0$$

$$D(a) = \frac{5}{2} \Omega_{m0} H(a) H^2 \int_0^a \frac{1}{a^3 H(\tilde{a})^2} d\tilde{a}$$

Calculations:

Parameters:

$$w = -1; \quad \Omega_{m0} = 0.734, \quad \Omega_{k0} = 0, \quad \Omega_{r0} = 0.2646$$

$$\Omega_{m0} = 0.09 \times 10^{-5}, \quad \lambda = 0.71$$

$$\Omega_{\text{M0}} \text{CDM} = 1.0 - \Omega_{\text{m0}} \Omega_{\text{r0}} b / (\text{Hubble H} * \text{Hubble H})$$

- Omega Vacuum - Omega Quintessence - Omega Radiation

\* Why MZ p is right?

Weinberg's theorem!

Neutrinos

\* Power Spectra in gauge

$$\textcircled{2} \quad R_p \propto$$

Weinberg's notation vs Lemaître's

$$W = 51.31 - 51.33$$

$$h_{00} \sim -E_T \quad \xrightarrow{\text{vector field}} \quad h_{00} \sim -\frac{1}{2} E_T$$

$$h_{0i} \sim a[-F_k Y_i - \bar{F}_k \bar{Y}_{\bar{i}}]$$

$$h_{ij} \sim a^2 [(A - \frac{1}{3} a' B) Y_{ij} + B \bar{k} Y_{\bar{i}\bar{j}} + \text{[redacted]}]$$

$$= a^2 2 A Y$$

$$= a^2 \bar{B} Y_{\bar{i}}$$

$$a^2 [2 H_u Y_{ij} + 2 \bar{H}_T Y_{\bar{i}\bar{j}}]$$

齊次齊次

23

Under this mapping from Weinberg to Kodama.

$$\bar{R}_g = H_L + \frac{1}{3}H_T + H\delta u_g \\ = H_L + \frac{1}{3}H_T + H(-\frac{V-B}{k})$$

Weinberg:

$$ST_i^0 := (\rho + p)(\partial_i \delta u) \quad (\text{Dropping vector part})$$

for T expansion

$$\hat{T}_i^0 := (\rho + p)(-\mathbf{k} \cdot \delta u \gamma_i) \rightarrow \rho \delta u_g = -\frac{1}{k}(V-B)$$

$$\hat{T}_i^0 := (\rho + p)(V-B)\gamma_i$$

$$\begin{aligned}\bar{R}_g &= \bar{H}_L + \frac{1}{3}\bar{H}_T + \cancel{\frac{H}{k}B} - \frac{H}{k}\bar{U} \\ &= H_L - \frac{1}{3}kL - \frac{a'}{a}T + \frac{1}{3}H_T + \frac{1}{3}kL + \frac{1}{k}H(B + L' + HT) \\ &\quad - \frac{H}{k}(V + L') \\ &= H_L + \frac{1}{3}H_T + \frac{1}{k}H(B - \cancel{\frac{1}{k}HV} - \frac{1}{3}kL - \frac{a'}{a}T + \frac{1}{3}kL \\ &\quad + \frac{1}{k}HL' + \cancel{\frac{1}{k}HKT}) - \frac{H}{k}L'\end{aligned}$$

Note that Weinberg used

$$ds^2 = -c^2 dt^2 + a^2(dx^2 + dy^2 + dz^2)$$

when calculating the gauge transformation.

Thus all  $a'$  in Kodama should be  $a$  when calculating.

In this way

$$\begin{aligned}\bar{R}_g &= H_L + \frac{1}{3}H_T + \frac{1}{k}HB - \frac{1}{k}HV - \frac{1}{3}kL - \frac{a}{a'}T + \\ &\quad \frac{1}{3}kL + \cancel{\frac{1}{k}HL} + \frac{1}{k}HKT - \frac{H}{k}L' \\ &= \circledast \bar{R}_g\end{aligned}$$

