8. Give a geometrical description of the images and kernels of each of the following linear maps on $\mathbb{R}^3$

(a) $T : (x, y, z) \mapsto (x + 2y + z,\ x + 2y + z,\ 2x + 4y + 2z)$,

(b) $S : (x, y, z) \mapsto (x + 2y + 3z,\ x - y + z,\ x + 5y + 5z)$.

9. A linear map $\mathbb{R}^4 \to \mathbb{R}^4$ is defined by $\mathbf{x} \mapsto M\mathbf{x}$ where

$$M = \begin{pmatrix} a & a & b & a \\ a & a & b & 0 \\ a & b & a & b \\ a & b & a & 0 \end{pmatrix}.$$

Find the image and kernel of this map for all real values of a and b.

1. Let $\alpha \in \mathbb{R}$, let $A_\alpha = \begin{pmatrix} 1 & 1 & \alpha \\ 1 & \alpha & \alpha \\ \alpha & 1 & 1 \end{pmatrix}$ and let $T_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by A_α, so that $T_\alpha(\mathbf{v}) = A_\alpha \mathbf{v}$ for any column vector $\mathbf{v} \in \mathbb{R}^3$.

(a) [8 marks] By using elementary row operations and making case distinctions about α, find all solutions to the system of linear equations

$$A_\alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha - 1 \\ -1 \\ 1 \end{pmatrix}.$$

(b) [7 marks] Rewrite your set of solutions (if non-empty, depending on α) in terms of $\ker T_\alpha$ and some fixed solution $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ (depending on α).

Relate the existence of solutions and their uniqueness to properties of $\operatorname{im} T_\alpha$ and $\ker T_\alpha$.

(c) [5 marks] Find a 4×4 matrix B_α and a vector $\mathbf{b}_\alpha \in \mathbb{R}^4$, both depending on the parameter $\alpha \in \mathbb{R}$, such that the equation

$$B_\alpha \mathbf{x} = \mathbf{b}_\alpha$$

has no solution $\mathbf{x} \in \mathbb{R}^4$ for some choice of α, has a unique solution for infinitely many choices of α, and has infinitely many solutions for a unique choice of α.

11.* A semi-magic square is a 3×3 matrix of (rational) numbers such that all rows and columns sum up to the same total. A simple example is

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix},$$

where all rows and columns sum up to 6.

(a) Why do the semi-magic squares form a vector space?

(b) Show that the matrices

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},\quad M_2 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad M_3 = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix},\quad M_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

form a basis of the semi-magic squares. (Therefore, the dimension of this space is five and all semi-magic squares can be written as linear combinations of the above five matrices.)

1. Let V be a finite-dimensional real vector space.

   (a) [12 marks] Let X and Y be subspaces of V.

   (i) State, without proof, a formula connecting the dimensions of

$$X + Y, \quad X \cap Y, \quad X, \quad Y.$$

   (ii) The subspaces X and Y of $\mathbb{R}^5$ are defined as

$$X = \langle (-1, 1, 0, 1, 0), (1, 0, 1, -1, 1), (0, 1, 1, 0, 1) \rangle$$

   and

$$Y = \{(x_1, x_2, x_3, x_4, x_5) : x_1 + 2x_3 + x_4 + x_5 = 0; \quad x_1 + x_2 + x_3 + x_5 = 0\}.$$

   Carefully determine the dimensions of $X + Y$, $X \cap Y$, X and Y, and verify the formula from (a)(i).

   (b) [8 marks] Let $X_1, \ldots, X_k$ be subspaces of V. We write

$$V = X_1 \oplus X_2 \oplus \cdots \oplus X_k$$

   if every $v \in V$ can be uniquely written as $v = x_1 + x_2 + \cdots + x_k$ where $x_i \in X_i$.

   (i) In this case show that

$$\dim V = \dim X_1 + \dim X_2 + \cdots + \dim X_k.$$

   (ii) Let $Y_1, \ldots, Y_k$ be subspaces of V. Show that there exist subspaces $Z_1, \ldots, Z_k$ of V such that

$$Z_1 \oplus Z_2 \oplus \cdots \oplus Z_i = Y_1 + Y_2 + \cdots + Y_i$$

   for each $i = 1, \ldots, k$.