*Proof.* The number of positions of the cop and the robber is $n(n-1)$, and an optimal winning strategy for the cop does not allow repeats of the same position. $\square$

**Remark 3.** *The same reasoning can be applied to $p$ cops. Also, if a graph $G$ displays some symmetry, we can be more precise. Indeed, the group of automorphisms $Aut(G)$ acts on the set of positions of the cop(s) and robber. Since optimal winning strategies for the cop(s) do not allow repeats of the same position up to automorphism, the capture time is bounded by the number of orbits of this action.*

**Example of a partially looped grid:** Consider the $2 \times n$ grid, with $n \geq 5$. Its vertices shall be represented by the elements of $\{1, 2\} \times \{1, \ldots, n\}$. After adding a loop on vertices $(1, 1)$ and $(1, n)$, the resulting graph will be denoted $\mathcal{G}_n$. We will show that the optimal strategy on $\mathcal{G}_n$, which is copwin, has two unusual properties: the capture time is quadratic (proposition 6), and the cop is required to move away from the robber several times in a row (corollary 3).

While these properties will seem evident to anyone who briefly examines the strategy of the cop described in proposition 7, they are surprisingly difficult to prove. Showing that this strategy is winning is easy, however, showing that it is optimal is not. The key point is to prove that the cop needs to stay on row 1 (corollary 2). Our attempts at proving it elegantly have failed. However, there is a brute force method which can be used to prove that a strategy is optimal: after computing the capture times obtained from applying this strategy for all possible starting position, it suffices to show that in any position, either the robber gets caught immediately, or the capture time decreses by 1 after each player makes their best move. While this is a simple matter of calculation, all the properties of the game can then be deduced from these capture times.

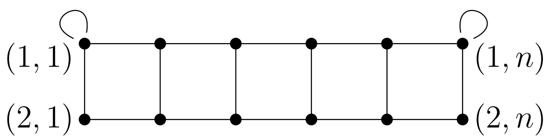



Figure 1.3: Partially looped $2 \times n$ grid

For any two vertices of $\mathcal{G}_n$, $(x, a)$ and $(y, b)$, we define $ct((x, a), (y, b))$ as follows:
$ct((x, a), (y, b)) = 0$ if $(x, a) = (y, b)$.
$ct((x, a), (y, b)) = 1$ if $(x, a)$ and $(y, b)$ are neighbours.
$ct((1, a), (1, b)) = n\frac{b-a+1}{2} - a$ if $a < b - 1$ and $a + b$ is odd.
$ct((1, a), (1, b)) = n\frac{a+b}{2} + a - 1$ if $a < b$, $a + b$ is even and $a + b \leq n$.
$ct((1, a), (1, b)) = n\frac{2n+2-a-b}{2} - a$ if $a < b$, $a + b$ is even and $a + b > n$.
$ct((1, a), (2, b)) = n\frac{b-a}{2} - a$ if $a < b$ and $a + b$ is even.
$ct((1, a), (2, b)) = n\frac{a+b-1}{2} + a - 1$ if $a < b$, $a + b$ is odd and $a + b \leq n + 1$.
$ct((1, a), (2, b)) = n\frac{2n+3-a-b}{2} - a$ if $a < b$, $a + b$ is odd and $a + b > n + 1$.
$ct((2, a), (1, b)) = n\frac{b-a+2}{2} - a + 1$ if $a < b$ and $a + b$ is even.
$ct((2, a), (1, b)) = n\frac{a+b+1}{2} + a$ if $a < b$, $a + b$ is odd and $a + b < n$.

$ct((2,a),(1,b)) = n\frac{n+1}{2} - a + 1$ if $a < b$, $a+b$ is odd and $a+b = n$.
$ct((2,a),(1,b)) = \frac{n^2}{2} + a$ if $a < b$, $a+b$ is odd and $a+b = n+1$.
$ct((2,a),(1,b)) = n\frac{2n+3-a-b}{2} - a + 1$ if $a < b$, $a+b$ is odd and $a+b > n+1$.
$ct((2,a),(2,b)) = n\frac{b-a+1}{2} - a + 1$ if $a < b-1$ and $a+b$ is odd.
$ct((2,a),(2,b)) = n\frac{a+b}{2} + a$ if $a < b-1$, $a+b$ is even and $a+b \leq n$.
$ct((2,a),(2,b)) = n\frac{2n+2-a-b}{2} - a + 1$ if $a < b-1$, $a+b$ is even and $a+b > n$.
$ct((x,a),(y,b)) = ct((x,n+1-a),(y,n+1-b))$.