Power Spectrum:

$$P = \langle |D_g|^2 \rangle$$

$$D_g = \delta - 3(H_u + \frac{1}{3}H_T) = \delta - (\beta H_L + H_T)$$

In synchronous gauge,

$$3H_u + H_T = \frac{3}{2} \frac{\chi^2 P a^2}{k} \delta - \frac{1}{2} \frac{a' h'_L}{a} . \quad h_L = 2x_3 H_u$$

$$(h'_L)' + \frac{a'}{a} h'_L = -\chi^2 P a^2 \delta$$

$$(3H_u + H_T)' = -\frac{3}{2} \chi^2 h a^2 \frac{v'}{k}, \quad h = p + \rho = p \text{ for matter}$$

$$v' + 9 \frac{a'}{a} v = 0 \Rightarrow v \propto a^{-1} \rightarrow v = \text{Const.} a^1$$

$$\Rightarrow (3H_u + H_T)' = -\frac{3}{2} \text{Const.} \chi^2 P a^2 \frac{1}{k} \frac{1}{a} = -\frac{3}{2} \text{Const.} \chi^2 P \frac{a}{k}$$

$$= 3H_L + H_T = -\frac{3}{2} \text{Const.} \chi^2 \frac{1}{k} \cdot \int P a dz$$

Synchronous Gauge

Interacting DE model ~~☆☆☆☆☆~~

1111, 3953

$$\ddot{\delta}_c = -2H \left[ 1 - \beta_c(t) \frac{\dot{\phi}}{H \dot{\phi}_0} \right] \dot{\delta}_c + 4\pi g \left[ \rho_b \delta_b + \rho_c \delta_c \left( 1 + \frac{4}{3} \beta_c^2(t) \right) \right]$$

friction

$$\dot{\delta}_c = B \dot{\delta}_c + C \delta_c + D$$

$\Rightarrow B > 0, C > 0, \text{ in home } \delta_c, D > 0, \text{ growth.}$

EZ diagram

$$\ddot{\delta}_c = -2H \left[ 1 - \beta_c(t) \frac{\dot{\phi}}{H \dot{\phi}_0} \right] \dot{\delta}_c + 4\pi g \left[ \rho_b \delta_b + \rho_c \delta_c \left( 1 + \frac{4}{3} \beta_c^2(t) \right) \right]$$

Fifth force  
always positive  
means attractive

and accelerates the growth.



$$\begin{cases} \ln D_a = \ln a(t) \\ \frac{D_a}{a} = \frac{D_a}{a(t)} \end{cases}$$

+ growth  
in atop

Continuous distributed DE. with  $\phi$

Large-scale instability in interacting dark energy and dark matter fluids  
stacks.ipap/JCAP /2008/ i=07/a=020  
sound speed of dark energy :  $P_6$

$$c_{sx}^2 = 1, c_{ax}^2 = W_x = \text{const} < 0$$

Sound speed problem of DE.

~~Formulas~~ Interacting DE (perturbed)

~~(1)~~ 
$$Q_m^v = \begin{bmatrix} \frac{3H}{\alpha^2}(\delta_1 P_m + \delta_2 P_d) \\ 0 \\ \vdots \end{bmatrix}, Q_d^v = \begin{bmatrix} -\frac{3H}{\alpha^2}(S_1 P_m + S_2 P_d) \\ 0 \\ \vdots \end{bmatrix}$$

For the simple case  ~~$\delta_1 = 0$~~ ,  ~~$\delta_2 = 0$~~

$$Q_m^v = \frac{1}{\alpha^2} 3H \delta_2 P_d, Q_d^v = -\frac{1}{\alpha^2} 3H \delta_2 P_d$$

Conservation eqns

$$\delta_m' + k V_m = -3H_L' - \delta_m \frac{1}{P_m} \alpha^2 \frac{3H}{\alpha^2} \delta_2 P_d + \frac{\alpha^2}{P_m} \frac{3H}{\alpha^2} \delta_2 P_d \delta_d$$

$$\delta_m' + k V_m = -3H_L' - 3\delta_2 H \cdot \frac{P_d}{P_m} \cdot \delta_m + 3H (\delta_2 \cdot \frac{P_d}{P_m} \cdot \delta_d + 3\delta_2 \delta_d)$$

