Review of Linear Algebra

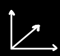
Vectors

$\vec{AB} = B - A$

Length of a vector : $\|\vec{a}\|$

Normalization : $\hat{a} = \vec{a} / \|\vec{a}\|$

Cartesian Coordinates



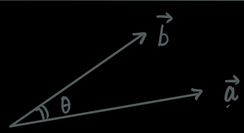
$A = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad A^T = (x, y)$ $\quad \|A\| = \sqrt{x^2 + y^2}$

x, y can be any vectors

Vector Multiplication 乘法

用法 / 意义

Dot <Scalar> Product 点乘



$\vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta$

UNIT VECTORS : $\cos\theta = \hat{a} \cdot \hat{b}$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$
$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$
$$(k\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b}) = k(\vec{a} \cdot \vec{b})$$

Dot Product in Cartesian Coordinates

- Component-wise multiplication, then adding up
  - In 2D

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \end{pmatrix} = x_a x_b + y_a y_b$$

  - In 3D

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = x_a x_b + y_a y_b + z_a z_b$$

① Find angle between two vectors

eg. cosine angle between light source and surface

② Find projection of one vector on another

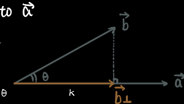
$\vec{b}_\perp$: projection of $\vec{b}$ onto $\vec{a}$

"b perp"

$\vec{b}_\perp$ must be along $\vec{a}$ or $\hat{a}$

$\vec{b}_\perp = k\hat{a}$

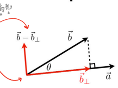
$k = \|\vec{b}_\perp\| = \|\vec{b}\| \cos\theta$

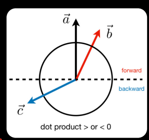


Dot Product in Graphics

观察镜面反射、高光

- Measure how close two directions are
- Decompose a vector
- Determine forward / backward

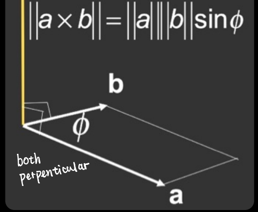




cos θ determines pos / neg

Cross Product 叉乘

$$a \times b = -b \times a$$
$$\|a \times b\| = \|a\| \|b\| \sin\phi$$



- Cross product is orthogonal to two initial vectors
- Direction determined by right-hand rule
- Useful in constructing coordinate systems (later)

不满足交换率

$\vec{x} \times \vec{y} = +\vec{z}$ 右手作标系

$\vec{y} \times \vec{x} = -\vec{z}$

$\vec{y} \times \vec{z} = +\vec{x}$

$\vec{z} \times \vec{y} = -\vec{x}$

$\vec{z} \times \vec{x} = +\vec{y}$

$\vec{x} \times \vec{z} = -\vec{y}$

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$\vec{a} \times \vec{a} = \vec{0}$ 0向量

$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$\vec{a} \times (k\vec{b}) = k(\vec{a} \times \vec{b})$

Cross Product: Cartesian Formula?

$$\vec{a} \times \vec{b} = \begin{pmatrix} y_a z_b - y_b z_a \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{pmatrix}$$

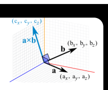
- Later in this lecture

$$\vec{a} \times \vec{b} = A^* b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

dual matrix of vector a

When **a** and **b** start at the origin point (0,0,0), the Cross Product will end at:

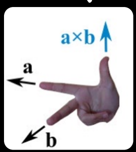
- $c_x = a_y b_z - a_z b_y$
- $c_y = a_z b_x - a_x b_z$
- $c_z = a_x b_y - a_y b_x$



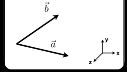
Example: The cross product of **a** = (2,3,4) and **b** = (5,6,7)

- $c_x = a_y b_z - a_z b_y = 3\times7 - 4\times6 = -3$
- $c_y = a_z b_x - a_x b_z = 4\times5 - 2\times7 = 6$
- $c_z = a_x b_y - a_y b_x = 2\times6 - 3\times5 = -3$

Answer: **a × b** = (−3,6,−3)



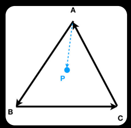
① determine left / right



$\vec{a} \times \vec{b} > 0$ $\vec{b}$ 在 $\vec{a}$ 左侧

$\vec{b} \times \vec{a} < 0$ $\vec{a}$ 在 $\vec{b}$ 右侧

② determine inside / outside



$\vec{AB} \times \vec{AP} > 0$ $\vec{AP}$ 在 $\vec{AB}$ 左侧

$\vec{BC} \times \vec{BP} > 0$ $\vec{BP}$ 在 $\vec{BC}$ 左侧

$\vec{CA} \times \vec{CP} > 0$ $\vec{CP}$ 在 $\vec{CA}$ 左侧

A 均左或均右 → P在三角内部

Coordinates

- Any set of 3 vectors (in 3D) that

$$\|\vec{u}\| = \|\vec{v}\| = \|\vec{w}\| = 1$$

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{w} = \vec{u} \cdot \vec{w} = 0$$ 互相垂直

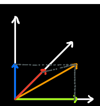
$$\underset{z}{\vec{w}} = \underset{x}{\vec{u}} \times \underset{y}{\vec{v}}$$ (right-handed)

$$\vec{p} = (\vec{p} \cdot \vec{u})\vec{u} + (\vec{p} \cdot \vec{v})\vec{v} + (\vec{p} \cdot \vec{w})\vec{w}$$ 在三轴上的投影

(projection)

$$\underbrace{\|\vec{p}\| \cdot \underset{1\ \text{单位向量}}{\|\vec{u}\|} \cdot \cos\theta}$$

$$\vec{p}_{\perp} = \underline{\|\vec{p}\| \cos\theta}\ \vec{u}$$



Matrices Multiplication

Matrix-Matrix Multiplication

- # (number of) columns in A must = # rows in B
  (M x N) (N x P) = (M x P)

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix} = \begin{pmatrix} 9 & \overset{27}{?} & 33 & 13 \\ 19 & 44 & 61 & 26 \\ 8 & 28 & 32 & \underset{12}{?} \end{pmatrix}$$

- Element (i, j) in the product is
  the dot product of row i from A and column j from B

- Properties
  - **Non-commutative**
    (AB and BA are different in general)
  - Associative and distributive
    - (AB)C=A(BC)
    - A(B+C) = AB + AC
    - (A+B)C = AC + BC

- Treat vector as a column matrix (m×1)
- Key for transforming points (next lecture)
- Official spoiler: 2D reflection about y-axis

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

矩阵的转置 transpose of matrix

Transpose of a Matrix

- Switch rows and columns (ij -> ji)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

- Property

$$(AB)^T = B^T A^T$$

Identity Matrix and Inverses

$$I_{3\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ 单位矩阵

$$AA^{-1} = A^{-1}A = I$$ 互逆矩阵

$$(AB)^{-1} = B^{-1}A^{-1}$$

Calculate Vector by Matrices

Vector multiplication in Matrix form

- Dot product?

$$\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$$

$$= \begin{pmatrix} x_a & y_a & z_a \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = \begin{pmatrix} x_a x_b + y_a y_b + z_a z_b \end{pmatrix}$$

- Cross product?

$$\vec{a} \times \vec{b} = A^* b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

dual matrix of vector a

dual matrix