

# Special Topics in Logic and Security 1

Variable and Memory Space Analysis

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# How to detect buffer overflow

Often exploited software defects can be reduced to the following snippet:

```
char buf[10];  
i = 0;  
while (i < 20) {  
    buf[i] = i;  
    i = i + 1;  
}
```

How do current tools behave when encountering this sequence?

How can we use static analysis to find such defects?

## Example: C routine

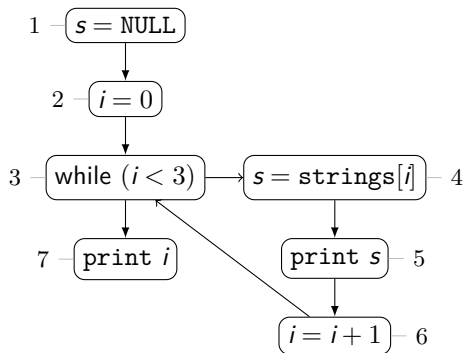
```
char *strings[] = { "One", "Two", "Three" };

char *s = NULL;
int i;

for (i = 0; i < 3; i++) {
    s = strings[i];
    printf("%s\n", s);
}

printf("%d\n", i);
```

## Example: IMP and CFG adaptation



# Preliminaries

We denote

- $v_n^i$  – the possible values of variable  $i$  in nodes  $n = \overline{1, 6}$
- $v_n^s$  – the memory addresses towards which  $s$  points to in nodes  $n = \overline{1, 6}$

We describe the values of  $i$  as an interval and those of  $s$  as an abstract set of addresses  $\mathcal{A}$ .

- `strings[i]` represents the address of string  $i$
- we store this address as  $s_i \in \mathcal{A}$  for  $i \in \{1, 2, 3\}$
- we denote void, zero, or uninitialized addresses with `NULL` or  $\emptyset$

For the constraints system we will need the operator

$$[\ell_1, u_1] \overline{\Upsilon} [\ell_2, u_2] = [\min(\ell_1, \ell_2), \max(u_1, u_2)]$$

that computes the minimum range containing both given ranges.

## Example: ranges

$$v_1^i = \perp$$

$$v_2^i = [0, 0]$$

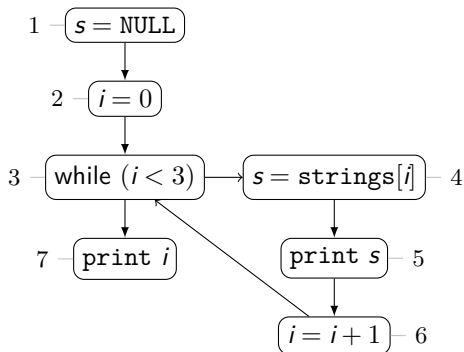
$$v_3^i = v_2^i \bar{\vee} v_6^i$$

$$v_4^i = v_3^i \cap [-2^{31}, 2]$$

$$v_5^i = v_4^i \cap [-2^{31}, 2]$$

$$v_6^i = \{v + 1 \mid v \in v_5^i\}$$

$$v_7^i = v_3^i \cap [3, 2^{31} - 1]$$



# Example: addresses

$$v_1^s = \{\text{NULL}\}$$

$$v_2^s = v_1^s$$

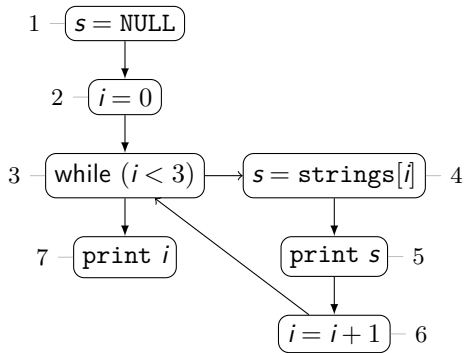
$$v_3^s = v_2^s \cup v_6^s$$

$$v_4^s = \{s_1 \mid 0 \in v_4^i\} \cup \\ \{s_2 \mid 1 \in v_4^i\} \cup \\ \{s_3 \mid 2 \in v_4^i\}$$

$$v_5^s = v_4^s$$

$$v_6^s = v_5^s$$

$$v_7^s = v_3^s$$



# Example: resulting equations

## Ranges

$$v_1^i = \perp$$

$$v_2^i = [0, 0]$$

$$v_3^i = v_2^i \bar{\vee} v_6^i$$

$$v_4^i = v_3^i \cap [-2^{31}, 2]$$

$$v_5^i = v_4^i \cap [-2^{31}, 2]$$

$$v_6^i = \{v + 1 \mid v \in v_5^i\}$$

$$v_7^i = v_3^i \cap [3, 2^{31} - 1]$$

## Addresses

$$v_1^s = \{\text{NULL}\}$$

$$v_2^s = v_1^s$$

$$v_3^s = v_2^s \cup v_6^s$$

$$v_4^s = \{s_1 \mid 0 \in v_4^i\} \cup$$

$$\{s_2 \mid 1 \in v_4^i\} \cup$$

$$\{s_3 \mid 2 \in v_4^i\}$$

$$v_5^s = v_4^s$$

$$v_6^s = v_5^s$$

$$v_7^s = v_3^s$$

## Remark

*Note the link between the values domain of  $i$  represented as ranges and the domain of pointer  $s$  represented as a set of addresses.*



