Linear Regression.

A number of slides and screenshots from : Andrew Ng's course on machine learning and Sebastian Raschka's course on deep learning
Both can be found for free on youtube !

What does regression mean ?

- Seen in intro, but :
- Regression means predictiong **real-valued** outputs.

labels

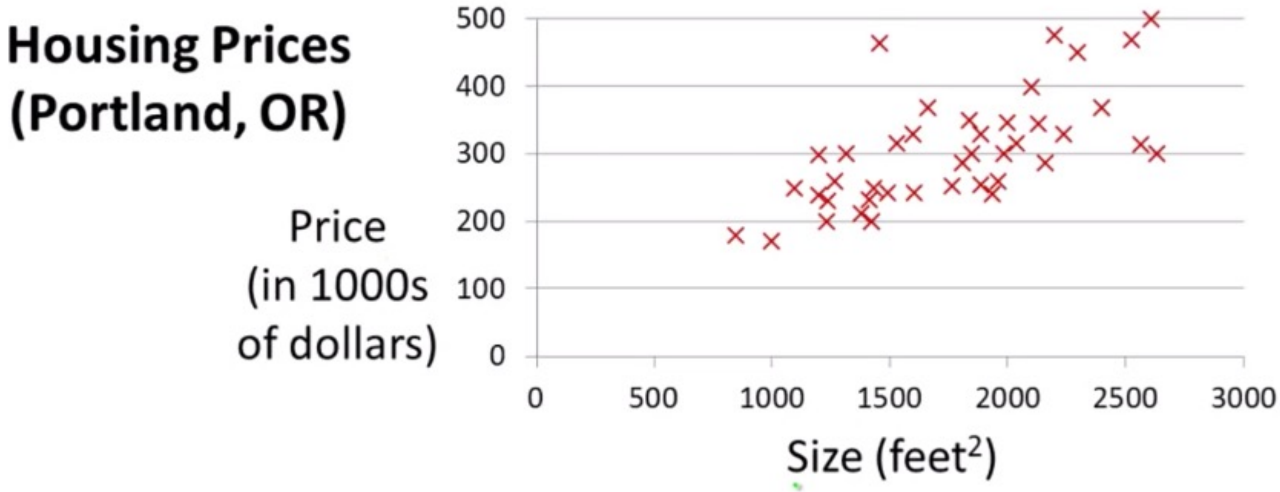
- An essential type of supervised machine learning task : for each example in the data, we want to get as close as possible to the real-valued label. $[0, 1]$
- Often contrasted with classification (**discrete** labels).

$\{0, 1, 2\}$

- Example :
  - Predicting height => many many real-valued outputs are possible…
  - Vs. Predicting a « height class » : short | medium-height | tall

Dataset and problem example

- Imagine we want to create an ML algorithm that predicts the price of a house using collected data, which only contains information about the size of the house.



Training Set and Notation

Training set of housing prices (Portland, OR)

Size in feet^2 (x)	Price ($) in 1000's (y)
2104	460
1416	232
1534	315
852	178
...	...

Feature 1

Label

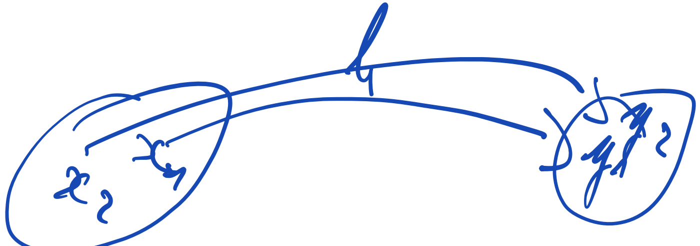
m

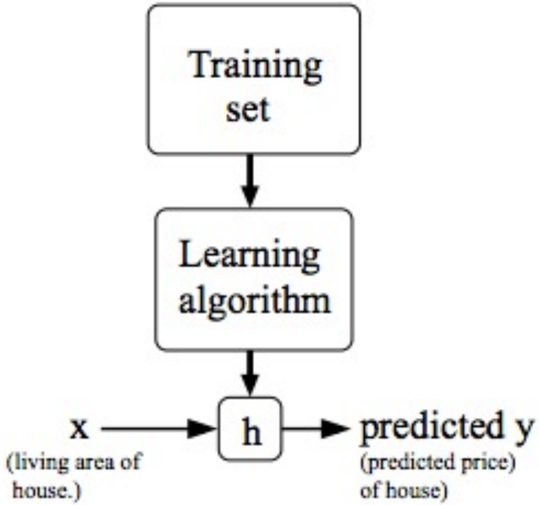
Notation:

m = Number of training examples
x's = "input" variable / features
y's = "output" variable / "target" variable

The supervised learning workflow

- **h**: hypothesis
- h is a function which **maps** x’s to y’s
- Our goal will be to find the function which takes x as input and predicts the correct y for that x.





How to model *h*

- To start with, we will use a simple model, a function which corresponds to the equation of a line (maybe you remember y = ax + b?)

$$(1)\quad h(x) = \theta_0 + \theta_1 x \Rightarrow$$

- This model will predict that y is some linear function (straight line) :

$J \Rightarrow$ cost function

If this seems a bit odd to you…

- Remember we want our function to predict the examples we have in our training set correctly, which our simple model will probably not do very well….

- What if we can’t get to all the points using a straight line ?

- Don’t worry for now, this is still a very decent starting point in practice !

Cost Function

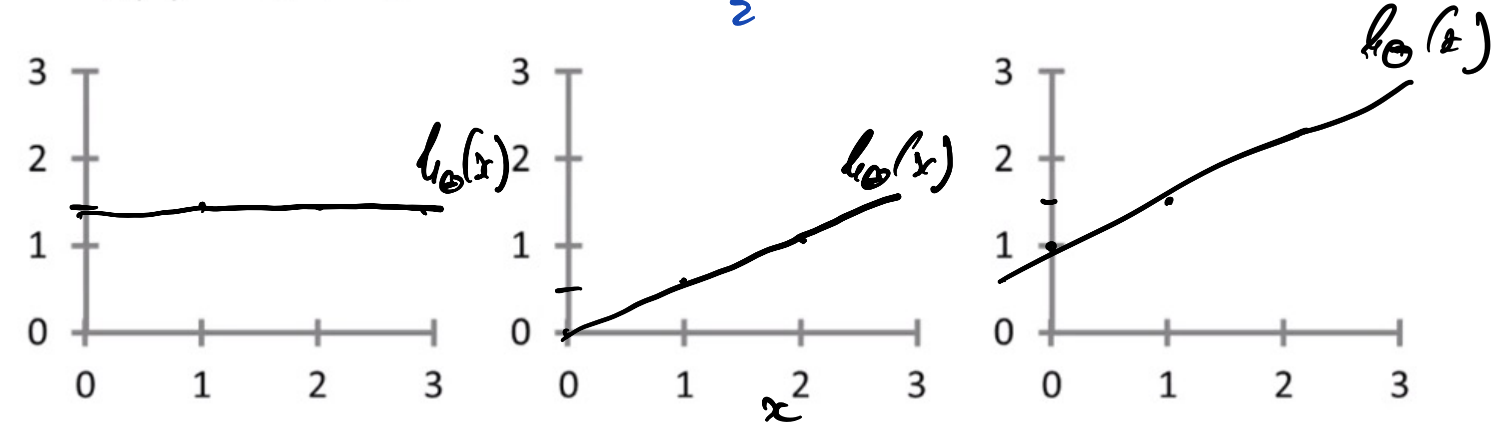
- This is a **second** function we will use to judge **how well** our straight **line fits** the data.
- In other words, this function will help us **find the best possible straight line.**

Motivating the Cost Function…

- To recap:
- $h(x) = \theta_0 + \theta_1 x$ is our **model**
- θ_i are what we call **parameters**
- We want to find the right combination of those parameters to get the best line.
- So **how do we choose the right parameters** ?

