Special Topics in Logic and Security 1

Variable and Memory Space Analysis

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How to detect buffer overflow

Often exploited software defects can be reduced to the following snippet:

```
char buf[10];
i = 0;
while (i < 20) {
    buf[i] = i;
    i = i + 1;
}
```

How do current tools behave when encountering this sequence?

How can we use static analysis to find such defects?

Example: C routine

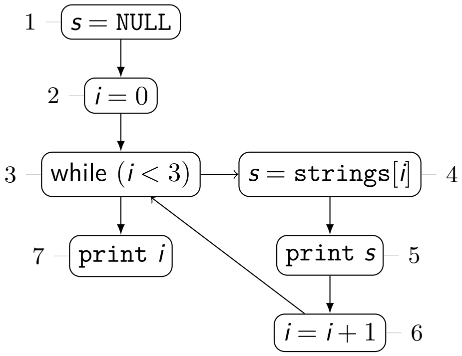
```
char *strings[] = { "One", "Two", "Three" };

char *s = NULL;
int i;

for (i = 0; i < 3; i++) {
    s = strings[i];
    printf("%s\n", s);
}

printf("%d\n", i);
```

Example: IMP and CFG adaptation



Preliminaries

We denote

- v_n^i – the possible values of variable i in nodes $n = \overline{1,6}$
- v_n^s – the memory addresses towards which s points to in nodes $n = \overline{1,6}$

We describe the values of i as an interval and those of s as an abstract set of addresses $\mathcal{A}$.

- `strings[i]` represents the address of string i
- we store this address as $s_i \in \mathcal{A}$ for $i \in \{1, 2, 3\}$
- we denote void, zero, or uninitialized addresses with `NULL` or $\emptyset$

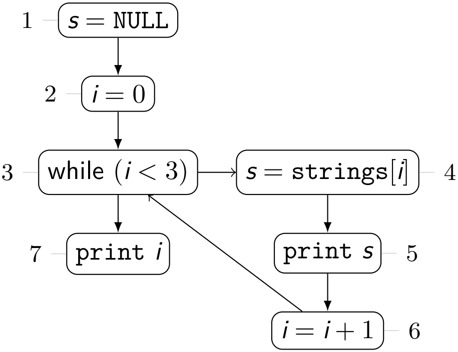
For the constraints system we will need the operator

$$[\ell_1, u_1] \, \overline{\Upsilon} \, [\ell_2, u_2] = [\min(\ell_1, \ell_2), \max(u_1, u_2)]$$

that computes the minimum range containing both given ranges.

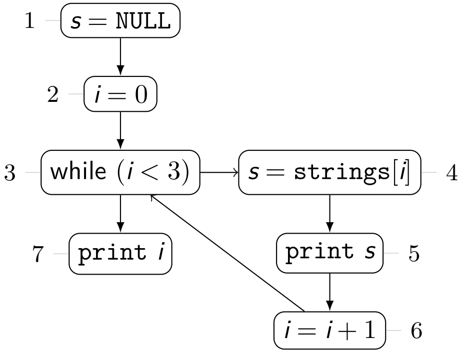
Example: ranges

$$
\begin{aligned}
v^i_1 &= \perp \\
v^i_2 &= [0,0] \\
v^i_3 &= v^i_2 \,\overline{\Upsilon}\, v^i_6 \\
v^i_4 &= v^i_3 \cap \left[-2^{31}, 2\right] \\
v^i_5 &= v^i_4 \cap \left[-2^{31}, 2\right] \\
v^i_6 &= \left\{ v+1 \mid v \in v^i_5 \right\} \\
v^i_7 &= v^i_3 \cap \left[3, 2^{31}-1\right]
\end{aligned}
$$



Example: addresses

$$
\begin{aligned}
v_1^s &= \{\texttt{NULL}\} \\
v_2^s &= v_1^s \\
v_3^s &= v_2^s \cup v_6^s \\
v_4^s &= \left\{s_1 \mid 0 \in v_4^i\right\} \cup \\
&\quad \left\{s_2 \mid 1 \in v_4^i\right\} \cup \\
&\quad \left\{s_3 \mid 2 \in v_4^i\right\} \\
v_5^s &= v_4^s \\
v_6^s &= v_5^s \\
v_7^s &= v_3^s
\end{aligned}
$$