# Solving with the Fixed Point Theorem

The solution to the above equations can be obtained through the fixed point theorem:

- start from the initial state  $\perp = (\emptyset, \dots, \emptyset)$
- iterates towards the top of the lattice with  $F^n(\perp) = F(F^{n-1}(\perp))$
- here each unknown  $x_j \in \{x_1, \dots, x_n\}$  represents a tuple consisting of the range  $x_j^i$  and the address set  $x_j^s$
- denote the initial state  $x_j^i = \perp$  and  $x_j^s = \emptyset$  such that  $x_j = \perp = \langle \perp, \emptyset \rangle$

**Exercise:** Determine the least fixed point:

	$x_1^i$	$x_1^s$	$x_2^i$	$x_2^s$	$x_3^i$	$x_3^s$	$x_4^i$	$x_4^s$	$x_5^i$	$x_5^s$	$x_6^i$	$x_6^s$	$x_7^i$	$x_7^s$
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
$F(\perp)$	$\perp$	$\{\text{NULL}\}$	$[0, 0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
$F^2(\perp)$	$\perp$	$\{\text{NULL}\}$	$[0, 0]$	$\{\text{NULL}\}$	$[0, 0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

# Solution: Determining the Least Fixed Point

	$x_1^i$	$x_1^s$	$x_2^i$	$x_2^s$	$x_3^i$	$x_3^s$	$x_4^i$	$x_4^s$	$x_5^i$	$x_5^s$	$x_6^i$	$x_6^s$	$x_7^i$	$x_7^s$
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$

# Solution: Determining the Least Fixed Point

	$x_1^j$	$x_1^s$	$x_2^j$	$x_2^s$	$x_3^j$	$x_3^s$	$x_4^j$	$x_4^s$	$x_5^j$	$x_5^s$	$x_6^j$	$x_6^s$	$x_7^j$	$x_7^s$
	$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\emptyset$
1	$\perp$	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{NULL}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	{NULL}

# Solution: Determining the Least Fixed Point

	$x_1^i$	$x_1^s$	$x_2^i$	$x_2^s$	$x_3^i$	$x_3^s$	$x_4^i$	$x_4^s$	$x_5^i$	$x_5^s$	$x_6^i$	$x_6^s$	$x_7^i$	$x_7^s$
	$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\emptyset$
1	$\perp$	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{NULL}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	{NULL}
5	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[1, 1]</u>	$\emptyset$	$\perp$	{NULL}

# Solution: Determining the Least Fixed Point

	$x_1^i$	$x_1^s$	$x_2^i$	$x_2^s$	$x_3^i$	$x_3^s$	$x_4^i$	$x_4^s$	$x_5^i$	$x_5^s$	$x_6^i$	$x_6^s$	$x_7^i$	$x_7^s$
	$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\emptyset$
1	$\perp$	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{NULL}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	{NULL}
5	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[1, 1]</u>	$\emptyset$	$\perp$	{NULL}
6	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 1]</u>	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	<u>{s<sub>1</sub>}</u>	$\perp$	{NULL}

# Solution: Determining the Least Fixed Point

	$x_1^i$	$x_1^s$	$x_2^i$	$x_2^s$	$x_3^i$	$x_3^s$	$x_4^i$	$x_4^s$	$x_5^i$	$x_5^s$	$x_6^i$	$x_6^s$	$x_7^i$	$x_7^s$
	$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\emptyset$
1	$\perp$	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{NULL}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	{NULL}
5	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[1, 1]</u>	$\emptyset$	$\perp$	{NULL}
6	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 1]</u>	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	<u>{s<sub>1</sub>}</u>	$\perp$	{NULL}
7	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	<u>{NULL, s<sub>1</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	$\perp$	{NULL}

