Def: An ordinary differential equation (ODE) is an equation related to $y(x)$ and its derivatives. The order of the ODE is the order of the highest der.

$\rightarrow (y''(x))^2 + \sin(y(x))$ is a second-order ODE

Note: If y had more independent variables, e.g. $y = f(x, z)$, then the resulting equation would be called a partial DE. ... because of partial derivatives

Def: An initial value problem (IVP) is an ODE with an initial condition.

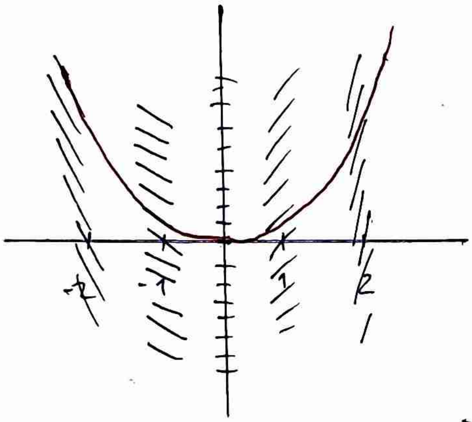
• **Direction Fields**

- a way of visualizing the set of solutions to $y' = f(x,y)$

$\rightarrow f(x,y)$ gives the slope of the solution at (x,y)

Examples:

① $y'(x) = 2x$



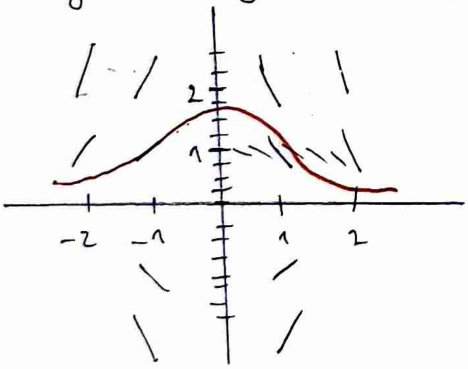
$\rightarrow$ choose a point and trace it to get the solution passing through that point

$\rightarrow$ this looks like

$y(x) = x^2 + B \qquad B \in \mathbb{R}$

$\rightarrow$ check: $y'(x) = 2x$ ✓

② $y'(x) = -xy$



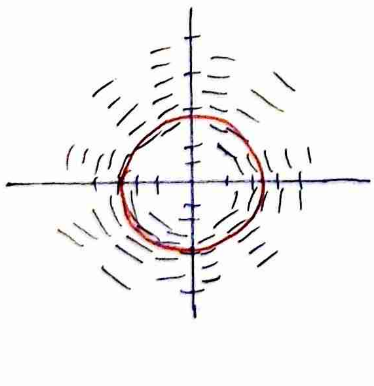
x	y	$-xy$
0	y	0
x	0	0
1	y	$-y$
x	1	$-x$
2	y	$-2y$

$\rightarrow$ looks like the bell curve

guess: $y = Ae^{-\frac{x^2}{2}} \qquad A \in \mathbb{R}$

check: $y' = A \cdot (-x) e^{-\frac{x^2}{2}} = -xy$ ✓

③ $y'(x) = -x/y$



x	y	$-x/y$
0	y	0
x	≈ 0	$\pm\infty$
x	$-x$	1
x	x	-1
1	y	$-1/y$

$\rightarrow$ circles around the origin

guess: $y = \pm\sqrt{r^2 - x^2} \qquad r \in \mathbb{R} \setminus \{0\}$

check: $y' = \pm \frac{-2x}{2\sqrt{r^2-x^2}} = \mp \frac{x}{\sqrt{r^2-x^2}} = -\frac{x}{y}$

• Separable Differential Equations

Def: $y'(x) = G(x)F(y)$

Method:

$$\frac{dy}{dx} = G(x)F(y) \Rightarrow \frac{dy}{F(y)} = G(x)\,dx \Rightarrow \int \frac{dy}{F(y)} = \int G(x)\,dx$$

Examples:

① Solve the IVP $y' = -x/y$, $y(0) = y_0 \in \mathbb{R}^-$

$$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow \int y\,dy = -\int x\,dx \Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

$$\Rightarrow y^2 = -x^2 + C \Rightarrow y = \pm\sqrt{C - x^2}$$

Note: $y_0 < 0 \Rightarrow y = -\sqrt{C - x^2}$,

IVP: $y_0 = -\sqrt{C - 0^2} \Rightarrow C = y_0^2 \Rightarrow \underline{y(x) = -\sqrt{y_0^2 - x^2}}$

Theorem (Existence + Uniqueness): Let $(x_0, y_0) \in \mathbb{R}^2$ be given and consider the IVP

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0$$

If f and $\frac{\partial f}{\partial y}$ are continuous in a neighborhood of (x_0, y_0), then the problem has a unique solution. More precisely $\exists y(x)$ and $\varepsilon > 0$ s.t. $y(x)$ is defined on $U^\varepsilon(x_0)$, solves the IVP and all functions which satisfy the IVP have the same values on $U^\varepsilon(x_0)$. In other words

$$(\exists \varepsilon > 0)(\exists y^* : U^\varepsilon(x_0) \to \mathbb{R}) : \left(y^{*\prime}(x) = f(x, y) \ \& \ y^*(x_0) = y_0\right)$$

$$(\forall y : U \subseteq \mathbb{R} \to \mathbb{R}) : \left(\mathrm{Rng}(y) \supseteq U^\varepsilon(x_0) \ \& \ y'(x) = f(x,y) \ \& \ y(x_0) = y_0\right) \Rightarrow y \restriction U^\varepsilon(x_0) = y^* \restriction U^\varepsilon(x_0)$$

The solution y is continuous ... it is differentiable

Note: A solution that is defined for all x is called global, otherwise it is local

Ex:

① Solve the IVP $\frac{dy}{dx} = \frac{y}{1+y^2}$, $y(3) = 0$

$$\int \frac{1+y^2}{y}\,dy = \int dx \Rightarrow x = \int \frac{1}{y} + y\,dy = \ln|y| + \frac{1}{2}y^2 + C$$

IVP: $y(3) = 0$: $\ln(0) + \ldots$ undefined

E+U: $f(x,y) = \frac{y}{1+y^2}$, $\frac{\partial f}{\partial y}$ are continuous near $(3, 0)$

$\Rightarrow$ unique solution exists

$y(x) = 0$ works ... $y' = \frac{0}{1+0} = 0$ & $y(3) = 0$ ✓

↳ dividing by y ... shouldn't be surprising that something happens when $y = 0$

(2) Solve the IVP $y' = y(1-y)$, $y(0) = \frac{1}{2}$

$\frac{dy}{dx} = y(1-y) \Rightarrow \int \frac{dy}{y(1-y)} = \int dx$

Special cases: $y = 0$ and $y = 1$

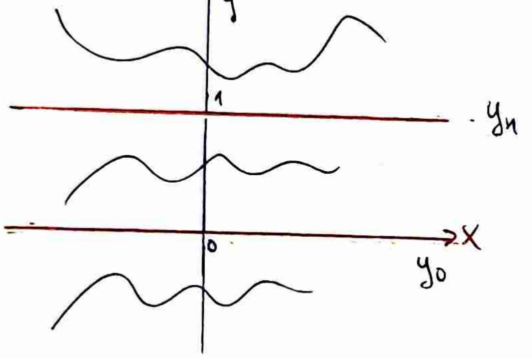
$\Rightarrow$ we need that every y which solves this equation has $\forall x: y(x) \notin \{0, 1\}$.

E+U: $f(x,y) = y(1-y)$ & $\frac{\partial f}{\partial y} = 1 - 2y$ are continuous (everywhere)

$\Rightarrow \exists!$ solution for every initial value

- $y(x_0) = 0 \rightsquigarrow y_0(x) = 0$ is the solution
- $y(x_1) = 1 \rightsquigarrow y_1(x) = 1$ is the solution

$\Big\} \; y' = 0$



$\to y_0$ is a solution and passes through all points $(x, 0)$

E+U $\Rightarrow$ no other sol. can pass through $(x, 0)$

$\to y_1$ ... E+U $\Rightarrow$ no other passes through $(x, 1)$

$\int \frac{dy}{y(1-y)} = \int \frac{1}{y} + \frac{1}{1-y} \, dy = \ln|y| - \ln|1-y| = x + c$

$\to$ IVP: the solution goes through $y(0) = \frac{1}{2} \Rightarrow 0 \leq y \leq 1 \Rightarrow$ no absolute values

$\ln(y) - \ln(1-y) = \ln\left(\frac{y}{1-y}\right) = x + C \Rightarrow \frac{y}{1-y} = e^{x+c}$

IVP: $y(0) = \frac{1}{2}$: $\frac{1}{2} / \frac{1}{2} = e^c \Rightarrow c = 0$

$\frac{y}{1-y} = e^x \Rightarrow y = e^x - ye^x \Leftrightarrow y(1 + e^x) = e^x \Rightarrow \underline{\underline{y(x) = \frac{e^x}{1+e^x}}}$

- Linear Differential Equations

Def: An ODE is called linear $\equiv$ it is a linear equation in the dependant variable and its derivatives, where the coefficients are functions of the independent variable; that is

$$\sum_{i=0}^{m} a_i(t) y^{(i)}(t) = b(t)$$

• First Order Linear Differential Equations

<u>Method</u>: Suppose an ODE of the form

$$y'(x) + p(x)\,y(x) = q(x)$$

The solution is

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)\,q(x)\,dx \quad , \quad \mu(x) = e^{\int p(x)\,dx}$$

where $\mu(x)$ is called the integrating factor. More precisely $\mu'(x) = p(x)\,\mu(x)$.

<u>Proof</u>: Multiply both sides by some carefully chosen $\mu(x)$

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)q(x)$$

<u>Note</u>: if $\mu'(x) = \mu(x)p(x)$ we have

$$\mu(x)y'(x) + \mu'(x)y(x) = [\mu(x)y(x)]' = \mu(x)q(x) \Rightarrow \mu(x)y(x) = \int \mu(x)q(x)\,dx$$

$\mu(x) = e^{P(x)}$ where $P'(x) = p(x)$ is what we need

<u>Ex</u>:

① $y' + 2y = e^x$

$$\mu = e^{\int 2\,dx} = e^{2x} \Rightarrow y(x) = \frac{1}{e^{2x}} \int e^{2x} e^x\,dx = e^{-2x}\left(\frac{e^{3x}}{3} + c\right) = \frac{1}{3}e^x + \frac{c}{e^{2x}}$$

② $(x+1)y' + (x+2)y = x \quad , \quad y(0) = a$

$$y' + \frac{x+2}{x+1}y = \frac{x}{x+1} \Rightarrow \mu(x) = e^{\int \frac{x+2}{x+1}dx} = e^{\int 1 + \frac{1}{x+1}dx} = e^{x + \ln|x+1|} = |x+1|e^x$$

→ what about the absolute value? We only need $\mu'(x) = p(x)\mu(x)$

$$\mu(x) = (x+1)e^x \Rightarrow \mu'(x) = e^x + (x+1)e^x = e^x(x+2) = e^x(x+1)\cdot\frac{x+2}{x+1} = \mu(x)p(x) \checkmark$$

$\Rightarrow \mu(x) = (x+1)e^x$ works

D		I
$+\ x$	→	e^x
$-\ 1$	→	e^x
$+\ 0$		e^x

$$\rightarrow y(x) = \frac{1}{(x+1)e^x}\int (x+1)e^x \frac{x}{x+1}\,dx = \frac{1}{(x+1)e^x}\int xe^x\,dx =$$

$$= \frac{1}{(x+1)e^x}\left(xe^x - e^x + c\right) = \frac{x-1}{x+1} + \frac{c}{(x+1)e^x}$$

<u>IVP</u>. $y(0) = a = \frac{-1}{1} + \frac{c}{1} \Rightarrow c = a + 1$

$$\Rightarrow y(x) = \frac{x-1}{x+1} + \frac{a+1}{(x+1)e^x}$$

- Homogeneous Differential Equations

Def: The function $f: \mathbb{R}^2 \to \mathbb{R}$ is called homogeneous of degree n $\equiv$

$$\forall \alpha \in \mathbb{R}: \quad f(\alpha x, \alpha y) = \alpha^n f(x,y)$$

All polynomials of the form $f(x,y) = \sum_{i=0}^{n} a_i x^{n-i} y^i$ are hom. of degree n.

Def: A first order ODE of the form

$$f(x,y)\, dy = g(x,y)\, dx$$

is homogeneous $\equiv$ f and g are both homogeneous functions of the same degree.

Method:

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} \implies \text{for any } \alpha: \quad \frac{g(\alpha x, \alpha y)}{f(\alpha x, \alpha y)} = \frac{g(x,y)}{f(x,y)} = \frac{dy}{dx}$$

→ let $\alpha = 1/x$ to get

$$\frac{dy}{dx} = \frac{g(1, y/x)}{f(1, y/x)} =: F(y/x)$$

→ substitute $u = y/x \implies y = xu \implies \frac{dy}{dx} = u + x\frac{du}{dx}$

$$u + x\frac{du}{dx} = F(u) \implies \frac{du}{dx} = \frac{F(u) - u}{x} \quad \Leftarrow \text{separable}$$

Ex:

① $\frac{dy}{dx} = \frac{y}{x} - e^{y/x}, \quad x > 1$

$u = y/x \implies y = xu \implies \frac{dy}{dx} = u + x\frac{du}{dx}$.

$\implies u + x\frac{du}{dx} = u - e^u \implies \frac{du}{dx} = -\frac{e^u}{x} \implies \int -e^{-u}\, du = \int \frac{dx}{x} \implies e^{-u} = \ln|x| + C$

→ since $x > 1$: $|x| = x$

$\implies -u = \ln(\ln(x) + C) \implies y = -x \ln(\ln(x) + C)$

② $x^2 dy = (y^2 + 3xy)\, dx, \quad x > 0$

x^2 and $y^2 + 3xy$ are both 2-homogeneous polynomials

$\frac{dy}{dx} = \frac{y^2 + 3xy}{x^2} = \frac{y^2}{x^2} + \frac{3y}{x} \implies u = \frac{y}{x} \quad \frac{dy}{dx} = u + x\frac{du}{dx}$

$u + x\frac{du}{dx} = u^2 + 3u \implies \frac{du}{dx} = \frac{u^2 + 2u}{x} \implies \int \frac{du}{u(u+2)} = \int \frac{dx}{x} = \ln|x| + C$

• Homogeneous Differential Equations

Def: The function $f: \mathbb{R}^2 \to \mathbb{R}$ is called homogeneous of degree n $\equiv$

$$\forall \alpha \in \mathbb{R}: \quad f(\alpha x, \alpha y) = \alpha^n f(x,y)$$

All polynomials of the form $f(x,y) = \sum_{i=0}^{n} a_i x^{n-i} y^i$ are hom. of degree n.

Def: A first order ODE of the form

$$f(x,y)\,dy = g(x,y)\,dx$$

is homogeneous $\equiv$ f and g are both homogeneous functions of the same degree

Method:

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} \implies \text{for any } \alpha: \frac{g(\alpha x, \alpha y)}{f(\alpha x, \alpha y)} = \frac{g(x,y)}{f(x,y)} = \frac{dy}{dx}$$

→ let $\alpha = 1/x$ to get

$$\frac{dy}{dx} = \frac{g(1, y/x)}{f(1, y/x)} =: F(y/x)$$

→ substitute $u = y/x \implies y = xu \implies \frac{dy}{dx} = u + x\frac{du}{dx}$

$$u + x\frac{du}{dx} = F(u) \implies \frac{du}{dx} = \frac{F(u) - u}{x} \quad \leftarrow \text{separable}$$

Ex:

① $\frac{dy}{dx} = \frac{y}{x} - e^{y/x}$, $x > 1$

$u = y/x \implies y = xu \implies \frac{dy}{dx} = u + x\frac{du}{dx}$.

$\implies u + x\frac{du}{dx} = u - e^u \implies \frac{du}{dx} = -\frac{e^u}{x} \implies \int -e^{-u}\,du = \int \frac{dx}{x} \implies e^{-u} = \ln|x| + C$

→ since $x > 1$: $|x| = x$

$\implies -u = \ln(\ln(x) + C) \implies \underline{\underline{y = -x\ln(\ln(x) + C)}}$

② $x^2 dy = (y^2 + 3xy)\,dx$, $x > 0$

x^2 and $y^2 + 3xy$ are both 2-homogeneous polynomials

$$\frac{dy}{dx} = \frac{y^2 + 3xy}{x^2} = \frac{y^2}{x^2} + \frac{3y}{x} \implies u = \frac{y}{x} \qquad \frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = u^2 + 3u \implies \frac{du}{dx} = \frac{u^2 + 2u}{x} \implies \int \frac{du}{u(u+2)} = \int \frac{dx}{x} = \ln|x| + C$$

The integral on the LHS can be solved as

$$\int \frac{du}{u(u+2)} = \int \frac{\frac{1}{2}}{u} - \frac{\frac{1}{2}}{u+2}\, du = \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2| = \frac{1}{2}\ln\left|\frac{u}{u+2}\right| = \ln|x| + C$$

$$\Rightarrow \ln\left|\frac{u}{u+2}\right| = 2\ln|x| + C \Rightarrow \left|\frac{u}{u+2}\right| = x^2 e^C = Ax^2, \quad A \in \mathbb{R}^+$$

Dealing with the absolute value:

$\rightarrow$ note the special cases $u=0$ and $u=-2$

the constant functions $u_0(x)=0$ and $u_1(x)=-2$ solve the ODE

E+U: $f(x,u) = \frac{u^2+2u}{x}$ & $\frac{\partial f}{\partial u} = \frac{2u+2}{x}$ are continuous $\forall x \neq 0$

$\Rightarrow$ all other solutions satisfy $u(x) \neq 0$ and $u(x) \neq -2$ for all x

$\Rightarrow$ 3 scenarios

$$\left.\begin{array}{l} 1)\ u>0 \Rightarrow u+2>0 \Rightarrow \left|\frac{u}{u+2}\right| = \frac{u}{u+2} \\ 2)\ u<-2 \Rightarrow u+2<0 \Rightarrow \left|\frac{u}{u+2}\right| = \frac{u}{u+2} \\ 3)\ -2<u<0 \Rightarrow u+2>0 \Rightarrow \left|\frac{u}{u+2}\right| = -\frac{u}{u+2} \end{array}\right\} \begin{array}{l} Ax^2 = \frac{u}{u+2} \text{ or } Ax^2 = -\frac{u}{u+2} \\ A \in \mathbb{R}^+ \end{array}$$

$\Rightarrow$ Hence $Kx^2 = \frac{u}{u+2}$ when $K \in \mathbb{R}\setminus\{0\}$ ... $K>0$ in 1 and 2 and $K<0$ in 3

$$uKx^2 + 2Kx^2 = u \Rightarrow u(Kx^2-1) = -2Kx^2$$

$$\Rightarrow u = -\frac{2Kx^2}{Kx^2-1} = \frac{2Kx^2}{1-Kx^2} = \underbrace{\frac{2x^2}{B-x^2}}_{B=\frac{1}{K}} \Rightarrow \underline{\underline{y = ux = \frac{2x^3}{B-x^2}, \quad B \in \mathbb{R}\setminus\{0\}}}$$

3) $\frac{dy}{dx} = \frac{y^2}{x^2-xy}$

$$\frac{dy}{dx} = \frac{y^2/x^2}{1-y/x} \Rightarrow u = y/x \Rightarrow \frac{du}{dx} = \frac{1}{x}\left(\frac{u^2}{1-u} - u\right) = \frac{1}{x}\left(\frac{2u^2-u}{1-u}\right)$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{1-u}{u(2u-1)}\, du \Rightarrow \ln|x| = \int \frac{-1}{u} + \frac{1}{2u-1}\, du = -\ln|u| + \frac{1}{2}\ln|2u-1| + C$$

$$\Rightarrow \ln|2u-1| - 2\ln|u| = 2\ln|x| + C$$

$$\Rightarrow \ln\left|\frac{2u-1}{u^2}\right| = \ln x^2 + C \Rightarrow \frac{|2u-1|}{u^2} = x^2 e^C = Ax^2, \quad A \in \mathbb{R}^+$$

E+U: $u=0$ and $u=\frac{1}{2}$ are solutions to $\frac{du}{dx} = \frac{2u^2-u}{x(1-u)} =: f(x,u)$

and $f(x,u)$ & $\frac{\partial f}{\partial u}$ are continuous when $x \neq 0$ and $u \neq 1$

$\Rightarrow$ 3 cases

$$\begin{array}{l} 1)\ u<0 \Rightarrow 2u-1<0 \\ 2)\ 0<u<1/2 \Rightarrow 2u-1<0 \end{array}\Big\} |2u-1| = -2u+1 \quad \Bigg\} \frac{2u-1}{u^2} = Kx^2, \quad K \in \mathbb{R}\setminus\{0\}$$

$$3)\ u>1/2 \Rightarrow 2u-1>0 \ \} \ |2u-1| = 2u-1$$

$\hookrightarrow$ isolate u ...

• Bernoulli Equations

Def: A Bernoulli equation is an ODE of the form

$$y'(x) + P(x)\,y(x) = Q(x)\,(y(x))^n \qquad n \in \mathbb{R}\setminus\{0,1\}$$

↳ otherwise it would be linear

Method: Substitute $w(x) = (y(x))^{1-n}$

$$\Rightarrow w'(x) = (1-n)(y(x))^{-n}\,y'(x)$$
$$= (1-n)(y(x))^{-n}\left(Q(x)(y(x))^n - P(x)y(x)\right)$$
$$= (1-n)\left(Q(x) - P(x)(y(x))^{1-n}\right)$$
$$= (1-n)\left(Q(x) - P(x)\,w(x)\right)$$

$$\Rightarrow w'(x) + (1-n)P(x)\,w(x) = (1-n)Q(x) \qquad \leftarrow \text{F.O. linear}$$

Ex:

① $y' - 2xy = x^3y^2$

$w = y^{1-2} = y^{-1} \quad \Rightarrow w' = -y^{-2}\cdot y'$

$\Rightarrow w' = -y^{-2}(x^3y^2 + 2xy) = -(x^3 + 2xy^{-1}) = -x^3 - 2xw$

$\Rightarrow w' + 2xw = -x^3$

↳ $\mu(x) = e^{\int 2x\,dx} = e^{x^2}$

$\Rightarrow w(x) = e^{-x^2}\int e^{x^2}(-x^3)\,dx = \left| \begin{matrix} u = x^2 \\ du = 2x\,dx \end{matrix} \right| = -\frac{1}{2}e^{-u}\int e^u\, u\, du$

	D	I
+	u	e^u
−	1	e^u
+	0	e^u

$= -\frac{1}{2}e^{-u}(ue^u - e^u + c)$

$= -\frac{1}{2}(u - 1 + ce^{-u}) = -\frac{x^2}{2} + \frac{1}{2} + \frac{c}{e^{x^2}}$

$$w = y^{-1} \Rightarrow y = w^{-1} = \frac{1}{\frac{1}{2} - \frac{x^2}{2} + ce^{-x^2}} = \underline{\underline{\frac{2}{1 - x^2 + c\cdot e^{-x^2}}}} \qquad c \in \mathbb{R}$$

Def: The interval of validity of an IVP with initial condition $y(t_0) = y_0$ is the largest possible interval containing t_0 on which a solution is defined.

EX:

(2) Solve the IVP $y' + \frac{4}{x}y = x^3y^2$, $y(2) = -1$ and find the IoV.

→ this is a Bernouli eq. Interval of Validity

$w = y^{1-2} = y^{-1}$

$\Rightarrow w' = -y^{-2}y' = -y^{-2}(x^3y^2 - \frac{4}{x}y) = -x^3 + \frac{4}{x}w$

$\Rightarrow w' - \frac{4}{x}w = -x^3$

$\hookrightarrow \mu(x) = e^{\int -\frac{4}{x}dx} = e^{-4\ln|x|} = e^{\ln|x|^{-4}} = |x|^{-4} = \frac{1}{x^4}$

$\Rightarrow w(x) = x^4 \int \frac{1}{x^4}(-x^3)\,dx = x^4 \int -\frac{1}{x}dx = -x^4(\ln|x| + c) = x^4(c - \ln|x|)$

$y(x) = \frac{1}{w(x)} = \underline{\underline{\frac{-1}{x^4(c-\ln|x|)}}}$, $c \in \mathbb{R}$

IVP: $y(2) = -1 = \frac{1}{16(c-\ln 2)} \Rightarrow -16 = \frac{1}{c-\ln 2} \Rightarrow 16\ln 2 - 16c = 1 \Rightarrow c = \ln 2 - \frac{1}{16}$

IoV: $x \neq 0 \Rightarrow$ either $x<0$ or $x>0$

→ because we have $y(2) = -1 \Rightarrow 2 \in IoV \Rightarrow x > 0 \Rightarrow \ln|x| = \ln(x)$

$\Rightarrow \underline{\underline{y(x) = \frac{1}{x^4(\ln 2 - 1/16 - \ln x)}}}$

IoV: We need $x^4(\ln 2 - 1/16 - \ln x) \neq 0$, keep in mind $x>0$

$\Rightarrow \ln x \neq \ln 2 - 1/16 \rightarrow x \neq 2e^{-1/16}$

$\Rightarrow \underline{\underline{IoV = (2e^{-1/16}, \infty)}}$

(3) Solve and find the IoV: $y' + \frac{y}{x} = \sqrt{y}$, $y(1) = 0$

$w = y^{1-\frac{1}{2}} = \sqrt{y} \Rightarrow w' = \frac{1}{2\sqrt{y}}y' = \frac{1}{2\sqrt{y}}(\sqrt{y} - \frac{y}{x}) = \frac{1}{2}(1 - \frac{\sqrt{y}}{x}) = \frac{1}{2} - \frac{w}{2x}$

$\Rightarrow w' + \frac{w}{2x} = \frac{1}{2} \rightarrow \mu = \exp(\int \frac{1}{2x}dx) = \exp(\frac{1}{2}\ln|x|) = \sqrt{|x|}$

$\Rightarrow \mu(x) = \sqrt{|x|}$, but IVP: $y(1) = 0$ and $x \neq 0 \Rightarrow IoV \subseteq (0,\infty)$

$\hookrightarrow 1 \in IoV$

$\hookrightarrow$ in the IoV we have $\mu(x) = \sqrt{x}$

$\Rightarrow w(x) = \frac{1}{\sqrt{x}}\int \sqrt{x}\,\frac{1}{2}\,dx = \frac{1}{2\sqrt{x}}\cdot\left(\frac{x^{3/2}}{3/2} + c\right) = \frac{1}{3}x + \frac{c}{\sqrt{x}}$

IVP: $\sqrt{y} = w \Rightarrow \sqrt{0} = \frac{1}{3} + \frac{c}{1} \Rightarrow c = -\frac{1}{3}$

$\Big\}\ w(x) = \frac{1}{3}(x - \frac{1}{\sqrt{x}})$

$\rightarrow \underline{\underline{y(x) = w(x)^2 = \frac{1}{9}(x - \frac{1}{\sqrt{x}})^2}}$

IoV: $x \neq 0$ & $1 \in IoV \Rightarrow \underline{\underline{IoV = (0,\infty)}}$

- Other substitutions
- Linear substitution

$$\frac{dy}{dx} = f(Ax+By+C) \rightsquigarrow u = Ax+By+C \Rightarrow \frac{du}{dx} = A + B\frac{dy}{dx}$$

Separable

Ex:

① $\frac{dy}{dx} = (x+y)^2 \rightarrow u = x+y \Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx}$

$\frac{du}{dx} - 1 = u^2 \Rightarrow \int \frac{du}{u^2+1} = \int dx \Rightarrow \operatorname{arctg}(u) = x + C$

$\Rightarrow x+y = \operatorname{tg}(x+C) \Rightarrow \underline{y = \operatorname{tg}(x+C) - x}, \quad C \in \mathbb{R}$

② $\frac{dy}{dx} = \frac{1}{2x-4y+7} \rightarrow u = 2x-4y+7 \Rightarrow \frac{du}{dx} = 2 - 4\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{4}\left(2 - \frac{du}{dx}\right)$

$\Rightarrow \frac{1}{4}\left(2 - \frac{du}{dx}\right) = \frac{1}{u} \Rightarrow \frac{du}{dx} = 2 - \frac{4}{u} = \frac{2(u-2)}{u}$

$\Rightarrow \int \frac{u}{u-2} du = 2\int dx \rightarrow$ Note: $u=2$ is a solution

$\rightarrow 2x = \int \frac{u-2+2}{u-2} du = \int 1 + \frac{2}{u-2} du = u + 2\ln|u-2| + C$

$\rightarrow 2x = 2x - 4y + 7 + 2\ln|2x-4y+5| + C$

$\Rightarrow 2\ln|2x-4y+5| = 4y - 7 + C = 4y + C \Rightarrow \ln|\cdots| = 2y + C$

$\Rightarrow 2x - 4y + 5 = Ae^{2y}, \quad A \in \mathbb{R}^+$

$\Rightarrow$ Solutions are

1) $u = 2 \Rightarrow 2x - 4y + 7 = 2 \Rightarrow \underline{y = \frac{1}{4}(2x+5)}$

2) $y(x)$ which satisfy: $\underline{2x - 4y + 5 = Ae^{2y}}, \quad A \in \mathbb{R}^+$

③ $y' + 3y\operatorname{cotg}(x) = 6\cos(x)\cdot y^{2/3}, \quad \sin(x) \neq 0 \rightarrow$ else undefined

$\rightarrow$ Bernoulli eq. $\Rightarrow w = y^{1-2/3} = y^{1/3}$

$w' = \frac{1}{3}y^{-2/3} \circledast y' = \frac{y^{-2/3}}{3}\left(6\cos(x)y^{2/3} - 3y\operatorname{cotg}(x)\right) = 2\cos(x) - w\operatorname{cotg}(x)$

$\Rightarrow w' + w\operatorname{cotg}(x) = 2\cos(x)$

$\hookrightarrow \mu = e^{\int \operatorname{cotg}(x)dx} = e^{\int \frac{\cos x}{\sin x}dx} = e^{\ln|\sin x|} = |\sin x|$

$\circledast$ Want $y(x) \neq 0 \ \forall x$
$\circledast$ $y_0(x) = 0$ is a solution
E+U: $y(x) \neq 0 \ \forall x$

$\Rightarrow w(x) = \frac{1}{|\sin x|}\int 2\cos(x)|\sin x| dx = \left| \begin{matrix} u = \sin x \\ du = \cos x dx \end{matrix} \right| = \frac{2}{|u|}\int |u| du$

$= \frac{2}{|u|}\left(A\frac{u^2}{2} + C\right) \rightarrow A = \begin{cases} 1, & u > 0 \\ -1, & u < 0 \end{cases} \Rightarrow A = \operatorname{sgn}(u).$

$= \frac{1}{|u|}\left(u^2\operatorname{sgn}(u) + C\right) = u + \frac{C}{|u|} \quad ; \quad y = w^3 \Rightarrow \underline{y = \left(\sin x + \frac{C}{|\sin x|}\right)^3}$ or $\underline{y=0}$

④ $xy' - y = \sqrt{xy + x^2}$

$\hookrightarrow$ divide by x : $y' - \frac{y}{x} = \sqrt{\frac{y}{x} + 1}$ $\longrightarrow$ homogeneous

$u = \frac{y}{x} \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

$\Rightarrow u + x\frac{du}{dx} - u = \sqrt{u+1} \Rightarrow x\frac{du}{dx} = \sqrt{u+1} \Rightarrow \int \frac{du}{\sqrt{u+1}} = \int \frac{dx}{x}$

$\ln|x| + C = \int (u+1)^{-1/2} du = \frac{(u+1)^{1/2}}{1/2} = 2\sqrt{u+1}$

$u \neq -1$

$\Rightarrow \sqrt{u+1} = \frac{1}{2}\ln|x| + C = \ln\sqrt{|x|} + C$

$u = -1$ is a solution

$\Rightarrow u = (\ln\sqrt{|x|} + C)^2 - 1$

$\Rightarrow \underline{y = xu = x(\ln\sqrt{|x|} + C)^2 - x}$ and $\underline{y = -x}$

⑤ $\cos(4y - 8x + 3)\, y' = 2 + 2\cos(4y - 8x + 3)$

$u = 4y - 8x + 3 \Rightarrow \frac{du}{dx} = -8 + 4\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{4}\left(\frac{du}{dx} + 8\right) = \frac{1}{4}\frac{du}{dx} + 2$

$\Rightarrow \cos(u)\left(\frac{1}{4}\frac{du}{dx} + 2\right) = 2 + 2\cos(u)$

$\frac{1}{4}\frac{du}{dx} + 2 = \frac{2(1 + \cos(u))}{\cos(u)} = \frac{2}{\cos(u)} + 2 \Rightarrow \frac{1}{4}\frac{du}{dx} = \frac{2}{\cos u}$

$u \neq \frac{\pi}{2} + k\pi$

$\hookrightarrow$ not a solution though.

$\Rightarrow \int \cos u \, du = \int 8\, dx \Rightarrow \sin(u) = 8x + C$

$\Rightarrow u = \arcsin(8x + C) = 4y - 8x + 3$

$\Rightarrow \underline{y = \frac{1}{4}(8x - 3 + \arcsin(8x + C))}$

⑥ $\cos(y)\, y' = e^{-x} - \sin(y)$

$\hookrightarrow u = \sin(y) \Rightarrow u' = \cos(y)\, y'$

$\Rightarrow u' = e^{-x} - u \Rightarrow u' + u = e^{-x}$ $\rightarrow \mu = e^{\int 1 dx} = e^{x}$

$\Rightarrow u = e^{-x}\int e^{x} e^{-x} dx = e^{-x}(x + C)$

$\Rightarrow \underline{y = \arcsin(u) = \arcsin\left(\frac{x + C}{e^{x}}\right)}$, $C \in \mathbb{R}$

⑦ $y' + 2x = 2\sqrt{y + x^2}$

$\hookrightarrow u = y + x^2 \Rightarrow u' = y' + 2x \rightsquigarrow u' = 2\sqrt{u}$ $\rightarrow$ separable

⑧ $y' = x\left(1 + \frac{2y}{x^2} + \frac{y^2}{x^4}\right)$

$\rightarrow u = \frac{y}{x^2} \Rightarrow y = ux^2 \Rightarrow y' = 2xu + x^2 u'$

$\Rightarrow 2xu + x^2 u' = x(1 + 2u + u^2)$

$x^2\frac{du}{dx} = x(1 + u^2) \Rightarrow x\frac{du}{dx} = 1 + u^2$ $\rightarrow$ separable

⑨ Solve the IVP and determine the IoV.

a) $xy' + (x+1)y = x \quad , \quad y(\ln 2) = 1 \quad , \quad x > 0$

$$y' + \frac{x+1}{x} y = 1 \quad \rightarrow \quad \mu = \exp\left(\int \frac{x+1}{x} dx\right) = \exp\left(\int 1 + \frac{1}{x} dx\right) = e^{x + \ln|x|} = |x| e^x$$

$|x| = x \ldots x > 0$

$$\Rightarrow y(x) = \frac{1}{xe^x} \int x e^x x \, dx = \frac{1}{xe^x}\left(x^2 e^x - 2xe^x + 2e^x + C\right)$$

$$= x - 2 + \frac{2}{x} + \frac{C}{xe^x}$$

D	I
$+\ x^2$	e^x
$-\ 2x$	e^x
$+\ 2$	e^x
$-\ 0$	e^x

IVP: $y(\ln 2) = 1 = \ln 2 - 2 + \frac{2}{\ln 2} + \frac{C}{2\ln 2}$

$$\Rightarrow C = 2\ln 2\left(3 - \ln 2 - \frac{2}{\ln 2}\right) = 6\ln 2 - 2\ln^2(2) - 4$$

IoV: $x > 0$, $x \neq 0 \Rightarrow \text{IoV} = (0, \infty)$

b) $y' = \frac{xy^3}{\sqrt{1+x^2}} \quad , \quad y(0) = 1$

$$\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}} \Rightarrow \int \frac{dy}{y^3} = \int \frac{x}{\sqrt{1+x^2}} dx \qquad \begin{aligned} u &= 1+x^2 \\ du &= 2x\,dx \end{aligned}$$

$$\Rightarrow \frac{y^{-2}}{-2} + C = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2} du = \frac{1}{2} \cdot \frac{u^{1/2}}{1/2} = \sqrt{u} = \sqrt{1+x^2}$$

$$\Rightarrow \frac{-1}{2y^2} = \sqrt{1+x^2} + C \Rightarrow \frac{1}{y^2} = -2\sqrt{1+x^2} + C$$

$$\Rightarrow y^2 = \frac{1}{C - 2\sqrt{1+x^2}} \Rightarrow y = \pm\left(C - 2\sqrt{1+x^2}\right)^{-1/2}, \quad y(0) = 1 \rightarrow +$$

IVP: $y(0) = 1 = (C-2)^{-\frac{1}{2}} \Rightarrow 1^{-2} = 1 = C - 2 \Rightarrow C = 3$

$$\Rightarrow y(x) = \left(3 - 2\sqrt{1+x^2}\right)^{-1/2} = \frac{1}{\sqrt{\ldots}}$$

IoV: $3 - 2\sqrt{1+x^2} > 0 \Rightarrow \sqrt{1+x^2} < \frac{3}{2} \Rightarrow 1 + x^2 < \frac{9}{4} \Rightarrow x^2 < \frac{5}{4} \Rightarrow |x| < \frac{\sqrt{5}}{2}$

$$\text{IoV} = \left(-\tfrac{1}{2}\sqrt{5}, \tfrac{1}{2}\sqrt{5}\right)$$

⑩ $(x+y)y' = y$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x+y} = \frac{y/x}{1+y/x} \quad \Rightarrow \begin{aligned} u &= y/x \\ y &= xu \end{aligned} \quad \rightarrow \frac{dy}{dx} \quad u + x\frac{du}{dx}$$

$$\Rightarrow u + x\frac{du}{dx} = \frac{u}{1+u} \Rightarrow \frac{du}{dx} = \frac{1}{x}\left(\frac{u - u(1+u)}{1+u}\right) = \frac{1}{x}\left(\frac{-u^2}{1+u}\right)$$

$$\Rightarrow \int \frac{1+u}{u^2} du = -\int \frac{dx}{x} \qquad \rightarrow u \neq 0 \quad \rightsquigarrow \text{Note: } u = 0 \text{ is a solution}$$

$$\Rightarrow -\ln|x| + C = \int u^{-2} + \frac{1}{u} du = \frac{u^{-1}}{-1} + \ln|u| = -\frac{1}{u} + \ln|u|$$

$$\Rightarrow e^{-\ln|x| + C} = e^{-\frac{1}{u} + \ln|u|} \Rightarrow \frac{A}{|x|} = |u| e^{-\frac{1}{u}}, \quad A \in \mathbb{R}^+$$

Absolute values:

E+U: $f(x,u) = -\frac{u^2}{x(1+u)}$ and $\frac{\partial f}{\partial u}$ are continuous when $u \neq 1$, $x \neq 0$

$\Rightarrow$ since $u(x) = 0$ is a solution, all other solutions satisfy $u(x) \neq 0 \ \forall x$

→ we either have

- $u > 0 \Rightarrow |u| = u$
- $u < 0 \Rightarrow |u| = -u$

$\Big\}\ \dfrac{A}{|x|} = k \cdot u\, e^{-\frac{1}{u}}$, $A \in \mathbb{R}^+$

$\hookrightarrow B/x$, $B \in \mathbb{R} \setminus \{0\}$

$$\Rightarrow \frac{B}{x} = u e^{-\frac{1}{u}}$$

$$\Rightarrow \frac{1}{u} e^{\frac{1}{u}} = Mx \quad , \quad M \in \mathbb{R} \setminus \{0\}$$

→ Lambert W function

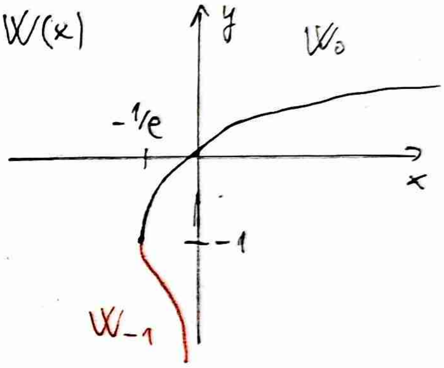
$$\Rightarrow W\left(\frac{1}{u} e^{\frac{1}{u}}\right) = W(Mx) \Rightarrow \frac{1}{u} = W(Mx)$$

$$y = ux \rightarrow \underline{\underline{y = \frac{x}{W(Mx)}}} \quad M \in \mathbb{R} \setminus \{0\} \quad \text{and} \quad \underline{\underline{y(x) = 0}} \leftarrow u = 0$$

<u>Def</u>: <u>Lambert W function</u> is a family of functions which satisfy

$$W(x) e^{W(x)} = x \Rightarrow y e^y = x \Leftrightarrow y = W(x)$$

<u>Note</u>: There are 2 branches if $x \in \mathbb{R}$ ⟶



(11) $\underline{\dfrac{dy}{dx} - \dfrac{1}{x} y = \dfrac{1}{y} \quad , \quad y(1) = 3}$

$y' - \frac{1}{x} y = y^{-1}$ ... Bernoulli eq.

$$\rightarrow w = y^{1-(-1)} = y^2$$

$$w' = 2yy' = 2y\left(\frac{y}{x} + y^{-1}\right) = 2\left(\frac{y^2}{x} + 1\right) = 2 + \frac{w}{x}$$

$$\Rightarrow w' - \frac{1}{x} w = 2 \rightarrow \mu(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln|x|} = \frac{1}{|x|}$$

<u>IVP</u>: $y(1) = 3$ & $x \neq 0 \Rightarrow x > 0 \Rightarrow |x| = x$

$$\rightarrow w(x) = x \int \frac{1}{x} 2\, dx = x(2\ln|x| + C) = x(2\ln x + C)$$

$$\rightarrow y = \pm\sqrt{w} = \pm\sqrt{x(2\ln x + C)} \quad \dots \ y(1) = 3 \Rightarrow +$$

<u>IVP</u>: $y(1) = 3 = \sqrt{1(0 + C)} \Rightarrow \sqrt{C} = 3 \Rightarrow C = 9$

$$\Rightarrow \underline{\underline{y(x) = \sqrt{x(2\ln x + 9)}}}$$

<u>IoV</u>: $x \neq 0$ & $1 \in \text{IoV} \Rightarrow \text{IoV} \subseteq (0, \infty)$

$$x(2\ln(x) + 9) > 0 \Rightarrow 2\ln x + 9 > 0 \Rightarrow \ln x > -\frac{9}{2} \Rightarrow x > e^{-\frac{9}{2}}$$

$$\Rightarrow \underline{\underline{\text{IoV} = (e^{-9/2}, \infty)}}$$

• Finding Bounds of Solutions

Theorem: Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is continuous. Then

$y(x)$ solves $\quad y'(x) = f(x, y(x)) \quad \& \quad y(x_0) = y_0$

$$\Updownarrow$$

$y(x)$ solves $\quad y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$

Proof: $\Uparrow$: $y'(x) = f(x, y(x))$ by the fundamental theorem of calculus

$y(x_0) = y_0 + 0$

$\Downarrow$: $\int y'(x)\,dx = \int f(x, y(x))\,dx \Rightarrow y(x) = F(x, y(x)) + C$

Note: $y_0 = y(x_0) = F(x_0, y_0) + C \Rightarrow C = y_0 - F(x_0, y_0)$

$\Rightarrow$ any antiderivative F will do, lets choose

$F(x, y(x)) = \int_{x_0}^{x} f(t, y(t))\,dt$ ... an antiderivative given by

$\Rightarrow F(x_0, y_0) = \int_{x_0}^{x_0} \ldots = 0 \quad \Rightarrow y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$ $\blacksquare$

Note: This formula is useful for proving that a solution exists and finding bounds of the solution. It's not so useful in finding the solution explicitly since the integral may not be evaluable.

