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Research Article

The Solution of a Class of Third-Order Boundary Value Problems by the Reproducing Kernel Method

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This paper expands the application of reproducing kernel method to a class of third-order boundary value problems with mixed nonlinear boundary conditions. The analytical solution is represented in the form of series in the reproducing kernel space. The n-term approximation is obtained and is proved to converge to the analytical solution. The numerical examples are given to demonstrate the computation efficiency of the presented method. Results obtained by the method indicate that the method is simple and effective.

1. Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, for example, in the deflection of a curved beam having a constant or varying cross section, a three layer beam, electromagnetic waves, or gravity driven flows. Third-order boundary value problems were discussed in many papers in recent years, for instance, see [1–6] and references therein. In [1–3], the authors used the spline functions to solve boundary value problems. In [4], the authors developed a second-order method for solving third-order three-point boundary value problems based on Padé approximant in a recurrence relation. In [5], the authors introduced Adomian decomposition method for multipoint boundary value problems (BVPs). In this paper, we use reproducing kernel to solve singular third-order boundary value problems with mixed boundary conditions. Recently, the reproducing kernel methods [7–10] emerge one after another. Using the reproducing kernel methods, the authors discussed two-point boundary value problems and periodic boundary value problems. For third-order boundary value problems with mixed nonlinear boundary conditions, however, this method has not yet been applied. In previous work, the reproducing kernel method cannot be used directly to solve third-order boundary value problems with nonlinear boundary conditions. Our work is to present a numerical algorithm for solving a class of singular third-order boundary value problems. By using this method, the analytical solution and approximate solution are given and uniformly converge to the exact solution and its corresponding derivatives. The algorithms are efficiently applied to solve some model problems.

Let us consider the following singular problems of third-order ordinary differential equations:

$$\begin{gathered} u'''(x) + p_1(x)u''(x) + p_2(x)u'(x) + p_3(x)u(x) = F(x), \quad x \in (0,1), \\ \lambda_i u = r_i, \quad (i = 1,2,3), \end{gathered} \tag{1.1}$$

where $p_j(x), f(x) \in L^2[0,1]$, $(j = 1,2,3)$ are known functions. $\lambda_i u$, $(i = 1,2,3)$ are linear independence conditions of determining the solution. We assume that (1.1) has a unique solution which belongs to $W_2^4[0,1]$, where $W_2^4[0,1]$ which is a reproducing kernel space is defined in the second section.

In order to solve (1.1), let $Lu = u'''(x) + p_1(x)u''(x) + p_2(x)u'(x) + p_3(x)u(x)$. It is easy to prove that $L : W_2^4[0,1] \to L^2[0,1]$ is bounded linear operator.On the other hand, we assume that the conditions of determining the solution can be homogenized; after homogenization of these conditions, we put the conditions into the reproducing kernel space $W_2^4[0,1]$ constructed in the following section. Equation (1.1) can be transformed into the following form in $W_2^4[0,1]$:

$$(Lu)(x) = F(x). \tag{1.2}$$

To solve problem (1.2), we give a space $W_2^4[0,1] = \{u \mid u \in \overline{W}_2^4[0,1] \text{ and } \lambda_i u = 0,\ i = 1,2,3\}$. The inner product in $W_2^4[0,1]$ is given by $\langle u(x), v(x)\rangle = \sum_{i=0}^{3} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(4)}(t)v^{(4)}(t)dt$. Like in [8], we can get the following reproducing kernel space.

Theorem 1.1. *The space $W_2^4[0,1]$ is a reproducing kernel space and its reproducing kernel is $K(x,y)$, and*

$$K(x,y) = R_y(x) - \frac{h_1(x)h_1(y)}{\|h_1(x)\|^2} - \frac{h_2(x)h_2(y)}{\|h_2(x)\|^2} - \frac{h_3(x)h_3(y)}{\|h_3(x)\|^2}, \tag{1.3}$$

where

$$R_y(x) = \begin{cases} \dfrac{5040 - y^7 + 35x^3y^3(4+y) - 21x^2y^2(-60+y^3) + 7xy(720+y^5)}{5040}, & x < y \\[2ex] \dfrac{5040 - x^7 + 5040xy + 7x^6y + 1260x^2y^2 - 21x^5y^2 + 140x^3y^3 + 35x^4y^3}{5040}, & y < x, \end{cases} \tag{1.4}$$

