$$g_{\mu\nu} = \begin{pmatrix} 1+2\psi & \\ & a^2(-1-2\phi)\delta_{ij} \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 1-2\psi & \\ & \frac{1}{a^2}(-1+2\phi)\delta^{ij} \end{pmatrix} \qquad \Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}\right)$$

$$\Gamma^0_{\alpha\beta} = \frac{1}{2} g^{0\nu}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}) \qquad \Gamma^0_{00} = 0, \quad \Gamma^0_{0i} = \Gamma^0_{i0} = 0$$

$$= \frac{1}{2} g^{00}(g_{\alpha0,\beta} + g_{\beta0,\alpha} - g_{\alpha\beta,0}) \qquad \Gamma^0_{ij} = \frac{1}{2} g^{00}(-g_{ij,0}) = \frac{1}{2}\, 2a\dot{a}\delta_{ij} = a\dot{a}\,\delta_{ij}$$

$$\Gamma^i_{\alpha\beta} = \frac{1}{2} g^{ii}(g_{\alpha i,\beta} + g_{\beta i,\alpha} - g_{\alpha\beta,i}) \qquad \Gamma^i_{00} = 0, \quad \Gamma^i_{0j} = \Gamma^i_{j0} = \frac{+1}{2} g^{ii} g_{ij,0} = \frac{-1}{2a^2}\cdot(-2a\dot{a})\delta_{ij} = \frac{\dot{a}}{a}\delta_{ij}$$

$$\Gamma^i_{jk} = 0$$

$$\delta\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\delta g^{\mu\nu}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}) + \frac{1}{2} g^{\mu\nu}(\delta g_{\alpha\nu,\beta} + \delta g_{\beta\nu,\alpha} - \delta g_{\alpha\beta,\nu})$$

$$\delta\Gamma^0_{\alpha\beta} = \frac{1}{2}\delta g^{00}(g_{\alpha0,\beta} + g_{\beta0,\alpha} - g_{\alpha\beta,0}) + \frac{1}{2} g^{00}(\delta g_{\alpha0,\beta} + \delta g_{\beta0,\alpha} - \delta g_{\alpha\beta,0})$$

$$\delta\Gamma^0_{00} = 0 + \frac{1}{2}\delta g_{00,0} = \frac{1}{2}\cdot 2\dot{\psi} = \dot{\psi}$$

$$\delta\Gamma^0_{0i} = \cancel{\tfrac{1}{2}}\, 0 + \frac{1}{2}(\delta g_{00,i} + \delta g_{i0,0} - \delta g_{0i,0}) = \frac{1}{2}\cdot 2\psi_{,i} = \partial_i\psi$$

$$\delta\Gamma^0_{ij} = \frac{1}{2}\delta g^{00}\cdot(-g_{ij,0}) + \frac{1}{2} g^{00}(-\delta g_{ij,0})$$

$$= \frac{1}{2}(-2\psi)\, 2a\dot{a}\,\delta_{ij} + \frac{1}{2}(4a\dot{a}\phi + 2a^2\dot{\phi})\delta_{ij} = (2a\dot{a}(\phi-\psi) + a^2\dot{\phi})\delta_{ij}$$

$$\delta\Gamma^i_{\alpha\beta} = \frac{1}{2}\delta g^{ii}(g_{\alpha i,\beta} + g_{\beta i,\alpha} - g_{\alpha\beta,i}) + \frac{1}{2} g^{ii}(\delta g_{\alpha i,\beta} + \delta g_{\beta i,\alpha} - \delta g_{\alpha\beta,i})$$

$$\delta\Gamma^i_{00} = 0 + \frac{1}{2}\frac{+1}{a^2}\delta g_{00,i} = \frac{1}{a^2}\psi_{,i} = \frac{1}{a^2}\partial_i\psi$$

$$\delta\Gamma^i_{0j} = \frac{1}{2}\frac{2\phi}{a^2} g_{ij,0} + \frac{1}{2}\frac{-1}{a^2}\delta g_{ij,0}$$

$$= \frac{1}{a^2}\phi\cdot\cancel{(-2a\dot{a})}\delta_{ij} + \frac{-1}{2a^2}(-\cancel{4a\dot{a}\phi} - 2a^2\dot{\phi})\delta_{ij} = \dot{\phi}\,\delta_{ij}$$

$$\delta\Gamma^i_{jk} = 0 + \frac{1}{2}\frac{-1}{a^2}(\delta g_{ij,k} + \delta g_{ik,j} - \delta g_{jk,i})$$

$$= \frac{+1}{2a^2} 2a^2(\phi_{,k}\delta_{ij} + \phi_{,j}\delta_{ik} - \phi_{,i}\delta_{jk})$$

$$= \partial_k\phi\,\delta_{ij} + \partial_j\phi\,\delta_{ik} - \partial_i\phi\,\delta_{jk}$$

$$\Gamma^0_{ij} = a\dot{a}\,\delta_{ij}$$

$$\Gamma^i_{0j} = \frac{\dot{a}}{a}\delta_{ij}$$

$$\delta\Gamma^0_{00} = \dot{\psi}$$

$$\delta\Gamma^0_{0i} = \partial_i\psi$$

$$\delta\Gamma^0_{ij} = \{2a\dot{a}(\phi-\psi) + a^2\dot{\phi}\}\,\delta_{ij}$$

$$\delta\Gamma^i_{00} = \frac{1}{a^2}\partial_i\psi$$

$$\delta\Gamma^i_{0j} = \dot{\phi}\,\delta_{ij}$$

$$\delta\Gamma^i_{jk} = \partial_k\phi\,\delta_{ij} + \partial_j\phi\,\delta_{ik} - \partial_i\phi\,\delta_{jk}$$

