算法设计与分析第一次作业 - 分治策略

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问题一 （1）

> You are interested in analyzing some hard-to-obtain data from two separate databases. Each database contains n numerical values, so there are $2n$ values total and you may assume that no two values are the same. You'd like to determine the median of this set of $2n$ values, which we will define here to be the n^{th} smallest value.
>
> However, the only way you can access these values is through *queries* to the databases. In a single query, you can specify a value k to one of the two databases, and the chosen database will return the k^{th} smallest value that it contains. Since queries are expensive, you would like to compute the median using as few queries as possible.
>
> Give an algorithm that finds the median value using at most $O(logn)$ queries.

1. 算法描述与伪代码

令分离的两个数据库分别为A和B。

基本思路：所求的中值必然在A的中值和B的中值之间。基于此，可以划分子问题，子问题也满足同样的规律。

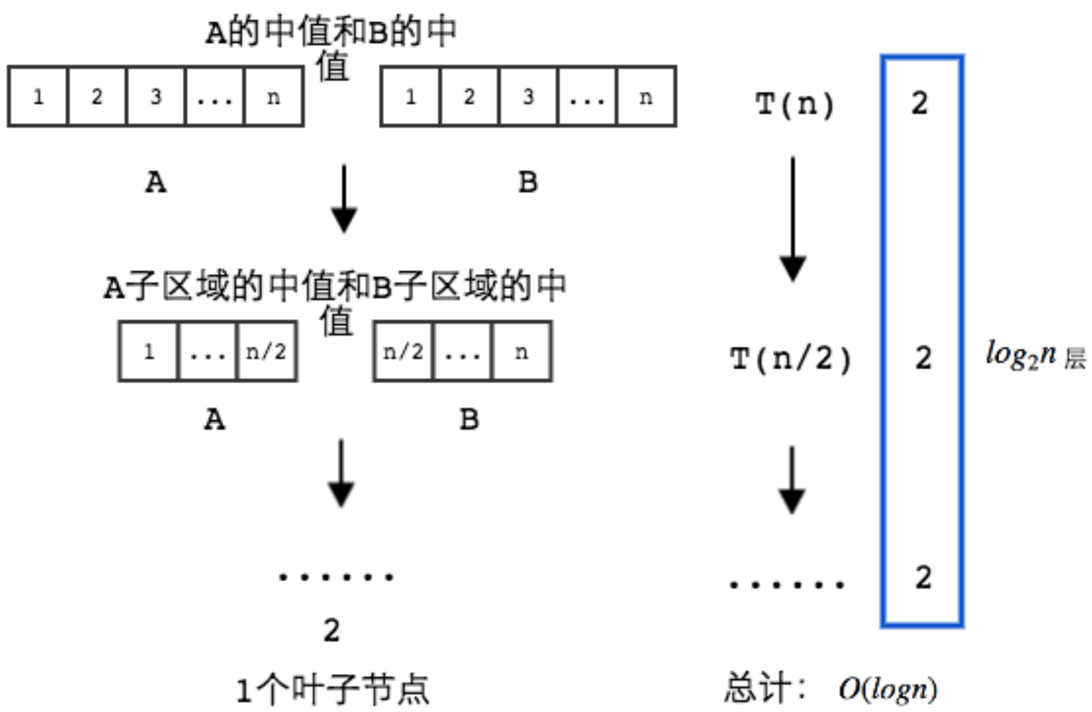
算法：

- 函数$Query(D, k)$用来返回数据库D中第k小的数。
- 函数$Median(A, low1, high1, B, low2, high2)$用来获取数据库$A$从$low1^{th}$到$high1^{th}$的数加上数据库$B$从$low2^{th}$到$high2^{th}$的数的中间值，比如：$Median(A, 1, n, B, 1, n)$即为问题所求。