Example: resulting equations

Ranges

$$
\begin{aligned}
v_1^i &= \perp \\
v_2^i &= [0, 0] \\
v_3^i &= v_2^i \,\overline{\Upsilon}\, v_6^i \\
v_4^i &= v_3^i \cap \left[-2^{31}, 2\right] \\
v_5^i &= v_4^i \cap \left[-2^{31}, 2\right] \\
v_6^i &= \left\{ v + 1 \mid v \in v_5^i \right\} \\
v_7^i &= v_3^i \cap \left[3, 2^{31} - 1\right]
\end{aligned}
$$

Addresses

$$
\begin{aligned}
v_1^s &= \{\mathtt{NULL}\} \\
v_2^s &= v_1^s \\
v_3^s &= v_2^s \cup v_6^s \\
v_4^s &= \left\{ s_1 \mid 0 \in v_4^i \right\} \cup \\
&\quad\ \left\{ s_2 \mid 1 \in v_4^i \right\} \cup \\
&\quad\ \left\{ s_3 \mid 2 \in v_4^i \right\} \\
v_5^s &= v_4^s \\
v_6^s &= v_5^s \\
v_7^s &= v_3^s
\end{aligned}
$$

Remark

Note the link between the values domain of i represented as ranges and the domain of `pointer` *s represented as a set of addresses.*

Solving with the Fixed Point Theorem

The solution to the above equations can be obtained through the fixed point theorem:

- start from the initial state $\perp = (\emptyset, \ldots, \emptyset)$
- iterates towards the top of the lattice with $F^n(\perp) = F(F^{n-1}(\perp))$
- here each unknown $x_j \in \{x_1, \ldots, x_n\}$ represents a tuple consisting of the range x_j^i and the address set x_j^s
- denote the initial state $x_j^i = \perp$ and $x_j^s = \emptyset$ such that $x_j = \perp = \langle \perp, \emptyset \rangle$

Exercise: Determine the least fixed point:

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
$F(\perp)$	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
$F^2(\perp)$	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{\texttt{NULL}\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{\texttt{NULL}\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{\texttt{NULL}\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
5	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[1,1]}$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\bot$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$
1	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$
2	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$
3	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{\texttt{NULL}\}}$	$\underline{[0,0]}$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$
4	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[0,0]}$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\{\texttt{NULL}\}$
5	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[1,1]}$	$\emptyset$	$\bot$	$\{\texttt{NULL}\}$
6	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,1]}$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\underline{\{s_1\}}$	$\bot$	$\{\texttt{NULL}\}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{\texttt{NULL}\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
5	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[1,1]}$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
6	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,1]}$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\underline{\{s_1\}}$	$\perp$	$\{\texttt{NULL}\}$
7	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\underline{\{\texttt{NULL}, s_1\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}\}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{\texttt{NULL}\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
5	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[1,1]}$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
6	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,1]}$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\underline{\{s_1\}}$	$\perp$	$\{\texttt{NULL}\}$
7	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\underline{\{\texttt{NULL}, s_1\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}\}$
8	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\underline{\{\texttt{NULL}, s_1\}}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{\texttt{NULL}\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
5	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[1,1]}$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
6	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,1]}$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\underline{\{s_1\}}$	$\perp$	$\{\texttt{NULL}\}$
7	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\underline{\{\texttt{NULL}, s_1\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}\}$
8	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\underline{\{\texttt{NULL}, s_1\}}$
9	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[1,2]}$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}, s_1\}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{\texttt{NULL}\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
5	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[1,1]}$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
6	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,1]}$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\underline{\{s_1\}}$	$\perp$	$\{\texttt{NULL}\}$
7	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\underline{\{\texttt{NULL}, s_1\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}\}$
8	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\underline{\{\texttt{NULL}, s_1\}}$
9	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[1,2]}$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}, s_1\}$
10	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,2]}$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$[1,2]$	$\underline{\{s_1, s_2\}}$	$\perp$	$\{\texttt{NULL}, s_1\}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\underline{\{\mathtt{NULL}\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\mathtt{NULL}\}$
5	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[1,1]}$	$\emptyset$	$\perp$	$\{\mathtt{NULL}\}$
6	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\underline{[0,1]}$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\underline{\{s_1\}}$	$\perp$	$\{\mathtt{NULL}\}$
7	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\underline{\{\mathtt{NULL}, s_1\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\mathtt{NULL}\}$
8	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\underline{\{\mathtt{NULL}, s_1\}}$
9	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[1,2]}$	$\{s_1\}$	$\perp$	$\{\mathtt{NULL}, s_1\}$
10	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\underline{[0,2]}$	$\{\mathtt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$[1,2]$	$\underline{\{s_1, s_2\}}$	$\perp$	$\{\mathtt{NULL}, s_1\}$
11	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,2]$	$\underline{\{\mathtt{NULL}, s_1, s_2\}}$	$\underline{[0,2]}$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\perp$	$\{\mathtt{NULL}, s_1\}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{\texttt{NULL}\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