Ex: Consider the IVP $y' = \sin(y^2) + 2x, \quad y(0) = y_0 \in \mathbb{R}$.
Show that the solution is unique + find bounds on it

- $f(x,y) := \sin(y^2) + 2x$ is continuous

  $\Rightarrow y(x) = y_0 + \int_0^x \sin(y(t)^2) + 2t\,dt \quad \Rightarrow$ a solution exists

- $\frac{\partial f}{\partial y} = \cos(y^2)2y$ is continuous $\Rightarrow$ the solution is unique

$\rightarrow$ so what is y? ... $\int \sin(y(t)^2)\,dt$ is non-elementary

$\rightarrow$ we can not evaluate it but can estimate it

$$|y(x)| \leq |y_0| + \left|\int_0^x \sin(y(t)^2) + 2t\,dt\right| \leq |y_0| + \int_0^x |\sin(y(t)^2)| + 2|t|\,dt$$

$$\leq |y_0| + \int_0^x 1 + 2|t|\,dt$$

→ 2 options

$\underline{x \geq 0}$: $|y(x)| \leq |y_0| + \int_0^x (1+2t)\,dt = |y_0| + x + x^2$

$\underline{x \leq 0}$: $|y(x)| \leq |y_0| + \left| -\int_x^0 \sin(y(t)^2) + 2t\,dt \right| \leq |y_0| + \int_x^0 (1+2|t|)\,dt$

$= |y_0| + \int_x^0 (1-2t)\,dt = |y_0| - x + x^2$

$\Rightarrow \forall x \in \mathbb{R}$: $\underline{\underline{|y(x)| \leq |y_0| + |x| + x^2}}$

Theorem (Gronwall Inequality): Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and $g \geq 0$.

$$\forall x \geq x_0 : f(x) \leq C + \int_{x_0}^x f(t) g(t)\,dt \implies \forall x \geq x_0 : f(x) \leq C \cdot e^{\int_x^{x_0} g(t)\,dt} \quad , C \in \mathbb{R}$$

Proof:

Let $H(x) := C + \int_{x_0}^x f(t) g(t)\,dt \Rightarrow f(x) \leq H(x)$

Note: $H'(x) = f(x) g(x) \leq H(x) g(x)$

Consider the ODE $H'(x) = H(x) g(x)$ — some antiderivative of $-g(x)$

$H'(x) - H(x) g(x) = 0 \rightsquigarrow \mu(x) = \exp\left(\int -g(x)\,dx\right) = \exp\left(-\int_{x_0}^x g(t)\,dt\right)$

$\Rightarrow [\mu(x) H(x)]' = \mu'(x) H(x) + \mu(x) H'(x) =$

$= -g(x) \mu(x) H(x) + \mu(x) H'(x) = \mu(x) [H'(x) - g(x) H(x)] < 0$

$\Rightarrow \mu(x) H(x)$ is decreasing

Note: $\mu(x_0) H(x_0) = e^0 (C + 0) = C$

$\Big\}$ $\forall x \geq x_0 : \mu(x) H(x) \leq C \Rightarrow H(x) \leq C \mu^{-1}$

→ we have $H(x) \leq C \mu^{-1} \Rightarrow f(x) \leq H(x) \leq C \cdot \exp\left(\int_{x_0}^x g(t)\,dt\right)$ ■

How to use this?

Consider an IVP $y' = f(x,y)$ and rewrite it as an integral

$$y(x) = y_0 + \int_{x_0}^x f(t, y(t))\,dt$$

Now examine the absolute value and try to simplify to get

$\forall x \geq x_0$: $|y(x)| \leq |y_0| + \int_{x_0}^x |f(t, y(t))|\,dt \leq \dots \leq |y_0| + \int_{x_0}^x g(t) |y(t)|\,dt$

Finally use the Gronwall inequality

$\forall x \geq x_0$: $|y(x)| \leq |y_0| \cdot \exp\left(\int_{x_0}^x g(t)\,dt\right)$

Ex: Let $y'(x) = \frac{x^2+3}{x^3+x}\, y(x) \sin(y(x))$, $y(1) = a \in \mathbb{R}^+$.

<u>Show that $\forall x \geq 1$: $|y(x)| \leq \frac{2ax^3}{x^2+1}$</u>

→ since $f(x,y) = \frac{x^2+3}{x^3+x}\, y \cdot \sin(y)$ is continuous when $x \neq 0$, $y(x)$ satisfies

$$y(x) = a + \int_1^x f(t, y(t))\, dt$$

→ consider only $x \geq 1$ and examine the absolute value

$$|y(x)| \leq |a| + \int_1^x |f(t,y(t))|\, dt = \underset{a \in \mathbb{R}^+}{a} + \int_1^x \left|\frac{t^2+3}{t^3+t}\right| \cdot |y(t)| \cdot \overbrace{|\sin(y(t))|}^{\leq 1}\, dt$$

$$\leq a + \int_1^x \frac{t^2+3}{t^3+t} \cdot |y(t)|\, dt \quad \text{... } t \in (0,x) \ \& \ x \geq 1$$

→ by the Gronwall inequality $\forall x \geq 1$:

$$|y(x)| \leq a \cdot \exp\left(\int_1^x \frac{t^2+3}{t^3+t}\, dt\right)$$

$$\Rightarrow \int_1^x \frac{t^2+3}{t(t^2+1)}\, dt = \int_1^x \frac{1}{t} + \frac{2}{t(t^2+1)}\, dt = \int_1^x \frac{1}{t} + \frac{2}{t} + \frac{\overset{-2}{A}t + \overset{0}{B}}{t^2+1}\, dt = \int_1^x \frac{3}{t} - \frac{2t}{t^2+1}\, dt$$

$$= \Big[3\ln|t| - \ln|t^2+1|\Big]_1^x = 3\ln x - \ln(x^2+1) + \ln 2 = \ln \frac{2x^3}{x^2+1}$$

$$\Rightarrow |y(x)| \leq a \cdot \exp\left(\ln \frac{2x^3}{x^2+1}\right) = \underline{\underline{a \cdot \frac{2x^3}{x^2+1}}}$$

- <u>Closer look at the E+U theorem</u>

<u>Theorem (E+U)</u>: Let $(t_0, y_0) \in \mathbb{R}^2$, suppose $f: \mathbb{R}^2 \to \mathbb{R}$ and $\frac{\partial f}{\partial y}$ are both continuous in the rectangle $R = \{(t,y) \in \mathbb{R}^2 \mid |t - t_0| < a,\ |y - y_0| < b\}$, $a, b \in \mathbb{R}^+$

Then there exists some $\varepsilon \in \mathbb{R}$, $0 < \varepsilon \leq a$ s.t. the IVP

$$y'(t) = f(t, y(t)) \quad , \quad y(t_0) = y_0$$

Has a unique solution defined in $(t_0 - \varepsilon, t_0 + \varepsilon)$.

<u>Proof</u>:

Without loss of generality let $(t_0, y_0) = (0,0)$

$\Rightarrow R = \{(t,y) \in \mathbb{R}^2 \mid |t| \leq a,\ |y| < b\}$ ... solution defined in $(-\varepsilon, \varepsilon)$

<u>Ex: Transform the IVP $y' = t^2 + y^2$, $y(1) = 2$ to an equivalent problem with the initial point at the origin</u>

→ want $u(0) = 0$ ⇒ define $u(t) := y(t+1) - 2$

$\Rightarrow u'(t) = y'(t+1) = (t+1)^2 + (y(t+1))^2$ ↳ $y(t+1) = u(t) + 2$

$$\Rightarrow \underline{\underline{u'(t) = (t+1)^2 + (u(t)+2)^2 \quad , \quad u(0) = 0 \quad , \quad y(t) = u(t-1) + 2}}$$

→ in general it might be impossible to find an explicit formula for $y(t)$

Idea: Construct a sequence of functions, all of them satisfying the initial condition, and show that the sequence converges to a limit function which satisfies the differential equation. Meaning

a) the sequence converges

b) the limit function satisfies the IVP

c) the limit function is a unique solution in some region of R

→ this is very difficult in general, we will give an idea with an example

f is continuous ⇒ can express y as an integral

$y(t) = 0 + \int_0^t f(s, y(s))\,ds$ → the solution satisfies $y(t) = \int_0^t f(s, y(s))\,ds$

- <u>Method of Successive Approximations</u> – Picard's iterative method

Want to solve $\underline{y'(t) = f(t, y(t)),\ y(0) = 0}$, f is continuous around $(0,0)$

→ find an initial guess for y, easiest is $y_0(t) = 0$

→ then use the integral formula to get better approximations of y

$y_0(t) = 0 \qquad y_1(t) = \int_0^t f(s, y_0(s))\,ds \qquad \underline{y_n(t) = \int_0^t f(s, y_{n-1}(s))\,ds}$

→ we have a sequence of functions $y_0, y_1, \dots$ satisfying $y_n(0) = 0 \ \forall n$

- if for some k we have $y_{k+1} = y_k$, then its a solution to the IVP

$y_k = y_{k+1} = \int_0^t f(s, y_k(s))\,ds \xrightarrow{\frac{d}{dt}} y_k'(t) = f(t, y_k(t))$ ... solution ✓

- in general this does not happen and we need to ask

1) do all members of the sequence exist?

2) does the sequence converge?

3) does the limit function satisfy the differential equation?

4) is this the only solution?

→ the general proof shows that if $\underline{f \text{ and } \frac{\partial f}{\partial y} \text{ are continuous}}$ on some rectangle, then this sequence does indeed converge for some sufficiently small inputs and that its indeed the only solution.

→ we can also get some bounds on how big ε is

→ lets do a concrete example and prove the E+U theorem in a special case

↳ Consider $y' = 2t(1+y)$, $y(0) = 0$

$f(t,y) := 2t(1+y)$ is continuous on $\mathbb{R}^2$

$\Rightarrow$ consider the corresponding integral equation

$y(t) = \int_0^t 2s(1+y(s))\,ds$ → and try to approximate y

→ let $y_0(t) = 0$

$y_1(t) = \int_0^t 2s(1+y_0(s))\,ds = \int_0^t 2s\,ds = t^2$

$y_2(t) = \int_0^t 2s(1+s^2)\,ds = \int_0^t 2s + 2s^3\,ds = t^2 + \frac{t^4}{2}$

$y_3(t) = \int_0^t 2s(1+s^2+\frac{s^4}{2})\,ds = \int_0^t 2s + 2s^3 + s^5\,ds = t^2 + \underbrace{\frac{t^4}{2}}_{2!} + \underbrace{\frac{t^6}{6}}_{3!}$

→ let $\frac{t^{2k}}{m}$ be the last term in y_k

↳ $y_{k+1} = y_k + \int_0^t 2s \frac{s^{2k}}{m}\,ds = y_k + \frac{2}{m} \cdot \frac{t^{2k+2}}{2k+2} = y_k + \frac{t^{2(k+1)}}{m(k+1)}$

→ coefficients:

$$\left.\begin{aligned} y_1 &: t^2/1 \Rightarrow 1 \\ y_2 &: t^4/2 \Rightarrow 2\cdot 1 = 2 \\ y_3 &: t^6/6 \Rightarrow 3\cdot 2 = 6 \end{aligned}\right\} \quad \begin{aligned} &y_1 : 1 \\ &y_{k+1} : (k+1)\cdot[\text{coeff } y_k] = (k+1)! \end{aligned}$$

$\Rightarrow y_m(t) = t^2 + \frac{t^2}{2!} + \frac{t^6}{3!} + \dots + \frac{t^{2m}}{m!} = \sum_{k=1}^{m} \frac{t^{2k}}{k!}$

Note: ① all y_m exist

→ does $y(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$ converge?

→ ratio test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{t^{2k+2}}{(k+1)!} \cdot \frac{k!}{t^{2k}}\right| = \lim_{k\to\infty} \left|\frac{t^2}{k+1}\right| = 0 < 1$

$\Rightarrow \sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$ converges for all $t \in \mathbb{R}$

Note: ② $\{y_m(t)\}_{m=1}^{\infty}$ converges

→ is there a closed formula? ... use $e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$

$\Rightarrow$ $y(t) = e^{t^2} - 1$ → does it satisfy the IVP?

$y'(t) = e^{t^2} \cdot 2t = (e^{t^2} - 1 + 1)2t = (y+1)\cdot 2t$ ✓

Note: ③ the limit satisfies the IVP

④ is the solution unique?

Suppose the IVP has two solutions $y(t)$ and $z(t)$

$$\left.\begin{array}{l} y(t) = \int_0^t 2s(1+y(s))\,ds \\ z(t) = \int_0^t 2s(1+z(s))\,ds \end{array}\right\} |y(t)-z(t)| = \left|\int_0^t 2s(1+y(s)) - 2s(1+z(s))\,ds\right|$$

$$\Rightarrow |y(t)-z(t)| \leq 2\int_0^t |s|\cdot|y(s)-z(s)|\,ds$$

- Suppose $t \geq 0$:

$$\Rightarrow |y(t)-z(t)| \leq 2\int_0^t s\,|y(s)-z(s)|\,ds \leq 2t\int_0^t |y(s)-z(s)|\,ds \quad \circledast$$

$\Rightarrow$ define $u(t) := \int_0^t |y(s)-z(s)|\,ds$ $\rightarrow$ want to show $u(t)=0 \Rightarrow y(t)=z(t)$

$u(0)=0$, $u(t)\geq 0$.

$u'(t) = |y(t)-z(t)| \Rightarrow \circledast : u'(t) \leq 2t\,u(t)$

$\Rightarrow u'(t) - 2t\,u(t) \leq 0 \quad \rightsquigarrow \mu(t) = e^{\int -2t\,dt} = e^{-t^2}$

$$e^{-t^2}u'(t) - 2te^{-t^2}\cdot u(t) = \left[e^{-t^2}\cdot u(t)\right]' \leq 0$$

Note: e^{-t^2} is continuous and $u(t)$ is differentiable $\Rightarrow$ continuous

$\Longrightarrow e^{-t^2}\cdot u(t)$ is decreasing

$u(0)=0 \Rightarrow e^{-t^2}u(t) = 0$ when $t=0$

$\Rightarrow \forall t>0 : \underbrace{e^{-t^2}}_{+}\cdot u(t) \leq 0 \Rightarrow u(t) \leq 0$

$\rightarrow$ but above we have $u(t)\geq 0 \Rightarrow \forall t>0 : u(t)=0 \Rightarrow \underline{\underline{y(t)=z(t)}}$

- Similarly, if $t \leq 0$ we also get $y(t) = z(t)$.

Note: (4) the solution $y(t) = e^{t^2}-1$ is unique.

$\rightarrow$ the general approch follows the same line of thought.

E+U: Domain of the solution

$f:\mathbb{R}^2 \to \mathbb{R}$ is continuous on $R = \{(t,y) \mid |t| \leq a, |y| \leq b\}$, $a, b \in \mathbb{R}^+$

→ recall $y_{n+1}(t) = \int_0^t f(s, y_n(s))\, ds$

→ we need $(s, y_n(s)) \in R$, otherwise f might not be continuous or even defined

⇒ the ε we chose will have to satisfy

$\forall t \in (-\varepsilon, \varepsilon)$: $y_n(t)$ is defined and $\forall s \in (0,t)$: $(s, y_n(s)) \in R$

→ in other words: $0 < \varepsilon \leq a$ and $\forall t \in (-\varepsilon, \varepsilon)$ $\forall n$: $|y_n(t)| \leq b$

→ to find ε, use the fact that continuous functions on closed intervals are bounded

$\Rightarrow \exists M \geq 0$ s.t. $\forall (t,y) \in R: |f(t,y)| \leq M$

→ assume ε is chosen in a way s.t. $\forall t \in (-\varepsilon, \varepsilon)$: $|y_n(t)| \leq b$

→ examine $y_{n+1}(t)$, $t \in (-\varepsilon, \varepsilon)$:

$y'_{n+1}(t) = f(t, y_n(t)) \Rightarrow |y'_{n+1}(t)| = |f(t, y_n(t))| \leq M$

⇒ the maximum slope of the graph of $y_{n+1}(t)$ is M

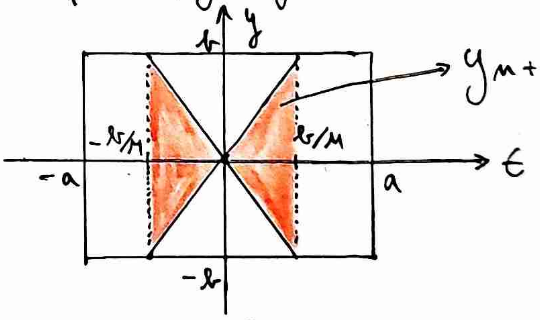
⇒ $y_{n+1}(t)$ must lie in the following region:

- **suppose $b/M < a$**

→ need to limit t s.t.

$tM \leq b \Rightarrow t \leq b/M$

& $-t \geq -b/M$

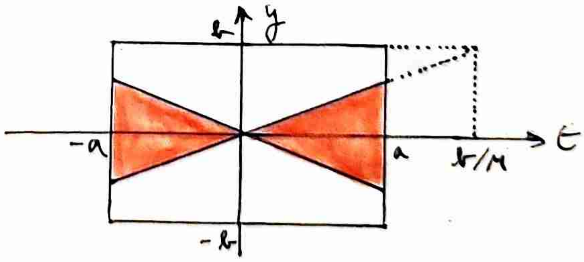


- **suppose $b/M > a$**

tM reaches b when

$t = b/M > a$

⇒ no need to limit t



⇒ $(t, y_n(t))$ stays within R as long as $|t| \leq \varepsilon$ where $\varepsilon = \min\{a, b/M\}$

→ if we set $y_0(t) = 0$: $f(t, y_n(t))$ is continuous on $[-\varepsilon, \varepsilon] \times [b, b]$ for $\forall n \in \mathbb{N}$

⇒ <u>conclusion</u>: $\varepsilon = \min\{a, \frac{b}{M}\}$ where M bounds $|f(t,y)|$ from above on R

$\Rightarrow \varepsilon \to 0$ as $M \to \infty$

Ex:

① Solve $y' = y + 1 - t$, $y(0) = 0$

a) using Picard's iterative method

$f(t,y) := y + 1 - t$ and $\frac{\partial f}{\partial y} = 1$ are continuous

$\rightarrow y(t) = \int_0^t f(s, y(s))\, ds = \int_0^t 1 - s + y(s)\, ds$

$\rightarrow y_0(t) := 0$

$y_1(t) = \int_0^t 1 - s + 0\, ds = t - \frac{1}{2}t^2$

$y_2(t) = \int_0^t 1 - s + s - \frac{1}{2}s^2\, ds = t - \frac{1}{3!}t^3$

$y_3(t) = \int_0^t 1 - \frac{1}{3!}s^3\, ds = t - \frac{1}{4!}t^4$

$\Bigg\}\ y_n(t) = t - \frac{t^{n+1}}{(n+1)!}$

$\Rightarrow \underline{\underline{y(t) = \lim_{n\to\infty} y_n(t) = t}}$

b) using other methods

$y' - y = 1 - t \rightsquigarrow \mu(t) = e^{-\int 1 dt} = e^{-t}$

$\rightarrow y(t) = e^t \int e^{-t}(1-t)\, dt$

	D	I
		e^{-t}
+	$1-t$	$-e^{-t}$
−	-1	e^{-t}
+	0	

$= e^t\left((t-1)e^{-t} + e^{-t} + C\right)$

$= t + C\cdot e^t$

IVP: $y(0) = 0 = 0 + C\cdot 1 \Rightarrow C = 0 \Rightarrow \underline{\underline{y(t) = t}}$

② Solve $y'(t) = 3t^2(y-1)$, $y(0) = 0$ using Picard's method

$f(t,y) := 3t^2(y-1)$ and $\frac{\partial f}{\partial y} = 3t^2$ are continuous

$\Rightarrow y(t) = \int_0^t f(s, y(s))\, ds = \int_0^t -3s^2 + 3s^2 y(s)\, ds$

$\rightarrow y_0(t) = 0$

$y_1(t) = \int_0^t -3s^2\, ds = -t^3$

$y_2(t) = \int_0^t -3s^2 + 3s^2(-s^3)\, ds = -t^3 - \frac{t^6}{2}$

$y_3(t) = \int_0^t -3s^2 + 3s^2\left(-s^3 - \frac{s^6}{2}\right) ds = -t^3 - \frac{t^6}{2} - \frac{t^9}{2\cdot 3}$

$\Rightarrow$ guess $y_n(t) = -\sum_{k=1}^{n} \frac{t^{3k}}{k!}$

Induction: $y_{n+1}(t) = \int_0^t -3s^2 - 3s^2 \sum_{k=1}^{n} \frac{s^{3k}}{k!}\, ds = -t^3 - 3\sum_{k=1}^{n} \frac{t^{3k+3}}{k!\,(3k+3)}$

$= -t^3 - \sum_{k=1}^{n} \frac{1}{(k+1)!} t^{3(k+1)} = -\sum_{k=1}^{n+1} \frac{t^{3k}}{k!}$ ✓

$\Rightarrow y(t) = \lim_{n\to\infty} y_n(t) = -\left(e^{t^3} - 1\right) = \underline{\underline{1 - e^{t^3}}}$

Ex: Uniqueness breaking down. $y' = 3y^{2/3}$, $y(0) = 0$

$f(t,y) = 3y^{2/3}$ is continuous on $\mathbb{R}^2$

$\frac{\partial f}{\partial y} = 3 \cdot \frac{2}{3} y^{-1/3} = \frac{2}{\sqrt[3]{y}}$ is not continuous if $y = 0$ } no E+U !

→ clearly $\underline{y(t) = 0}$ is a solution but it might not be unique

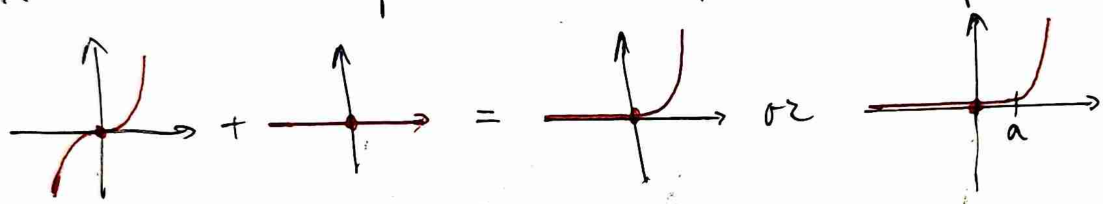
→ this is a separable ODE:

$dy = 3y^{2/3}\,dt \Rightarrow \frac{1}{3}\int y^{-2/3}\,dy = \int dt$

$\Rightarrow t + C = \frac{1}{3}\frac{y^{1/3}}{1/3} = \sqrt[3]{y}$

IVP: $y(0) = 0 \Rightarrow C = \sqrt[3]{0}$ } $\underline{y(t) = x^3}$

There is an infinite number of solutions of the form



$$y(t) = \begin{cases} 0, & t \leq a \\ (t-a)^3, & t > a \end{cases}, \quad a \in \mathbb{R}^+$$

Ex: E+U breaking down: x_0, y_0 are given, $xy' = y$, $y(x_0) = y_0$

→ we have $y' = y/x$, there are 3 cases for x_0 and y_0

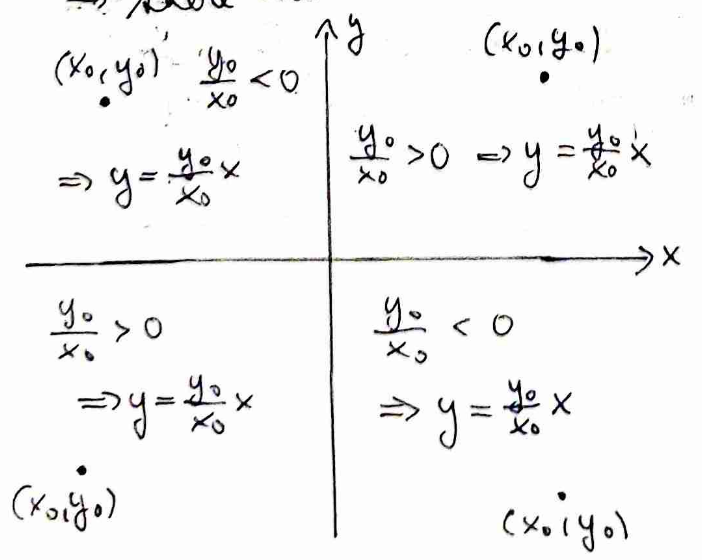
a) $\underline{x_0 \neq 0}$: $f(x,y) := y/x$ & $\frac{\partial f}{\partial y}$ continuous in a neighborhood of (x_0, y_0)

→ $\exists$ unique solution

$\int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \ln|y| = \ln|x| + C \Rightarrow |y| = |x| \cdot e^C$

IVP: $|y_0| = |x_0| \cdot e^C \Rightarrow e^C = \left|\frac{y_0}{x_0}\right| \Rightarrow \boxed{|y| = |x| \cdot \left|\frac{y_0}{x_0}\right|}$

⇒ there are 4 cases to consider for the initial condition:



⇒ in general, $\underline{y = x \cdot \frac{y_0}{x_0}}$ is a solution unique in a neighborhood of (x_0, y_0)

→ more precisely, $f(x,y)$ is not continuous at $(0,0)$, so the solution is unique in the quadrant containing (x_0, y_0)

Consider: $x_0 > 0$, $y_0 > 0$

⇒ $y = \frac{y_0}{x_0}x$ is unique in top-right quadrant

⇒ $y = \frac{y_0}{x_0}|x|$ which is also a solution has the same values there

b) $x_0 = 0$ & $y_0 \neq 0$: sub into $xy' = y$ to get $0 = 0y' = y_0 \neq 0$ ↯
↳ solution doesnt exist

c) $x_0 = 0$ & $y_0 = 0$:

$f(x,y) = y/x$ not continuous at (x_0, y_0)

$\Rightarrow$ no guruntee of solutions existing or being unique

→ solving as before we get

$|y| = |x| \cdot e^c \rightsquigarrow |y_0| = |x_0| \cdot e^c \Rightarrow 0 = 0 \Rightarrow C \in \mathbb{R}$

$\Rightarrow \underline{y = \pm |x| e^c = Ax}$ is a valid solution for any $A \in \mathbb{R}$

Theorem (Continuation of solutions): Let $(x_0, y_0) \in \mathbb{R}^2$ and suppose $f: \mathbb{R}^2 \to \mathbb{R}$, $\frac{\partial f}{\partial y}$ are continuous and bounded in an open region $A \subseteq \mathbb{R}^2$, $(x_0, y_0) \in A$.
Then the IVP $y'(x) = f(x, y(x))$, $y(x_0) = y_0$ has a unique solution which is defined on a maximal interval (α_1, α_2) ... interval of validity.
Moreover there are 3 possible scenarios:

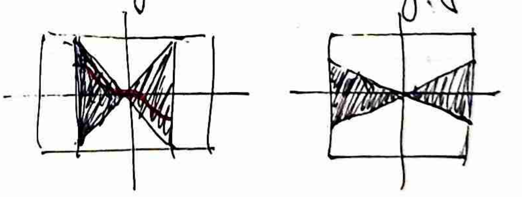
i) $(\alpha_1, \alpha_2) = (-\infty, \infty)$ ... global solution

ii) The end point α_i is finite and there is an infinite discontinuity at α_i

$|y(x)| \to \infty$ as $x \to \alpha_i$

→ the solution is said to blow up in finite time

iii) The endpoint α_i is finite and $(x, y(x)) \to$ boundary of A as $x \to \alpha_i$.

recall from the proof of E+U: 

Ex:

① Let $y_0 > 0$ and consider $y' = y^2$, $y(0) = y_0$

$f(x,y) := y^2$ & $\frac{\partial f}{\partial y} = 2y$ are both defined and continuous on $\mathbb{R}^2$

$\Rightarrow \int \frac{dy}{y^2} = \int dx \Rightarrow -y^{-1} = x + C \Rightarrow \frac{-1}{y_0} = 0 + C \Rightarrow C = -\frac{1}{y_0}$

$\Rightarrow -y = \frac{1}{x+C} \Rightarrow y = \frac{-1}{x - \frac{1}{y_0}} = \underline{\underline{\frac{y_0}{1 - xy_0}}}$

IoV: $1 - xy_0 \neq 0 \Rightarrow x \neq \frac{1}{y_0}$

→ $(0, y_0)$ must be included $\Rightarrow 0 \in$ IoV, $y_0 > 0 \Rightarrow \underline{\underline{\text{IoV} = (-\infty, \frac{1}{y})}}$

→ what happens at $\frac{1}{y_0}$? $\lim_{x \to 1/y_0^-} |y(x)| = \infty \rightsquigarrow$ ii)

(2) $y' = 1 + y^2$, $y(0) = 0$

$f(x,y) = 1+y^2$ & $\frac{\partial f}{\partial y} = 2y$ ... continuous on $\mathbb{R}^2$

$\int \frac{dy}{1+y^2} = \int dx \Rightarrow \arctan(y) = x + C \Rightarrow 0 = 0 + C \Rightarrow \underline{\underline{y = \tan(x)}}$

IoV: $y(0) = 0 \Rightarrow$ IoV $= (-\frac{\pi}{2}, \frac{\pi}{2})$

→ both end points are finite and as $x \to -\frac{\pi}{2}^+$, $y \to -\infty$
$x \to \frac{\pi}{2}^-$, $y \to \infty$

(3) $y' = -\frac{x}{y}$, $y(0) = y_0 < 0$.

circle $\Rightarrow y = \pm\sqrt{y_0^2 - x^2}$

note $y(0) < 0 \Rightarrow \underline{\underline{y = -\sqrt{y_0^2 - x^2}}}$ ← unique solution

IoV: $y_0^2 - x^2 \geq 0 \Rightarrow |x| \leq |y_0| \Rightarrow x \in [-|y_0|, |y_0|]$

↳ y is defined for $x \in [-|y_0|, |y_0|]$, but y' is not defined at the endpoints $\Rightarrow$ IoV $= (-|y_0|, |y_0|)$

→ f and $\frac{\partial f}{\partial y}$ are continuous in IoV $\times [-|y_0|, 0) =: A$
and the solution approaches the boundary of A :

$x \to -|y_0|^+$, $y(x) \to 0^-$
$x \to |y_0|^-$, $y(x) \to 0^-$

Note: y never blows up, it stays bounded the whole time

Theorem (Continuous dependence on initial data): Let $(x_0, y_0) \in \mathbb{R}^2$ and suppose $f: \mathbb{R}^2 \to \mathbb{R}$ and $\frac{\partial f}{\partial y}$ are continuous and bounded on an open region $A \subseteq \mathbb{R}^2$ which contains (x_0, y_0). Then from the E+U theorem there $\exists \varepsilon > 0$ s.t. $\exists!$ unique solution y defined on $U^\varepsilon(x_0)$ to the IVP $y'(x) = f(x, y(x))$, $y(x_0) = y_0$. Moreover, this solution depends continuously on the initial conditions x_0, y_0.

Idea: Slight variation in the initial value will slightly vary the solution.
↳ this is ofcourse only guaranteed for variations within A

Formally: Suppose that we know x_0, but we dont know the precise value of y_0, only that $|y_0 - \tilde{y}_0| \leq \varepsilon_0$ for some known $\tilde{y}_0$.

→ consider: $y' = f(x,y)$, $y(x_0) = y_0$ & $y' = f(x,y)$, $y(x_0) = \tilde{y}_0$

$(\forall \delta > 0)(\exists \varepsilon > 0)(\forall x): |x - x_0| < \delta \Rightarrow |y(x) - \tilde{y}(x)| < \varepsilon$

Ex: $y' = y$, $y(x_0) = y_0$

$$y = Ae^x \rightsquigarrow y_0 = Ae^{x_0} \Rightarrow A = y_0 e^{-x_0}$$

$\Rightarrow \underline{y(x) = y_0 e^{x - x_0}}$ which is continuous in both x_0 and y_0

Ex:

① Solve $y' = -\frac{1}{2}y + t$, $y(0) = 0$ using Picards iteration

$\to f(t,y) := -\frac{1}{2}y + t$ & $\frac{\partial f}{\partial y} = -\frac{1}{2}$ are both continuous on $\mathbb{R}^2$

$\Rightarrow y(t) = \int_0^t f(s, y(0))\, ds \Rightarrow$ let $y_0(t) := 0$

$$y_1(t) = \int_0^t s - \frac{1}{2}\cdot y_0(s)\, ds = \int_0^t s\, ds = \frac{1}{2}t^2$$

$$y_2(t) = \int_0^t s - \frac{1}{2}\cdot\frac{s^2}{2}\, ds = \frac{1}{2}t^2 - \frac{1}{2}\cdot\frac{t^3}{3!}$$

$$y_3(t) = \int_0^t s - \frac{1}{2}\cdot\frac{s^2}{2} + \frac{1}{4}\cdot\frac{s^3}{3!}\, ds = \frac{t^2}{2} - \frac{1}{2}\frac{t^3}{3!} + \frac{1}{4}\frac{t^4}{4!}$$

$$y_4(t) = \frac{t^2}{2!} - \frac{1}{2}\cdot\frac{t^3}{3!} + \frac{1}{2^2}\cdot\frac{t^4}{4!} - \frac{1}{2^3}\cdot\frac{t^5}{5!}$$

$\Rightarrow$ guess $y_n(t) = \sum_{k=2}^{n+1} \left(-\frac{1}{2}\right)^{k-2}\cdot\frac{t^k}{k!}$

Induction: $y_{n+1}(t) = \int_0^t s - \frac{1}{2}\cdot y_n(s)\, ds = \frac{t^2}{2} + \int_0^t \sum_{k=2}^{n+1}\left(-\frac{1}{2}\right)^{k-1}\cdot\frac{s^k}{k!}\, ds =$

$$= \frac{t^2}{2} + \sum_{k=2}^{n+1}\left(-\frac{1}{2}\right)^{k-1}\cdot\frac{t^{k+1}}{(k+1)!} = \sum_{k=2}^{n+2}\left(-\frac{1}{2}\right)^{k-2}\cdot\frac{t^k}{k!} \quad \checkmark$$

Closed form:

$$y(t) = \sum_{k=2}^{\infty}\left(-\frac{1}{2}\right)^{k-2}\frac{t^k}{k!} = 4\cdot\sum_{k=2}^{\infty}\left(-\frac{1}{2}\right)^k\frac{t^k}{k!} = 4\cdot\sum_{k=2}^{\infty}\frac{1}{k!}\left(-\frac{t}{2}\right)^k$$

$$= 4\cdot\left(e^{-\frac{t}{2}} - 1 - \left(-\frac{t}{2}\right)^1\right) = 4\left(e^{-t/2} - 1 + \frac{t}{2}\right) = \underline{4e^{-\frac{t}{2}} + 2t - 4}$$

② Suppose the IVP $y' = y^3 - y$, $y(0) = y_0 \in \mathbb{R}$. For which y_0 is the solution global?

$$\frac{dy}{dx} = y^3 - y \Rightarrow \int dx = \int\frac{dy}{y(y^2-1)} = \int\frac{dy}{y(y-1)(y+1)}$$

$$\Rightarrow x + c = \int \frac{-1}{y} + \frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1}\, dy = -\ln|y| + \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1|$$

$$\Rightarrow \ln|y-1| - \ln|y+1| - 2\ln|y| = 2x + c \Rightarrow \ln\left|\frac{y-1}{(y+1)y^2}\right| = 2x + c$$

$$\Rightarrow \left|\frac{y-1}{(y+1)y^2}\right| = e^{2x}\cdot e^c \text{ , IVP: } e^c = \left|\frac{y_0 - 1}{(y_0+1)y_0^2}\right| = \frac{|y_0 - 1|}{y_0^2(y_0+1)}$$

- if $y_0 \notin \{0, 1, -1\}$: $y(x)$ satisfying this is a global unique solution
- if $y_0 \in \{0, 1, -1\}$: $y(x) := y_0$ is a global unique solution

$\Rightarrow$ the solution is global for every $y_0 \in \mathbb{R}$

• **Systems of linear differential equations**

Def: A first order linear system of differential equations is a system of the form

$$x_1'(t) = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \dots + a_{1n}(t)x_n(t) + g_1(t)$$
$$x_2'(t) = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \dots + a_{2n}(t)x_n(t) + g_2(t)$$
$$\vdots$$
$$x_n'(t) = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \dots + a_{nn}(t)x_n(t) + g_n(t)$$

Which can be expresed as

$$\tilde{x}'(t) = A(t)\tilde{x}(t) + \tilde{g}(t)$$

Where $\tilde{x}: \mathbb{R} \to \mathbb{R}^n$, $\tilde{g}: \mathbb{R} \to \mathbb{R}^n$, $A: \mathbb{R} \to \mathbb{R}^{n\times n}$.

The system is said to be homogeneous $\equiv \tilde{g}(t) = 0$, otherwise its inhomogeneous.

Def: We define the derivative of vector and matrix functions to be component-wise, that is for $A: \mathbb{R} \to \mathbb{R}^{n\times m}$ define

$$\frac{dA}{dt}(t) := \lim_{h\to 0} \frac{A(t+h) - A(t)}{h} \equiv \left(A'(t)\right)_{ij} = a_{ij}'(t)$$

Def: An n^{th}-order linear differential equation is an ODE of the form

$$\sum_{k=0}^{n} a_k(t)\, y^{(k)}(t) = g(t)$$

Every n^{th}-order linear ODE is equivalent to a system of n first order linear ODEs.