# Solution: Determining the Least Fixed Point

	$x_1^j$	$x_1^s$	$x_2^j$	$x_2^s$	$x_3^j$	$x_3^s$	$x_4^j$	$x_4^s$	$x_5^j$	$x_5^s$	$x_6^j$	$x_6^s$	$x_7^j$	$x_7^s$
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{NULL}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	{NULL}
5	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[1, 1]</u>	$\emptyset$	$\perp$	{NULL}
6	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 1]</u>	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	<u>{s<sub>1</sub>}</u>	$\perp$	{NULL}
7	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	<u>{NULL, s<sub>1</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	$\perp$	{NULL}
8	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	$\perp$	{NULL, s <sub>1</sub> }

# Solution: Determining the Least Fixed Point

	$x_1^j$	$x_1^s$	$x_2^j$	$x_2^s$	$x_3^j$	$x_3^s$	$x_4^j$	$x_4^s$	$x_5^j$	$x_5^s$	$x_6^j$	$x_6^s$	$x_7^j$	$x_7^s$
	$\perp$	$\perp$	$\emptyset$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{NULL}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	{NULL}
5	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[1, 1]</u>	$\emptyset$	$\perp$	{NULL}
6	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 1]</u>	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	<u>{s<sub>1</sub>}</u>	$\perp$	{NULL}
7	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	<u>{NULL, s<sub>1</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	$\perp$	{NULL}
8	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	$\perp$	<u>{NULL, s<sub>1</sub>}</u>
9	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[1, 2]</u>	{s <sub>1</sub> }	$\perp$	{NULL, s <sub>1</sub> }



# Solution: Determining the Least Fixed Point

	$x_1^j$	$x_1^s$	$x_2^j$	$x_2^s$	$x_3^j$	$x_3^s$	$x_4^j$	$x_4^s$	$x_5^j$	$x_5^s$	$x_6^j$	$x_6^s$	$x_7^j$	$x_7^s$
	$\perp$	$\perp$	$\emptyset$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{NULL}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	{NULL}
5	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[1, 1]</u>	$\emptyset$	$\perp$	{NULL}
6	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 1]</u>	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	<u>{s<sub>1</sub>}</u>	$\perp$	{NULL}
7	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	<u>{NULL, s<sub>1</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	$\perp$	{NULL}
8	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	$\perp$	<u>{NULL, s<sub>1</sub>}</u>
9	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[1, 2]</u>	{s <sub>1</sub> }	$\perp$	{NULL, s <sub>1</sub> }
10	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 2]</u>	{NULL, s <sub>1</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	$\perp$	{NULL, s <sub>1</sub> }

# Solution: Determining the Least Fixed Point

	$x_1^j$	$x_1^s$	$x_2^j$	$x_2^s$	$x_3^j$	$x_3^s$	$x_4^j$	$x_4^s$	$x_5^j$	$x_5^s$	$x_6^j$	$x_6^s$	$x_7^j$	$x_7^s$
	$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\emptyset$
1	$\perp$	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{NULL}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	{NULL}
5	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[1, 1]</u>	$\emptyset$	$\perp$	{NULL}
6	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 1]</u>	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	<u>{s<sub>1</sub>}</u>	$\perp$	{NULL}
7	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	<u>{NULL, s<sub>1</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	$\perp$	{NULL}
8	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	$\perp$	<u>{NULL, s<sub>1</sub>}</u>
9	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[1, 2]</u>	{s <sub>1</sub> }	$\perp$	{NULL, s <sub>1</sub> }
10	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 2]</u>	{NULL, s <sub>1</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	$\perp$	{NULL, s <sub>1</sub> }
11	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 2]	<u>{NULL, s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[0, 2]</u>	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	{s <sub>1</sub> , s <sub>2</sub> }	$\perp$	{NULL, s <sub>1</sub> }

