Infinite Sets

Course notes from NAIL063 by doc. Kynčl

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Contents

1 ZF basics recap

We will be dealing with the Zermelo–Fraenkel (ZF) set theory; after adding the axiom of choice, we obtain the much stronger ZFC theory. Let us first recall some important axioms and definitions.

Remark. Note on notation: I always use the symbol $\subset$ for a proper subset and $\subseteq$ for a general subset.

1.1 Sets vs. classes

Definition 1 (Class). If $\varphi(x)$ is a formula, then the expression $\{x \mid \varphi(x)\}$ is called a *class term*. It defines the "collections" of all sets x satisfying $\varphi(x)$. We call this collection the *class* determined by $\varphi(x)$.

Every set is a class but not all classes are sets (take the class of all sets). Such a class is called a *proper class*. The major difference between sets and classes is that classes cannot be members of other classes or sets, while sets can.

We can substitute class terms into logical formulas in place of free variables, but unlike sets, we cannot quantify them using $\forall$ and $\exists$. It isn't hard to show that for every formula with class terms (but without quantified class variables), there is an equivalent formula in the base language without class terms.

We will usually denote sets using small letters $a, b, c, x, y, \dots$ and classes using capital letters A, B, C, etc. The exception from this are well-ordered sets which will often be denoted as W. Finally, the class of all sets, also called the *universal class* is denoted by V.

1.2 Axiom schema of replacement

Axiom. *When we take any map f and a preimage set a, then the class of images $b = f[a]$ is also a set. Formally, if $\psi(x,y)$ is a formula, without free variables y_1, y_2 and b, then the formula*

$$(\forall x)(\forall y_1,y_2)\big((\psi(x,y_1) \wedge \psi(x,y_2)) \Rightarrow y_1 = y_2\big) \Rightarrow \\ (\forall a)(\exists b) : (\forall y)\big(y \in b \Leftrightarrow (\exists x)(x \in a \wedge \psi(x,y))\big)$$

is an axiom. The formula $\psi(x,y_1)$, resp. $\psi(x,y_2)$ are created from $\psi(x,y)$ by substituting y_1, resp. y_2 for y.

The first part of this axiom says that $\psi(x,y)$ should behave like a map $y = f(x)$. In the second part, a denotes the set of preimages and b the set of corresponding images.

1.3 Axiom of choice

Definition 2 (Choice function). A choice function on the set x is any function $f : x \to \bigcup x$ s.t.

$$(\forall t \in x)(t \neq \varnothing \Rightarrow f(t) \in t).$$

We can WLOG assume that the choice function is defined on $x \setminus \{\varnothing\}$ and all $t \in \mathrm{Dom}(f)$ satisfy $f(t) \in t$.

Axiom (of choice)**.** *Every set has a choice function.*

Other famous conditions equivalent [1] to the axiom of choice are

- The well-ordering principle,
- Zorn's lemma,
- The cartesian product of a nonempty indexed family of sets is nonempty.

1.4 Natural numbers and the axiom of infinity

We use Von Neumann ordinals, meaning that natural numbers are defined as

$$0 := \varnothing, 1 := \{0\}, 2 := \{0,1\}, \dots\ n+1 := \{0,1,\dots n\} = n \cup \{n\}.$$

Definition 3. We can define the successor function as $S : V \to V$, mapping $S : v \mapsto v \cup \{v\}$. We will usually express this with the following notation

$$v + 1 := S(v) = v \cup \{v\}.$$

Definition 4. The set w is inductive, $\mathrm{Ind}(w) \equiv 0 \in w \wedge (\forall n \in w)(n+1 \in w)$.

Axiom (of infinity)**.** *There exists an inductive set.*

Definition 5. We define the set of all natural numbers as $\omega := \bigcap\{w \mid \mathrm{Ind}(w)\}$. Or equivalently, as the smallest inductive set.

1.5 Well-ordering and initial segments

Let us recall a very important definition in set theory, that is the notion of a well-ordered set.

Definition 6. The binary relation R on the class X is a

- trichotomy $\equiv (\forall x,y \in X)(x\,R\,y \vee y\,R\,x \vee x = y)$,
- strict order $\equiv$ it is anti-reflexive, strongly anti-symmetric and transitive on X; (note that strong anti-symmetry follows from the other two),
- (partial) order $\equiv$ it is reflexive, weakly anti-symmetric and transitive on X,
- total (or linear) order $\equiv$ is a trichotomous partial order on X.

If R is an ordering, then instead of $x\,R\,y$ we write $x \leq_R y$ and we call $(X, \leq_R)$ an ordered class. Similarly, if R is a strict ordering, then we write $x <_R y$ and we call $(X, <_R)$ a strictly ordered class.

We call an element minimal if there is no smaller one, and we call it a minimum if it is smaller than all others. Similarly, we define a maximal and maximum element.

