Convex Optimization

SIE 449/549: Optimization for Machine Learning

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Convexity

Definition 1 (Convex Function)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function if it satisfies

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y), \quad \forall \alpha \in (0,1), x, y \in \mathbb{R}^n$$

Moreover, f is called strictly convex if the inequality is strict for any $x \neq y$.

- Examples of convex functions:

  (a) Exponential:

  (b) Negative logarithm:

Convex Function

(c) Affine functions:

(d) Norms:

(e) Nonnegative weighted sums of convex functions:

Convex Function

Example 1

Use the fact that $2ab \leq a^2 + b^2$ *and prove* $f(x) = x^2$ *is a convex function.*

Theorem 2 (Young's Inequality)

If $a \geq 0$ *and* $b \geq 0$ *are nonnegative real numbers and if* $p > 1$ *and* $q > 1$ *are real numbers such that* $\frac{1}{p} + \frac{1}{q} = 1$*, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Convexity

- It can be shown that a continuous function is convex if and only if
$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \quad \forall x, y$$
- It means that for any $x \in \mathbb{R}^n$, the graph of linear approximation of a convex function f around x should lie below the graph of f.

Convex Set

Definition 3 (Convex Set)

A set S is convex if for any $x, y \in S$, $\alpha x + (1-\alpha)y \in S$ for all $\alpha \in [0,1]$.

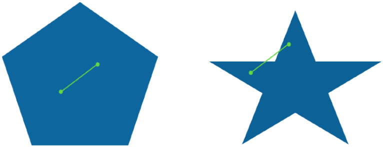

Figure 1: Convex vs. nonconvex sets

- Some examples of convex sets:

  (a) n-dimensional Euclidean space $\mathbb{R}^n$

  (b) Nonnegative orthant $\mathbb{R}^n_+$:

Convex set

(c) Balls defined by an arbitrary norm:

(d) Affine subspace:

(e) Intersections of convex sets:

Convex Optimization

Definition 4 (Convex Optimization)

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & x \in X \end{aligned}$$

is convex if f is a convex function and X is a convex set.

- For convex optimization problems, local optimality implies global optimality.

Convex Optimization

- If f is twice continuously differentiable, then

$$f \text{ is convex} \iff \nabla^2 f(x) \succeq 0 \text{ for all } x$$

- Function $f(x)$ is convex, then $\nabla f(x^*) = 0$ implies that x^* is a global minimum.
- The set of solutions of a convex optimization problem is convex.
- If the objective function is strictly convex, then the minimum is unique.

Example

$$\min_{x,y} \; f(x,y) = \frac{1}{2}(\alpha x^2 + \beta y^2) - x$$
$$\text{s.t.} \; (x,y) \in \mathbb{R}^2$$

- The Gradient is

- The Hessian is