# Solution: Determining the Least Fixed Point

	$x_1^i$	$x_1^s$	$x_2^i$	$x_2^s$	$x_3^i$	$x_3^s$	$x_4^i$	$x_4^s$	$x_5^i$	$x_5^s$	$x_6^i$	$x_6^s$	$x_7^i$	$x_7^s$
⊥	⊥	∅	⊥	∅	⊥	∅	⊥	∅	⊥	∅	⊥	∅	⊥	∅
1	⊥	{NULL}	[0, 0]	∅	⊥	∅	⊥	∅	⊥	∅	⊥	∅	⊥	∅
2	⊥	{NULL}	[0, 0]	{NULL}	[0, 0]	∅	⊥	∅	⊥	∅	⊥	∅	⊥	∅
3	⊥	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{NULL}</u>	<u>[0, 0]</u>	∅	⊥	∅	⊥	∅	⊥	∅
4	⊥	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[0, 0]</u>	∅	⊥	∅	⊥	{NULL}
5	⊥	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[1, 1]</u>	∅	⊥	{NULL}
6	⊥	{NULL}	[0, 0]	{NULL}	<u>[0, 1]</u>	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	<u>{s<sub>1</sub>}</u>	⊥	{NULL}
7	⊥	{NULL}	[0, 0]	{NULL}	[0, 1]	<u>{NULL, s<sub>1</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	⊥	{NULL}
8	⊥	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	⊥	<u>{NULL, s<sub>1</sub>}</u>
9	⊥	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[1, 2]</u>	{s <sub>1</sub> }	⊥	{NULL, s <sub>1</sub> }
10	⊥	{NULL}	[0, 0]	{NULL}	<u>[0, 2]</u>	{NULL, s <sub>1</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	⊥	{NULL, s <sub>1</sub> }
11	⊥	{NULL}	[0, 0]	{NULL}	[0, 2]	<u>{NULL, s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[0, 2]</u>	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	{s <sub>1</sub> , s <sub>2</sub> }	⊥	{NULL, s <sub>1</sub> }
12	⊥	{NULL}	[0, 0]	{NULL}	[0, 2]	{NULL, s <sub>1</sub> , s <sub>2</sub> }	[0, 2]	<u>{s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>	<u>[0, 2]</u>	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	{s <sub>1</sub> , s <sub>2</sub> }	⊥	<u>{NULL, s<sub>1</sub>, s<sub>2</sub>}</u>

# Solution: Determining the Least Fixed Point

	$x_1^j$	$x_1^s$	$x_2^j$	$x_2^s$	$x_3^j$	$x_3^s$	$x_4^j$	$x_4^s$	$x_5^j$	$x_5^s$	$x_6^j$	$x_6^s$	$x_7^j$	$x_7^s$
	$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\emptyset$
1	$\perp$	$\{\text{NULL}\}$	$[0, 0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\text{NULL}\}$	$[0, 0]$	$\{\text{NULL}\}$	$[0, 0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\text{NULL}\}$	$[0, 0]$	$\{\text{NULL}\}$	$[0, 0]$	<u><math>\{\text{NULL}\}</math></u>	<u><math>[0, 0]</math></u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\text{NULL}\}$	$[0, 0]$	$\{\text{NULL}\}$	$[0, 0]$	$\{\text{NULL}\}$	$[0, 0]$	<u><math>\{s_1\}</math></u>	<u><math>[0, 0]</math></u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\text{NULL}\}$
5	$\perp$	$\{\text{NULL}\}$	$[0, 0]$	$\{\text{NULL}\}$	$[0, 0]$	$\{\text{NULL}\}$	$[0, 0]$	$\{s_1\}$	$[0, 0]$	<u><math>\{s_1\}</math></u>	<u><math>[1, 1]</math></u>	$\emptyset$	$\perp$	$\{\text{NULL}\}$
6	$\perp$	$\{\text{NULL}\}$	$[0, 0]$	$\{\text{NULL}\}$	<u><math>[0, 1]</math></u>	$\{\text{NULL}\}$	$[0, 0]$	$\{s_1\}$	$[0, 0]$	$\{s_1\}$	$[1, 1]$	<u><math>\{s_1\}</math></u>	$\perp$	$\{\text{NULL}\}$
7	$\perp$	$\{\text{NULL}\}$	$[0, 0]$	$\{\text{NULL}\}$	$[0, 1]$	<u><math>\{\text{NULL}, s_1\}</math></u>	<u><math>[0, 1]</math></u>	$\{s_1\}$	$[0, 0]$	$\{s_1\}$	$[1, 1]$	$\{s_1\}$	$\perp$	$\{\text{NULL}\}$
8	$\perp$	$\{\text{NULL}\}$	$[0, 0]$	$\{\text{NULL}\}$	$[0, 1]$	$\{\text{NULL}, s_1\}$	$[0, 1]$	<u><math>\{s_1, s_2\}</math></u>	<u><math>[0, 1]</math></u>	$\{s_1\}$	$[1, 1]$	$\{s_1\}$	$\perp$	<u><math>\{\text{NULL}, s_1\}</math></u>
9	$\perp$	$\{\text{NULL}\}$	$[0, 0]$	$\{\text{NULL}\}$	$[0, 1]$	$\{\text{NULL}, s_1\}$	$[0, 1]$	$\{s_1, s_2\}$	$[0, 1]$	<u><math>\{s_1, s_2\}</math></u>	<u><math>[1, 2]</math></u>	$\{s_1\}$	$\perp$	$\{\text{NULL}, s_1\}$
10	$\perp$	$\{\text{NULL}\}$	$[0, 0]$	$\{\text{NULL}\}$	<u><math>[0, 2]</math></u>	$\{\text{NULL}, s_1\}$	$[0, 1]$	$\{s_1, s_2\}$	$[0, 1]$	$\{s_1, s_2\}$	$[1, 2]$	<u><math>\{s_1, s_2\}</math></u>	$\perp$	$\{\text{NULL}, s_1\}$
11	$\perp$	$\{\text{NULL}\}$	$[0, 0]$	$\{\text{NULL}\}$	$[0, 2]$	<u><math>\{\text{NULL}, s_1, s_2\}</math></u>	<u><math>[0, 2]</math></u>	$\{s_1, s_2\}$	$[0, 1]$	$\{s_1, s_2\}$	$[1, 2]$	$\{s_1, s_2\}$	$\perp$	$\{\text{NULL}, s_1\}$
12	$\perp$	$\{\text{NULL}\}$	$[0, 0]$	$\{\text{NULL}\}$	$[0, 2]$	$\{\text{NULL}, s_1, s_2\}$	$[0, 2]$	<u><math>\{s_1, s_2, s_3\}</math></u>	<u><math>[0, 2]</math></u>	$\{s_1, s_2\}$	$[1, 2]$	$\{s_1, s_2\}$	$\perp$	<u><math>\{\text{NULL}, s_1, s_2\}</math></u>
13	$\perp$	$\{\text{NULL}\}$	$[0, 0]$	$\{\text{NULL}\}$	$[0, 2]$	$\{\text{NULL}, s_1, s_2\}$	$[0, 2]$	$\{s_1, s_2, s_3\}$	$[0, 2]$	<u><math>\{s_1, s_2, s_3\}</math></u>	<u><math>[1, 3]</math></u>	$\{s_1, s_2\}$	$\perp$	$\{\text{NULL}, s_1, s_2\}$

