Homework 2 – Due by 11:59 PM on Sunday Feb 16

(1) For each of the following functions, find all the stationary points and classify them as saddle points, minimum points, maximum points:

(a) $f(x_1, x_2, x_3) = x_1^4 - 2x_1^2 + x_2^2 + 2x_2x_3 + 2x_3^2$

(b) $f(x_1, x_2) = x_1^4 + 2x_1^2x_2 + x_2^2 - 4x_1^2 - 8x_1 - 8x_2$

(2) (a) Let $C_1, \dots, C_k \subseteq \mathbb{R}^n$ be convex sets and $\mu_1, \dots, \mu_k \in \mathbb{R}$. Then the set $C \triangleq \sum_{i=1}^k \mu_i C_i$ is convex.

(b) Let $M \subseteq \mathbb{R}^n$ be a convex set and $A \in \mathbb{R}^{m\times n}$. Then the set $\mathbf{A}(M) = \{Ax : x \in M\}$ is convex.

(c) Let $D \subseteq \mathbb{R}^m$ be a convex set and let $A \in \mathbb{R}^{m\times n}$. Then the set

$$A^{-1}(D) = \{x \in \mathbb{R}^n \mid Ax \in D\}$$

is convex.

(3) Show that the set of all positive semidefinite matrices is a convex set.

(4) Is $f(x) = -x_1x_2$ a convex function or not? Why?

(5) (a) Let A be a symmetric 2×2 matrix. Then A is positive semidefinite if and only if $\mathrm{tr}(A) \geq 0$ and $\det(A) \geq 0$.

(b) Use part (a) and prove that $f(x) = \dfrac{x_1^2}{x_2}$ is a convex function for $x_2 > 0$.

Students enrolled in SIE 549 must solve the following problems.
Students in SIE 449 will get extra credit by solving them.

(6) Function $f : \mathbb{R}^n \to \mathbb{R}$ is called a *strongly convex function*, if there exists $\alpha > 0$ such that $f(x) - \alpha\|x\|^2$ is convex. Use this definition and answer the following questions.

(a) Show that if f is strongly convex then f is convex.

(b) Show that x^4 is a convex function but it is not strongly convex.

(1a) $f(x_1,x_2,x_3) = x_1^4 - 2x_1^2 + x_2^2 + 2x_2x_3 + 2x_3^2$

$$\nabla f(x_1,x_2,x_3) = \begin{bmatrix} 4x_1^3 - 4x_1 \\ 2x_2 + 2x_3 \\ 2x_2 + 4x_3 \end{bmatrix}$$

$$\begin{cases} 4x_1^3 - 4x_1 = 0 = 4x_1(x_1^2 - 1) & x_1 = 0, -1, 1 \\ 2x_2 + 2x_3 = 0 = x_2 = -x_3 & x_2 = -x_3 \\ 2x_2 + 4x_3 = 0 = x_3 = -\tfrac{1}{2}x_2 & x_2 = -2x_3 \end{cases} \Rightarrow \begin{matrix} x_2 = 0 \\ x_3 = 0 \end{matrix}$$

Stationary Points: $(0,0,0)$, $(-1,0,0)$, $(1,0,0)$

$$\nabla^2 f(x_1,x_2,x_3) = \begin{bmatrix} 12x_1^2 - 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\nabla^2 f(0,0,0) = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$ eigenvalues: $-4, 1, 6$

Saddle Point

$$\nabla^2 f(-1,0,0) = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$ eigenvalues: $8, 1, 6$

minimum point

$$\nabla^2 f(1,0,0) = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$ eigenvalues: $8, 1, 6$

minimum point

(1b) $f(x_1,x_2) = x_1^4 + 2x_1^2x_2 + x_2^2 - 4x_1^2 - 8x_1 - 8x_2$

$$\nabla f(x_1,x_2) = \begin{bmatrix} 4x_1^3 + 4x_2x_1 - 8x_1 - 8 \\ 2x_1^2 + 2x_2 - 8 \end{bmatrix}$$

$$\nabla^2 f(x_1,x_2) = \begin{bmatrix} 12x_1^2 + 4x_2 - 8 & 4x_1 \\ 4x_1 & 2 \end{bmatrix}$$

$$\begin{cases} 4x_1^3 + 4x_2x_1 - 8x_1 - 8 = 0 \rightarrow 4x_1^3 + 4x_1(-x_1^2 + 4) - 8x_1 - 8 \rightarrow 4x_1^3 - 4x_1^3 + 16x_1 - 8x_1 - 8 \rightarrow 8x_1 - 8 \rightarrow x_1 = 1 \\ 2x_1^2 + 2x_2 - 8 = 0 \rightarrow x_2 = -x_1^2 + 4 \rightarrow x_2 = 3 \end{cases}$$

Stationary Point: $(1,3)$

$$\nabla^2 f(1,3) = \begin{bmatrix} 16 & 4 \\ 4 & 2 \end{bmatrix}$$ det $= 32 - 16 = 16 > 0$

trace $= 16 + 2 = 18 > 0$ $\succ 0$ Minimum point

(2)

a: $C \triangleq \sum_{i=1}^{k} \mu_i C_i$

let $u = \sum_{i=1}^{k} \mu_i x_i$, $v = \sum_{i=1}^{k} \mu_i y_i$ where $x_i, y_i \in C_i$

$$\alpha u + (1-\alpha)v = \alpha(\sum \mu_i x_i) + (1-\alpha)(\sum \mu_i y_i)$$

$$= \sum \mu_i (\alpha x_i) + \sum \mu_i ((1-\alpha) y_i)$$

$$= \sum \mu_i (\alpha x_i + (1-\alpha) y_i)$$

Since C_i is convex and $x_i, y_i \in C_i$

then for $\alpha \in [0,1]$, $\alpha x_i + (1-\alpha) y_i$ is also convex

■

b: $M \subseteq \mathbb{R}^n$ convex set and $A \in \mathbb{R}^{m \times n}$. Show $A(M) = \{Ax : x \in M\}$ is convex.

let $u = Ax$ $v = Ay$ where $x, y \in M$

$$\alpha u + (1-\alpha)v = \alpha Ax + (1-\alpha)Ay$$

$= A(\alpha x + (1-\alpha)y)$ since $x, y \in M$, $\alpha x + (1-\alpha)y$ for $\alpha \in [0,1]$ also $\in M$

therefore, $A(M)$ is convex ■

c: $D \subseteq \mathbb{R}^m$ convex, $A \in \mathbb{R}^{m \times n}$. Prove $A^{-1}(D) = \{x \in \mathbb{R}^n \mid Ax \in D\}$ is convex

let $Ax \in D$, $Ay \in D$

Show $A(\alpha x + (1-\alpha)y) \in D$ s.t. $\alpha x + (1-\alpha)y \in A^{-1}(D)$

$$= \alpha x A + (1-\alpha) y A$$

$= \alpha(Ax) + (1-\alpha)(Ay)$ Since $Ax, Ay \in D$, $\alpha x + (1-\alpha)y$ also exists in D

Due to property of $A^{-1}(D)$, $\alpha x + (1-\alpha)y \in A^{-1}(D)$ and $A^{-1}(D)$ is convex

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(3) Set of PSD $n \times n$ matrices: S^{+n}

let $A, B \in S^{+n}$

$x^T A x \geq 0$
$x^T B x \geq 0$ for vector $x \in R^n$

Show that $\alpha A + (1-\alpha)B$ is PSD (for $\alpha \in [0,1]$)

$$x^T(\alpha A + (1-\alpha)B)x = x^T \alpha A x + x^T (1-\alpha) B x$$
$$= \alpha(x^T A x) + (1-\alpha)(x^T B x)$$

Since $x^T A x \geq 0$, $x^T B x \geq 0$, and $\alpha \geq 0$, $1-\alpha \geq 0$
$\alpha(x^T A x) + (1-\alpha)(x^T B x) \geq 0$ and PSD

therefore S^{+n} is convex ■

(4) Definition of convex function:

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

let $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $\alpha = 1/2$

$f(x) = -1 \quad f(y) = -1$

$$f(\alpha x + (1-\alpha)y) = f\left(\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}\right)$$
$$= f(0, 0) = 0$$

$$\alpha f(x) + (1-\alpha) f(y) = -1/2 - 1/2 = -1$$

$0 \leq -1$ is invalid, so $f(x)$ is not convex ■

(5)

a: Show A is PSD $\iff tr(A) \geq 0$ and $det(A) \geq 0$

proof for $\Rightarrow$:

assume A is PSD.

All eigenvalues are positive, and $det(A) = \lambda_1 \cdot \lambda_2$
then $det(A) \geq 0$

Similarly, $tr(A) = \lambda_1 + \lambda_2$
so $tr(A) \geq 0$ since $\lambda_1, \lambda_2 \geq 0$

proof for $\Leftarrow$:

Assume $tr(A) \geq 0$, $det(A) \geq 0$

then $\lambda_1 + \lambda_2 \geq 0$

$\lambda_1 \lambda_2 \geq 0$

if one eigenvalue was negative, $\lambda_1 \lambda_2 \geq 0$ would be invalid
if both eigenvalues were negative, $\lambda_1 + \lambda_2 \geq 0$ would be invalid
so eigenvalues must be positive to satisfy $tr(A) \geq 0$ and $det(A) \geq 0$

b: Prove $f(x) = \frac{x_1^2}{x_2}$ is convex for $x_2 > 0$

$$\nabla f(x) = \begin{bmatrix} 2x_1/x_2 \\ -x_1^2/x_2^2 \end{bmatrix} \quad \nabla^2 f(x) = \begin{bmatrix} 2/x_2 & -2x_1/x_2^2 \\ -2x_1/x_2^2 & 2x_1^2/x_2^3 \end{bmatrix}$$

$$tr(\nabla^2 f(x)) = 2/x_2 + 2x_1^2/x_2^3 \geq 0$$

$$det(\nabla^2 f(x)) = (2/x_2)(2x_1^2/x_2^3) - (-2x_1/x_2^2)^2$$

$$= 4x_1^2/x_2^4 - 4x_1^2/x_2^2 = 0 \geq 0$$

Since $tr(\nabla^2 f(x)) \geq 0$ and $det(\nabla^2 f(x)) \geq 0$, $\nabla^2 f(x)$ is PSD meaning $f(x)$ is convex. $\blacksquare$