EX: Rewrite the following ODE as a system:

a) $2y'' - 5y' + y = 0$, $y(3) = 6$, $y'(3) = -1$

→ let $x_1(t) = y(t)$ and $x_2(t) = y'(t)$

$$\left.\begin{aligned} \Rightarrow x_1'(t) &= y'(t) = x_2(t) \\ x_2'(t) &= y''(t) = \tfrac{1}{2}(5y' - y) = \tfrac{5}{2}x_2(t) - \tfrac{1}{2}x_1(t) \end{aligned}\right\}$$

$$\begin{array}{l|l} x_1'(t) = x_2(t) & x_1(3) = 6 \\ x_2'(t) = -\tfrac{1}{2}x_1(t) + \tfrac{5}{2}x_2(t) & x_2(3) = -1 \end{array}$$

b) $y^{(4)} + 3y'' - \sin(t)y' + 8y = t^2$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 3$, $y'''(0) = 4$

$$\left.\begin{aligned} x_1 &:= y \Rightarrow x_1' = y' = x_2 \\ x_2 &:= y' \Rightarrow x_2' = y'' = x_3 \\ x_3 &:= y'' \Rightarrow x_3' = y''' = x^4 \\ x_4 &:= y''' \Rightarrow x_4' = y^{(4)} = t^2 - 3x^3 + \sin(t)x_2 - 8x_1 \end{aligned}\right\} \tilde{x}'(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & \sin(t) & -3 & 0 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{bmatrix}$$

$$\hookrightarrow \tilde{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

<u>To generalize</u>: The equation

$$y^{(n)}(t) + a_{n-1}(t)\,y^{(n-1)}(t) + \dots + a_1(t)\,y'(t) + a_0(t)\,y(t) = g(t)$$

Is equivalent to the system

$$\forall k:\ x_k := y^{(k-1)} \rightsquigarrow \tilde{x}'(t) = \left[\begin{array}{c|c} \vec{0} & I_{n-1} \\ \hline -a_0\ -a_1 & \cdots\ -a_{n-1} \end{array}\right] \tilde{x}(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g \end{bmatrix}$$

<u>Theorem</u> (<u>Superposition principle</u>) The set of solutions of a linear homogeneous system is <u>closed</u> under <u>addition</u> and <u>scalar multiplication</u>. Meaning:

$$\tilde{x}_1(t)\ \&\ \tilde{x}_2(t) \text{ solve } \tilde{x}' = Ax \implies c_1\tilde{x}_1 + c_2\tilde{x}_2 \text{ is also a solution}$$

<u>Proof</u>: Let $c_1, c_2 \in \mathbb{R}$, we have

$$(c_1\tilde{x}_1 + c_2\tilde{x}_2)' = c_1\tilde{x}_1' + c_2\tilde{x}_2' = c_1A\tilde{x}_1 + c_2A\tilde{x}_2 = A(c_1\tilde{x}_1 + c_2\tilde{x}_2) \quad \blacksquare$$

<u>Collolary</u>: The set of solutions of $\tilde{x}' = A\tilde{x}$ form a vector space over the real numbers.

<u>Fact</u>: We will later show that the dimension of this space $= n$, when $A \in \mathbb{R}^{n\times n}$.

<u>Def</u>: The functions $\tilde{x}_1, \dots \tilde{x}_n : I \subseteq \mathbb{R} \to \mathbb{R}^n$ are

- <u>linearly dependent</u> $\equiv (\exists c \in \mathbb{R}^n \setminus \{\vec{0}\})(\forall t \in I): \sum_{i=1}^{n} c_i\tilde{x}_i(t) = \vec{0}$
- <u>linearly independent</u> $\equiv$ otherwise

If there is a point $t_0 \in I$ such that the <u>vectors</u> $\{\tilde{x}_1(t_0), \dots \tilde{x}_n(t_0)\}$ are <u>linearly independent</u> in $\mathbb{R}^n$, then the <u>functions</u> $\{\tilde{x}_1, \dots \tilde{x}_n\}$ are <u>linearly independent</u> as well.

<u>Pf</u>: Lets assume that the functions are not lin. ind

$$\implies \exists \tilde{c} \in \mathbb{R}^n \setminus \{\vec{0}\} \text{ s.t. } \tilde{c}\cdot\tilde{\mathbf{X}}(t) = \vec{0} \text{ for } \forall t \in I \Rightarrow \tilde{c}\,\tilde{\mathbf{X}}(t_0) = \vec{0}\ \text{↯}$$

<u>Ex</u>: <u>Are the following functions linearly ind</u>?

$$x_1(t) = \begin{bmatrix} e^t \\ t \end{bmatrix},\ x_2(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ x_3(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow c_1x_1 + c_2x_2 + c_3x_3 = 0\ ?$$

$$\implies \begin{array}{l} c_1e^t + c_2t + c_3 = 0 \\ c_1t + c_2t + c_3 = 0 \end{array} \quad \text{for } \forall t \in \mathbb{R}\ \&\ (c_1, c_2, c_3) \neq (0,0,0)$$

- <u>$t = 0$</u>: $c_1 + c_3 = 0\ \&\ c_3 = 0 \implies c_1 = c_3 = 0$
- <u>$t = 1$</u>: $0 + c_2 + 0 = 0 \implies c_2 = 0$

$\left.\right\} (c_1, c_2, c_3) = (0,0,0) \implies$ <u>lin. ind</u>

Ex: Find a basis for the set of solutions of the linear homogeneous system

$$\tilde{x}'(t) = \begin{bmatrix} 1 & 0 \\ e^t & 2 \end{bmatrix} \tilde{x}(t) \quad \Rightarrow \quad \begin{array}{l} \boxed{1}\ x_1'(t) = x_1(t) \\ \boxed{2}\ x_2'(t) = e^t x_1(t) + 2x_2(t) \end{array}$$

$\boxed{1}$ $x_1(t) = Ae^t$, $A \in \mathbb{R}$

$\boxed{2}$ $x_2'(t) = Ae^{2t} + 2x_2(t) \quad \rightsquigarrow \mu(t) = e^{\int -2\,dt} = e^{-2t}$

$$\Rightarrow x_2(t) = e^{2t} \int e^{-2t} A e^{2t}\, dt = e^{2t}(At + B)$$

$$\Rightarrow \tilde{x}(t) = \begin{bmatrix} Ae^t \\ Ate^{2t} + Be^{2t} \end{bmatrix} = \underline{\underline{A \begin{bmatrix} e^t \\ te^{2t} \end{bmatrix} + B \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}}} \qquad A, B \in \mathbb{R}$$

↳ linear independence?

Suppose:
$$\left.\begin{array}{ll} Ae^t + B\cdot 0 = 0 & \rightsquigarrow t = 0 : A = 0 \\ Ate^t + Be^{2t} = 0 & \rightsquigarrow t = 0 : 0 + B = 0 \end{array}\right\} A = B = 0$$

$\Rightarrow$ lin. ind. $\Rightarrow \begin{bmatrix} e^t \\ te^{2t} \end{bmatrix}, \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}$ form a basis

Def: If the set of unctions $\{\tilde{x}_1, \tilde{x}_2, \dots \tilde{x}_n\}$ form a basis for the set of all solutions of the linear homogeneous system $\tilde{x}' = A\tilde{x}$, then they are called a fundamental set of solutions of that system.

Linear algebra recap

- Change of basis
  - → the canonical basis of $\mathbb{R}^n$ is $\mathcal{E} := \{e^1, e^2, \dots e^n\}$, where $e^i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}$
  - → suppose $B = \{v_1, \dots v_n\}$ is also a basis of $\mathbb{R}^n$
  - → lets write the vector $u \in \mathbb{R}^n$ using this basis:

$$u = \sum_{i=1}^{n} c_i v_i \quad \rightsquigarrow \quad [u]_B := (c_1, \dots c_n)^T \leftarrow \text{coordinate vector of } u$$

if we are expressing a vector using the canonical basis, then $[u]_{\mathcal{E}} = u$

→ usually when thinking about a linear transformation (matrix) A, we think about $w = Av$, where the input (v) and output (w) vectors are with respect to the canonical basis.

→ we want to expres A in such a way, that it works for input vectors with respect to some basis X and it outputs vectors with respect to basis Y.

→ we also want some linear transformation to translate between $[v]_X$ and $[v]_Y$

Def: Suppose $X = \{x_1, \dots x_n\}$ and $Y = \{y_1, \dots y_n\}$ are basis of $\mathbb{R}^n$.
The change of basis matrix from X to Y is

$${}_Y[id]_X := ([x_1]_Y, [x_2]_Y, \dots [x_n]_Y)$$

coordinate vectors of X with respect to Y

$[u]_Y = {}_Y[id]_X \cdot [u]_X$

$[x_i]_X = e^i$

$\Rightarrow {}_X[id]_X = I_n$

Pf: $u = \sum_{i=1}^{n} a_i y_i \Rightarrow [u]_Y = (a_1, \dots a_n)^T$

$u = \sum_{i=1}^{n} b_i x_i \Rightarrow [u]_X = (b_1, \dots b_n)^T$

$\hookrightarrow [u]_Y = \left[\sum_i b_i x_i\right]_Y = \sum_i b_i [x_i]_Y = {}_Y[id]_X \cdot [u]_X$

Def: To generalize, suppose $X = \{x_1, \dots x_n\}$ and $Y = \{y_1, \dots y_n\}$ are basis of $\mathbb{R}^n$ and consider $A \in \mathbb{R}^{n \times n}$ and the associated linear transformation $f: x \mapsto Ax$.
The matrix of this linear transformation with respect to X and Y is

$${}_Y[A]_X := ([Ax_1]_Y, [Ax_2]_Y, \dots [Ax_n]_Y)$$

$\rightarrow {}_K[A]_K = A$

$[Au]_Y = {}_Y[A]_X \cdot [u]_X$

Pf: $u = \sum_i a_i x_i \Rightarrow [u]_X = (a_1, \dots a_n)^T$

$\hookrightarrow Au = A\left(\sum_i a_i x_i\right) = \sum_i a_i Ax_i \Rightarrow [Au]_Y = \sum_i a_i [Ax_i]_Y = {}_Y[A]_X \cdot [u]_X$

- Eigenvalues and eigenvectors

Def: $\lambda \in \mathbb{R}$ is an eigen-value of $A \in \mathbb{R}^{n \times n}$ $\equiv \exists\, x \in \mathbb{R}^n \setminus \{0\} : Ax = \lambda x$.
The vector x is called an eigen-vector associated with λ.

Def: The characteristic polynomial of $A \in \mathbb{R}^{n \times n}$ is $p_A(t) := \det(A - tI_n)$.

Theorem: $\lambda \in \mathbb{R}$ is an eigen value of $A \iff p_A(\lambda) = 0$.

$\hookrightarrow$ eigenvector solves $A - \lambda I = 0$

Proof: $\lambda \in \mathbb{R}$ is an eigenvalue of $A \iff$

$\exists x \in \mathbb{R}^n \setminus \{0\} : Ax = \lambda x \iff Ax - \lambda x = (A - \lambda I_n)x = 0$

$\iff (A - \lambda I_n)$ is singular $\iff \det(A - \lambda I_n) = 0$.

Corollary: The eigenvalues of a Δ-matrix are the values on its diagonal.

The eigenvectors associated with the eigenvalue λ form a vector space.

$\hookrightarrow$ Suppose $Ax = \lambda x$ & $Ay = \lambda y$ $\Rightarrow A(\alpha x + \beta y) = \alpha Ax + \beta Ay = \alpha\lambda x + \beta\lambda y = \lambda(\alpha x + \beta y)$

Def: The geometric multiplicity of λ := dim (space generated by λ-eigenvectors)
The algebraic multiplicity of λ := its multiplicity as a root of p_A.

• Diagonalization

The matrix form of a linear transformation is not unique

$${}_X[f]_X = {}_X[id]_Y \, {}_Y[f]_Y \, {}_Y[id]_X$$

Note: ${}_Y[id]_X \, {}_X[id]_Y = I_n \Rightarrow {}_X[id]_Y = {}_Y[id]_X^{-1}$) motivation

Def: The matrices $A, B \in \mathbb{R}^{n \times n}$ are similar $\equiv$

$\exists$ non-singular $R \in \mathbb{R}^{n \times n}$: $A = R^{-1} B R$. → note $(R^{-1})^{-1} = R$

Similar matrices have identical eigenvalues

Suppose $Ax = \lambda x$, we want to find y s.t. $By = \lambda y$

↳ $y = Rx$ works: $B(Rx) = RAR^{-1}(Rx) = RAx = R\lambda x = \lambda(Rx)$

Def: A is diagonalizable $\equiv$ it's similar to a diagonal matrix.

Theorem: $A \in \mathbb{R}^{n \times n}$ is diagonalizable

$\iff$ its possible to form a basis of $\mathbb{R}^n$ from its eigenvectors.

$\iff$ the sum of geometric multiplicities of its eigenvalues $= n$

Let this basis be $\{v_1, \dots v_n\}$ with the corresponding eigenvalues $\lambda_1, \dots \lambda_n$.

Then A can be expressed as

$$A = RDR^{-1}, \text{ where } D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \text{ and } R = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

Proof: Denote the basis as $F := \{v_1, \dots v_n\}$. We can clearly express A as

$$A = {}_E[A]_E = {}_E[id]_F \cdot {}_F[A]_F \cdot {}_F[id]_E$$

• show ${}_F[A]_F = D$:

→ the i^{th} column of ${}_F[A]_F$ is $[Av_i]_F = [\lambda v_i]_F = \lambda [v_i]_F = \lambda e^i$

• show ${}_E[id]_F = R$:

→ the i^{th} column of ${}_E[id]_F$ is $[v_i]_E = v_i$ $\blacksquare$

If $A = RDR^{-1}$, then $A^k = RD^kR^{-1}$

↳ $A^k = (RDR^{-1})(RDR^{-1}) \cdots (RDR^{-1}) = R(D)(D) \cdots (D) R^{-1} = R D^k R^{-1}$

↳ this is useful because D^k is very easy to compute

- **Jordan normal form**
  - → what if it's no possible to form a basis from the eigenvectors of A?
  - → we can still express $A = RJR^{-1}$, where J is almost diagonal

Def: A Jordan block is a matrix of the form $J_\lambda = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & \ddots & 1 \\ & & & & \lambda \end{bmatrix}$

Def: Suppose $A \in \mathbb{R}^{n\times n}$ has an eigenvalue λ.
The vector $v_m \in \mathbb{R}^n$ is a generalized eigenvector of rank m $\equiv$
$(A-\lambda I_n)^m v_m = 0$ but $(A-\lambda I_n)^{m-1} v_m \neq 0$.

↳ idea: if v is eigenvector of λ: $Av = \lambda v \Rightarrow (A-\lambda I)v = 0$
if $(A-\lambda I)^m v = 0$, then $(A-\lambda I)^{m+1} v = 0$, hence the condition on $m-1$.

Def: The null space (or kernel) of a matrix $A \in \mathbb{R}^{m\times n}$ is $\mathcal{N}(A) := \{x \in \mathbb{R}^n \mid Ax = 0\}$

We can use v_m to form a Jordan chain as follows:

$v_m \in \mathcal{N}(A-\lambda I)^m =: N_m$

$v_{m-1} := (A-\lambda I)v_m \quad \Rightarrow v_{m-1} \in \mathcal{N}(A-\lambda I)^{m-1} =: N_{m-1}$

$v_{m-2} := (A-\lambda I)v_{m-1} \quad \Rightarrow v_{m-2} \in \mathcal{N}(A-\lambda I)^{m-2} =: N_{m-2}$

$\vdots$

$v_1 := (A-\lambda I)v_2 \quad \Rightarrow v_1 \in \mathcal{N}(A-\lambda I) =: N_1$ → v_1 is an eigenvector

$v_0 := (A-\lambda I)v_1 = (A-\lambda I)^m v_m = 0$

In general: $v_j = (A-\lambda I)v_{j+1} = (A-\lambda I)^{m-j} v_m \in \mathcal{N}(A-\lambda I)^j$

Fact: $\dim(N_1) = 1$, $\dim(N_2) = 2$ ... $\dim(N_m) = m$ & $N_1 \subset N_2 \subset \dots \subset N_m$

↳ if we pick $v_m \in N_m \setminus N_{m-1}$, then $\{v_m, \dots v_1\}$ are linearly independent.

Theorem: Suppose $A \in \mathbb{R}^{m\times m}$ has only one eigenvalue λ, with geom. multiplicity 1 and that we form the corresponding Jordan chain $\{v_1, v_2, \dots v_m\}$, which forms a basis for $\mathbb{R}^m$. Then A can be expressed as

$$A = RJ_\lambda R^{-1}, \quad \text{where } J_\lambda \text{ is a Jordan block and } R = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_m \\ | & | & & | \end{bmatrix}$$

Proof: Denote the basis as $F := \{v_1, \dots v_m\}$. We can write A as

$A = {}_\mathcal{E}[A]_\mathcal{E} = {}_\mathcal{E}[id]_F \cdot {}_F[A]_F \cdot {}_F[id]_\mathcal{E}$

- ${}_\mathcal{E}[id]_F = R$: the i^{th} column of ${}_\mathcal{E}[id]_F$ is $[v_i]_\mathcal{E} = v_i$
- ${}_F[A]_F = J_\lambda$: first column: $[Av_1]_F = [\lambda v_1]_F = \lambda [v_1]_F \, \lambda e^1$
  - → j^{th} column, $j \geq 2$: $v_{j-1} = (A-\lambda I)v_j = Av_j - \lambda v_j \Rightarrow Av_j = \lambda v_j + v_{j-1}$

  ↳ $[Av_j]_F = [\lambda v_j + v_{j-1}]_F = \lambda [v_j]_F + [v_{j-1}]_F = \lambda e^j + e^{j-1}$ ∎

Theorem: Every $A \in \mathbb{C}^{n \times n}$ is similar to a matrix in <u>Jordan normal form</u>,

$$A = RJR^{-1}, \quad \text{where} \quad J = \begin{pmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_k} \end{pmatrix}$$

→ every Jordan block J_{λ_i} corresponds to the eigenvalue λ_i of A

- # blocks with λ_i = $\text{Geom}(\lambda_i)$
- $\sum$ of sizes of blocks with $\lambda_i = \text{Alg}(\lambda_i)$.

→ the matrix R is contains generalized eigenvectors forming Jordan chains corresponding to the Jordan blocks.

→ basically if $\text{Geom}(\lambda_i) = 2$, but $\text{Alg}(\lambda_i) = 4$, then we can use the two lin. ind. eigenvectors, but need to suplement them with 2 additional lin. ind. generalized aigenvectors, to cover the size 4.

<u>Note:</u> We need $\mathbb{C}$, because its algebraically closed $\Rightarrow \sum \text{Alg}(\lambda) = n$

- <u>Matrix inverse using determinants</u>

<u>Def:</u> Let $A \in \mathbb{R}^{n \times n}$ and define its <u>adjugate matrix</u> $\text{Adj}(A)$ as

$$(\text{Adj}(A))_{ij} := (-1)^{i+j} \det(A^{ji}),$$

where A^{ji} := submatrix made from A by omitting the j^{th} row and i^{th} column.

<u>Theorem:</u> If $A \in \mathbb{R}^{n \times n}$ is invertible, then $A^{-1} = \frac{1}{\det(A)} \cdot \text{Adj}(A)$.

<u>Proof:</u> Uses Laplace expansion of the determinant to show $A \cdot \text{Adj}(A) = \det(A) \cdot I_n$.

<u>Ex:</u> If $A \in \mathbb{R}^{2 \times 2}$, then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightsquigarrow A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

<u>Ex:</u> $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 0 \\ 2 & 1 & 1 \end{bmatrix} \rightsquigarrow B^{-1} = ?$

$\det(B) = -3 + 2 + 6 - 2 = 3$

$$B^{-1} = \frac{1}{\det(B)} \text{Adj}(B) = \frac{1}{3} \begin{bmatrix} -3 & 0 & 3 \\ -2 & -1 & 2 \\ 8 & 1 & -5 \end{bmatrix}$$

- Eigendecomposition method

Suppose $A \in \mathbb{R}^{n\times n}$ is diagonalizable and consider $\tilde{x}' = A\tilde{x}$

$\hookrightarrow \lambda_1, \dots \lambda_n$ eigenvalues
$v_1, \dots v_n$ eigenvectors $\rightsquigarrow D = RAR^{-1}$

Note: Denote $F := \{v_1, \dots v_n\}$ and note: $x = [x]_\varepsilon = {}_\varepsilon[id]_F \cdot [x]_F = R \cdot [x]_F$

$\hookrightarrow z := [x]_F \Rightarrow x = Rz \Rightarrow z = R^{-1}x$

$\Rightarrow z' = R^{-1}x' = R^{-1}(Ax) = R^{-1}ARz = Dz$

easy

Idea: We changed basis and transformed the problem into $z' = Dz$

$$\tilde{z}' = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \tilde{z} = \begin{bmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \\ \vdots \\ \lambda_n z_n \end{bmatrix}$$

i^{th} equation: $z_i'(t) = \lambda_i z_i(t)$

$\Rightarrow z_i(t) = c_i e^{\lambda_i t}$, $c_i \in \mathbb{R}$

$$\Rightarrow \tilde{z}(t) = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix} \Rightarrow \tilde{x}(t) = R\tilde{z}(t) = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \cdot \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$

$$\Rightarrow \tilde{x}(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \dots + c_n e^{\lambda_n t} v_n, \quad c \in \mathbb{R}^n$$

Theorem: $\{e^{\lambda_1 t} v_1, \dots e^{\lambda_n t} v_n\}$ form a fundamental set of solutions of $\tilde{x} = Ax$.

$\hookrightarrow$ meaning that their lin. comb. cover all of the solutions

Proof: We will show that they are lin. ind $\rightarrow$

$\hookrightarrow$ suppose $\exists b \in \mathbb{R}^n$: $\sum_{k=1}^n b_k e^{\lambda_k t} v_k = 0$ for $\forall t \in I$

$\hookrightarrow$ evaluate at $t = 0$: $\sum_{k=1}^n b_k v_k = 0$, but $\{v_1, \dots v_n\}$ forms a basis for $\mathbb{R}^n$. $\lightning$ $\blacksquare$

Ex: Solve $\tilde{x}'(t) = \begin{bmatrix} 8 & 15 \\ -2 & -3 \end{bmatrix} \tilde{x}(t)$

eigenvalues: $\det\begin{pmatrix} 8-t & 15 \\ -2 & -3-t \end{pmatrix} = (8-t)(-3-t) + 30 = 0$

$\Rightarrow -24 + t^2 - 5t + 30 = t^2 - 5t + 6 = (t-2)(t-3) = 0 \Rightarrow \lambda_1 = 3, \lambda_2 = 2$

eigenvectors:

$\lambda_1 = 3$: $\begin{bmatrix} 8-3 & 15 \\ -2 & -3-3 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$\lambda_2 = 2$: $\begin{bmatrix} 6 & 15 \\ -2 & -5 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 \\ 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 \\ 0 & 0 \end{bmatrix} \Rightarrow v_2 = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

$$\Rightarrow \tilde{x}(t) = c_1 e^{2t} \begin{bmatrix} 5 \\ -2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

Ex: Solve $\tilde{x}' = A\tilde{x}$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 1 & 4 \end{bmatrix}$

→ A is triangular ⇒ eigenvalues are on the main diagonal

$$\lambda_1 = 1: \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix} \Rightarrow v_1 = \begin{pmatrix} 3 \\ -3 \\ -2 \end{pmatrix}$$

$$\lambda_2 = 3: \begin{bmatrix} -2 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\lambda_3 = 4: \begin{bmatrix} -3 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \tilde{x}(t) = c_1 e^{t} \begin{pmatrix} 3 \\ -3 \\ -2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad c_1, c_2, c_3 \in \mathbb{R}$$

→ what if A is not diagonalizable?

Method: Suppose $A \in \mathbb{R}^{n \times n}$ and consider $\tilde{x}'(t) = A\tilde{x}(t)$. There are 3 possibilities

(1) All eigenvalues are real and $\sum \text{Geom}(\lambda) = n$.

↳ $\tilde{x}(t) = c_1 e^{\lambda_1 t} v_1 + \dots + c_n e^{\lambda_n t} v_n$, $c \in \mathbb{R}^n$

(2) Some eigenvalues are complex and $\sum \text{Geom}(\lambda) = n$.

↳ A is still diagonalizable.

↳ but we will have to deal with the complex valued solutions

(3) $\sum \text{Geom}(\lambda) < n$ ⇒ A is not diagonalizable

↳ Fact: Allways $\text{Geom}(\lambda) \leq \text{Arit}(\lambda)$

→ we will again use a change of basis to make the problem easier, but instead of diagonalization we will use Jordan normal form

- Dealing with complex eigenvalues ← (2)

Method: Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$ and $v \in \mathbb{C}^n$ is its corresponding eigenvector. Now consider the system $\tilde{x}'(t) = A\tilde{x}(t)$.

→ note: $\overline{Av} = A\overline{v}$ & $\overline{Av} = \overline{\lambda v} = \overline{\lambda}\,\overline{v}$ ⇒ $A\overline{v} = \overline{\lambda}\,\overline{v}$

$\overline{\lambda}$ is also an eigenvalue with eigenvector $\overline{v}$

⇒ complex eigenvalues come in pairs $\lambda, \overline{\lambda}$ ⇝ $v, \overline{v}$.

→ suppose $\tilde{z}(t): \mathbb{R} \to \mathbb{C}^n$ solves $\tilde{x}' = A\tilde{x}$.

↳ $\tilde{z}(t) = \tilde{a}(t) + i\tilde{b}(t)$

• $\tilde{z}' = A\tilde{z} \Rightarrow (\tilde{a} + i\tilde{b})' = A(\tilde{a} + i\tilde{b}) \Rightarrow \tilde{a}' + i\tilde{b}' = A\tilde{a} + iA\tilde{b}$

$\Rightarrow \tilde{a}' = A\tilde{a}$ & $\tilde{b}' = A\tilde{b}$

$\Rightarrow \tilde{a}, \tilde{b}: \mathbb{R} \to \mathbb{R}^n$ are real valued solutions to $\tilde{x}' = A\tilde{x}$.

Corollary: $\tilde{x} = Ax$, $\lambda + i\mu \in \mathbb{C}$ is an eigenvalue of A and $v \in \mathbb{C}^n$ is its eigenvector.

note: $v \cdot e^{(\lambda + i\mu)t}$ is a complex valued solution ↳ $v = a + bi$

$$v e^{(\lambda+i\mu)t} = v e^{\lambda t} e^{i\mu t} = (a+bi)\, e^{\lambda t}(\cos(\mu t) + i\sin(\mu t))$$

$$= e^{\lambda t}(a\cos(\mu t) - b\sin(\mu t) + i\,b\cos(\mu t) + ia\sin(\mu t))$$

$$= e^{\lambda t}(a\cos(\mu t) - b\sin(\mu t)) + i\, e^{\lambda t}(a\sin(\mu t) + b\cos(\mu t))$$

↳ this can be split to get two real solutions

note: $\overline{\lambda + i\mu} = \lambda - i\mu$ is also an eigenvalue with eigenvector $\bar{v} = a - bi$

$$\bar{v} e^{\overline{(\lambda+i\mu)}t} = \bar{v}\, \overline{e^{(\lambda+i\mu)t}} = \overline{v\, e^{(\lambda+i\mu)t}} =$$

$$= e^{\lambda t}(a\cos(\mu t) - b\sin(\mu t)) - i\, e^{\lambda t}(a\sin(\mu t) + b\cos(\mu t))$$

$\Rightarrow$ this doesn't give us a new ind. solution, the $\ominus$ is just a constant multiple

Summary: Complex eigenvalues come in pairs $\lambda, \bar{\lambda}$ w/ $v, \bar{v}$. The complex valued solution can be broken down to get two real solutions and its sufficient to do this for only one of the eigenvalues in the pair.

Ex: Find the real valued solutions of $\tilde{x}'(t) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tilde{x}(t)$

→ eigenvalues: $p_A(t) = (1-t)(1-t) + 1 = t^2 - 2t + 2 \Rightarrow \lambda_{1,2} = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

→ we need only one ($\bar{\lambda}_1 = \lambda_2$) → choose $\lambda = 1 + i$

eigenvector: $\begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \sim \begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ } note: $\lambda_2 = 1 - i$, $v_2 = \bar{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

→ complex solution: $z(t) = e^{(1+i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^t e^{it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^t(\cos t + i\sin t)\begin{bmatrix} 1 \\ -i \end{bmatrix}$

$$= e^t \begin{bmatrix} \cos t + i\sin t \\ \sin t - i\cos t \end{bmatrix} = e^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + i e^t \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$$

note: $\lambda_2 = \bar{\lambda}_1 = 1 - i$ would give us $\overline{z(t)} = e^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} - i e^t \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$

$\Rightarrow$ real solution: $x(t) = c_1 e^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$, $c_1, c_2 \in \mathbb{R}$

Ex: Find a basis for the set of solutions of the linear system

$$x'(t) = A(t)\,x(t) \quad , \qquad A(t) = \begin{bmatrix} 3t^2 & 3t^2 \\ 0 & \frac{3t^2}{t^3+5} \end{bmatrix}$$

Solution:

$\boxed{1}$ $x_1'(t) = 3t^2 x_1(t) + 3t^2 x_2(t)$

$\boxed{2}$ $x_2'(t) = \frac{3t^2}{t^3+5} x_2(t)$

Note: $\boxed{2}$ is also separable

$\hookrightarrow \mu(t) = e^{-\int \frac{3t^2}{t^3+5}dt} = e^{-\ln|t^3+5|} = \frac{1}{|t^3+5|}$

$\nearrow$ need: $\mu(t)\cdot\frac{-3t^2}{t^3+5} = \mu'(t)$

$\hookrightarrow \mu(t) = \frac{1}{3t^2+5}$ works ✓

$$\Rightarrow x_2(t) = (t^3+5)\int 0\,dt = \underline{\underline{C_1\cdot(t^3+5)}} \quad , \quad C_1 \in \mathbb{R}$$

$\boxed{1}$: $x_1'(t) = 3t^2 x_1(t) + 3t^2\cdot C_1(t^3+5)$:

$$\Rightarrow x_1'(t) - 3t^2 x_1(t) = 3C_1 t^2(t^3+5) \quad \rightarrow \mu(t) = e^{-\int 3t^2 dt} = e^{-t^3}$$

$$\Rightarrow x_1(t) = e^{t^3}\int e^{-t^3}\cdot 3C_1 t^2(t^3+5)\,dt = \left|\begin{array}{l} u = t^3 \\ du = 3t^2 dt \end{array}\right.$$

$$= C_1 e^{u}\int e^{-u}(u+1)\,du$$

	D	I
+	$u+1$	e^{-u}
−	1	$-e^{-u}$
+	0	e^{-u}

$$= C_1 e^{u}\left(-(u+1)e^{-u} - e^{-u} + C_2\right)$$

$$= -C_1(u+1+1 - C_2 e^{u})$$

$$= -C_1(u+2) + \underbrace{C_1 C_2}_{\in\mathbb{R}} e^{u} \doteq \underline{-C_1(t^3+2) + C_2 e^{t^3}}, \quad C_1, C_2 \in \mathbb{R}$$

$$\Rightarrow \underline{\underline{x(t) = C_1\begin{bmatrix} -t^3-2 \\ t^3+5 \end{bmatrix} + C_2\begin{bmatrix} e^{t^3} \\ 0 \end{bmatrix}}} \quad , \quad C_1, C_2 \in \mathbb{R}$$

? are $\begin{bmatrix} -t^3-2 \\ t^3+5 \end{bmatrix}$ and $\begin{bmatrix} e^{t^3} \\ 0 \end{bmatrix}$ linearly independent?

$$\begin{array}{l} -C_1(t^3+2) + C_2 e^{t^3} = 0 \\ C_1(t^3+5) = 0 \end{array} \rightsquigarrow t=0: \; 5C_1 = 0 \Rightarrow C_1 = 0$$

$$\Rightarrow C_2 e^{t^3} = 0 \Rightarrow C_2 = 0$$

} YES ✓

Ex: Solve $\tilde{x}' = A\tilde{x}$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$

$$p_A(t) = \begin{vmatrix} 1-t & 0 & 0 \\ 2 & 1-t & -2 \\ 3 & 2 & 1-t \end{vmatrix} = (1-t)^3 + 4(1-t) = (1-t)(t^2-2t+5)$$

$$\Rightarrow \lambda_1 = 1, \quad \lambda_{2,3} = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

eigenvectors:

$$\lambda_1 = 1: \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 3 & 2 & 0 \end{bmatrix} \Rightarrow v_1 = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

$$\lambda_2 = 1+2i: \begin{bmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -1 & i & 1 \\ 0 & 1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow v_2 = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$$

complex: $$z(t) = e^{\lambda_2 t} v_2 = e^{(1+2i)t} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = e^t \left(\cos(2t) + i\sin(2t)\right) \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

$$= e^t \begin{bmatrix} 0 \\ -\sin(2t) + i\cos(2t) \\ \cos(2t) + i\sin(2t) \end{bmatrix} = e^t \begin{bmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{bmatrix} + i e^t \begin{bmatrix} 0 \\ \cos 2t \\ \sin 2t \end{bmatrix}$$

real: $$x(t) = c_1 e^t \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{bmatrix} + c_3 e^t \begin{bmatrix} 0 \\ \cos 2t \\ \sin 2t \end{bmatrix}$$

Ex: Solve $\tilde{x}'(t) = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \tilde{x}(t)$

$$p_A(t) = (1-t)(3-t) + 4 = t^2 - 4t + 7 \Rightarrow \lambda_{1,2} = \frac{4 \pm \sqrt{16-28}}{2} = 2 \pm i\sqrt{3}$$

$$\lambda = 2+i\sqrt{3}: \begin{bmatrix} -1-i\sqrt{3} & -2 \\ 2 & 1-i\sqrt{3} \end{bmatrix} \sim \begin{bmatrix} 1+i\sqrt{3} & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow v = \begin{bmatrix} -2 \\ 1+i\sqrt{3} \end{bmatrix}$$

↳ i know that the second row will cancel

complex: $$z(t) = e^{\lambda t} v = e^{(2+i\sqrt{3})t} v = e^{2t} \left(\cos(\overbrace{\sqrt{3}t}^{u}) + i\sin(\sqrt{3}t)\right) \begin{bmatrix} -2 \\ 1+i\sqrt{3} \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} -2\cos(u) - 2i\sin(u) \\ \cos(u) + i\sqrt{3}\cos(u) + i\sin(u) - \sqrt{3}\sin(u) \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} -2\cos(u) \\ \cos(u) - \sqrt{3}\sin(u) \end{bmatrix} + i e^{2t} \begin{bmatrix} -2\sin(u) \\ \sqrt{3}\cos(u) + \sin(u) \end{bmatrix}$$

real: $$x(t) = c_1 e^{2t} \begin{bmatrix} -2\cos(\sqrt{3}t) \\ \cos(\sqrt{3}t) - \sqrt{3}\sin(\sqrt{3}t) \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -2\sin(\sqrt{3}t) \\ \sin(\sqrt{3}t) + \sqrt{3}\cos(\sqrt{3}t) \end{bmatrix}$$

- ## Matrix Exponential

Def: Let $A \in \mathbb{C}^{n\times n}$ and define its exponential as

$$e^A := I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

It can be shown that this series converges for all matrices A.

Note: When calculating e^A, one has to take powers of matrices.
$\Rightarrow$ use the Jordan normal form $J = R^{-1} A R$

$\hookrightarrow A^k = (RJR^{-1})^k = RJ^kR^{-1}$ ... recall: J^k is easy to compute

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} RJ^kR^{-1} = R\left(\sum_{k=0}^{\infty} \frac{1}{k!} J^k\right)R^{-1} = \underline{R e^J R^{-1}}$$

Theorem: Suppose $A, B \in \mathbb{C}^{n\times n}$ and define $\Phi(t) := e^{tA}$.

(1) If $\underline{A\cdot B = B\cdot A}$, then $\underline{e^{A+B} = e^A \cdot e^B}$

(2) $\underline{\frac{d}{dt} e^{tA} = A e^{tA}}$ $\Rightarrow$ e^{tA} is a matrix solution to $\tilde{x}' = A\tilde{x}$.

$\hookrightarrow$ recall: $\left(\frac{d}{dt}\Phi(t)\right)_{ij} = \frac{d}{dt}\left(\Phi(t)\right)_{ij}$

(3) The columns of $\Phi(t) = e^{tA}$ are vector solutions of $\tilde{x}' = A\tilde{x}$ and they form a basis for the space of all solutions U $\Rightarrow$ $\underline{\dim(U) = n}$.
$\hookrightarrow$ in fact $\dim(U) = n$ even if $A: \mathbb{R} \to \mathbb{C}^{n\times n}$

Proof:

(1) $e^{A+B} = \sum_{m=0}^{\infty} \frac{1}{m!}(A+B)^m$ ... Note: if $AB = BA$, then we can use the binomial theorem

$$= \sum_{m=0}^{\infty} \frac{1}{m!}\left(\sum_{k=0}^{m} \binom{m}{k} A^k B^{m-k}\right) = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} A^k B^{m-k}$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{1}{k!(m-k)!} A^k B^{m-k}$$

$e^A e^B = \left(\sum_{i=0}^{\infty} \frac{1}{i!} A^i\right)\left(\sum_{j=0}^{\infty} \frac{1}{j!} B^j\right)$ ... multiply out and collect "like" terms

$= \sum_{m=0}^{\infty} \sum_{i+j=m} \frac{1}{i!}\cdot\frac{1}{j!} A^i B^j$ ... note $j = m - i$

$$= \sum_{m=0}^{\infty} \sum_{i=0}^{m} \frac{1}{i!} \frac{1}{(m-i)!} A^i B^{m-i}$$

(2) $\Phi(t) = e^{tA}$

$$\Rightarrow \Phi'(t) = \frac{d}{dt} e^{tA} = \frac{d}{dt} \sum_{k=0}^{\infty} \frac{1}{k!} (tA)^k = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d}{dt} (t^k A^k)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d}{dt} (t^k) A^k = \sum_{k=0}^{\infty} \frac{1}{k!} k t^{k-1} A^k = \sum_{\underline{k=1}}^{\infty} \frac{1}{(k-1)!} t^{k-1} A^{k-1} A$$

$$= A \cdot \sum_{k=1}^{\infty} \frac{1}{(k-1)!} (tA)^{k-1} = A \cdot \sum_{k=0}^{\infty} \frac{1}{k!} (tA)^k = A \cdot e^{tA} = A \Phi(t)$$

(3) We have $(e^{tA})' = A e^{tA}$. Recall that differentiation of matrix / vector functions is done element-wise.

$$e^{tA} = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \Rightarrow (e^{tA})' = \begin{pmatrix} | & & | \\ v_1' & \cdots & v_n' \\ | & & | \end{pmatrix}$$

- j^{th} column of $(e^{tA})' = v_j'$
- j^{th} column of $A e^{tA} = A v_j$

$\left.\right\}$ since $(e^{tA})' = A e^{tA} \Rightarrow v_j' = A v_j$

$\Rightarrow v_j$ solves $\tilde{x}' = Ax$

Next we will show that

i) $\{v_1, v_2, \ldots v_n\}$ span the solution space $U := \{x \mid \tilde{x}' = Ax\}$

ii) $\{v_1, v_2, \ldots v_n\}$ form a linearly independent set

$\rightarrow$ we will need a generalization of the E+U theorem

<u>Fact</u>: The n-initial value problem $\tilde{x}'(t) = A\tilde{x}(t)$, $\tilde{x}(t_0) = \tilde{x}_0 \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ has a unique solution. $\tilde{x}(t) : \mathbb{R} \to \mathbb{R}^n$.

☼ $t = 0$: $e^{0A} = I_n \Rightarrow v_i(0) = e^i$

$\Rightarrow$ ii) $\mathcal{B} := \{v_1, \ldots v_n\}$ are lin. ind because $\{v_1(0), \ldots v_n(0)\} = \{e^1, \ldots e^n\}$ is lin. ind

$\rightarrow$ to show i) consider n initial value problems

$$\tilde{x}' = A\tilde{x}, \quad \tilde{x}(0) = e^i \quad \text{for } i = 1, \ldots n$$

$\rightarrow$ by uniqueness $\exists$ n unique solutions $\tilde{x}_1 = v_1, \ldots \tilde{x}_n = v_n$

$\rightarrow$ now let $\tilde{x}(t)$ be a solution $\tilde{x}' = A\tilde{x}$ with initial condition $\tilde{x}(0) = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}$

↳ we want to express $\tilde{x}$ as a lin. comb. of $v_1, \ldots v_n$

$\Rightarrow$ consider $\tilde{y}(t) := x_1^0 v_1(t) + x_2^0 v_2(t) + \ldots + x_n^0 v_n(t)$

<u>note</u>: $\tilde{y}(t)$ is a solution of $\tilde{x}' = A\tilde{x}$, because of the superposition principle

☼ $\tilde{y}(0) = x_1^0 e^1 + x_2^0 e^2 + \ldots + x_n^0 e^n = \begin{bmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{bmatrix}$

$\Rightarrow \tilde{y}(0) = \tilde{x}(0)$ & E+U: the solution is unique $\Rightarrow \tilde{x}(t) = \tilde{y}(t)$

$\Rightarrow \{v_1, \ldots v_n\}$ spans U & its independent $\Rightarrow \dim(U) = n$. ■

Ex:

① Compute the matrix exponential e^{tA} of $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

$$p_A(t) = (4-t)(3-t) - 2 = t^2 - 7t + 10 = (t-2)(t-5)$$

$$\Rightarrow \lambda_1 = 2: \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\Rightarrow \lambda_2 = 5: \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left.\right\} A = \underbrace{\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}}_{\overset{!!}{R}} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}^{-1}}_{\overset{\shortparallel}{\frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}}}$$

$$\Rightarrow e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} R D^k R^{-1} = R \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} D^k \right) R^{-1} = R e^{tD} R^{-1}$$

$$\Rightarrow e^{tD} = \sum_{k=0}^{\infty} \frac{t^k}{k!} D^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{bmatrix} 2^k & 0 \\ 0 & 5^k \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(5t)^k}{k!} \end{bmatrix} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}$$

$$\Rightarrow e^{tA} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} e^{2t} & e^{5t} \\ -2e^{2t} & e^{5t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} e^{2t} + 2e^{5t} & -e^{2t} + e^{5t} \\ -2e^{2t} + 2e^{5t} & 2e^{2t} + e^{5t} \end{bmatrix}$$

note: The columns of e^{tA} are solutions of $\tilde{x}' = A\tilde{x}$

column 1: $\frac{1}{3}(e^{2t} v_1 + 2e^{5t} v_2)$
column 2: $\frac{1}{3}(-e^{2t} v_1 + e^{5t} v_2)$
} they are lin. ind. and form a basis for the set of all solutions to $\tilde{x}' = A\tilde{x}$

If A is a diagonal matrix

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \Longrightarrow e^{tA} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$$

↳ take the sum the inside of the matrix and use $\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} = e^{\lambda t}$

Fact: Suppose J_λ is a Jordan block. Then

$$J_\lambda = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix} \Rightarrow J_\lambda^k = \begin{bmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \cdots & \binom{k}{j}\lambda^{k-j} \\ & \lambda^k & \ddots & \\ & & \lambda^k & \binom{k}{1}\lambda^{k-1} \\ & & & \lambda^k \end{bmatrix}$$

→ on the diagonal with distance j from the main diagonal are elements $\binom{k}{j}\lambda^{k-j}$

↳ note: $\binom{k}{j} = 0$ for $j > k$

② Find e^{tA} and hence solve $\tilde{x}' = A\tilde{x}$, where $A = \begin{bmatrix} 4 & 1 \\ -4 & 0 \end{bmatrix}$

$p_A(t) = (4-t)(-t)+4 = t^2 - 4t + 4 = (t-2)^2 \Rightarrow \lambda = 2$

$\Rightarrow \lambda = 2$ is the only eigenvalue. What is its geometric multiplicity?