# Solution: Determining the Least Fixed Point

	$x_1^i$	$x_1^s$	$x_2^i$	$x_2^s$	$x_3^i$	$x_3^s$	$x_4^i$	$x_4^s$	$x_5^i$	$x_5^s$	$x_6^i$	$x_6^s$	$x_7^i$	$x_7^s$
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{NULL}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	{NULL}
5	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[1, 1]</u>	$\emptyset$	$\perp$	{NULL}
6	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 1]</u>	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	<u>{s<sub>1</sub>}</u>	$\perp$	{NULL}
7	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	<u>{NULL, s<sub>1</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	$\perp$	{NULL}
8	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	$\perp$	<u>{NULL, s<sub>1</sub>}</u>
9	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[1, 2]</u>	{s <sub>1</sub> }	$\perp$	{NULL, s <sub>1</sub> }
10	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 2]</u>	{NULL, s <sub>1</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	$\perp$	{NULL, s <sub>1</sub> }
11	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 2]	<u>{NULL, s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[0, 2]</u>	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	{s <sub>1</sub> , s <sub>2</sub> }	$\perp$	{NULL, s <sub>1</sub> }
12	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 2]	{NULL, s <sub>1</sub> , s <sub>2</sub> }	[0, 2]	<u>{s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>	<u>[0, 2]</u>	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	{s <sub>1</sub> , s <sub>2</sub> }	$\perp$	<u>{NULL, s<sub>1</sub>, s<sub>2</sub>}</u>
13	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 2]	{NULL, s <sub>1</sub> , s <sub>2</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	<u>{s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>	<u>[1, 3]</u>	{s <sub>1</sub> , s <sub>2</sub> }	$\perp$	{NULL, s <sub>1</sub> , s <sub>2</sub> }
14	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 3]</u>	{NULL, s <sub>1</sub> , s <sub>2</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[1, 3]	<u>{s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>	$\perp$	{NULL, s <sub>1</sub> , s <sub>2</sub> }

# Solution: Determining the Least Fixed Point

	$x_1^i$	$x_1^s$	$x_2^i$	$x_2^s$	$x_3^i$	$x_3^s$	$x_4^i$	$x_4^s$	$x_5^i$	$x_5^s$	$x_6^i$	$x_6^s$	$x_7^i$	$x_7^s$
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{NULL}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	{NULL}
5	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[1, 1]</u>	$\emptyset$	$\perp$	{NULL}
6	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 1]</u>	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	<u>{s<sub>1</sub>}</u>	$\perp$	{NULL}
7	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	<u>{NULL, s<sub>1</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	$\perp$	{NULL}
8	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	$\perp$	<u>{NULL, s<sub>1</sub>}</u>
9	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[1, 2]</u>	{s <sub>1</sub> }	$\perp$	{NULL, s <sub>1</sub> }
10	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 2]</u>	{NULL, s <sub>1</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	$\perp$	{NULL, s <sub>1</sub> }
11	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 2]	<u>{NULL, s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[0, 2]</u>	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	{s <sub>1</sub> , s <sub>2</sub> }	$\perp$	{NULL, s <sub>1</sub> }
12	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 2]	{NULL, s <sub>1</sub> , s <sub>2</sub> }	[0, 2]	<u>{s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>	<u>[0, 2]</u>	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	{s <sub>1</sub> , s <sub>2</sub> }	$\perp$	<u>{NULL, s<sub>1</sub>, s<sub>2</sub>}</u>
13	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 2]	{NULL, s <sub>1</sub> , s <sub>2</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	<u>{s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>	<u>[1, 3]</u>	{s <sub>1</sub> , s <sub>2</sub> }	$\perp$	{NULL, s <sub>1</sub> , s <sub>2</sub> }
14	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 3]</u>	{NULL, s <sub>1</sub> , s <sub>2</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[1, 3]	<u>{s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>	$\perp$	{NULL, s <sub>1</sub> , s <sub>2</sub> }
15	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 3]	<u>{NULL, s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[1, 3]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	<u>[3, 3]</u>	{NULL, s <sub>1</sub> , s <sub>2</sub> }

