A quantitative analyis of the Lion and Man Game

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Pursuit-Evasion Games: The Lion and Man Game

The lion and man problem, going back to R. Rado, is one of the most challenging pursuit-evasion games. **Littlewood's Miscellany** it is described as follows:

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This fact, as well as the potential applications in different fields such as robotics, biology and random processes.

Many variants of the game:

- continuous and discrete,
- one or more evaders hunted by one or more pursuers,
- physical capture or ε-capture,
- different degrees of freedom in the movement of the lion.

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Fix upper bound $\boldsymbol{D} > \boldsymbol{0}$ on the distance the lion and the man may jump.

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After n steps, the lion moves from $\boldsymbol{L_n}$ to $\boldsymbol{L_{n+1}}$ **along a geodesic** from $\boldsymbol{L_n}$ to $\boldsymbol{M_n}$, i.e. $\boldsymbol{d(L_n, M_n) = d(L_n, L_{n+1}) + d(L_{n+1}, M_n)}$, s.t. its distance to $\boldsymbol{L_n}$ equals $\boldsymbol{\min\{D, d(L_n, M_n)\}}$.

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Given a **metric space** $\boldsymbol{X}$, we say that the lion wins if $\boldsymbol{\lim_{n\to\infty} d(L_{n+1}, M_n) = 0}$ for **any pair** of sequences $\boldsymbol{(L_n), (M_n)}$ satisfying the previous **metric conditions** for any $\boldsymbol{D > 0}$. Otherwise the man wins.

The point of departure of our research

Let (X, d) be a **uniquely** geodesic space. Then the move of the lion is **uniquely determined**

$$L_{n+1} := (1-\lambda_n)L_n + \lambda_n M_n, \ \ \lambda_n := \min\{D, d(L_n, M_n)\}/d(L_n, M_n).$$

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The proof proceeds by an induction along an **iterated use of sequential compactness** i.e. of **arithmetical comprehension**!

Proof-Mining

- Based on **general logical metatheorems**: one can **extract an explicit rate of convergence** if one upgrades 'uniquely geodesic' and 'betweenness property' to **'uniform uniquely geodesic (with modulus)'** and **'uniform betweenness property (with modulus)'**.

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- Proof mining provides an explicit **rate of convergence** which only depends on a given **modulus of uniform betweenness** $\boldsymbol{\Theta}$ (in addition to $b \geq \mathrm{diam}(A), \varepsilon > 0, D$).
- **Crucial:** $\boldsymbol{\lim d(L_{n+1}, M_n) = 0} \in \boldsymbol{\Pi^0_2}$ since the sequence is **nonincreasing**.

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- **Ptolemy spaces** (based on the proof of betweenness by A. Nicolae)
- A **particular nonstrictly normed space** $(\mathbb{R}^3, \|\cdot\|_{\mathrm{DW}})$ (based on the proof of betweenness by Diminnie and White).

Basics of Proof Mining

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- effective bounds,
- algorithms,
- continuous dependency or full independence from certain parameters,
- generalizations of proofs: weakening of premises.

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PA$^{\omega,X}$ is the extension of Peano Arithmetic to all types.

$\mathcal{A}^{\omega,X}$:=**PA**$^{\omega,X}$+**DC**, where

DC: axiom of dependent choice for all types

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$\mathcal{A}^{\omega}[X, d, \ldots]$ results by adding constants $d_X, \ldots$ with axioms expressing that $(X, d, \ldots)$ is a nonempty metric, hyperbolic $\ldots$ space.

Majorization

y, x functionals of types $\rho, \widehat{\rho} := \rho[\mathbb{N}/X]$ and a^X of type X:

$$x^{\mathbb{N}} \gtrsim^a_{\mathbb{N}} y^{\mathbb{N}} :\equiv x \geq y$$
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Example:

$$\mathbf{f^* \gtrsim^a_{X \to X} f \equiv \forall n \in \mathbb{N}, x \in X[n \geq d(a, x) \to f^*(n) \geq d(a, f(x))].}$$

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$f : X \to X$ is **nonexpansive (n.e.)** if $d(f(x), f(y)) \leq d(x, y)$.

Then $\lambda n.n + b \gtrsim^a_{X\to X} f$, if $d(a, f(a)) \leq b$.

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Then $\boldsymbol{\lambda n.n + b \gtrsim^{a}_{X \to X} f}$, if $\boldsymbol{d(a, f(a)) \leq b.}$

Normed linear case: $\boldsymbol{a := 0_X}$.

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Also **several** metric structures $X_1, \ldots, X_n$ simultaneously (Günzel/K.).

