Special Topics in Logic and Security 1

Variable and Memory Space Analysis

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How to detect buffer overflow

Often exploited software defects can be reduced to the following snippet:

```
char buf[10];
i = 0;
while (i < 20) {
    buf[i] = i;
    i = i + 1;
}
```

How do current tools behave when encountering this sequence?

How can we use static analysis to find such defects?

Example: C routine

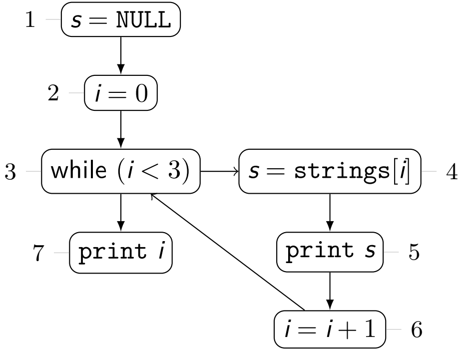
```
char *strings[] = { "One", "Two", "Three" };

char *s = NULL;
int i;

for (i = 0; i < 3; i++) {
    s = strings[i];
    printf("%s\n", s);
}

printf("%d\n", i);
```

Example: IMP and CFG adaptation



Preliminaries

We denote

- v_n^i – the possible values of variable i in nodes $n = \overline{1,6}$
- v_n^s – the memory addresses towards which s points to in nodes $n = \overline{1,6}$

We describe the values of i as an interval and those of s as an abstract set of addersses $\mathcal{A}$.

- `strings[i]` represents the address of string i
- we store this address as $s_i \in \mathcal{A}$ for $i \in \{1, 2, 3\}$
- we denote void, zero, or uninitialized addresses with `NULL` or $\emptyset$

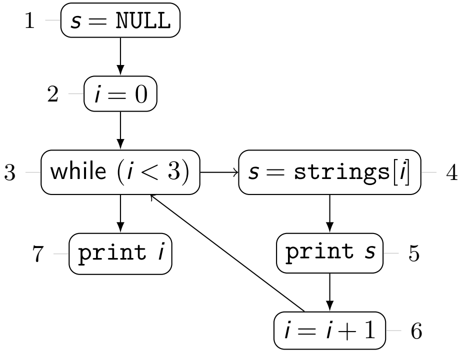
For the constraints system we will need the operator

$$[\ell_1, u_1] \, \overline{\Upsilon} \, [\ell_2, u_2] = [\min(\ell_1, \ell_2), \max(u_1, u_2)]$$

that computes the minimum range containing both given ranges.

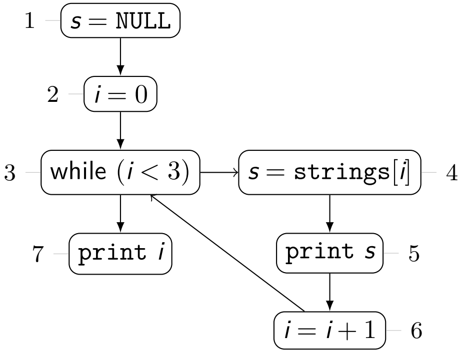
Example: ranges

$$
\begin{aligned}
v^i_1 &= [0,0] \\
v^i_2 &= [0,0] \\
v^i_3 &= v^i_2 \,\overline{\Upsilon}\, v^i_6 \\
v^i_4 &= v^i_3 \cap \left[-2^{31}, 2\right] \\
v^i_5 &= v^i_4 \cap \left[-2^{31}, 2\right] \\
v^i_6 &= \left\{ v + 1 \mid v \in v^i_5 \right\} \\
v^i_7 &= v^i_3 \cap \left[3, 2^{31} - 1\right]
\end{aligned}
$$



Example: addresses

$$
\begin{aligned}
v_1^s &= \emptyset \\
v_2^s &= v_1^s \\
v_3^s &= v_2^s \cup v_6^s \\
v_4^s &= \{s_1 \mid 0 \in v_4^i\} \cup \\
&\quad \{s_2 \mid 1 \in v_4^i\} \cup \\
&\quad \{s_3 \mid 2 \in v_4^i\} \\
v_5^s &= v_4^s \\
v_6^s &= v_5^s \\
v_7^s &= v_3^s
\end{aligned}
$$



