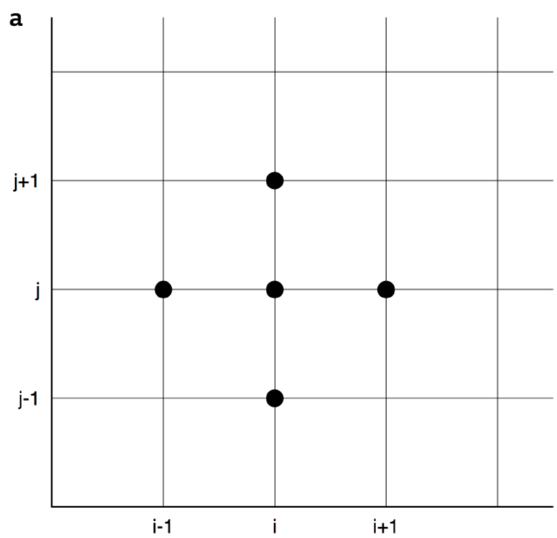
笛卡尔坐标下静电场的二维泊松方程可以表示为

$$\nabla^2\varphi(x,y)=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=-\frac{\rho(x,y)}{\varepsilon_0}$$

在二维网格中的差分格式可以写作



$$\nabla^2\varphi\left(x_i,y_j\right)\overset{\mathrm{d}}{=}-\frac{\rho_{i,j}}{\varepsilon_0}$$

$$=\frac{1}{h^2}\left(\varphi_{i+1,j}+\varphi_{i-1,j}+\varphi_{i,j+1}+\varphi_{i,j-1}-4\varphi_{i,j}\right)$$

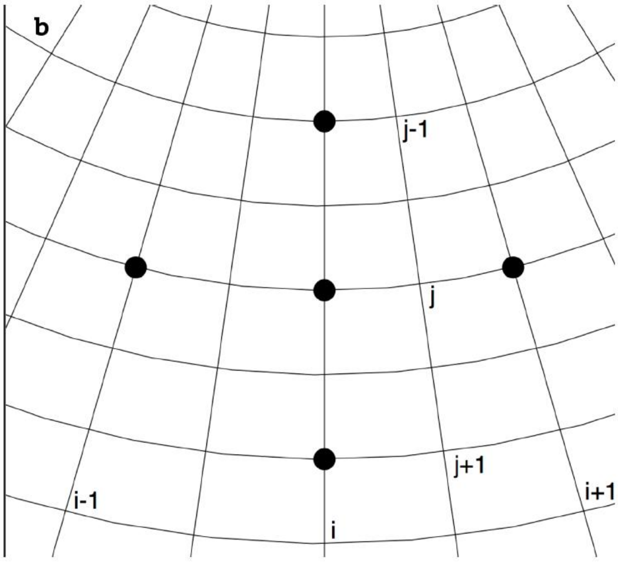
顾名思义，矩阵法是将未知量和已知量之间的线性关系表示为矩阵的形式，通过对矩阵进行消元和变换直接求出未知量的值。矩阵的边长等于总格点个数。可将矩阵表示为如下形式

$$
\left(\begin{array}{c:c:c}
\mathbf{A} & \begin{matrix} e & & \\ & \ddots & \\ & & e \end{matrix} & \\
\hdashline
\begin{matrix} d & & \\ & \ddots & \\ & & d \end{matrix} & \begin{matrix} \bullet & & \\ & \bullet & \\ & & \bullet \end{matrix} & \begin{matrix} e & & \\ & \ddots & \\ & & e \end{matrix} \\
\hdashline
 & \begin{matrix} d & & \\ & \ddots & \\ & & d \end{matrix} & \mathbf{A}
\end{array}\right)
$$

$$
\mathrm{A} = \begin{pmatrix} a & c & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & c \\ & & b & a \end{pmatrix}
$$

$$
\begin{cases} a = 4 \\ b = c = -1 \\ d = e = -1 \end{cases}
$$

2 极坐标系下的带状矩阵初始化



在极坐标系下，直接应用极坐标形式的泊松方程进行求解较为繁琐，

$$
\begin{aligned}
\frac{Q}{\varepsilon_0} &= \int_V \frac{\rho}{\varepsilon_0}\mathrm{d}V = -\int_V \nabla^2\varphi \mathrm{d}V \\
&= \int_V \nabla\cdot\vec{E}\mathrm{d}V = \oint_S \vec{E}\cdot\mathrm{d}\vec{S}
\end{aligned}
$$

$$
\begin{aligned}
&\int_V \frac{\rho}{\varepsilon_0}\mathrm{d}V = \frac{1}{\varepsilon_0}\left(\rho_{i,j} r_i \Delta\theta\Delta r\right) = \oint_S \vec{E}\cdot\mathrm{d}\vec{S} \\
=& E^r_{i+\frac{1}{2},j}\cdot r_{i+\frac{1}{2}}\Delta\theta - E^r_{i-\frac{1}{2},j}\cdot r_{i-\frac{1}{2}}\Delta\theta \\
&+ E^\theta_{i,j+\frac{1}{2}}\Delta r - E^\theta_{i,j-\frac{1}{2}}\Delta r
\end{aligned}
$$