$h_1(x) = \lambda_{1y}R_y(x)$, $h_2(x) = \lambda_{2y}(R(x,y) - (h_1(x)h_1(y)/\|h_1(x)\|^2))$, $h_3(x) = \lambda_{3y}(R(x,y) - (h_1(x)h_1(y)/\|h_1(x)\|^2) - (h_2(x)h_2(y)/\|h_2(x)\|^2))$, the symbol λ_{iy} indicates that the operator λ_i applies to the function of y.

2. The Reproducing Kernel Method

Let $\psi_i(x) = (L_y K_x(y))(x_i)$, $i = 1, 2, \ldots$, Practise Gram-Schmidt orthonomalization for $\{\psi_i(x)\}_{i=1}^{\infty}$, we get

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x), \tag{2.1}$$

where β_{ik} are coefficients of Gram-Schmidt orthonormalization.

Theorem 2.1. *If $\{x_i\}_{i=1}^{\infty}$ is distinct points dense in $[0,1]$ and L^{-1} is existent, then*

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F(x_k) \overline{\psi}_i(x) \tag{2.2}$$

is an analytical solution of the problem (1.2).

Proof. Since $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is an orthonormal systems, $u(x)$ is expressed as

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty} \langle u(x), \overline{\psi}_i(x) \rangle \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), (L_s K_x(s))(x_k) \rangle \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} (L_s \langle u(x), K_x(s) \rangle)(x_k) \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} (L_s u(s))(x_k) \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F(x_k) \overline{\psi}_i(x). \end{aligned} \tag{2.3}$$

□

We denote the approximate solution of $u_n(x)$ by

$$u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} F(x_k) \overline{\psi}_i(x). \tag{2.4}$$

Theorem 2.2. *Let $\varepsilon_n^2 = \|u(x) - u_n(x)\|^2$ where $u(x)$, $u_n(x)$ are given by* (2.2) *and* (2.4), *then the sequence of number $\varepsilon_n(x)$ is monotone decreasing and $\varepsilon_n(x) \to 0$.*

Proof. Because

$$\varepsilon_n^2 = \|u(x) - u_n(x)\|^2 = \sum_{i=n+1}^{\infty} \langle u(x), \overline{\varphi}_i(x) \rangle \overline{\varphi}_i(x) = \sum_{i=n+1}^{\infty} \left(\langle u(x), \overline{\varphi}_i(x) \rangle \right)^2, \tag{2.5}$$

clearly $\varepsilon_{n-1} \geq \varepsilon_n$ and consequently $\{\varepsilon_n\}$ is monotone decreasing in the sense of $\|\cdot\|_{W_2^4}$. By Theorem 2.1, we know $\sum_{i=1}^{\infty} \langle u(x), \overline{\varphi}_i(x) \rangle \overline{\varphi}_i(x)$ is convergent in the norm of $\|\cdot\|_{W_2^4}$, then we have

$$\varepsilon_n^2 = \|u(x) - u_n(x)\|^2 \longrightarrow 0. \tag{2.6}$$

Hence, $\varepsilon_n \to 0$. □

Theorem 2.3 (convergence analysis). *If $u(x)$, $u_n(x)$ are given by* (2.2) *and* (2.4), *then $u_n(x)$ and $u_n^{(k)}(x)$ uniformly convergent to $u(x)$ and $u^{(k)}(x)$, where $k = 0, 1, 2, 3$.*

Proof. For any $x \in [0,1]$, $k = 0, 1, 2, 3$

$$\left| u_n^{(k)}(x) - u^{(k)}(x) \right| = \left| \left\langle u_n(t) - u(t), \frac{\partial^k K(x,t)}{\partial x^k} \right\rangle \right| \leq \|u_n(t) - u(t)\| \cdot \left\| \frac{\partial^k K(x,t)}{\partial x^k} \right\|, \tag{2.7}$$

then there exists $C_k > 0$ such that, $|u_n^{(k)}(x) - u^{(k)}(x)| \leq C_k \|u_n(t) - u(t)\| = C_k \varepsilon_n \to 0$. □

