An Elementary Proof of Stirling's Formula

Stirling's formula $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$ is a powerful asymptotic approximation of the factorial function. Many well-known proofs of this formula are grounded in integral calculus. In this paper we present an alternative proof of Stirling's formula using only limits and the Wallis product.

Proposition 1. *The sequence* $a_n := \dfrac{n!}{\sqrt{n}\, n^n e^{-n}}$ *converges to a positive number.*

Proof. First we will show, that the sequence (a_n) is decreasing and then prove, that it is bounded from below by a positive number. To begin, note that

$$\frac{a_n}{a_{n+1}} = \frac{n!}{\sqrt{n}\, n^n e^{-n}} \cdot \frac{\sqrt{n+1}\,(n+1)^{n+1}\, e^{-n-1}}{(n+1)n!} = \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{(n+1)^n}{n^n e} = \frac{1}{e}\left(\frac{n+1}{n}\right)^{\frac{2n+1}{2}}.$$

Now define $b_n := \ln(a_n)$. Then

$$b_n - b_{n+1} = \ln\left(\frac{a_n}{a_{n+1}}\right) = \frac{2n+1}{2}\ln\left(\frac{n+1}{n}\right) - 1.$$

Next we introduce a new variable, k, such that $\frac{n+1}{n} = \frac{1+k}{1-k}$. This choice of k proves useful, as it allows us to utilize a Taylor series expansion. To satisfy this condition we set $k := \frac{1}{2n+1}$, leading to the following:

$$b_n - b_{n+1} = \frac{2n+1}{2}\ln\left(\frac{n+1}{n}\right) - 1 = \frac{1}{2k}\ln\left(\frac{1+k}{1-k}\right) - 1.$$

Using a Taylor series expansion, we get

$$\begin{aligned}
\ln\left(\frac{1+k}{1-k}\right) &= \ln(1+k) - \ln(1-k) = \\
&= \left(k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \cdots\right) - \left(-k - \frac{k^2}{2} - \frac{k^3}{3} - \frac{k^4}{4} - \cdots\right) = \\
&= 2\left(k + \frac{k^3}{3} + \frac{k^5}{5}\cdots\right) = 2\sum_{i=0}^{\infty}\frac{k^{2i+1}}{2i+1}.
\end{aligned}$$

Now we have

$$b_n - b_{n+1} = \frac{1}{2k}\ln\left(\frac{1+k}{1-k}\right) - 1 = \sum_{i=0}^{\infty}\frac{k^{2i}}{2i+1} - 1 = \sum_{i=1}^{\infty}\frac{k^{2i}}{2i+1} > 0.$$

Therefore, the sequence (b_n) is decreasing; the natural logarithm is a monotonic function and so (a_n) is decreasing as well. In order to show that it is bounded from bellow, we resume the calculation, noting that $0 < k < 1$:

$$\begin{aligned}
b_n - b_{n+1} &= \sum_{i=1}^{\infty}\frac{k^{2i}}{2i+1} < \sum_{i=1}^{\infty}k^{2i} = k^2\sum_{i=1}^{\infty}k^{2i-2} = k^2\sum_{i=0}^{\infty}k^{2i} = \\
&= \frac{k^2}{1-k^2} = \frac{1}{(2n+1)^2\left(1 - \frac{1}{(2n+1)^2}\right)} = \frac{1}{(2n+1)^2 - 1} = \\
&= \frac{1}{2n(2n+2)} = \frac{1}{4n(n+1)} = \frac{1}{4n} - \frac{1}{4(n+1)}.
\end{aligned}$$

Hence

$$b_n - \frac{1}{4n} < b_{n+1} - \frac{1}{4(n+1)}.$$

We see that the sequence $\left(b_n - \frac{1}{4n}\right)$ is increasing, therefore

$$b_n > b_n - \frac{1}{4n} > b_1 - \frac{1}{4} = \frac{3}{4} \Rightarrow a_n > e^{0.75},$$

bounding (a_n) from bellow. This completes the proof.

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We have shown, that $n!$ grows up to a constant multiple as does $\sqrt{n}\, n^n e^{-n}$. We will need the following lemma to find this constant.

Definition 1. *Define* $(2n)!! := 2\cdot 4\cdot 6\cdots(2n)$ *and* $(2n-1)!! := 1\cdot 3\cdot 5\cdots(2n-1)$.

Observation. It holds, that $(2n)!!(2n-1)!! = (2n)!$ and $(2n)!! = 2^n\, n!$.

Lemma 2. $\displaystyle\lim_{n\to\infty} \frac{4^n\, n!^2}{\sqrt{n}\,(2n)!} = \sqrt{\pi}$.

Proof. Rewrite the famous Wallis product $\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots = \frac{\pi}{2}$ as a limit:

$$\lim_{n\to\infty} \frac{(2n)!!^2}{(2n-1)!!^2(2n+1)} = \frac{\pi}{2}.$$

By expanding the fraction and simplifying the double factorials, we obtain

$$\lim_{n\to\infty} \frac{(2n)!!^2(2n)!!^2}{(2n)!!^2(2n-1)!!^2(2n+1)} = \lim_{n\to\infty} \frac{2^{4n}\, n!^4}{(2n)!^2(2n+1)} = \frac{\pi}{2}.$$

After taking the square root, we get

$$\lim_{n\to\infty} \frac{4^n\, n!^2}{(2n)!\sqrt{2n+1}} = \sqrt{\frac{\pi}{2}} \Rightarrow \lim_{n\to\infty} \frac{4^n\, n!^2\sqrt{2}}{(2n)!\sqrt{2n+1}} = \lim_{n\to\infty} \frac{4^n\, n!^2}{(2n)!\sqrt{n}} = \sqrt{\pi}.$$

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Theorem 3. $\displaystyle A := \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!}{\sqrt{n}\, n^n e^{-n}} = \sqrt{2\pi}$.

Proof. We start with the limit derived in the preceding lemma. We have

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{4^n\, n!^2}{\sqrt{n}\,(2n)!} = \lim_{n\to\infty} \frac{4^n \left(\dfrac{n!}{\sqrt{n}\, n^n e^{-n}}\right)^2 \left(\sqrt{n}\, n^n e^{-n}\right)^2}{\sqrt{n}\, \dfrac{(2n)!}{\sqrt{2n}\,(2n)^{2n} e^{-2n}} \sqrt{2n}\,(2n)^{2n} e^{-2n}} =$$

$$= \lim_{n\to\infty} \frac{4^n\, A^2\, n\, n^{2n} e^{-2n}}{\sqrt{n}\, A\, \sqrt{2n}\,(2n)^{2n} e^{-2n}} = \lim_{n\to\infty} \frac{A}{\sqrt{2}}.$$

Therefore $A = \sqrt{2\pi}$.

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Corollary. Thus, we obtain Stirling's formula $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

References

I learned about this proof from Dominik Beck, who showed me how the Wallis product can be used to prove Stirling's formula. It could be argued here, that the Wallis product is typically proven using integral calculus, which undermines the essence of this proof. Fortunately for us, Johan Wästlund was able to prove the Wallis product using only elementary algebra, see [2]. The proof of the convergence of the limit comes from [1].

[1] Charles H.C. Little, Kee L. Teo, and Bruce van Brunt. *Real Analysis via Sequences and Series*. 1st ed. 2015. Springer New York, pp. 371–373. ISBN: 1-4939-2651-9.

[2] Johan Wastlund. "An Elementary Proof of the Wallis Product Formula for pi". In: *The American mathematical monthly* (2007), pp. 914–917. ISSN: 0002-9890.

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