$$\mathcal{L}_\varphi = -\frac{1}{2}(\partial\varphi)^2 + V = -\frac{1}{2}\varphi^{,\mu}\varphi_{,\mu} + V$$

$$T^{\mu\nu}_{(\varphi)} = \frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}\,\mathcal{L}_\varphi}{\delta g_{\mu\nu}} = \frac{2}{\sqrt{-g}}\left(\frac{\delta\sqrt{-g}}{\delta g_{\mu\nu}}\mathcal{L}_\varphi + \sqrt{-g}\,\frac{\delta\mathcal{L}_\varphi}{\delta g_{\mu\nu}}\right) = g^{\mu\nu}\mathcal{L}_\varphi + \varphi^{\mu}\varphi^{\nu}$$

$$T_{(\varphi)}{}^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}\mathcal{L}_\varphi + \varphi^{\mu}\varphi_{\nu} .$$

$$T_{(\varphi)}{}^{00} = \frac{1}{2}\dot\varphi^2 + V = \rho_\varphi \quad , \quad T_{(\varphi)}{}^{ij} = \frac{1}{a^2}\left(\frac{1}{2}\dot\varphi^2 - V\right)\delta^{ij} = \frac{1}{a^2}p_\varphi\,\delta^{ij}$$

$$T_{(\varphi)}{}^{0}{}_{0} = \frac{1}{2}\dot\varphi^2 + V = \rho_\varphi \qquad T_{(\varphi)}{}^{i}{}_{j} = \left(-\frac{1}{2}\dot\varphi^2 + V\right)\delta^i_j = -p_\varphi$$

$$\delta T_{(\varphi)}{}^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}\,\delta\left(-\frac{1}{2}g^{\alpha\beta}\varphi_\alpha\varphi_\beta + V\right) + \delta\left(g^{\mu\sigma}\varphi_\sigma\varphi_\nu\right)$$

$$= \delta g^{\mu\sigma}\varphi_\sigma\varphi_\nu + g^{\mu\sigma}(\varphi_\sigma\delta\varphi_\nu + \varphi_\nu\delta\varphi_\sigma) - \frac{1}{2}\delta^{\mu}{}_{\nu}\left\{\delta g^{\alpha\beta}\varphi_\alpha\varphi_\beta + g^{\alpha\beta}(\varphi_\alpha\delta\varphi_\beta + \varphi_\beta\delta\varphi_\alpha)\right\} + \delta^{\mu}{}_{\nu}V'\delta\varphi$$

$$\delta T_{(\varphi)}{}^{0}{}_{0} = -\dot\varphi^2\phi + \dot\varphi\,\delta\dot\varphi + V'\delta\varphi . \quad \delta T_{(\varphi)}{}^{0}{}_{i} = +\dot\varphi\,\partial_i\delta\varphi \quad , \quad \delta T_{(\varphi)}{}^{i}{}_{0} = \frac{-1}{a^2}\dot\varphi\,\partial_i\delta\varphi$$

$$\delta T_{(\varphi)}{}^{i}{}_{j} = \left(\dot\varphi^2\phi - \dot\varphi\,\delta\dot\varphi + V'\delta\varphi\right)\delta^i_j$$

$$T_{(m)}{}^{\mu}{}_{\nu} = (\rho + p)u^\mu u_\nu - p\,\delta^{\mu}{}_{\nu}$$

$$T_{(m)}{}^{0}{}_{0} = \rho .$$

$$\delta T_{(m)}{}^{0}{}_{0} = \delta\rho . \quad \delta T_{(m)}{}^{0}{}_{i} = -a\rho v_i \quad , \quad \delta T_{(m)}{}^{i}{}_{0} = \frac{+1}{a}\rho v^i . \quad \delta T^i{}_j = 0$$

$$v_i = v^i$$

$$S_m = \int d^4x \sqrt{-g}\,\bar{\mathcal{L}}_m$$

$$\delta S_m = \int d^4x\, \delta(\sqrt{-g}\,\bar{\mathcal{L}}_m) = \int d^4x \frac{\sqrt{-g}}{2} T^{\mu\nu}_{(m)}\, \delta g_{\mu\nu} = \int d^4x \frac{\sqrt{-g}}{2} T^{\mu\nu}_{(m)} \frac{\partial g_{\mu\nu}}{\partial \bar g_{\mu\nu}}\, \delta \bar g_{\mu\nu}$$

$$= \int d^4x \frac{\sqrt{-g}}{2C} T^{\mu\nu}_m \left(\bar g_{\mu\lambda}\,\varepsilon^\lambda{}_{,\nu} + \bar g_{\lambda\nu}\,\varepsilon^\lambda{}_{,\mu} + \bar g_{\mu\nu,\lambda}\,\varepsilon^\lambda\right)$$

$$= \int d^4x \left\{ -\partial_\nu\left(\frac{\sqrt{-g}}{2C}T^{\mu\nu}_m \bar g_{\mu\lambda}\right) - \partial_\mu\left(\frac{\sqrt{-g}}{2C} T^{\mu\nu}_m \bar g_{\lambda\nu}\right) + \frac{\sqrt{-g}}{2C}T^{\mu\nu}_m \bar g_{\mu\nu,\lambda}\right\}\varepsilon^\lambda = 0.$$