Theorem 2.4. *If $\{x_k\}_{k=1}^{\infty}$ is distinct points dense in $[0,1]$ and $u(x)$, $u_n(x)$ are given by* (2.2) *and* (2.4), *then $Lu(x_k) = Lu_n(x_k)$.*

Proof. We may set projective operator $P_n : W_2^4[0,1] \to \{\sum_{m=1}^{n} c_m \psi_m(x), c_m \in R\}$. Hence,

$$\begin{aligned} Lu_n(x_k) &= \langle u_n(\xi), L_{x_k} K_{x_k}(\xi) \rangle = \langle u_n(\xi), \psi_k(\xi) \rangle = \langle P_n u(\xi), \psi_k(\xi) \rangle \\ &= \langle u(\xi), P_n \psi_k(\xi) \rangle = \langle u(\xi), \psi_k(\xi) \rangle = \langle u(\xi), L_{x_k} K_{x_k}(\xi) \rangle \\ &= L_{x_k} \langle u(\xi), K_{x_k}(\xi) \rangle = L_{x_k} u(x_k) = Lu(x_k). \end{aligned} \tag{2.8}$$

□

Theorem 2.5 (error estimate). *If $\{x_k\}_{k=1}^{\infty}$ is distinct points dense in $[0,1]$ and $u(x)$, $u_n(x)$ are given by* (2.2) *and* (2.4), *then $|u(x) - u_n(x)| < (M/n)$, where $M = \| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F(x_k) \overline{\varphi}_i(x) \| \cdot \|\partial K_\eta(\xi)/\partial \eta\|$.*

Proof. For every given $x \in [0,1]$, there is always $x_i \in \{x_k\}_{k=1}^{\infty}$ satisfying $x_i < x$ and $x - x_i = 1/n$. By Theorem 2.5 and $x_i \in \{x_k\}_{k=1}^{\infty}$ implying $Lu(x_i) = Lu_n(x_i)$, so we obtain

$$|Lu(x) - Lu_n(x)| = |Lu(x) - Lu(x_i) - [Lu_n(x) - Lu_n(x_i)]|. \tag{2.9}$$

For application reproducing kernel property, we have

$$u(x) = \langle u(\xi), K_x(\xi) \rangle, \qquad Lu(x) = \langle u(\xi), LK_x(\xi) \rangle. \tag{2.10}$$

We also have

$$\begin{aligned} Lu(x) - Lu_n(x) &= Lu(x) - Lu(x_i) - [Lu_n(x) - Lu_n(x_i)] \\ &= \langle u(\xi), LK_x(\xi) - LK_{x_i}(\xi) \rangle - \langle u_n(\xi), LK_x(\xi) - LK_{x_i}(\xi) \rangle \\ &= \langle u(\xi) - u_n(\xi), LK_x(\xi) - LK_{x_i}(\xi) \rangle. \end{aligned} \tag{2.11}$$

Moreover,

$$\begin{aligned} |u(x) - u_n(x)| &= \left| L^{-1}[Lu(x) - Lu_n(x)] \right| \\ &\leq \left| \left\langle u(\xi) - u_n(\xi), L^{-1}LK_x(\xi) - L^{-1}LK_{x_i}(\xi) \right\rangle \right| \\ &= |\langle u(\xi) - u_n(\xi), K_x(\xi) - K_{x_i}(\xi) \rangle| \\ &\leq \|u - u_n\| \|K_x(\xi) - K_{x_i}(\xi)\|. \end{aligned} \tag{2.12}$$