Visualizing different parameter choices/hypotheses

$$\overset{y}{h_\theta(x)} = \theta_0 + \theta_1 x \Rightarrow 1 + \frac{1}{2}x$$



$\theta_0 = 1.5$, $\theta_1 = 0$ | $\theta_0 = 0$, $\theta_1 = 0.5$ | $\theta_0 = 1$, $\theta_1 = 0.5$

Exercise

$0{,}5 = \theta_0 + 0$

- Look at the plot of $h(x) = \theta_0 + \theta_1 x$ $= 0$

$0{,}5$

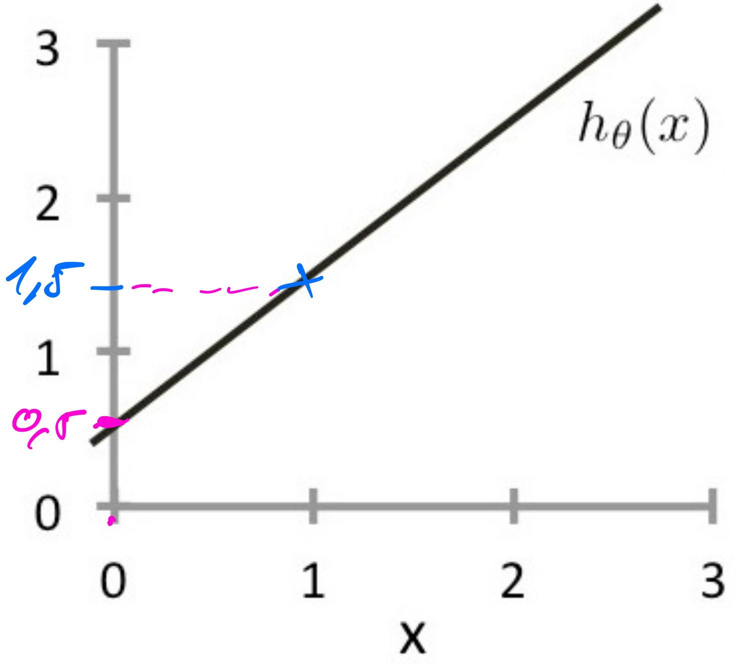
$1{,}5 = 0{,}5 + \theta_1 \cdot 1$

$\Leftrightarrow 1{,}5 - 0{,}5 = \theta_1$

- Just by eyeballing the plot, what seem to be the values of θ_0 and θ_1 ?

$\theta_1 = 1$

$0{,}5$ 1

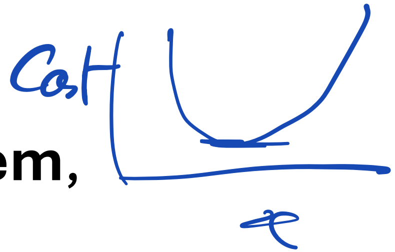


Finding the Cost as a Minimization Problem

- We want to choose θ_0 and θ_1 so that

- $h(x)$ is close to y for our training examples (x, y)…

- This actually comes down to a **minimization problem**,



- where we want to **minimize** $(\boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{y})^2$ for example, by tweaking our parameters θ_0 and θ_1

Cost function = Quantifying the model's error

	sq feet	price
m	300	300k

- For all of our examples m the **average error** is :

250k 300k

$$J(\theta_0,\theta_1) = \frac{1}{2m}\sum_{i=1}^{m}\left(h\left(x^{(i)}\right) - y^{(i)}\right)^2$$

300

Picking $\frac{1}{2m}$ makes the math easier later on, but you can regard this as just an averaging constant.