$$0 = \partial_\nu\left(\frac{\sqrt{-g}}{\cancel{2}C} T^{\mu\nu}_m \bar g_{\mu\lambda}\right) - \frac{\sqrt{-g}}{2C}T^{\mu\nu}_m \bar g_{\mu\nu,\lambda}$$

$$= \partial_\nu\left[\frac{\sqrt{-g}}{\cancel{2}C} T^{\mu\nu}_m (C g_{\mu\lambda} + D\varphi_\mu\varphi_\lambda)\right] - \frac{\sqrt{-g}}{2C}T^{\mu\nu}_m\, \partial_\lambda(C g_{\mu\nu} + D\varphi_\mu\varphi_\nu)$$

$$= \underline{\partial_\nu(\sqrt{-g}\,T^{\mu\nu}_m g_{\mu\lambda})} + \partial_\nu\left(\sqrt{-g}\,\frac{D}{C}T^{\mu\nu}_m\varphi_\mu\varphi_\lambda\right) - \sqrt{-g}\,\frac{C'}{2C}T^{\mu\nu}_m g_{\mu\nu}\varphi_\lambda - \underline{\frac{\sqrt{-g}}{2}T^{\mu\nu}_m g_{\mu\nu,\lambda}} - \frac{\sqrt{-g}}{2C}T^{\mu\nu}_m\, \partial_\lambda(D\varphi_\mu\varphi_\nu)$$

$$\cancel{(\partial_\nu\sqrt{-g})T^{\mu\nu}g_{\mu\lambda} + \sqrt{-g}\,\partial_\nu T^{\mu\nu}} \qquad (\partial_\nu\sqrt{-g})T^\nu{}_\lambda + (\partial_\nu T^\nu{}_\lambda)\sqrt{-g}$$

$$\partial_\nu(\sqrt{-g}\,T^{\mu\nu}_m g_{\mu\lambda}) - \frac{\sqrt{-g}}{2}T^{\mu\nu}_m g_{\mu\nu,\lambda} = \sqrt{-g}\,\nabla_\nu\left(T_{(m)}{}^\nu{}_\lambda\right) \qquad \boxed{\Gamma^\sigma_{\sigma\nu} = \frac{1}{\sqrt{-g}}\partial_\nu\sqrt{-g}}$$

$$\partial_\nu\left(\sqrt{-g}\,\frac{D}{C}T^{\mu\nu}_m\varphi_\mu\varphi_\lambda\right) = \partial_\nu(\sqrt{-g})\frac{D}{C}T^{\mu\nu}_m\varphi_\mu\varphi_\nu + \sqrt{-g}\,\partial_\nu\left(\frac{D}{C}T^{\mu\nu}_m\varphi_\mu\varphi_\lambda\right) = \sqrt{-g}\left[\frac{D}{C}\Gamma^\sigma_{\sigma\nu}T^{\mu\nu}_{(m)}\varphi_\mu\varphi_\lambda + \partial_\nu\left(\frac{D}{C}T^{\mu\nu}_m\varphi_\mu\varphi_\lambda\right)\right]$$

$$\frac{1}{\sqrt{-g}}\left\{\partial_\nu\left(\sqrt{-g}\,\frac{D}{C}T^{\mu\nu}_m\varphi_\mu\varphi_\lambda\right) - \sqrt{-g}\,\frac{C'}{2C}T^{\mu\nu}_m g_{\mu\nu}\varphi_\lambda - \frac{\sqrt{-g}}{2C}T^{\mu\nu}_m\,\partial_\lambda(D\varphi_\mu\varphi_\nu)\right\}$$

$$= \frac{D}{C}\Gamma^\sigma_{\sigma\nu}T^{\mu\nu}_m\varphi_\mu\varphi_\lambda + \partial_\nu\left(\frac{D}{C}T^{\mu\nu}_m\varphi_\mu\varphi_\lambda\right) - \frac{C'}{2C}T^{\mu\nu}_m g_{\mu\nu}\varphi_\lambda - \frac{1}{2C}T^{\mu\nu}_m\,\partial_\lambda(D\varphi_\mu\varphi_\nu)$$

$$= \frac{D}{C}\Gamma^\sigma_{\sigma\nu}T^{\mu\nu}_m\varphi_\mu\varphi_\lambda + \partial_\nu\left(\frac{D}{C}T^{\mu\nu}_m\varphi_\mu\right)\varphi_\lambda + \cancel{\frac{D}{C}T^{\mu\nu}_m\varphi_\mu\,\partial_\nu\partial_\lambda\varphi} - \frac{C'}{2C}T_m\varphi_\lambda - \frac{D'}{2C}T^{\mu\nu}_m\varphi_\mu\varphi_\nu\varphi_\lambda$$

$$- \frac{D}{2C}T^{\mu\nu}_m\left(\cancel{\varphi_\mu\,\partial_\lambda\partial_\nu\varphi + \varphi_\nu\,\partial_\lambda\partial_\mu\varphi}\right)$$

$$= \left\{\Gamma^\sigma_{\sigma\nu}\frac{D}{C}T^{\mu\nu}_m\varphi_\mu + \partial_\nu\left(\frac{D}{C}T^{\mu\nu}_m\varphi_\mu\right) - \frac{C'}{2C}T_m - \frac{D'}{2C}T^{\mu\nu}_m\varphi_\mu\varphi_\nu\right\}\varphi_{,\lambda}$$