5	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[1,1]}$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
6	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,1]}$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\underline{\{s_1\}}$	$\perp$	$\{\texttt{NULL}\}$
7	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\underline{\{\texttt{NULL}, s_1\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}\}$
8	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\underline{\{\texttt{NULL}, s_1\}}$
9	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[1,2]}$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}, s_1\}$
10	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,2]}$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$[1,2]$	$\underline{\{s_1, s_2\}}$	$\perp$	$\{\texttt{NULL}, s_1\}$
11	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,2]$	$\underline{\{\texttt{NULL}, s_1, s_2\}}$	$\underline{[0,2]}$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\perp$	$\{\texttt{NULL}, s_1\}$
12	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,2]$	$\{\texttt{NULL}, s_1, s_2\}$	$[0,2]$	$\underline{\{s_1, s_2, s_3\}}$	$\underline{[0,2]}$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\perp$	$\underline{\{\texttt{NULL}, s_1, s_2\}}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\bot$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$
1	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$
2	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$
3	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{\texttt{NULL}\}}$	$\underline{[0,0]}$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$
4	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[0,0]}$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\{\texttt{NULL}\}$
5	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[1,1]}$	$\emptyset$	$\bot$	$\{\texttt{NULL}\}$
6	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,1]}$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\underline{\{s_1\}}$	$\bot$	$\{\texttt{NULL}\}$
7	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\underline{\{\texttt{NULL}, s_1\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\bot$	$\{\texttt{NULL}\}$
8	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\bot$	$\underline{\{\texttt{NULL}, s_1\}}$
9	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[1,2]}$	$\{s_1\}$	$\bot$	$\{\texttt{NULL}, s_1\}$
10	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,2]}$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$[1,2]$	$\underline{\{s_1, s_2\}}$	$\bot$	$\{\texttt{NULL}, s_1\}$
11	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,2]$	$\underline{\{\texttt{NULL}, s_1, s_2\}}$	$\underline{[0,2]}$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\bot$	$\{\texttt{NULL}, s_1\}$
12	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,2]$	$\{\texttt{NULL}, s_1, s_2\}$	$[0,2]$	$\underline{\{s_1, s_2, s_3\}}$	$\underline{[0,2]}$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\bot$	$\underline{\{\texttt{NULL}, s_1, s_2\}}$
13	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,2]$	$\{\texttt{NULL}, s_1, s_2\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\underline{\{s_1, s_2, s_3\}}$	$\underline{[1,3]}$	$\{s_1, s_2\}$	$\bot$	$\{\texttt{NULL}, s_1, s_2\}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\bot$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$
1	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$
2	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$
3	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{\texttt{NULL}\}}$	$\underline{[0,0]}$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$
4	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[0,0]}$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\{\texttt{NULL}\}$
5	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[1,1]}$	$\emptyset$	$\bot$	$\{\texttt{NULL}\}$
6	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,1]}$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\underline{\{s_1\}}$	$\bot$	$\{\texttt{NULL}\}$
7	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\underline{\{\texttt{NULL}, s_1\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\bot$	$\{\texttt{NULL}\}$
8	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\bot$	$\underline{\{\texttt{NULL}, s_1\}}$
9	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[1,2]}$	$\{s_1\}$	$\bot$	$\{\texttt{NULL}, s_1\}$
10	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,2]}$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$[1,2]$	$\underline{\{s_1, s_2\}}$	$\bot$	$\{\texttt{NULL}, s_1\}$
11	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,2]$	$\underline{\{\texttt{NULL}, s_1, s_2\}}$	$\underline{[0,2]}$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\bot$	$\{\texttt{NULL}, s_1\}$
12	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,2]$	$\{\texttt{NULL}, s_1, s_2\}$	$[0,2]$	$\underline{\{s_1, s_2, s_3\}}$	$\underline{[0,2]}$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\bot$	$\underline{\{\texttt{NULL}, s_1, s_2\}}$
13	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,2]$	$\{\texttt{NULL}, s_1, s_2\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\underline{\{s_1, s_2, s_3\}}$	$\underline{[1,3]}$	$\{s_1, s_2\}$	$\bot$	$\{\texttt{NULL}, s_1, s_2\}$
