Sinple bar finite element method

Consider a bar of length l and using 2 nodes at each end. So we will have $x_1 = 0, x_2 = l$

At each node the displacement ond force function are $q_i(t), Q_i(t)$ respectfully.

Now, assume a simple shape function for the bar $u(x,t) = a_0 + a_1 x$

We want to convert the differential equation

$\ddot{u}(x,t) = c^2 u''(x,t)$

where $\ddot{u}, u''$ are second derivatives relative to time and displacement respectfully.

To a system of ordinary differential equations

$[m]\ddot{u}(t) + [k]u(t) = \vec{F}(t)$

that can be easily solved by a computer.

With the previously shape function, we can combine the boundry conditions

$x = 0, u(0,t) = q_1(t)$

to get

$a_0 = q_1(t)$

and

$x = l, u(l,t) = q_2(t)$

to get

$a_1 = \frac{q_2(t) - q_1(t)}{l}$

Now replug the value back to the orginal shape function we can get

$$\begin{aligned} u(x,t) &= q_1 + \frac{q_2 - q_1}{l} x \\ &= \begin{bmatrix} 1 - \frac{x}{l} & \frac{x}{l} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \end{aligned}$$

We will rewrite the two parts as

$u(x,t) = \overrightarrow{\Phi^T}(x) \cdot \vec{q}(t)$ Where $\overrightarrow{\Phi^T}$ is the shape function and $\vec{q}$ is the nodal coordinates.