$$V_m' + H V_m = -3\delta_2 H \left( \frac{P_d}{P_m} \right) V_m$$

$$V_d' + H((1-3w_d) V_d) = \frac{k}{1+w_d} \delta_d - 3\delta_2 H V_d$$

$$V_d' + H((1-3w_d + 3\delta_2) V_d) = \frac{k}{1+w_d} \delta_d$$

$$\frac{P_d}{P_m} = \lambda$$

$$\begin{aligned}\delta_m'' + H(1+3\delta_2)\lambda\delta_m' &= -3H_L'' - 3H_L' H(1+3\delta_2)\lambda + 3\delta_2[H(\lambda)(\delta_d - \delta_m)]' \\ &\quad + 3\delta_2 H(\lambda)(\delta_d - \delta_m) \cdot H(1+3\delta_2) \\ \delta_d'' + 3(1-W_d)(H\delta_d)' + H(1-3W_d+3\delta_2)\delta_d' + 3H^2(1-3W_d+3\delta_2)\delta_d(1-W_d) \\ &\quad - k^2\delta_d - 3(1+W_d)H_L'' - 3(1+W_d)H(1-3W_d+3\delta_2)H_L'\end{aligned}$$

$$\left\{ \begin{array}{l} k(2(\Phi - \Phi')) = 4\pi G a^2 \sum_i (P_i + p_i) V_i \\ \Phi \sim \frac{1}{k^2}, \Phi' \sim \frac{1}{k^2} \end{array} \right. \Rightarrow V \sim \frac{1}{k} \text{ is a} \\ \text{order 1 small quantity} \\ \text{Drop order 2 small quantities only.}$$

It's weird that I get different eqns from OP02,0660  
eqn 21 & and eqn 22

eqn 21:  $\Rightarrow$

$$\begin{aligned}\delta_m'' + H((1+6\delta_2)\lambda)\delta_m' &= 3\delta_2 H^2 (\delta_d - \delta_m)(1+3\delta_2\lambda) \\ &\quad - 3\delta_2(H\lambda)' \delta_m + 3\delta_2(-\lambda\delta_d)' + \frac{1}{2}k^2 a^2 (P_m \delta_m + P_d \delta_d)\end{aligned}$$

~~$\Rightarrow$~~  RG:  $\begin{cases} P_m' + 3(P_m = 3H(\delta_1 P_m + \delta_2 P_d)) \\ P_d' + 3H(1+W_d)P_d = -3H(\delta_1 P_m + \delta_2 P_d) \end{cases} \begin{matrix} \delta_1 = 0 \\ \delta_2 = 0 \end{matrix}$

$$\begin{aligned}\delta_d'' &= [3H'(W_d-1) + 3H^2(W_d-1)(1-3\delta_2 \frac{1}{1+W_d}) - k^2 + \frac{1}{2}k^2 a^2 (1+W_d)P_d] \delta_d \\ &\quad - H[4-3W_d-3\delta_2 \frac{1}{1+W_d}] \delta_d' + \frac{1}{2}k^2 a^2 (1+W_d)P_m \delta_m ?\end{aligned}$$

$$C_{st}^2 = 1, C_{ad}^2 = W_d, C_{st}^2 = \frac{\delta P_d}{\delta P_m} \text{ rest-frame}, C_{ad}^2 = \frac{P_d}{P_m} = W_d + \frac{W_d'}{P_d/P_m/a}$$

This only holds  
for dark energy  
rest frame.

- From: Large-scale instability

in interacting dark energy and dark matter  
fluids. JCAP07(2007)020

We have to set  $C_{st}^2 < 1$  for  
interacting case. On page 166.

$$\frac{P_d}{P_d} = -3\lambda(\delta_2 + w_a + 1), \quad \frac{P_m'}{P_m} = 3\lambda(\delta_2\lambda - 1)$$

$$\Rightarrow \lambda' = \frac{P_d'}{P_d}\lambda - \lambda \frac{P_m'}{P_m} = 3\lambda(\lambda(-\delta_2 - w_a - 1 - \delta_2\lambda + 1)) \\ = 3\lambda(-\delta_2 - 3\lambda(\delta_2 + \lambda\delta_2 + w_a))$$