又

$$
\begin{cases}
E^r_{i+\frac{1}{2},j} = -\dfrac{\varphi_{i+1,j} - \varphi_{i,j}}{\Delta r} \\[2em]
E^r_{i-\frac{1}{2},j} = -\dfrac{\varphi_{i,j} - \varphi_{i-1,j}}{\Delta r} \\[2em]
E^\theta_{i,j+\frac{1}{2}} = -\dfrac{\varphi_{i,j+1} - \varphi_{i,j}}{r_i \Delta \theta} \\[2em]
E^\theta_{i,j-\frac{1}{2}} = -\dfrac{\varphi_{i,j} - \varphi_{i,j-1}}{r_i \Delta \theta}
\end{cases}
$$

可以得到非圆点处的五点差分公式

$$
\begin{aligned}
&-\frac{\rho_{i,j}}{\varepsilon_0} = \frac{1}{r_i^2 \Delta\theta^2}\left(\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}\right) \\
&+\frac{1}{r_i \Delta r^2}\left(\varphi_{i+1,j} r_{i+\frac{1}{2}} - 2\varphi_{i,j} r_i + \varphi_{i-1,j} r_{i-\frac{1}{2}}\right) \\
&= -\left(\frac{2}{\Delta r^2} + \frac{2}{r_i^2 \Delta\theta^2}\right)\varphi_{i,j} + \frac{r_{i+\frac{1}{2}}}{r_i \Delta r^2}\varphi_{i+1,j} \\
&+\frac{r_{i-\frac{1}{2}}}{r_i \Delta r^2}\varphi_{i-1,j} + \frac{1}{r_i^2 \Delta\theta^2}\varphi_{i,j+1} + \frac{1}{r_i^2 \Delta\theta^2}\varphi_{i,j-1}
\end{aligned}
$$

对于坐标原点，同样可解得其差分公式为

$$
\frac{\rho_0}{\varepsilon_0} = \frac{2\Delta\theta}{\pi \Delta r^2} \sum_{j=1}^{N_\theta} \left(\varphi_{1,j} - \varphi_0\right)
$$

其矩阵为

$$\left(\begin{array}{c|c|c|c}
m & n \quad \ldots \quad n & & \\ \hline
\begin{matrix} d_1 \\ \vdots \\ d_1 \end{matrix} & \mathbf{A} & \begin{matrix} e_1 & & \\ & \ddots & \\ & & e_1 \end{matrix} & \\ \hline
& \begin{matrix} d_2 & & \\ & \ddots & \\ & & d_i \end{matrix} & \begin{matrix} \bullet & & \\ & \bullet & \\ & & \bullet \end{matrix} & \begin{matrix} e_2 & & \\ & \ddots & \\ & & e_i \end{matrix} \\ \hline
& & \begin{matrix} d_{i+1} & & \\ & \ddots & \\ & & d_{i+1} \end{matrix} & \mathbf{A}
\end{array}\right)$$

$$\mathrm{A} = \begin{pmatrix} a_i & c_i & & b_i \\ b_i & \ddots & \ddots & \\ & \ddots & \ddots & c_i \\ c_i & & b_i & a_i \end{pmatrix}$$

其中，各系数的取值为

$$\begin{cases} a_i = \dfrac{2}{\Delta r^2} + \dfrac{2}{r_i^2 \Delta\theta^2} \\ b_i = c_i = -\dfrac{1}{r_i^2 \Delta\theta^2} \\ d_i = -\dfrac{r_{i-\frac{1}{2}}}{r_i \Delta r^2} \qquad\qquad e_i = -\dfrac{r_{i+\frac{1}{2}}}{r_i \Delta r^2} \\ m = \dfrac{2N_\theta \Delta\theta}{\pi \Delta r^2} \qquad\qquad n = -\dfrac{2\Delta\theta}{\pi \Delta r^2} \end{cases}$$

考试1：利用消元法（或者LU法），雅克比迭代、高斯迭代、超松弛迭代分别求解矩形和圆形接地导体边界情况下，点电荷在空间中的电势分布。并不断增加矩阵尺寸，比较各种方法计算速度。例如: