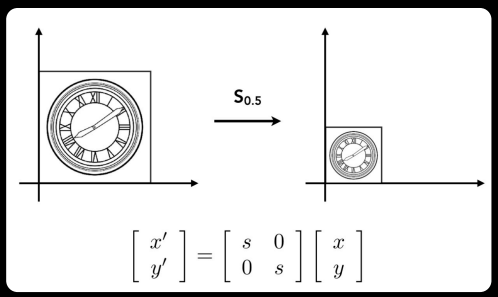
Modeling Transformation

Viewing Transformation

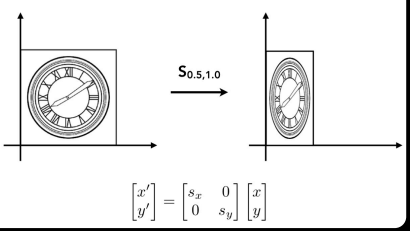
2D Transformation

Linear Transformation

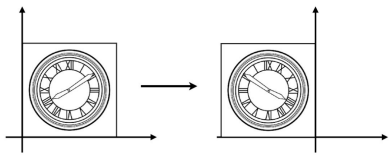
$x \Rightarrow -x$



Scale (Non-Uniform)



Reflection Matrix



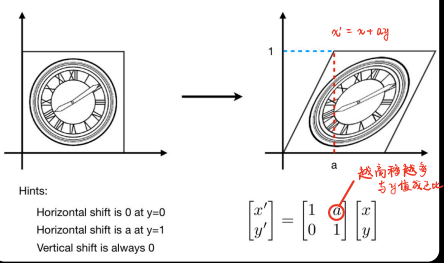
Horizontal reflection:

$x' = -x$

$y' = y$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Shear Matrix



Hints:

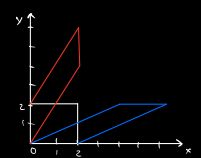
Horizontal shift is 0 at y=0

Horizontal shift is a at y=1

Vertical shift is always 0

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

越向物越多 a 值越大



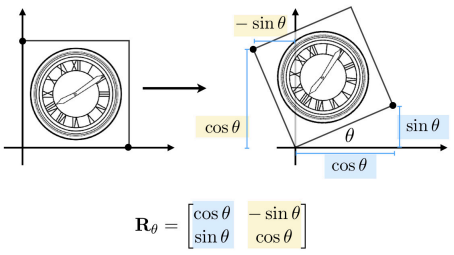
Shear 2 unit

(0,0) (0,2) (2,0) (2,2)

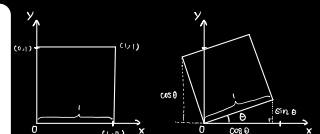
X (0,0) (4,2) (2,0) (6,2)

Y (0,0) (0,2) (2,4) (2,6)

Rotation Matrix



$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



$(x, y) \Rightarrow (x', y')$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

① $(1,0) \Rightarrow (\cos\theta, \sin\theta)$

$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \Rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\cos\theta = A \cdot 1 + B \cdot 0 = A$

$\sin\theta = C \cdot 1 + D \cdot 0 = C$

② $(0,1) \Rightarrow (-\sin\theta, \cos\theta)$

△ 旋转永远是以原点为中心

$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R_\theta^T$

$R_\theta^T = R_\theta^{-1}$ <By definition> 转置 = 逆 → 正交矩阵 Orthogonal Matrix

$R_\theta \times R_\theta^{-1} = I \quad R_\theta^T \Rightarrow R_\theta \cdot y = R_{\theta y}^T$

Linear Transformation Conclusion

Linear Transforms = Matrices

(of the same dimension)

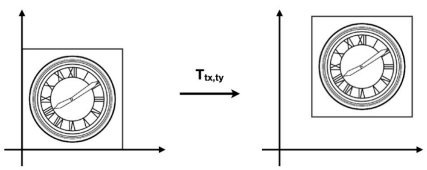
$$x' = a\,x + b\,y$$
$$y' = c\,x + d\,y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{M}\,\mathbf{x}$$

Homogeneous Coordinates 齐次坐标

Translation??



$$x' = x + t_x$$
$$y' = y + t_y$$

Why Homogeneous Coordinates

- Translation cannot be represented in matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

(So, translation is NOT linear transform!)

- But we don't want translation to be a special case
- Is there a unified way to represent all transformations? (and what's the cost?)

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Solution: Homogenous Coordinates

Add a third coordinate (*w-coordinate*)

- 2D point = (x, y, 1)^T 通过增加一个坐标
- 2D vector = (x, y, 0)^T 来使平移变成矩阵乘法

↳ 向量具有平移不变性。

Matrix representation of translations

$$\begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}$$

What if you translate a vector?

Affine Transformations

Affine map = linear map + translation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

Using homogenous coordinates:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

translation

always 0 0 1 in an affine transformation

27

2D Transformations

Scale

$$\mathbf{S}(s_x, s_y) = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation

$$\mathbf{R}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Translation

$$\mathbf{T}(t_x, t_y) = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

仿射变换

two matrices

one matrix

Homogenous Coordinates

Valid operation if w-coordinate of result is 1 or 0

- vector + vector = vector
- point − point = vector
- point + vector = point
- point + point = ??

In homogeneous coordinates,

$\begin{pmatrix} x \\ y \\ w \end{pmatrix}$ is the 2D point $\begin{pmatrix} x/w \\ y/w \\ 1 \end{pmatrix}$, $w \neq 0$

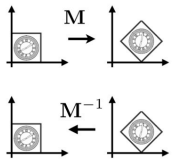
两点中点

Inverse Transformation

Inverse Transform

$$\mathbf{M}^{-1}$$

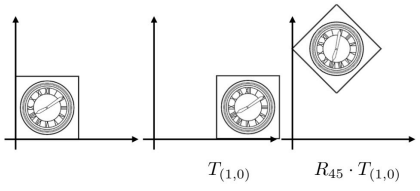
$\mathbf{M}^{-1}$ is the inverse of transform $\mathbf{M}$ in both a matrix and geometric sense



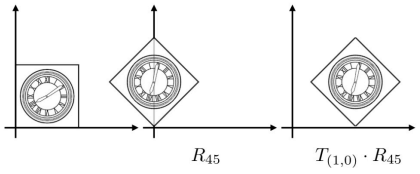
Composing Transform

ORDER MATTERS !

Translate Then Rotate?



Rotate Then Translate



Transform Ordering Matters!

Matrix multiplication is not commutative

$$R_{45} \cdot T_{(1,0)} \neq T_{(1,0)} \cdot R_{45}$$

Note that matrices are applied right to left:

$$T_{(1,0)} \cdot R_{45} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 45° & -\sin 45° & 0 \\ \sin 45° & \cos 45° & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

TRANSLATION ROTATE

Composing Transforms

Sequence of affine transforms A_1, A_2, A_3, ...

- Compose by matrix multiplication
  - Very important for performance!

$$A_n(\dots A_2(A_1(\mathbf{x}))) = \underbrace{\mathbf{A}_n \cdots \mathbf{A}_2 \cdot \mathbf{A}_1}_{} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

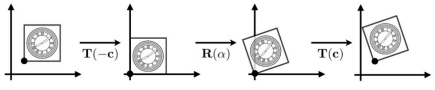
Pre-multiply *n* matrices to obtain a single matrix representing combined transform

Decomposing Transformation

Decomposing Complex Transforms

How to rotate around a given point c?

1. Translate center to origin
2. Rotate
3. Translate back



Matrix representation?

$$\mathbf{T}(\mathbf{c}) \cdot \mathbf{R}(\alpha) \cdot \mathbf{T}(-\mathbf{c})$$

3D Transformation

3D Transformations

Use homogeneous coordinates again:

- 3D point = (x, y, z, 1)^T
- 3D vector = (x, y, z, 0)^T

In general, (x, y, z, w) (w != 0) is the 3D point:

(x/w, y/w, z/w)

Homogeneous Coordinate in 3D

3D Transformations

Use 4×4 matrices for affine transformations

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

What's the order?
Linear Transform first or Translation first?

① ②

Scale and Translation

Scale

$$\mathbf{S}(s_x, s_y, s_z) = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Translation

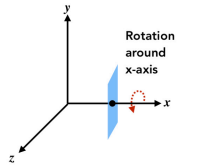
$$\mathbf{T}(t_x, t_y, t_z) = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation around x-, y-, or z-axis

$$\mathbf{R}_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



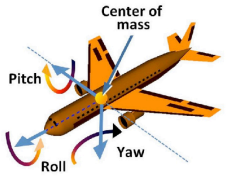
Anything strange about R_y?

Z × x ⇒ y
not counter-clockwisely x × z
so flip x and z (T)

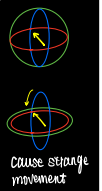
Compose any 3D rotation from $\mathbf{R}_x, \mathbf{R}_y, \mathbf{R}_z$?

$$\mathbf{R}_{xyz}(\alpha, \beta, \gamma) = \mathbf{R}_x(\alpha)\,\mathbf{R}_y(\beta)\,\mathbf{R}_z(\gamma)$$

- So-called *Euler angles*
- Often used in flight simulators: roll, pitch, yaw



gimble-lock



cause strange movement

Rodrigues' Rotation Formula

Rodrigues

Rotation by angle α around axis n

x, y or z

$$\mathbf{R}(\mathbf{n}, \alpha) = \cos(\alpha)\,\mathbf{I} + (1 - \cos(\alpha))\,\mathbf{n}\mathbf{n}^T + \sin(\alpha)\underbrace{\begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}}_{\mathbf{N}}$$

How to prove this magic formula?

