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BILATERAL GENERATING FUNCTIONS FOR A NEW CLASS OF GENERALIZED LEGENDRE POLYNOMIALS

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ABSTRACT. Recently Chatterjea (1) has proved a theorem to deduce a bilateral generating function for the Ultraspherical polynomials. In the present paper an attempt has been made to give a general version of Chatterjea's theorem. Finally, the theorem has been specialized to obtain a bilateral generating function for a class of polynomials $\{P_n(x; \alpha, \beta)\}$ introduced by Bhattacharjya (2).

KEY WORDS AND PHRASES. Bilateral generating function, Ultraspherical polynomials, Legendre polynomials, Orthogonal polynomials, Weight function, Rodrigue's formula.

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1. INTRODUCTION.

Using the following differential formula for the Ultraspherical polynomials $P_n^\lambda(x)$ due to Tricomi,

$$P_n^\lambda [x(x^2-1)^{-1/2}] = \frac{(-1)^n}{n!} (x^2-1)^{\lambda+\frac{n}{2}} D^n (x^2-1)^{-\lambda}, \tag{1.1}$$

Chatterjea (1) has recently obtained a bilateral generating function for the Ultraspherical polynomials in the form of following theorem.

THEOREM 1. If

$$F(x,t) = \sum_{m=o}^{\infty} a^m t^m P_m^\lambda (x),$$

then

$$\rho^{-2\lambda} F\left(\frac{x-t}{\rho}, \frac{ty}{\rho}\right) = \sum_{r=o}^{\infty} t^r b_r(y) P_r^\lambda (x), \tag{1.2}$$

where

$$b_r(y) = \sum_{m=o}^{\infty} \binom{r}{m} a_m y^m, \text{ and } \rho = (1-2xt+t^2)^{1/2}.$$

A closer look at the above relation (1.2) suggests the following interesting general version of Chatterjea's theorem:

2. Let $F \circ G$ be used to denote the composition $F \circ G(x) = F(G(x))$. In terms of this notation, we state

THEOREM 2. Suppose that there exist functions f, g, h and X and a sequence of constants $\{c_n\}$ such that the sequence of functions $\{Q_n\}$ is generated by the formula

$$c_n\, f\, g^n\, Q_n \circ X = D^n h, \qquad n = 0,1,2,\dots, \tag{2.1}$$

where $D \equiv d/dx$. Define the generating function

$$F(x,t) = \sum_{n=o}^{\infty} a_n t^n Q_n(x). \tag{2.2}$$

Then

$$fF(X,gtz)\,\big|_{x+t} = f \sum_{n=o}^{\infty} c_n (gt)^n Q_n \circ X\; b_n(z),$$

where

$$b_n(z) = \sum_{k=o}^{\infty} \frac{a_k}{c_k\,(n-k)!}\, z^k.$$

PROOF. By Taylor's theorem,

$$fF(X,gtz)\,\big|_{x+t} = e^{tD} fF(X,gtz). \tag{2.3}$$

To evaluate the right hand side of (2.3), we shall use as our starting point the relations (2.1) and (2.2), and the series expansion for e^{tD}. Thus

$$\begin{aligned}
e^{tD} fF(X,gtz) &= e^{tD} f \sum_{n=o}^{\infty} a_n (gtz)^n Q_n \circ X \\
&= e^{tD} \sum_{n=o}^{\infty} \frac{a_n}{c_n} (tz)^n D^n h \\
&= \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} (a_n/c_n)\, t^{n+m} z^n D^{n+m} h/m! \\
&= \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} (a_n/c_n)\, (gt)^{n+m} c_{n+m} f\, Q_{n+m} \circ X/m! \\
&= f \sum_{n=o}^{\infty} c_n (gt)^n Q_n \circ X\, b_n(z),
\end{aligned}$$

where

$$b_n(z) = \sum_{k=o}^{\infty} \frac{a_k}{c_k (n-k)!}\, z^k.$$

It is worthwhile to remark here that if we choose $Q_n(x) = P_n^{\lambda}(x)$, $f(x) = (x^2-1)^{-\lambda}$, $g(x) = (x^2-1)^{-1/2}$, $X(x) = x(x^2-1)^{-1/2}$, $h(x) = (x^2-1)^{-\lambda}$ and $c_n = n!\,/(-1)^n$ then Theorem 2 would correspond to Chatterjea's theorem.

APPLICATIONS: Earlier, Bhattacharjya (2) introduced a new class of generalized Legendre polynomials $\{P_n(x;\ \alpha,\ \beta)\}$ which are orthogonal with the weight function $\frac{|x|^{\beta}}{(1-x^2)^{(\beta-\alpha)/2}}$. The Rodrigue's formulae for these polynomials are (2 , (6.6) and (6.8)):

$$P_{2m}(x^{-1/2};\alpha,\beta) = \frac{x^{m+(\alpha+1)/2}(1-x)^{(\beta-\alpha)/2}}{(-2m-(\alpha-1)/2)_m} \cdot D^m[(1-x)^{m-(\beta-\alpha)/2}x^{-m-(\alpha+1)/2}], \tag{2.4}$$

and

$$P_{2m+1}(x^{-1/2};\alpha,\beta) = \frac{x^{m+1+\alpha/2}(1-x)^{(\beta-\alpha)/2}}{(-2m-(\alpha+1)/2)_m} \cdot D^m[(1-x)^{m-(\beta-\alpha)/2}x^{-m-(\alpha+3)/2}] \tag{2.5}$$

Here we note that the sequences $\{P_{2n}(x^{-1/2};\alpha-2n,\beta)\}$ and $\{P_{2n+1}(x^{-1/2};\alpha-2n,\beta)\}$ are amenable to a method of Theorem 2 for finding bilateral generating functions.