# Solution: Determining the Least Fixed Point

	$x_1^i$	$x_1^s$	$x_2^i$	$x_2^s$	$x_3^i$	$x_3^s$	$x_4^i$	$x_4^s$	$x_5^i$	$x_5^s$	$x_6^i$	$x_6^s$	$x_7^i$	$x_7^s$
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{NULL}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[0, 0]</u>	$\emptyset$	$\perp$	$\emptyset$	$\perp$	{NULL}
5	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[1, 1]</u>	$\emptyset$	$\perp$	{NULL}
6	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 1]</u>	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	<u>{s<sub>1</sub>}</u>	$\perp$	{NULL}
7	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	<u>{NULL, s<sub>1</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	$\perp$	{NULL}
8	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	$\perp$	<u>{NULL, s<sub>1</sub>}</u>
9	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[1, 2]</u>	{s <sub>1</sub> }	$\perp$	{NULL, s <sub>1</sub> }
10	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 2]</u>	{NULL, s <sub>1</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	$\perp$	{NULL, s <sub>1</sub> }
11	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 2]	<u>{NULL, s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[0, 2]</u>	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	{s <sub>1</sub> , s <sub>2</sub> }	$\perp$	{NULL, s <sub>1</sub> }
12	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 2]	{NULL, s <sub>1</sub> , s <sub>2</sub> }	[0, 2]	<u>{s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>	<u>[0, 2]</u>	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	{s <sub>1</sub> , s <sub>2</sub> }	$\perp$	<u>{NULL, s<sub>1</sub>, s<sub>2</sub>}</u>
13	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 2]	{NULL, s <sub>1</sub> , s <sub>2</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	<u>{s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>	<u>[1, 3]</u>	{s <sub>1</sub> , s <sub>2</sub> }	$\perp$	{NULL, s <sub>1</sub> , s <sub>2</sub> }
14	$\perp$	{NULL}	[0, 0]	{NULL}	<u>[0, 3]</u>	{NULL, s <sub>1</sub> , s <sub>2</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[1, 3]	<u>{s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>	$\perp$	{NULL, s <sub>1</sub> , s <sub>2</sub> }
15	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 3]	<u>{NULL, s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[1, 3]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	<u>[3, 3]</u>	{NULL, s <sub>1</sub> , s <sub>2</sub> }
16	$\perp$	{NULL}	[0, 0]	{NULL}	[0, 3]	{NULL, s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[1, 3]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[3, 3]	<u>{NULL, s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>