$$\boxed{\begin{aligned}\nabla_\nu A^\nu &= \partial_\nu A^\nu + \Gamma^\nu_{\nu\sigma}A^\sigma \\ &= \partial_\nu A^\nu + \Gamma^\sigma_{\sigma\nu}A^\nu\end{aligned}}$$

$$= \left\{\nabla_\nu\left(\frac{D}{C}T^{\mu\nu}_m\varphi_\mu\right) - \frac{C'}{2C}T_m - \frac{D'}{2C}T^{\mu\nu}_{(m)}\varphi_\mu\varphi_\nu\right\}\varphi_{,\lambda} \equiv Q\,\varphi_{,\lambda}$$

$$\nabla_\nu T_{(m)}{}^\nu{}_\lambda = -Q\,\varphi_{,\lambda} \qquad \boxed{\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\sigma_{\mu\nu}A_\sigma}$$

$$\boxed{\begin{aligned}Q &\equiv \nabla_\nu\left(\frac{D}{C}T^{\mu\nu}_m\varphi_\mu\right) - \frac{C'}{2C}T_m - \frac{D'}{2C}T^{\mu\nu}_{(m)}\varphi_\mu\varphi_\nu \\ &= \partial_\nu\left(\frac{D}{C}T^{\mu\nu}_m\varphi_\mu\right) + \Gamma^\sigma_{\sigma\nu}\frac{D}{C}T^{\mu\nu}_m\varphi_\mu - \frac{C'}{2C}T_m - \frac{D'}{2C}T^{\mu\nu}_m\varphi_\mu\varphi_\nu\end{aligned}}$$

$$\nabla_\mu T_{(\varphi)}^{\mu\nu} = \nabla_\mu\left[\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_\varphi)}{\delta g_{\mu\nu}}\right] = \nabla_\mu\left(\frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g_{\mu\nu}}\mathcal{L}_\varphi + 2\mathcal{L}_{\varphi,X}\frac{\delta X}{\delta g_{\mu\nu}}\right)$$

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu}$$

$\mathcal{L}(\varphi, X)$

$$X = -\frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi$$

$$\delta X = \frac{1}{2}\varphi^\mu\varphi^\nu\delta g_{\mu\nu}$$

$$\nabla_\mu X = -\frac{1}{2}(\nabla_\mu\nabla^\nu\varphi\,\nabla_\nu\varphi + \nabla^\nu\varphi\,\nabla_\mu\nabla_\nu\varphi) = -\nabla_\mu\nabla^\nu\varphi\,\nabla_\nu\varphi$$

$$= \nabla_\mu(\mathcal{L}_\varphi g^{\mu\nu} + \mathcal{L}_{\varphi,X}\varphi^\mu\varphi^\nu)$$

$$= g^{\mu\nu}\nabla_\mu\mathcal{L}_\varphi + \underline{\nabla_\mu(\mathcal{L}_{\varphi,X})\varphi^\mu\varphi^\nu} + \mathcal{L}_{\varphi,X}\nabla_\mu(\varphi^\mu\varphi^\nu)$$

$$= g^{\mu\nu}(\mathcal{L}_{\varphi,\varphi}\varphi_\mu - \mathcal{L}_{\varphi,X}\varphi_\sigma\nabla_\mu\nabla^\sigma\varphi)$$

$$+ (\mathcal{L}_{\varphi,X\varphi}\varphi_\mu - \mathcal{L}_{\varphi,XX}\varphi^\sigma\nabla_\mu\nabla_\sigma\varphi)\varphi^\mu\varphi^\nu + \mathcal{L}_{\varphi,X}(\nabla^2\varphi\,\varphi^\nu + \varphi^\mu\nabla_\mu\nabla^\nu\varphi)$$

$$= \mathcal{L}_{\varphi,\varphi}\varphi^\nu - \cancel{\mathcal{L}_{\varphi,X}\varphi_\sigma\nabla^\nu\nabla^\sigma\varphi} + \mathcal{L}_{\varphi,X\varphi}(\partial\varphi)^2\varphi^\nu - \mathcal{L}_{\varphi,XX}\varphi^\mu\varphi^\sigma\nabla_\mu\nabla_\sigma\varphi\,\varphi^\nu + \mathcal{L}_{\varphi,X}\nabla^2\varphi\,\varphi^\nu + \cancel{\mathcal{L}_{\varphi,X}\varphi_\mu\nabla^\mu\nabla^\nu\varphi}$$

$$= \{\mathcal{L}_{\varphi,\varphi} + \mathcal{L}_{\varphi,X\varphi}(\partial\varphi)^2 + \mathcal{L}_{\varphi,X}\nabla^2\varphi - \mathcal{L}_{\varphi,XX}\varphi^\mu\varphi^\nu\nabla_\mu\nabla_\nu\varphi\}\varphi^\nu$$

$$\varphi^\nu\nabla_\mu T_{(\varphi)}{}^\mu{}_\nu = \{\mathcal{L}_{\varphi,\varphi} + \mathcal{L}_{\varphi,X\varphi}(\partial\varphi)^2 + \mathcal{L}_{\varphi,X}\nabla^2\varphi - \mathcal{L}_{\varphi,XX}\varphi^\mu\varphi^\nu\nabla_\mu\nabla_\nu\varphi\}(\partial\varphi)^2$$

$$\varphi^\nu\nabla_\mu T_{m}{}^\mu{}_\nu = -\{\nabla_\nu\left(\frac{D}{C}T_m^{\mu\nu}\varphi_\mu\right) - \frac{C'}{2C}T_m - \frac{D'}{2C}T_m^{\mu\nu}\varphi_\mu\varphi_\nu\}(\partial\varphi)^2$$

$$= \{\frac{D}{C}\varphi^\nu\nabla_\mu T_m{}^\mu{}_\nu + \nabla_\alpha\left(\frac{D}{C}\varphi_\beta\right)T_m^{\alpha\beta} - \frac{C'}{2C}T_m - \frac{D'}{2C}T_m^{\alpha\beta}\varphi_\alpha\varphi_\beta\}(+2X)$$

$$\left(\frac{1}{+2X} - \frac{D}{C}\right)\varphi^\nu\nabla_\mu T_m{}^\mu{}_\nu = \frac{D'}{C}\varphi_\alpha\varphi_\beta T_m^{\alpha\beta} - \frac{C'D}{C^2}\varphi_\alpha\varphi_\beta T_m^{\alpha\beta} - \frac{C'}{2C}T_m - \frac{D'}{2C}\varphi_\alpha\varphi_\beta T_m^{\alpha\beta} + \frac{D}{C}\nabla_\alpha\nabla_\beta\varphi\,T_m^{\alpha\beta}$$

$$\frac{C-2XD}{2XC}\varphi^\nu\nabla_\mu T_m{}^\mu{}_\nu = \{\left(\frac{D'}{2C} - \frac{C'D}{C^2}\right)\varphi_\alpha\varphi_\beta - \frac{C'}{2C}g_{\alpha\beta}\}T_m^{\alpha\beta} + \frac{D}{C}\nabla_\alpha\nabla_\beta T_m^{\alpha\beta}$$

$$= QT_m^{\alpha\beta} + \frac{D}{C}\nabla_\alpha\nabla_\beta\varphi\,T_m^{\alpha\beta} \qquad Q \equiv \left(\frac{D'}{2C} - \frac{C'D}{C^2}\right)\varphi_\alpha\varphi_\beta - \frac{C'}{2C}g_{\alpha\beta}$$

$$\frac{1}{+2X}\varphi^\nu\nabla_\mu T_m{}^\mu{}_\nu = \frac{C}{C-2DX}QT_m^{\alpha\beta} + \frac{D}{C-2DX}\nabla_\alpha\nabla_\beta\varphi\,T_m^{\alpha\beta}$$

$$0 = \frac{1}{-2X}\varphi^\nu\nabla_\mu(T_m{}^\mu{}_\nu + T_\varphi{}^\mu{}_\nu)$$

$$= -\frac{C}{C+2DX}QT_m^{\alpha\beta} - \frac{D}{C+2DX}\nabla_\alpha\nabla_\beta\varphi\,T_m^{\alpha\beta}$$

$$+ \mathcal{L}_{\varphi,\varphi} + \mathcal{L}_{\varphi,X\varphi}(-2X) + \mathcal{L}_{\varphi,X}g^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi - \mathcal{L}_{\varphi,XX}\varphi^\alpha\varphi^\beta\nabla_\alpha\nabla_\beta\varphi$$

$$= \frac{-C}{C-2DX}QT_m^{\alpha\beta} + \left(\mathcal{L}_{\varphi,X}g^{\alpha\beta} - \mathcal{L}_{\varphi,XX}\varphi^\alpha\varphi^\beta - \frac{D}{C+2DX}T_m^{\alpha\beta}\right)\nabla_\alpha\nabla_\beta\varphi + \left(\mathcal{L}_{\varphi,\varphi} - 2X\mathcal{L}_{\varphi,X\varphi}\right)$$

$$\nabla_\mu T^{\mu}_{(m)\,\nu} = -Q\varphi_{,\nu} \qquad \nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\sigma} T^\sigma{}_\nu - \Gamma^\sigma_{\mu\nu} T^\mu_\sigma$$

$$\nabla_\mu T^\mu_{(m)\,0} = \partial_\mu T^\mu_{(m)0} + \Gamma^\mu_{\mu\sigma} T^\sigma_0 - \Gamma^\sigma_{\mu 0} T^\mu_\sigma = \dot\rho + 3H\rho = -Q\dot\varphi$$

$$\mathcal{L}_\varphi = X + V \qquad \mathcal{L}_{\varphi,\varphi} = V' \ , \ \mathcal{L}_{\varphi,X} = 1$$

$$0 = \frac{-C}{C-2DX} Q\, T^{\alpha\beta}_{(m)} + g^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi - \frac{D}{C-2DX} T^{\alpha\beta}_{(m)} \nabla_\alpha\nabla_\beta\varphi + V'$$

$$= \frac{-C}{C-2DX}\left(\frac{D'}{2C} - \frac{C'D}{C^2}\right)\varphi_\alpha\varphi_\beta T^{\alpha\beta}_m + \frac{C'}{2(C-2DX)} g_{\alpha\beta} T^{\alpha\beta}_m - \frac{D}{C-2DX} T^{\alpha\beta}_m \nabla_\alpha\nabla_\beta\varphi + g^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi + V'$$

$$= \frac{C}{C-2DX}\left(\frac{C'D}{C^2} - \frac{D'}{2C}\right)\dot\varphi^2\rho + \frac{C'}{2(C-2DX)}\rho - \frac{D}{C-2DX}\rho\ddot\varphi + \underline{\ddot\varphi + 3H\dot\varphi + V'}$$

$$(C-2DX-D\rho)(\ddot\varphi + 3H\dot\varphi + V') = -D\rho(3H\dot\varphi + V') + \left(\frac{D'}{2} - \frac{C'D}{C}\right)\dot\varphi^2\rho - \frac{C'}{2}\rho$$

