Instructor: Afrooz Jalilzadeh SIE 449/549

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Homework 1 – Due by 11:59 PM on Sunday Feb 9

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(1) Compute $\nabla f(x)$ and $\nabla^2 f(x)$ for f where f is defined as one of the following.

(a) $f(x_1, x_2, x_3) = x_2^2$ (b) $f(x_1, x_2, x_3) = x_1^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$.

(2) Consider the function

$$f(x) = (a^\top x)(b^\top x),$$

where a, b, and x are $n-$dimensional vectors. Find $\nabla_x f(x)$ *(Hint: use product rule)*

(3) Are the following vectors linearly independent or dependent? Why?

$$a_1 = \begin{bmatrix} 0.2 \\ -7 \\ 8.6 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -0.1 \\ 2 \\ -1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 0 \\ -1 \\ 2.2 \end{bmatrix}$$

(4) Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A \in \mathbb{R}^{n\times n}$. Show that the eigenvalues of the matrix $I_n - A$ are $(1-\lambda_1), \ldots, (1-\lambda_n)$.

(5) Show that for any $x, y, z \in \mathbb{R}^n$, $\|x - z\| \le \|x - y\| + \|y - z\|$.

(6) Show that $\|\cdot\|_{1/2}$ is not a norm. *(Hint: Show that it may not satisfy the triangle inequality)*

Students enrolled in SIE 549 must solve the following problems.
Students in SIE 449 will get extra credit by solving them.

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(7) Show that $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$.

(8) Consider matrix A, from the definition of matrix norms, we know that $\|A\|_2 = \sigma_1$ where σ_1 is the largest singular value of A, and $\|A\|_\infty = \|A^T\|_1$. Show that $\|A\|_2^2 \le \|A\|_1\|A\|_\infty$.

(Hint: Consider first singular vector v of A, we know that $A^TAv = \sigma_1^2 v$)

(1)

(a) $\nabla f(x_1, x_2, x_3) = \begin{bmatrix} 0 \\ 2x_2 \\ 0 \end{bmatrix}$ $\quad \nabla^2 f(x_1, x_2, x_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\nabla f(x_1, x_2, x_3) = \begin{bmatrix} 2x_1 + 2x_2 + 2x_3 \\ 2x_1 + 2x_3 \\ 2x_3 + 2x_1 + 2x_2 \end{bmatrix}$ $\quad \nabla^2 f(x_1, x_2, x_3) = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

(2) $\nabla_x f(x) = b^T(a^T x) + a^T(b^T x)$

$= b^T x a^T + a^T x b^T$

$= x(b^T a^T + a^T b^T)$

(3) $a_1 = \begin{bmatrix} 0.2 \\ -7 \\ 8.6 \end{bmatrix}$ $\quad a_2 = \begin{bmatrix} -0.1 \\ 2 \\ -1 \end{bmatrix}$ $\quad a_3 = \begin{bmatrix} 0 \\ -1 \\ 2.2 \end{bmatrix}$

$c_1 a_1 + c_2 a_2 + c_3 a_3 = 0$

$$\begin{cases} c_1 = \tfrac{1}{2} c_2, \; c_2 = 2c_1 \\ c_2 = \tfrac{7}{2} c_1 + \tfrac{1}{4} c_3 \quad = c_2 = \tfrac{7}{4} c_2 + \tfrac{1}{2} c_3 \\ 2.2 c_3 = -8.6 c_1 + c_2 = c_2 = -4.3 c_2 + 2.2 c_3 \end{cases}$$

$-4.3 c_2 + 2.2 c_3 = \tfrac{7}{4} c_2 + \tfrac{1}{2} c_3$

$0.7 c_3 = (1.75 + 4.3) c_2$

$= 6.05 c_2$

$\boxed{c_3 = 8.64}$

$2c_1 = \tfrac{7}{2} c_1 + \tfrac{1}{2}(8.64)$

$\tfrac{1}{2} c_1 = 4.32$ $\quad \boxed{c_1 = 2.16}$

$\boxed{c_2 = 4.32}$

Linearly independant b/c all constants are unique.

(4) $Av = \lambda v$

$(A - \lambda I) v = 0$

$Av = \lambda v$

$-Av = -\lambda v$ divide by -1

$v - Av = v - \lambda v$ add v

$(I - A)v = (1 - \lambda)v$ distribute $\blacksquare$

(5) $\|x-z\| \leq \|x-y\| + \|y-z\|$

$A = (x-y)$
$B = (y-z)$
$A+B = x-y+y-z$
$= x-z$

$\|A+B\| \leq \|A\| + \|B\|$
$\|x-z\| \leq \|x-y\| + \|y-z\|$ ∎

(6) $\|\cdot\|_{1/2} = \left(\sum_{ij}^{n} |\cdot_{ij}|^{1/2}\right)^2$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$A+B = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$

$\|A\|_{1/2} = (\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4})^2$
$= (6.15)^2 = 37.82$

$\|B\|_{1/2} = (\sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{8})^2$
$= (10.16)^2 = 103.23$

$\|A+B\|_{1/2} = (\sqrt{6} + \sqrt{8} + \sqrt{10} + \sqrt{12})^2$
$= (11.90)^2 = 141.71$

$\|A\| + \|B\| = 141.05$

$\|A+B\| = 141.71$

$\|A+B\| > \|A\| + \|B\|$ ∎