$Av = \lambda v \Rightarrow Av - \lambda v = (A - \lambda I)v = 0 \Rightarrow v \in N(A - \lambda I)$

• $N(A-2I)$: $\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow N(A-2I) = \left\{ \alpha \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \ \alpha \in \mathbb{R} \right\}$

$\Rightarrow \dim(N(A-2I)) = 1 \Rightarrow \text{Geom}(\lambda) = 1 \Rightarrow$ we will need Jordan normal form

→ recall that $\dim(N(A-2I)^2) = \dim(N(A-2I)) + 1 = 2$

↳ this means $N(A-2I)^2 = \mathbb{R}^2$

$\Rightarrow$ to start the Jordan chain we need $v_2 \in N(A-2I)^2 - N(A-2I)$

↳ pick $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ← generalized eigenvector of rank 2

$\Rightarrow v_1 = (A-2I)v_2 = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

$\Rightarrow$ form a basis from the Jordan chain: $\mathcal{F} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right\}$

↳ $A = \underbrace{\begin{bmatrix} 2 & 1 \\ -4 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}}_{J} \cdot \underbrace{\begin{bmatrix} 2 & 1 \\ -4 & 0 \end{bmatrix}^{-1}}_{R^{-1}}$, $\quad \begin{bmatrix} 2 & 1 \\ -4 & 0 \end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix} 0 & -1 \\ 4 & 2 \end{bmatrix}$

$$\Rightarrow e^{tJ} = \sum_{k=0}^{\infty} \frac{t^k}{k!} J^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{bmatrix} 2^k & \binom{k}{1} 2^{k-1} \\ 0 & 2^k \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} & \sum_{k=0}^{\infty} \frac{t^k}{k!} k 2^{k-1} \\ 0 & \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

Note: $\sum_{k=0}^{\infty} \frac{t^k}{k!} k 2^{k-1} = \sum_{k=1}^{\infty} \frac{t^k}{(k-1)!} 2^{k-1} = t \sum_{k=0}^{\infty} \frac{t^k}{k!} 2^k = t \cdot e^{2t}$

$$\Rightarrow e^{tA} = R e^{tJ} R^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 4 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2e^{2t} & 2te^{2t} + e^{2t} \\ -4e^{2t} & -4te^{2t} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 4 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8e^{2t} + 4te^{2t} & 4te^{2t} \\ -16te^{2t} & 4e^{2t} - 8te^{2t} \end{bmatrix} = \underline{\begin{bmatrix} 2e^{2t} + te^{2t} & te^{2t} \\ -4te^{2t} & e^{2t} - 2te^{2t} \end{bmatrix}}$$

$\Rightarrow$ solutions to $\tilde{x}' = Ax$: $\underline{\underline{x(t) = c_1 e^{2t}\begin{bmatrix} 2+t \\ -4t \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} t \\ 1-2t \end{bmatrix}}}$, $\quad c_1, c_2 \in \mathbb{R}$

If J is a Jordan block, then

$e^{tJ} = \sum_{k=0}^{\infty} \frac{t^k}{k!} J^k$ → looking at an entry in J^k of the form $\binom{k}{j}\lambda^{k-j}$

$$\sum_{k=0}^{\infty} \frac{t^k}{k!}\binom{k}{j}\lambda^{k-j} = \sum_{k=j}^{\infty} \frac{t^k}{k!}\binom{k}{j}\lambda^{k-j} = \sum_{k=j}^{\infty} \frac{t^k}{k!} \cdot \frac{k!}{j!(k-j)!}\lambda^{k-j}$$

$$= \frac{t^j}{j!}\sum_{k=j}^{\infty} \frac{t^{k-j}}{(k-j)!}\lambda^{k-j} = \underline{\frac{1}{j!} t^j e^{\lambda t}}$$

③ $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}$, find e^{tA} and solve $\tilde{x}' = A\tilde{x}$.

→ the only eigenvalue is 3

$\mathcal{N}(A-3I) : \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathcal{N}(A-3I) = \left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} ; x \in \mathbb{R} \right\}$ ... dim = 1

→ Jordan chain:

$\mathcal{N}_1$ ... $\dim(\mathcal{N}(A-3I)) = 1$

$\mathcal{N}_2$ ... $\dim(\mathcal{N}(A-3I)^2) = 2$

$\mathcal{N}_3$ ... $\dim(\mathcal{N}(A-3I)^3) = 3$

pick $v_3 \in \mathcal{N}_3 - \mathcal{N}_2$, $\mathcal{N}_3 = \mathbb{R}^3$

$\Rightarrow (A-3I)^2 = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathcal{N}_2 = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} ; x, y \in \mathbb{R} \right\}$

$\Rightarrow$ pick $v_3 = (0,0,1)^T \in \mathbb{R}^3 - \mathcal{N}_2$

$v_2 = (A-3I)v_3 = (2,4,0)^T$

$v_1 = (A-3I)v_2 = (4,0,0)^T$

$\Rightarrow A = RJR^{-1}$, $J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, $R = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R^{-1} = \begin{bmatrix} 1/4 & -1/8 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow e^{tA} = Re^{tJ}R^{-1} = R \begin{bmatrix} e^{3t} & te^{3t} & \frac{1}{2}t^2e^{3t} \\ 0 & e^{3t} & te^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix} R^{-1} = e^{3t} \begin{bmatrix} 4 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & -1/8 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= e^{3t} \begin{bmatrix} 4 & 4t+2 & 2t^2+2t \\ 0 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & -1/8 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\underline{e^{3t} \begin{bmatrix} 1 & t & 2t^2+2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}}}$$

Solutions to $\tilde{x}' = A\tilde{x}$: $\tilde{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 2t^2+2 \\ 4 \\ 1 \end{bmatrix}$

(4) $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$, find e^{tA} and solve $\tilde{x}' = A\tilde{x}$

$p_A(t) = (3-t)(-1-t) + 8 = t^2 - 2t + 5 \Rightarrow \lambda_{1,2} = \frac{2 \pm \sqrt{4-20}}{2} = \underline{\underline{1 \pm 2i}}$

$\rightarrow$ 2 distinct eigenvalues $\Rightarrow$ A is diagonalizable

$\lambda_1 = 1-2i : \begin{bmatrix} 2+2i & -2 \\ 4 & -2+2i \end{bmatrix} \sim \begin{bmatrix} 1+i & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

$\lambda_2 = 1+2i = \overline{\lambda_1} \Rightarrow v_2 = \overline{v_1} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$

$\Rightarrow A = RDR^{-1}$, $D = \begin{bmatrix} 1+2i & 0 \\ 0 & 1-2i \end{bmatrix}$, $R = \begin{bmatrix} 1 & 1 \\ 1-i & 1+i \end{bmatrix}$, $R^{-1} = \frac{1}{2i}\begin{bmatrix} 1+i & -1 \\ -1+i & 1 \end{bmatrix}$

$\Rightarrow e^{tA} = Re^{tD}R^{-1} = R\begin{bmatrix} e^{(1+2i)t} & 0 \\ 0 & e^{(1-2i)t} \end{bmatrix} R^{-1} = \begin{bmatrix} 1 & 1 \\ 1-i & 1+i \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} 1+i & -1 \\ -1+i & 1 \end{bmatrix} \cdot \frac{1}{2i}$

$= \frac{1}{2i}\begin{bmatrix} A & B \\ (1-i)A & (1+i)B \end{bmatrix}\begin{bmatrix} 1+i & -1 \\ -1+i & 1 \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} (1+i)A - (1-i)B & -A+B \\ 2A - 2B & (-1+i)A + (1+i)B \end{bmatrix}$

$= \frac{e^t}{2i}\begin{bmatrix} (1+i)e^{2it} - (1-i)e^{-2it} & -e^{2it} + e^{-2it} \\ 2e^{2it} - 2e^{-2it} & (-1+i)e^{2it} + (1+i)e^{-2it} \end{bmatrix}$

Note: $e^{2it} = \cos(2t) + i\sin(2t) = C + iS$

$e^{-2it} = \cos(2t) - i\sin(2t) = C - iS$

(1,1) entry $= C + iS + iC - S - C + iS + iC + S = 2i(S+C)$

(1,2) entry $= -C - iS + C - iS = -2iS$

(2,1) entry $= 2C + 2iS - 2C + 2iS = 4iS$

(2,2) entry $= -C - iS + iC - S + C - iS + iC + S = 2i(C-S)$

$\Rightarrow e^{tA} = \frac{e^t}{2i}\begin{bmatrix} 2i(S+C) & 4iS \\ -2iS & 2i(C-S) \end{bmatrix} = \underline{\underline{e^t\begin{bmatrix} \sin(2t) + \cos(2t) & 2\sin(2t) \\ -\sin(2t) & \cos(2t) - \sin(2t) \end{bmatrix}}}$

$\Rightarrow$ solutions of $\tilde{x}' = A\tilde{x}$: $x'(t) = C_1 e^t \begin{bmatrix} \sin(2t) + \cos(2t) \\ -\sin(2t) \end{bmatrix} + C_2 e^t \begin{bmatrix} 2\sin(2t) \\ \cos(2t) - \sin(2t) \end{bmatrix}$

Note: A is real $\Rightarrow$ we expect e^{tA} to be real as well

- **Inhomogeneous Linear Systems**

Theorem: Consider the inhomogeneous linear system

$$(*) \quad \tilde{x}'(t) = A(t)\,\tilde{x}(t) + \tilde{b}(t), \qquad A: \mathbb{R} \to \mathbb{R}^{n \times n}$$

and suppose $\tilde{b}: J \subseteq \mathbb{R} \to \mathbb{R}^n$ is continuous.
Then every solution of $(*)$ can be written in the form

$$\tilde{x}(t) = \tilde{x}_h(t) + \tilde{x}_p(t),$$

where $\tilde{x}_h(t) = \sum c_i \tilde{x}_i(t)$ is the general solution of $\tilde{x}' = A\tilde{x}$ and $\tilde{x}_p(t)$ is a particular solution of $(*)$.

This also means that the set of solutions of $\tilde{x}' = A\tilde{x} + \tilde{b}$ forms an affine vector space based on the space of solutions of $\tilde{x}' = A\tilde{x}$

Proof: Suppose $\phi(t) =$ general solution of $(*)$ and $v(t)$ is a particular solution of $(*)$.

$\to$ need to show $u(t) := \phi(t) - v(t) =$ general solution of $\tilde{x}' = Ax$

1) every solution of the ILS gives a solution to the HLS

$$u'(t) = \phi'(t) - v'(t) = A\phi(t) + b(t) - (Av(t) + b(t)) = A(\phi(t) - v(t)) = Au(t)$$

2) every solution of the HLS is covered

$\to$ suppose $u(t)$ solves $\tilde{x}' = A\tilde{x}$, we will show that $u(t) + v(t)$ solves $(*)$

$$[u+v]' = Au + Av + b = A(u+v) + b$$

$\blacksquare$

Method: Suppose $A \in \mathbb{R}^{n \times n}$ is diagonalizable and consider $\tilde{x}' = A\tilde{x} + \tilde{b}$

$\hookrightarrow \lambda_1, \dots \lambda_n$ eigenvalues
$\hookrightarrow v_1, \dots v_n$ eigenvectors $\quad \rightsquigarrow D = RAR^{-1}$

$\to$ denote $F := \{v_1, \dots v_n\}$ and note $x = [x]_E = {}_E[id]_F \cdot [x]_F = R \cdot [x]_F$

$\hookrightarrow \tilde{z} := [x]_F \Rightarrow \tilde{x} = R\tilde{z} \Rightarrow \tilde{z} = R^{-1}\tilde{x}$

$$\Rightarrow \tilde{z}' = R^{-1}\tilde{x}' = R^{-1}(A\tilde{x} + \tilde{b}) = R^{-1}(AR\tilde{z} + \tilde{b}) = D\tilde{z} + R^{-1}\tilde{b}$$

$\to$ meaning we need to solve the uncoupled (there is one unknown function per row) system $\underline{\tilde{z}' = D\tilde{z} + \tilde{h}}$, where $\tilde{h} := R^{-1}\tilde{b}$

$$z_j'(t) = \lambda_j z_j(t) + h_j(t) \quad \rightsquigarrow \mu = e^{-\lambda_j t}$$

$$\Rightarrow z_j(t) = e^{\lambda_j t} \int e^{-\lambda_j t} h_j(t)\,dt = e^{\lambda_j t} \int e^{-\lambda_j t} h_j(t)\,dt + c_j e^{\lambda_j t}, \quad c_j \in \mathbb{R}$$

$$\Rightarrow \tilde{z}(t) = \begin{bmatrix} e^{\lambda_1 t}\int e^{-\lambda_1 t} h_1(t)\,dt \\ \vdots \\ e^{\lambda_m t}\int e^{-\lambda_m t} h_m(t)\,dt \end{bmatrix} + \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_m e^{\lambda_m t} \end{bmatrix} \qquad \& \quad \tilde{x} = R\tilde{z} = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{bmatrix} \cdot \tilde{z}$$

$$\Rightarrow \tilde{x}(t) = \underbrace{\sum_{i=1}^{m} \left(e^{\lambda_i t} \int e^{-\lambda_i t} h_i(t)\,dt \right) v_i}_{\text{particular solution to } \circledast} + \underbrace{\sum_{i=1}^{m} c_i e^{\lambda_i t} v_i}_{\text{general sol. of } \tilde{x}' = A\tilde{x}}$$

<u>Ex</u>: $x'(t) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$

1, <u>general solution to the homogeneous system</u>

$$p_A(t) = (-2-t)(-2-t) - 1 = (2+t)^2 - 1 = t^2 + 4t + 3 = (t+1)(t+3)$$

$$\left.\begin{aligned} \lambda_1 = -1 &: \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \lambda_2 = -3 &: \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}\right\} \quad \underset{A}{\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}} = \underset{R}{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}} \underset{D}{\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{-1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow x_h(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

2, <u>change basis and find a particular solution of the inhomogeneous</u>

$$x = Rz \Rightarrow z = R^{-1}x \Rightarrow z' = R^{-1}x' = R^{-1}(Ax + b) = R^{-1}(ARz + b) = Dz + R^{-1}b$$

$$\Rightarrow z' = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} z + \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} = \begin{bmatrix} -z_1 + e^{-t} + \frac{3}{2}t \\ -3z_2 + e^{-t} - \frac{3}{2}t \end{bmatrix}$$

$\boxed{1}$ $z_1' = -z_1 + e^{-t} + \frac{3}{2}t \quad \rightsquigarrow \mu_1(t) = e^{\int dt} = e^t$

$$\Rightarrow z_1(t) = e^{-t}\int e^t\left(e^{-t} + \frac{3}{2}t\right)dt = e^{-t}\left(t + \frac{3}{2}(te^t - e^t)\right) = \underline{\underline{te^{-t} + \frac{3}{2}t - \frac{3}{2}}}$$

$\boxed{2}$ $z_2' = -3z_2 + e^{-t} - \frac{3}{2}t \quad \rightsquigarrow \mu_1(t) = e^{3t}$

$$\begin{array}{c|cc} & D & I \\ \hline + & -\frac{3}{2}t & e^{3t} \\ - & -\frac{3}{2} & \frac{1}{3}e^{3t} \\ + & 0 & \frac{1}{9}e^{3t} \end{array}$$

$$\Rightarrow z_2(t) = e^{-3t}\int e^{3t}\left(e^{-t} - \frac{3}{2}t\right)dt = e^{-3t}\int e^{2t} - \frac{3}{2}te^{3t}\,dt =$$

$$= e^{-3t}\left(\frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t}\right) = \underline{\underline{\frac{1}{2}e^{-t} - \frac{1}{2}t + \frac{1}{6}}}$$

$$\Rightarrow x(t) = x_h(t) + x_p(t) = x_h(t) + Rz(t) = x_h(t) + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} te^{-t} + \frac{3}{2}t - \frac{3}{2} \\ \frac{1}{2}e^{-t} - \frac{1}{2}t + \frac{1}{6} \end{bmatrix}$$

$$= x_h(t) + \begin{bmatrix} te^{-t} + \frac{3}{2}t - \frac{3}{2} + \frac{1}{2}e^{-t} - \frac{1}{2}t + \frac{1}{6} \\ te^{-t} + \frac{3}{2}t - \frac{3}{2} - \frac{1}{2}e^{-t} + \frac{1}{2}t - \frac{1}{6} \end{bmatrix}$$

$$= \underbrace{te^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 4 \\ 5 \end{bmatrix}}_{\text{particular}} + \underbrace{c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{gen. to hom.}}$$

→ <u>What if A is not diagonalizable?</u>

→ then use the Jordan normal form $A = RDR^{-1}$

→ after a change of basis we will get $\tilde{z}' = J\tilde{z} + R^{-1}\tilde{b}$

→ this system is not uncoupled since some rows of J have two non-zero entries, however we can solve for $z_1, \dots z_n$ consecutively, starting with z_n and working backwards

<u>Ex: Solve $\tilde{x}' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \tilde{x} + \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix}$, $t > 0$</u>

$p_A(t) = (4-t)(-4-t) + 16 = t^2 \Rightarrow \lambda = 0$ is the only eigenvalue

→ what is the geometric multiplicity of $\lambda = 0$?

$N(A - \lambda I)$: $\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow N(A - \lambda I) = \left\{ \alpha \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} ; \alpha \in \mathbb{R} \right\}$ ... dim = 1

$\Rightarrow$ geom$(\lambda) = 1 \Rightarrow$ we need a Jordan chain

$$-\frac{1}{2}\begin{bmatrix} -1 & 0 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 2 & -1 \end{bmatrix}$$

$N(A - \lambda I)^2 = \mathbb{R}^2 \Rightarrow$ pick $v_2 \in \mathbb{R}^2 - N(A - \lambda I) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$$\left.\begin{aligned} v_2 &= (0, -1)^T \\ v_1 &= (A - \lambda I) v_2 = A v_2 = (2, 4)^T \end{aligned}\right\} \underbrace{\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2 & 0 \\ 4 & -1 \end{bmatrix}}_{R} \cdot \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{J} \cdot \underbrace{\begin{bmatrix} 2 & 0 \\ 4 & -1 \end{bmatrix}^{-1}}_{R^{-1}}$$

$\Rightarrow x = Rz \quad \Rightarrow z = R^{-1}x$

$\Rightarrow z' = R^{-1}x' = R^{-1}(Ax + b) = R^{-1}(ARz + b) = Jz + R^{-1}b$

$$\Rightarrow \begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1/2 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix} = \begin{bmatrix} z_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/2\, t^{-3} \\ 2t^{-3} + t^{-2} \end{bmatrix}$$

$\boxed{1}$ $z_2' = 2t^{-3} + t^{-2} \Rightarrow z_2 = \int 2t^{-3} + t^{-2}\, dt = \underline{-t^{-2} - t^{-1} + C_2}$

$\boxed{2}$ $z_1' = z_2 + \frac{1}{2}t^{-3} = \frac{1}{2}t^{-3} - t^{-2} - t^{-1} + C_2$

$\Rightarrow z_1 = -\frac{1}{4}t^{-2} + t^{-1} - \ln|t| + C_2 t + C_1$, note $\ln|t| = \ln t$

$$\Rightarrow x(t) = Rz(t) = \begin{bmatrix} 2 & 0 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} t^{-1} - \frac{1}{4}t^{-2} - \ln t + C_2 t + C_1 \\ -t^{-1} - t^{-2} + C_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2t^{-1} - \frac{1}{2}t^{-2} - 2\ln t + 2C_2 t + 2C_1 \\ 4t^{-1} - t^{-2} - 4\ln t + 4C_2 t + 4C_1 + t^{-1} + t^{-2} - C_2 \end{bmatrix}$$

$$= \frac{1}{t}\begin{bmatrix} 2 \\ 5 \end{bmatrix} - \frac{1}{2t^2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\ln t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 2t \\ 4t - 1 \end{bmatrix} + 2C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} , \quad C_1, C_2 \in \mathbb{R}$$

• Variation of parameters

Def: We say that $\phi(t): \mathbb{R} \to \mathbb{R}^{n\times n}$ is a fundamental matrix for the linear system

$$\tilde{x}'(t) = A(t)\tilde{x}(t) \quad , \quad A: \mathbb{R} \to \mathbb{R}^{n\times n}$$

$\equiv$ the columns of $\phi(t)$ form a basis for the set of solutions.

Ex: When $A \in \mathbb{R}^{n\times n}$ then e^{tA} is a fundamental matrix for $\tilde{x}'(t) = A\tilde{x}(t)$.

Let $\phi(t)$ be a fundamental matrix for $\tilde{x}'(t) = A(t)\tilde{x}(t)$

1. $\forall$ solution = lin. comb. of colums of $\phi(t)$ $\Rightarrow$ $\tilde{x}(t) = \phi(t)\tilde{c}$, $c \in \mathbb{R}^n$
2. $\phi(t)$ is a matrix solution to the system i.e. $\phi'(t) = A(t)\phi(t)$

Method: Consider ⊛ $\tilde{x}'(t) = A(t)\tilde{x}(t) + \tilde{b}(t)$, where $A: J \subseteq \mathbb{R} \to \mathbb{R}^{n\times n}$, $b: J \subseteq \mathbb{R} \to \mathbb{R}^n$ are continuous on J. Let $\phi(t)$ be a fundamental matrix of the corresponding homogeneous system $\tilde{x}'(t) = A(t)\tilde{x}(t)$.

Note: $\phi(t)\tilde{c}$, $\tilde{c} \in \mathbb{R}^n$ gives the general solution of $\tilde{x}'(t) = A(t)\tilde{x}(t)$

$\Rightarrow$ we seek a solution of ⊛ of the form $\tilde{x}(t) = \phi(t)\tilde{u}(t)$, $\tilde{u}: J \subseteq \mathbb{R} \to \mathbb{R}^n$.

• $\tilde{x}(t) = \phi(t)\tilde{u}(t) \Rightarrow \tilde{x}'(t) = \phi'(t)\tilde{u}(t) + \phi(t)\tilde{u}'(t)$

↳ does this work?

• i^{th} element of $\phi(t)\tilde{u}(t) = \sum_{k=1}^{n} \phi_{ik} u_k$

$$\Rightarrow \frac{d}{dt} = \sum_{k=1}^{n} \phi_{ik}' u_k + \phi_{ik} u_k' = i^{th} \text{ element of } \phi' \cdot \tilde{u} + i^{th} \text{ element of } \phi \cdot \tilde{u}'$$

• since $\tilde{x}(t)$ solves ⊛, we have $\tilde{x}'(t) = A(t)\tilde{x}(t) + \tilde{b}(t)$

$$\Rightarrow A\tilde{x} + \tilde{b} = \phi'\tilde{u} + \phi\tilde{u}' \Rightarrow A\phi\tilde{u} + \tilde{b} = A\phi\tilde{u} + \phi\tilde{u}' \Rightarrow \tilde{b}(t) = \phi(t)\tilde{u}'(t)$$

Note: The columns of $\phi(t)$ are lin. ind. on J $\Rightarrow$ $\phi(t)$ is invertible

$$\tilde{u}'(t) = \phi^{-1}(t) b(t) \implies \tilde{u}(t) = \int \phi^{-1}(t)\tilde{b}(t)\, dt$$

$$\Rightarrow \tilde{x} = \phi\tilde{u} \Rightarrow \tilde{x}(t) = \underbrace{\phi(t)\cdot\tilde{c}}_{\substack{\text{gen. sol.} \\ \text{of } \tilde{x}' = A\tilde{x}}} + \underbrace{\phi(t)\int \phi^{-1}(t)\tilde{b}(t)\, dt}_{\text{particular sol. of ⊛}} \quad , \quad c \in \mathbb{R}^n$$

When to use this?

→ if $A \in \mathbb{R}^{n\times n}$, i.e. it has constant entries, then we can use change of basis and diagonalization to solve the inhomogeneous system

→ however if A does have non-constant entries, then we cant do that ::

Ex:

① Use the method of variation of parameters to solve $\tilde{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \tilde{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$

→ earlier we have shown that $\phi(t) = \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix}$

⇒ let's assume that we can write $x(t) = \phi(t)\, u(t)$

$\Rightarrow u(t) = \int \phi^{-1}(t) f(t)\, dt \quad \dots \quad \phi(t) = \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix} \Rightarrow \phi^{-1}(t) = \frac{-1}{2e^{-4t}} \begin{bmatrix} -e^{-3t} & -e^{-3t} \\ -e^{-t} & e^{-t} \end{bmatrix}$

$$\Rightarrow u(t) = \frac{1}{2} \int e^{4t} \begin{bmatrix} e^{-3t} & e^{-3t} \\ e^{-t} & -e^{-t} \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} dt = \frac{1}{2} \int \begin{bmatrix} e^{t} & e^{t} \\ e^{3t} & -e^{3t} \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix} dt = \frac{1}{2} \int \begin{bmatrix} 2 + 3te^{t} \\ 2e^{2t} - 3te^{3t} \end{bmatrix} dt$$

$$= \frac{1}{2} \begin{bmatrix} 2t + 3(te^{t} - e^{t}) + c_1 \\ e^{2t} - 3\left(\frac{1}{3}te^{3t} - \frac{1}{9}e^{3t}\right) + c_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2t + 3te^{t} - 3e^{t} \\ e^{2t} - te^{3t} + \frac{1}{3}e^{3t} \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$\Rightarrow x(t) = \phi(t)\, u(t) = \frac{1}{2} \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix} \begin{bmatrix} 2t + 3te^{t} - 3e^{t} \\ e^{2t} - te^{3t} + \frac{1}{3}e^{3t} \end{bmatrix} + \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \quad d_1, d_2 \in \mathbb{R}$$

② $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} t^{-1}$ is the gen sol of $t\tilde{x}'(t) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \tilde{x}(t)$, $t > 0$

↳ find the gen. sol. of $t\tilde{x}'(t) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} 1 - t^2 \\ 2t \end{bmatrix}$

$\rightarrow \phi(t) = \begin{bmatrix} t & t^{-1} \\ t & 3t^{-1} \end{bmatrix} \quad \Rightarrow \phi^{-1}(t) = \frac{1}{2} \begin{bmatrix} 3t^{-1} & -t^{-1} \\ -t & t \end{bmatrix}$ $\quad / :t$

$$\Rightarrow \text{let } \tilde{u}(t) = \int \phi^{-1}(t) f(t)\, dt = \frac{1}{2} \int \begin{bmatrix} 3t^{-1} & -t^{-1} \\ -t & t \end{bmatrix} \begin{bmatrix} t^{-1} - t \\ 2 \end{bmatrix} dt =$$

$$= \frac{1}{2} \int \begin{bmatrix} 3t^{-2} - 3 - 2t^{-1} \\ t^2 - 1 + 2t \end{bmatrix} dt = \frac{1}{2} \begin{bmatrix} -3t^{-1} - 3t - 2\ln t \\ \frac{1}{3}t^3 - t + t^2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \tilde{x}(t) = \phi(t)\tilde{u}(t) = \frac{1}{2} \begin{bmatrix} t & t^{-1} \\ t & 3t^{-1} \end{bmatrix} \begin{bmatrix} -3t^{-1} - 3t - 2\ln t \\ \frac{1}{3}t^3 - t + t^2 \end{bmatrix} + \begin{bmatrix} t & t^{-1} \\ t & 3t^{-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -3 - 3t^2 - 2t\ln t + \frac{1}{3}t^2 - 1 + t \\ -3 - 3t^2 - 2t\ln t + t^2 - 3 + 3t \end{bmatrix} + c_1 \begin{bmatrix} t \\ t \end{bmatrix} + c_2 \begin{bmatrix} t^{-1} \\ 3t^{-1} \end{bmatrix}$$

$$= -\frac{1}{3}t^2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \frac{1}{2}t \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} - t\ln t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_1 t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 t^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

- ## Higher order linear differential equations

- we know: n^{th} order linear ODE $\rightsquigarrow$ n first order linear ODEs
  ↳ valid strategy but not very efficient

Theorem (E+U): Consider the IVP

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t), \quad y(t_0) = y_0 \quad \& \quad y'(t_0) = y_0',$$

where $p, q, g : J \subseteq \mathbb{R} \to \mathbb{R}$ are continuous on $J \ni t_0$.
Then there exists a unique solution $\phi : J \to \mathbb{R}$.

→ **Scalar coefficients**

Method: Consider $\underline{ay'' + by' + cy = 0}$, where $a, b, c \in \mathbb{R}$.

↳ we look for solutions of the form $y(t) = e^{rt}$

$$\left.\begin{array}{l} y = e^{rt} \\ y' = re^{rt} \\ y'' = r^2 e^{rt} \end{array}\right\} \quad e^{rt} \text{ is a solution} \iff ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

$$\iff \underline{ar^2 + b \cdot r + c = 0}$$

↳ characteristic equation

- There are 3 cases

i) 2 distinct real roots - r_1, r_2
ii) 1 repeating real root - r
iii) 2 conjugate complex roots - $\lambda \pm i\mu$

Method: We will show that the following are general solutions:

i) $\underline{y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$

ii) $\underline{y(t) = (c_1 + c_2 t)e^{rt}}$

iii) $\underline{y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)}$

↳ complex: $y(t) = e^{(\lambda + i\mu)t} = e^{\lambda t}(\cos(\mu t) + i\sin(\mu t))$

→ we will later generalize this for n^{th} order linear equations

Def: The <u>Wronskian determinant</u> of $y_1, y_2 : \mathbb{R} \to \mathbb{R}$ is the function

$$W(y_1, y_2)(t) := \det \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} = y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t)$$

Theorem: Consider ⊛ $y'' + p y' + q y = 0$, where $p, q : J \subseteq \mathbb{R} \to \mathbb{R}$ are continuous on J.

↳ $\{y_1, y_2\}$ form a fundamental set of solutions to ⊛

$\iff \exists\, t_0 \in J : W(y_1, y_2)(t_0) \neq 0$

Proof: $\Rightarrow$: holds, but it isnt important to us

$\Leftarrow$: suppose $\phi(t)$ is a solution → we will show $\phi(t) = c_1 y_1 + c_2 y_2$

Let t_0 be the point where $W(y_1, y_2) \neq 0$

↳ denote $y_0 := \phi(t_0)$, $y_0' := \phi'(t_0)$ and consider the IVP:

$$y'' + p y' + q y = 0, \quad y(t_0) = y_0 \quad \& \quad y'(t_0) = y_0'$$

ϕ solves this IVP

→ if we find c_1, c_2 s.t. $c_1 y_1 + c_2 y_2$ also solves it, then by uniqueness $c_1 y_1 + c_2 y_2 = \phi$.

→ <u>need</u>: $c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$
$c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$ $\rightsquigarrow \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$

↳ $\det \neq 0 \Rightarrow$ nonsingular $\Rightarrow c_1, c_2$ exist

∎

• <u>Case (i)</u>: r_1, r_2 real distinct roots

<u>claim</u>: All solutions to $ay'' + by' + cy = 0$ are given by $c_1 e^{r_1 t} + c_2 e^{r_2 t}$

↳ $W(e^{r_1 t}, e^{r_2 t}) = \det \begin{bmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{bmatrix} = r_2 e^{(r_1 + r_2)t} - r_1 e^{(r_1 + r_2)t}$

↳ $W(\dots) \neq 0$ because $r_1 \neq r_2$

<u>Ex</u>:

① <u>Find the general solution of $y'' + 5y' + 6y = 0$</u>

<u>char eq.</u> $r^2 + 5r + 6 = 0 \Rightarrow r_1 = -2,\ r_2 = -3$

$\Rightarrow$ <u>$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$</u>, $c_1, c_2 \in \mathbb{R}$

② <u>Solve the IVP $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 3$</u>

↳ $y = c_1 e^{-2t} + c_2 e^{-3t}$ $\Rightarrow y(0) = 2 = c_1 + c_2$
$y' = -2c_1 e^{-2t} - 3c_2 e^{-3t}$ $\Rightarrow y'(0) = 3 = -2c_1 - 3c_2$ $\Big\}$ $7 = -c_2 \Rightarrow c_2 = -7 \Rightarrow c_1 = 9$

$\Rightarrow$ <u>$y(t) = 9e^{-2t} - 7e^{-3t}$</u>

• Case (iii): $\lambda \pm i\mu$ complex conjugate roots

↳ solutions to ⊛ $ay'' + by' + cy = 0$ are given by $e^{(\lambda \pm i\mu)t}$

we again care only about one of the roots → $\lambda + i\mu$

$\Rightarrow y(t) := e^{(\lambda + i\mu)t} = e^{\lambda t}(\cos \mu t + i \sin \mu t)$

$y = u + iv$ is a complex solution of ⊛ $\Rightarrow u, v$ are real solutions of ⊛

$$(u+iv)'' + p(u+iv)' + q(u+iv) = 0$$
$$u'' + iv'' + pu' + piv' + qu + qiv = 0$$
$$(\underline{u'' + pu' + qu}) + i(\underline{v'' + pv' + qv}) = \underline{0} + \underline{0}i$$

$\Rightarrow$ real-valued solutions of ⊛ are $\{e^{\lambda t}\cos(\mu t),\ e^{\lambda t}\sin(\mu t)\}$

Lemma: These form a fundamental set of solutions

Pf: $$W(e^{\lambda t}\cos \mu t,\ e^{\lambda t}\sin \mu t) = \det \begin{bmatrix} e^{\lambda t}\cos(\mu t) & e^{\lambda t}\sin(\mu t) \\ \lambda e^{\lambda t}\cos(\mu t) - e^{\lambda t}\sin(\mu t)\mu, & \lambda e^{\lambda t}\sin(\mu t) + \mu e^{\lambda t}\cos(\mu t) \end{bmatrix}$$

$$t = 0:\ \det \begin{bmatrix} 1 & 0 \\ \lambda & \mu \end{bmatrix} = \mu \neq 0 \quad \checkmark$$

Ex: Solve $16y'' - 8y' + 145y = 0$, $y(0) = -2$, $y'(0) = 1$

char eq: $16r^2 - 8r + 145 = 0 \Rightarrow r_{1,2} = \frac{8 \pm \sqrt{64 - 64 \cdot 145}}{32} = \frac{1}{4}(1 \pm 12i) = \underline{\underline{\frac{1}{4} \pm 3i}}$

$\Rightarrow$ general solution: $\underline{\underline{y(t) = e^{t/4}(C_1\cos(3t) + C_2\sin(3t))}}$

↳ $y' = \frac{1}{4}e^{t/4}(C_1\cos(3t) + C_2\sin(3t)) + e^{t/4}(-3C_1\sin(3t) + 3C_2\cos(3t))$

$$y(0) = -2 = C_1 + 0 \Rightarrow C_1 = -2$$
$$y'(0) = 1 = \tfrac{1}{4}(C_1) + 3C_2 \Rightarrow 3C_2 - \tfrac{1}{2} = 1 \Rightarrow C_2 = \tfrac{1}{2}$$

$\Rightarrow \underline{\underline{y(t) = -2e^{t/4}\cos(3t) + \frac{1}{2}e^{t/4}\sin(3t)}}$

• Case (ii): r - one repeated real root

$ar^2 + br + c = 0$, suppose $b^2 - 4ac = 0 \Rightarrow r = -\frac{b}{2a}$

$\circledast$ $y(t) = e^{rt}$ is a solution $\qquad\qquad ay'' + by' + cy = 0$

→ lets assume $y(t) = v(t)\, e^{rt}$ is also a solution of $\circledast$ for some $v : \mathbb{R} \to \mathbb{R}$

$$y'(t) = v' e^{rt} + r v e^{rt} = e^{rt}(v' + rv)$$
$$y''(t) = e^{rt}(v'' + rv') + r e^{rt}(v' + rv) = e^{rt}(v'' + 2rv' + r^2 v)$$

→ subbing to $\circledast$:

$$a e^{rt}(v'' + 2rv' + r^2 v) + b e^{rt}(v' + rv) + c v e^{rt} = 0 \qquad \text{, recall } r = -\frac{b}{2a}$$
$$a\left(v'' - \frac{b}{a} v' + \frac{b^2}{4a^2} v\right) + b\left(v' - \frac{b}{2a} v\right) + cv = 0$$
$$a v'' + \frac{b^2}{4a} v - \frac{b^2}{2a} v + cv = 0$$
$$a v'' + v\left(c - \frac{b^2}{4a}\right) = a v'' + v \frac{\overbrace{4ac - b^2}^{0}}{4a} = 0$$
$$\Rightarrow a v'' = 0 \Rightarrow v'' = 0 \Rightarrow v(t) = C_1 + C_2 t$$

$\Rightarrow$ $y(t) = (C_1 + C_2 t)\, e^{rt}$ is a solution

Lemma: $\{e^{rt}, t e^{rt}\}$ form a fundamental set of solutions

Pf: $W = \det \begin{bmatrix} e^{rt} & t e^{rt} \\ r e^{rt} & e^{rt} + rt e^{rt} \end{bmatrix} \rightsquigarrow t = 0 : \det \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} = 1 \neq 0$

Examples:

① $y'' + 4y' + 4y = 0 \rightarrow r = -2 \Rightarrow y(t) = (C_1 + C_2 t)\, e^{-2t}$

② $9y'' - y = 0 \rightarrow 9r^2 - 1 = 0 \Rightarrow r = \pm\frac{1}{3} \Rightarrow y(t) = C_1 e^{t/3} + C_2 e^{-t/3}$

③ $9y'' + y = 0 \rightarrow 9r^2 + 1 = 0 \Rightarrow r = \pm i/3$

complex: $e^{(i/3)t} = \cos(t/3) + i \sin(t/3)$

real: $y(t) = C_1 \cos(t/3) + C_2 \sin(t/3)$

④ $9y'' - y' = 0 \rightarrow 9r^2 - r = 0 \Rightarrow r_1 = 0,\ r_2 = \frac{1}{9}$

$\Rightarrow y(t) = C_1 + C_2 e^{t/9}$

⑤ $y'' + 10y' + 25y = 0 \rightarrow r^2 + 10r + 25 = 0 \Rightarrow r = -5$

$\Rightarrow y(t) = (C_1 + C_2 t)\, e^{-5t}$

Ex:

① Find the general solution to $\tilde{x}'(t) = \begin{bmatrix}-4 & 2\\ 2 & -1\end{bmatrix}\tilde{x}(t) + \begin{bmatrix}t^{-1}\\ 2t^{-1}+4\end{bmatrix}$, $t > 0$

$$p_A(t) = (-4-t)(-1-t)-4 = t^2+5t = t(t+5)$$

$$-\frac{1}{5}\begin{bmatrix}-1 & -2\\ -2 & 1\end{bmatrix}$$

$$\lambda_1 = 0: \begin{bmatrix}-4 & 2\\ 2 & -1\end{bmatrix} \sim \begin{bmatrix}2 & -1\\ 0 & 0\end{bmatrix} \Rightarrow v_1 = \begin{pmatrix}1\\2\end{pmatrix}$$

$$\lambda_2 = -5: \begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix} \Rightarrow v_2 = \begin{pmatrix}2\\-1\end{pmatrix}$$

$$\Rightarrow \underset{A}{\begin{bmatrix}-4 & 2\\ 2 & -1\end{bmatrix}} = \underset{R}{\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}}\underset{D}{\begin{bmatrix}0 & 0\\ 0 & -5\end{bmatrix}}\underset{R^{-1}}{\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}^{-1}}$$

→ change of basis: $x = [x]_E = {}_E[id]_F \cdot [x]_F = Rz$

$$z = R^{-1}x \Rightarrow z' = R^{-1}x' = R^{-1}(Ax+b) = R^{-1}(ARz+b) = Dz + R^{-1}b$$

$$\Rightarrow \begin{bmatrix}z_1'\\ z_2'\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & -5\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}t^{-1}\\ 2t^{-1}+4\end{bmatrix} = \begin{bmatrix}0\\ -5z_2\end{bmatrix} + \frac{1}{5}\begin{bmatrix}t^{-1}+4t^{-1}+8\\ 2t^{-1}-2t^{-1}-4\end{bmatrix}$$

$$\boxed{1}: z_1' = t^{-1} + \frac{8}{5} \Rightarrow z_1 = \ln t + \frac{8}{5}t + C_1$$

$$\boxed{2}: z_2' = -5z_2 - \frac{4}{5} \Rightarrow \mu = e^{5t} \Rightarrow z_2 = e^{-5t}\int e^{5t}\left(-\frac{4}{5}\right)dt = e^{-5t}\left(-\frac{4}{25}e^{5t} + C_2\right)$$

$$\Rightarrow x = Rz = \begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}\ln t + \frac{8}{5}t + C_1\\ -4/25 + C_2e^{-5t}\end{bmatrix} = \begin{bmatrix}\ln t + \frac{8}{5}t + C_1 - 8/25 + 2C_2e^{-5t}\\ 2\ln t + \frac{16}{5}t + 2C_1 + 4/25 - C_2e^{-5t}\end{bmatrix}$$

$$x(t) = \ln t\begin{bmatrix}1\\2\end{bmatrix} + \frac{1}{5}t\begin{bmatrix}8\\16\end{bmatrix} + \frac{1}{25}\begin{bmatrix}-8\\4\end{bmatrix} + C_1\begin{bmatrix}1\\2\end{bmatrix} + C_2e^{-5t}\begin{bmatrix}2\\-1\end{bmatrix}$$

② $\tilde{x}'(t) = \begin{bmatrix}1 & 1\\ 4 & 1\end{bmatrix}\tilde{x}(t) + \begin{bmatrix}2\\-1\end{bmatrix}e^t$

$$p_A(t) = (1-t)^2 - 4 = t^2 - 2t - 3 = (t-3)(t+1)$$

$$\lambda_1 = 3: \begin{bmatrix}-2 & 1\\ 4 & -2\end{bmatrix} \sim \begin{bmatrix}2 & -1\\ 0 & 0\end{bmatrix} \Rightarrow v_1 = \begin{pmatrix}1\\2\end{pmatrix}$$

$$\lambda_2 = -1: \begin{bmatrix}2 & 1\\ 4 & 2\end{bmatrix} \sim \begin{bmatrix}2 & 1\\ 0 & 0\end{bmatrix} \Rightarrow v_2 = \begin{pmatrix}1\\-2\end{pmatrix}$$

$$\phi^{-1}(t) = \frac{1}{-4e^{2t}}\begin{bmatrix}-2e^{-t} & -e^{-t}\\ -2e^{3t} & e^{3t}\end{bmatrix} = \frac{1}{4}\begin{bmatrix}2e^{-3t} & e^{-3t}\\ 2e^{t} & -e^{t}\end{bmatrix}$$

→ variation of parameters: $\phi(t) = \begin{bmatrix}e^{3t} & e^{-t}\\ 2e^{3t} & -2e^{-t}\end{bmatrix}$ is a fundamental matrix for $x' = Ax$

↳ general solution $= \phi(t)\cdot u(t)$ & $b(t) = \phi(t)u'(t)$

$$\Rightarrow u(t) = \int \phi^{-1}(t)b(t)\,dt = \frac{1}{4}\int\begin{bmatrix}2e^{-3t} & e^{-3t}\\ 2e^{t} & -e^{t}\end{bmatrix}\begin{bmatrix}2e^t\\ -e^t\end{bmatrix}dt = \frac{1}{4}\int\begin{bmatrix}4e^{-2t} - e^{-2t}\\ 4e^{2t} + e^{2t}\end{bmatrix}dt$$

$$= \frac{1}{4}\int\begin{bmatrix}3e^{-2t}\\ 5e^{2t}\end{bmatrix}dt = \frac{1}{4}\begin{bmatrix}-3/2\, e^{-2t}\\ 5/2\, e^{2t}\end{bmatrix} + \begin{bmatrix}c_1\\ c_2\end{bmatrix}$$

$$\Rightarrow x(t) = \phi(t)u(t) = \frac{1}{8}\begin{bmatrix}e^{3t} & e^{-t}\\ 2e^{3t} & -2e^{-t}\end{bmatrix}\begin{bmatrix}-3e^{-2t}\\ 5e^{2t}\end{bmatrix} + \phi(t)\begin{bmatrix}c_1\\ c_2\end{bmatrix} =$$

$$= \frac{1}{8}\begin{bmatrix}-3e^t + 5e^t\\ -6e^t - 10e^t\end{bmatrix} + \phi(t)\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \frac{1}{8}\begin{bmatrix}2e^t\\ -16e^t\end{bmatrix} + \phi(t)\begin{bmatrix}c_1\\ c_2\end{bmatrix}$$

$$= \frac{1}{4}e^t\begin{bmatrix}1\\-8\end{bmatrix} + c_1e^{3t}\begin{bmatrix}1\\2\end{bmatrix} + c_2e^{-t}\begin{bmatrix}1\\-2\end{bmatrix}$$

• n^{th} order linear differential equations

→ lets generalize what we just did for second order ODEs

⊛ $y^{(n)}(t) + p_{n-1}(t)\,y^{(n-1)}(t) + \dots + p_1(t)\,y'(t) + p_0(t)\,y(t) = g(t)$

Theorem (E+U): Suppose $p_i, g: J \subseteq \mathbb{R} \to \mathbb{R}$ are continuous on J, then there is a unique solution to the IVP ⊛ ∧ $\forall i: y^{(i)}(t_0) = y_0^i$, defined on J.