$$= \left\{\frac{C'}{2} - \frac{D'}{2}\dot\varphi^2 + D\left(3H\dot\varphi + V' + \frac{C'}{C}\dot\varphi^2\right)\right\}(-\rho)$$

$$\ddot\varphi + 3H\dot\varphi + V' = -\frac{C' + 2D\left(3H\dot\varphi + V' + \frac{C'}{C}\dot\varphi^2\right) - D'\dot\varphi^2}{2[C - D(\rho - \dot\varphi^2)]}\rho \quad = Q_0$$

$$\nabla_\mu T^\mu_{(\varphi)0} = \partial_\mu T^\mu_{(\varphi)0} + \Gamma^\nu_{\mu\sigma} T^\sigma_0 - \Gamma^\sigma_{\mu 0} T^\mu_\sigma = \dot\varphi\ddot\varphi + V'\dot\varphi + 3H\dot\varphi^2 = Q\dot\varphi$$

$$\ddot\varphi + 3H\dot\varphi + V' = Q.$$

$$Q = \partial_0\left(\frac{D}{C}\dot\varphi\rho\right) + 3H\frac{D}{C}\dot\varphi\rho - \frac{C'}{2C}\rho - \frac{D'}{2C}\dot\varphi^2\rho$$

$$= \frac{D'}{C}\dot\varphi^2\rho - \frac{C'D}{C^2}\dot\varphi^2\rho + \frac{D}{C}\ddot\varphi\rho + \frac{D}{C}\dot\varphi\dot\rho + 3H\frac{D}{C}\dot\varphi\rho - \frac{C'}{2C}\rho - \frac{D'}{2C}\dot\varphi^2\rho$$

$$= \frac{D'}{2C}\dot\varphi^2\rho - \frac{C'D}{C^2}\dot\varphi^2\rho - \frac{C'}{2C}\rho + \frac{D}{C}(\ddot\varphi + 3H\dot\varphi)\rho + \frac{D}{C}\dot\varphi\dot\rho$$

$$= -\frac{C'}{2C}\rho + \frac{D'}{2C}\dot\varphi^2\rho - \frac{C'D}{C^2}\dot\varphi^2\rho + \frac{D}{C}(Q - V')\rho + \frac{D}{C}(-Q\dot\varphi - 3H\rho)\dot\varphi$$

$$\left[1 - \frac{D}{C}(\rho - \dot\varphi^2)\right]Q = -\frac{C'}{2C}\rho + \left(\frac{D'}{2C} - \frac{C'D}{C^2}\right)\dot\varphi^2\rho - \frac{D}{C}(3H\dot\varphi + V')\rho$$

$$2[C - D(\rho - \dot\varphi^2)]Q = \left\{C' - D'\dot\varphi^2 + 2D\left(\frac{C'}{C}\dot\varphi^2 + 3H\dot\varphi + V'\right)\right\}(-\rho)$$

$$Q = -\frac{C' + 2D\left(\frac{C'}{C}\dot\varphi^2 + 3H\dot\varphi + V'\right) - D'\dot\varphi^2}{2[C - D(\rho - \dot\varphi^2)]}\rho$$

$$\nabla_\mu \delta T^\mu_{(\varphi)0} = \partial_\mu \delta T^\mu_{(\varphi)0} + \delta\Gamma^\mu_{\mu\sigma} T^\sigma_{(\varphi)0} + \Gamma^\mu_{\mu\sigma}\delta T^\sigma_{(\varphi)0} - \delta\Gamma^\sigma_{\mu 0} T^\mu_{(\varphi)\sigma} - \Gamma^\sigma_{\mu 0}\delta T^\mu_{(\varphi)\sigma}$$

$$= \partial_0(-\dot\varphi^2\psi + \dot\varphi\delta\dot\varphi + V'\delta\varphi) + \partial_i \frac{-1}{a^2}\dot\varphi\partial_i\delta\varphi + (\dot\psi + 3\dot\phi)(\tfrac{1}{2}\dot\varphi^2 + V) + 3H(-\dot\varphi^2\psi + \dot\varphi\delta\dot\varphi + V'\delta\varphi)$$

$$- \dot\psi(\tfrac{1}{2}\dot\varphi + V) - 3\dot\phi(-\tfrac{1}{2}\dot\varphi^2 + V) - 3H(\dot\varphi^2\psi - \dot\varphi\delta\dot\varphi + V'\delta\varphi)$$

$$= (-2\dot\varphi\ddot\varphi\psi - \dot\varphi^2\dot\psi + \ddot\varphi\delta\dot\varphi + \dot\varphi\delta\ddot\varphi + V'\delta\dot\varphi + V''\dot\varphi\delta\varphi) - \frac{1}{a^2}\dot\varphi\partial_i^2\delta\varphi + 3\dot\phi(\tfrac{1}{2}\dot\varphi^2 + V) - 3\dot\phi(-\tfrac{1}{2}\dot\varphi^2 + V)$$

$$+ 6H(-\dot\varphi^2\psi + \dot\varphi\delta\dot\varphi)$$

$$= \dot\varphi\delta\ddot\varphi + (\ddot\varphi + V' + 6H\dot\varphi)\delta\dot\varphi + V''\dot\varphi\delta\varphi - \frac{1}{a^2}\dot\varphi\partial_i^2\delta\varphi$$