# Solution: Determining the Least Fixed Point

	$x_1^i$	$x_1^s$	$x_2^i$	$x_2^s$	$x_3^i$	$x_3^s$	$x_4^i$	$x_4^s$	$x_5^i$	$x_5^s$	$x_6^i$	$x_6^s$	$x_7^i$	$x_7^s$
⊥	⊥	∅	⊥	∅	⊥	∅	⊥	∅	⊥	∅	⊥	∅	⊥	∅
1	⊥	{NULL}	[0, 0]	∅	⊥	∅	⊥	∅	⊥	∅	⊥	∅	⊥	∅
2	⊥	{NULL}	[0, 0]	{NULL}	[0, 0]	∅	⊥	∅	⊥	∅	⊥	∅	⊥	∅
3	⊥	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{NULL}</u>	<u>[0, 0]</u>	∅	⊥	∅	⊥	∅	⊥	∅
4	⊥	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[0, 0]</u>	∅	⊥	∅	⊥	{NULL}
5	⊥	{NULL}	[0, 0]	{NULL}	[0, 0]	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	<u>{s<sub>1</sub>}</u>	<u>[1, 1]</u>	∅	⊥	{NULL}
6	⊥	{NULL}	[0, 0]	{NULL}	<u>[0, 1]</u>	{NULL}	[0, 0]	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	<u>{s<sub>1</sub>}</u>	⊥	{NULL}
7	⊥	{NULL}	[0, 0]	{NULL}	[0, 1]	<u>{NULL, s<sub>1</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[0, 0]	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	⊥	{NULL}
8	⊥	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[0, 1]</u>	{s <sub>1</sub> }	[1, 1]	{s <sub>1</sub> }	⊥	<u>{NULL, s<sub>1</sub>}</u>
9	⊥	{NULL}	[0, 0]	{NULL}	[0, 1]	{NULL, s <sub>1</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[1, 2]</u>	{s <sub>1</sub> }	⊥	{NULL, s <sub>1</sub> }
10	⊥	{NULL}	[0, 0]	{NULL}	<u>[0, 2]</u>	{NULL, s <sub>1</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	<u>{s<sub>1</sub>, s<sub>2</sub>}</u>	⊥	{NULL, s <sub>1</sub> }
11	⊥	{NULL}	[0, 0]	{NULL}	[0, 2]	<u>{NULL, s<sub>1</sub>, s<sub>2</sub>}</u>	<u>[0, 2]</u>	{s <sub>1</sub> , s <sub>2</sub> }	[0, 1]	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	{s <sub>1</sub> , s <sub>2</sub> }	⊥	{NULL, s <sub>1</sub> }
12	⊥	{NULL}	[0, 0]	{NULL}	[0, 2]	{NULL, s <sub>1</sub> , s <sub>2</sub> }	[0, 2]	<u>{s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>	<u>[0, 2]</u>	{s <sub>1</sub> , s <sub>2</sub> }	[1, 2]	{s <sub>1</sub> , s <sub>2</sub> }	⊥	<u>{NULL, s<sub>1</sub>, s<sub>2</sub>}</u>
13	⊥	{NULL}	[0, 0]	{NULL}	[0, 2]	{NULL, s <sub>1</sub> , s <sub>2</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	<u>{s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>	<u>[1, 3]</u>	{s <sub>1</sub> , s <sub>2</sub> }	⊥	{NULL, s <sub>1</sub> , s <sub>2</sub> }
14	⊥	{NULL}	[0, 0]	{NULL}	<u>[0, 3]</u>	{NULL, s <sub>1</sub> , s <sub>2</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[1, 3]	<u>{s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>	⊥	{NULL, s <sub>1</sub> , s <sub>2</sub> }
15	⊥	{NULL}	[0, 0]	{NULL}	[0, 3]	<u>{NULL, s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[1, 3]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	<u>[3, 3]</u>	{NULL, s <sub>1</sub> , s <sub>2</sub> }
16	⊥	{NULL}	[0, 0]	{NULL}	[0, 3]	{NULL, s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[1, 3]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	<u>[3, 3]</u>	<u>{NULL, s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub>}</u>
17	⊥	{NULL}	[0, 0]	{NULL}	[0, 3]	{NULL, s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[0, 2]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	[1, 3]	{s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }	<u>[3, 3]</u>	{NULL, s <sub>1</sub> , s <sub>2</sub> , s <sub>3</sub> }



- program execution can be traced in the equations semantic
- a join operation of two paths is depicted in the equations as a union operation
- a fork of two paths (or meet) can be identified in the equations as a an intersection operation
- the  $\perp$  symbol denotes states that can not be reached (yet); this point in the program state is also called unreachable
- the  $x_7^s$  points of iterations 4–14 do not make sense because the corresponding code is unreachable due to  $x_7^i$  that is  $\perp$  for 4–14  $\implies x_7^s = \perp$  for 4–14
- the join operations are domain specific: simple sets union for addresses or the complex  $\overline{\Upsilon}$  operation for ranges
- in the former example the meet operations exist only for intervals

**Conclusion:** simultaneously using two domains at a time leads to new observations and new types of analysis that are based on the interaction between the two (domain interaction).

# Num Lattice

In general the domains and associated operations form a lattice.

## Definition

*Num* is the numerical domain bounding the possible values a variable can have.

## Theorem

$(Num, \leq_N, \vee_N, \wedge_N)$  forms a lattice.

- $\leq_N$  is the inclusion operator  $\subset$
- $\vee_N = \overline{\Upsilon}$  is the join operator for ranges
- $\wedge_N$  is the meet operation for ranges

$$\begin{array}{lll} \vee_N : Num \rightarrow Num \times Num, & \forall x, y \in Num & x \vee_N y := \sup\{x, y\} \\ \wedge_N : Num \rightarrow Num \times Num, & \forall x, y \in Num & x \wedge_N y := \inf\{x, y\} \end{array}$$

# Proof *Num* Lattice

## Theorem

$(Num, \subset, \overline{\Upsilon}, \wedge_N)$  forms a lattice.