$Median(A, low1, high1, B, low2, high2)$：

```
k1 = (low1 + high1) / 2;     // k1 is the index of median
1. k2 = (low2 + high2) / 2;
median1 = Query(A, k1);
median2 = Query(B, k2);

if Median1 < Median2 then
    if low1 == high1 && low2 == high2 then
        return median1;
    else
        if (high1 - low1) is even
            return Median(A, k1, h1, B, l2, k2);
        // make sure left part and right part have equal number of  elements
        else
            return Median(A, k1 + 1, h1, B, l2, k2);
else
    if low1 == high1 && low2 == high2 then
        return median2;
    else
        if (high1 - low1) is even
            return Median(A, l1, k1, B, k2, h2);
        // make sure left part and right part have equal number of  elements
        else
            return Median(A, l1, k1, B, k2 + 1, h2);
```

2.子问题缩减图



3. 正确性证明：（归纳法）

1. 当 $n=1$，A和B中都只有一个元素，算法返回较小的数，满足问题的要求。
2. 假设对于$n \leq k-1$算法得到正确结果。假设n是偶数，那么对于$n=k$。 令A中的中值为m_1，B中的中值为m_2，$A+B$的中值为m。假设$m_1 > m_2$，算法返回$A[1,\frac{n}{2}]$和$B[\frac{n}{2}+1,n]$中所有数的中值。首先，有$m_1 \geq m \geq m_2$，因为如果$m > m_1$，那么B中$[1,n/2]$的数都要比m小，加上B中$\frac{n}{2}$个数，至少有n个数小于m，违背了中值的定义，同理$m < m_2$也不成立。其次，根据中值的定义，在$A[1,\frac{n}{2}]$和$B[\frac{n}{2}+1,n]$中，有$\frac{n}{2}$个数是小于或等于m的，有$\frac{n}{2}$个数是大于m的。而$\frac{n}{2} \in \{1,2,...,n-1\}$，对于$n$是偶数，$m_1 < m_2$的情况同理可证。$m$满足题目的要求。所以$n=k$时满足问题要求。

根据归纳法定义，算法正确。

4. 运行时间分析

每次只会产生一个子问题，问题的大小减半，每个子问题中使用两次查询。所以有：

$$T(n)=\begin{cases}2, & n=1\\ T(n/2)+2, & n\geq 2\end{cases}$$

根据主定理，有：

$$T(n)=O(logn)$$

问题二 （2）

> Find the k^{th} largest element in an unsorted array. Note that it is the k^{th} largest element in the sorted order, not the k^{th} distinct element.
>
> INPUT: An unsorted array A and k.
>
> OUTPUT: The k^{th} largest element in the unsorted array A.