$-3\lambda\lambda'$

$$\delta_m'' + \lambda((1+6\delta_2)\lambda)\delta_m' = \{3\delta_2\lambda^2[(1+w_a) + 3\delta_2(1+2\lambda)] + \frac{1}{2}\lambda^2 a^2 P_m\}\delta_m \\ + \{3\delta_2\lambda^2[1 - 3(\delta_2 + w_a)] + 3\delta_2\lambda'\lambda + \frac{1}{2}\lambda^2 a^2 P_d\}\delta_d + 3\delta_2\lambda\lambda'\delta_d'$$

$$\lambda = \frac{a'}{a} = \dot{a}, \quad \lambda' = \frac{d}{dt} = a \cdot \frac{d}{da} = a \frac{d}{dt}$$

~~Write all of them in terms of  $a$~~

$$\text{BG} \Rightarrow P_d = \text{Const. } a^{-3(\delta_2 + 1 + w_a)}$$

Change  $\delta_2$  to  $\tilde{\delta}_2$  ~~to~~ in order to avoid confusion.  
One can infer from this that this kind of interaction looks just like expansion (which is ~~an~~ equally effect in diluting the energy density of dark energy) (And I guess if the interaction is proportional to  $\delta_m$ , this becomes ~~an~~ equally effective in diluting dark matter.)

Use the first background equation

$$P_m = P_{m0} a^{\frac{3}{1-\delta_2}} = \frac{3}{4} \tilde{\delta}_2 P_{d0} a^{-3(\tilde{\delta}_2 + 1 + w_a)} \\ + P_{m0} - \frac{3}{4} \tilde{\delta}_2 P_{d0}$$

$$P_d = P_{d0} a^{-3(\tilde{\delta}_2 + 1 + w_a)}$$

Change  $\delta_2 \rightarrow \tilde{\delta}_2$

$$\text{Leaving } P_m' + \frac{3}{P_m} (P_m = 3)(\tilde{\delta}_2 P_{d0}) a^{-3(\tilde{\delta}_2 + 1 + w_a)}$$

$$P_m = \text{Constant} \cdot e^{-\int P(z) dz} + e^{-\int P(z) dz} \int Q(z) e^{\int P(z) dz} dz$$

$$P_m = \text{Constant} \cdot a^{-3} - \frac{\delta_2}{\delta_2 + W_d} \cdot P_{d0} a^{-3} (\delta_2 + W_d)$$

$$= \text{Constant } a^{-3} - \frac{\delta_2}{\delta_2 + W_d} P_d$$

$$\text{Constant} = P_{m0} + \frac{\delta_2}{\delta_2 + W_d} P_{d0}$$

$$P_m = P_{m0} a^{-3} + \frac{\delta_2}{\delta_2 + W_d} P_{d0} a^{-3} [1 - a^{-3(\delta_2 + W_d)}]$$

$$H^2 = H_0^2 [P_r + P_m + P_d] \quad \sim \text{Hubble function}$$

$$= H_0^2 \left[ \Omega_{r0} a^{-4} + \Omega_{m0} a^{-3} + \frac{\delta_2}{\delta_2 + W_d} \Omega_{de0} a^{-3} (1 - a^{-3(\delta_2 + W_d)}) \right. \\ \left. + \Omega_{de0} \cdot a^{-3(\delta_2 + W_d)} \right]$$

$$= H_0^2 \left[ \Omega_{r0} a^{-4} + \Omega_{m0} a^{-3} + \frac{1}{\delta_2 + W_d} \Omega_{de0} a^{-3} (\delta_2 + W_d a^{-3(\delta_2 + W_d)}) \right]$$