Small types (over $\mathbb{N}, X$): $\mathbb{N}$, $\mathbb{N} \to \mathbb{N}$, X, $\mathbb{N} \to X$, $X \to X$.

Theorem (K., Trans.AMS 2005, Gerhardy/K.,Trans.AMS 2008)

Let P, K be Polish resp. compact metric spaces, $A_\exists$ $\exists$-formula, $\underline{\tau}$ small. If $\mathcal{A}^\omega[X, d]$ **proves**

$$\forall \mathbf{x} \in \mathbf{P} \forall \mathbf{y} \in \mathbf{K} \forall \underline{\mathbf{z}}^{\underline{\tau}} \exists \mathbf{v}^{\mathbb{N}} \mathbf{A}_\exists(\mathbf{x}, \mathbf{y}, \underline{\mathbf{z}}, \mathbf{v}),$$

then one can extract a **computable** $\varphi : \mathbb{N}^{\mathbb{N}} \times \underline{\mathbb{N}}^{(\mathbb{N})} \to \mathbb{N}$ s.t. the following holds in every nonempty metric space: for all representatives $r_x \in \mathbb{N}^{\mathbb{N}}$ of $x \in P$ and all $\underline{z}^{\underline{\tau}}$ and $\underline{z}^* \in \mathbb{N}^{(\mathbb{N})}$ s.t. $\exists a \in X(\underline{z}^* \gtrsim^a_{\underline{\tau}} \underline{z})$:

$$\forall \mathbf{y} \in \mathbf{K} \exists \mathbf{v} \leq \varphi(\mathbf{r_x}, \underline{\mathbf{z}}^*)\, \mathbf{A}_\exists(\mathbf{x}, \mathbf{y}, \underline{\mathbf{z}}, \mathbf{v}).$$

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Recent Survey:

K., Proof-Theoretic Methods in Nonlinear Analysis. Proc. ICM 2018.

Metric spaces with the betweenness and uniform betweenness properties

The concept of 'betweenness' can be formulated in arbitrary metric spaces:

Definition (Diminnie and White 1981)

Let (X, d) be a metric space. X satisfies the betweenness property if for any distinct points $x, y, z, w \in X$

$$\left.\begin{array}{l} d(x,y)+d(y,z) \leq d(x,z) \\ d(y,z)+d(z,w) \leq d(y,w) \end{array}\right\} \Rightarrow d(x,z)+d(z,w) \leq d(x,w).$$

Logical form (put in prenex normal form):

$$\forall x, y, z, w \in X\, \forall k, m \in \mathbb{N}\, \exists n \in \mathbb{N}$$

$$\left(\left. \begin{array}{l} \mathrm{sep}\{x, y, z, w\} \geq 2^{-k} \wedge \\ d(x,y) + d(y,z) \leq d(x,z) + 2^{-n} \wedge \\ d(y,z) + d(z,w) \leq d(y,w) + 2^{-n} \end{array} \right\} \to d(x,z) + d(z,w) < d(x,w) + 2^{-m} \right)$$

where $\Big(\dots \Big)$ is a purely existential formula $A_\exists$.

Logical form (put in prenex normal form):

$$\forall x, y, z, w \in X \, \forall k, m \in \mathbb{N} \, \exists n \in \mathbb{N}$$
$$\left(\left. \begin{array}{l} \mathrm{sep}\{x, y, z, w\} \geq 2^{-k} \wedge \\ d(x, y) + d(y, z) \leq d(x, z) + 2^{-n} \wedge \\ d(y, z) + d(z, w) \leq d(y, w) + 2^{-n} \end{array} \right\} \to d(x, z) + d(z, w) < d(x, w) + 2^{-m} \right)$$

where $\Big(\ldots \Big)$ is a purely existential formula $A_\exists$.

Logic bound extraction theorems extract from (suitable) proofs of X satisfying the betweenness property, a bound (and hence **realizer**) for $\exists n \in \mathbb{N}$ which only depends on k, m and **majorants** for x, y, z, w.

In metric setting (taking as reference point e.g. x) any $b \in \mathbb{N}$ s.t. $b \geq diam\{x, y, z, w\}$ provides such a majorant. This gives rise to the following notion (expressed for convenience in ε/δ-style):

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Definition (K., Lopéz-Acedo, Nicolae 2019)

A metric space (X, d) satisfies the uniform betweenness property with modulus $\Theta : (0, \infty)^3 \to (0, \infty)$ if

$$\forall \varepsilon, a, b > 0 \, \forall x, y, z, w \in X$$
$$\left(\begin{array}{c} \left\{ \begin{array}{l} \mathrm{sep}\{x, y, z, w\} \geq a \wedge \mathrm{diam}\{x, y, z, w\} \leq b \\ d(x, y) + d(y, z) \leq d(x, z) + \Theta(\varepsilon, a, b) \\ d(y, z) + d(z, w) \leq d(y, w) + \Theta(\varepsilon, a, b) \end{array} \right\} \\ \Rightarrow d(x, z) + d(z, w) \leq d(x, w) + \varepsilon \end{array} \right).$$