👁 If $y_1, \dots y_n$ are solutions of the homogeneous equation $\sum_{k=0}^{n} p_k(t)\,y^{(k)}(t) = 0$; then $\sum_{i=0}^{n} c_i y_i(t)$ is also a solution for $\forall\, \tilde{c} \in \mathbb{R}^n$.

Def: The Wronskian determinant of the functions $y_1, \dots y_n : J \subseteq \mathbb{R} \to \mathbb{R}$ is

$$W(y_1, \dots y_n)(t) := \det \begin{bmatrix} y_1(t) & y_2(t) & \dots & y_n(t) \\ y_1'(t) & y_2'(t) & & y_n'(t) \\ & \vdots & & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & & y_n^{(n-1)}(t) \end{bmatrix}$$

Theorem: If $\{y_1, \dots y_n\}$ are solutions of $\sum_{k=0}^{n} p_k(t)\,y^{(k)}(t)$, then they form a fundamental set of solutions $\iff \exists t_0 \in J: W(y_1, \dots y_n)(t_0) \neq 0$.

Pf: ⇐: Consider the IVP $\sum_{k=0}^{n} p_k(t)\,y^{(k)}(t) = 0$, $\forall i: y^{(i)}(t_0) = y_0^i$, $t_0 \in J$.

↳ the solution can be written as a linear comb. of $y_1, \dots y_n \iff \exists \tilde{c} \in \mathbb{R}^n$:

$$\left.\begin{aligned} &\sum_i c_i y_i(t_0) = y_0 \\ &\quad\vdots \\ &\sum_i c_i y_i^{(n-1)}(t_0) = y_0^{n-1} \end{aligned}\right\} \iff W(y_1, \dots y_n)(t_0) \neq 0$$

→ then by uniqueness this is the only solution ∎

• Homogeneous equations with constant coefficients

Consider: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$, $\tilde{a} \in \mathbb{R}^n$

👁 e^{rt} is a solution $\iff$ r is a root of $\sum_i a_i r^i = 0$

Method:

i) $\exists\, n$ real distinct roots $r_1, \dots r_n \Rightarrow \{e^{r_1 t}, \dots e^{r_n t}\}$ = fundamental set of solutions

ii) $\exists$ complex roots. If $\lambda \pm i\mu$ are roots of the char. eq, then $e^{\lambda t}\cos(\mu t)$, $e^{\lambda t}\sin(\mu t)$ are real valued solutions of the ODE

iii) $\exists$ repeated roots. If r is a root with multiplicity k, then $(1 + t + t^2 + \dots + t^{k-1})e^{rt}$ gives k linearly independent solutions

↳ note e^{rt} can be broken down if r is complex

Case (i): n real distinct roots $r_1, \dots r_n$

$$W(e^{r_1 t}, \dots e^{r_n t}) = \det \begin{bmatrix} e^{r_1 t} & \dots & e^{r_n t} \\ r_1 e^{r_1 t} & & r_n e^{r_n t} \\ \vdots & \ddots & \vdots \\ r_1^{n-1} e^{r_1 t} & \dots & r_n^{n-1} e^{r_n t} \end{bmatrix} \Rightarrow W(0) = \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ r_1 & r_2 & & r_n \\ r_1^2 & r_2^2 & & r_n^2 \\ \vdots & \vdots & & \vdots \\ r_1^{n-1} & r_2^{n-1} & \dots & r_n^{n-1} \end{bmatrix}$$

→ when $t=0$ we get the determinant of the Vandermond matrix, which is $\neq 0 \iff$ all r_i are distinct

→ since $r_1, \dots r_n$ are distinct, $\{e^{r_1 t}, \dots e^{r_n t}\}$ form a fundamental set of sols.

Ex: $2y''' - 4y'' - 2y' + 4y = 0$

char eq: $2r^3 - 4r^2 - 2r + 4 = 0$

$r^3 - 2r^2 - r + 2 = 0$ → guess: $r = 1, -1, 2, -2$ ✓ ✓ ✓ ×

$\Rightarrow \underline{y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{2t}}$, $c_1, c_2, c_3 \in \mathbb{R}$

Case (ii): $\exists$ complex conjugate roots $\lambda \pm i\mu$

$\Rightarrow e^{(\lambda + i\mu)t} = e^{\lambda t}(\cos \mu t + i \sin \mu t)$ is a complex solution

→ $e^{\lambda t} \cos(\mu t)$ and $e^{\lambda t} \sin(\mu t)$ are real solutions

Ex: $y^{(4)} - y = 0$

char eq: $r^4 - 1 = 0 \Rightarrow (r^2-1)(r^2+1) = 0 \Rightarrow r = \pm 1, \pm i$

complex: $e^{it} = \cos t + i \sin t$

$\Rightarrow \underline{y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t}$

Case (iii): r is a repeated root with multiplicity k

→ it can be shown that $(c_0 + c_1 t + c_2 t^2 + \dots + c_{k-1} t^{k-1}) e^{rt}$ is a solution and that $t^j e^{rt}$ are independent

→ note: if $r = \lambda + i\mu$, then there are $2k$ real valued solutions

$$(c_0^1 + c_1^1 t + c_2^1 t^2 + \dots + c_{k-1}^1 t^{k-1}) e^{\lambda t} \cos(\mu t)$$
$$(c_0^2 + c_1^2 t + c_2^2 t^2 + \dots + c_{k-1}^2 t^{k-1}) e^{\lambda t} \sin(\mu t)$$

↳ because $\lambda - i\mu$ is also a root

Ex: $y^{(4)} + 2y'' + y = 0$

char eq: $r^4 + 2r^2 + 1 = 0 \Rightarrow (r^2+1)^2 \Rightarrow r = \pm i$ with multiplicity 2

complex: $(1+t)e^{it} = (1+t)(\cos t + i \sin t)$

real: $\underline{y(t) = (c_1 + c_2 t) \cos t + (c_3 + c_4 t) \sin t}$

Examples:

① $y''' + y'' + y' + y = 0$ $\quad \rightarrow r = -1, \pm i$

$$r^3 + r^2 + r + 1 = 0 \Rightarrow r(r^2+1) + r^2 + 1 = (r^2+1)(r+1) = 0$$

$$y(t) = c_1 e^{-t} + c_2 \cos(t) + c_3 \sin(t)$$

② $y^{(6)} - 3y^{(4)} + 3y'' - y = 0$

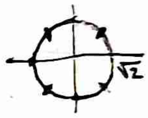
$$r^6 - 3r^4 + 3r^2 - 1 = 0 \Rightarrow (r^2-1)^3 = (r-1)^3(r+1)^3 \rightarrow r = \pm 1, \text{ multiplicity} = 3$$

$$y(t) = (c_1 + c_2 t + c_3 t^2) e^{t} + (c_4 + c_5 t + c_6 t^2) e^{-t}$$

③ $y^{(8)} + 8y^{(4)} + 16y = 0$

$r^8 + 8y^4 + 16 = (r^4+4)^2 \rightarrow r^4 = -4$, everything multiplicity 2

$$r = \sqrt{2} e^{i\theta} \rightarrow (e^{i\theta})^4 = -1 \Rightarrow e^{4i\theta} = e^{i(\pi + 2k\pi)}$$



$$\Rightarrow 4\theta = \pi + 2k\pi \Rightarrow \theta = \frac{\pi}{4} + k\cdot\frac{\pi}{2}, \quad k = 0,1,2,3$$

$$\Rightarrow r = 1 \pm i, \quad -1 \pm i$$

$$\Rightarrow y(t) = (c_1 + c_2 t) e^{t}\cos t + (c_3 + c_4 t) e^{t} \sin t + (c_5 + c_6 t) e^{-t} \cos t + (c_7 + c_8 t) e^{-t} \sin t$$

• Inhomogeneous linear equations

Method: Consider the equation

$$(*) \quad a_n(t) y^{(n)}(t) + a_{n-1}(t) y^{(n-1)}(t) + \cdots + a_1(t) y'(t) + a_0(t) y(t) = g(t)$$

The solution of $(*)$ can be written as $y(t) = y_h(t) + y_p(t)$, where

- $y_h(t)$ = general solution to the homogeneous equation
- $y_p(t)$ = particular solution of $(*)$

↳ check: $y(t) - y_p(t)$ is a solution of the homogeneous equation

⇒ we need a way of finding a particular solution

• **Undetermined Coeffients Method**

Method: Consider $a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \ldots + a_1 y'(t) + a_0 y(t) = g(t)$,

- $a_n \in \mathbb{R}^n$
- $g(t)$ = linear comb. of $t^j, t^j e^{\lambda t}, t^j e^{\lambda t} \sin(\mu t), t^j e^{\lambda t} \cos(\mu t)$

→ then the particular solution can be expressed as a lin. comb. of the following:

$\boxed{1}$ $g(t)$ contains a degree k polynomial $\Rightarrow \sum_{j=0}^{k} A_j t^j \in y_p$

$\boxed{2}$ $g(t)$ contains the term $t^k e^{\lambda t}$ $\Rightarrow e^{\lambda t} \sum_{j=0}^{k} A_j t^j \in y_p$

$\boxed{3}$ $g(t)$ contains $t^k e^{\lambda t} \sin(\mu t)$ or $t^k e^{\lambda t} \cos(\mu t)$, but not necessarily both

$$\Rightarrow e^{\lambda t} \sin(\mu t) \sum_{j=0}^{k} A_j t^j + e^{\lambda t} \cos(\mu t) \sum_{j=0}^{k} B_j t^j \in y_p$$

Furthermore, if any function in $y_p(t)$ repeats a solution of the homogeneous equation, then it needs to be multiplied by t repeatedly until it no longer contains terms which appear in the homogeneous solution.

Exercises:

① $y'' - 3y' + 2y = 2t + 5$

char eq: $r^2 - 3r + 2 = (r-2)(r-1) = 0 \Rightarrow y_h(t) = c_1 e^{2t} + c_2 e^t$

particular sol.:
$$\left.\begin{aligned} y_p(t) &= At + B \\ y_p'(t) &= A \\ y_p''(t) &= 0 \end{aligned}\right\} \begin{aligned} &-3A + 2At + 2B = 2t + 5 \\ &t: 2A = 2 \Rightarrow A = 1 \\ &1: -3A + 2B = 5 \Rightarrow -3 + 2B = 5 \Rightarrow B = 4 \end{aligned}$$

$$\Rightarrow y(t) = y_h(t) + y_p(t) = c_1 e^{2t} + c_2 e^t + t + 4$$

② $y'' + 5y' + 6y = \sin t$

char eq: $r^2 + 5r + 6 = (r+2)(r+3) = 0 \Rightarrow y_h(t) = c_1 e^{-2t} + c_2 e^{-3t}$

particular:
$$\left.\begin{aligned} y_p &= A\sin t + B\cos t \\ y_p' &= A\cos t - B\sin t \\ y_p'' &= -A\sin t - B\cos t \end{aligned}\right\} \begin{aligned} &-A\sin - B\cos + 5A\cos - 5B\sin + 6A\sin + 6B\cos = \sin \\ &\left.\begin{aligned} \sin: -5B + 5A &= 1 \\ \cos: 5B + 5A &= 0 \end{aligned}\right\} \begin{aligned} 10A = 1 &\Rightarrow A = \tfrac{1}{10} \\ &\Rightarrow B = -\tfrac{1}{10} \end{aligned} \end{aligned}$$

$$\Rightarrow y(t) = c_1 e^{-2t} + c_2 e^{-3t} + \tfrac{1}{10}\sin t - \tfrac{1}{10}\cos t$$

③ $y^{(4)} + 2y'' + y = 3\sin t - 5\cos t$

char eq: $r^4 + 2r^2 + 1 = (r^2+1)^2 \Rightarrow r = \pm i$, multiplicity $= 2$

$\hookrightarrow y_h(t) = (C_1 + C_2 t)\cos t + (C_3 + C_4 t)\sin t$

$\Rightarrow y_p(t) = \cancel{A\sin t + B\cos t}$ ↑ included

$= At^2 \sin t + Bt^2 \cos t = t^2(A\sin t + B\cos t)$

	D	D
0	t^2	$A\sin + B\cos$
1	$2t$	$A\cos - B\sin$
2	2	$-A\sin - B\cos$
3	0	$-A\cos + B\sin$
4	0	$A\sin + B\cos$

$y_p''(t) = -t^2(A\sin t + B\cos t) + 4t(A\cos t - B\sin t) + 2(A\cdot s + B\cdot c)$

$y_p^{(4)}(t) = t^2(A\sin t + B\cos t) + 8t(-A\cos t + B\sin t) - 12(A\cdot s + B\cdot c)$

$\Rightarrow y^{(4)} + 2y'' + y = 3\sin t - 5\cos t$

$t^2(A\cdot s + B\cdot c - 2A\cdot s - 2B\cdot c + A\cdot s + B\cdot c)$

$+t(-8A\cdot c + 8B\cdot s + 8A\cdot c - 8B\cdot s)$

$-12A\cdot s - 12B\cdot c + 4A\cdot s + 4B\cdot c = 3s - 5c$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$\Rightarrow s: -8A = 3 \Rightarrow A = -3/8$

$c: -8B = -5 \Rightarrow B = 5/8$

$\Rightarrow \underline{\underline{y(t) = (C_1 + C_2 t)\cos t + (C_3 + C_4 t)\sin t - \frac{3}{8}t^2 \sin t + \frac{5}{8}t^2 \cos t}}$

④ $y''' - 3y'' + 3y' - y = 4e^t$

$r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0 \Rightarrow y_h(t) = (C_1 + C_2 t + C_3 t^2)e^t$

$\Rightarrow y_p(t) = At^3 e^t$

$y_p'(t) = Ae^t(t^3 + 3t^2)$

$y_p''(t) = Ae^t(t^3 + 6t^2 + 6t)$

$y_p'''(t) = Ae^t(t^3 + 9t^2 + 18t + 6)$

	D	D
0:	t^3	e^t
1:	$3t^2$	e^t
2:	$6t$	e^t
3:	6	e^t

$e^t: A(t^3 + 9t^2 + 18t + 6 - 3t^3 - 18t^2 - 18t + 3t^3 + 9t^2 - t^3) = 4 \Rightarrow 6A = 4 \Rightarrow A = \frac{2}{3}$

$\Rightarrow \underline{\underline{y(t) = (C_1 + C_2 t + C_3 t^2)e^t + \frac{2}{3}t^3 e^t}}$

⑤ $y'' + y = 2\sin t + 4e^t$

$r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y_h(t) = C_1 \cos t + C_2 \sin t$

$\Rightarrow y_p(t) = At\cos t + Bt\sin t + Ce^t$

$y_p'(t) = A\cos t - At\sin t + B\sin t + Bt\cos t + Ce^t = A\cos t + B\sin t + t(B\cos t - A\sin t) + Ce^t$

$y_p''(t) = -A\sin t + B\cos t + B\cos t - A\sin t + t(-B\sin t - A\cos t) + Ce^t$

$\Rightarrow -2A\cdot s + 2B\cdot c - t(B\cdot s + A\cdot c) + C\cdot e^t + At\cdot c + Bt\cdot s + Ce^t = 2\cdot s + 4e^t$

$-2A\sin t + 2B\cos t + 2Ce^t = 2\sin t + 4e^t$

$\Rightarrow A = -1,\ B = 0,\ C = 2$

$\Rightarrow \underline{y(t) = C_1\cos t + C_2 \sin t - t\cos t + 2e^t}$

① $x'(t) = \begin{bmatrix}-6 & -10\\ 5 & 9\end{bmatrix} x(t) + \begin{bmatrix}8t+1\\ e^{-2t}\end{bmatrix}$

eigenvalues: $f_A(t) = (-6-t)(9-t)+50 = t^2-3t-4 \rightarrow \lambda_1=-1\ ,\ \lambda_2=4$

$$\lambda_1=-1: \begin{bmatrix}-5 & -10\\ 5 & 10\end{bmatrix} \sim \begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix} \Rightarrow v_1 = \begin{bmatrix}-2\\ 1\end{bmatrix}$$

$$\lambda_2=4: \begin{bmatrix}-10 & -10\\ 5 & 5\end{bmatrix} \sim \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} \Rightarrow v_2 = \begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\rightarrow B = \begin{bmatrix}-2 & 1\\ 1 & -1\end{bmatrix},\quad B^{-1} = \begin{bmatrix}-1 & -1\\ -1 & -2\end{bmatrix}$$

$$B^{-1} = -\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$$

$$\Rightarrow \text{sol. to hom: } c_1 e^{-t}\begin{bmatrix}-2\\ 1\end{bmatrix} + c_2 e^{4t}\begin{bmatrix}1\\ -1\end{bmatrix}$$

Let $z := Bx$

$$\Rightarrow z' = \begin{bmatrix}-1 & 0\\ 0 & 4\end{bmatrix} z + B^{-1}\begin{bmatrix}8t+1\\ e^{-2t}\end{bmatrix} \Rightarrow \begin{bmatrix}z_1'\\ z_2'\end{bmatrix} = \begin{bmatrix}-z_1\\ 4z_2\end{bmatrix} - \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}8t+1\\ e^{-2t}\end{bmatrix}$$

$\boxed{1}$ $z_1' = -z_1 - e^{-2t} - 8t - 1 \rightsquigarrow \mu_1(t) = e^{t}$

$\boxed{2}$ $z_2' = 4z_2 - 2e^{-2t} - 8t - 1 \rightsquigarrow \mu_2(t) = e^{-4t}$

$$\begin{array}{cc} D & I \\ +8t+1 & e^{t} \\ -8 & e^{t} \\ +0 & e^{t} \end{array}$$

$\boxed{1} \Rightarrow z_1(t) = e^{-t}\int e^{t}(-e^{-2t}-8t-1)\,dt = -e^{-t}\left(\int e^{-t}dt + \int e^{t}(8t+1)\,dt\right)$

$= -e^{-t}\left(-e^{-t} + (8t+1)e^{t} - 8e^{t}\right) + c_1 e^{-t}$

$= e^{-2t} - (8t-7) + c_1 e^{-t} = e^{-2t} - 8t + 7 + c_1 e^{-t}$

$\boxed{2} \Rightarrow z_2(t) = e^{4t}\int e^{-4t}(-2e^{-2t}-8t-1)\,dt = -e^{4t}\left(\int 2e^{-6t}dt + \int e^{-4t}(8t+1)\,dt\right)$

$= -e^{4t}\left(-\frac{1}{3}e^{-6t} - \frac{1}{4}(8t+1)e^{-4t} - \frac{1}{2}e^{-4t}\right) + c_2 e^{4t}$

$= \frac{1}{3}e^{-2t} + \frac{1}{4}(8t+1) + \frac{1}{2} + c_2 e^{4t}$

$= \frac{1}{3}e^{-2t} + 2t + \frac{3}{4} + c_2 e^{4t}$

$$\begin{array}{cc} D & I \\ +8t+1 & e^{-4t} \\ -8 & -\frac{1}{4}e^{-4t} \\ +0 & \frac{1}{16}e^{-4t} \end{array}$$

$$\Rightarrow x(t) = Bz(t) = \begin{bmatrix}-2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}e^{-2t} - 8t + 7 + c_1 e^{-t}\\ \frac{1}{3}e^{-2t} + 2t + \frac{3}{4} + c_2 e^{4t}\end{bmatrix}$$

$$= \begin{bmatrix}-2e^{-2t} + 16t - 14 - 2c_1 e^{-t} + \frac{1}{3}e^{-2t} + 2t + \frac{3}{4} + c_2 e^{4t}\\ e^{-2t} - 8t + 7 + c_1 e^{-t} - \frac{1}{3}e^{-2t} - 2t - \frac{3}{4} - c_2 e^{4t}\end{bmatrix}$$

$$= c_1 e^{-t}\begin{bmatrix}-2\\ 1\end{bmatrix} + c_2 e^{4t}\begin{bmatrix}1\\ -1\end{bmatrix} + \frac{1}{3}e^{-2t}\begin{bmatrix}-5\\ 2\end{bmatrix} + t\begin{bmatrix}18\\ -10\end{bmatrix} + \frac{1}{4}\begin{bmatrix}-53\\ 25\end{bmatrix},\quad c_1, c_2 \in \mathbb{R}$$

② a) $\underline{y''' + 24y' + 160y = 0}$

char eq: $r^3 + 24r + 160 = 0$ ... $r_1 = -4$ is a root

$(r+4)(r^2 - 4r + 40) = 0 \rightarrow r_{2,3} = \frac{4 \pm \sqrt{16-160}}{2} = 2 \pm 6i$

complex: $e^{(2+6i)t} = e^{2t} e^{i6t} = e^{2t}(\cos 6t + i \sin 6t)$

real: $\underline{\underline{y(t) = C_1 e^{-4t} + C_2 e^{2t}\cos(6t) + C_3 e^{2t}\sin(6t)}}$, $C_1, C_2, C_3 \in \mathbb{R}$

b) $\underline{y''' + 24y' + 160y = 72e^{-4t} + 320t^2}$

- $y_h(t) = C_1 e^{-4t} + C_2 e^{2t}\cos(6t) + C_3 e^{2t}\sin(6t)$
- $y_n(t) = At^2 + Bt + C + kte^{-4t}$

$y_n'(t) = 2At + B + ke^{-4t}(1 - 4t)$

$y_n''(t) = 2A +$ don't care

$y_n'''(t) = ke^{-4t}(-64t + 48)$

0:	t	e^{-4t}	1
1:	1	$-4e^{-4t}$	1 1
2:	0	$16e^{-4t}$	1 2 1
3:	0	$-64e^{-4t}$	1 3 3 1

$\Rightarrow ke^{-4t}(-64t+48) + 24(ke^{-4t}(1-4t) + 2At + B) + 160(kte^{-4t} + At^2 + Bt + C) = 72e^{-4t} + 320t^2$

constants: $24B + 160C = 0 \sim 3B + 20C = 0 \rightarrow -5 + 20C = 0 \Rightarrow C = \frac{1}{4}$

t : $48A + 160B = 0 \sim 3A + 10B = 0 \Big\} \; 6 + 10B = 0 \Rightarrow B = -\frac{10}{6} = -\frac{5}{3}$

t^2 : $160A = 320 \Rightarrow A = 2$

e^{-4t}: $48K + 24K = 72 \Rightarrow K = 1$

$\Rightarrow y_n(t) = te^{-4t} + 2t^2 - \frac{5}{3}t + \frac{1}{4}$

$\Rightarrow y(t) = y_h(t) + y_n(t) = \underline{\underline{C_1 e^{-4t} + C_2 e^{2t}\cos(6t) + C_3 e^{2t}\sin(6t) + te^{-4t} + 2t - \frac{5}{3}t + \frac{1}{4}}}$

③ $\underline{y^{(6)} - 3y^{(4)} + 3y'' - y = 0}$

$r^6 - 3y^4 + 3y^2 - 1 = 0 \rightarrow (r^2 - 1)^3 = 0 \Rightarrow (r-1)^3(r+1)^3 = 0$

$\Rightarrow r = \pm 1$, multiplicity $= 3$

$\Rightarrow \underline{\underline{y(t) = (C_1 + C_2 t + C_3 t^2)e^t + (C_4 + C_5 t + C_6 t^2)e^{-t}}}$

④ $\underline{2y^{(4)} - y''' - 9y'' + 4y' + 4y = 0, \; y(0) = -2, \; y'(0) = 0, \; y''(0) = -2, \; y'''(0) = 0}$

$2r^4 - r^3 - 9r^2 + 4r + 4 = 0$, guess: $1 ✓, -1 ✗, 2 ✓, -2 ✓$

$= (r-1)(r-2)(r+2)(2r+A) \rightarrow (-1)(-2)2A = 4 \Rightarrow A = 1$

$\Rightarrow$ roots: $1, 2, -2, -\frac{1}{2}$

$\Rightarrow y(t) = C_1 e^t + C_2 e^{2t} + C_3 e^{-2t} + C_4 e^{-\frac{1}{2}t}$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -2 & -\frac{1}{2} \\ 1 & 4 & 4 & \frac{1}{4} \\ 1 & 8 & -8 & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \end{bmatrix} \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -2 \\ 0 & 1 & -3 & -3/2 & 2 \\ 0 & 3 & 3 & -3/4 & 0 \\ 0 & 7 & -9 & -9/8 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -2 \\ 0 & 1 & -3 & -3/2 & 2 \\ 0 & 0 & 12 & 15/4 & -6 \\ 0 & 0 & 12 & 75/8 & -12 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -2 \\ 0 & 2 & -6 & -3 & 4 \\ 0 & 0 & 16 & 5 & -8 \\ 0 & 0 & 0 & 15 & -16 \end{array}\right] \rightarrow C_4 = -\frac{16}{15}$$

⊛ $16C_3 = -8 + \frac{16}{3}$

$\Rightarrow C_3 = -\frac{1}{2} + \frac{1}{3} = -\frac{1}{6}$

- $2C_2 = 4 - 1 - \frac{16}{5}$
- $C_2 = \frac{3}{2} - \frac{8}{5} = -\frac{1}{10}$
- $C_1 = -2 + \frac{1}{10} + \frac{1}{6} + \frac{16}{15}$

$= \frac{-60 + 3 + 5 + 32}{30} = -\frac{20}{30}$

$= -\frac{2}{3}$

$\Rightarrow \underline{\underline{y(t) = -\frac{2}{3}e^t - \frac{1}{10}e^{2t} - \frac{1}{6}e^{-2t} - \frac{16}{15}e^{-\frac{1}{2}t}}}$

⑥ $y'' - 2y' + y = 2e^t + 3t + 4$

$r^2 - 2r + 1 = (r-1)^2 \quad \Rightarrow \quad y_h(t) = (c_1 + tc_2)e^t$

$\Rightarrow y_p(t) = At^2e^t + Bt + C$

$y_p'(t) = At^2e^t + 2Ate^t + B$

$y_p''(t) = At^2e^t + 4Ate^t + 2Ae^t$

$\Rightarrow At^2e^t + 4Ate^t + 2Ae^t - 2At^2e^t - 4Ate^t - 2B + At^2e^t + Bt + C = 2e^t + 3t + 4$

$2Ae^t + Bt + C - 2B = 2e^t + 3t + 4 \quad \Rightarrow A = 1, \; B = 3, \; C = 10$

$\Rightarrow y(t) = (c_1 + c_2t)e^t + t^2e^t + 3t + 10$

• **Variation of Parameters Method**

– more general – doesn't require scalar coefficients or special form of $g(t)$

Method: Suppose y_1, y_2 are independent solutions of

$$y'' + py' + qy = 0, \quad \text{where } p, q \text{ are continuous.}$$

A particular solution of

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t)$$

is given by

$$y(t) = -y_1(t)\int \frac{g(t)}{W(t)} y_2(t)\,dt + y_2(t)\int \frac{g(t)}{W(t)} y_1(t)\,dt, \qquad W(t) := W(y_1, y_2)(t)$$

In general: Suppose $\{y_1, \dots y_n\}$ form a fundamental set of solutions for

$$y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \dots + p_1(t)y'(t) + p_0(t)y(t) = 0, \quad p_0, \dots p_{n-1} \text{ continuous}$$

We will abbreviate this equation as $P(t)Y(t) = 0$.

A particular solution of $P(t)Y(t) = g(t)$ is given by

$$y_p = \sum_{j=1}^{n} u_j y_j, \quad \text{where} \quad u_j(t) = (-1)^{n+j}\int \frac{g(t)}{W(t)} W^j(t)\,dt,$$

where $W(t) := W(y_1, \dots y_n)(t)$, $\quad W^j(t) := W(y_1, \dots, y_{j-1}, y_{j+1}, \dots y_n)(t)$

↳ y_j is missing

Note: For efficiency its best to compute $W^1, \dots W^n$ first and the comute W using Laplace expansion of the last row:

$$W = \sum_{j=1}^{n} (-1)^{n+j} y_j^{(n-1)} W^j$$

Proof: We seek a particular solution of $P(t)Y(t) = g(t)$ in the form

$$y_p = u_1 y_1 + u_2 y_2 + \dots + u_n y_n$$

→ the idea is to differentiate this n times and substitute into $PY = g$.

↳ this would get very messy ⇒ we impose some conditions:

$$(*) \quad \forall k \in \{0, \dots n-2\} : \sum_{j=1}^{n} u_j' y_j^{(k)} = u_1' y_1^{(k)} + u_2' y_2^{(k)} + \dots + u_n' y_n^{(k)} = 0$$

→ <u>differentiating</u>:

$$y_p' = \sum_{j=1}^{n} u_j' y_j + u_j y_j' = \overbrace{\sum_{j=1}^{n} u_j' y_j}^{0} + \sum_{j=0}^{n} u_j y_j' = \sum_{j=0}^{n} u_j y_j'$$

$$y_p'' = \sum_{j=1}^{n} u_j' y_j' + u_j y_j'' = \sum_{j=1}^{n} u_j y_j''$$

$$\Rightarrow \forall k \in \{0, \dots n-1\} : \quad y_p^{(k)} = \sum_{j=1}^{n} u_j y_j^{(k)} \quad \& \quad y_p^{(n)} = \sum_{j=1}^{n} \left(u_j' y_j^{(n-1)} + u_j y_j^{(n)}\right)$$

→ <u>subbing back in</u>:

<u>denote</u>: $Y_j(t) := (y_j^{(n)}, y_j^{(n-1)}, \dots y_j', y_j)$

<u>recall</u>: $Y(t) = (y^{(n)}, y^{(n-1)}, \dots y', y)$

<u>note</u>: $P(t) Y_j(t) = 0$ ... because y_j are solutions of $P(t)Y(t) = 0$

$$\Rightarrow PY = y_p^{(n)} + \sum_{k=0}^{n-1} p_k y_p^{(k)} = \sum_{j=1}^{n} u_j' y_j^{(n-1)} + \sum_{j=1}^{n} u_j y_j^{(n)} + \sum_{k=0}^{n-1} p_k \sum_{j=1}^{n} u_j y_j^{(k)}$$

$$= \sum_{j=1}^{n} u_j' y_j^{(n-1)} + \sum_{j=1}^{n} u_j \underbrace{\left(y_j^{(n)} + \sum_{k=0}^{n-1} p_k y_j^{(k)}\right)}_{0} = \sum_{j=1}^{n} u_j' y_j^{(n-1)}$$

$$\Rightarrow u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \dots + u_n' y_n^{(n-1)} = g$$

$$(*) : u_1' y_1^{(k)} + u_2' y_2^{(k)} + \dots + u_n' y_n^{(k)} = 0 \quad \dots \forall \in \{0, \dots n-2\}$$

$$\Rightarrow \underbrace{\begin{bmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & & & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix}}_{A} \cdot \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g \end{bmatrix}}_{v}$$

<u>note</u>: the determinant of this system is $W(y_1, \dots y_n)$

<u>notation</u>: $A^{i \to v}$ = i^{th} column of A was replaced by v

→ using Cramers rule: $u_j' = \dfrac{\det(A^{j \to v})}{W}$ → what is $\det(A^{j \to v})$?