It is noted that we take norm of $\|K_x(\xi) - K_{x_i}(\xi)\|$ for variable ξ. The function $k_x(\xi)$ is derived on x in the interval of $[0, 1]$, so we have $K_x(\xi) - K_{x_i}(\xi) = (\partial K_\eta(\xi)/\partial \eta)(x - x_i)$. Hence,

$$\begin{aligned} |u(x) - u_n(x)| &\leq \|u - u_n\| \left\| \frac{\partial K_\eta(\xi)}{\partial \eta}(x - x_i) \right\| \\ &= \|u - u_n\| \left\| \frac{\partial K_\eta(\xi)}{\partial \eta} \right\| (x - x_i) = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F(x_k) \overline{\varphi}_i(x) \right\| \left\| \frac{\partial K_\eta(\xi)}{\partial \eta} \right\| (x - x_i) \leq \frac{M}{n}. \end{aligned} \tag{2.13}$$

□

3. Numerical Experiment

For showing the effectiveness of our method, we consider the following problems.

Example 3.1 (see [2, 3]). Considering the following third-order boundary values problem

$$\begin{gathered} u'''(x) - xu(x) = \left(x^3 - 2x^2 - 5x - 3\right)e^x, \quad 0 \leq x \leq 1, \\ u(0) = u(1) = 0, \qquad u'(0) = 1, \end{gathered} \tag{3.1}$$

where the exact solution is $u_T(x) = x(1-x)e^x$. By the homogeneous process of the boundary condition, let $v(x) = u(x) - x(1-x)$, problem (3.8) can be transformed into the equivalent form

$$v'''(x) - xv(x) = (1-x)x^2 + \left(x^3 - 2x^2 - 5x - 3\right)e^x, \quad 0 \le x \le 1,$$
$$v(0) = v(1) = 0, \qquad v'(0) = 0. \tag{3.2}$$

The numerical results are presented in Tables 1, 2, and 3.

Example 3.2 (see [11, 14]). Considering the following third-order obstacle problems:

$$u'''(x) = \begin{cases} 0, & 0 \le x \le \dfrac{1}{4}, \ \dfrac{1}{4} \le x \le \dfrac{1}{4}, \\ u(x) - 1, & \dfrac{3\pi}{4} \le x \le 1, \end{cases} \tag{3.3}$$

where $u(0) = u'(0) = u'(1) = 0$, the exact solution is

$$u_T(x) = \begin{cases} \dfrac{1}{2}a_1 x^2, & 0 \le x \le \dfrac{1}{4}, \\ 1 + a_2 e^x + e^{-x/2}\left[a_3 \cos\dfrac{\sqrt{3}x}{2} + a_4 \sin\dfrac{\sqrt{3}x}{2}\right], & \dfrac{1}{4} \le x \le \dfrac{3}{4} \\ \dfrac{1}{2}a_5 x(x-2) + a_6, & \dfrac{3}{4} \le x \le 1, \end{cases} \tag{3.4}$$

where

$$a_1 = 0.24391096222648, \quad a_2 = -0.17847234452746, \quad a_3 = -0.81893573565615,$$
$$a_4 = -0.30266818001866, \quad a_5 = -0.24213890868443, \quad a_6 = -0.06537630092111. \tag{3.5}$$

The numerical results are presented in Table 4.

Example 3.3. Considering the following boundary value problems with nonclassical condition:

$$u'''(x) + \frac{1}{x\sqrt{1-x}}u''(x) + \frac{1}{x\sqrt{x(1-x)}}u'(x) + \frac{1}{x^2\sqrt{1-x}}u(x) = f(x), \quad 0 < x < 1,$$
$$u(0) = 0, \qquad u'(0) = \int_0^1 u(x)dx, \quad 2u(1) = u'(1). \tag{3.6}$$

We determine $f(x)$ to get the true solution, given by $u_T(x) = xe^x$. The numerical results are presented in Table 5.

Table 1: The absolute error of Example 3.1.