At  $\delta_2 \rightarrow 0$  limit

$$H^2 = H_0^2 \left[ \Omega_{r0} a^{-4} + \Omega_{m0} a^{-3} + \frac{1}{W_d} \Omega_{de0} a^{-3} W_d a^{-3 W_d} \right] \\ = H_0^2 \left[ \Omega_{r0} a^{-4} + \Omega_{m0} a^{-3} + \Omega_{de0} a^{-3(1+W_d)} \right] \checkmark$$

$$\square \rightarrow \square : \frac{d}{dt} C = \frac{d}{da} \cdot \frac{da}{dt} = \frac{d}{da} \cdot a \dot{a} = \frac{d}{da} \cdot a H$$

$$\dot{\delta}_m' = aH\ddot{\delta}_m \quad \ddot{\delta}_m'' = aH(H+a\dot{H})\ddot{\delta}_m + (aH)^2 \ddot{\delta}_m'''$$

$$H' = aH\dot{H}, \quad \lambda = \frac{P_d}{P_m} = \frac{\Omega_{de0} a^{-3(\delta_2 + W_d)}}{\Omega_{m0} + \frac{\delta_2}{\delta_2 + W_d} \cdot \Omega_{de0} (1 - a^{-3(\delta_2 + W_d)})}$$

$$\ddot{\delta}_m + \left( \frac{2}{a} + \frac{H}{H'} + \frac{6\delta_2 \lambda}{a} \right) \dot{\delta}_m = \left\{ 3\delta_2 \lambda \frac{1}{a^2} [ (1+3W_d) + 3\delta_2 (1+2\lambda) ] \right.$$

$$- 3\delta_2 \lambda \frac{1}{a} \frac{H}{H'} + \left. \frac{1}{2} \dot{H}^2 \frac{1}{H^2} P_m \right\} \delta_m + \left\{ 3\delta_2 \lambda \frac{1}{a^2} [ 1 - 3(\delta_2 + W_d) ] \right.$$

$$+ 3\delta_2 \lambda \frac{1}{a} \frac{H}{H'} + \left. \frac{1}{2} \dot{H}^2 \frac{1}{H^2} P_d \right\} \delta_d + \cancel{3\delta_2 \lambda \frac{1}{a} \dot{\delta}_d}$$

$$\Rightarrow \left[ \frac{1}{2} \dot{H}^2 \frac{1}{H^2} P_m = \frac{3}{2} \frac{1}{a^2} \left[ \Omega_{m0} a^{-3} + \frac{\delta_2}{\delta_2 + W_d} \Omega_{de0} a^{-3} (1 - a^{-3(\delta_2 + W_d)}) \right] \right]$$

$$\cancel{\frac{1}{2} \dot{H}^2 \frac{1}{H^2} P_d} = \frac{3}{2} \frac{1}{a^2} \Omega_{de0} a^{-3(\delta_2 + 1 + W_d)}$$

$$\delta'_d = \alpha H \delta_d^{\circ}, \quad \delta''_d = \alpha((H + \alpha \dot{H})\delta_d^{\circ} + (H^2)^2 \delta_d^{\circ}, \quad H' = \alpha H^2$$

$$\ddot{\delta}_d + [\frac{1}{\alpha} + \frac{H}{H'} + \frac{1}{\alpha}(4 - 3W_d) - 3\delta_2 \frac{1}{1+W_d}] \dot{\delta}_d$$

$$= [3\frac{H}{H'}(W_d + 1)\frac{1}{\alpha} + 3\frac{1}{\alpha^2}(W_d + 1)(1 - 3\delta_2 \frac{1}{1+W_d}) - k^2 \cancel{(\textcircled{2})}$$

$$+ \frac{1}{2} \chi^2 \frac{1}{H^2} (1 + W_d) P_d] \ddot{\delta}_d + \frac{1}{2} \chi^2 \frac{1}{H^2} (1 + W_d) P_m \delta_m$$

## Some knowledge on Interacting DE

Testing coupled dark energy  
with future large-scale observations

by Valeria Pettorino SISSA, Trieste

Ob. 10, 11, The Dark Universe, Heidelberg

An example: Coupling to the trace of  $T_{(m)}$  AS as a scalar field  
 $T_{(m)}^{\mu}_{\nu; \mu} = \beta T_{(m)} \phi_{;\nu}, \quad T_{(\phi)}^{\mu}_{\nu; \mu} = -\beta T_{(m)} \phi_{;\nu}$

