(5) Practice Problems

(1) Write a MATLAB script to solve the following least squares optimization problem using gradient descent with different step-size strategies:

$$\min_x \|Ax - b\|^2$$

- A is an $m \times n$ matrix with entries drawn uniformly at random.
- x is an $n \times 1$ decision variable.
- b is an $m \times 1$ vector with entries drawn uniformly at random.
- Set $m = n = 5$.
- The algorithm should start from $x = 0$ and stop when the norm of the gradient is less than 10^{-5} or after a maximum of 400 iterations.

Your implementation should:

- Implement three different step-size selection methods:
  - **Fixed step-size using the Lipschitz constant**: Set $\alpha = \frac{1}{L}$, where L is the Lipschitz constant of the gradient.
  - **Diminishing step-size**: Set $\alpha = 0.1/k$ at iteration k.
  - **Armijo rule (backtracking line search)**: Start with $\alpha = 2$ and reduce it using backtracking until the Armijo condition is satisfied. Set $\sigma = 0.25$ and $\beta = 0.5$.
- Allow the user to select the step-size method via an input prompt.
- Plot the function values over iterations.

(2) Solve the following constrained least squares optimization problem using the projected gradient method with a step size of $1/L$, where L is the Lipschitz constant of the gradient, and set the tolerance to $\epsilon = 1e - 5$:

$$\min_x \|Ax - b\|^2, \quad \text{s.t. } -0.5 \leq x \leq 0.5.$$

Fix the seed to 123, generate $A \in \mathbb{R}^{4\times 3}$ and $b \in \mathbb{R}^4$ uniformly at random, and let $x_0 = \mathbf{0}$. Display the final iteration point and its objective value.