	Absolute error						
x	Present ($n=20$)	Present ($n=40$)	Present ($n=80$)	Present ($n=160$)	Ref. [3]	Ref. [2]	Present ($n=500$)
0.0	0	0	0	0	0	0	0
0.1	1.52298×10^{-5}	2.85196×10^{-6}	5.80026×10^{-7}	1.27547×10^{-7}	1×10^{-7}	0	1.18126×10^{-8}
0.3	7.42332×10^{-5}	1.5986×10^{-5}	3.66544×10^{-6}	8.74382×10^{-7}	3×10^{-7}	5×10^{-3}	8.65902×10^{-8}
0.4	1.08810×10^{-4}	2.42167×10^{-5}	5.67330×10^{-6}	1.37023×10^{-6}	5×10^{-7}	7×10^{-3}	1.36944×10^{-7}
0.5	1.40256×10^{-4}	3.19274×10^{-5}	7.58382×10^{-6}	1.84582×10^{-6}	6×10^{-7}	8×10^{-3}	1.85506×10^{-7}
0.7	1.68693×10^{-4}	3.95170×10^{-5}	9.54427×10^{-6}	2.3440×10^{-6}	7×10^{-7}	7×10^{-3}	2.37089×10^{-7}
0.9	9.78854×10^{-5}	2.33293×10^{-5}	5.68883×10^{-6}	1.40422×10^{-6}	4×10^{-7}	4×10^{-3}	1.42538×10^{-7}
1.0	1.50887×10^{-15}	1.74768×10^{-15}	1.57219×10^{-15}	1.90655×10^{-15}	0	1.4×10^{-2}	1.49019×10^{-15}
	$0\,\text{s} < \text{CPU runtime} \le 1\,\text{s}$		CPU runtime $\le 10\,\text{s}$				CPU runtime $= 180\,\text{s}$

Table 2: The numerical results of Example 3.1.

x	$u_T'(x)$	$u_{500}'(x)$	$\lvert u_T'(x) - u_{500}'(x)\rvert$	$u_T''(x)$	$u_{500}''(x)$	$\lvert u_T''(x) - u_{500}''(x)\rvert$
0	1	1	0	0	0.0000123873	0.0000123873
0.1	0.983602	0.983602	2.217025×10^{-7}	−0.342603	−0.342601	1.904429×10^{-6}
0.3	0.823414	0.823414	4.842037×10^{-7}	−1.33636	−1.33636	6.462399×10^{-7}
0.5	0.41218	0.412181	4.459369×10^{-7}	−2.88526	−2.88526	1.127018×10^{-6}
0.7	−0.382613	−0.382613	1.172353×10^{-8}	−5.21562	−5.21562	3.578872×10^{-6}
0.9	−1.74632	−1.74632	1.0452710×10^{-6}	−8.63321	−8.63321	6.928269×10^{-6}
1.	−2.71828	−2.71828	1.8404898×10^{-6}	−10.8731	−10.8731	9.029439×10^{-6}

Table 3: The numerical results of Example 3.1.

x	$u_T^{(3)}(x)$	$u_{500}^{(3)}(x)$	$\lvert u_T^{(3)}(x) - u_{500}^{(3)}(x)\rvert$	$u_T^{(4)}(x)$	$u_{500}^{(4)}(x)$	$\lvert u_T^{(4)}(x) - u_{500}^{(4)}(x)\rvert$
0	−3	−3.01001	0.010006	−8	−3.01001	4.98999
0.2	−4.93447	−4.93447	8.518423×10^{-9}	−11.53	−11.5489	0.0188419
0.4	−7.69782	−7.69782	5.477735×10^{-8}	−16.3504	−16.3771	0.0266921
0.6	−11.5887	−11.5887	1.335948×10^{-8}	−22.8858	−22.9229	0.0370827
0.8	−17.0031	−17.0031	1.724353×10^{-8}	−31.6917	−31.7423	0.050624
1.	−24.4645	−24.4645	2.771117×10^{-13}	−43.4925	3.021444×10^{-16}	43.4925

Table 4: The observed maximum errors of Example 3.2.