$$\boxed{\ddot\varphi + 3H\dot\varphi + V' = Q}$$

$$- (2\dot\varphi\ddot\varphi + 6H\dot\varphi^2)\psi - \dot\varphi^2\dot\psi + 3\dot\varphi^2\dot\phi$$

$$= \dot\varphi\delta\ddot\varphi + (Q + 3H\dot\varphi)\delta\dot\varphi + V''\dot\varphi\delta\varphi - \frac{1}{a^2}\dot\varphi\partial_i^2\delta\varphi - 2(Q - V')\dot\varphi\psi - \dot\varphi^2\dot\psi + 3\dot\varphi^2\dot\phi$$

$$\nabla_\mu\delta T^\mu_{(\varphi)0} = \delta(Q\dot\varphi) = \delta Q\dot\varphi + Q\delta\dot\varphi$$

$$\delta\ddot\varphi + 3H\delta\dot\varphi + V''\delta\varphi - \frac{1}{a^2}\dot\varphi\partial_i^2\delta\varphi = 2(Q - V')\psi + \dot\varphi\dot\psi - 3\dot\varphi\dot\phi + \delta Q$$

Mota

$\psi \to \Phi$

$\phi \to -\Psi$

$$\nabla_\mu\delta T^\mu_{(m)0} = \partial_\mu\delta T^\mu_{(m)0} + \delta\Gamma^\mu_{\mu\sigma}T^\sigma_{(m)0} + \Gamma^\mu_{\mu\sigma}\delta T^\sigma_{(m)0} - \delta\Gamma^\sigma_{\mu 0}T^\mu_{(m)\sigma} - \Gamma^\sigma_{\mu 0}\delta T^\mu_{(m)\sigma}$$

$$= \delta\dot\rho + \frac{1}{a}\rho\,\partial_i v^i + (\dot\psi + 3\dot\phi)\rho + 3H\delta\rho - \dot\psi\rho$$

$$= \delta\dot\rho + 3H\delta\rho + \frac{1}{a}\rho\partial_i v^i + 3\dot\phi\rho = \rho\dot\delta + \dot\rho\delta + 3H\rho\delta + \frac{1}{a}\rho\partial_i v^i + 3\dot\phi\rho$$

$$= \rho\dot\delta - Q\dot\varphi\delta + \frac{1}{a}\rho\partial_i v^i + 3\dot\phi\rho = -\delta(Q\dot\varphi) = -\delta Q\dot\varphi - Q\delta\dot\varphi$$

$$\dot\delta + \frac{1}{a}\partial_i v^i - \frac{Q}{\rho}\dot\varphi\delta = -3\dot\phi - \frac{Q}{\rho}\delta\dot\varphi - \frac{\delta Q}{\rho}\dot\varphi$$

$$\nabla_\mu\delta T^\mu_{(m)i} = \partial_\mu\delta T^\mu_{(m)i} + \delta\Gamma^\mu_{\mu\sigma}T^\sigma_{(m)i} + \Gamma^\mu_{\mu\sigma}\delta T^\sigma_{(m)i} - \delta\Gamma^\sigma_{\mu i}T^\mu_{(m)\sigma} - \Gamma^\sigma_{\mu i}\delta T^\mu_{(m)\sigma}$$

$$= \partial_0(-a\rho v_i) + 3H(-a\rho v_i) - \partial_i\psi\rho - a\dot a\,\cancel{\delta_{ij}\frac{1}{a}\rho v^j} - \frac{\dot a}{a}\,\cancel{\delta^j_i(-a\rho v_j)}$$

$$= -a\rho\dot v_i - \dot a\rho v_i - a\dot\rho v_i - 3H a\rho v_i - \partial_i\psi\rho$$

$$\boxed{\dot\rho + 3H\rho = -Q\dot\varphi}$$

$$= -a\rho\dot v_i - \dot a\rho v_i + aQ\dot\varphi v_i - \partial_i\psi\rho$$

$$\partial_i\nabla_\mu\delta T^\mu_{(m)i} = -a\rho\,\partial_i\dot v_i - \dot a\rho\,\partial_i v_i + aQ\dot\varphi\,\partial_i v_i - \partial_i^2\psi\rho = -\partial_i\delta(Q\varphi_i) = -Q\partial_i^2\delta\varphi$$

$$a\,\partial_i\dot v_i + \dot a\,\partial_i v_i - a\frac{Q}{\rho}\dot\varphi\,\partial_i v_i = -\partial_i^2\psi + \frac{Q}{\rho}\partial_i^2\delta\varphi$$

$$H^2 = \frac{8\pi G}{3}\left(\rho + \frac{\dot\varphi^2}{2} + V\right), \quad \dot H + H^2 = -\frac{4\pi G}{3}\left(\rho + 2\dot\varphi^2 - 2V\right) \qquad \underline{\kappa^2 \equiv 8\pi G}$$

$$\dot\rho + 3H\rho = Q_0\dot\varphi, \quad \ddot\varphi + 3H\dot\varphi + V_{,\varphi} = -Q_0 \qquad ' \equiv \frac{d}{d\ln a}, \quad \frac{d}{dt} = H\frac{d}{d\ln a}$$

$$x \equiv \frac{\kappa}{\sqrt6}\varphi', \quad y = \frac{\kappa}{H}\sqrt{\frac{V}{3}}, \quad z = \frac{\kappa}{H}\sqrt{\frac{\rho}{3}}, \quad g = \frac{\kappa}{H^2}Q_0. \qquad \Omega_\varphi = x^2+y^2, \quad \Omega_m = z^2, \quad w_\varphi = \frac{x^2-y^2}{x^2+y^2}$$

