HIGHER SCHOOL OF ECONOMICS
NATIONAL RESEARCH UNIVERSITY

Project presentation

A Divide and Conquer Method for the Symmetric Tridiagonal Eigenproblem

Konstantin Leladze

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Algorithm summary

$$\begin{pmatrix} 2i & 1+i & 4+i & -2i \\ -1+i & 3i & 4+7i & 1-i \\ -4+i & -4+7i & -9i & 3-4i \\ -2i & -1-i & -3-4i & 5i \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & c_{n-1} & a_n \end{pmatrix} \longrightarrow$$

$T \rightarrow Q \Lambda Q^T$

$\lambda_1 \approx 6.05765$

$\lambda_2 \approx -1.31398$

$\lambda_3 \approx 1.25634$

$v_1 \approx (1.7707, 0.643472, 1)$

$v_2 \approx (-0.175117, -2.06943, 1)$

$v_3 \approx (-0.948525, -0.397569, 1)$

How to find an eigencomposition of an arbitrary hermitian matrix?

1. The hermitian matrix should be tridiagonalized first
2. Then, eigencomposition can be easily built using the implemented **divide and conquer algorithm**

Divide

$$\left(\begin{array}{cc|cc} 1 & 6 & & \\ 6 & 3 & 6 & \\ \hline & 6 & 2 & 6 \\ & & 6 & 9 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 6 & & \\ 6 & -3 & & \\ \hline & & -4 & 6 \\ & & 6 & 9 \end{array}\right) + \underbrace{6}_{\rho} \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}} \cdot \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}$$

$$T_1 \qquad T_2$$

$$T = \left(\begin{array}{c|c} T_1 & \\ \hline & T_2 \end{array}\right) + \rho v v^T$$

Conquer

$$T_1 = Q_1 D_1 Q_1^T$$
$$T_2 = Q_2 D_2 Q_2^T$$

$$\Downarrow$$

$$T = \begin{pmatrix} Q_1 D_1 Q_1^T & \\ & Q_2 D_2 Q_2^T \end{pmatrix} + \rho v v^T = \begin{pmatrix} Q_1 & \\ & Q_2 \end{pmatrix} \cdot \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix} \cdot \begin{pmatrix} Q_1^T & \\ & Q_2^T \end{pmatrix} + \rho v v^T =$$

$$= \begin{pmatrix} Q_1 & \\ & Q_2 \end{pmatrix} \cdot \left[\begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix} + \rho u u^T \right] \cdot \begin{pmatrix} Q_1^T & \\ & Q_2^T \end{pmatrix}$$

where $u = \begin{pmatrix} Q_1^T & \\ & Q_2^T \end{pmatrix} v$

Proof:

let $Q_e = \begin{pmatrix} Q_1 & \\ & Q_2 \end{pmatrix}$

Q_e is orthogonal

then:

$Q_e \cdot u u^T Q_e^T = Q_e \cdot Q_e^T \cdot v v^T Q_e \cdot Q_e^T = v v^T$

Conquer

Eigenvalues of T

$\parallel$

Eigenvalues $\underbrace{\begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix}}_{= D \text{ is diag.}} + \rho u u^T$

$\det(D + \rho u u^T - \lambda E) = 0$

$\Updownarrow$

$\det(D - \lambda E) = 0$

$\det(E + \rho (D - \lambda E)^{-1} \cdot u u^T) = 0$

1-st corner case:

is $\rho = 0$, then

$T = \underbrace{Q_e}\cdot\underbrace{D}\cdot Q_e^T$

eigen vectors and values of T

$\Longrightarrow$

deflation

$\exists j : d_i = d_j$

d_i is an eigen val of T

$u_i = 0 \Rightarrow$

else:

$$1 + \rho \cdot \sum_{j=1}^{n} \frac{u_j^2}{d_j - \lambda} = 0$$

Solving the equation

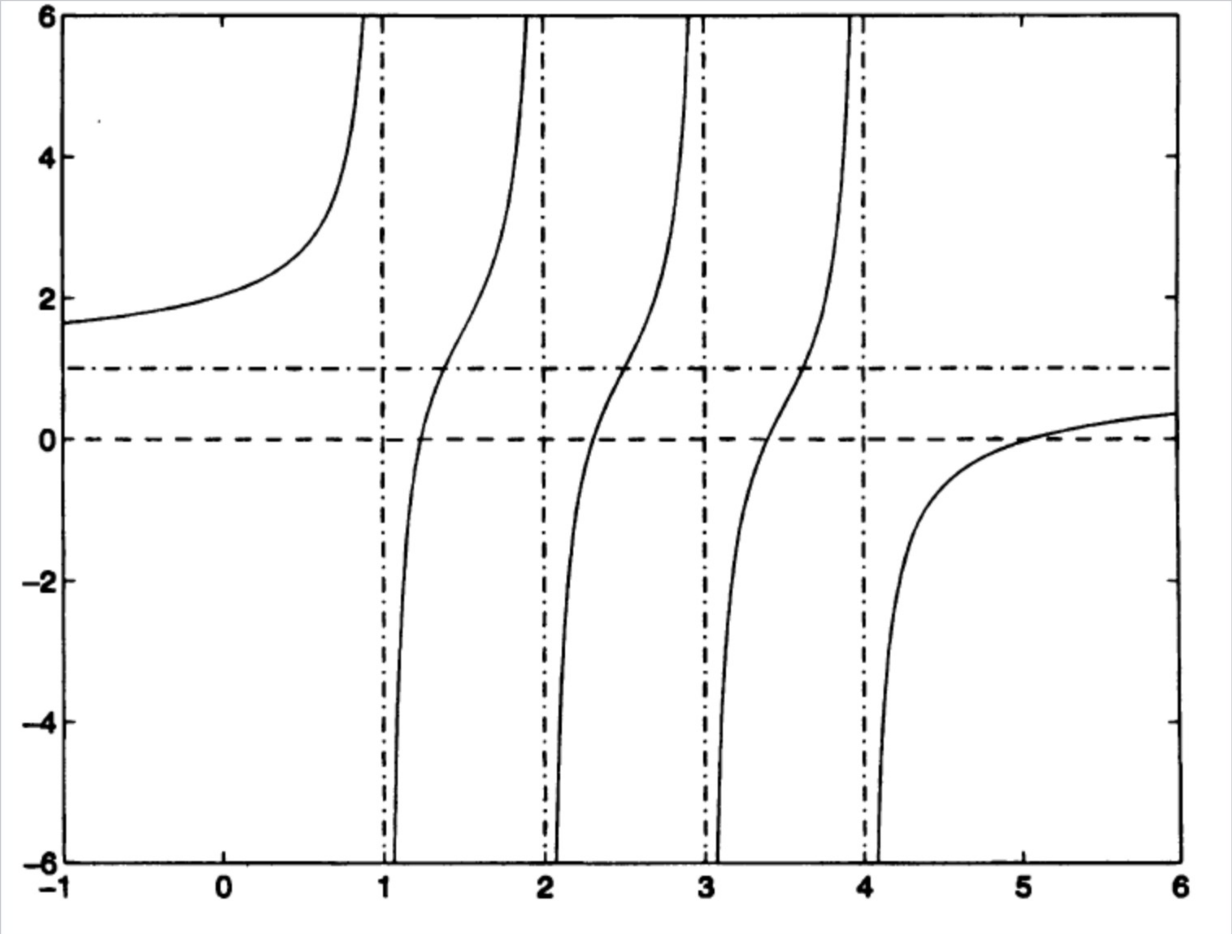


Рис. 1. График функции $f(\lambda) = 1 + \frac{0.5}{1-\lambda} + \frac{0.5}{2-\lambda} + \frac{0.5}{3-\lambda} + \frac{0.5}{4-\lambda}$

Obtaining the eigen vectors

$$d_{i_1} = d_{i_2} \Rightarrow e'_i = \begin{array}{c} \\ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -u_{i1}/u_{i2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow i_1 \\ \\ \\ \\ \leftarrow i_2 \\ \\ \\ \\ \end{array}$$

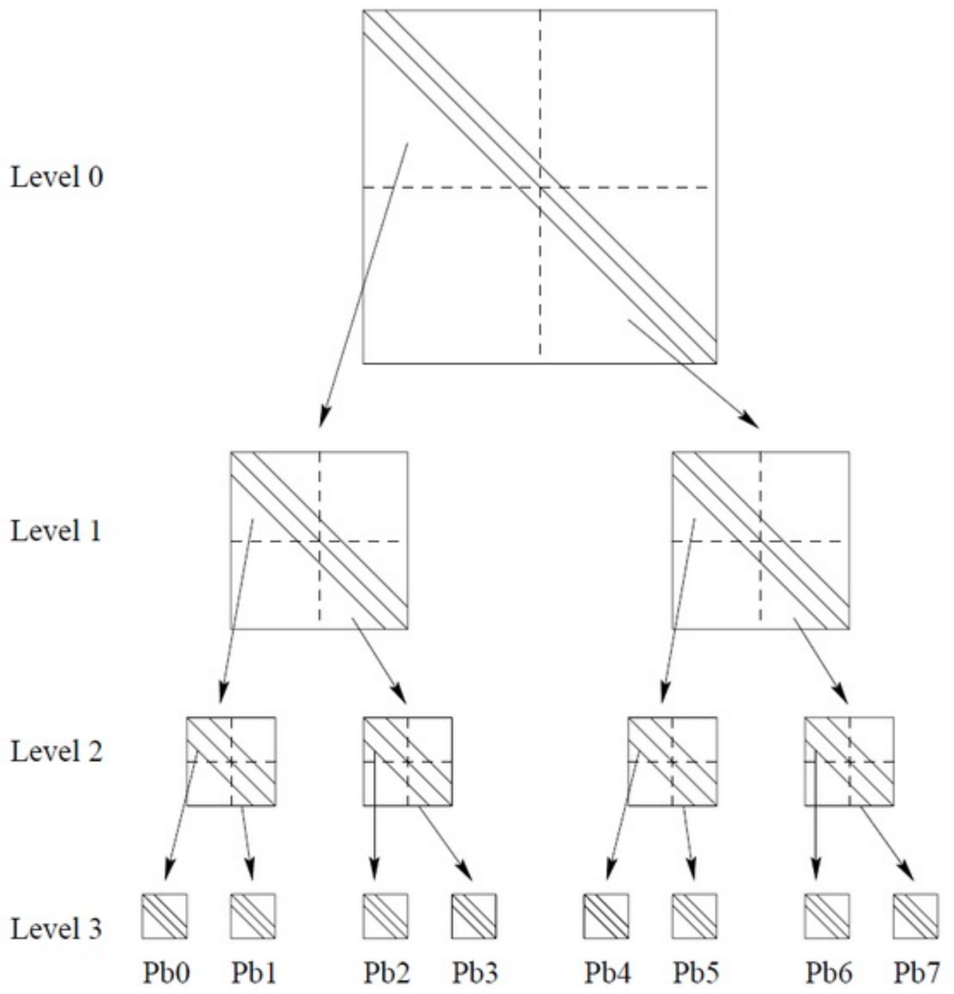
$$u_i = 0 \Rightarrow e'_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \end{array}$$

else:

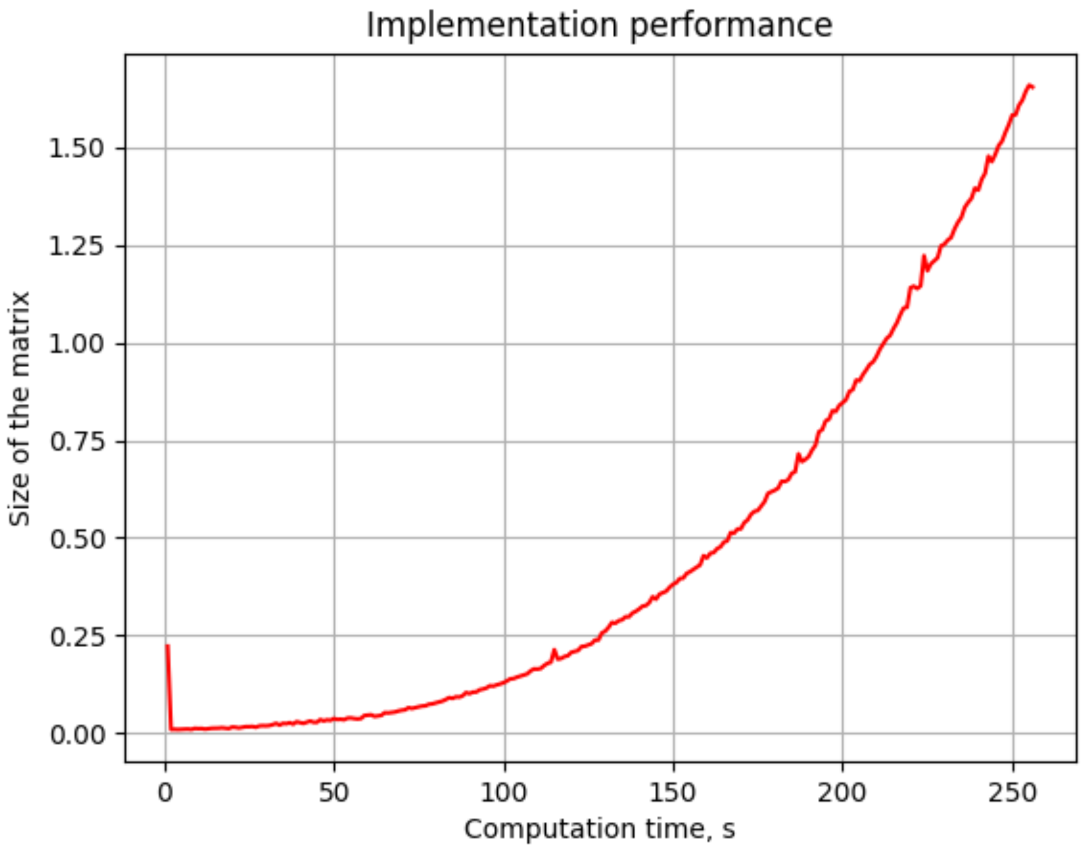
$$e_i = \frac{u_j}{d_j - \lambda_i}$$

$$e_i = Q_e \cdot e'_i$$

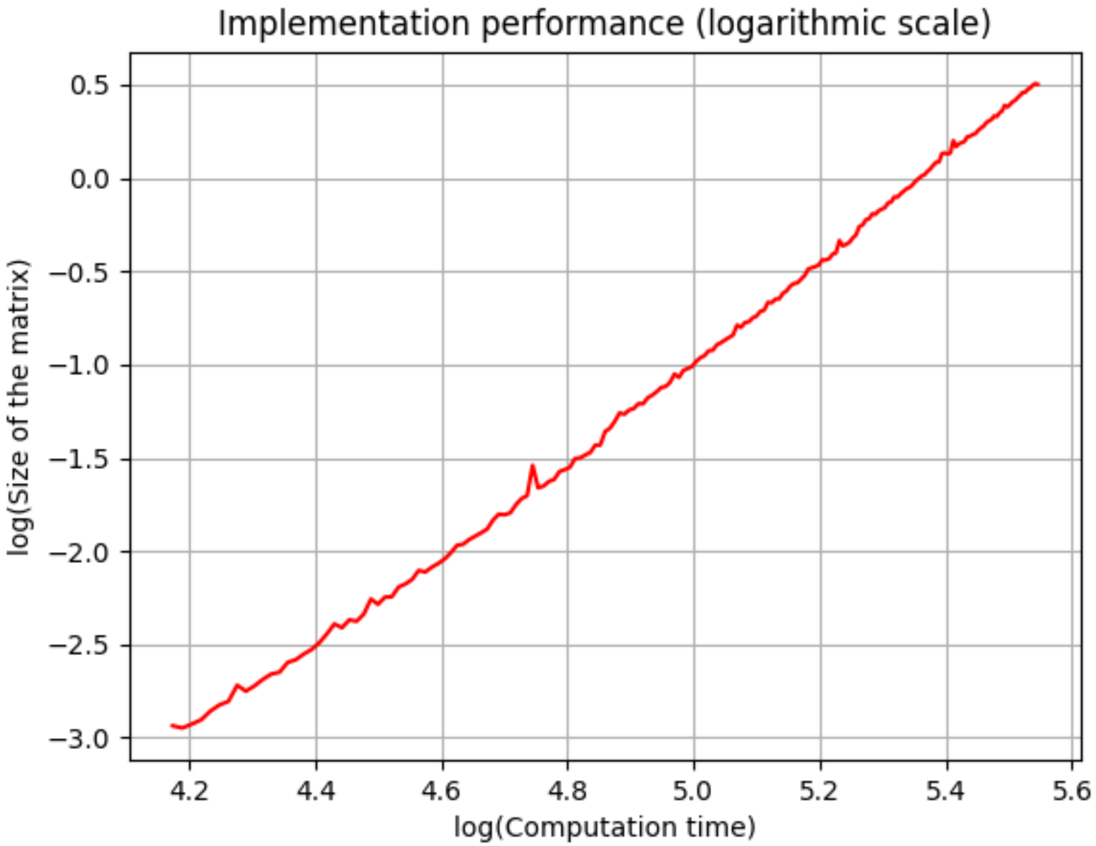
Multithread implementation



Performance



Performance



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