14	$\bot$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,3]}$	$\{\texttt{NULL}, s_1, s_2\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[1,3]$	$\underline{\{s_1, s_2, s_3\}}$	$\bot$	$\{\texttt{NULL}, s_1, s_2\}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\underline{\{\mathtt{NULL}\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\mathtt{NULL}\}$
5	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[1,1]}$	$\emptyset$	$\perp$	$\{\mathtt{NULL}\}$
6	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\underline{[0,1]}$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\underline{\{s_1\}}$	$\perp$	$\{\mathtt{NULL}\}$
7	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\underline{\{\mathtt{NULL}, s_1\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\mathtt{NULL}\}$
8	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\underline{\{\mathtt{NULL}, s_1\}}$
9	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[1,2]}$	$\{s_1\}$	$\perp$	$\{\mathtt{NULL}, s_1\}$
10	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\underline{[0,2]}$	$\{\mathtt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$[1,2]$	$\underline{\{s_1, s_2\}}$	$\perp$	$\{\mathtt{NULL}, s_1\}$
11	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,2]$	$\underline{\{\mathtt{NULL}, s_1, s_2\}}$	$\underline{[0,2]}$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\perp$	$\{\mathtt{NULL}, s_1\}$
12	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,2]$	$\{\mathtt{NULL}, s_1, s_2\}$	$[0,2]$	$\underline{\{s_1, s_2, s_3\}}$	$\underline{[0,2]}$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\perp$	$\underline{\{\mathtt{NULL}, s_1, s_2\}}$
13	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,2]$	$\{\mathtt{NULL}, s_1, s_2\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\underline{\{s_1, s_2, s_3\}}$	$\underline{[1,3]}$	$\{s_1, s_2\}$	$\perp$	$\{\mathtt{NULL}, s_1, s_2\}$
14	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\underline{[0,3]}$	$\{\mathtt{NULL}, s_1, s_2\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[1,3]$	$\underline{\{s_1, s_2, s_3\}}$	$\perp$	$\{\mathtt{NULL}, s_1, s_2\}$
15	$\perp$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,3]$	$\underline{\{\mathtt{NULL}, s_1, s_2, s_3\}}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[1,3]$	$\{s_1, s_2, s_3\}$	$\underline{[3,3]}$	$\{\mathtt{NULL}, s_1, s_2\}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\bot$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$
1	$\bot$	$\{\mathtt{NULL}\}$	$[0,0]$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$
2	$\bot$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$
3	$\bot$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\underline{\{\mathtt{NULL}\}}$	$\underline{[0,0]}$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\emptyset$
4	$\bot$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[0,0]}$	$\emptyset$	$\bot$	$\emptyset$	$\bot$	$\{\mathtt{NULL}\}$
5	$\bot$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[1,1]}$	$\emptyset$	$\bot$	$\{\mathtt{NULL}\}$
6	$\bot$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\underline{[0,1]}$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\underline{\{s_1\}}$	$\bot$	$\{\mathtt{NULL}\}$
7	$\bot$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\underline{\{\mathtt{NULL}, s_1\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\bot$	$\{\mathtt{NULL}\}$
8	$\bot$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\bot$	$\underline{\{\mathtt{NULL}, s_1\}}$
9	$\bot$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,1]$	$\{\mathtt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[1,2]}$	$\{s_1\}$	$\bot$	$\{\mathtt{NULL}, s_1\}$
10	$\bot$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\underline{[0,2]}$	$\{\mathtt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$[1,2]$	$\underline{\{s_1, s_2\}}$	$\bot$	$\{\mathtt{NULL}, s_1\}$
11	$\bot$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,2]$	$\underline{\{\mathtt{NULL}, s_1, s_2\}}$	$\underline{[0,2]}$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\bot$	$\{\mathtt{NULL}, s_1\}$
12	$\bot$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,2]$	$\{\mathtt{NULL}, s_1, s_2\}$	$[0,2]$	$\underline{\{s_1, s_2, s_3\}}$	$\underline{[0,2]}$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\bot$	$\underline{\{\mathtt{NULL}, s_1, s_2\}}$
13	$\bot$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,2]$	$\{\mathtt{NULL}, s_1, s_2\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\underline{\{s_1, s_2, s_3\}}$	$\underline{[1,3]}$	$\{s_1, s_2\}$	$\bot$	$\{\mathtt{NULL}, s_1, s_2\}$
14	$\bot$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$\underline{[0,3]}$	$\{\mathtt{NULL}, s_1, s_2\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[1,3]$	$\underline{\{s_1, s_2, s_3\}}$	$\bot$	$\{\mathtt{NULL}, s_1, s_2\}$
15	$\bot$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,3]$	$\underline{\{\mathtt{NULL}, s_1, s_2, s_3\}}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[1,3]$	$\{s_1, s_2, s_3\}$	$\underline{[3,3]}$	$\{\mathtt{NULL}, s_1, s_2\}$
16	$\bot$	$\{\mathtt{NULL}\}$	$[0,0]$	$\{\mathtt{NULL}\}$	$[0,3]$	$\{\mathtt{NULL}, s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[1,3]$	$\{s_1, s_2, s_3\}$	$[3,3]$	$\underline{\{\mathtt{NULL}, s_1, s_2, s_3\}}$