$$C = C_0 e^{\alpha\kappa\varphi}, \quad D = D_0 e^{\beta\kappa\varphi}, \quad \tilde D = \frac{H^2}{\kappa^2}D = \tilde D_0 e^{\beta\kappa\varphi}, \quad V = V_0 e^{-\gamma\kappa\varphi}$$

$$H^2 = \frac{\kappa^2}{3}\left(\rho + \frac12 H^2\varphi'^2 + V\right) \Rightarrow 1 = x^2 + y^2 + z^2$$

$$\dot H = -H^2 - \frac{4\pi G}{3}\left(\rho + 2\dot\varphi^2 - 2V\right) = -4\pi G\left(\rho + \dot\varphi^2\right)$$

$$H' = \frac{\dot H}{H} = -\frac{\kappa^2}{2}H\left(\frac{\kappa^2}{H^2}\rho + \kappa^2\varphi'^2\right) = -\frac{H}{2}\left(3z^2 + 6x^2\right) = -\frac{H}{2}\left(3x^2 - 3y^2 + 3\right)$$

$$H^2\varphi'' + HH'\varphi' + 3H^2\varphi' + V_{,\varphi} = -Q_0 \qquad \frac{d^2}{dt^2} = \frac{d}{dt}\left(H\frac{d}{d\ln a}\right) = \dot H\frac{d}{d\ln a} + H^2\frac{d^2}{d\ln a^2} = HH'\frac{d}{d\ln a} + H^2\frac{d^2}{d\ln a^2}$$

$$x' = \frac{\kappa}{\sqrt6}\varphi'' = \frac{\kappa}{\sqrt6}\left(-\frac{H'}{H}\varphi' - 3\varphi' + \frac{1}{H^2}\gamma\kappa V - \frac{Q_0}{H^2}\right)$$

$$= -\frac{H'}{H}\frac{\kappa}{\sqrt6}\varphi' - 3\frac{\kappa}{\sqrt6}\varphi' + \frac{3}{\sqrt6}\gamma\frac{\kappa^2}{H^2}\frac{V}{3} - \frac{1}{\sqrt6}\frac{\kappa}{H^2}Q_0$$

$$= -x\left(-\frac32x^2 + \frac32y^2 - \frac32 + 3\right) + \sqrt{\frac32}\gamma y^2 - \frac{1}{\sqrt6}g$$

$$= \frac{x}{2}\left(3x^2 - 3y^2 - 3\right) + \sqrt{\frac32}\gamma y^2 - \frac{1}{\sqrt6}g$$

$$y' = \frac12\frac{\kappa}{H}\sqrt{\frac{1}{3V}}V_{,\varphi}\varphi' - \frac{\kappa}{H^2}\sqrt{\frac{V}{3}}H'$$

$$= -\frac{\sqrt6}{2}\frac{\kappa}{H}\sqrt{\frac{1}{3V}}\gamma V\frac{\kappa}{\sqrt6}\varphi' - \frac{\kappa}{H}\sqrt{\frac{V}{3}}\frac{H'}{H}$$

$$= -\frac{\sqrt6}{2}\gamma xy + y\cdot\frac12\left(3x^2 - 3y^2 + 3\right)$$

$$= \frac12 y\left(3x^2 - 3y^2 + 3\right) - \sqrt{\frac32}\gamma xy$$

$$g = \frac{\kappa}{H^2}Q_0 = -\frac{\kappa}{H^2}\rho\cdot\frac{C_\varphi + 2D\left(\frac{C_\varphi}{C}\dot\varphi^2 + 3H\dot\varphi + V_\varphi\right) - D_\varphi\dot\varphi^2}{2\left[C - D\left(\rho - \dot\varphi^2\right)\right]}$$

$$= -\left(\frac{\kappa^2}{H^2}\rho\right)\frac{1}{\kappa}\cdot\frac{\alpha\kappa C + 2D\left(\alpha\kappa H^2\varphi'^2 + 3H^2\varphi' - \gamma\kappa V\right) - \beta\kappa D H^2\varphi'^2}{2\left[C - D\left(\rho - H^2\varphi'^2\right)\right]}$$

$$= -3z^2\cdot\frac{\alpha C + 2\frac{H^2}{\kappa^2}D\left(\alpha\kappa^2\varphi'^2 + 3\kappa\varphi' - \gamma\frac{\kappa^2}{H^2}V\right) - \beta\frac{H^2}{\kappa^2}D\kappa^2\varphi'^2}{2\left[C - \frac{H^2}{\kappa^2}D\left(\frac{\kappa^2}{H^2}\rho - \kappa^2\varphi'^2\right)\right]}$$

$$= -3z^2\,\frac{\alpha C + 2\tilde D\left(6\alpha x^2 + \sqrt{\frac32}x - 3\gamma y^2\right) - 6\beta\tilde D x^2}{2\left[C - \tilde D\left(3z^2 - 6x^2\right)\right]}$$

$$= -3z^2\cdot\frac{\alpha C + 6\tilde D(2\alpha - \beta)x^2 + \sqrt6\tilde D x - 6\tilde D\gamma y^2}{2\left[C - \tilde D\left(3z^2 - 6x^2\right)\right]}$$

$$= 3\left(x^2 + y^2 - 1\right)\frac{\alpha C + 6\tilde D(2\alpha - \beta)x^2 + \sqrt6\tilde D x - 6\tilde D\gamma y^2}{2\left[C - \tilde D\left(3 - 9x^2 - 3y^2\right)\right]}$$