Check out the supplementary material on the course website!

Viewing Transformation

View / camera transformation

- What is view transformation?
- Think about how to take a photo
  - Find a good place and arrange people (**model** transformation)
  - Find a good "angle" to put the camera (**view** transformation)
  - Cheese! (**projection** transformation)

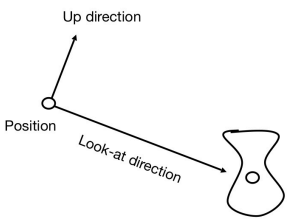
滑动鼠标屏幕变成一张图像

- Viewing (观测) transformation
  - View (视图) / Camera transformation
  - Projection (投影) transformation
    - Orthographic (正交) projection
    - Perspective (透视) projection

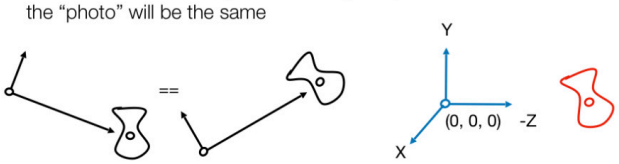
MVP
渲染过程

View / Camera Transformation

- How to perform view transformation?
- Define the camera first
  - Position $\vec{e}$
  - Look-at / gaze direction $\hat{g}$
  - Up direction $\hat{t}$ (assuming perp. to look-at)



- Key observation
  - If the camera and all objects move together, the "photo" will be the same



- How about that we always transform the camera to
  - The origin, up at Y, look at -Z
  - And transform the objects along with the camera

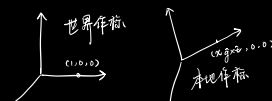
- M_{view} in math?
  - Let's write $M_{view} = R_{view}T_{view}$
  - Translate e to origin

$$T_{view} = \begin{bmatrix} 1 & 0 & 0 & -x_e \\ 0 & 1 & 0 & -y_e \\ 0 & 0 & 1 & -z_e \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - Rotate g to -Z, t to Y, (g x t) To X (g: gaze, t: up)
  - Consider its inverse rotation: X to (g x t), Y to t, Z to -g

$$R_{view}^{-1} = \begin{bmatrix} x_{\hat{g}\times\hat{t}} & x_t & x_{-g} & 0 \\ y_{\hat{g}\times\hat{t}} & y_t & y_{-g} & 0 \\ z_{\hat{g}\times\hat{t}} & z_t & z_{-g} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{WHY?} \quad R_{view} = \begin{bmatrix} x_{\hat{g}\times\hat{t}} & y_{\hat{g}\times\hat{t}} & z_{\hat{g}\times\hat{t}} & 0 \\ x_t & y_t & z_t & 0 \\ x_{-g} & y_{-g} & z_{-g} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