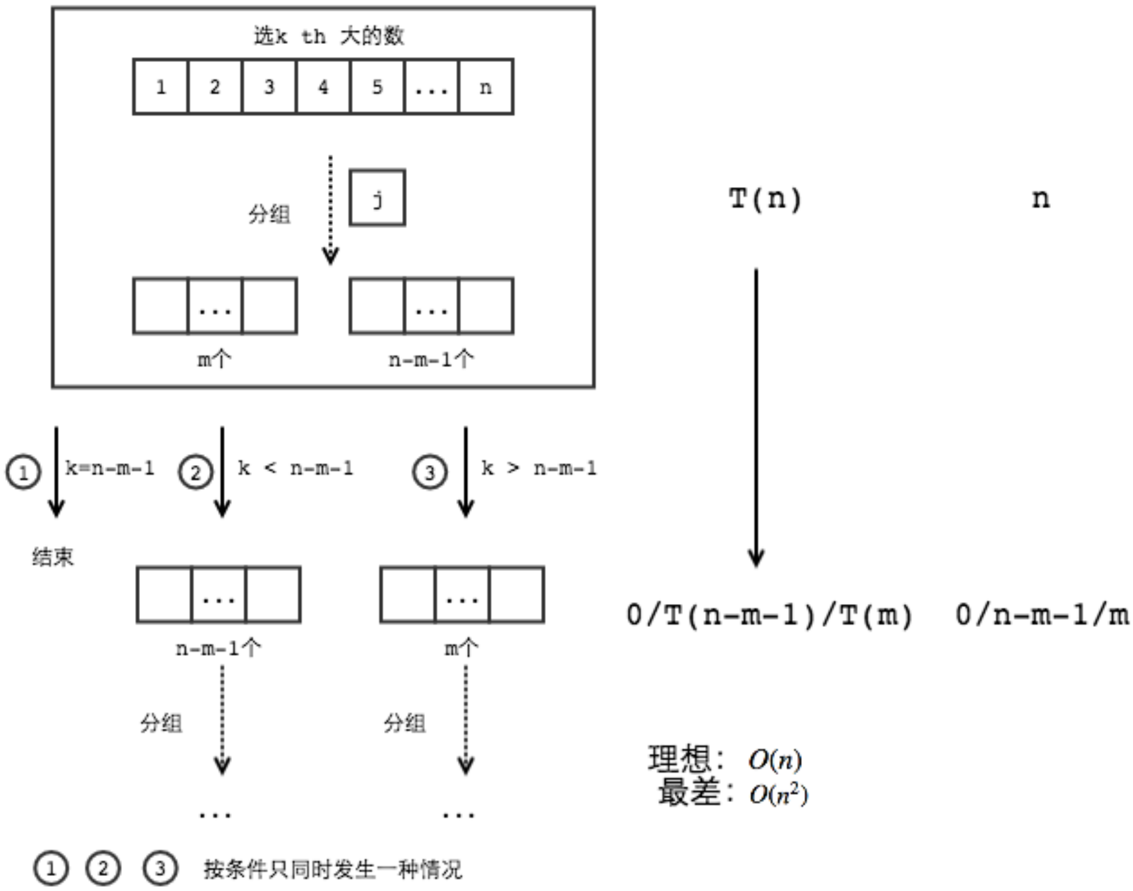
1. 算法描述与伪代码

基本思路：基于快速排序思想的快速选择方法。首先随机选择一个数作为轴点，所有的数与这个轴点进行比较，分为大小两组。则会出现三种情况：
1. 大于轴点的数有$k-1$个，返回这个轴点
2. 如果大于轴点的数大于$k-1$个，递归调用返回大的那组的第k大的数
3. 如果大于轴点的数小于$k-1$个，令大的那组个数为m，递归调用返回小的那组的第$k-m-1$大的数。
算法：

$QuickSelect(A,k)$

```
if |A| == 1 and k == 1 then
    return A[1];
else
    ERROR;
end if

choose a splitter A[j] randomly;
for i = 0 to n - 1 do
    if i != j then
        Put A[i] in S- if A[i] < A[j];
        Put A[i] in S+ if A[i] >= A[j];
    end if
end for

if |S+| == k then
    return A[j];
else if |S+| > k then
    return QuickSelect(S+, k);
else
    return QuickSelect(S-, k - |S+| - 1);
end if
```

2. 子问题缩减图



3. 正确性证明（归纳法）

1. $|A| = 1, k = 1$时，返回数组第一个元素，满足问题要求。
2. $|A| \le n, k \le d$时，假设算法正确。

   - 当$|A| = n + 1, k = d$时，根据算法，第一次运行后出现三种情况。第一种，直接返回轴点，满足问题要求。后两种情况，必然有$|S+| \le n, |S-| \le n, (k - |S+| - 1) < d$ ，根据假设，算法返回正确结果。
   - 当$|A| = n, k = d + 1$时，根据算法，递归时出现第一种，直接返回轴点，满足问题要求。出现第三种，必然有$|S-| \le n, (k - |S+| - 1) < d$，根据假设，算法返回正确结果。假设第二种情况持续发生，因为轴点排除在数组之外，数组大小持续减少，必然在某一时刻，使得$|S+| = k$，导致第二种情况不再满足。进而进入第一种或第三种情况，最终得到正确结果。