Method	$n = 16$	$n = 32$	$n = 64$	$n = 128$
Our method	1.18×10^{-3}	5.47×10^{-4}	2.62×10^{-4}	1.29×10^{-4}
Nonpolynomial spline [11]	7.12×10^{-4}	4.05×10^{-4}	2.22×10^{-4}	1.15×10^{-4}
Quartic spline [12]	1.15×10^{-3}	5.32×10^{-4}	2.56×10^{-4}	1.26×10^{-4}
Finite difference [13]	1.96×10^{-4}	4.89×10^{-5}	1.22×10^{-5}	3.06×10^{-6}
Cubic spline [14]	1.23×10^{-3}	5.53×10^{-4}	2.61×10^{-4}	1.27×10^{-4}
Colloc. quantic spline [15]	1.26×10^{-3}	5.60×10^{-4}	3.10×10^{-4}	1.67×10^{-4}
Finite difference [16]	6.89×10^{-3}	7.11×10^{-3}	7.27×10^{-3}	7.36×10^{-3}
Quartic B spline [17]	1.13×10^{-3}	5.30×10^{-4}	5.52×10^{-4}	1.23×10^{-4}

Table 5: The numerical results of Example 3.3.

x	$u_T(x)$	$u_{100}(x)$	$\lvert u_T(x) - u_{100}(x)\rvert$	$\lvert u_T'(x) - u_{100}'(x)\rvert$	$\lvert u_T''(x) - u_{100}''(x)\rvert$	$\lvert u_T'''(x) - u_{100}'''(x)\rvert$
0	0	0	0	2.11892×10^{-5}	5.34582×10^{-5}	2.49157×10^{-2}
0.1	0.110517	0.110518	1.06299×10^{-6}	6.59012×10^{-6}	4.42396×10^{-6}	2.85088×10^{-4}
0.3	0.404958	0.404962	4.25342×10^{-6}	3.27006×10^{-5}	2.16465×10^{-4}	1.15677×10^{-3}
0.5	0.824361	0.824376	1.51582×10^{-5}	7.46045×10^{-5}	1.67531×10^{-4}	8.58014×10^{-4}
0.7	1.40963	1.40966	3.24736×10^{-5}	9.40379×10^{-5}	3.75442×10^{-5}	5.12073×10^{-4}
0.8	1.78043	1.78047	4.20348×10^{-5}	9.71987×10^{-5}	3.89027×10^{-5}	5.59346×10^{-4}
0.9	2.21364	2.21369	5.20699×10^{-5}	1.05201×10^{-4}	1.40742×10^{-4}	1.08743×10^{-3}
1.	2.71828	2.71835	6.36065×10^{-5}	1.27213×10^{-4}	1.9082×10^{-4}	1.07270×10^{-2}

Table 6: The numerical results of Example 3.4, $p = q = \beta = 1$, $\alpha = 2$, $k = 2$, $n = 20$.

x	u_T	u_{20}	$\lvert u_T - u_{20}\rvert$	u_T'	u_{20}'	$\lvert u_T' - u_{20}'\rvert$	$\lvert u_T'' - u_{20}''\rvert$
0.08	−0.0286650	−0.0286650	5.20529×10^{-9}	0.0273871	0.0273873	2.32501×10^{-7}	7.75366×10^{-6}
0.16	−0.0255710	−0.0255710	6.94285×10^{-9}	0.0490631	0.0490632	7.15527×10^{-8}	2.10654×10^{-6}
0.24	−0.0209523	−0.0209523	9.70246×10^{-9}	0.0655842	0.0655842	1.41703×10^{-8}	5.37958×10^{-7}
0.32	−0.0152036	−0.0152035	9.39437×10^{-9}	0.0773741	0.0773741	1.84526×10^{-8}	3.37592×10^{-7}
0.48	−0.0017588	−0.0017588	1.71495×10^{-9}	0.0878568	0.0878567	8.12846×10^{-8}	4.48111×10^{-7}
0.56	0.00525584	0.00525583	6.04974×10^{-9}	0.0868185	0.0868184	1.11842×10^{-7}	4.02874×10^{-7}
0.64	0.0120206	0.0120205	1.65310×10^{-8}	0.0815939	0.0815937	1.53109×10^{-7}	6.25311×10^{-7}
0.72	0.0181958	0.0181957	3.10494×10^{-8}	0.0720489	0.0720486	2.13478×10^{-7}	9.11482×10^{-7}
0.88	0.0273346	0.0273346	8.10037×10^{-8}	0.0389010	0.0389005	4.47861×10^{-7}	1.32613×10^{-6}
0.96	0.0295066	0.0295065	1.25247×10^{-7}	0.0144477	0.0144469	8.40832×10^{-7}	9.84025×10^{-6}