△ 旋转矩阵的逆就是它的转置 <正交矩阵的性质>
inverse transpose

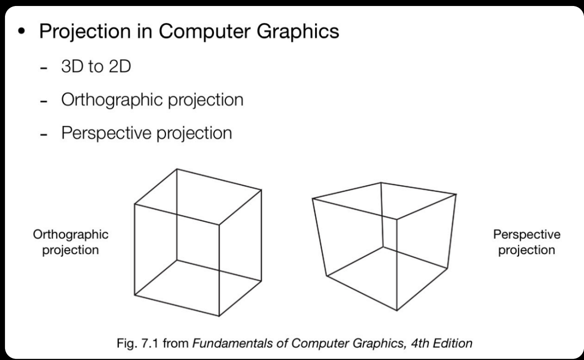


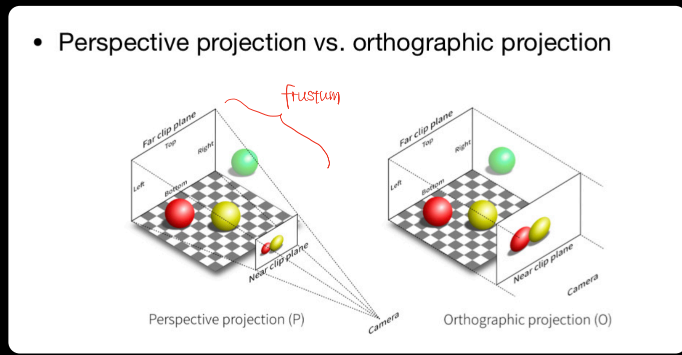
△ 免去了繁杂的三角运算计算

将摄像机从环境变换至原点位置
视图变换

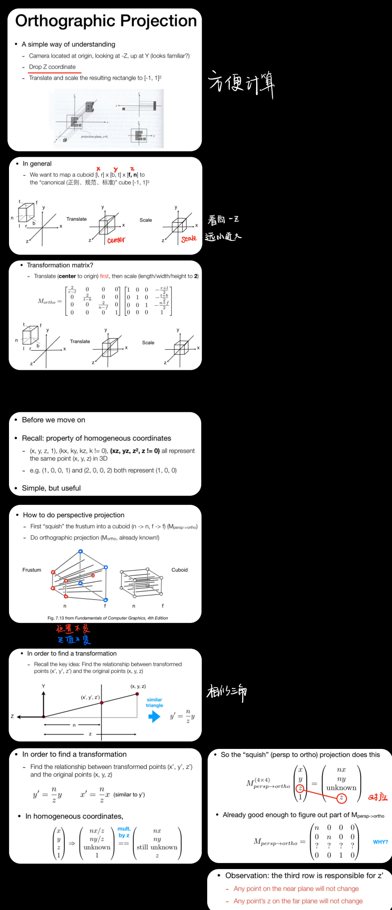
- Summary
  - Transform objects together with the camera
  - Until camera's at the origin, up at Y, look at -Z
- Also known as ModelView Transformation
- But why do we need this?
  - For projection transformation!

Projection Transformation





Projection
- Orthographic
- Perspective



方便计算

以不变点为前提条件推算

the 3rd row must be (0 0 A B)

Premise ① NEAR $\begin{pmatrix} x \\ y \\ n \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ n \\ 1 \end{pmatrix} == \begin{pmatrix} nx \\ ny \\ n^2 \\ n \end{pmatrix}$ 对应

Therefore $(0\ 0\ A\ B)\begin{pmatrix} x \\ y \\ n \\ 1 \end{pmatrix} = An + B = n^2$

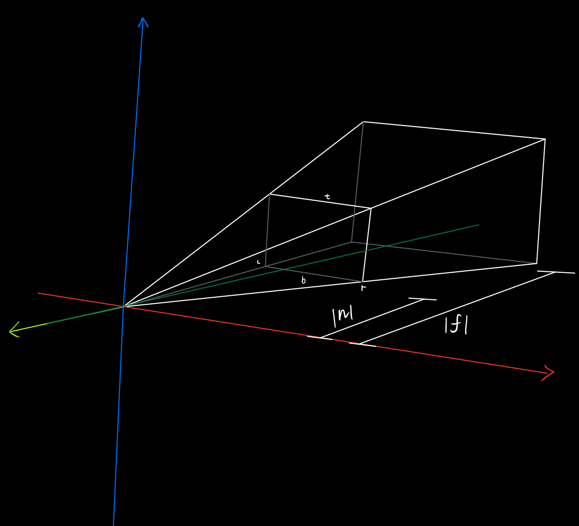
Premise ② CENTER OF FAR $\begin{pmatrix} 0 \\ 0 \\ f \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ f \\ 1 \end{pmatrix} == \begin{pmatrix} 0 \\ 0 \\ f^2 \\ f \end{pmatrix}$ 对应

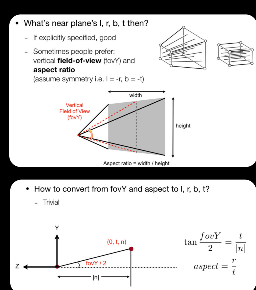
Therefore $(0\ 0\ A\ B)\begin{pmatrix} x \\ y \\ f \\ 1 \end{pmatrix} = Af + B = f^2$

$$\begin{cases} An + B = n^2 \\ Af + B = f^2 \end{cases} \Rightarrow \begin{cases} A = n + f \\ B = -nf \end{cases}$$

☆ n、f 均为坐标（有正负）

$$M_{persp\to ortho} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -nf \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

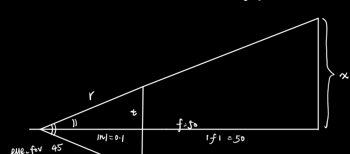




宽高比

可视角度 + 宽高比 即可推出所有参数

Homework / Note



aspect_ratio = $\frac{r}{t}$ = 1

$\tan(\frac{eye\text{-}fov}{2}) = \frac{t}{|n|} = \frac{x}{|f|}$

$\tan(\frac{eye\text{-}fov}{2}) =$