Thus we have the background as

$$\rho_\phi' = -3H((1 + w_\phi)P_\phi + \beta\phi'(1 - 3w_\phi)P_\phi)$$

$$P_\phi' = -3H((1 + w_\phi)P_\phi - \beta\cdot\phi'(1 - 3w_\phi)P_\phi)$$

Then Lagrange is

$$\mathcal{L} = \cancel{\frac{1}{2}\partial^\mu\phi\partial_\mu\phi} - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - U(\phi) - m(\phi)\bar{\psi}\psi + \mathcal{L}_{kin}[\bar{\psi}\psi]$$

$$Q_{(\phi)\mu} = \frac{\partial \ln m(\phi)}{\partial \phi} (\rho - 3P) \partial_\mu \phi, \quad \text{in which, } m(\phi) = m_0 e^{-\beta\frac{\phi}{M}}$$

$m(\phi)$  is just like a mass term.

$$\text{EOM for } \bar{\psi}\psi: t_{,\mu}^{\nu} = \phi_{,\mu} \frac{\partial \mathcal{L}}{\partial \phi_{,\nu}} - \delta_{\mu}^{\nu} f$$

- $DE \sim \text{Baryons interaction}$
- $DM \sim \text{Gravities interaction}$ : Dario Pettorino
- $DE \sim DM$  interaction:
  - \* Scalar tensor theories
- $DE \sim \text{neutrinos}$

What's:  
 Scalar tensor theory  
 Self-acceleration

#  $DE \sim DM$  interacting

Paper: PRD 77, 103003 (2008) , Valeria Pettorino et al.  
 About coupled quintessence

Paper: ~~arxiv.org~~: 0804.0232

~~RG: 91~~ {  $P_c' = -3\mathcal{H}(P_c - aQ)$  Large-scale instabilities...  
 $P_x' = -3\mathcal{H}(1+w_x)P_x + aQ$   $\mathcal{H} = \frac{d\ln a}{dt}$

$$\Leftrightarrow \left\{ \begin{array}{l} P_c' + 3\mathcal{H}(1+w_{c,\text{eff}})P_c = 0 \\ P_x' + 3\mathcal{H}(1+w_{x,\text{eff}})P_x = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} w_{c,\text{eff}} = \frac{aQ}{3\mathcal{H}P_c} \\ w_{x,\text{eff}} = -w_x - \frac{aQ}{3\mathcal{H}P_x} \end{array} \right.$$

Attention: ① It should be in perturbation theory,

we should find a invariant expression for

dark  $a$

② Energy perturbation should be stable. This

concerns about the sound speed.

Perturbation { 91 Metric:

$$ds^2 = a^2 \left\{ -(1+2\phi)dt^2 + 2a^2 B dx^i dx^j + [(1-2\psi)\delta_{ij} + 2a^2 \partial^i E] dx^i dx^j \right\}$$

12

~~B~~)  $\partial_i B$ : shift function  
~~C~~  $\star$ : lapse function ( $A_{i2}$  is added to  $(1+\phi)$ )  
 $-2(\partial_i \delta_{ij} - 2\partial_j E)$ : curvature perturbation because when the lapse is actually a expansion  
 curvature perturbation  
 $\partial_i \partial_j E$  is traceless



Four velocity :  $U^A$  starts for different component  
 background four velocity  
 $\bar{U}^{\mu} = a^{-1} \bar{U}^M$   
 $\begin{cases} U_A^{\mu} = a^{-1} (1-\phi, \partial_i v_A) \\ U_M^{\mu} = a (-1-\phi, \partial_i [v_A + B]) \end{cases}$

$v_A$ : peculiar velocity potential

Through this we can see the effect of  $\phi$  which is the lapse function.

Also, for a vector (which has a upper script) is relative to the threading!  
 for a covector (which has a lower script) is relative to the slice (that means we have to consider the normal of slice) !