综上所述，根据双重归纳法定义可以证明该算法正确。

4. 运行时分析

与快速排序运行时计算方式同理，可得：

- 最差情况：（取最大/最小值，每次随机选择的是最小/最大值）

$$T(n) \le T(n-1) + cn \Rightarrow T(n) = O(n^2)$$

- 理想情况：（轴点总选在在中点处）

$$T(n) \leq T(n/2) + cn \Rightarrow T(n) = O(n)$$

- 大部分情况：通常轴点在中心点附近，此时依然有

$$T(n) = O(n)$$

问题三 （3）

> Consider an n-node complete binary tree T, where $n = 2^d - 1$ for some d. Each node v of T is labeled with a real number x_v. You may assume that the real numbers labeling the nodes are all distinct. A node v of T is a local minimum if the label x_v is less than the label x_w for all nodes w that are joined to v by an edge.
>
> You are given such a complete binary tree T , but the labeling is only specified in the following *implicit* way: for each node v, you can determine the value x_v by probing the node v. Show how to find a local minimum of T using only $O(log\, n)$ probes to the nodes of T .

1. 算法描述与伪代码

基本思路：对于寻找一个局部最小点的问题，可以转化为探究每个节点的值和两个叶子节点值的最小值，即 $min(this, leave_l, leave_r)$。如果该节点值为最小，返回该节点为局部最小点。否则，探究值最小的那个节点。

算法：
- root 表示根节点
- this表示当前节点，this->left表示当前节点的左叶子节点，this->right表示当前节点的右叶子节点
- porbe(node)返回该节点的值
- min(a,b,c)返回a,b,c三个节点中值最小的节点
- $FindLocalMin(node)$返回以node为根节点的一个局部最小点，$FindLocalMin(root)$即为所求

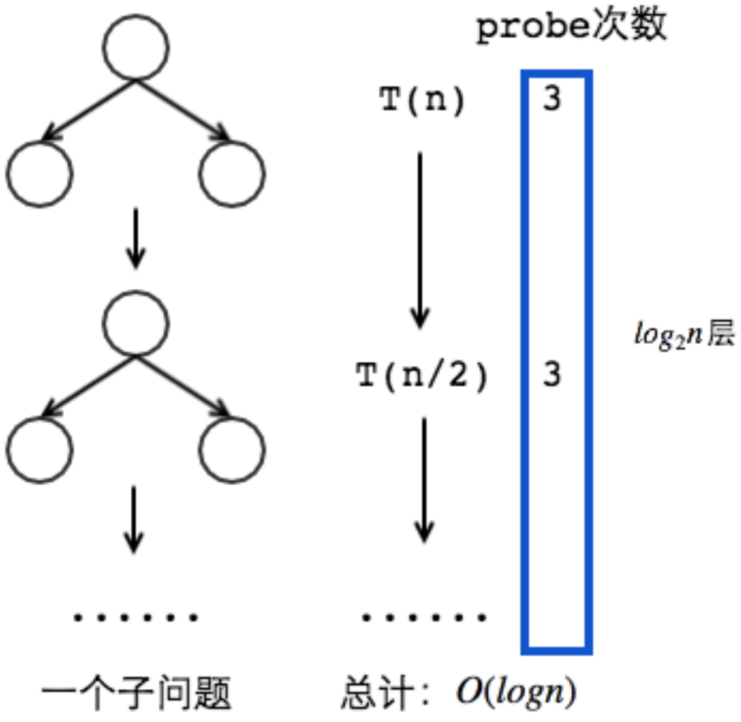
$FindLocalMin(node)$

```
if node has no leave then
    return node;
min_node = min(node, node->left, node->right);
if min_node == node then
    return node;
else
    return FindLocalMin(min_node);
```

$min(a, b, c)$

```
valuea = probe(a);
valueb = probe(b);
valuec = probe(c);

minimum = a;
min_value = valuea;

if valueb < min_value then
    minimum = b;
    min_value = valueb;

if valuec < min_value then
    minimum = c;

return minimum;
```