Example: resulting equations

Ranges

$$
\begin{aligned}
v^i_1 &= [0,0]\\
v^i_2 &= [0,0]\\
v^i_3 &= v^i_2 \overline{\Upsilon} v^i_6\\
v^i_4 &= v^i_3 \cap \left[-2^{31}, 2\right]\\
v^i_5 &= v^i_4 \cap \left[-2^{31}, 2\right]\\
v^i_6 &= \left\{v+1 \mid v \in v^i_5\right\}\\
v^i_7 &= v^i_3 \cap \left[3, 2^{31}-1\right]
\end{aligned}
$$

Addresses

$$
\begin{aligned}
v^s_1 &= \emptyset\\
v^s_2 &= v^s_1\\
v^s_3 &= v^s_2 \cup v^s_6\\
v^s_4 &= \left\{s_1 \mid 0 \in v^i_4\right\} \cup\\
&\quad\ \left\{s_2 \mid 1 \in v^i_4\right\} \cup\\
&\quad\ \left\{s_3 \mid 2 \in v^i_4\right\}\\
v^s_5 &= v^s_4\\
v^s_6 &= v^s_5\\
v^s_7 &= v^s_3
\end{aligned}
$$

Remark

Note the link between the values domain of i represented as ranges and the domain of `pointer` *s represented as a set of addresses.*

Solving with the Fixed Point Theorem

The solution to the above equations can be obtained through the fixed point theorem:

- start from the initial state $\perp = (\emptyset, \ldots, \emptyset)$
- iterates towards the top of the lattice with $F^n(\perp) = F(F^{n-1}(\perp))$
- here each unknown $x_j \in \{x_1, \ldots, x_n\}$ represents a tuple consisting of the range x_j^i and the address set x_j^s
- denote the initial state $x_j^i = \perp$ and $x_j^s = \emptyset$ such that $x_j = \perp = \langle \perp, \emptyset \rangle$

Exercise: Determine the least fixed point:

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
$F(\perp)$	$\perp$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
$F^2(\perp)$	$\perp$	$\{\texttt{NULL}\}$	$[0, 0]$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
5	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
5	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
6	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}\}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
5	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
6	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}\}$
7	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}\}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\mathtt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\mathtt{NULL}\}$
5	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\{\mathtt{NULL}\}$
6	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\mathtt{NULL}\}$
7	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\mathtt{NULL}\}$
8	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$[1,1]$	$\underline{\{s_1, s_2\}}$	$\perp$	$\underline{\{\mathtt{NULL}, s_1}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\mathtt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\mathtt{NULL}\}$
5	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\{\mathtt{NULL}\}$
6	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\mathtt{NULL}\}$
7	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\mathtt{NULL}\}$
8	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$[1,1]$	$\underline{\{s_1, s_2\}}$	$\perp$	$\underline{\{\mathtt{NULL}, s_1}$
9	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$\underline{[1,2]}$	$\{s_1, s_2\}$	$\perp$	$\{\mathtt{NULL}, s_1$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\mathtt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\mathtt{NULL}\}$
5	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\{\mathtt{NULL}\}$
6	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\mathtt{NULL}\}$
7	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\mathtt{NULL}\}$
8	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$[1,1]$	$\underline{\{s_1, s_2\}}$	$\perp$	$\underline{\{\mathtt{NULL}, s_1}$
9	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$\underline{[1,2]}$	$\{s_1, s_2\}$	$\perp$	$\{\mathtt{NULL}, s_1$
10	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\underline{[0,2]}$	$\underline{\{\mathtt{NULL}, s_1, s_2\}}$	$[0,2]$	$\{s_1, s_2\}$	$[0,2]$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\perp$	$\{\mathtt{NULL}, s_1,$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
5	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
6	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}\}$
7	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}\}$
8	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$[1,1]$	$\underline{\{s_1, s_2\}}$	$\perp$	$\underline{\{\texttt{NULL}, s_1}$
9	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$\underline{[1,2]}$	$\{s_1, s_2\}$	$\perp$	$\{\texttt{NULL}, s_1$
10	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,2]}$	$\underline{\{\texttt{NULL}, s_1, s_2\}}$	$[0,2]$	$\{s_1, s_2\}$	$[0,2]$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\perp$	$\{\texttt{NULL}, s_1,$
11	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,2]$	$\{\texttt{NULL}, s_1, s_2, s_3\}$	$[0,2]$	$\underline{\{s_1, s_2, s_3\}}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[1,2]$	$\{s_1, s_2, s_3\}$	$\perp$	$\{\texttt{NULL}, s_1,$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\mathtt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\mathtt{NULL}\}$
5	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\{\mathtt{NULL}\}$
6	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\mathtt{NULL}\}$
7	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\mathtt{NULL}\}$
8	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$[1,1]$	$\underline{\{s_1, s_2\}}$	$\perp$	$\underline{\{\mathtt{NULL}, s_1}$
9	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$\underline{[1,2]}$	$\{s_1, s_2\}$	$\perp$	$\{\mathtt{NULL}, s_1$
10	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\underline{[0,2]}$	$\underline{\{\mathtt{NULL}, s_1, s_2\}}$	$[0,2]$	$\{s_1, s_2\}$	$[0,2]$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\perp$	$\{\mathtt{NULL}, s_1,$
11	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,2]$	$\{\mathtt{NULL}, s_1, s_2, s_3\}$	$[0,2]$	$\underline{\{s_1, s_2, s_3\}}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[1,2]$	$\{s_1, s_2, s_3\}$	$\perp$	$\{\mathtt{NULL}, s_1,$
12	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,2]$	$\{\mathtt{NULL}, s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$\underline{[1,3]}$	$\{s_1, s_2, s_3\}$	$\perp$	$\{\mathtt{NULL}, s_1,$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
5	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
6	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}\}$
7	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}\}$
8	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$[1,1]$	$\underline{\{s_1, s_2\}}$	$\perp$	$\underline{\{\texttt{NULL}, s_1}$
9	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$\underline{[1,2]}$	$\{s_1, s_2\}$	$\perp$	$\{\texttt{NULL}, s_1$
10	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,2]}$	$\underline{\{\texttt{NULL}, s_1, s_2\}}$	$[0,2]$	$\{s_1, s_2\}$	$[0,2]$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\perp$	$\{\texttt{NULL}, s_1,$
11	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,2]$	$\{\texttt{NULL}, s_1, s_2, s_3\}$	$[0,2]$	$\underline{\{s_1, s_2, s_3\}}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[1,2]$	$\{s_1, s_2, s_3\}$	$\perp$	$\{\texttt{NULL}, s_1,$
12	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,2]$	$\{\texttt{NULL}, s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$\underline{[1,3]}$	$\{s_1, s_2, s_3\}$	$\perp$	$\{\texttt{NULL}, s_1,$
13	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,3]}$	$\{\texttt{NULL}, s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[1,3]$	$\{s_1, s_2, s_3\}$	$\underline{[3,3]}$	$\{\texttt{NULL}, s_1,$

Remarks

- program execution can be traced in the equations semantic
- a join operation of two paths is depicted in the equations as a union operation
- a fork of two paths (or meet) can be identified in the equations as a an intersection operation
- the $\perp$ symbol denotes states that can not be reached (yet); this point in the program state is also called unreachable
- the x_7^s points of iterations 4–12 do not make sense because the corresponding code is unreachable due to x_7^i that is $\perp$ for 4–12 $\implies x_7^s = \perp$ for 4–12
- the join operations are domain specific: simple sets union for addresses or the complex $\overline{\Upsilon}$ operation for ranges
- in the former example the meet operations exist only for intervals

Conclusion: simultaneously using two domains at a time leads to new observations and new types of analysis that are based on the interaction between the two (domain interaction).

Num Lattice

In general the domains and associated operations form a lattice.

Definition

Num is the numerical domain bounding the possible values a variable can have.