Definition (Lion-Man Game in general metric spaces)

Let X be a metric space, $D > 0$ and Let $(M_n), (L_n)$ be sequences in X s.t. for all $n \in \mathbb{N}$

$$d(M_n, M_{n+1}) \leq D, \ d(L_{n+1}, L_n) + d(L_{n+1}, M_n) = d(L_n, M_n),$$

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Then $\langle (M_n), (L_n) \rangle$ is called a **Lion-Man game** with speed $D > 0$.

Let X be a **b-bounded** metric space with the uniform betweenness property with modulus Θ satisfying

$$\Theta(\varepsilon) := \Theta(\varepsilon, \varepsilon, b) \leq \varepsilon \quad \text{for all } \varepsilon > 0.$$

For $D > 0$ let $N \in \mathbb{N}$ be s.t. $b + 1 < ND$.

Theorem (K./Lopéz-Acedo/Nicolae 2019)

Let X be a bounded metric space with the uniform betweenness property and $\langle (M_n), (L_n) \rangle$ be an arbitrary Lion-Man game with speed $D > 0$. Then the Lion approaches the man arbitrarily close.

Theorem (K./Lopéz-Acedo/Nicolae 2019)

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Moreover with $b \geq \mathrm{diam}(X)$, Θ, N as above:

$$\forall \varepsilon > 0 \, \forall n \geq \Omega_{D,b,\Theta}(\varepsilon) \; (d(L_{n+1}, M_n) < \varepsilon),$$

where

$$\Omega_{D,b,\Theta}(\varepsilon) = N + N \left\lceil \frac{b}{\Theta^{(N)}(\alpha)} \right\rceil$$

with

$$0 < \alpha \leq \min\left\{ \frac{1}{N}, \frac{D}{2}, \frac{\varepsilon}{2} \right\}.$$

Uniform betweenness in normed spaces

Let $(X, \|\cdot\|)$ be a normed space.

Proposition (Diminnie, White 1981)

The betweennes property (BW) is equivalent to

(BW)′: for all $\boldsymbol{x, y, z \in X}$

$$\|x\| = \|y\| = \|z\| = \left\|\frac{x+y}{2}\right\| = \left\|\frac{y+z}{2}\right\| = 1 \to \|x+y+z\| = 3.$$

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(BW)′ also has an obvious **uniformization** (UBW)′: for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y, z \in X$ with $\|x\| = \|y\| = \|z\| = 1$:

$$\left\|\frac{x+y}{2}\right\|, \left\|\frac{y+z}{2}\right\| \geq 1 - \delta \to \|x+y+z\| \geq 3 - \varepsilon$$

together with the corresponding concept of a modulus

Proposition (K.,Lopéz-Acedo,Nicolae 2019)

Let $(X, \|\cdot\|)$ be a normed space. Then X satisfies (UBW) iff it satisfies (UBW)$'$. Moreover, respective moduli can be transformed into each other by the transformations

$$\Theta(\varepsilon, a, b) := 2a \cdot \delta\left(\frac{\varepsilon}{2b}\right), \; \delta(\varepsilon) := \frac{1}{2}\min\left\{\Theta\left(\frac{\varepsilon}{2}, \frac{1}{2}, 3\right), \frac{1}{2}, \frac{\varepsilon}{2}\right\}.$$

Examples of uniquely geodesic spaces with uniform betweenness

Definition (K./Lopéz-Acedo/Nicolae 2019)

We say that X is **uniformly uniquely geodesic** if for all $\varepsilon, b > 0$ there exists $\varphi > 0$ such that for all $x, y, z_1, z_2 \in X$ with $d(x,y) \le b$ and all $t \in [0,1]$ we have

$$\left.\begin{array}{l} d(x,z_1) \le td(x,y), d(y,z_1) \le (1-t)d(x,y) + \varphi \\ d(x,z_2) \le d(x,y), d(y,z_2) \le (1-t)d(x,y) + \varphi \end{array}\right\} \Rightarrow d(z_1,z_2) < \varepsilon.$$

A mapping $\Phi : (0,\infty) \times (0,\infty) \to (0,\infty)$ providing for given $\varepsilon, b > 0$ such a $\varphi = \Phi(\varepsilon, b)$ is called a **modulus of uniform uniqueness**.

