BILATERAL GENERATING FUNCTIONS FOR A NEW CLASS OF GENERALIZED LEGENDRE POLYNOMIALS

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ABSTRACT. Recently Chatterjea (1) has proved a theorem to deduce abilateral generating function for the Ultraspherical polynomials. In the present paper an attempt has been made to give a general version of Chatterjea's theorem. Finally, the theorem has been specialized to obtain a bilateral generating function for a class of polynomials $\{P_n \ (x; \alpha, \beta)\}$ introduced by Bhattacharjya (2).

KEY WORDS AND PHRASES. Bilateral generating function, Ultraspherical polynomials, Legendre polynomials, Orthogonal polynomials, Weight function, Rodrigue's formula.

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1. INTRODUCTION.

Using the following differential formula for the Ultraspherical polynomials P_n^{λ} (x) due to Tricomi,

$$P_{n}^{\lambda} \left[x(x^{2}-1)^{-1/2} \right] = \frac{(-1)^{n}}{n!} (x^{2}-1)^{\lambda + \frac{n}{2}} D^{n} (x^{2}-1)^{-\lambda}, \qquad (1.1)$$

Chatterjea (1) has recently obtained a bilateral generating function for the Ultraspherical polynomials in the form of following theorem.

THEOREM 1. If
$$F(x,t) = \int\limits_{m=0}^{\infty} a^m \ t^m \ P_m^{\lambda} \ (x) \, ,$$

then

$$\rho^{-2\lambda} F(\frac{x-t}{\rho}, \frac{ty}{\rho}) = \sum_{r=0}^{\infty} t^r b_r(y) P_r^{\lambda}(x), \qquad (1.2)$$

where

$$b_r(y) = \sum_{m=0}^{\infty} {r \choose m} a_m y^m$$
, and $\rho = (1-2xt+t^2)^{1/2}$.

A closer look at the above relation (1.2) suggests the following interesting general version of Chatterjea's theorem:

- 2. Let F o G be used to denote the composition F o G(x) = F(G(x)). In terms of this notation, we state
- THEOREM 2. Suppose that there exist functions f, g, h and X and a sequence of constants $\{c_n\}$ such that the sequence of functions $\{Q_n\}$ is generated by the formula

$$c_n f g^n Q_n \circ X = D^n h,$$
 $n = 0, 1, 2, ...,$ (2.1)

= d/dx. Define the generating function

$$F(x,t) = \sum_{n=0}^{\infty} a_n t^n Q_n(x).$$
 (2.2)

Then

$$fF(X,gtz)$$
 | x+t = $f \sum_{n=0}^{\infty} c_n (gt)^n Q_n \circ X b_n (z)$,

where

$$b_n(z) = \sum_{k=0}^{\infty} \frac{a_k}{c_k (n-k)!} z^k.$$

By Taylor's theorem PROOF.

$$fF(X,gtz)|_{x+t} = e^{tD} fF(X,gtz).$$
 (2.3)

To evaluate the right hand side of (2.3), we shall use as our starting point the

relations (2.1) and (2.2), and the series expansion for
$$e^{tD}$$
. Thus
$$e^{tD} fF(X,gtz) = e^{tD} f \sum_{n=0}^{\infty} a_n (gtz)^n Q_n \circ X$$

$$= e^{tD} \sum_{n=0}^{\infty} \frac{a_n}{c_n} (tz)^n D^n h$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (a_n/c_n) t^{n+m} z^n D^{n+m} h/m!$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (a_n/c_n) (gt)^{n+m} c_{n+m} f Q_{n+m} \circ X/m!$$

$$= f \sum_{n=0}^{\infty} c_n (gt)^n Q_n \circ X b_n (z),$$
here
$$b_n \cdot (z) = \sum_{k=0}^{\infty} \frac{a_k}{c_k (n-k)!} z^k.$$

where

It is worthwhile to remark here that if we choose Q_n (x) = P_n^{λ} (x), $f(x) = (x^2-1)^{-\lambda}$, $g(x) = (x^2-1)^{-1/2}$, $X(x) = x(x^2-1)^{-1/2}$, $h(x) = (x^2-1)^{-\lambda}$ and $c_n = n! / (-1)^n$ then Theorem ? would correspond to Chatterjea's theorem.

APPLICATIONS: Earlier, Bhattacharjya (2) introduced a new class of generalized Legendre polynomials $\{P_{n}(x;\;\alpha,\;\beta)\,\}$ which are orthogonal with the weight function $\frac{|x|^{\beta}}{(1-x^2)^{(\beta-\alpha)/2}}$. The Rodrigue's formulae for these polynomials are (2, (6.6) and (6.8)):

$$P_{2m} (x^{-1/2}; \alpha, \beta) = \frac{x^{m+(\alpha+1)/2} (1-x)^{(\beta-\alpha)/2}}{(-2m-(\alpha-1)/2)_{m}},$$

$$D^{m}[(1-x)^{m-(\beta-\alpha)/2} x^{-m-(\alpha+1)/2}], \qquad (2.4)$$

and

$$P_{2m+1} (x^{-1/2}; \alpha, \beta) = \frac{x^{m+1+\alpha/2} (1-x)^{(\beta-\alpha)/2}}{(-2m-(\alpha+1)/2)_{m}} \cdot D^{m}[(1-x)^{m-(\beta-\alpha)/2} x^{-m-(\alpha+3)/2}]$$
(2.5)

Here we note that the sequences $\{P_{2n}(x^{-1/2}; \alpha-2n, \beta)\}$ and $\{P_{2n+1}(x^{-1/2}; \alpha-2n, \beta)\}$ are amenable to a method of Theorem 2 for finding bilateral generating functions.