$(Num, \subset)$  POSET: Let  $a \leq b \leq c \leq d \in \mathbb{N}$

- reflexive:  $x \in Num \implies x \subset x$  ;  
 $[a, b] \subset [a, b]$  ex.  $[1, 3] \subset [1, 3]$
- anti-symmetric:  $x, y \in Num$  and  $x \subset y, y \subset x \implies x = y$  ;  
 $[a, b] \subset [a, c]$  ,  $[a, c] \not\subset [a, b]$ , except  $[a, c] \subset [a, c] \implies [a, c] = [a, c]$   
ex.  $[1, 3] \subset [1, 5]$  ,  $[1, 5] \not\subset [1, 3]$ , except  $[1, 5] \subset [1, 5] \implies [1, 5] = [1, 5]$
- transitivity:  $x, y, z \in Num$  and  $x \subset y, y \subset z \implies x \subset z$  ;  
 $[a, b] \subset [a, c]$  ,  $[a, c] \subset [a, d] \implies [a, b] \subset [a, d]$   
ex.  $[1, 3] \subset [1, 4]$  ,  $[1, 4] \subset [1, 5] \implies [1, 3] \subset [1, 5]$

Lattice: Let  $\vee_N$  (i.e.  $\overline{\Upsilon}$ ),  $\wedge_N$  with  $x, y, z \in Num$  ([show your work!](#))

- associative:  $(x \vee_N y) \vee_N z = x \vee_N (y \vee_N z)$  ;  $(x \wedge_N y) \wedge_N z = x \wedge_N (y \wedge_N z)$
- commute:  $x \vee_N y = y \vee_N x$  ;  $x \wedge_N y = y \wedge_N x$
- absorb:  $x \vee_N (x \wedge_N y) = x$  ;  $x \wedge_N (x \vee_N y) = x$

## Definition

$Pts$  is the address pointer domain ([points-to](#)) used to represent the address spaces towards which a [pointer](#) can point.

## Theorem

$(Pts, \leq_A, \vee_A, \wedge_A)$  forms a lattice.

- what is  $\leq_A$ ?
- $\vee_A = \cup$  is the [join](#) operation for addresses
- what is  $\wedge_A$ ?

# The Pts Abstract Domain

## Definition

Define  $\mathcal{X}$  the finite set of variables of a program  $P$  and  $\mathcal{A}$  the finite set of addresses towards which these variables can point.

Then  $Pts = \mathcal{X} \rightarrow \mathcal{P}(\mathcal{A})$  represents the set of maps that tie each variable  $x \in \mathcal{X}$  to a subset of addresses  $A(x) \in \mathcal{A}$ .

Let  $A_1, A_2, A' \in Pts$ .

**Update:**  $A \in Pts$  becomes  $A' = \{A \cup [x \rightarrow a] \mid a \in \mathcal{A}\}$  such that  $A'(x) = a$  and  $A'(y) = A(y), \forall y \neq x$ .

**Order:**  $A_1 \leq_A A_2 \iff A_1(x) \subseteq A_2(x), \forall x \in \mathcal{X}$

**Join:**  $A' = A_1 \vee_A A_2$  s.t.  $A'(x) = A_1(x) \cup A_2(x), \forall x \in \mathcal{X}$ .

**Meet:** The [meet](#) operation can be seen as an update operation that helps us filter the elements of  $A$ .

# Properties of the Pts Domain

Let  $NULL \in \mathcal{A}$  be a special tag that is different from the regular addresses.

If  $x \in \mathcal{X}$  contains a value instead of an address, then we say that  $NULL \in A(x)$ .

**Projection:** Define  $\exists_X : Pts \rightarrow Pts$  as the projection operator that resets the points-to set of each  $x \in X$  such that:

$$A' = \exists_X(A) \implies A'(x) = \{NULL\}; \quad A'(y) = y, \quad \forall y \notin X.$$

**Unsatisfiable:** We say that  $A \in Pts$  is unsatisfiable if  $A(x) = \emptyset$  for some  $x \in X$ .  
The unsatisfiable domain  $A$  at a certain program location implies that this location is unreachable.

**Conclusion:**  $(Pts, \leq_A)$  forms a CPO: for any subset configuration  $B \in \mathcal{P}(Pts)$  there exists  $A \in Pts$  such that  $A = \bigvee_A B \implies$  we can apply [Kleene](#) iterations.  
([prove it!](#))

# Exercise

What can we say about `p`? What does static analysis tell us?

```
int a, b, *p;  
p = NULL;  
if (rand())  
    p = & a;  
if (p)  
    *p = 42;
```