$\boxed{\text{MSE}}$

- This function is known as the **Mean Squared Error** (we'll see how it works in a few slides) and is the most **commonly** used

To recap

(1) Hypothesis:

Input: feat² → feature
Output: price → label (ℝ)

$$h_\theta(x) = \theta_0 + \theta_1 x$$

price

Parameters:

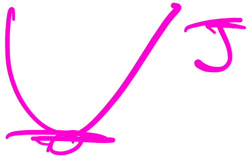
$$\theta_0, \theta_1$$

loss

(2) Cost Function:

$$J(\theta_0, \theta_1) = \frac{1}{2m} \sum_{i=1}^{m} \left(h_\theta(x^{(i)}) - y^{(i)}\right)^2$$

Goal: minimize $J(\theta_0, \theta_1)$
θ_0, θ_1



+ −

h_1 | h_4

⇓ ⇓

score$_1$ > score$_4$

https://www.coursera.org/learn/machine-learning/home/welcome

Cost Function Intuition

- Let's use a **simplified model hypothesis** to understand what's going on a bit better:

$$h(x) = \theta_1 x$$

$h(x) = \theta_0 + \boxed{\theta_1} x$

$= 0$

- Our objective is now to minimize

$$J(\theta_1)$$

- Which is equal to

$h(x)$

$$\frac{1}{2m}\sum_{i=1}^{m}(\theta_1 x^i - y^i)^2$$

Hypothesis function vs. Cost function

- If the points on the graph represent our training data and $\theta_1 = 1$, what does our **hypothesis** (line) look like ?

$\boxed{h(x) = \theta_1 x} = x$

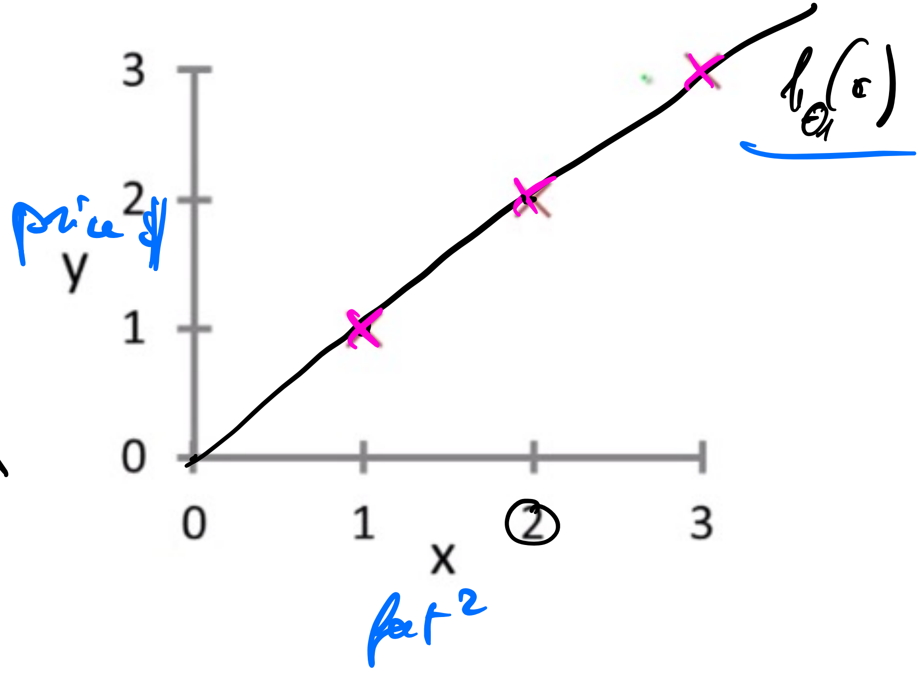
- What is the cost ? $= 0$ $\boxed{\theta_1 = 0}$

- Remember : $\left[\frac{1}{2m}\sum_{i=1}^{m}(\boxed{\theta_1 x^i} - y^i)^2\right]$