Theorem

$(Num, \leq_N, \vee_N, \wedge_N)$ *forms a lattice.*

- $\leq_N$ *is the inclusion operator* $\subset$
- $\vee_N = \overline{\Upsilon}$ *is the join operator for ranges*
- $\wedge_N$ *is the meet operation for ranges*

$$
\begin{aligned}
\vee_N : Num \rightarrow Num \times Num, \qquad & \forall x, y \in Num \qquad & x \vee_N y := \sup\{x, y\} \\
\wedge_N : Num \rightarrow Num \times Num, \qquad & \forall x, y \in Num \qquad & x \wedge_N y := \inf\{x, y\}
\end{aligned}
$$

Proof *Num* Lattice

Theorem

$(Num, \subset, \overline{\Upsilon}, \wedge_N)$ *forms a lattice.*

$(Num, \subset)$ POSET: Let $a \leq b \leq c \leq d \in \mathbb{N}$

- reflexive: $x \in Num \implies x \subset x$;
  $[a, b] \subset [a, b]$ ex. $[1, 3] \subset [1, 3]$
- anti-symmetric: $x, y \in Num$ and $x \subset y, y \subset x \implies x = y$;
  $[a, b] \subset [a, c]$, $[a, c] \not\subset [a, b]$, but $[a, c] \subset [a, c] \implies [a, c] = [a, c]$
  ex. $[1, 3] \subset [1, 5]$, $[1, 5] \not\subset [1, 3]$, but $[1, 5] \subset [1, 5] \implies [1, 5] = [1, 5]$
- transitivity: $x, y, z \in Num$ and $x \subset y, y \subset z \implies x \subset z$;
  $[a, b] \subset [a, c]$, $[a, c] \subset [a, d] \implies [a, b] \subset [a, d]$
  ex. $[1, 3] \subset [1, 4]$, $[1, 4] \subset [1, 5] \implies [1, 3] \subset [1, 5]$

Lattice: Let $\overline{\Upsilon}, \wedge_N$ with $x, y, z \in Num$

- associative: $(x \vee_N y) \vee_N z = x \vee_N (y \vee_N z)$; $(x \wedge_N y) \wedge_N z = x \wedge_N (y \wedge_N z)$
- commute: $x \vee_N y = y \vee_N x$; $x \wedge_N y = y \wedge_N x$
- absorb: $x \vee_N (y \wedge_N z) = x$; $x \wedge_N (y \vee_N z) = x$

Pts Lattice

Definition

Pts is the address pointer domain (points-to) used to represent the address spaces towards which a pointer can point.

Theorem

$(Pts, \leq_A, \vee_A, \wedge_A)$ *forms a lattice.*

- *what is* $\leq_A$*?*
- $\vee_A = \cup$ *is the join operation for addresses*
- *what is* $\wedge_A$*?*

The Pts Abstract Domain

Definition

Define $\mathcal{X}$ the finite set of variables of a program P and $\mathcal{A}$ the finite set of addresses towards which these variables can point.

Then $Pts = \mathcal{X} \rightarrow \mathcal{P}(\mathcal{A})$ represents the function that ties each variable $x \in \mathcal{X}$ to a subset of addresses $A(x) \in \mathcal{A}$.

Let $A_1, A_2, A' \in Pts$.

Update: $A \in Pts$ becomes $A' = \{A \cup [x \rightarrow a] \mid a \in \mathcal{A}\}$ such that $A'(x) = a$ and $A'(y) = A(y), \forall y \neq x$.

Order: $A_1 \leq_A A_2 \iff A_1(x) \subseteq A_2(x), \; \forall x \in \mathcal{X}$

Join: $A' = A_1 \vee_A A_2$ s.t. $A'(x) = A_1(x) \cup A_2(x), \; \forall x \in \mathcal{X}$.

Meet: The meet operation can be seen as an update operation that helps us filter the elements of A.

Conclusion: $(Pts, \leq_A)$ forms a CPO: for any subset configuration $B \in \mathcal{P}(Pts)$ there exists $A \in Pts$ such that $A = \bigvee_A B \implies$ we can apply Kleene iterations.

Exercise

What can we say about `p`? What does static analysis tell us?

```
int a, b, *p;
p = NULL;
if (rand())
    p = & a;
if (p)
    *p = 42;
```