Also define a volume expansion rate  
 $\theta_A = -k^2 (v_A + B)$

EM Tensor  
~~T~~  $\text{so}$

- ① Find a good velocity to set up as the energy-frame four-velocity
- ② Calculate the we must have the energy density defined :  $T_A^{\mu} v^{\nu} \cdot U_A^{\nu} = -\rho_A u_A^{\mu}$

$\mu$  is a script to identify species. [62]

For species A:  $T_A^{\mu\nu} = (P_A + P_A)U_A^\mu U_A^\nu + P_A \delta^{\mu\nu} + \Pi_A^{\mu\nu}$

 $P_A = \bar{P}_A + \delta P_A, \quad P_A = \bar{P}_A + \delta P_A$ 
 $\Pi_{A\nu}^i = 0, \quad \Pi_{A,i}^j = (\partial^i \partial_j - \frac{1}{3} \delta^i_j \nabla^2) \Pi_A \quad \text{Tracedless}$ 
 $\text{Tr}(\square^i_j) = \delta^i_j \square^i_i$

We can define a set of variables for total matter.

$$(P+P)U^\mu U_\nu + P \delta^{\mu\nu} + \Pi^{\mu\nu} + q^\mu U_\nu + q_\nu U^\mu$$
 $= \sum_A (P_A + P_A) U_A^\mu U_A^\nu + \sum_A P_A + \sum_A \Pi_A^{\mu\nu}$

Here  $q^\mu$  emerged because there might be momentum flux relative to the total four-velocity  $U^\mu$ . (which can have the form  $U^\mu = a^{-1}(1-\eta^i, \partial^i)$ )

Now question is - What  $\mathcal{V}$  to choose

We choose this one:  $(P+P)\mathcal{V} = \sum_A (P_A + P_A) \mathcal{V}_A$

Because this one eliminates  $q^i$ , i.e.,  $q^i = 0$ .

 $q^i = a^{-1} \sum_A (P_A + P_A) \partial^i \mathcal{V}_A - a^{-1} (P+P) \partial^i \mathcal{V}$ 

## Energy transfer / Energy-momentum balance

$$\nabla_\nu T_A^{\mu\nu} = Q_A^\mu, \quad \sum_A Q_A^\mu = 0 \quad (\text{Total EM tensor conserved})$$

Thus the transfer can be expanded using the velocity

$$Q_A^\mu = Q_A U^\mu + F_A^\mu; \quad Q_A = \bar{Q}_A + \sum Q_A, \quad U_\mu F_A^\mu = 0$$

↑  
energy density transfer rate

↑  
momentum transfer rate

relative to  $U^\mu$

Also  $F_A^\mu$  can be written as  
 $F_A^\mu = a^{-1}(0, \partial^i f_A)$   
 $f_A$ : momentum transfer potential

$$Q_i^A = a \partial_i [f_A + Q_A(v+B)] , Q_0^A = -a [Q_A(1+\phi) + \delta Q_A]$$

That is to say, total energy momentum conservation

$$0 = \sum_A Q_A = \sum_A \delta Q_A = \sum_A f_A$$

Now we can write down the conservation equations  
Considering the problem of sound speed.

They used a new kind of interacting term:

$$Q = \Gamma P_c, \quad Q_c^{\mu} = -Q_x^{\mu} = \Gamma T_c^{\mu}, \quad U_c^{\mu} = -[P_c]U_c^{\mu}$$

$\Gamma$ : constant (interaction rate)

This constant property of  $\Gamma$  leads to the result that

- this interaction could be ~~only~~ functionally at early time.

$$\textcircled{B} \quad P_c' = -3HP_c - aQ$$

when  $|aQ| \ll 3HP_c$ , this interaction could be dropped for ~~cold~~ cold dark matter.

Thus ~~some~~  $d = \frac{aQ}{3H P_{\text{de}}(1+w)}$  is the DOOM factor

\*: When  $Q \propto P_{\text{de}}$ , stable for  $w < -1$  and  $w = \text{constant}$  and  $Q < 0$  (So try some  $w = -1.1, Q = -Q_0/10$ )

Example:  $Q = \frac{1}{2}HP_{\text{de}} < 0$  : HE 09, Gavaia 09

When  $Q \propto P_{\text{dm}}$ , to be need Jackson 09  
stable for  $w = w(a)$

Damour 90, Wetterich 95 Amendola 00 Bean 08

$$Q \propto \alpha P_{\text{dm}} \nabla_2 \phi / M_p$$

$\alpha < 0.08$  from WMAP, SDSS, HST

Other studies in Valiviita 09 Majoratto 09

$$Q^{\nu} = -\alpha \Gamma P_{\text{dm}} U_{\text{dm}}^{\nu} \quad \text{and } w = w_0 + w_e(1-a)$$