$(1-1) + (2-2) + (3-3)$

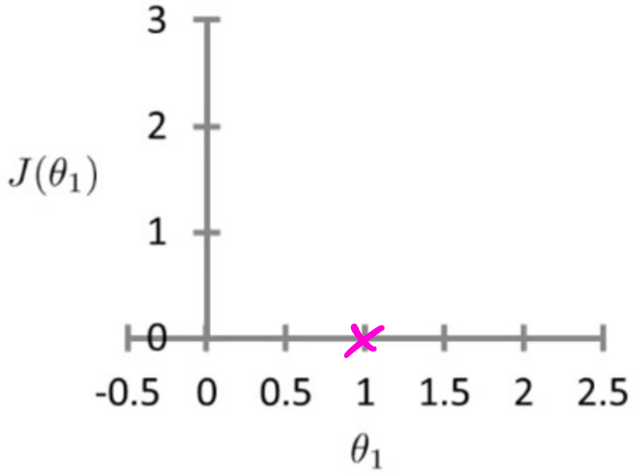
$0 \quad 0 \quad 0$

$\frac{1}{2m} \times \sum 0 = 0$



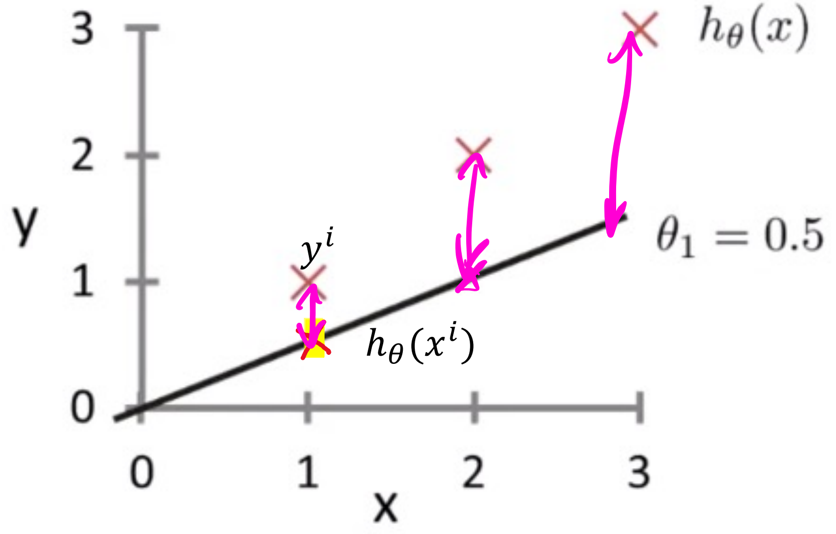
Hypothesis function vs. Cost function

- $J(\theta_1 = 1) = \boxed{\mathbf{0}}$
- We can now **plot** our error rate
- Notice that the values for $\boldsymbol{\theta_1}$ **are on the horizontal axis**. This is **not the same** plot as before !!
- This is a plot for the **cost function** :



Hypothesis function vs. Cost function

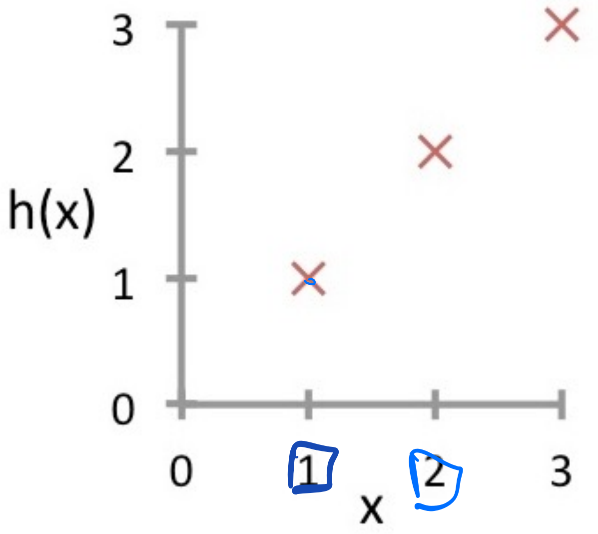
- Now let's look at $\theta_1 = 0.5$
- And compute $J(\theta_1 = 0.5)$ (approx. 0.58)
- The error for each point is actually the **height** which **seperates** the **data point from the line** for a giver



Your turn !