↳ when we replace the j^{th} column of A with v, we will get a column full of zeros, except for the last element $= g$

⇒ calculate the determinant using Laplace expansion of this column

$$\det(A^{j \to v}) = (-1)^{n+j} g \cdot W^j, \quad \text{where } W^j = \text{Wronskian of } \{y_1, \dots y_n\} \setminus \{y_j\}$$

Examples:

① $y'' + y = \sec(t)$

$\hookrightarrow r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y_h(t) = C_1 \underset{y_1}{\cos t} + C_2 \underset{y_2}{\sin t}$

$$W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

$$\Rightarrow y_p(t) = -y_1 \int \frac{\sec(t)}{1} y_2 \, dt + y_2 \int \frac{\sec(t)}{1} y_1 \, dt$$

$u = \cos t$
$du = -\sin t \, dt$

$$= -\cos t \int \sec t \sin t \, dt + \sin t \int \sec t \cos t \, dt = -\cos t \int \frac{\sin t}{\cos t} dt + \sin t \int 1 \, dt$$

$$= \cos(t) \cdot \ln|\cos t| + \sin(t) \cdot t$$

$$\Rightarrow \underline{\underline{y(t) = C_1 \cos t + C_2 \sin t + \cos(t) \cdot \ln|\cos t| + t \sin t}}$$

② $y'' - 2y' + y = e^t \ln t$

$\hookrightarrow r^2 - 2r + 1 = (r-1)^2 \Rightarrow y_h(t) = (C_1 + C_2 t) \cdot e^t$

$$\rightarrow y_1 = e^t, \; y_2 = te^t \Rightarrow W = \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} = e^{2t}(t+1) - e^{2t} t = e^{2t}$$

$$\Rightarrow y_p(t) = -e^t \int \frac{e^t \ln t}{e^{2t}} te^t \, dt + te^t \int \frac{e^t \ln t}{e^{2t}} e^t \, dt$$

$$= -e^t \int t \ln t \, dt + te^t \int \ln t \, dt$$

	D	I		D	I
+	$\ln t$	t	+	$\ln t$	1
−	$1/t$	$\frac{1}{2}t^2$	−	$1/t$	t

$$= -e^t \left(\tfrac{1}{2}t^2 \ln t - \tfrac{1}{2}\int t \, dt\right) + te^t \left(t \ln t - \int 1 \, dt\right)$$

$$= -e^t \left(\tfrac{1}{2}t^2 \ln t - \tfrac{1}{4}t^2\right) + te^t (t \ln t - t)$$

$$= t^2 e^t \left(-\tfrac{1}{2}\ln t + \tfrac{1}{4} + \ln t - 1\right) = \tfrac{1}{4}t^2 e^t (2 \ln t - 3)$$

$$\Rightarrow \underline{\underline{y(t) = (C_1 + C_2 t)e^t + \tfrac{1}{4}t^2 e^t (2 \ln t - 3)}}$$

③ $y'' + 4y = \csc(2t)$ $\qquad e^{2it} = \cos(2t) + i \sin(2t)$

$\hookrightarrow r^2 + 4 = 0 \Rightarrow r = \pm 2i \Rightarrow y_h(t) = C_1 \cos(2t) + C_2 \sin(2t)$

$$\Rightarrow y_1 = \cos(2t), \; y_2 = \sin(2t) \Rightarrow W(y_1, y_2) = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} = 2C^2 + 2S^2 = 2$$

$$\Rightarrow y_p(t) = -\cos(2t) \int \frac{\csc(2t)}{2} \sin(2t) \, dt + \sin(2t) \int \frac{\csc(2t)}{2} \cos(2t) \, dt$$

$$= -\tfrac{1}{2}\cos(2t) \int 1 \, dt + \tfrac{1}{2}\sin(2t) \int \frac{\cos(2t)}{\sin(2t)} dt \qquad \Big| \; \begin{aligned} u &= \sin(2t) \\ du &= 2\cos(2t)\, dt \end{aligned}$$

$$= -\tfrac{1}{2}\cos(2t)(t + C) + \tfrac{1}{4}\sin(2t)\left(\ln|\sin(2t)| + D\right)$$

$$\Rightarrow \underline{\underline{y(t) = C_1 \sin(2t) + C_2 \cos(2t) + \tfrac{1}{4}\sin(2t) \cdot \ln|\sin(2t)| - \tfrac{1}{2}t \cos(2t)}}$$

(4) Solve $xy''' - y'' - xy' + y = 8x^2e^x$

↳ given: $y_1 = x$, $y_2 = e^x$, $y_3 = e^{-x}$ form a fundamental set of solutions

$$W(x) = \begin{vmatrix} x & e^x & e^{-x} \\ 1 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = 0\cdot W_1 - e^x W_2 + e^{-x} W_3 = x+1 + x-1 = 2x$$

$$W_1(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2, \quad W_2(x) = \begin{vmatrix} x & e^{-x} \\ 1 & -e^{-x} \end{vmatrix} = -e^{-x}(x+1), \quad W_3(x) = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = e^x(x-1)$$

$$\rightarrow g(x) = \frac{8x^2e^x}{x} = 8xe^x \Rightarrow \frac{g(x)}{W(x)} = \frac{8xe^x}{2x} = 4e^x$$

$$\Rightarrow u_1' = 4e^x \cdot W_1 = 4e^x(-2) \Rightarrow u_1 = -\int 8e^x dx = -8e^x$$

$$u_2' = -4e^x W_2 = -4e^x(-e^{-x}(x+1)) \Rightarrow u_2 = 4\int x+1\,dx = 2(x+1)^2$$

$$u_3' = 4e^x W_3 = 4e^x e^x(x-1) \Rightarrow u_3 = 4\int e^{2x}(x-1)dx = 4\left((x-1)\tfrac{1}{2}e^{2x} - \tfrac{1}{4}e^{2x}\right)$$

$$= e^{2x}(2x-2-1) = e^{2x}(2x-3)$$

$$\Rightarrow y_p(x) = u_1y_1 + u_2y_2 + u_3y_3$$

$$= -8e^x x + 2e^x(x+1)^2 + e^x(2x-3) = e^x(-8x + 2x^2 + 4x + 2 + 2x - 3)$$

$$= e^x(2x^2 - 2x - 1) \approx e^x(2x^2 - 2x) = 2xe^x(x-1)$$

↳ note: $-e^x$ is a solution of the homogenous equation

$$\Rightarrow \underline{y(x) = C_1x + C_2e^x + C_3e^{-x} + 2xe^x(x-1)}$$

(5) $y^{(4)} - 4y'' = t^2 + e^t$

$$r^4 - 4r^2 = r^2(r^2-4) = r^2(r-2)(r+2) \Rightarrow y_h(t) = c_1e^{2t} + c_2e^{-2t} + (c_3 + c_4t)\cancel{e^{0t}}$$

⇒ undetermined coefficients

$$y_p(t) = Ae^t + \underbrace{Bt^2 + Ct + D}_{\text{parts are included}} = Ae^t + t^2(Bt^2 + Ct + D) = Ae^t + Bt^4 + Ct^3 + Dt^2$$

$$y_p''(t) = Ae^t + 4\cdot3Bt^2 + 3\cdot2Ct + 2D$$

$$y_p^{(4)}(t) = Ae^t + 4\cdot3\cdot2B$$

$$\Rightarrow Ae^t + 24B - 4(Ae^t + 12Bt^2 + 6Ct + 2D) = t^2 + e^t$$

e^t: $A - 4A = 1 \Rightarrow A = -\frac{1}{3}$

1: $24B - 8D = 0 \Rightarrow 24(-\frac{1}{48}) = 8D \Rightarrow D = \frac{1}{8}(-\frac{1}{2}) = -\frac{1}{16}$

t: $-24C = 0 \Rightarrow C = 0$

t^2: $-48B = 1 \Rightarrow B = -\frac{1}{48}$

$$\Rightarrow \underline{y(t) = C_1e^{2t} + C_2e^{-2t} + C_3 + C_4t - \tfrac{1}{3}e^t - \tfrac{1}{16}t^2 - \tfrac{1}{48}t^4}$$

⑥ $y'' + 4y' + 4y = e^{-2t} t^{-2}, \quad t > 0$

$r^2 + 4r + 4 = (r+2)^2 \Rightarrow r = -2 \Rightarrow y_h(t) = (C_1 + C_2 t) e^{-2t}$

$\Rightarrow$ variation of parameters: $y_1 = e^{-2t}, \quad y_2 = t e^{-2t}$

$$\Rightarrow W(t) = \begin{vmatrix} e^{-2t} & t e^{-2t} \\ -2e^{-2t} & -2te^{-2t} + e^{-2t} \end{vmatrix} = e^{-2t} \cdot e^{-2t}(1-2t) + 2t e^{-4t} = e^{-4t}$$

$$\Rightarrow g(t)/W(t) = e^{-2t} t^{-2} \cdot e^{4t} = e^{2t}/t^2$$

$$\Rightarrow u_1' = -\frac{e^{2t}}{t^2} \cdot t e^{-2t} = -\frac{1}{t} \Rightarrow u_1 = -\ln|t| = -\ln t$$

$$u_2' = +\frac{e^{2t}}{t^2} \cdot e^{-2t} = \frac{1}{t^2} \Rightarrow u_2 = -\frac{1}{t}$$

$$\Rightarrow y_p(t) = u_1 y_1 + u_2 y_2 = -e^{-2t} \ln t - \frac{1}{t} t e^{-2t} = -e^{-2t}(1 + \ln t)$$

$$\Rightarrow y(t) = (C_1 + C_2 t) e^{-2t} - e^{-2t}(1 + \ln t) \approx \underline{\underline{(C_1 + C_2 t - \ln t) e^{-2t}}}$$

⑦ $y'' - 3y' + 2y = 2\cos t - 3t + 4e^{2t}$

homogeneous: $r^2 - 3r + 2 = (r-2)(r-1) \Rightarrow y_h(t) = C_1 e^t + C_2 e^{2t}$

undetermined coefficients:

$$y_p = A\sin t + B\cos t + Ct + D + Et e^{2t}$$

$$y_p' = A\cos t - B\sin t + C + E(e^{2t} + 2te^{2t}) \quad \ldots \; E \cdot e^{2t}(1+2t)$$

$$y_p'' = -A\sin t - B\cos t + E(4e^{2t} + 4te^{2t}) \quad \ldots \; 4E e^{2t}(1+t)$$

coefficients:

$\sin t: \; -A + 3B + 2A = 0$

$\cos t: \; -B - 3A + 2B = 2$

$\Rightarrow \left.\begin{matrix} A + 3B = 0 \\ -3A + B = 2 \end{matrix}\right\} \; 10B = 2 \Rightarrow B = \frac{1}{5} \Rightarrow A = -\frac{3}{5}$

$e^{2t}: \; 4E - 3E = 4 \Rightarrow E = 4$

$t^1: \; 2C = -3 \Rightarrow C = -\frac{3}{2}$

$t^0: \; -3C + 2D = 0 \Rightarrow 9/2 + 2D = 0 \Rightarrow D = -\frac{9}{4}$

$$\Rightarrow \underline{\underline{y(t) = C_1 e^t + C_2 e^{2t} + 4te^{2t} - \frac{3}{5}\sin t + \frac{1}{5}\cos t - \frac{3}{2}t - \frac{9}{4}}}$$

- **Autonomous Systems**

→ we have solved systems $\begin{aligned} x_1' &= f(t, x_1, x_2) \\ x_2' &= g(t, x_1, x_2) \end{aligned}$

Def: A system of (not necessarily linear) ODEs is said to be autonomous ≡ the functions x_i' don't explicitly depend on t.

↳ in this case, the system can be written as $\begin{aligned} x_1' &= f(x_1, x_2) \\ x_2' &= g(x_1, x_2) \end{aligned}$

$\tilde{x}' = A\tilde{x}$, where $A \in \mathbb{R}^{n\times n}$ is an autonomous system.

Theorem (E+U): If f, g are continuously differentiable then
→ for $\forall$ initial condition $\tilde{x}(t_0) = \tilde{x}_0$ $\exists!$ solution $(x_1(t), x_2(t))$

Remark: The solution $(x_1(t), x_2(t))$ forms a parametric curve in $\mathbb{R}^2$
↳ distinct curves do not intersect (by uniqueness)

Idea: Autonomous systems are easier to visualize
↳ we can trace the trajectory of $\tilde{x}(t)$ as t changes
↳ a picture of the trajectories is called the phase portrait of the system

→ there are 3 basic cases: $(A \in \mathbb{R}^{2\times 2})$

- A has 2 distinct eigenvalues λ_1, λ_2
  - $\lambda_1, \lambda_2 < 0 \Rightarrow$ nodal sink
  - $\lambda_1, \lambda_2 > 0 \Rightarrow$ nodal source
  - $\lambda_1 \cdot \lambda_2 < 0 \Rightarrow$ saddle
- A has 1 eigenvalue
  - diagonalizable
  - not diag.
- A has complex eigenvalues
  - $Re = 0 \Rightarrow$ circle
  - $Re \neq 0 \Rightarrow$ spiral

Phase portraits $\tilde{x}(t) = A\tilde{x}(t)$, $A \in \mathbb{R}^{2\times 2}$

- Case 1: A has 2 distinct eigenvalues λ_1, λ_2

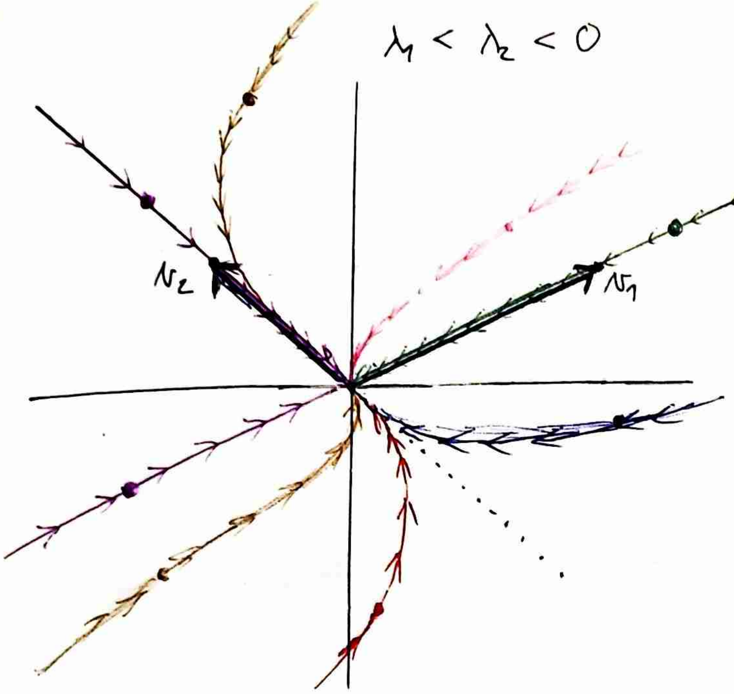
$$\Rightarrow \tilde{x}(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

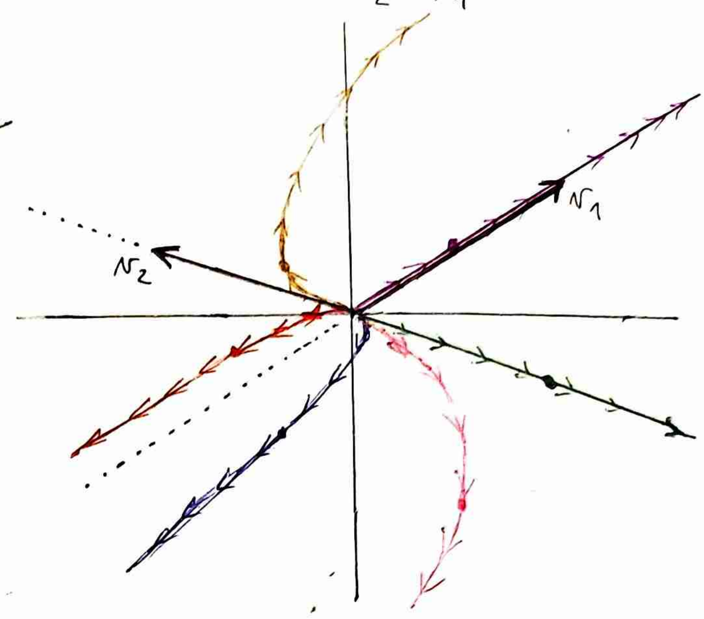
a) $\lambda_1 < \lambda_2 < 0 \Rightarrow \tilde{x}(t) \to \sigma$ as $t \to \infty$, but how?

→ the initial point $\tilde{x}(t_0) = \tilde{x}_0$ determines c_1 and c_2

- $\tilde{x}_0 = a \cdot v_1 \Rightarrow c_2 = 0 \Rightarrow \tilde{x}(t) \to \sigma$ by the half line $(\sigma, \tilde{x}_0) \sim v_1$
- $\tilde{x}_0 = b \cdot v_2 \Rightarrow c_1 = 0 \Rightarrow \tilde{x}(t) \to 0$ by the half line $(\sigma, \tilde{x}_0) \sim v_2$
- else $\tilde{x}(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$ & recall $\lambda_1 < \lambda_2 < 0$
  - $t \to \infty$: $c_2 e^{\lambda_2 t} v_2$ dominates $\Rightarrow \tilde{x}(t)$ approaches σ parallel to v_2
  - $t \to -\infty$ $c_1 e^{\lambda_1 t} v_1$ dominates $\Rightarrow \tilde{x}(t)$ diverges in parallel to v_1

$\lambda_1 < \lambda_2 < 0$ $\qquad\qquad$ $0 < \lambda_2 < \lambda_1$





origin = nodal sink (improper node) $\qquad$ origin = nodal source (improper node)

b) $\underline{0 < \lambda_2 < \lambda_1}$ $\Rightarrow$ $\tilde{x}(t) \to \vec{0}$ as $t \to -\infty$

$\to$ $\tilde{x}_0 = a \cdot v_1$ or $\tilde{x}_0 = b \cdot v_2$ $\Rightarrow$ $\tilde{x}(t)$ is a half line through $(\vec{0}, \tilde{x}_0)$

$\to$ else $\tilde{x}(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$ & $0 < \lambda_2 < \lambda_1$

- $t \to \infty$: $c_1 e^{\lambda_1 t} v_1$ dominates $\Rightarrow$ $\tilde{x}(t)$ diverges in parallel to v_1
- $t \to -\infty$: $c_1 e^{\lambda_2 t} v_2$ dominates $\Rightarrow$ $\tilde{x}(t)$ approaches $\vec{0}$ in parallel to v_2

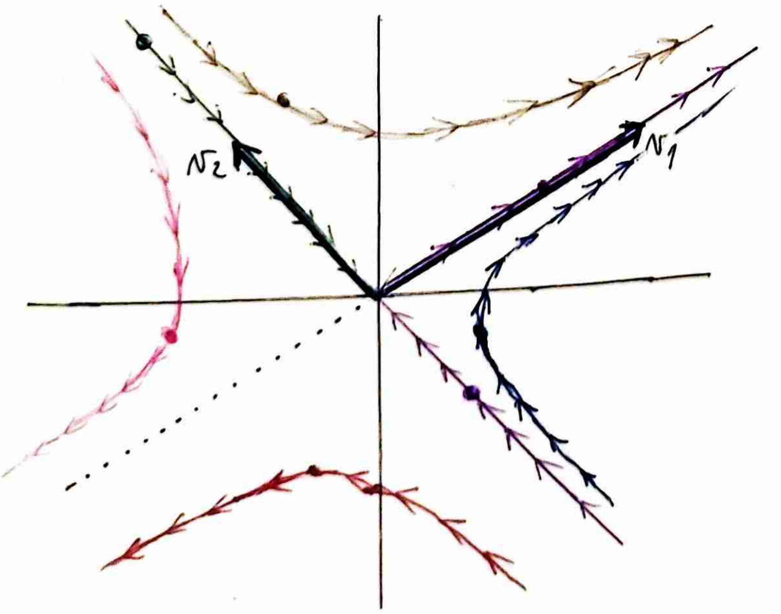
c) $\underline{\lambda_2 < 0 < \lambda_1}$

$\to$ $\tilde{x}_0 = a v_1$ $\Rightarrow$ $\tilde{x}(t)$ is a halfline through $\tilde{x}_0$ from the origin

$\tilde{x}_0 = b v_2$ $\Rightarrow$ $\tilde{x}(t)$ is a halfline through $\tilde{x}_0$ to the origin

$\to$ else $\tilde{x}(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$ & $\lambda_2 < 0 < \lambda_1$

- $t \to \infty$: $c_1 e^{\lambda_1 t} v_1$ dominates $\Rightarrow$ $\tilde{x}(t)$ diverges in parallel to v_1
- $t \to -\infty$: $c_2 e^{\lambda_2 t} v_2$ dominates $\Rightarrow$ $\tilde{x}(t)$ diverges in parallel to v_2



origin = saddle point

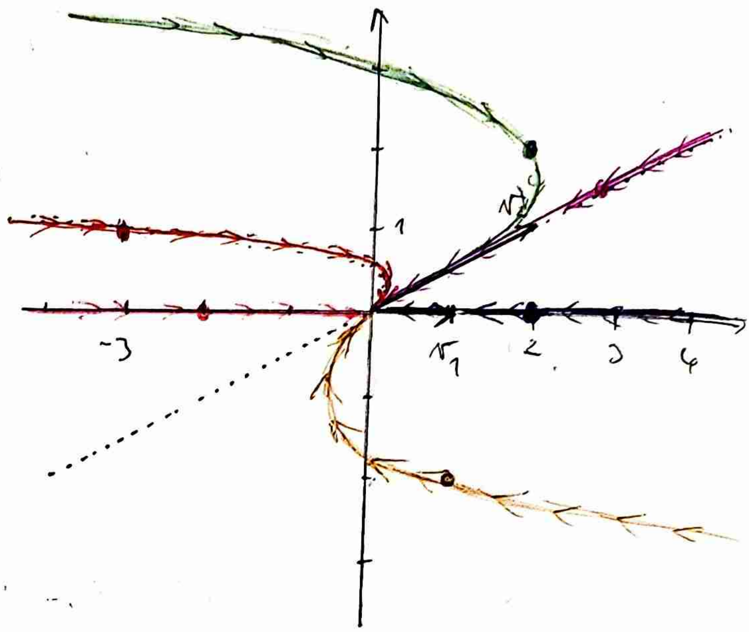
Ex: Sketch the phase field of $\tilde{x}' = \begin{bmatrix}-3 & 4\\ 0 & -1\end{bmatrix}\tilde{x}$

$p_A(t) = (3+t)(1+t) \rightarrow \lambda_1 = -3 \Rightarrow v_1 = \begin{bmatrix}1\\0\end{bmatrix}$

$ \searrow \lambda_2 = -1 \Rightarrow v_2 = \begin{bmatrix}2\\1\end{bmatrix}$

$\Rightarrow \underset{-3}{\lambda_1} < \underset{-1}{\lambda_2} < 0$

$\Rightarrow \tilde{x}(t) = c_1 e^{-3t}\begin{bmatrix}1\\0\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}2\\1\end{bmatrix}$



1) $\underline{\tilde{x}(0) = \begin{bmatrix}2\\0\end{bmatrix}} = 2\cdot v_1$ 2) $\underline{\tilde{x}(0) = \begin{bmatrix}-2\\0\end{bmatrix}} = -2v_1$

3) $\underline{\tilde{x}(0) = \alpha\begin{bmatrix}2\\1\end{bmatrix}} = \alpha v_2$

4) $\underline{\tilde{x}(0) = \begin{bmatrix}-3\\1\end{bmatrix}} = v_2 - 5v_1$

↳ $t \to \infty$: $\tilde{x} \to 0$, λ_2 dominates

$t \to -\infty$: $\tilde{x} \to \infty$, λ_1 dominates

5) $\underline{\tilde{x}(0) = \begin{bmatrix}2\\2\end{bmatrix}} = 2v_2 - 2v_1$

6) $\underline{\tilde{x}(0) = \begin{bmatrix}1\\-2\end{bmatrix}} = -2v_2 + 5v_1$

Case 2: repeated eigenvalue λ

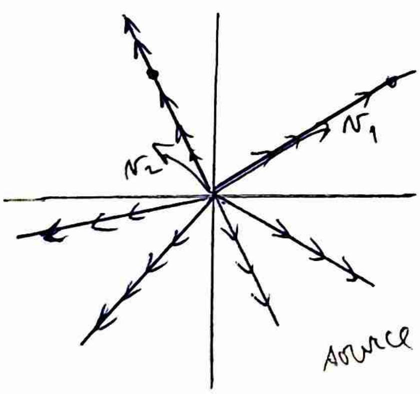
a) $\underline{\text{Geom}(\lambda) = 2}$ $\Rightarrow$ 2 linearly independent eigenvectors v_1, v_2

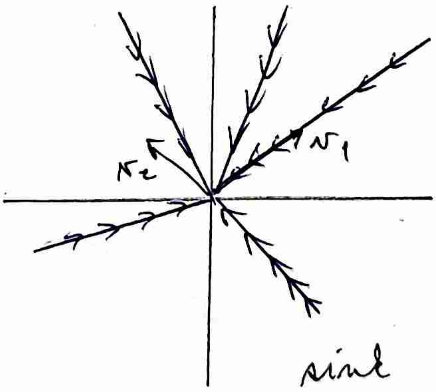
$\tilde{x}(t) = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t} v_2 = e^{\lambda t}(c_1 v_1 + c_2 v_2)$

→ the initial point $\tilde{x}(t_0) = \tilde{x}_0$ determines the vector $v_0 = c_1 v_1 + c_2 v_2$

↳ v_0 then gets scaled by $e^{\lambda t}$

$\underline{\lambda > 0}$ $\underline{\lambda < 0}$





origin = proper node

b) $\underline{\text{Geom}(\lambda) = 1}$ $\Rightarrow$ have to use Jordan normal form

$\underline{\text{Ex}}$: $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow p_A(t) = (1-t)(3-t)+1 = t^2-4t+4 = (t-2)^2 \Rightarrow \lambda = 2$

$\lambda = 2$: $\mathcal{N}(A-2I)$: $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathcal{N}(A-2I) = \left\{ \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \alpha \in \mathbb{R} \right\} \Rightarrow \dim = 1$

$v_2 \in \mathbb{R} - \mathcal{N}(A-2I) \Rightarrow$ let $v_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow v_1 = (A-2I)v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} = RJR^{-1}$ ↳ $R = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \Rightarrow R^{-1} = \frac{1}{-1}\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$

$\tilde{x}' = A\tilde{x} \rightarrow \tilde{x} = R\tilde{z} \Rightarrow \tilde{z} = R^{-1}\tilde{x}$

$\Rightarrow \tilde{z}' = R^{-1}\tilde{x}' = R^{-1}A\tilde{x} = R^{-1}\underline{RJR^{-1}}\,\underline{R\tilde{z}} = J\tilde{z} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\tilde{z}$

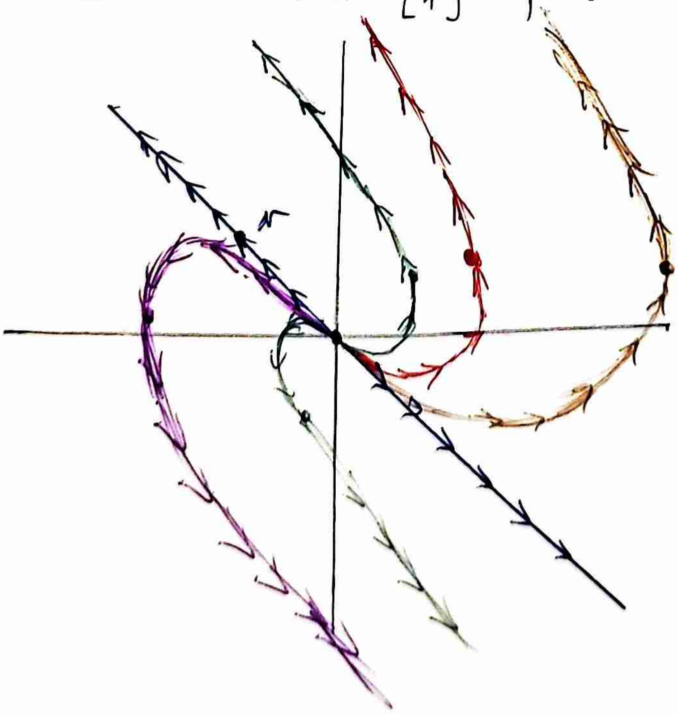
$\Rightarrow$ $z_1' = 2z_1 + z_2 = 2z_1 + c_2e^{2t} \rightarrow \mu_1 = e^{-2t} \Rightarrow z_1 = e^{2t}\int c_2\,dt = e^{2t}(c_2t + c_1)$

$z_2' = 2z_2 \Rightarrow z_2 = c_2e^{2t}$

$$\Rightarrow \tilde{x} = R\tilde{z} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} (c_1+c_2t)e^{2t} \\ c_2e^{2t} \end{bmatrix} = \underline{\underline{(c_1+c_2t)e^{2t}\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{= v} + c_2e^{2t}\underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{= w}}}$$

- $\tilde{x}_0 = a\cdot\tilde{v} \Rightarrow c_2 = 0 \Rightarrow \tilde{x}(t) \sim$ halfline from origin through $\tilde{x}_0$
- $\tilde{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = v_1 - 2v_2 \Rightarrow \tilde{x}(t) = (1-2t)e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} - 2e^{2t}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$
  - $t \to \infty$: $-2te^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ dominates but $-2e^{t}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ is not insignificant ↗ diverges
  - $t \to -\infty$: $-2te^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ dominates and $-2e^{t}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ is insignificant ↘ $\tilde{x}(t) \to 0$

$\rightarrow t \to \infty$: $\tilde{x}(t) \parallel \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $t \to -\infty$, $\tilde{x}(t) \parallel \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



origin = improper node

Case 3: $A \in \mathbb{R}^{2\times 2}$ has complex eigenvalues, $\lambda \pm i\mu$, $v_1 = \overline{v_2}$ eigenvectors

↳ complex solution: $e^{(\lambda + i\mu)t} v_1$, let $\tilde{v}_1 = \tilde{a} + i\tilde{b}$

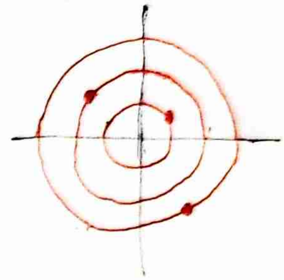
$$= e^{\lambda t}(\cos\mu t + i\sin\mu t)(a + ib)$$
$$= e^{\lambda t}(a\cos\mu t - b\sin\mu t + ib\cos\mu t + ia\sin\mu t)$$

$\Rightarrow$ real solution: $\tilde{x}(t) = c_1 e^{\lambda t}(\tilde{a}\cos\mu t - \tilde{b}\sin\mu t) + c_2 e^{\lambda t}(\tilde{b}\cos\mu t + \tilde{a}\sin\mu t) \in \mathbb{R}^2$

→ distance from origin:

$$|\tilde{x}(t)| = \sqrt{x_1(t)^2 + x_2(t)^2} = \dots = e^{\lambda t}\sqrt{c_1^2 + c_2^2}$$

origin = centre



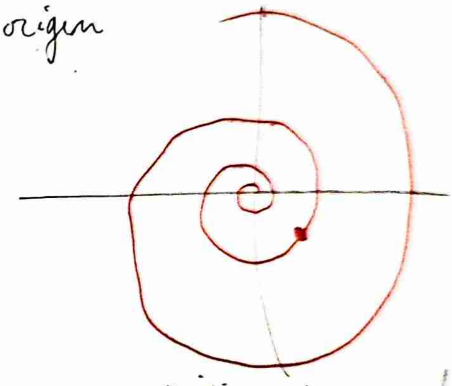
a) if $\mathrm{Re} = \lambda = 0$, then $|\tilde{x}(t)| = \sqrt{c_1^2 + c_2^2}$

$\Rightarrow \tilde{x}(t)$ is a circle centered at $\begin{bmatrix}0\\0\end{bmatrix}$ passing through $\begin{bmatrix}c_1\\c_2\end{bmatrix}$

b) if $\lambda \neq 0$, then $|\tilde{x}(t)| = e^{\lambda t}\sqrt{c_1^2 + c_2^2}$

$\Rightarrow \tilde{x}(t)$ is a spiral that winds around the origin

- $\lambda > 0$: away from the origin $\Rightarrow$ source
- $\lambda < 0$: towards the origin $\Rightarrow$ sink



origin = proper node

Ex: $\tilde{x}'(t) = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \tilde{x}(t)$

$p_A(t) = (\frac{1}{2} + t)^2 + 1 \Rightarrow (t + \frac{1}{2})^2 = -1 \Rightarrow t + \frac{1}{2} = \pm i$

$\Rightarrow \lambda_{1,2} = -\frac{1}{2} \pm i \Rightarrow v_{1,2} = \begin{pmatrix}1\\ \pm i\end{pmatrix}$

complex solution: $e^{(-\frac{1}{2} + i)t}\begin{bmatrix}1\\ i\end{bmatrix} = e^{-\frac{1}{2}t}(\cos t + i\sin t)\begin{bmatrix}1\\ i\end{bmatrix}$

real: $\tilde{x}(t) = c_1 e^{-\frac{1}{2}t}\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix} + c_2 e^{-\frac{1}{2}t}\begin{bmatrix}\sin t\\ \cos t\end{bmatrix} = e^{-\frac{1}{2}t}\begin{bmatrix} c_1\cos t + c_2\sin t \\ -c_1\sin t + c_2\cos t\end{bmatrix}$

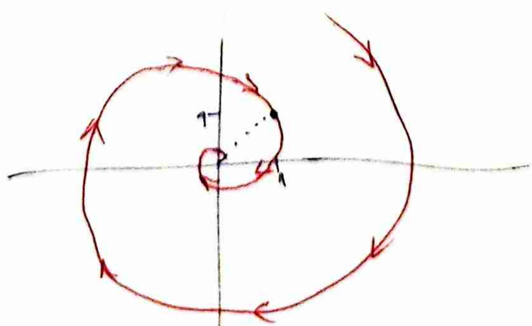
→ suppose $\tilde{x}(0) = \begin{bmatrix}1\\1\end{bmatrix} \Rightarrow \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

$$\Rightarrow |\tilde{x}(t)|^2 = x_1(t)^2 + x_2(t)^2 = e^{-t}(\cos t + \sin t)^2 + e^{-t}(\cos t - \sin t)^2$$
$$= e^{-t}(1 + 2\cos t\sin t) + e^{-t}(1 - 2\cos t\sin t) = \underline{2e^{-t}}$$

$\Rightarrow$ using trig formulas: $\tilde{x}(t) = \sqrt{2}\, e^{-t/2}\begin{bmatrix}\cos(t - \frac{\pi}{4})\\ -\sin(t - \frac{\pi}{4})\end{bmatrix}$, $|\tilde{x}(t)| = \sqrt{2}\, e^{-t/2}$

$t \to \infty$: $\tilde{x}(t) \to \begin{pmatrix}0\\0\end{pmatrix}$

$t \to -\infty$: $\tilde{x}(t) \to \infty$



• The Stability of Autonomous Systems

Def: An autonomous system is a system of ODEs of the form

$$\tilde{x}'(t) = F(\tilde{x}(t)), \quad \text{where } \tilde{x}: \mathbb{R} \to \mathbb{R}^n, \; F: \mathbb{R}^n \to \mathbb{R}^n$$

Def: $\tilde{x}_0 = \tilde{x}(t_0) \in \mathbb{R}^n$ is a critical point of the system $\tilde{x}' = F(\tilde{x}) \equiv F(\tilde{x}_0) = \tilde{0}$.

The constant function $\tilde{x}_c : t \mapsto \tilde{x}_0$ is a solution of $\because \tilde{x}_c' = \tilde{0} = F(\tilde{x}_0)$

$\Rightarrow$ C.P. are called equilibriums solutions of the system

↳ the solution begins and stays at the C.P. as t changes

The C.P. of the autonomous system $\tilde{x}' = A\tilde{x}$, $A \in \mathbb{R}^{n \times n}$ are $\{c \in \mathbb{R}^n \mid Ac = \tilde{0}\}$

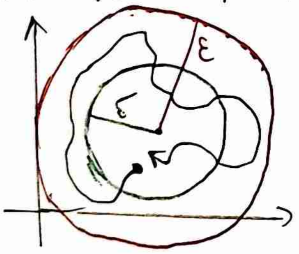
$\Rightarrow$ if A is invertible then $\tilde{c} = \tilde{0}$ is the only C.P.

Ex: Find the C.P. of the system

$$\begin{aligned} x_1' &= 1 - x_2 = f_1(x_1, x_2) = 0 \Rightarrow x_2 = 1 \\ x_2' &= 4 - (x_1)^2 = f_2(x_1, x_2) = 0 \Rightarrow x_1 = \pm 2 \end{aligned} \quad \Rightarrow \text{C.P.} = \begin{bmatrix} \pm 2 \\ 1 \end{bmatrix}$$

Def: A critical point $\tilde{c}$ of the system $\tilde{x}' = F(\tilde{x})$ is said to be stable $\equiv$

$$(\forall \varepsilon > 0)(\exists \delta > 0): \tilde{x}' = F(\tilde{x}) \Rightarrow \left(\|\tilde{x}(0) - \tilde{c}\| < \delta \Rightarrow \forall t > 0 : \|\tilde{x}(t) - \tilde{c}\| < \varepsilon \right)$$



→ if every solution which starts sufficiently close to the C.P. stays close to the C.P. for $\forall t > 0$

↳ then the C.P. is stable

Def: A stable C.P. is said to be asymptotically stable $\equiv$

$$\exists \delta > 0 : \tilde{x}' = F(\tilde{x}) \Rightarrow \left(\|\tilde{x}(0) - \tilde{c}\| < \delta \Rightarrow \lim_{t \to \infty} \tilde{x}(t) = \tilde{c} \right)$$

for example $\tilde{x}'(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tilde{x}(t)$

↳ phase field

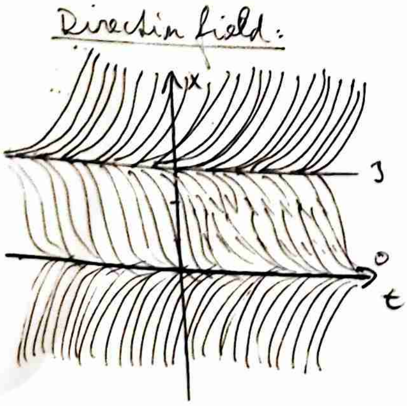
as. stable $(0,0)^T$ C.P.

Def: A C.P. is said to be unstable $\equiv$ it is not stable.

Ex: $\frac{dx}{dt} = x^2 - 3x$ → find the C.P. and classify them

↳ $x^2 - 3x = x(x-3) = 0 \Rightarrow x = 0 \lor x = 3$

Direction field:



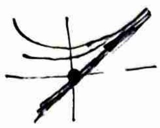
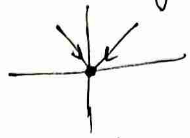
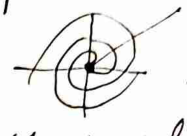
Are the C.P. stable?

$x = 0$: asymptotically stable

$x = 3$: unstable

Def: A critical point $\tilde{c}$ is called a <u>node</u> $\equiv$ every trajectory aproaches $\tilde{c}$ as $t \to \infty$ or every trajectory recedes from $\tilde{c}$ as $t \to \infty$.

- a node is <u>proper</u> $\equiv$ no two distinct solution curves are tangent to the same line through the critical point

 – improper  proper  proper

- a node is called a <u>sink</u> $\equiv$ all trajectories close to it approach it
- a node is called a <u>source</u> $\equiv$ all trajectories close to it recede from it

- <u>Stability of linear systems</u>

$\to$ Consider $\tilde{x}'(t) = A\tilde{x}(t)$, $A \in \mathbb{R}^{n \times n}$ is invertible $\Rightarrow \tilde{x}_0 = \tilde{0}$ is the only C.P.

$\hookrightarrow$ suppose $\{\lambda_i\}$ are the eigenvalues of A

from the analysis of phase fields for $A \in \mathbb{R}^{2\times 2}$ we can see that $\tilde{0}$ is stable $\Leftrightarrow \lambda_1 \le 0$ & $\lambda_2 \le 0$ and this is true in general

<u>Theorem</u>: The C.P. $\tilde{x} = \tilde{0}$ of the <u>linear system</u> $\tilde{x}' = A\tilde{x}$ is

- stable $\Leftrightarrow \forall i: \underline{Re(\lambda_i) \le 0}$ ... $Re = 0 \Rightarrow$ circle
- as. stable $\Leftrightarrow \forall i: \underline{Re(\lambda_i) < 0}$

<u>Idea</u>: The solutions of $\tilde{x}' = A\tilde{x}$ can be written as $e^{At} \cdot \tilde{c}$, $c \in \mathbb{R}^n$

$\hookrightarrow$ the entries in e^{At} involve expressions of the form

$$t^k e^{\lambda_i t} \sin \mu_i t, \; t^k e^{\lambda_i t} \cos \mu_i t \to 0 \text{ as } t \to \infty \Leftrightarrow Re(\lambda_i) < 0$$

$\to$ if one of the eigenvalues has positive real part, then some solutions grow exponentially and the zero C.P. is unstable

<u>Ex: Consider $\tilde{x}'(t) = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix} \tilde{x}(t)$. Determine the stability based on b.</u>

$p_A(t) = (b-t)^2 + 1 \Rightarrow (b-t)^2 = -1 \Rightarrow b - t = \pm i \Rightarrow \lambda_{1,2} = b \pm i$

$\hookrightarrow Re(\lambda_{1,2}) = b$ & the C.P. is $\tilde{x} = \binom{0}{0}$

$\Rightarrow (0,0)^T$ is stable $\Leftrightarrow b \le 0$

$(0,0)^T$ is as. stable $\Leftrightarrow b < 0$

$(0,0)^T$ is unstable $\Leftrightarrow b > 0$

$\lambda_1 = b+1$
$\lambda_2 = b-1$

<u>Ex: Do the same for $\tilde{x}'(t) = \begin{bmatrix} b & 1 \\ 1 & b \end{bmatrix} \tilde{x}(t)$</u> $\to p_A(t) = (b-t)^2 - 1 \Rightarrow b - t = \pm 1 \Rightarrow \lambda_{1,2} = b \pm 1$

$(0,0)$ is stable $\Leftrightarrow b \pm 1 \le 0 \Leftrightarrow b \le -1$

$(0,0)$ is as. stable $\Leftrightarrow b < -1$

$(0,0)$ is unstable $\Leftrightarrow b > -1$

Ex: Consider the system $\tilde{x}'(t) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tilde{x}(t)$

↳ let $T := tr(A) = a_{11} + a_{22}$ & $D := det(A) = a_{11}a_{22} - a_{12}a_{21}$

→ $p_A(t) = (a_{11} - t)(a_{22} - t) - a_{12}a_{21} = t^2 - tT + D \Rightarrow \lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2}$

a) $D < 0 \Rightarrow$ origin = saddle point

$D < 0 \Rightarrow 4D < 0 \Rightarrow -4D > 0 \Rightarrow T^2 - 4D > T^2 \Rightarrow \sqrt{T^2-4D} > |T|$

$$\left.\begin{aligned} \lambda_1 &= \tfrac{1}{2}(T + \sqrt{\cdots}) > \tfrac{1}{2}(T + |T|) \geq 0 \\ \lambda_2 &= \tfrac{1}{2}(T - \sqrt{\cdots}) < \tfrac{1}{2}(T - |T|) \leq 0 \end{aligned}\right\} \lambda_1 > 0 \ \& \ \lambda_2 < 0$$

b) $D > 0$ & $T = 0 \Rightarrow$ origin = centre ($\tilde{x}(t)$ form circles around the origin)

↳ want to show that $Re(\lambda_i) = 0$

$T = 0 \Rightarrow \lambda_{1,2} = \pm \frac{1}{2}\sqrt{-4D} = \pm i\sqrt{D}$ ... purely imaginable

c) $D > 0$ & $T < 0 \Rightarrow$ origin = asym. stable

$D > 0 \Rightarrow 4D > 0 \Rightarrow -4D < 0 \Rightarrow T^2 - 4D < T^2$

- suppose $T^2 - 4D > 0$: $\sqrt{T^2-4D} < |T| = -T$

$$\left.\begin{aligned} &\to T + \sqrt{\cdots} < T + |T| = T - T = 0 \\ &\to \underbrace{T}_{\ominus} - \underbrace{\sqrt{\cdots}}_{\oplus} < 0 \end{aligned}\right\} \text{both eigenvalues } < 0$$

- $T^2 - 4D = 0$: $\lambda = T/2 < 0$ ✓
- $T^2 - 4D < 0$:

$\lambda_{1,2} = \frac{1}{2}(T + \sqrt{T^2-4D}) \Rightarrow Re(\lambda_{1,2}) = \frac{1}{2}T < 0 \Rightarrow$ asym. stable

Ex: Is the zero solution of the following systems stable? Asymptotically stable?

a) $x' = -2y, \quad y' = 2x$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow p_A(t) = t^2 + 4 \Rightarrow \lambda = \pm 2i \Rightarrow Re(\lambda) = 0$

$\Rightarrow \sigma$ is stable, but not asymptotically

b) $x' = x + 2y, \quad y' = x$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow p_A(t) = (1-t)(-t) - 2 = t^2 - t - 2 = (t-2)(t+1) \Rightarrow \lambda_1 = 2, \lambda_2 = -1$

→ σ is unstable because $Re(\lambda_1) > 0$

c) $x' = x - 5y, \quad y' = x - 3y$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow p_A(t) = (1-t)(-3-t) + 5 = t^2 + 2t + 2 \Rightarrow \lambda_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2}$

$\Rightarrow Re(\lambda_{1,2}) = -1 \Rightarrow \sigma$ is asymptotically stable

• Stability of more complicated systems

Ex: Find the C.P. of (*) $x'(t) = -2x - xy^2$, $y'(t) = -2y^3 + x^2 y$ and determine stability

$\boxed{1}$ $x' = -x(2+y^2) = 0 \Rightarrow x = 0 \lor \cancel{y = \pm i\sqrt{2}}$ → we want $\mathbb{R}$

$\boxed{2}$ $y' = -y(2y^2 - x^2) = 0 \Rightarrow -y(2y^2) = 0 \Rightarrow y = 0 \Rightarrow CP = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

→ let $z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ be a solution of the system and examine stability

↳ what happens to $\|z(t) - 0\|$ as $t \to \infty$?