Solution: Determining the Least Fixed Point

	x_1^i	x_1^s	x_2^i	x_2^s	x_3^i	x_3^s	x_4^i	x_4^s	x_5^i	x_5^s	x_6^i	x_6^s	x_7^i	x_7^s
$\perp$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
1	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
2	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
3	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{\texttt{NULL}\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\emptyset$
4	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[0,0]}$	$\emptyset$	$\perp$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
5	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\underline{\{s_1\}}$	$\underline{[1,1]}$	$\emptyset$	$\perp$	$\{\texttt{NULL}\}$
6	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,1]}$	$\{\texttt{NULL}\}$	$[0,0]$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\underline{\{s_1\}}$	$\perp$	$\{\texttt{NULL}\}$
7	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\underline{\{\texttt{NULL}, s_1\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[0,0]$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}\}$
8	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[0,1]}$	$\{s_1\}$	$[1,1]$	$\{s_1\}$	$\perp$	$\underline{\{\texttt{NULL}, s_1\}}$
9	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,1]$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\underline{\{s_1, s_2\}}$	$\underline{[1,2]}$	$\{s_1\}$	$\perp$	$\{\texttt{NULL}, s_1\}$
10	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,2]}$	$\{\texttt{NULL}, s_1\}$	$[0,1]$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$[1,2]$	$\underline{\{s_1, s_2\}}$	$\perp$	$\{\texttt{NULL}, s_1\}$
11	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,2]$	$\underline{\{\texttt{NULL}, s_1, s_2\}}$	$\underline{[0,2]}$	$\{s_1, s_2\}$	$[0,1]$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\perp$	$\{\texttt{NULL}, s_1\}$
12	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,2]$	$\{\texttt{NULL}, s_1, s_2\}$	$[0,2]$	$\underline{\{s_1, s_2, s_3\}}$	$\underline{[0,2]}$	$\{s_1, s_2\}$	$[1,2]$	$\{s_1, s_2\}$	$\perp$	$\underline{\{\texttt{NULL}, s_1, s_2\}}$
13	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,2]$	$\{\texttt{NULL}, s_1, s_2\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\underline{\{s_1, s_2, s_3\}}$	$\underline{[1,3]}$	$\{s_1, s_2\}$	$\perp$	$\{\texttt{NULL}, s_1, s_2\}$
14	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$\underline{[0,3]}$	$\{\texttt{NULL}, s_1, s_2\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[1,3]$	$\underline{\{s_1, s_2, s_3\}}$	$\perp$	$\{\texttt{NULL}, s_1, s_2\}$
15	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,3]$	$\underline{\{\texttt{NULL}, s_1, s_2, s_3\}}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[1,3]$	$\{s_1, s_2, s_3\}$	$\underline{[3,3]}$	$\{\texttt{NULL}, s_1, s_2\}$
16	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,3]$	$\{\texttt{NULL}, s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[1,3]$	$\{s_1, s_2, s_3\}$	$[3,3]$	$\underline{\{\texttt{NULL}, s_1, s_2, s_3\}}$
17	$\perp$	$\{\texttt{NULL}\}$	$[0,0]$	$\{\texttt{NULL}\}$	$[0,3]$	$\{\texttt{NULL}, s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[0,2]$	$\{s_1, s_2, s_3\}$	$[1,3]$	$\{s_1, s_2, s_3\}$	$[3,3]$	$\{\texttt{NULL}, s_1, s_2, s_3\}$

Remarks

- program execution can be traced in the equations semantic
- a join operation of two paths is depicted in the equations as a union operation
- a fork of two paths (or meet) can be identified in the equations as a an intersection operation
- the $\perp$ symbol denotes states that can not be reached (yet); this point in the program state is also called unreachable
- the x_7^s points of iterations 4–14 do not make sense because the corresponding code is unreachable due to x_7^i that is $\perp$ for 4–14 $\implies x_7^s = \perp$ for 4–14
- the join operations are domain specific: simple sets union for addresses or the complex $\overline{\Upsilon}$ operation for ranges
- in the former example the meet operations exist only for intervals

Conclusion: simultaneously using two domains at a time leads to new observations and new types of analysis that are based on the interaction between the two (domain interaction).

Num Lattice

In general the domains and associated operations form a lattice.

Definition

Num is the numerical domain bounding the possible values a variable can have.

Theorem

$(Num, \leq_N, \vee_N, \wedge_N)$ *forms a lattice.*

- $\leq_N$ *is the inclusion operator* $\subset$
- $\vee_N = \overline{\Upsilon}$ *is the join operator for ranges*
- $\wedge_N$ *is the meet operation for ranges*

$$\begin{aligned} \vee_N : Num \to Num \times Num, \qquad & \forall x, y \in Num \qquad & x \vee_N y := \sup\{x, y\} \\ \wedge_N : Num \to Num \times Num, \qquad & \forall x, y \in Num \qquad & x \wedge_N y := \inf\{x, y\} \end{aligned}$$