[1] "The Axiom of Choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma?" — Jerry Bona

Observation. *Every minimum is minimal. Furthermore, if R is a total order, then there is at most one minimal element, and if it exists, then it is also the minimum. There is always at most one minimum.*

Definition 7. The ordered class $(A, \leq_R)$ is well-ordered $\equiv$ every non-empty $B \subseteq A$ has a minimum. We can similarly define a strict well-ordering.

Observation. *If $(X, \leq_R)$ is a well-ordered class and $Y \subseteq X$, then $(Y, \leq_R)$ is also a well-ordered class.*

Note that every well-ordered class is also totally ordered, because we can take any two elements, and one of them has to be the minimum and is therefore smaller.

Definition 8 (initial segment)**.** Let $(A, <_R)$ be a (strictly) ordered class. Then $X \subseteq A$ is an initial segment of $A \equiv (\forall x \in X)(\forall a \in A)(a <_R x \Rightarrow a \in X)$. Note that every element $a \in A$ determines a special initial segment

$$(\leftarrow, a) \coloneqq \{x \mid x \in A \wedge x <_R a\}$$

This is similar to the notion of intervals of real numbers of the form $(-\infty, a]$ and $(-\infty, a)$.

Observation. *Note that $a \notin (\leftarrow, a)$ and $x <_R y \iff (\leftarrow, x) \subset (\leftarrow, y)$.*

Lemma 1. *Let $(W, <_R)$ be a strictly well-ordered set and suppose that $X \subset W$ is an initial segment of W. Then there exists a unique $x \in W$ such that $X = (\leftarrow, x)$.*

Proof. We define x as the minimum of $a \setminus X$. Then every $y <_R x$ belongs to X. So $(\leftarrow, x) \subseteq X$. We want also the opposite inclusion. For contradiction, suppose there is a $y \in X$ s.t. $y \notin (\leftarrow, x)$. If y is not smaller, then it has to be bigger, that is $x <_R y$. But this means that $x \in X$ because $y \in X$ and X is an initial segment. But this is a contradiction because $x \notin X$.

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2 Ordinal numbers

Ordinal numbers are the extension of natural numbers and ω further. We will first do a quick recap of ordinal number basics and then prove a theorem that deeply links ordinals and well-ordered sets.

2.1 Ordinals as an extension of naturals

Definition 9. The class X is transitive $\equiv y \in x \in X \implies y \in X$.

Observation. *Equivalently: $x \in X \implies x \subset X$.*

Theorem 2. *Every natural number and the set of all natural numbers ω are*

1. *transitive,*

2. *(strictly) well-ordered by the membership relation* $\in$.

Remark. From now on, we will denote the (strictly) well-ordered set $(\omega, \in)$ as $(\omega, <)$ instead and write $n < m$ instead of $n \in m$ when talking about natural numbers.

Definition 10. The set α is an ordinal number $\equiv$

1. α is transitive,

2. α is (strictly) well-ordered by the membership relation $\in$.

The class of all ordinals is denoted as O_n,

Observation. *Every* $n \in \omega$ *and* ω *itself are ordinals.*

Remark. The class O_n itself is transitive and strictly well-ordered by $\in$. This also implies that it is not a set, because otherwise $O_n \in O_n$. In fact, any proper class X that is transitive and well-ordered by $\in$ is identical to O_n.

As for notation, we will use symbols $\alpha, \beta, \gamma, \ldots$ to denote ordinals and the symbol $<$ to compare them, similar to natural numbers:

$$\alpha < \beta \equiv \alpha \in \beta \iff \alpha \subset \beta.$$

Observation. *If* $\beta < \alpha$*, then* β *is an initial segment of* α*.*

Definition 11. If $\alpha \in O_n$, then we call all $\beta < \alpha$ the predecessors of α. The successor of α is the ordinal $\alpha + 1 := \alpha \cup \{\alpha\}$. We say that α is the direct predecessor of $\alpha + 1$.

Remark. It is easy to show that $\alpha + 1$ is the smallest ordinal larger than α.

Definition 12. The ordinal α is an

- isolated ordinal $\equiv \alpha = 0$ or α has a direct predecessor.

- limit ordinal $\equiv$ it is not isolated.

Example. Every $n \in \omega$ or $\omega + 1$ is isolated, but ω is a limit ordinal. In fact, ω is the smallest infinite ordinal.

2.2 Ordinals as types of well-ordered sets

The definition of ordinals we just saw is one by Von Neumann in 1923. However, Cantor originally defined ordinals in 1895 as types of well-ordered sets. We will prove a theorem linking these two concepts together.

Lemma 3. *Every initial segment of* $(O_n, <)$ *is an ordinal.*

Proof. Let X be an initial segment of $(O_n, <)$. We need to show that

1. X is transitive. Suppose $\alpha \in \beta \in X$, that is $\alpha < \beta \in X$. Then because X is an initial segment we have $\alpha \in X$.