$\|z(t)\| = \sqrt{x^2(t) + y^2(t)} =: r(t) \Rightarrow r^2(t) = x^2(t) + y^2(t)$

$\frac{d}{dt}$: $2rr' = 2xx' + 2yy'$

$\Rightarrow rr' = x(-2x - xy^2) + y(-2y^3 + x^2 y)$

$= -2x^2 - x^2y^2 - 2y^4 + x^2y^2 = -2(x^2 + y^4) \leq 0$

$\Rightarrow rr' \leq 0$ & $r = \text{dist} \geq 0 \Rightarrow r' \leq 0$

$\Rightarrow r = \text{dist}$ is decreasing $\Rightarrow C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a stable critical point

Ex: Find the C.P of $x' = -2x + 4xy^3$, $y' = -y - 2x^2$ and determine stability

$x' = -2x(1 - 2y^3) = 0 \Rightarrow x = 0 \lor 1 = 2y^3 \Rightarrow y = 2^{-1/3}$

$y' = -(y + 2x^2) = 0 \Rightarrow y = 0 \quad \lor \quad 2^{-1/3} + 2x^2 \neq 0 \quad \Rightarrow CP = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the only C.P. Again let $z(t)$ be a solution and examine $\|z(t) - 0\|$

$\|z(t)\| = \sqrt{x^2(t) + y^2(t)} =: r(t) \Rightarrow r^2(t) = x^2(t) + y^2(t)$

$\frac{d}{dt}$: $2rr' = 2xx' + 2yy'$

$rr' = x(-2x + 4xy^3) + y(-y - 2x^2)$

$= -2x^2 + 4x^2y^3 - y^2 - 2yx^2$ ⇒ not clear what is happening

Let's consider the function

$H(t) = x^2(t) + y^4(t)$

$H'(t) = 2xx' + 4y^3y' = 2x(-2x + 4xy^3) + 4y^3(-y - 2x^2)$

$= -4x^2 + 8x^2y^3 - 4y^4 - 8x^2y^3$

$= -4(x^2 + y^4) \leq 0 \Rightarrow H$ decreasing

Moreover $H' = -4H \Rightarrow H(t) = C \cdot e^{-4t} = H(0) \cdot e^{-4t}$

$\lim_{t\to\infty} H(t) = 0$ & $\lim_{t\to\infty} H(t) = 0 \Leftrightarrow \lim_{t\to\infty} r(t) = 0 \Leftrightarrow \lim_{t\to\infty} \|z(t) - 0\| = 0$

$\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is asymptotically stable

What exactly have we done?

→ we constructed functions $r(t) = \sqrt{x^2(t) + y^2(t)}$ and $H(t) = x^2(t) + y^4(t)$ that satisfied:

i) $r \geq 0$ & $r = 0 \iff (x, y) = (0,0)$

ii) $\frac{dr}{dt} = \frac{\partial r}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial r}{\partial y} \cdot \frac{dy}{dt} \leq 0$ i.e. $\nabla r \cdot F(\tilde{z}(t)) \leq 0$,

where $\tilde{z}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is a solution of the system

→ if i) and ii) hold then σ is stable and if $\lim_{t \to 0} z(t) = \sigma$, then it is assymptotically stable

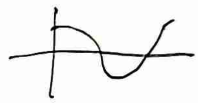
⇒ think about r as a positive-valued function that lies over the xy plane (in which the solution curves lie) and touches it at the origin

→ $r(t)$ increases in all directions pointing away from the origin

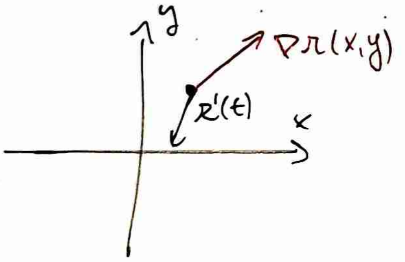
⇒ the gradient vectors $\nabla r(t)$ point away from the origin

$$\nabla r \cdot F(z(t)) = \nabla \cdot r \cdot z'(t) = |\nabla r| \cdot |z'(t)| \cdot \cos\theta \leq 0$$

→ θ = acute angle in between ∇r and $z'(t)$



⇒ $\frac{\pi}{2} \leq \theta \leq \pi$ ⇒ vectors ∇r and $z'(t)$ don't point in the same direction



⇒ since ∇r points away from the origin,
$z'(t)$ = direction vector of the solution must point toward the origin
⇒ the origin is a stable critical point

⇒ Goal: construct more functions like r and H

Def: Let $f: \mathbb{R}^n \to \mathbb{R}$ and let $D \subseteq \mathbb{R}^n$ with $\sigma \in D$.
Then f is said to be

1. positive definite on $D \equiv \forall \tilde{x} \in D: f(\tilde{x}) \geq 0$ & $f(\tilde{x}) = 0 \iff \tilde{x} = \sigma$
2. positive semi-definite on $D \equiv \forall \tilde{x} \in D: f(\tilde{x}) \geq 0$
3. negative definite on $D \equiv \forall \tilde{x} \in D: f(\tilde{x}) \leq 0$ & $f(\tilde{x}) = 0 \iff \tilde{x} = \sigma$
4. negative semi-definite on $D \equiv \forall \tilde{x} \in D: f(\tilde{x}) \leq 0$
5. indefinite $\equiv$ otherwise

Ex:

- $\sin(x^2 + y^2)$ → positive definite on $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq \frac{\pi}{2}\}$
- $(x+y)^2$ → positive semi-definite on $D = \{(x,y) \in \mathbb{R}^2 / x^2 + y^2 \leq 1\}$ but also $D = \mathbb{R}^2$

Theorem: Suppose an $n \times n$ autonomous system

$$\tilde{x}'(t) = F(\tilde{x}(t)) \quad , \quad F: \mathbb{R}^n \to \mathbb{R}^n$$

has a critical point at the origin i.e. $F(\sigma) = \sigma$.

If there exists a function $V: \mathbb{R}^n \to \mathbb{R}$ s.t.

i) V is continuous
ii) V has continuous first order partial derivatives $\Big\} \; V \in C^1$
iii) V is positive definite on some open region $\sigma \in D \subseteq \mathbb{R}^n$

Then if on D is

1. $\nabla V \cdot \tilde{x}'(t)$ negative definite $\Rightarrow \sigma$ is asymptotically stable
2. $\nabla V \cdot \tilde{x}'(t)$ negative semi-definite $\Rightarrow \sigma$ is stable
3. $\nabla V \cdot \tilde{x}'(t)$ positive definite $\Rightarrow \sigma$ is unstable

Proof: We will show 2.

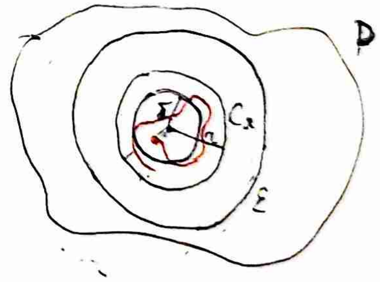
$\to$ let $\varepsilon > 0$, we will show that $\exists \delta$: $\underset{\substack{t=0\\ \downarrow}}{\tilde{x}(0)} \in D$ & $\|\tilde{x}(0)\| < \delta$,
then $\forall t \geq 0 : \|\tilde{x}(t)\| < \varepsilon$

since D is open, if we shrink δ enough we have $\|\tilde{x}(0)\| < \delta \Rightarrow \tilde{x}(0) \in D$

$\Rightarrow$ choose $0 < r < \varepsilon$ small enough s.t. $C_r := \{\tilde{x} \in \mathbb{R}^n \mid \|\tilde{x}\| = r\} \subseteq D$

Since V is continuous and C_r is closed and bounded, V has a max and min on C_r

$\Rightarrow$ define $m := \min_{\tilde{x} \in C_r} V(\tilde{x})$



$m > 0$ $\because$ V is positive definite and $\sigma \notin C_r$

Since V is continuous and $V(\sigma) = 0$, $\exists \delta > 0$, s.t.

$\circledast$ $\|\tilde{x} - \sigma\| = \|\tilde{x}\| < \delta \Rightarrow \|V(\tilde{x}) - V(\sigma)\| = \|V(\tilde{x})\| < \frac{m}{2}$

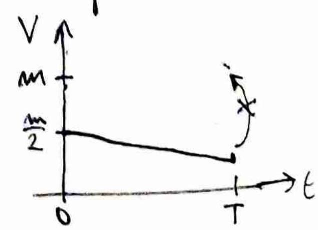
$\delta \leq r$ ... suppose $\delta > r$, then $\|\tilde{x}\| < \delta$ holds for points in C_r
and $\|V(\tilde{x})\| < \frac{m}{2}$ is not true $\lightning$

$\to$ we will show that $\|\tilde{x}(0)\| < \delta \Rightarrow \forall t \geq 0 : \|\tilde{x}(t)\| < r < \varepsilon$

$\Rightarrow$ suppose T is the first point to violate this. Therefore $\|\tilde{x}(T)\| = r$, $T \in C_r$
and for $0 \leq t < T$, $\tilde{x}(t)$ lies within C_r.

$$\frac{d}{dt}\left(V(\tilde{x}(t))\right) = \nabla V(\tilde{x}(t)) \cdot \tilde{x}'(t) \leq 0 \quad \dots \quad \nabla V \cdot \tilde{x}' \text{ is negative semi-definite}$$

$\Rightarrow V(\tilde{x}(t))$ is decreasing on $[0, T)$, therefore using $\circledast$



$\|\tilde{x}(0)\| < \delta \Rightarrow \forall 0 \leq t < T : V(\tilde{x}(t)) \leq V(\tilde{x}(0)) < \frac{m}{2}$

$\to$ since V is continuous, $V(\tilde{x}(T)) < m$ (see picture).

$\hookrightarrow$ but $T \in C_r$, therefore m is not the minimum $\lightning$

Def: A Lyapunov function of the autonomous system $\tilde{x}'(t) = F(\tilde{x}(t))$, $F: \mathbb{R}^n \to \mathbb{R}^n$ is a function $V: \mathbb{R}^n \to \mathbb{R}$ satisfying i) ii) iii) and (2)

Ex: Show that the zero equilibrium solution is stable for $\begin{array}{l} x' = y^2 - x^3 \\ y' = -y - 2xy \end{array}$

→ check that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a C.P. ... ✓

⇒ consider $V(x,y) = ax^2 + by^2$, $a, b > 0$

↳ V is C^1 and $V \geq 0$

$$\frac{dV}{dt} = \nabla V \cdot F = 2axx' + 2byy' = 2ax(y^2 - x^3) + 2by(-y - 2xy)$$

$$= 2axy^2 - 2ax^4 - 2by^2 - 4bxy^2 = 2xy^2(a - 2b) - 2(ax^4 + by^2)$$

→ we want this ≤ 0 ⇒ let $a = 2b$

$$\Rightarrow \frac{dV}{dt} = -2(ax^4 + by^2) \leq 0$$

⇒ $\nabla V \cdot F$ is negative definite ⇒ $\underline{\underline{0 \text{ is asymptotically stable}}}$

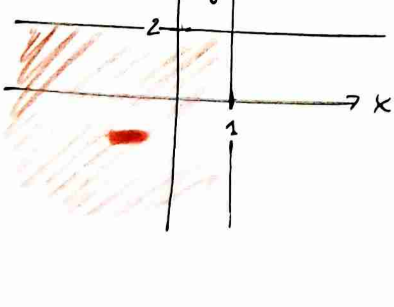
Ex: By constructing a Lyapunov function of the form $V(x,y) = ax^2 + by^2$ show that the zero solution of $\begin{array}{l} x' = y^2 - 2x \\ y' = x^2 - y \end{array}$ is asymptotically stable

$V(x,y) = ax^2 + by^2$

$$\Rightarrow \frac{dV}{dt} = 2axx' + 2byy' = 2ax(y^2 - 2x) + 2by(x^2 - y)$$

$$= 2axy^2 - 4ax^2 + 2byx^2 - 2by^2 = 2x^2(by - 2a) + 2y^2(ax - b)$$

→ let $a = 1$, $b = 1$ ⇒ $\frac{dV}{dt} = 2x^2(y - 2) + 2y^2(x - 1)$



→ let $D = \{(x,y) \in \mathbb{R}^2 \mid y < 2 \ \& \ x < 1\}$

↳ V is positive definite

↳ $\frac{dV}{dt}$ is negative definite ⇒ 0 is asym. stable

Def: If $V: \mathbb{R}^n \to \mathbb{R}$ is a negative definite Lyapunov function, the it is said to be a strict Lyapunov function.

Ex: Find the C.P. of $\begin{array}{l} x' = -x - \frac{x^3}{3} - x\sin y \\ y' = -y - y^3/3 \end{array}$ and determine its stability

$\boxed{1}$ $-x - \frac{x^3}{3} - x\sin y = 0 \iff -x\left(1 + \frac{x^2}{3} + \cancel{\sin y}\right) = 0$

$\Rightarrow x = 0 \quad \Rightarrow \text{C.P.} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \sigma$

$\boxed{2}$ $y' = -y\left(1 + \frac{y^2}{3}\right) = 0 \Rightarrow y = 0$

Let $V(x,y) = ax^2 + by^2$, $a, b > 0$

$\frac{dV}{dt} = 2ax\left(-x - \frac{x^3}{3} - x\sin y\right) + 2by\left(-y - \frac{y^3}{3}\right)$

$= -2ax^2\left(1 + \frac{x^2}{3} + \sin y\right) - 2by^2\left(1 + \frac{y^2}{3}\right) \leq 0$

$\Rightarrow$ two options how to approch this

1) realize that $1 + \sin y \geq 0 \Rightarrow \frac{dV}{dt}$ negative definite on $\mathbb{R}^2$

2) we dont need the whole $\mathbb{R}^2$, just need an open region containing σ

$\hookrightarrow \sin y \geq 0$ when $|y| \leq \frac{\pi}{2} \quad \Rightarrow D = \{(x,y) \mid x \in \mathbb{R}, |y| \leq \frac{\pi}{2}\}$

$\Rightarrow$ since $\frac{dV}{dt}$ is negative definite, σ is asymptotically stable

Ex: Let $a > 0$. Show that $V(x,y) = x^2 + 2y^2$ is a strict Lyapunov function for the system $x' = ay^2 - x$, $y' = -y - ax^2$.

$\frac{dV}{dt} = 2xx' + 4yy' = 2x(ay^2 - x) - 4y(y + ax^2)$

$= 2axy^2 - 2x^2 - 4y^2 - 4ayx^2$

$= -2x^2(1 + 2ay) - 2y^2(2 - ax) \leq 0$

$\Rightarrow$ need $\begin{array}{l} 1 + 2ay > 0 \Rightarrow y > \frac{-1}{2a} \\ 2 - ax > 0 \Rightarrow x < \frac{2}{a} \end{array} \quad \Rightarrow D = \{(x,y) \mid x < \frac{2}{a} \text{ \& } y > -\frac{1}{2a}\}$

$\hookrightarrow \sigma \in D$ since $1 > 0$ & $2 > 0 \quad \Rightarrow V$ is negative definite on D

Ex: Show that $V(x,y) = y^2 + \ln(1 + x^2)$ is a Lyapunov function for the system $x'(t) = x(y-1)$, $y'(t) = -\frac{x^2}{1+x^2}$

$\left.\begin{array}{l} V(x,y) = \underbrace{y^2 + \ln(1+x^2)}_{=0 \iff y = 0 \text{ \& } x = 0} \text{ and } \ln(x) > 0 \text{ for } x > 1 \end{array}\right\}$ V is positive definite

$\frac{dV}{dt} = 2yy' + \frac{2xx'}{1+x^2} = -\frac{2yx^2}{1+x^2} + \frac{2x^2(y-1)}{1+x^2} \approx \leftarrow$ preserve sign

$\approx -yx^2 + x^2(y-1) = -x^2 \leq 0$ and $\frac{dV}{dt} = 0 \iff x = 0, y \in \mathbb{R}$

$\Rightarrow \frac{dV}{dt}$ is neg. semi-def $\Rightarrow V(x,y)$ is a Lyapunov function

Ex: Show that the zero solution of the system $\begin{aligned} x' &= x^3 - y^3 \\ y' &= 2xy^2 + 4x^2y + 2y^3 \end{aligned}$ is unstable

$V(x,y) = ax^2 + by^2$

$\frac{dV}{dt} = 2axx' + 2byy' = 2ax(x^3 - y^3) + 2by(2xy^2 + 4x^2y + 2y^3)$

$\approx ax^4 - axy^3 + 2b\,xy^3 + 4b\,x^2y^2 + 2b\,y^4$

$= x^2(ax^2 + 4by^2) + 2by^4 + xy^3(2b - a)$

→ let $2b = a > 0$, then

$\frac{dV}{dt} = x^2(ax^2 + 4by^2) + 2by^4 \geq 0$ is positive definite $\Rightarrow 0$ is unstable

• Reduction of Order

Suppose y_1 is a solution of the homogeneous equation

$$a_m(t)y^{(m)}(t) + a_{m-1}(t)y^{(m-1)}(t) + \dots + a_1(t)y'(t) + a_0(t)y(t) = \sum_{i=0}^{m} a_i(t)y^{(i)}(t) = 0$$

and we want to solve the associated inhomogeneous equation

$$\circledast \quad \sum_{i=0}^{m} a_i(t)y^{(i)}(t) = g(t)$$

Method: Consider $z(t) = v(t)y_1(t)$ ... what must v satisfy for z to solve $\circledast$?

$z' = vy' + v'y$

$z'' = vy'' + 2v'y' + v''y$

$\rightarrow z^{(m)} = \sum_{k=0}^{m} \binom{m}{k} y^{(k)} v^{(m-k)}$

$\Rightarrow \sum_{i=0}^{m} a_i(t) \sum_{k=0}^{i} \binom{i}{k} y^{(k)} v^{(i-k)} = g(t)$ → take out $\binom{i}{i} y^{(i)} v^{(i-i)} = y^{(i)}v$

$$\sum_{i=0}^{m} a_i(t)\left(y^{(i)}v + \sum_{k=0}^{i-1} \binom{i}{k} y^{(k)} v^{(i-k)}\right) =$$

$$= \underbrace{v\sum_{i=0}^{m} a_i(t)y^{(i)}(t)}_{0} + \underbrace{\sum_{i=0}^{m} a_i(t) \sum_{k=0}^{i-i} \binom{i}{k} y^{(k)} v^{(i-k)} = g(t)}_{\text{ODE in } v' \text{ of order } m-1}$$

⇒ solve the $m-1$ order ODE for v' to get $z = yv$

↳ if we get the gen. solution with $m-1$ parameters, then $v = \int v'$ gives the mth parameter ⇒ z is general

Usage:

① solving inhomogeneous equations

② consider that we have 1 solution of a homogeneous ODE and we want a new, independent one

⇒ use reduction of order with $g(t) = 0$

Ex:

① Solve $t^2y'' - 2ty' + 2y = t\sqrt{t}$, $t>0$

→ to use RoO we first need a solution of the hom. eq.

$t^2y'' - 2ty' + 2y = 0$ ... Euler eq.

↳ guess: $y = t^r \Rightarrow y' = rt^{r-1} \Rightarrow y'' = r(r-1)t^{r-2}$

$\Rightarrow (r(r-1) - 2r + 2)t^r = 0$

$r^2 - 3r + 2 = (r-1)(r-2) = 0 \Rightarrow y_h = C_1 t + C_2 t^2$

– lets take $y = t$ and perform reduction of order

↳ let $z = t \cdot v(t) \Rightarrow z' = tv' + v \Rightarrow z'' = tv'' + 2v'$

$\Rightarrow t^2(tv'' + 2v') - 2t(tv' + v) + 2tv = t\sqrt{t}$

$v''(t^3) + v'(2t^2 - 2t^2) + v(-2t + 2t) = t\sqrt{t}$

$\Rightarrow t^3 v'' = t\sqrt{t} \Rightarrow v'' = t^{-3/2}$

$\Rightarrow v' = \int t^{-3/2} dt = \frac{t^{-1/2}}{-1/2} + C = -2t^{-1/2} + C$

$\Rightarrow v = \int -2t^{-1/2} + C \, dt = -2\frac{t^{1/2}}{1/2} + Ct + D = -4\sqrt{t} + Ct + D$

$$\Rightarrow z(t) = t\,v(t) = \underbrace{-4t\sqrt{t}}_{y_p(t)} + \underbrace{Ct^2 + Dt}_{y_h(t)}$$

② Solve $y'' - t^{-1}y' + t^{-2}y = 0$, $t>0$

↳ $t^2y'' - ty' + y = 0$ ... → guess $y = t^r$

$\Rightarrow$ indicial eq. $r(r-1) - r + 1 = r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow r = 1$

$\Rightarrow y(t) = t$ is a solution

RoO: $z(t) = tv \Rightarrow z' = tv' + v \Rightarrow z'' = tv'' + 2v'$

$\Rightarrow t^2(tv'' + 2v') - t(tv' + v) + tv = 0$

$v''(t^3) + v'(2t^2 - t^2) + v(-t + t) = 0$

$\Rightarrow t^3 v'' + t^2 v' = 0$ ... $w := v'$

$t^3 w' + t^2 w = 0 \iff t w' + w = 0 \iff \frac{dw}{dt} = -\frac{w}{t}$

$\Rightarrow \int \frac{dw}{w} = \int -\frac{dt}{t} \Rightarrow \ln|w| = -\ln|t| + C = \ln t^{-1} + C$, $t>0$

$\Rightarrow |w| = A \cdot t^{-1}$, $A \in \mathbb{R}^+ \Rightarrow w = B \cdot t^{-1}$, $B \in \mathbb{R}$ ... $w = 0$ is also a soluti

$\Rightarrow v = \int w = \int Bt^{-1} dt = B \ln t + C$

$\Rightarrow \underline{y(t) = tv = Bt\ln t + Ct}$

③ Verify that $y(t) = e^{2t}$ solves $y'' - (4+\frac{2}{t})y' + (4+\frac{4}{t})y = 0$, $t>0$ and hence find the general solution.

$$y = e^{2t}, \quad y' = 2e^{2t}, \quad y'' = 4e^{2t} \quad \Rightarrow 4 - (4+\tfrac{2}{t})2 + (4+\tfrac{4}{t}) = 0 \ \checkmark$$

RoO: $z = e^{2t}v \Rightarrow z' = e^{2t}v' + 2e^{2t}v \Rightarrow z'' = e^{2t}v'' + 4e^{2t}v' + 4e^{2t}v$

$$\Rightarrow v'' + 4v' + 4v - (4+\tfrac{2}{t})(v'+2v) + (4+\tfrac{4}{t})v = 0$$

$$v'' + v'(4 - 4 - \tfrac{2}{t}) + v(4 - 8 - \tfrac{4}{t} + 4 + \tfrac{4}{t}) = 0$$

$$\Rightarrow v'' - \tfrac{2}{t}v' = 0 \quad \dots \quad w := v'$$

$$w' - \tfrac{2}{t}w = 0 \quad \rightarrow \mu = e^{\int -\frac{2}{t}dt} = e^{-2\ln t} = t^{-2}$$

$$\Rightarrow w = t^2 \cdot \int 0\,dt = C \cdot t^2$$

$$\Rightarrow v = \int w = \int C t^2\,dt = C_1 t^3 + C_2$$

$$\Rightarrow \underline{\underline{z(t) = e^{2t}v = (C_1 + C_2 t^3)e^{2t}}}$$

Check for lin. independence: $W(e^{2t}, t^3e^{2t}) = \begin{vmatrix} e^{2t} & t^3e^{2t} \\ 2e^{2t} & 3t^2e^{2t} + 2t^3e^{2t} \end{vmatrix} = e^{2t}\begin{vmatrix} 1 & t^3 \\ 2 & 3t^2+2t^3 \end{vmatrix}$

$$\rightarrow t=1: \ e^2 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 3+2 \end{vmatrix} = e^2(5-2) \neq 0 \quad \checkmark$$

④ Solve $t^3y'' - ty' + y = 0$, $t>0$

guess: $y(t) = t^r \Rightarrow t^3 \cdot r(r-1)t^{r-2} - t r \cdot t^{r-1} + t^r = 0$

$$t \cdot r(r-1) - r + 1 = 0 \qquad \Rightarrow y_1(t) = t$$

$$\Leftrightarrow r = 0 \ \vee \ r = 1 \ \rightarrow \ r = 1 \text{ works}$$

RoO: $z(t) = y_1(t)v(t) = tv \Rightarrow z' = tv' + v \Rightarrow z'' = tv'' + 2v'$

$$\rightarrow t^3(tv'' + 2v') - t(tv' + v) + tv = 0$$

$$t^4v'' + v'(2t^3 - t^2) = 0 \quad \dots \quad w := v'$$

$$\rightarrow t^4w' + w(2t^3 - t^2) = 0 \iff t^2w' + (2t-1)w = 0$$

$$\Rightarrow \frac{dw}{dt} = \frac{(1-2t)w}{t^2} \Rightarrow \int \frac{dw}{w} = \int \frac{1-2t}{t^2}dt = \int t^{-2} - 2t^{-1}\,dt$$

$$\Rightarrow \ln|w| = -t^{-1} - 2\ln t + C \Rightarrow |w| = A \cdot e^{-t^{-1}}t^{-2}, \quad A \in \mathbb{R}^+$$

$$\Rightarrow w = B \cdot e^{-t^{-1}} \cdot t^{-2}, \quad B \in \mathbb{R} \quad \dots \quad w = 0 \text{ works as well}$$

$$\Rightarrow v = \int w = B\int t^{-2}e^{-t^{-1}}dt = \left| \begin{matrix} u = -t^{-1} \\ du = t^{-2} \end{matrix} \right| = B\int e^u du = B(e^u + D)$$

$$= C_1 e^{-t^{-1}} + C_2$$

$$\Rightarrow \underline{\underline{z = tv = C_1 t e^{-t^{-1}} + C_2 t}}$$

- **Solving ODEs using Power Series**

Def: A power series about the point x_0 is a series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad , \quad a_i \in \mathbb{R}$$

Def: The radius of convergence of a power series is $R \in \mathbb{R}^+ \cup \{\infty\}$ s.t.

$|x - x_0| < R \Rightarrow$ the power series converges at x — we say the x is in the radius of convergence

$|x - x_0| > R \Rightarrow$ the power series diverges at x

Fact: Power series converge absolutely within the radius of convergence

$\Rightarrow$ they can be added, multiplied, differentiated and integrated term by term and it will hold inside of the radius

→ we can find the radius of convergence using $\frac{1}{R} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

Def: The function $f : \mathbb{R} \to \mathbb{R}$ is analytic at the point x_0 $\equiv \exists F$, power series about x_0 with radius of convergence $R > 0$ s.t. $|x - x_0| < R \Rightarrow F(x) = f(x)$.

Convergent power series about x_0 form a vector space with basis $\{(x - x_0)^n \mid n \in \mathbb{N}_0\}$

Polynomials are a special type of power series. Since polynomials converge everywhere, they are analytic and have ∞ radius of convergence.

↳ degree of a polynomial $= d \quad \equiv \quad a_d \neq 0 \;\&\; n > d \Rightarrow a_n = 0$.

Example: Find a powerseries solution of the ODE $(4 - x^2) y'' + 6y = 0$

Let $y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \Rightarrow y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$

↳ $n = 0,1 \Rightarrow \sum \overset{0}{\cdots}$

$$\Rightarrow (4 - x^2) \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

coefficients:

$$\forall n \geq 0: \; x^n: \; 4(n+2)(n+1) a_{n+2} - \overbrace{n(n-1) a_n}^{0 \text{ if } n \in \{0,1\}} + 6 a_n = 0$$

$$\Rightarrow a_{n+2} = a_n \frac{n(n-1) - 6}{4(n+2)(n+1)} = a_n \frac{n^2 - n - 6}{4(n+2)(n+1)} = a_n \frac{(n-3)(n+2)}{4(n+2)(n+1)} = a_n \frac{n-3}{4(n+1)}$$

→ we can use generating functions to find an exact formula for a_n using a_0 and a_1

↳ and from this perhaps an exact formula for y

→ but we can easily get a particular solution of the ODE:

$a_5 = a_3 \cdot \frac{3-3}{4\cdot(4)} = 0 \Rightarrow a_5, a_7, a_9, \ldots = 0$ $\rightarrow a_3 = a_1 \frac{1-3}{4\cdot 2} = -\frac{1}{4}a_1$

if we let $a_0 = 0$, we have $a_2, a_4, \ldots = 0$

⇒ a non-zero polynomial solution is given by

$$y(x) = a_1 x + a_3 x^3 = a_1 x - \frac{1}{4} a_1 x^3 = \underline{\underline{a_1 x\left(1 - \frac{x^2}{4}\right)}}$$

→ if this was an IVP and we knew $y(0)$, we could figure out a_0 and $y'(0)$ would give us a_1

⇒ consider $y(0) = 1$ & $y'(0) = 0$

$y(0) = 1 = \sum_{n=0}^{\infty} a_n x^n \Rightarrow a_0 = 1$

$y'(0) = 0 = \sum_{n=0}^{\infty} n a_n x^{n-1} \Rightarrow a_1 = 0 \Rightarrow$ all odd terms $= 0$

$\rightarrow a_2 = -\frac{3}{4} a_0 = -\frac{3}{4} \Rightarrow a_4 = \frac{2-3}{4(2+1)} a_2 = -\frac{1}{12} \cdot \left(-\frac{3}{4}\right) = \frac{1}{16} \quad \ldots$

$\Rightarrow y(x) = 1 - \frac{3}{4}x^2 + \frac{1}{16}x^4 + \ldots$

• Ordinary and singular points

→ the forms of a powerseries solution depends on the type of point the expansion is about

Def: Consider the linear 2nd order ODE in standard form

$$y'' + p y' + q y = g \quad , \quad p, q, g : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$$

The point $x_0 \in I$ is an

i) ordinary point $\equiv p, q, r$ are analytic at x_0

ii) singular point $\equiv$ otherwise

If this is a homogeneous eq. $r(x) = 0$, then x_0 is a

iii) regular singular point $\equiv x_0$ is a singular point, but $(x-x_0)p$ and $(x-x_0)^2 q$ are analytic at x_0

iv) essential singularity $\equiv$ otherwise

If $f: \mathbb{R} \rightarrow \mathbb{R}$ is analytic at x_0, then it is infinitely differentiable at x_0

↳ it can be expressed as a powerseries

⇒ if f is not differentiable at x_0, it is not analytical there

Ex: $|x|$ at 0, $\frac{1}{x}$ at 0, $\frac{x}{(x-2)^2}$ at 2, $f: x \mapsto \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$ at 0

no derivative at $x = 0$

Series Solutions about an Ordinary Point

$y'' + py' + q = r$, x_0 is an ordinary point

1) Substitute $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, y' and y'' into the equation
2) If p, q or r are something like $\frac{x^2}{1-x}$, then multiply it out so there are no fractions, it will make everything easier
3) Expand the resulting coefficients to be powerseries about x_0 (Taylor series)
4) Group the terms by powers of $(x-x_0)$ and find a formula for the coefficients
   $\Rightarrow$ there will be two undetermined coefficients and the rest expressed recursively
   $\Rightarrow$ we can use generating functions to find an expression for them
   $\Rightarrow$ the two unknown coefficients can be found if given $y(x_0)$ and $y'(x_0)$
5) The radius of convergence of the resulting series is the largest radius which avoids any singular points $\rightarrow (x_0-R, x_0+R) \not\ni$ sing. point

Series Solutions about a Regular Singular Point

Theorem (Fuch): Consider the second order linear ODE

$$(x-x_0)^2 y'' + (x-x_0) p y' + q y = 0, \quad p, q \text{ analytic at } x_0$$

Divide by x^2 to obtain

$$y'' + \frac{p}{x-x_0} y' + \frac{q}{(x-x_0)^2} y = 0 \quad \Rightarrow x_0 \text{ is a regular singular point}$$

There exists a solution of the form

$$y(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n, \quad r \in \mathbb{C}, \quad a_0 \neq 0$$

Def: An Euler equation is an equation which can be written in the form

$$ax^2 y'' + bxy' + cy = 0, \quad a, b, c \in \mathbb{R}, \quad a \neq 0$$

Method: We will focus on finding solutions defined on the interval $(0, \infty)$
$\rightarrow$ they will have the form x^r, which is defined for $x \in (0, \infty)$

guess: $y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$

$$\Rightarrow ax^2 r(r-1)x^{r-2} + bxrx^{r-1} + cx^r = x^r(ar(r-1) + br + c) = 0$$

$$\Leftrightarrow p(r) := ar(r-1) + br + c = 0$$

→ The polynomial $p(r)$ is called the indicial polynomial

$\Rightarrow x^r$ is a solution $\Leftrightarrow$ $ar(r-1) + br + c = 0$

Theorem: Suppose the roots of the indicial equation $ar(r-1) + br + c = 0$ are r_1 and r_2. Then the general solution of the Euler equation

$$(*) \quad ax^2y'' + bxy' + cy = 0$$

on $(0,\infty)$ is

i) $y = C_1 x^{r_1} + C_2 x^{r_2}$ ... $r_1 \neq r_2$ distinct real roots

ii) $y = (C_1 + C_2 \ln x) x^r$ ... $r_1 = r_2$ repeated root

iii) $y = x^\lambda (C_1 \cos(\mu \ln x) + C_2 \sin(\mu \ln x))$ ... $r_{1,2} = \lambda \pm i\mu$ complex roots

Proof. Consider $Y(t) := y(e^t) = y(x)$ where $x = e^t$.

$$\Rightarrow Y'(t) = y'(x)x' = xy'(x)$$

$$\Rightarrow Y''(t) = x'y'(x) + x \cdot y''(x)x' = xy'(x) + x^2y''(x) = Y'(t) + x^2y''(x)$$

Substitute to $(*)$:

$$a(Y'' - Y') + bY' + cY = 0 \quad \Rightarrow aY'' + (b-a)Y' + cY = 0 \quad \boxtimes$$

This is a 2^{nd} order linear ODE with constant coefficients.

char eq: $ar^2 + (b-a)r + c = 0 \Leftrightarrow ar(r-1) + br + c = 0$ indicial eq.

We solve $\boxtimes$ for $Y(t) = y(e^t)$ ⇒ substitute $t = \ln x$ to find $y(x)$

i) $Y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ $\Rightarrow$ $y(x) = C_1 e^{r_1 \ln x} + C_2 e^{r_2 \ln x} = C_1 x^{r_1} + C_2 x^{r_2}$

ii) $Y(t) = (C_1 + C_2 t)e^{rt}$ $\Rightarrow$ $y(x) = (C_1 + C_2 \ln x) e^{r \ln x} = (C_1 + C_2 \ln x) x^r$

iii) $Y(t) = e^{\lambda t}(C_1 \cos(\mu t) + C_2 \sin(\mu t))$ $\Rightarrow$ $y(x) = x^\lambda (C_1 \cos(\mu \ln x) + C_2 \sin(\mu \ln x))$

Examples:

(1) $x^2y'' - xy' - 8y = 0$ → $y = x^r$ ⇒ $r(r-1) - r - 8 = r^2 - 2r - 8 = (r-4)(r+2) = 0$

⇒ $r_1 = 4$, $r_2 = -2$ ⇒ $y(x) = C_1 x^4 + C_2 x^{-2}$ $\quad x > 0$

(2) $x^2y'' - 5xy' + 9y = 0$ → $y = x^r$ ⇒ $r(r-1) - 5r + 9 = r^2 - 6r + 9 = (r-3)^2 = 0$

⇒ repeated root $r = 3$ ⇒ $y(x) = (C_1 + C_2 \ln x) x^3$ $\quad x > 0$

(3) $x^2y'' + 3xy' + 2y = 0$ → $y = x^r$ ⇒ $r(r-1) + 3r + 2 = r^2 + 2r + 2 = (r+1)^2 + 1 = 0$

⇒ $(r+1)^2 = -1$ ⇒ $r + 1 = \pm i$ ⇒ $r_{1,2} = -1 \pm i$

$Y(t) = e^{-t}(C_1 \cos t + C_2 \sin t)$ ⇒ $y(x) = \frac{1}{x}(C_1 \cos(\ln x) + C_2 \sin(\ln x))$ $\quad x > 0$

- The Frobenius method

$(x-x_0)^2 y'' + (x-x_0) p y' + q y = 0$, p, q analytic at x_0 $\Rightarrow$ x_0 regular singular point

$\Rightarrow$ change variables $z := x - x_0$ $\Rightarrow$ $y(x) = y(z + x_0)$ $\Rightarrow$ WLOG: $x_0 = 0$

1) Fuch's theorem sais that $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ gives a solution

$\Rightarrow$ substitute $y(x)$, $y'(x)$ and $y''(x)$ into the equation

$$0 = x^2 y'' + x p y' + q y = x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + x p \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + q \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= \sum_{n=0}^{\infty} a_n \left[(n+r)(n+r-1) + p(x)(n+r) + q(x)\right] x^{n+r}$$

$$= \left[r(r-1) + r p(x) + q(x)\right] a_0 x^r + \sum_{n=1}^{\infty} a_n \left[(n+r)(n+r-1) + p(x)(n+r) + q(x)\right] x^{n+r}$$

2) Expand $p(x)$ and $q(x)$ as power series about 0 – Taylor expansion

3) Group terms by powers of x. Since the RHS $= 0$, all of the coefficients $= 0$.

- we get a recurrent relation for a_n with a_0 undetermined
  - $\Rightarrow$ the general solution needs two independent solutions
- the lowest power of x after expanding $p(x)$ and $q(x)$ will be x^r

$$p(x) = \sum_{k=0}^{\infty} \frac{p^{(k)}(0)}{k!} x^k, \quad q(x) = \sum_{k=0}^{\infty} \frac{q^{(k)}(0)}{k!} x^k$$

$\Rightarrow$ coefficient of x^r: $\left[r(r-1) + r p(0) + q(0)\right] a_0 = 0$

$\Rightarrow$ indicial equation $r(r-1) + r p(0) + q(0) = 0$

$\rightarrow$ $\sum_{n=0}^{\infty} a_n x^{n+r}$ is a solution $\rightarrow$ $r(r-1) + r p(0) + q(0) = 0$ $\longrightarrow$ which root to choose?