Proof *Num* Lattice

Theorem

$(Num, \subset, \overline{\Upsilon}, \wedge_N)$ *forms a lattice.*

$(Num, \subset)$ POSET: Let $a \leq b \leq c \leq d \in \mathbb{N}$

- reflexive: $x \in Num \implies x \subset x$;
  $[a, b] \subset [a, b]$ ex. $[1, 3] \subset [1, 3]$
- anti-symmetric: $x, y \in Num$ and $x \subset y, y \subset x \implies x = y$;
  $[a, b] \subset [a, c]$, $[a, c] \not\subset [a, b]$, except $[a, c] \subset [a, c] \implies [a, c] = [a, c]$
  ex. $[1, 3] \subset [1, 5]$, $[1, 5] \not\subset [1, 3]$, except $[1, 5] \subset [1, 5] \implies [1, 5] = [1, 5]$
- transitivity: $x, y, z \in Num$ and $x \subset y, y \subset z \implies x \subset z$;
  $[a, b] \subset [a, c]$, $[a, c] \subset [a, d] \implies [a, b] \subset [a, d]$
  ex. $[1, 3] \subset [1, 4]$, $[1, 4] \subset [1, 5] \implies [1, 3] \subset [1, 5]$

Lattice: Let $\vee_N$ (i.e.$\overline{\Upsilon}$), $\wedge_N$ with $x, y, z \in Num$ (show your work!)

- associative: $(x \vee_N y) \vee_N z = x \vee_N (y \vee_N z)$; $(x \wedge_N y) \wedge_N z = x \wedge_N (y \wedge_N z)$
- commute: $x \vee_N y = y \vee_N x$; $x \wedge_N y = y \wedge_N x$
- absorb: $x \vee_N (y \wedge_N z) = x$; $x \wedge_N (y \vee_N z) = x$

Pts Lattice

Definition

Pts is the address pointer domain (points-to) used to represent the address spaces towards which a pointer can point.

Theorem

$(Pts, \leq_A, \vee_A, \wedge_A)$ *forms a lattice.*

- *what is* $\leq_A$*?*
- $\vee_A = \cup$ *is the join operation for addresses*
- *what is* $\wedge_A$*?*

The Pts Abstract Domain

Definition

Define $\mathcal{X}$ the finite set of variables of a program P and $\mathcal{A}$ the finite set of addresses towards which these variables can point.

Then $Pts = \mathcal{X} \to \mathcal{P}(\mathcal{A})$ represents the set of maps that tie each variable $x \in \mathcal{X}$ to a subset of addresses $A(x) \in \mathcal{A}$.

Let $A_1, A_2, A' \in Pts$.

Update: $A \in Pts$ becomes $A' = \{A \cup [x \to a] \mid a \in \mathcal{A}\}$ such that $A'(x) = a$ and $A'(y) = A(y), \forall y \neq x$.

Order: $A_1 \leq_A A_2 \iff A_1(x) \subseteq A_2(x), \; \forall x \in \mathcal{X}$

Join: $A' = A_1 \vee_A A_2$ s.t. $A'(x) = A_1(x) \cup A_2(x), \; \forall x \in \mathcal{X}$.

Meet: The meet operation can be seen as an update operation that helps us filter the elements of A.

Properties of the Pts Domain

Let $\texttt{NULL} \in \mathcal{A}$ be a special tag that is different from the regular addresses.

If $x \in \mathcal{X}$ contains a value instead of an address, then we say that $\texttt{NULL} \in A(x)$.

Projection: Define $\exists_X : Pts \to Pts$ as the projection operator that resets the points-to set of each $x \in X$ such that:

$$A' = \exists_X(A) \implies A'(x) = \{NULL\}; \quad A'(y) = y, \;\; \forall y \notin X.$$

Unsatisfiable: We say that $A \in Pts$ is unsatisfiable if $A(x) = \emptyset$ for some $x \in X$.

The unsatisfiable domain A at a certain program location implies that this location is unreachable.

Conclusion: $(Pts, \leq_A)$ forms a CPO: for any subset configuration $B \in \mathcal{P}(Pts)$ there exists $A \in Pts$ such that $A = \bigvee_A B \implies$ we can apply Kleene iterations. (prove it!)

Exercise

What can we say about `p`? What does static analysis tell us?

```
int a, b, *p;
p = NULL;
if (rand())
    p = & a;
if (p)
    *p = 42;
```