- Suppose this is our training set. $m = 3$
- Given the same hypothesis and cost functions as before, what is $\boxed{J(0)}$? ie. $\theta_1 = 0$

$\boxed{m = 3}$



- Should be approx. 2.3

$$J(\theta_1 = 0) = \frac{1}{2m} \sum_{i=1}^{m} \left(h(x^i) - y^i\right)^2$$

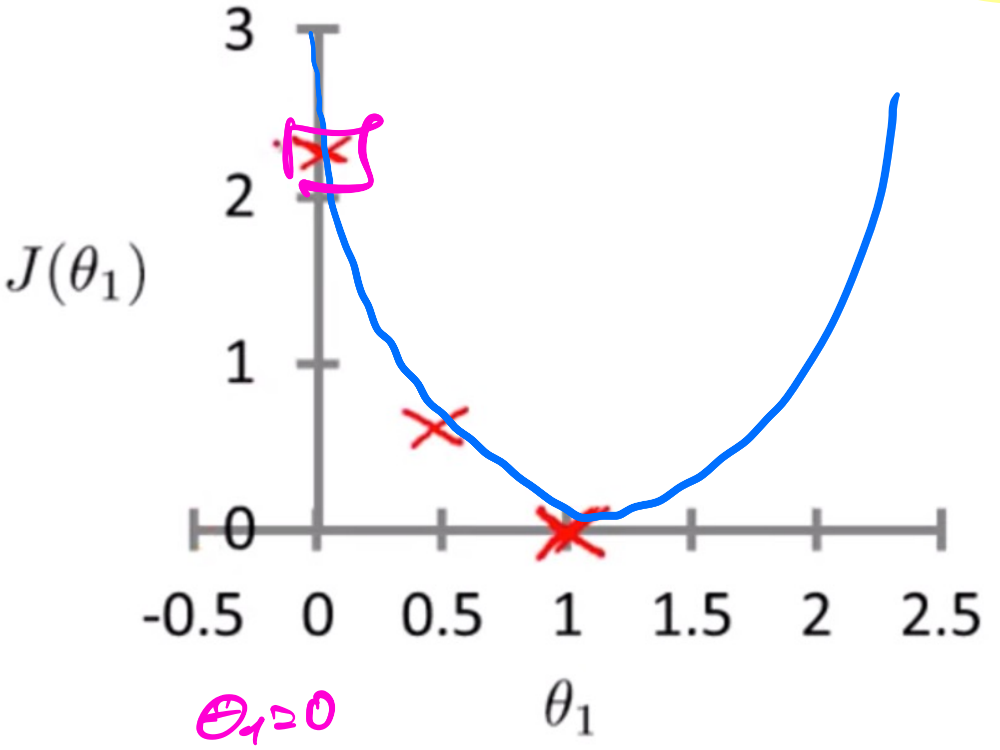
$$= \text{"} \left(\theta_1 x^i - y^i\right)^2 = \frac{1}{2m}\left[(0-1)^2 + (0-2)^2 + (0-3)^2\right]$$

$$= \frac{1}{2m}\left[1 + 4 + 9\right] = \frac{7}{m} = \frac{7}{3}$$

$m = 3$

Hypothesis function vs. Cost function

- We could continue plotting points but we'll stop here.
- With the error calculated for the different values of θ_1, we start to see part of the **general shape** of the function.
- It turns out the function is convex/looks like a parabola.