4) The radius of convergence of the series solution is the largest radius that avoids any other singular points

<u>Theorem:</u> Suppose the roots of the indicial equation $r(r-1) + r p(0) + q(0) = 0$ are r_1 and r_2. Then the fundamental set of solutions of

$$x^2 y'' + x p(x) y' + q(x) y = 0$$

on $(0, \infty)$ is given by

a_n and b_n obey the same recurrence relation but a_0 and b_0 may be chosen arbitrarily

i) $r_1 > r_2$ & $r_1 - r_2 \notin \mathbb{N}$

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1}, \quad y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

$\Rightarrow$ both roots give independent solutions

ii) $r_1 > r_2$ & $r_1 - r_2 \in \mathbb{N}$

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1}, \quad y_2(x) = K y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

→ y_2 will have undetermined b_0 which then determines K and b_n up to but not including $b_{r_1-r_2}$, which can be set arbitrarily. This then determines the rest of b_n

→ note that K can be zero

iii) $r_1 = r_2 =: r$

essentially $K=0$ $\quad$ b_n depends on a_0 and b_0

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r}$$

iv) $r_{1,2} = \lambda \pm i\mu$ → a_n and b_n obey the same recurrence relation but a_0 and b_0 may be chosen arbitrarily

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\lambda} \cos(\mu \ln x), \quad y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\lambda} \sin(\mu \ln x)$$

↳ complex: $\sum_{n=0}^{\infty} a_n x^{n+\lambda+i\mu} = \sum_{n=0}^{\infty} a_n x^{n+\lambda} \underset{\underset{\displaystyle e^{i\mu \ln(x)}}{\|}}{x^{i\mu}} = \sum_{n=0}^{\infty} a_n x^{n+\lambda} (\cos(\mu \ln x) + i \sin(\mu \ln x))$

Ex: $x^2 y'' - x y' + (1-x) y = 0$

Standard form: $y'' - \frac{1}{x} y' + \frac{1-x}{x^2} y = 0$

- $x = 0$ is a regular singular point
- $x \neq 0$ are ordinary points

Frobenius: $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$, $y'(x) = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$, $y''(x) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$$x^2 y'' - x y' + (1-x) y = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} (1-x)$$

$$= \sum_{n=0}^{\infty} a_n x^{n+r} \big((n+r)(n+r-1) - (n+r) + 1 \big) - \sum_{n=0}^{\infty} a_n x^{n+r+1}$$

$$= \sum_{n=0}^{\infty} a_n x^{n+r} \big((n+r)(n+r-2) + 1 \big) - \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Indicial eq: x^r: $a_n (r(r-2)+1) = 0 \Rightarrow r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow r = 1$

Recurrence: $n \geq 1$: x^{n+r}: $a_n((n+r)(n+r-2)+1) - a_{n-1} = 0$

$$\rightarrow r = 1 \Rightarrow a_n(\underbrace{(n+1)(n-1)}_{n^2-1} + 1) = a_{n-1} \Rightarrow a_n = \frac{a_{n-1}}{n^2}$$

↳ a_0, $a_1 = \frac{a_0}{1^2}$, $a_2 = \frac{a_0}{1^2 \cdot 2^2}$, $a_3 = \frac{a_0}{1^2 \cdot 2^2 \cdot 3^2} \Rightarrow a_n = \frac{a_0}{(n!)^2}$

$$\Rightarrow y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_0 \frac{1}{(n!)^2} x^{n+1} = a_0 x \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}, \quad a_0 \in \mathbb{R}$$

$\Rightarrow$ we have repeated root $r=1$ and solution $y_1(x) = a_0 \sum_{m=0}^{\infty} \frac{x^{m+1}}{m!^2}$

<u>Second solution:</u> $y_2(x) = \ln(x)\, y_1(x) + \sum_{m=0}^{\infty} b_m x^{m+1}$

$$y_2 = \ln x \sum_{m=0}^{\infty} a_m x^{m+1} + \sum_{m=0}^{\infty} b_m x^{m+1}$$

$$y_2' = \ln x \sum_{m=0}^{\infty} (m+1) a_m x^m + \sum_{m=0}^{\infty} a_m x^m + \sum_{m=0}^{\infty} b_m (m+1) x^m$$

$$y_2'' = \ln x \sum_{m=0}^{\infty} (m+1) m\, a_m x^{m-1} + \sum_{m=0}^{\infty} (m+1) a_m x^{m-1} + \sum_{m=0}^{\infty} m\, a_m x^{m-1} + \sum_{m=0}^{\infty} b_m (m+1) m\, x^{m-1}$$

<u>Note:</u> $x^2 y'' - x y' + (1-x) y = 0 \Rightarrow$ get coefficients

$$x^{m+1}: (m+1) a_m + m a_m + b_m (m+1) m - \left(a_m + b_m (m+1) \right) + b_m - b_{m-1} = 0$$

$$\Rightarrow a_m (m+1+m-1) + b_m \left((m+1) m - (m+1) + 1 \right) = b_{m-1}$$

$$\Rightarrow 2m\, a_m + m^2 b_m = b_{m-1} \implies b_m = \frac{b_{m-1}}{m^2} - 2 \frac{a_m}{m} = \frac{b_{m-1}}{m^2} - 2 \frac{a_{m-1}}{m^3}$$

$$a_m = \frac{a_{m-1}}{m^2}$$

We have derived $b_m = \frac{b_{m-1}}{m^2} - 2 \frac{a_{m-1}}{m^3}$

$$b_1 = \frac{b_0}{1^2} - 2 \frac{a_0}{1^3} \qquad \rightarrow a_1 = \frac{a_0}{1^2}$$

$$b_2 = \frac{b_0}{1^2 \cdot 2^2} - 2 \frac{a_0}{1^3 2^2} - 2 \frac{a_0}{1^2 \cdot 2^3} = \frac{b_0}{2!^2} - 2 \frac{a_0}{2!^2} \left(\frac{1}{1} + \frac{1}{2} \right) \qquad \rightarrow a_2 = \frac{a_0}{2!^2}$$

$$b_3 = \frac{b_0}{2!^2 3^2} - 2 \frac{a_0}{2!^2 3^2} \left(\frac{1}{1} + \frac{1}{2} \right) - 2 \frac{a_0}{2!^2 \cdot 3^3} = \frac{b_0}{3!^2} - 2 \frac{a_0}{3!^2} \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} \right)$$

$\Rightarrow$ in general $b_m = \frac{b_0}{m!^2} - 2 \frac{a_0}{m!^2} H_m$

$$\Rightarrow y_2 = a_0 \ln x \sum_{m=0}^{\infty} \frac{x^{m+1}}{m!^2} + \sum_{m=0}^{\infty} \left(\frac{b_0}{m!^2} - 2 \frac{a_0}{m!^2} H_m \right) x^{m+1}$$

$$= \underline{\underline{(a_0 \ln x + b_0) \sum_{m=0}^{\infty} \frac{x^{m+1}}{m!^2} - 2 a_0 \sum_{m=0}^{\infty} \frac{x^{m+1}}{m!^2} H_m}}$$

y_1 is in fact contained here in $b_0 \sum \frac{x^{m+1}}{m!^2}$

$a_0, b_0 \in \mathbb{R}$

The general solution of $x^2 y'' - x y' + (1-x) y = 0$ is

Ex: $x^2y'' + 6xy' + (4x^2+6)y = 0$ *

Standard form: $y'' + \frac{6}{x}y' + \frac{4x^2+6}{x^2}y = 0$

- $x=0$ is a regular singular point
- $x \neq 0$ are ordinary points

Frobenius: $y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \Rightarrow y' = \sum_{m=0}^{\infty} a_m (m+r) x^{m+r-1} \Rightarrow y'' = \sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^{m+r-2}$

*: $\sum a_m (m+r)(m+r-1) x^{m+r} + 6\sum a_m (m+r) x^{m+r} + (4x^2+6)\sum a_m x^{m+r} = 0$

$\sum_{m=0}^{\infty} a_m x^{m+r}\left[(m+r)(m+r-1) + 6(m+r) + 6\right] + 4\sum_{m=0}^{\infty} a_m x^{m+r+2} = 0$

$\sum_{m=0}^{\infty} a_m x^{m+r}\left[(m+r)(m+r+5) + 6\right] + 4\sum_{m=2}^{\infty} a_{m-2} x^{m+r} = 0$

$m=0$: x^r: $a_0(r(r+5)+6) = 0 \Rightarrow r^2+5r+6 = (r+2)(r+3) = 0 \rightarrow r_1 = -2,\ r_2 = -3$

$m=1$: x^{r+1}: $a_1((r+1)(r+6)+6) = a_1(r^2+7r+12) = a_1(r+3)(r+4) = 0 \Rightarrow a_1 = 0 \vee r=-3 \vee r=-4$

↳ since $r_1 = -2 > -3 = r_2$, we choose -2 as our root $\Rightarrow a_1 = 0$

$m\geq 2$: x^{m+r}: $a_m[(m+r)(m+r+5)+6] + 4a_{m-2} = 0$

$r=-2$: $a_m[(m-2)(m+3)+6] + 4a_{m-2} = 0$ ... $(m-2)(m+3)+6 = m^2+m = m(m+1)$

$\Rightarrow 4a_{m-2} + m(m+1)a_m = 0 \Rightarrow a_m = \frac{-4a_{m-2}}{m(m+1)}$, $m\geq 2$

Finding a_m: $a_1 = 0 \Rightarrow a_3 = 0 \Rightarrow \dots\ a_{ODD} = 0$

$a_2 = \frac{-4}{2\cdot 3}a_0 \Rightarrow a_4 = \frac{(-4)^2}{2\cdot3\cdot4\cdot5}a_0 \Rightarrow a_6 = \frac{(-4)^3}{7!}a_0 \Rightarrow a_{2m} = \frac{(-4)^m}{(2m+1)!}a_0$

$\Rightarrow y_1(x) = \sum_{m=0}^{\infty} a_m x^{m-2} = \sum_{k=0}^{\infty} a_{2k} x^{2k-2} = \underline{\frac{a_0}{x^2}\sum_{m=0}^{\infty}\frac{(-4)^m}{(2m+1)!}x^{2m}}$

→ we have real roots $r_1 = -2$, $r_2 = -3 \Rightarrow r_1 - r_2 = 1 \in \mathbb{N}$

$\Rightarrow y_2 = K\ln(x)y_1 + \sum_{m=0}^{\infty} b_m x^{m+r_2} = K\ln(x)\sum_{m=0}^{\infty} a_m x^{m-2} + \sum_{m=0}^{\infty} b_m x^{m-3}$

$y_2' = K\ln x\sum_{m=0}^{\infty} a_m(m-2)x^{m-3} + K\sum_{m=0}^{\infty} a_m x^{m-3} + \sum_{m=0}^{\infty} b_m(m-3)x^{m-4}$

$y_2'' = K\ln x\sum a_m(m-2)(m-3)x^{m-4} + K\sum a_m(m-2)x^{m-4} + K\sum a_m(m-3)x^{m-4} + \sum b_m(m-3)(m-4)x^{m-5}$

Coefficients: $x^2y'' + 6xy' + (4x^2+6)y = 0$

x^{m-2}: $Ka_m(m-2) + Ka_m(m-3) + b_{m+1}(m-2)(m-3) + 6(Ka_m + b_{m+1}(m-2)) + 6b_{m+1} + 4b_{m-1} = 0$

$m\geq 1$ $\quad Ka_m(m-2+m-3+6) + b_{m+1}(m^2-5m+6+6m-12+6) + 4b_{m-1} = 0$

* $Ka_m(2m+1) + b_{m+1}(m^2+m) + 4b_{m-1} = 0$

x^{-3}: $b_0(-3)(-4) + 6b_0(-3) + 6b_0 = 0 \Rightarrow 12b_0 - 18b_0 + 6b_0 = 0 \Rightarrow b_0 \in \mathbb{R}$

x^{-2}: as x^{m-2} but without $4x^2y \Rightarrow$ without $4b_{m-1}$, $m=0$ *

$\Rightarrow Ka_0 + b_1\cdot 0 = 0 \Rightarrow Ka_0 = 0$ and we know $a_0 \neq 0 \Rightarrow K = 0$

↳ b_1 can be anything $\Rightarrow$ pick $b_1 = 0$

Therefore * becomes $b_{m+1}(m+1)m + 4b_{m-1} = 0 \Rightarrow b_{m+1} = \frac{-4b_{m-1}}{m(m+1)} \Rightarrow b_m = \frac{-4b_{m-2}}{m(m-1)}$, $m\geq 2$

$\hookrightarrow b_1 = 0 \Rightarrow b_3 = 0 \Rightarrow \dots b_{\text{ODD}} = 0$

$b_2 = \frac{-4}{2\cdot 1} b_0 \Rightarrow b_4 = \frac{(-4)^2}{4\cdot 3\cdot 2\cdot 1} b_0 \Rightarrow b_{2n} = \frac{(-4)^n}{(2n)!} b_0$

$\rightarrow$ since $k=0$ we have $y_2(x) = x^{-3} \sum_{n=0}^{\infty} b_n x^n = x^{-3} \sum_{n=0}^{\infty} b_{2n} x^{2n} = \underline{\underline{\frac{b_0}{x^3} \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n)!} x^{2n}}}$

The general solution of $x^2 y'' + 6xy' + (4x^2+6)y = 0$ is

$$\underline{\underline{y(x) = \frac{a_0}{x^2} \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n+1)!} x^{2n} + \frac{b_0}{x^3} \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n)!} x^{2n}}}, \quad a_0, b_0 \in \mathbb{R}$$

Ex: Solve $x^2y'' + 3xy' + y = 0$ as an Euler equation and using method of Frobenius

a) Euler equation: guess $y = x^r \Rightarrow y' = rx^{r-1} \Rightarrow y'' = r(r-1)x^{r-2}$

indicial eq: $r(r-1) + 3r + 1 = 0$

$r^2 + 2r + 1 = (r+1)^2 = 0 \Rightarrow$ repeated root $r = -1$

$$\Rightarrow y(x) = (c_1 + c_2 \ln x)\, x^{-1}$$

b) Frobenius method

$x = 0$ is a regular singular point and $x \neq 0$ are ordinary points

$y = \sum a_m x^{m+r}$, $\quad y' = \sum (m+r) a_m x^{m+r-1}$, $\quad y'' = \sum (m+r)(m+r-1) a_m x^{m+r-2}$

Coefficients: $x^2 y'' + 3xy + y = 0$

$x^r \Rightarrow m = 0$: $(0+r)(0+r-1)a_0 + 3(0+r)a_0 + a_0 = 0$

$r(r-1) + 3r + 1 = r^2 + 2r + 1 = (r+1)^2 = 0 \Rightarrow$ repeated $r = -1$

x^{m+r}: $a_m(m+r)(m+r-1) + 3(m+r)a_m + a_m = 0$

$m \geq 1$ $\quad a_m[(m+r)(m+r+2)+1] \overset{r=-1}{=\!=} a_m[\underbrace{(m-1)(m+1)}_{m^2-1} + 1] = m^2 a_m = 0 \Rightarrow a_m = 0$

$$\Rightarrow y_1(x) = \sum_{m=0}^{\infty} a_m x^{m-1} = a_0 x^{-1}$$

Second solution: $y_2(x) = \ln(x)\, y_1(x) + \sum_{m=0}^{\infty} b_m x^{m-1}$

$y_2(x) = a_0 \ln(x)\, x^{-1} + \sum_{m=0}^{\infty} b_m x^{m-1}$

$y_2'(x) = a_0 \ln(x)(-1x^{-2}) + a_0 x^{-2} + \sum_{m=0}^{\infty} b_m (m-1) x^{m-2}$

$y_2''(x) = a_0 \ln(x) \cdot 2x^{-3} - a_0 x^{-3} - 2a_0 x^{-3} + \sum_{m=0}^{\infty} b_m (m-1)(m-2) x^{m-3}$

Coefficients: $x^2y'' + 3xy' + y = 0$

x^{-1}: $-a_0 - 2a_0 + b_0(-1)(-2) + 3(a_0 + b_0(-1)) + b_0 = 0$

$a_0(-3+3) + b_0(2-3+1) = 0 \Rightarrow a_0 \in \mathbb{R}, \quad b_0 \in \mathbb{R}$

x^{m-1}: $b_m(m-1)(m-2) + b_m(m-1) + b_m = 0$

$m \geq 1$ $\quad \Rightarrow b_m[(m-1)(m-1)+1] = b_m[\underbrace{(m-1)^2}_{\geq 0} + \underbrace{1}_{>0}] = 0 \Rightarrow b_m = 0, \quad m \geq 1$

$\Rightarrow y_2(x) = \ln(x)\, y_1(x) + b_0 x^{-1}$

$= a_0 \ln(x) x^{-1} + b_0 x^{-1} = (b_0 + a_0 \ln x)\, x^{-1}, \quad a_0, b_0 \in \mathbb{R}$

↳ this also gives the general solution

Ex: Solve $y'' + y = 0$ using Frobenius method and otherwise

a) Normally: char eq: $r^2 + 1 = 0 \Rightarrow r = \pm i$

complex: $y = e^{it} = \cos t + i \sin t$

real: $\underline{\underline{y(x) = C_1 \cos x + C_2 \sin x}}$

b) Frobenius

$\forall x \in \mathbb{R}$ is an ordinary point of this equation $\Rightarrow$ don't need Frobenius

$$y(x) = \sum_{m=0}^{\infty} a_m x^m \Rightarrow y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \Rightarrow y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$\underline{y'' + y = 0}$:
$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0$$
$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$x^m: \quad (m+2)(m+1) a_{m+2} + a_m = 0 \Rightarrow a_{m+2} = -a_m \frac{1}{(m+1)(m+2)} \qquad m \geq 0$$

$\underline{\text{Even}}$: a_0, $a_2 = a_0 \frac{-1}{1 \cdot 2}$, $a_4 = a_0 \frac{(-1)^2}{1 \cdot 2 \cdot 3 \cdot 4} \Rightarrow a_{2m} = a_0 \frac{(-1)^m}{(2m)!}$

$\underline{\text{Odd}}$: a_1, $a_3 = a_1 \frac{-1}{2 \cdot 3}$; $a_5 = a_1 \frac{(-1)^2}{2 \cdot 3 \cdot 4 \cdot 5} \Rightarrow a_{2m+1} = a_1 \frac{(-1)^m}{(2m+1)!}$

$$y(x) = \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_{2m} x^{2m} + \sum_{m=0}^{\infty} a_{2m+1} x^{2m+1} =$$
$$= a_0 \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} (-1) \frac{x^{2m+1}}{(2m+1)!} = \underline{\underline{a_0 \cos x + a_1 \sin x}}, \quad a_0, a_1 \in \mathbb{R}$$

Ex: Solve $4xy'' + 2y' + y = 0$ using Frobenius method

$\underline{\text{Standard form}}$: $y'' + \frac{1}{2x} y' + \frac{1}{4x} y = 0$

- $x \neq 0$ are ordinary points
- $x = 0$ is a singular point $\rightarrow$ is it regular?

  $x^2 \left(\frac{1}{4x}\right) = \frac{1}{4} x$ and $x \left(\frac{1}{2x}\right) = \frac{1}{2}$ are both analytical $\Rightarrow$ it is regular

$\underline{\text{Frobenius}}$: $y = \sum_{m=0}^{\infty} a_m x^{m+r}$, $y' = \sum_{m=0}^{\infty} a_m (m+r) x^{m+r-1}$, $y'' = \sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^{m+r-2}$

$\underline{\text{Coefficients}}$ $4xy'' + 2y' + y = 0$

$$x^{r-1}: \quad 4(a_0 (r)(r-1)) + 2(a_0 (r)) = 0 \Rightarrow 2 a_0 r [2(r-1) + 1] = 0$$

$\Rightarrow \underline{\text{indicial eq}}$: $r(2r-1) = 0 \Rightarrow r_1 = \frac{1}{2} > r_2 = 0$ & $r_1 - r_2 = \frac{1}{2} \notin \mathbb{N}$

$m \geq 0$: x^{m+r}: $4[a_{m+1} (m+1+r)(m+r)] + 2[a_{m+1}(m+r+1)] + a_m = 0$

$\underline{\text{note}}$: $4(m+r+1)(m+r) + 2(m+r+1) = 2(m+r+1)(2(m+r)+1) = \overbrace{2(m+r+1)(2m+2r+1)}^{\circledast}$

$$\Rightarrow a_{m+1} \cdot \circledast + a_m = 0 \Rightarrow a_{m+1} = a_m \frac{-1}{\circledast} = a_m \frac{-1}{2(m+r+1)(2m+2r+1)}$$

$r_1 = \frac{1}{2}$: $a_0 \in \mathbb{R}$, $a_{m+1} = \frac{-1}{2(m+\frac{3}{2})(2m+2)} a_m = \frac{-1}{(2m+3)(2m+2)} a_m \Rightarrow a_m = \frac{-1}{2m(2m+1)} a_{m-1}$

$r_2 = 0$: $b_0 \in \mathbb{R}$, $b_{m+1} = \frac{-1}{2(m+1)(2m+1)} b_m = \frac{-1}{(2m+1)(2m+2)} b_m \Rightarrow b_m = \frac{-1}{2m(2m-1)} b_{m-1}$

$\rightarrow$ solving $4xy'' + 2y' + y = 0$, $\quad y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r \sum_{n=0}^{\infty} a_n x^n$

$r_1 = \frac{1}{2} \Rightarrow a_m = \frac{-1}{2m(2m+1)} a_{m-1}$ $\quad | \quad$ $r_2 = 0 \Rightarrow b_m = \frac{-1}{2m(2m-1)} b_{m-1}$

- $a_0 \in \mathbb{R}$, $a_1 = \frac{-1}{2\cdot 3} a_0$, $a_2 = \frac{(-1)^2}{2\cdot 3\cdot 4\cdot 5} a_0$, $a_3 = \frac{(-1)^3}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} a_0 \Rightarrow a_m = \frac{(-1)^m}{(2m+1)!} a_0$
- $b_0 \in \mathbb{R}$, $b_1 = \frac{-1}{2\cdot 1} b_0$, $b_2 = \frac{(-1)^2}{4\cdot 3\cdot 2\cdot 1} b_0 \Rightarrow b_m = \frac{(-1)^m}{(2m)!} b_0$

$\Rightarrow$ General solution of $4xy'' + 2y' + y = 0$ is

$$y(x) = a_0 \sqrt{x} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^m + b_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^m = \underline{\underline{a_0 \sin(\sqrt{x}) + b_0 \cos(\sqrt{x})}}$$

$\cos(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} \Rightarrow \cos(\sqrt{x}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^m$

$\sin(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} \Rightarrow \sin(\sqrt{x}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{m+\frac{1}{2}}$

Ex: $y'' - 2xy' + 2y = 0$

a) Find the general solution about $x_0 = 0$

b) Solve the initial value problem with $y(0) = 1$ and $y'(0) = 2$

- $\forall x \in \mathbb{R}$ are ordinary points $\Rightarrow$ no singular points

$y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \Rightarrow y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$ $\quad \hookrightarrow 0$ for $n = 0, 1$

$$\Rightarrow y'' - 2xy' + 2y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2\sum_{n=0}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n$$
$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2\sum_{n=0}^{\infty} a_n x^n (n-1)$$

coefficients: x^n: $(n+2)(n+1) a_{n+2} - 2a_n(n-1) = 0 \Rightarrow a_{n+2} = \frac{2a_n(n-1)}{(n+1)(n+2)}$

$a_0, a_1 \in \mathbb{R}$ $\rightarrow$ ODD: $a_3 = a_1 \frac{2\cdot 0}{2\cdot 3} = 0 \Rightarrow a_5 = 0 \Rightarrow \ldots a_{odd} = 0$

$\hookrightarrow$ EVEN: $a_2 = \frac{2a_0(-1)}{1\cdot 2} = -a_0 \Rightarrow a_4 = \frac{2a_2\cdot 1}{3\cdot 4} = -\frac{2}{3\cdot 4} a_0$

$a_6 = -\frac{2}{3\cdot 4} a_0 \cdot \frac{2\cdot 3}{5\cdot 6} = -4a_0 \frac{1\cdot 3}{3\cdot 4\cdot 5\cdot 6}$

$a_8 = -8a_0 \frac{1\cdot 3}{3\cdot 4\cdot 5\cdot 6} \cdot \frac{5}{7\cdot 8} = -16 a_0 \cdot \frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8} = -2^4 a_0 \cdot \frac{5!!}{8!}$

$a_{10} = -2^5 a_0 \frac{5!!}{8!} \cdot \frac{7}{9\cdot 10} = -2^5 a_0 \frac{7!!}{10!}$, $\quad a_{12} = -2^6 a_0 \frac{9!!}{12!}$

$$\Rightarrow a_{2n} = -2^n a_0 \frac{(2n-3)!!}{(2n)!} = -2^n a_0 \frac{(2n-3)!!}{(2n)!!(2n-1)!!} = -a_0 \frac{2^n (2n-3)!!}{(2^n n!)(2n-1)(2n-3)!!} = -\frac{a_0}{(2n-1)n!}$$

$$\Rightarrow y(x) = \sum_{n=0}^{\infty} a_n x^n = a_1 x + \sum_{n=0}^{\infty} a_{2n} x^{2n} = \underline{\underline{a_1 x - a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n-1)n!}}}, \quad a_0, a_1 \in \mathbb{R}$$

b) $y(0)=1 \Rightarrow \left(\sum a_n x^n\right)(0) = 1 \Rightarrow a_0 + 0 = 1 \Rightarrow a_0 = 1$

$y'(0) = 2 \Rightarrow \sum_{n=1}^{\infty} n a_n \underbrace{x^{n-1}}_{=0} = 2 \Rightarrow 1 \cdot a_1 = 2 \Rightarrow a_1 = 2$

$$\Rightarrow \underline{\underline{y(x) = 2x - \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n-1)n!}}}$$

• <u>Solving ODEs using Fourier Transforms</u>

<u>Ex:</u> Solve $y' + 2y = h(x)$, where $h(x) = \begin{cases} e^{-x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$

<u>Recall:</u> $\frac{d^n}{dx^n} f(x) \overset{\mathcal{F}}{\longleftrightarrow} (ik)^n \hat{f}(k)$

$\Rightarrow$ take the F.T. on both sides:

$$ik\,\hat{y} + 2\hat{y} = \hat{h} \Rightarrow \hat{y}(ik+2) = \hat{h} \Rightarrow \hat{y} = \frac{1}{ik+2}\hat{h}$$

$\Rightarrow$ we need to find $\hat{h}$

$$\hat{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x} e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x(ik+1)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{-1}{ik+1} e^{-x(ik+1)} \Big|_0^{\infty} = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{ik+1}$$

$$\Rightarrow \hat{y}(k) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(ik+2)(ik+1)} = \frac{1}{\sqrt{2\pi}} \left(\frac{-1}{ik+2} + \frac{1}{ik+1} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{ik+1} - \frac{1}{\sqrt{2\pi}} \frac{\frac{1}{2}}{i\frac{k}{2}+1} = \hat{h}(k) - \frac{1}{2}\hat{h}\left(\frac{k}{2}\right).$$

$$\Rightarrow \underline{\underline{y(x) = h(x) - h(2x) = \begin{cases} e^{-x} - e^{-2x} & , x \geq 0 \\ 0 & ; x < 0 \end{cases}}}$$

<u>General idea:</u>

1) F.T. the whole equation, utilizing the derivative rule to get rid of the derivatives
2) express the F.T. of the unknown function
3) inverse F.T. to get the unknown function

REVISION FOR EXAM

① Solve the IVP $y''' - 4y'' - 3y' + 18y = 0, \quad y(0) = y'(0) = 0, \quad y''(0) = 25$

char eq. $r^3 - 4r^2 - 3r + 18 = 0 \quad \rightarrow$ guess $\pm 1, 2, -2$ (× × ✓) $\rightarrow -2$ is a root

$$(r+2)(r^2 - 6r + 9) = 0$$
$$(r+2)(r-3)^2 = 0 \quad \Rightarrow r_1 = -2, \quad r_{2,3} = 3$$

$$\Rightarrow y(t) = C_1 e^{-2t} + (C_2 + C_3 t)e^{3t}$$
$$y'(t) = -2C_1 e^{-2t} + 3C_2 e^{3t} + C_3(3te^{3t} + e^{3t})$$
$$y''(t) = 4C_1 e^{-2t} + 9C_2 e^{3t} + C_3(9te^{9t} + 6e^{3t})$$

$$\Rightarrow \begin{aligned} y(0) &= 0 = C_1 + C_2 \\ y'(0) &= 0 = -2C_1 + 3C_2 + C_3 \\ y''(0) &= 25 = 4C_1 + 9C_2 + 6C_3 \end{aligned} \quad \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 \\ 4 & 9 & 6 & 25 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 5 & 6 & 25 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 25 \end{array}\right]$$

$$\Rightarrow C_3 = 5, \quad 5C_2 + 5 = 0 \Rightarrow C_2 = -1, \quad C_1 - 1 = 0 \Rightarrow C_1 = 1$$

$$\Rightarrow \underline{\underline{y(t) = e^{-2t} - e^{3t} + 5te^{3t}}}$$

② Solve $x'(t) = \begin{bmatrix} -6 & -10 \\ 5 & 9 \end{bmatrix} x(t) + \begin{bmatrix} 8t+1 \\ e^{-2t} \end{bmatrix}$

$$p_A(t) = (-6-t)(9-t) + 50 = t^2 - 3t - 4 = (t+1)(t-4)$$

$$\lambda_1 = -1 : \begin{bmatrix} -5 & -10 \\ 5 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$\lambda_2 = 4 : \begin{bmatrix} -10 & -10 \\ 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\Rightarrow y_h(t) = c_1 e^{-t}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow A = RDR^{-1} \Rightarrow \begin{bmatrix} -6 & -10 \\ 5 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}^{-1} \rightarrow \frac{1}{-1}\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\rightarrow \text{let } \tilde{x} = R\tilde{z} \Rightarrow z = R^{-1}x \Rightarrow z' = R^{-1}x' = R^{-1}(Ax + b) = R^{-1}(RDR^{-1}x + b) = Dz + R^{-1}b$$

$$\Rightarrow \begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 8t+1 \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} -z_1 + 8t + 1 + e^{-2t} \\ 4z_2 - 8t - 1 - 2e^{-2t} \end{bmatrix}$$

$\boxed{1}$ $z_1' + z_1 = 8t + 1 + e^{-2t} \Rightarrow \mu_1 = e^{\int dt} = e^t$

$$\Rightarrow z_1 = e^{-t}\int e^t(8t+1+e^{-2t})\,dt = e^{-t}\int (8t+1)e^t + e^{-t}\,dt$$
$$= e^{-t}\left((8t+1)e^t - 8e^t - e^{-t}\right) = \underline{\underline{8t - 7 - e^{-2t}}}$$

D	I
$+\ 8t+1$	e^t
$-\ 8$	e^t
0	e^t

$\boxed{2}$ $z_2' - 4z_2 = -8t - 1 - 2e^{-2t} \rightarrow \mu_2 = e^{-4t}$

$$z_2 = -e^{4t}\int e^{-4t}(8t+1+2e^{-2t})\,dt = -e^{4t}\int (8t+1)e^{-4t} + 2e^{-6t}\,dt$$
$$= -e^{4t}\left(-\tfrac{1}{4}(8t+1)e^{-4t} - \tfrac{1}{2}e^{-4t} + \tfrac{2}{-6}e^{-6t}\right) = \underline{\underline{2t + \tfrac{3}{4} + \tfrac{1}{3}e^{-2t}}}$$

D	I
$+\ 8t+1$	e^{-4t}
$-\ 8$	$-\frac{1}{4}e^{-4t}$
0	$\frac{1}{16}e^{-4t}$

$$\tilde{x} = R\tilde{z} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 8t - 7 - e^{-2t} \\ 2t + \frac{3}{4} + \frac{1}{3}e^{-2t} \end{bmatrix} = \begin{bmatrix} 16t - 14 - 2e^{-2t} + 2t + \frac{3}{4} + \frac{1}{3}e^{-2t} \\ -8t + 7 + e^{-2t} - 2t - \frac{3}{4} - \frac{1}{3}e^{-2t} \end{bmatrix}$$

$$\Rightarrow \underline{\underline{\tilde{x}(t) = c_1 e^{-t}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 18 \\ -10 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} -53 \\ 25 \end{bmatrix} + \frac{1}{3}e^{-2t}\begin{bmatrix} -5 \\ 2 \end{bmatrix}}}$$

③ Solve $y y' - x y^2 = 6x e^{4x^2}$, $y(1) = 3$

$$y' - xy = 6xe^{4x^2}y^{-1} \quad \dots \text{ Bernoulli equation}$$

$$\Rightarrow w = y^{1-(-1)} = y^2 \Rightarrow w' = 2yy' = 2y(xy + 6xe^{4x^2}y^{-1}) = 2xw + 12xe^{4x^2}$$

$$\Rightarrow w' - 2xw = 12xe^{4x^2} \quad \to \mu = e^{\int -2x\,dx} = e^{-x^2}$$

$$\to w = e^{x^2}\int e^{-x^2} 12xe^{4x^2} dx = e^{x^2}\int 12x\,e^{3x^2}dx = \left| \begin{matrix} u = 3x^2 \\ du = 6x\,dx \end{matrix} \right|$$

$$= e^{x^2}\int 2e^u du = e^{x^2}(2e^u + c) = e^{x^2}(2e^{3x^2} + c)$$

$$y(1) = 3 \Rightarrow w(1) = y^2 = 9 \Rightarrow 9 = e(2e^3 + c) \Rightarrow \frac{9}{e} - 2e^3 = c$$

$$\Rightarrow \underline{\underline{y(x) = \sqrt{w} = e^{x^2/2}\sqrt{2e^{3x^2} + \frac{9}{e} - 2e^3}}}$$

④ $x^3 + y^3 - xy^2y' = 0$

$$1 + \left(\frac{y}{x}\right)^3 - y'\left(\frac{y}{x}\right)^2 = 0 \qquad \Rightarrow u = \frac{y}{x} \Rightarrow y = xu \Rightarrow y' = xu' + u$$

$$1 + u^3 - (xu' + u)\cdot u^2 = 0 \Rightarrow \frac{1 + u^3}{u^2} = xu' + u \Rightarrow xu' = \frac{1 + u^3}{u^2} - u = \frac{1}{u^2}$$

$$\Rightarrow x\frac{du}{dx} = \frac{1}{u^2} \Rightarrow \int u^2 du = \int\frac{dx}{x} \Rightarrow \frac{u^3}{3} = \ln|x| + C$$

$$\Rightarrow u = \sqrt[3]{3\ln|x| + C} \qquad \Rightarrow \underline{\underline{y = x\sqrt[3]{\ln|x| + C}}}$$

⑤ $y''' - 6y'' + 13y' - 10y = 0$

$$r^3 - 6r^2 + 13r - 10 = 0 \quad \dots \text{ guess } \underset{\times}{\pm 1},\ \underset{\checkmark}{2},\ \underset{\times}{-2}$$

$$(r-2)(r^2 - 4r + 5) = 0$$

$$\hookrightarrow r = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i$$

complex: $e^{(2+i)t} = e^{2t}(\cos t + i\sin t)$

$$\Rightarrow \underline{\underline{y(t) = c_1e^{2t} + c_2e^{2t}\cos t + c_3e^{2t}\sin t}}$$

⑥ Solve $y''' - 6y'' + 13y' - 10y = 17e^{-2t} - 10t + 33$

from ⑤: $y_h(t) = e^{2t}(C_1 + C_2\cos t + C_3 \sin t)$

undetermined coeff:

$$\left.\begin{aligned} y_u(t) &= Ae^{-2t} + Bt + C \\ y_u'(t) &= -2Ae^{-2t} + B \\ y_u''(t) &= 4Ae^{-2t} \\ y_u'''(t) &= -8Ae^{-2t} \end{aligned}\right\} \quad \begin{aligned} &e^{-2t}: -8A - 24A - 26A - 10A = 17 \\ &\qquad \Rightarrow A = -\frac{17}{68} = -\frac{1}{4} \\ &t: -10B = -10 \Rightarrow B = 1 \\ &t^0: 13B - 10C = 33 \Rightarrow 13 - 10C = 33 \to C = -2 \end{aligned}$$

$$\Rightarrow y(t) = e^{2t}(C_1 + C_2\cos t + C_3 \sin t) - \frac{1}{4}e^{-2t} + t - 2$$

⑦ Use the method of successive approximations to solve $y' = -3t^2(y+1)$, $y(0) = 0$

$y' = f(t,y) = -3t^2(y+1)$ and $\frac{\partial f}{\partial y} = -3t^2$ are both continuous

→ initial guess $y_0(t) = 0$ satisfies $y(0) = 0$

$$y_1(t) = \int_0^t f(s, y(s))\,ds = -3\int_0^t s^2(0+1)\,ds = -3\left[\frac{s^3}{3}\right]_0^t = -t^3$$

$$y_2(t) = -3\int_0^t s^2(-s^3+1)\,ds = -3\int_0^t -s^5 + s^2\,ds = -3\left[\frac{s^3}{3} - \frac{s^6}{6}\right]_0^t = -t^3 + \frac{t^6}{2}$$

$$y_3(t) = -3\int_0^t s^2\left(1 - s^3 + \frac{s^6}{2}\right)ds = -3\left[\frac{s^3}{3} - \frac{s^6}{6} + \frac{s^9}{2\cdot 9}\right]_0^t = -t^3 + \frac{t^6}{2} - \frac{t^9}{2\cdot 3}$$

$$\Rightarrow y_n(t) = \sum_{i=1}^{n} (-1)^i \frac{t^{3i}}{i!}$$

Induction: $y_{n+1}(t) = -3\int_0^t s^2 \sum_{i=1}^{n} (-1)^i \frac{s^{3i}}{i!}\,ds = -3\sum_{i=1}^{n} (-1)^i \frac{s^{3i+3}}{i!(3i+3)}$ ✓

$$\Rightarrow y(t) = \lim_{n\to\infty} y_n(t) = \sum_{i=1}^{\infty} (-1)^i \frac{t^{3i}}{i!} = \sum_{i=1}^{\infty} \frac{(-t^3)^i}{i!} = e^{-t^3} - 1$$

⑧ Let $t_0, y_0 \in \mathbb{R}$ and consider $y'(t) = 5(y(t))^{\frac{4}{5}}$, $y(t_0) = y_0$.

a) for which t_0, y_0 is the solution unique?

$f(t,y) = 5y^{\frac{4}{5}}$ and $\frac{\partial f}{\partial y} = 5\cdot\frac{4}{5}\cdot y^{-\frac{1}{5}} = \frac{4}{y^{1/5}}$ have to be continuous

↳ problem when $y = 0$

b) find 3 solutions when $y_0 = 0$

1) $y = 0$ is a solution

$y(t_0) = 0 \Rightarrow 0 = 5t_0 + c$

2) $\frac{dy}{dt} = 5y^{\frac{4}{5}} \Rightarrow \int y^{-\frac{4}{5}}dy = \int 5\,dt \Rightarrow 5y^{\frac{1}{5}} = 5t + c = 5t - 5t_0$

$\Rightarrow y^{1/5} = t - t_0 \Rightarrow y = (t-t_0)^5$

3)  → compound functions $y(t) = \begin{cases} 0, & t < b \\ (t-b)^5, & t \geq b \end{cases}$ ... $b > t_0$

(9) Solve $y''' + y' = \sec t$, $y(0)=2$, $y'(0)=1$, $y''(0)=-2$, $|t| < \frac{\pi}{2}$

let $w := y'$, then $w'' + w = \sec t$

$$r^2 + 1 = 0 \Rightarrow r = \pm i$$

$$\Rightarrow w = c_1 \cos t + c_2 \sin t$$

Variation of parameters: $w_1 = \cos t$, $w_2 = \sin t$ $\rightarrow W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$

$$\Rightarrow \frac{g(t)}{W(t)} = \sec t / (\cos^2 t + \sin^2 t) = \sec t$$

$\int u = \cos$, $du = -\sin dt$

$|t| < \frac{\pi}{2}$

$$\Rightarrow u_1' = -y_2 \cdot (\sec t) = -\sin t \sec t \Rightarrow u_1 = -\int \frac{\sin t}{\cos t} dt = \ln|\cos t| = \ln(\cos t)$$

$$u_2' = +y_1 \cdot (\sec t) = \cos t \sec t = u_2 = \int 1 \, dt = t$$

$$\Rightarrow w_p = w_1 u_1 + w_2 u_2 = \cos t \ln(\cos t) + t \sin t$$

$$\Rightarrow w(t) = c_1 \cos t + c_2 \sin t + \cos t \ln(\cos t) + t \sin t$$

$$\Rightarrow y = \int w = c_1 \sin t - c_2 \cos t + \int \cos t \ln(\cos t)\, dt + \int t \sin t \, dt + c_3$$

	D	I
+	$\ln \cos t$	$\cos t$
−	$-\frac{\sin t}{\cos t}$	$\sin t$

	D	I
+	t	$\sin t$
−	1	$-\cos t$
	0	$-\sin t$

$$I_1 = \sin t \ln \cos t + \int \frac{\sin^2 t}{\cos t} dt = \sin t \ln \cos t + \int \frac{1 - \cos^2 t}{\cos t}$$

$$= \sin t \ln \cos t + \int \sec t \, dt - \int \cos t \, dt$$

$$= \sin t \ln(\cos t) - \sin t + \ln|\sec t + \operatorname{tg} t|$$

$$I_2 = -t \cos t + \sin t$$

$\int \sec t \frac{\sec t + \operatorname{tg} t}{\sec t + \operatorname{tg} t} dt$ $\quad u = \sec t + \operatorname{tg} t$, $du = \sec t \operatorname{tg} t + \sec^2 t$

$= \int \frac{du}{u} = \ln|\sec t + \operatorname{tg} t|$

$$\Rightarrow y = c_1 \sin t - c_2 \cos t - t \cos t + \sin t \ln(\cos t) + \ln|\sec t + \operatorname{tg} t| + c_3$$

$$y' = w = c_1 \cos t + c_2 \sin t + t \sin t + \cos t \ln(\cos t)$$

$$y'' = -c_1 \sin t + c_2 \cos t + \sin t + t \cos t - \sin t \ln(\cos t) + \cos t \frac{-\sin t}{\cos t}$$

$$= -c_1 \sin t + c_2 \cos t + t \cos t - \sin t \ln(\cos t)$$

$$\Rightarrow y(0) = 2 = -c_2 + c_3$$

$$y'(0) = 1 = c_1$$

$$y''(0) = -2 = c_2$$

$$2 = 2 + c_3 \Rightarrow c_3 = 0$$

$$\Rightarrow \underline{y(t) = \sin t + 2\cos t - t \cos t + \sin t \ln(\cos t) + \ln|\sec t + \operatorname{tg} t|}$$

(10) $y' = \frac{y}{1+t} - \frac{y}{t} + t^2(1+t)$

substitute $u = \frac{y}{1+t} \Rightarrow y = (1+t)u \rightarrow y' = (1+t)u' + u$

$(1+t)u' + u = u - \frac{1+t}{t}u + t^2(1+t)$

$(1+t)u' + \frac{1+t}{t}u = t^2(1+t) \quad \Rightarrow u' + t^{-1}u = t^2 \quad \rightarrow \mu = e^{\int t^{-1}dt} = e^{\ln|t|} = |t|$

$\Rightarrow u = \frac{1}{|t|} \cdot \int |t| t^2 dt$

$\left.\begin{array}{l} t>0: \; u = \frac{1}{t}\int t^3 dt \\ t<0: \; u = -\frac{1}{t}\int -t^3 dt \end{array}\right\} u = \frac{1}{t}\left(\frac{t^4}{4} + c\right) = \frac{t^3}{4} + \frac{c}{t}$

$y = (1+t)u = \frac{1}{4}t^3(1+t) + C \cdot \frac{t+1}{t}$

(11) Show that the zero solution of the system $\begin{array}{l} x' = y \\ y' = -4x - 0.1y \end{array}$ is stable

$\rightarrow$ construct a Lyapunov function $V(x,y) = ax^2 + by^2$, $a, b > 0$

$\frac{dV}{dt} = 2axx' + 2byy' = 2axy - 8bxy - 0.2by^2$

$= 2xy(a - 4b) - 0.2by^2 \leq 0$

$\rightarrow$ let $a = 4$, $b = 1$

$\frac{dV}{dt} = -0.2y^2 \leq 0$ and $= 0$ when $y = 0$, $x \in \mathbb{R}$

$\Rightarrow \frac{dV}{dt}$ is semi-negative definite $\Rightarrow 0$ is stable