Table 7: The numerical results of Example 3.5.

x	u_T	u_{20}	$\lvert u_T - u_{20}\rvert$	u_T'	u_{20}'	$\lvert u_T' - u_{20}'\rvert$	$\lvert u_T'' - u_{20}''\rvert$
0.08	0.000256	0.00025618	1.87335×10^{-7}	0.0096	0.009597	3.13184×10^{-6}	1.14007×10^{-5}
0.16	0.002048	0.0020481	1.72967×10^{-7}	0.0384	0.038399	8.39162×10^{-7}	2.89347×10^{-5}
0.24	0.006912	0.0069121	1.38265×10^{-7}	0.0864	0.0864	1.89713×10^{-7}	1.24163×10^{-5}
0.32	0.016384	0.0163841	1.13116×10^{-7}	0.1536	0.1536	2.05885×10^{-7}	5.02804×10^{-7}
0.4	0.032000	0.0320001	9.39829×10^{-8}	0.2400	0.2400	2.21979×10^{-7}	1.40284×10^{-6}
0.48	0.055296	0.0552961	7.77508×10^{-8}	0.3456	0.3456	1.93052×10^{-7}	5.56607×10^{-7}
0.56	0.087808	0.0878081	6.38527×10^{-8}	0.4704	0.4704	1.60088×10^{-7}	3.15123×10^{-7}
0.64	0.131072	0.131072	5.21361×10^{-8}	0.6144	0.6144	1.33773×10^{-7}	3.10240×10^{-7}
0.72	0.186624	0.186624	4.23781×10^{-8}	0.7776	0.7776	1.13440×10^{-7}	1.75032×10^{-7}
0.8	0.256000	0.256000	3.44071×10^{-8}	0.9600	0.9600	9.20785×10^{-8}	3.49872×10^{-7}
0.88	0.340736	0.340736	2.78679×10^{-8}	1.1616	1.1616	4.41977×10^{-8}	3.00596×10^{-8}
0.96	0.442368	0.442368	2.08726×10^{-8}	1.3824	1.3824	2.08167×10^{-8}	9.08609×10^{-6}

Example 3.4. Considering the following singular third-order three points boundary value problems with nonlinear condition

$$\begin{gathered} u'''(x) + \frac{p}{x^{\alpha}}u''(x) - \frac{k^2}{x^{\alpha}(1-x)^{\beta}}u'(x) + \frac{q\sin(x)}{x^{\alpha}(1-x)^{\beta}}u(x) = f(x), \quad 0 < x < 1, \\ u'(1) = 0, \qquad 2u'(0) = \left(u''(0)\right)^2, \qquad u\left(\frac{1}{2}\right) = 0. \end{gathered} \tag{3.7}$$

We determine $f(x)$ to get the true solution, given by $u_T(x) = r(k(2x-1) - 2\sinh(kx) + 2(\cosh(kx))\tanh(k/2))/2k^3$. The numerical results are presented in Table 6.

Example 3.5. Considering the following boundary value problems with nonlinear condition:

$$u'''(x) - \frac{1}{\sqrt{1+x}} u'(x) + 2u(x) = f(x),$$
$$u(0) = u'(0) = 0, \qquad (u'(1))^2 = 2\int_0^1 u(x)dx. \tag{3.8}$$

We determine $f(x)$ to get the true solution, given by $u_T(x) = (1/2)x^3$. The numerical results are presented in Table 7.

4. Conclusions and Remarks

In this work, we present an algorithm for solving third-order mixed boundary value problems (BVPs) based on the reproducing kernel method. The method can be generalized to get reproducing kernel of problem with linear conditions. All computations are performed by the Mathematica 7.0 software package.

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