2. X is well-ordered by $\in$ because O_n is well-ordered by $\in$ and $X \subseteq O_n$. ❑

Definition 13 (Isomorphism)**.** Let $(A, \leq_R)$ and $(B, \leq_S)$ be ordered classes. The bijection $F : A \to B$ is an isomorphism of $(A, \leq_R)$ and $(B, \leq_S) \equiv$

$$(\forall x,y \in A)\big(x \leq_R y \iff F(x) \leq_S F(y)\big).$$

Theorem 4 (about comparing well-orderings)**.** *If $(a, \leq_R)$ and $(b, \leq_S)$ are well-ordered sets, then exactly one of the following holds:*

1. *either a and b are isomorphic, or*
2. *a is isomorphic to an initial segment of b, or*
3. *b is isomorphic to an initial segment of a.*

In each case, the isomorphism is unique.

Corollary 5. *No two distinct ordinals can be isomorphic.*

Proof. Suppose $\alpha < \beta$, that is $\alpha \in \beta$ and $\alpha \subset \beta$. Clearly α is an initial segment of β. Which means that we are in case 2 of the previous theorem. ❑

Theorem 6 (about the type of well-ordering)**.** *Every well-ordered set is isomorphic to a unique ordinal number. This ordinal is called the **type** of the ordering.*

The following proof is taken from [2].

Proof. Let $(W, <_R)$ be a well-ordered set. We want to show that there is a unique ordinal $(\alpha, <)$ isomorphic to it. Define X as the set of all $x \in W$ for which $(\leftarrow, x)$ is isomorphic to an ordinal. As no two distinct ordinals are isomorphic, this ordinal is uniquely determined, and we denote it α_x; we denote the isomorphism as $i_x : (\leftarrow, x) \to \alpha_x$.

Suppose that there exists a set S such that $S = \{\alpha_x \mid x \in X\} \subseteq O_n$. We claim that S is an initial segment of $(O_n, <)$ and therefore by lemma 3 it is an ordinal, let's call it α. Indeed, suppose $\beta < \alpha_x \in S$, we want $\beta \in S$. Note that β is an initial segment of α_x. This implies that $i_x^{-1}[\beta]$ is an initial segment of W. Because W is well-ordered, $i_x^{-1}[\beta]$ is equal to $(\leftarrow, b)$ for some $b \in W$, (using lemma 1). So $\beta = \alpha_b \in S$ by the definition of S. More precisely, $i_x \restriction (\leftarrow, b)$ is an isomorphism between $(\leftarrow, b)$ and β.

A similar argument shows that X is an initial segment of W. To show this, suppose $x \in X$ and take $y \in a$ such that $y <_R x$. We want $y \in X$. Because $y <_R x$, $(\leftarrow, y)$ is an initial segment of $(\leftarrow, x)$. Because isomorphisms conserve all ordering properties, $i_x \restriction (\leftarrow, y)$ is an isomorphism between $(\leftarrow, y)$ and an initial segment of α_x. By lemma 3, this is an ordinal; by our previous notation, α_y. Therefore $y \in X$.

We conclude that either $X = W$ or $X = (\leftarrow, c) \subset W$ for some $c \in W$, (using lemma 1). We now define a function $f : X \to S = \alpha$ by $f : x \mapsto \alpha_x$. From the definition of S and the fact that $x < y$ implies $(\leftarrow, x) \subset (\leftarrow, y)$ and therefore $\alpha_x < \alpha_y$ it is obvious that f is an isomorphism of $(X, <_R)$ and $(\alpha, <)$. If

- $X = (\leftarrow, c)$, then by the definition of the set X, $c \in X$ because $(\leftarrow, c)$ is isomorphic to an ordinal $\alpha_c = \alpha$. But this is a contradiction because $c \notin (\leftarrow, c) = X$.
- Therefore $X = W$ and α is the sought-after ordinal isomorphic to $(W, <_R)$.

The uniqueness of α follows from the simple observation that if W was isomorphic to two distinct α_1 and α_2, then by transitivity of isomorphism, α_1 would be isomorphic to α_2, which is impossible by corollary 5.

This would complete the proof if we were justified to make the assumption that the class S is a set and therefore an ordinal. In fact, we have to use the axiom of replacement to guarantee it. If we assume this axiom then S is a set because it is the image of the set X by the map f.

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Example. Is there a well-ordered proper class not isomorphic to $(O_n, <)$?.

Sources

This document is mostly my notes from the class NMAI074 taught at MFF CUNI by doc. Kynčl. The web of the course is HERE. The course mostly follows parts of the second and third chapters of [1], which is in Czech. My notes from the introductory set theory course can be found HERE, also in Czech.

If you found any mistakes or errors, please contact me at smolikj@matfyz.cz.

[1] Petr Balcar Bohuslav a Štěpánek. *Teorie množin*. Vydání 2., opravené a rozšířené. Praha: Academia, 2001. ISBN: 80-200-0470-X.

[2] Karel Hrbáček and Tomáš Jech. *Introduction to set theory*. eng. Third edition, revised and expanded. Pure and applied mathematics. A series of monographs and textbooks ; 220. Boca Raton: Taylor & Francis, 1999. ISBN: 0-8247-7915-0.