**Lemma 5.** *If $(x,a)$ and $(y,b)$ are neither equal nor adjacent, then:*

$$ct((x,a),(y,b)) = 1 + \min_{(x',a')\in N((x,a))} \max_{(y',b')\in N((y,b))} ct((x',a'),(y',b')),$$

*where $ct$ is the function defined above.*

The proof of lemma 5 consists of a simple case by case verification of the formula. The detailed calculations will not be fully included here, as they are both trivial and very lengthy. A couple of cases will be detailed in the appendix (section 1.6) in order to give the reader an idea of what the full version might look like. Yet, the idea that the problem can be solved in this way is more interesting than the verification itself. While this is not fully satisfying, my projects for future works include two possible alternatives: either finding a more elegant way to prove that the the cop needs to stay on the first row, or creating a machine-aided proof using an algorithm which verifies the formula in all cases.

**Proposition 5.** *If the cop starts in $(x,a)$ and the robber starts in $(y,b)$, then $ct((x,a),(y,b))$ is the capture time.*

*Proof.* The capture time can be defined by induction using the fact that if the cop makes her best move, then the robber makes his best move, the capture time decreases by one. This property is described by the minmax formula in lemma 5. Thus $ct$ is the capture time. □

**Corollary 1.** *The partially looped grid $\mathcal{G}_n$ is copwin.*

*Proof.* The capture time $ct$ is finite. Hence, $\mathcal{G}_n$ is copwin. □

**Corollary 2.** *Any optimal strategy for the cop requires her to start and stay on row 1, except for the very last move whence she may catch the robber on row 2.*

*Proof.* For all $a < b$, $ct((1,a),(1,b)) < ct((2,a),(2,b))$ and $ct((1,a),(2,b)) < ct((2,a),(1,b))$. Hence, when the cop moves to the second row, the robber can increase the capture time by making a vertical movement. □

**Proposition 6.** *The capture time for $\mathcal{G}_n$ is $\left\lfloor \frac{n^2}{2} \right\rfloor$.*

*Proof.* The capture time for $\mathcal{G}_n$ is given by the following formula:

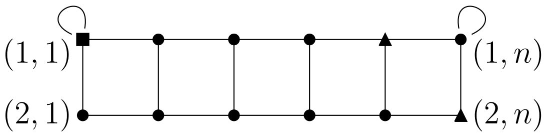
$$ct(\mathcal{G}_n) = \min_{(x,a)} \max_{(y,b)} ct((x,a),(y,b)).$$

Using the symmetry of the graph and corollary 2, we may consider only the cases where the cop starts in $(1,a)$ with $a \leq \frac{n}{2}$. If $n$ is even, the formulas for $ct$ give $ct((1,a),(y,b)) \leq$

$\frac{n^2}{2} + a - 1$, and this boundary is reached for $(y, b) = (1, n - a)$ or $(2, n + 1 - a)$. If $n$ is odd, then $ct((1, a), (y, b)) \leq n\frac{n+1}{2} - a$, which is reached in $(1, n + 1 - a)$ and $(2, n + 2 - a)$.

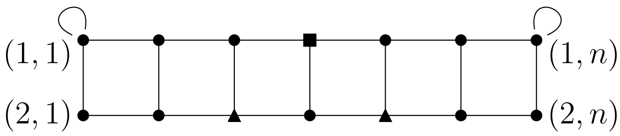
Hence, if $n$ is even, the cop should start in the corner $(1, 1)$, and the robber in $(1, n - 1)$ or $(2, n)$, for a capture time of $\frac{n^2}{2}$ (see fig. 1.4). And if $n$ is odd, the cop should start in the middle in $(1, \frac{n+1}{2})$ and the robber in $(2, \frac{n-1}{2})$ or $(2, \frac{n+3}{2})$, for a capture time of $\frac{n^2-1}{2}$ (see fig. 1.5).