Let $Q_n(x) = P_{2n}(x;\alpha-2n,\beta) \equiv P_{2n}(x)$. For simplicity of notation, set $y = -(\alpha+1)/2$ and $\delta = (\alpha-\beta)/2$. Then (2.1) holds with $f(x) = x^y(1-x)^{\delta}$, $g(x) = (1-x)^{-1}$, $X(x) = x^{-1/2}$ and $c_n = \phi(n) = (-n-(\alpha-1)/2)_n$. Upon replacing t by $-t$ and z by $-y$, we get

$$\left(\frac{x-t}{x}\right)^y\left(\frac{1-(x-t)}{1-x}\right)^{\delta} F\left(\frac{1}{(x-t)^{1/2}}, -\frac{yt}{(1-(x-t))}\right) = \sum_{r=0}^{\infty}\left(\frac{-t}{1-x}\right)^r \phi(r) \cdot P_{2r}(x^{-1/2})\, b_r(-y), \tag{2.6}$$

where

$$F\left(\frac{1}{x^{1/2}}, \frac{t}{1-x}\right) = \sum_{m=0}^{\infty} a_m\left(\frac{t}{1-x}\right)^m P_{2m}(x^{-1/2})$$

and

$$b_r(-y) = \sum_{m=0}^{\infty}\frac{a_m(-y)^m}{\phi(m)\,(r-m)!}. \tag{2.7}$$

Now replacing $x^{-1/2}$ by s and $t/(1-x)$ by t in (2.6), we are led to the following bilateral generating function for gerneralized even Legendre polynomials:

COROLLARY. 1: If

$$F(x,t) = \sum_{m=0}^{\infty} a_m t^m P_{2m}(x),$$

then

$$[1-(x^2-1)t]^{y} \quad (1+t)^{\delta} \quad F\left(\frac{x}{(1-t(x^2-1))^{1/2}}, \frac{yt}{1+t}\right) = \sum_{r=o}^{\infty} (-t)^r \phi(r) \cdot$$

$$\cdot\ P_{2r}(x)\ b_r(-y),$$

where $b_r(-y)$ is given by (2.7).

In the same way, let $Q_n(x) = P_{2n+1}(x;\ \alpha-2n,\ \beta) \equiv P_{2n+1}(x)$, and set $y = -(\alpha+2)/2$, $\delta = (\alpha-\beta)/2$. Then (2.1) holds with $f(x) = x^{y}\ (1-x)^{\delta}$, $g(x) = (1-x)^{-1}$, $\chi(x) = x^{-1/2}$ and $c_n = \psi(n) = (-n-(\alpha+1)/2)_n$. Replacing t by -t and z by -y and making the same substitution as before in (2.7), we are led to the following bilateral generating function for generalized odd Legendre polynomials.

COROLLARY 2: If

$$F(x,t) = \sum_{m=o}^{\infty} a_m t^m P_{2m+1}(x),$$

then

$$[1-(x^2-1)t]^{y} (1-t)^{\delta} F\left(\frac{x}{(1-t(x^2-1))^{1/2}}, \frac{ty}{1+t}\right) = \sum_{r=o}^{\infty} (-t)^r \psi(r) \cdot$$

$$\cdot\ P_{2r+1}(x)\ c_r(-y),$$

where

$$c_r(-y) = \sum_{m=o}^{r} \frac{a_m (-y)^m}{\Psi(m)\ (r-m)!} .$$

Taking $\alpha = \beta$ in Corollary 1 and 2, we can obtain bilateral generating functions for generalized Legendre polynomials due to Dutta and More (3).

Next, we note that (2),

$$P_{2m}(x;o,o) = \frac{(-1)^m\ m!\ P_{2m}(x)}{(-2m+\tfrac{1}{2})_m}, \tag{2.8}$$

and

$$P_{2m}(x;o,o) = \frac{(-1)^m\ m!\ P_{2m+1}(x)}{(-2m-\tfrac{1}{2})_m}, \tag{2.9}$$

where $P_{2m}(x)$ and $P_{2m+1}(x)$ are even and odd Legendre polynomials. Therefore, by (2.8), (2.9) and the above two corollories we can obtain bilateral generating functions for Legendre polynomials.

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REFERENCES

1. CHATTERJEA, S.K., A Bilateral generating Function for the Ultraspherical Polynomials, Pacific J. Math. 29 (1969) 73-76.

2. BHATTACHARJYA, M., On Some Generalisation of Legendre Polynomials, Bull. Cal. Math. Soc. 66 (1974) 77-85.

3. DUTTA, M. and MORE, K.L., A New Class of Generalised Legendre Polynomials, Mathematica (Cluj) 7(30) (1965) 33-41.