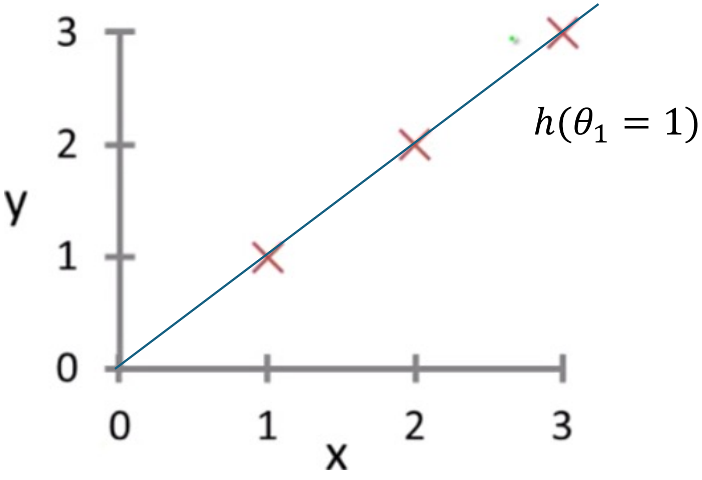
Quick cost function recap

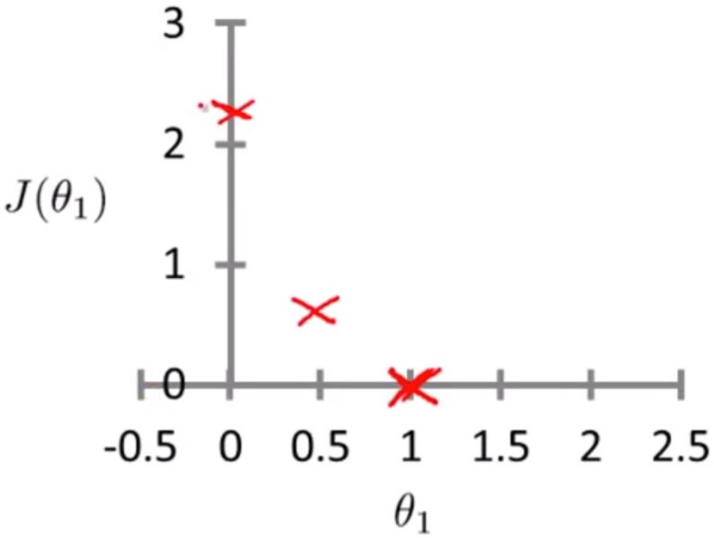
- Each **value** of θ_1 plotted corresponds to a **different hypothesis / model**

- For each value of θ_1 we can compute a value $J(\theta_1)$ to trace out the cost function.

- Now remember, we wanted to find the value of θ_1 which **minimized** $J(\theta_1)$… Looking at the graph we can now do so !

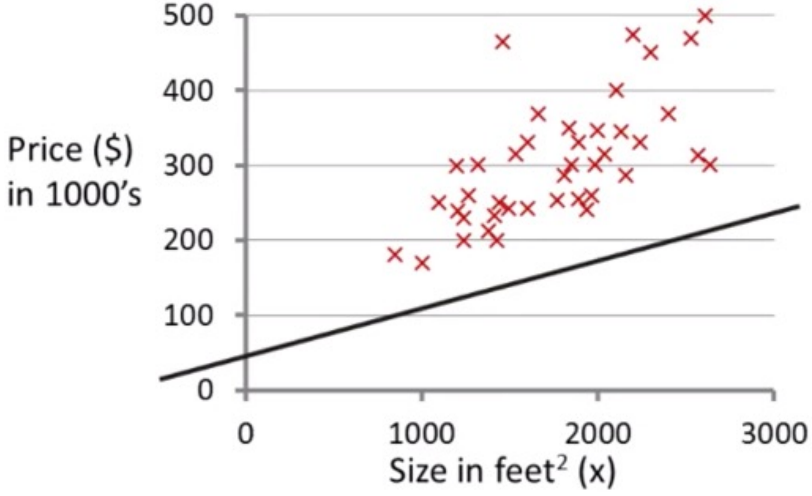
- No surprise, the value of θ_1 which minimizes the error is associated with the **model which fits the data perfectly**





Back to 2 parameters

- Now, going back to our original data and model, we use a 2 parameter hypothesis to draw our line
- For :
- $\theta_0 = 50$
- $\theta_1 = 0.06$
- We get this straight line as our model