Proposition (K.,Lopéz-Acedo,Nicolae 2019)

Let X be a uniformly uniquely geodesic space with modulus $\mathbf{\Phi}$ which satisfies the convexity condition

$$d(z,(1-t)x+ty) \leq (1-t)d(z,x)+td(z,y).$$

Then

$$\Theta(\varepsilon,a,b) = \min\left\{\Phi\left(\min\left\{\frac{a\cdot\varepsilon}{8b},\frac{a}{2}\right\},b\right),a\right\}$$

is a modulus of uniform betweenness.

Moduli $\mathbf{\Phi}$ and hence $\mathbf{\Theta}$ can be **explicitly computed** for L^p $(1 < p < \infty)$ and **CAT**(κ)**-spaces,** $\kappa > 0$.

Moduli Φ and hence Θ can be **explicitly computed** for L^p $(1 < p < \infty)$ and CAT(κ)-spaces, $\kappa > 0$.

For L^p:

$$\Phi(\varepsilon, b) = \begin{cases} \frac{p-1}{8} \frac{\varepsilon^2}{(b+\varepsilon)}, & \text{if } 1 < p \leq 2, \\ \frac{1}{p2^p} \frac{\varepsilon^p}{(b+\varepsilon)^{p-1}}, & \text{if } 2 < p < \infty. \end{cases}$$

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For $\mathrm{CAT}(\kappa)$-spaces X, $\kappa > 0$, with $\mathrm{diam}(X) < \pi/(2\sqrt{\kappa})$:

$$\Phi(\varepsilon, b) = \frac{c}{16}\frac{\varepsilon^2}{b+\varepsilon}, \text{ where}$$

$$c = (\pi - 2\sqrt{\kappa}\,\beta)\tan(\sqrt{\kappa}\,\beta) \text{ for any } 0 < \beta \leq \pi/(2\sqrt{\kappa}) - \mathrm{diam}(X).$$

Examples of (nonuniquely) geodesic spaces with uniform betweenness

Ptolemy spaces

Definition

A metric space (X, d) is a **Ptolemy** space if for all $x, y, z, w \in X$

$$d(x, z)d(y, w) \leq d(x, y)d(z, w) + d(x, w)d(y, z).$$

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Proposition (Foertsch, Lytchak, Schroeder 2007)

There are complete bounded Ptolemy spaces which are geodesic but **not uniquely geodesic**.

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There are complete bounded Ptolemy spaces which are geodesic but **not uniquely geodesic**.

Proposition (Nicolae 2013)

Every Ptolemy metric space **satisfies the betweenness property**.

Being Ptolemy is a purely universal axiom which, therefore, is admissible to be used in uniform bound extraction theorems for metric spaces. Hence the extractability of a modulus Θ is guaranteed!

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Indeed an easy analysis gives:

Proposition (K., Lopéz-Acedo, Nicolae 2019)

Let (X, d) be a Ptolemy space. Then $\Theta(\varepsilon, a, b) := \sqrt{b^2 + \varepsilon a} - b$ is a modulus for the uniform betweenness property.

A nonstrictly normed space with the uniform betweenness property

Definition (Diminnie, White 1981)

Consider $\mathbb{R}^3$ with the norm

$$\|(x,y,z)\|_{\mathrm{DW}} := \sqrt{|z^2-(x^2+y^2)|+3z^2+x^2+y^2}.$$

Proposition (Diminnie, White 1981)

$(X, \|\cdot\|_{\mathrm{DW}})$ is not strictly normed (and hence not uniquely geodesic) but satisfies the betweenness property.

Guaranteed by logical bound extraction metatheorems (this time we use that $\boldsymbol{K := \{x \in \mathbb{R}^3 : \|x\|_{\mathrm{DW}} \leq b\}}$ is compact): there must be a modulus for the uniform betweenness property extractable from the proof (by some affine shift we may assume that e.g. $x := 0$).

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Indeed, the (this time complicated) logical analysis of the proof by Diminnie and White gives:

Proposition (K., Lopéz-Acedo, Nicolae 2019)

Let $\eta(\varepsilon) := \varepsilon^2/8$ and $0 < \varepsilon \leq 1/2$.

$$\Theta(\varepsilon, a, b) := 2a \cdot \delta(\varepsilon/2b) \text{ with}$$
$$\delta(\varepsilon) := \min\left\{ \frac{\eta\left(\frac{\sqrt{2}\cdot\varepsilon}{256}\right)}{\sqrt{2}}, \frac{\varepsilon}{128} \right\}$$

is a **modulus for the uniform betweennes property** of $(X, \|\cdot\|_{\mathrm{DW}})$.