Let $Q_n(x) = P_{2n}(x; \alpha - 2n, \beta) \equiv P_{2n}(x)$. For simplicity of notation, set $y = -(\alpha + 1)/2$ and $\delta = (\alpha - \beta)/2$. Then (2.1) holds with $f(x) = x^y (1-x)^{\delta}$, $g(x) = (1-x)^{-1}$, $X(x) = x^{-1/2}$ and $c_n = \phi(n) = (-n-(\alpha - 1)/2)_n$. Upon replacing t by -t and z by -y, we get

$$\left(\frac{x-t}{x}\right)^{y} \left(\frac{1-(x-t)^{\delta}}{1-x}\right) \quad F\left(\frac{1}{(x-t)^{1/2}}, -\frac{yt}{(1-(x-t))}\right) = \sum_{r=0}^{\infty} \left(\frac{-t}{1-x}\right)^{r} \phi \ (r)^{\bullet}$$

where

$$F(\frac{1}{x^{1/2}}, \frac{t}{1-x}) = \sum_{m=0}^{\infty} a_m (\frac{t}{1-x})^m P_{2m} (x^{-1/2})$$

and

$$b_{r}(-y) = \sum_{m=0}^{\infty} \frac{a_{m}(-y)^{m}}{\phi(m)(r-m)!}$$
 (2.7)

Now replacing $x^{-1/2}$ by s and t/(1-x) by t in (2.6), we are led to the following

bilateral generating function for gerneralized even Legendre polynomials:

COROLLARY. 1: If

$$F(x,t) = \sum_{m=0}^{\infty} a_m t^m P_{2m}(x),$$

then

$$[1-(x^{2}-1)t]^{y} (1+t)^{\delta} F\left(\frac{x}{(1-t(x^{2}-1))^{1/2}}, \frac{yt}{1+t}\right) = \sum_{r=0}^{\infty} (-t)^{r} \phi(r).$$

$$P_{2r}(x) b_{r}(-y),$$

where b_r (-y) is given by (2.7).

In the same way, let $Q_n(x) = P_{2n+1}(x; \alpha-2n, \beta) \equiv P_{2n+1}(x)$, and set $y = -(\alpha+2)/2$, $\delta = (\alpha-\beta)/2$. Then (2.1) holds with $f(x) = x^y (1-x)^{\delta}$, $g(x) = (1-x)^{-1}$, $\chi(x) = x^{-1/2}$ and $c_n = \psi(n) = (-n-(\alpha+1)/2)_n$. Replacing t by -t and z by -y and making the same substitution as before in (2.7), we are led to the following bilateral generating function for generalized odd Legendre polynomials.

COROLLARY 2: If

$$F(x,t) = \sum_{m=0}^{\infty} a_m t^m P_{2m+1}(x),$$

then

$$[1-(x^2-1)t]^y (1-t)^{\delta} F\left(\frac{x}{(1-t(x^2-1))^{1/2}}, \frac{ty}{1+t}\right) = \sum_{r=0}^{\infty} (-t)^r \psi(r)$$

 $P_{2r+1}(x) c_r(-y)$

where

$$c_r (-y) = \sum_{m=0}^{r} \frac{a_m (-y)^m}{\Psi(m) (r-m)!}$$

Taking $\alpha = \beta$ in Corollary 1 and 2, we can obtain bilateral generating functions for generalized Legendre polynomials due to Dutta and More (3).

Next, we note that (2),

$$P_{2m}(x;o,o) = \frac{(-1)^m m! P_{2m}(x)}{(-2m + \frac{1}{2})_m}, \qquad (2.8)$$

and

$$P_{2m}(x;0,0) = \frac{(-1)^m m! P_{2m+1}(x)}{(-2m-\frac{1}{2})_m},$$
 (2.9)

where P_{2m} (x) and P_{2m+1} (x) are even and odd Legendre polynomials. Therefore, by

(2.8), (2.9) and the above two corollories we can obtain bilateral generating functions for Legendre polynomials.

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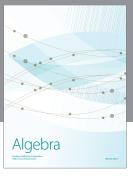
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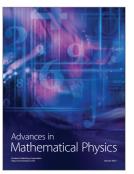


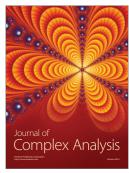




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