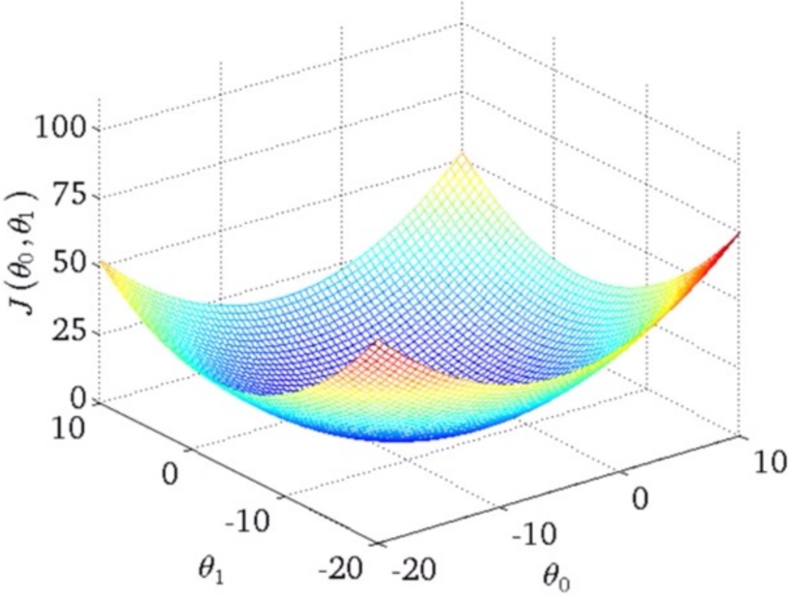
$$h_\theta(x) = 50 + 0.06x$$

Corresponding Cost function

- Now we have **two parameters**, the error graph will be slightly harder to plot as it has **3 dimensions**:

$$\theta_1, \theta_2, cost$$

- Indeed , $J(\theta_1, \theta_2)$ now has 2 inputs,
- So it will look like this in 3D:

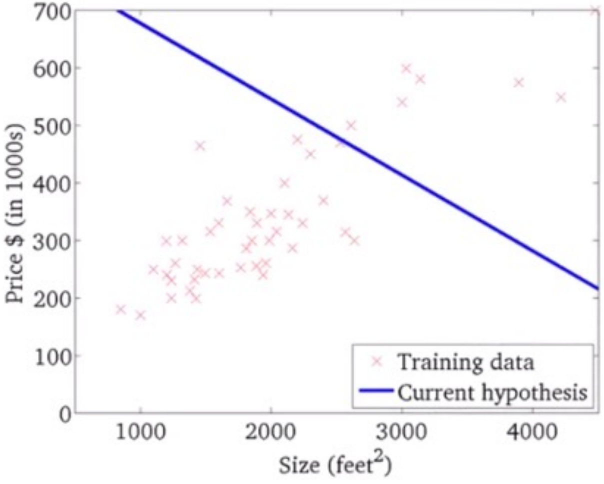


Contour Plots

- You will sometimes see the cost function represented by a contour plot .

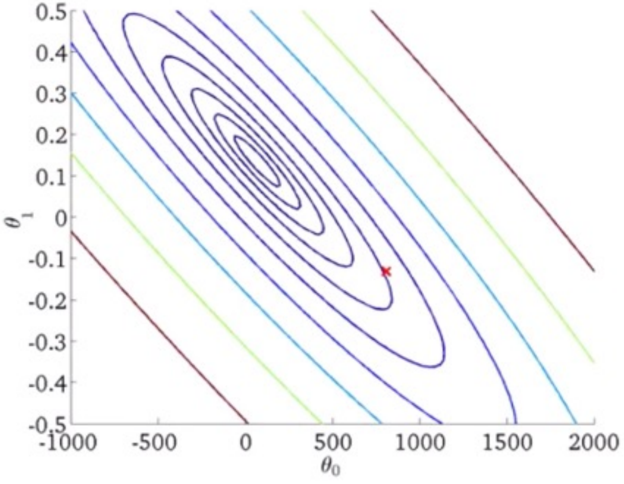
$h_\theta(x)$

(for fixed θ_0, θ_1, this is a function of x)



$J(\theta_0, \theta_1)$

(function of the parameters θ_0, θ_1)



The ovals/ellipses show the set of points which take on the same value for given values of θ_0, θ_1

Countour Plots

- The **minimum** is at the **center** of all the « ellipses ».
- This plot shows a model very close to the minimum.