□




■ optimal starting point for the cop
▲ optimal starting point for the robber

Figure 1.4: Optimal starting points for $n$ even




■ optimal starting point for the cop
▲ optimal starting point for the robber

Figure 1.5: Optimal starting points for $n$ odd

**Proposition 7.** *The optimal strategy for the cop on $\mathcal{G}_n$ is to go back and forth between the loops, following the loop once each time, until she can move to catch the robber. Assuming that the cop and robber both start in their optimal starting positions, the cop's first move should be to follow the loop if $n$ is even, or to move on the first row towards the robber if $n$ is odd.*

*Proof.* Corollary 2 states that in optimal play, the cop stays on the first row. Also, in optimal play, the cop can never move along the same edge twice in a row, as it would allow the robber to repeat the position by doing the same. This only leaves the cop a single possibility at each turn after the first. On the first move, if $n$ is even, she can either follow the loop or move to $(1, 2)$. If she moves to $(1, 2)$, the robber can move to $(2, n - 1)$. Since $ct((1, 2), (2, n - 1)) = \frac{n^2}{2} + 1$, which is greater than the capture time for $\mathcal{G}_n$, this is not optimal for the cop. If $n$ is odd and the robber starts in $(2, \frac{n+3}{2})$, the cop may go either to $(1, \frac{n-1}{2})$ or $(\frac{n+3}{2})$. In the former case, the robber may then go to $(1, \frac{n+3}{2})$, and

$ct((1,\frac{n-1}{2}),(1,\frac{n+3}{2})) = n\frac{n+1}{2} - \frac{n-1}{2} = \frac{n^2+1}{2}$. Hence, going to $(1,\frac{n-1}{2})$ is not optimal for the cop. $\square$

**Remark 4.** *The cop's optimal strategy is almost independent of how the robber plays. Only her first and last moves are influenced by the robber. In fact, if we assume that the cop has limited visibility, meaning that she can only see the robber if he is in her neighbourhood, the graph $\mathcal{G}_n$ remains copwin. If the cop with limited visibility is informed of the robber's starting point, then the capture time remains $\left\lfloor \frac{n^2}{2} \right\rfloor$.*

**Proposition 8.** *If the cop is using her optimal strategy, an optimal strategy for the robber is to go back and forth along the path $(1,1)$, $(2,1)$, $(2,2)$, $\ldots$, $(2,n)$, $(1,n)$, following the loop once each time. If $n$ is even, the robber will go to $(1,n)$ on his first move, then follow the loop. If $n$ is odd, the robber will go to $(2,\frac{n+1}{2})$ on his first move.*

*Proof.* Since the cop's optimal strategy is known and independent of how the robberplays, we can easily verify that this strategy allows the robber to get caught in exactly $\left\lfloor \frac{n^2}{2} \right\rfloor$ moves. Since that is the maximum capture time, this strategy is optimal. $\square$

**Remark 5.** *When it is the cop's turn to play, an even distance between the cop and the robber is favourable to the robber, while an odd distance is favourable to the cop. The reason why both players go back and forth between the looped vertices is that they are fighting to set the parity in their favour. However, the cop is able to follow the shortest path between the two loops, whereas the robber is forced to go via the second row in order to dodge her. So every time the cop reaches the next loop, the distance between the cop and the robber decreases by two.*

**Corollary 3.** *On the graph $\mathcal{G}_n$, in optimal play, the cop moves away from the robber $\left\lfloor \frac{n-3}{2} \right\rfloor$ times in a row.*

*Proof.* If $n$ is odd, in optimal play, the cop moves away from the robber on turns 2 to $\frac{n-1}{2}$ when she reaches the loop. If $n$ is even, the robber first crosses to the left of the cop on turn $\frac{n}{2}+2$; she then moves away from him until she reaches the loop on the $n$th turn. $\square$

**Remark 6.** *Non-looped graphs with the same properties as $\mathcal{G}_n$ may be obtained either by replacing the looped vertices with pairs of adjacent twin vertices, or by replacing the loops with triangles.*

**Remark 7.** *Since $\mathcal{G}_n$ has $2n$ vertices and a symmetry, the capture time $\left\lfloor \frac{n^2}{2} \right\rfloor$ tends to a quarter of the upper-bound given in remark 3 when $n \to +\infty$. When a loop is added to vertex $(2,1)$, the robber now goes back and forth between $(2,1)$ and $(1,n)$, so the path followed by the cop is shorter by $1$ instead of $2$. Thus the capture time is nearly doubled, for large values of $n$, by adding that loop. Since this also removes the symmetry, again, a quarter of the theoretical upper-bound is reached.*