Now since we have two equations about  $\delta_{de}$  and  $\delta_m$  and they are both ~~two~~ order-two differential equation, we have to find four initial equations.

$$\delta_m'[\dots] = \dots$$

$$\delta_m[\dots] = \dots$$

$$\delta_{de}'[\dots] = \dots$$

$$\delta_{de}[\dots] = \dots$$

We can easily expose adiabatic initial conditions

$$\phi = A\phi = \text{const} \quad , \quad \dot{\phi} = (1 + \frac{3}{5}R_v)\phi, \quad R_v := \frac{P_v}{P_v + \rho_p}$$

$$\Rightarrow \delta p = \delta v = \frac{4}{3}\delta_b = \frac{4}{3}\delta_c = -2\phi \quad , \quad \theta_p = \theta_v = \theta_b = \theta_c = \frac{1}{2}kT \propto \phi$$

$$\zeta_2 = \frac{1}{15}(kT)^2 \phi \quad , \quad \boxed{\delta_x = \frac{1}{2}\delta p, \quad \theta_x = \theta_p.}$$

The last two <sup>maybe</sup> ~~are~~ useful ~~x~~

When  $G^2 < 1$ , First things to mention:

① I just drop  $-\frac{P_d}{P_a} \frac{v_d}{k}$  in  $\delta - \frac{P_d}{P_a} \frac{v_d}{k}$  regardless of where they are (however, when  $\Lambda C_e^2$  even they are ~~not~~ multiplied by  $k^2 C_e^2$ )

② Since the interacting term only affects the conservation equation of dark energy. ~~we~~ we can still use George Ellis, III-1.25, to calculate  $H_T'' + H_T' \cdot \frac{a'}{a}$ .

③ I drop  $\Pi$ -term and  $\frac{v_d}{k}$  term in III-1.25.

④  $\frac{P_d}{P_a} = -3(\delta_2 - 3(1+w_2))$  according to background.

$$\delta_d'' = \left[ (1+3c_e^2 - 6w - 3\delta_2) \right] \frac{1}{HW} \frac{\partial \delta_d'}{\partial a} \delta_d'$$

$$\begin{aligned}\delta_d''' &= - \left( 1+3c_e^2 - 3w - \cancel{3\delta_2 + 3\frac{\partial \delta_d'}{\partial a}} - \frac{3}{1+w} \delta_2 \right) \times \delta_d' \\ &+ \left\{ -3c_e^2 + 3(c_e \delta_2 + w) \right\} \left[ -k^2 c_e^2 + \frac{3}{2}(1+w) \right] a^2 \alpha^{-3(1+w+\delta_2)} \\ &+ 3(w - c_e^2) \left( 1 - \frac{3\delta_2}{1+w} \right) \} \delta_d\end{aligned}$$

Here  $' = \frac{d}{da}$ , we need  $\frac{d}{da}$ . And  $\frac{''}{''} = \frac{d}{da} = a \times \frac{d}{da} = a \times \boxed{\frac{d}{da}}$

$$\begin{aligned}(aH)^2 \delta_d''' &= - \left( 2+3c_e^2 - 3w - \frac{3\delta_2}{w+1} + a \frac{\partial}{\partial a} \right) (aH)^2 \delta_d' \\ &+ \left\{ -3c_e^2 + 3(c_e \delta_2 + w) aH \left( \frac{\partial}{\partial a} - k^2 c_e^2 + \frac{3}{2} \right) \right. \\ &+ \left. \left\{ -3c_e^2 a \cdot \frac{\partial}{\partial a} + 3(c_e \delta_2 + w) \frac{\partial}{\partial a} a - k^2 c_e^2 \frac{1}{H^2} + \frac{3}{2}(1+w) \right\} a^{-3(1+w+\delta_2)} \right. \\ &\left. + 3(w - c_e^2) \left( 1 - \frac{3\delta_2}{1+w} \right) \right\} H \delta_d\end{aligned}$$

Another equation is on page 59.