2. 子问题缩减图



3. 正确性证明（归纳法）

1. 当$d = 1$，根据算法，返回根节点为局部最小点。
2. 当$d = n - 1$时，假设算法成立。对于$d = n$，根据算法，如果根节点的两个叶子节点的值都比根节点值要大，返回根节点，符合问题要求。如果不是，则寻找以值最小的那个节点为根节点的二叉树，的局

部最小点。问题化为求$d = n - 1$时的问题，根据假设，可以得到局部最小点。

根据归纳法定义，算法正确。

4. 运行时间分析

每次产生一个子问题，每个子问题需要进行三次probe，即获取三个节点的值，有：

$$T(n) = \begin{cases} 1, & d = 1 \\ T(n/2) + 3, & d \geq 2 \end{cases}$$
$$(n = 2^d - 1)$$

根据主定理，可得：

$$T(n) = O(log\,n)$$

问题四 （8）

> The attached file Q8.txt contains 100,000 integers between 1 and 100,000 (each row has a single integer), the order of these integers is random and no integer is repeated.
>
> 1. Write a program to implement the Sort-and-Count algorithms in your favorite language, find the number of inversions in the given file.
> 2. In the lecture, we count the number of inversions in O(n log\,n) time, using the Merge-Sort idea. Is it possible to use the Quick-Sort idea instead ?
>   If possible, implement the algorithm in your favourite language, run it over the given file, and compare its running time with the one above.
>   If not, give a explanation.

1. python 实现排序计逆序数算法

```python
from random import randint


def sort_and_count(array):
    """ Use Merge-Sort Idea to count inversions.

    :param array: target for inversions counting
    :return:
        inversion_count: count of inversion pair in array
        array_sorted: array sorted from small to large
    """

    size = len(array)

    # Stop condition of recursive call:
    # 1. When array only contains one element, number of inversions is 0
    #  and return itself.
    if size == 1:
        return 0, array
    # 2. Compare them when array contains two elements
    #  if contains a inversion, reverse two elements and number set to 1.
    if size == 2:
        if array[0] > array[1]:
            return 1, [array[1], array[0]]  # or array.reverse()
        else:
            return 0, array

    # Divide array into left and right parts from the middle.
    split_location = size / 2
    array_left = array[:split_location]
    array_right = array[split_location:]

    # Recursively call to sort_and_count left and right parts.
    count_left, array_left_sorted = sort_and_count(array_left)
    count_right, array_right_sorted = sort_and_count(array_right)

    # Merge after recursively call.
    count_merge, array_sorted \
        = merge_and_count(array_left_sorted, array_right_sorted)

    # Sum the inversion count of three parts.
    inversion_count = count_left + count_right + count_merge

    return inversion_count, array_sorted
```

```
def merge_and_count(array_x, array_y):
    """ Merge two sorted array with order and count the number of inversions
```

```
    :param array_x: sorted array from small to large order
    :param array_y: sorted array from small to large order
    :return:
        merge_inversion_count: number of inversions between array_x and array_y
        array_sorted: merge array_x and array_y into one array with order from
            small to large
    """
    size_x, size_y = len(array_x), len(array_y)
    index_x = index_y = 0
    merge_inversion_count = 0
    array_sorted = []

    while True:
        # In this situation, there is no element hasn't been compared
        #  in array_x, merge the elements reserved in array_y
        if index_x >= size_x:
            array_sorted += array_y[index_y:]
            break

        # Like the situation above.
        if index_y >= size_y:
            array_sorted += array_x[index_x:]
            break

        # Compare element of array_x and array_y and put smaller element into
        #  sorted array.
        # When put an element of array_y into array_sorted, it means every
        #  element reserved in array_x is bigger than it. Additional
        #  array_x_reserved_count inversions.
        if array_x[index_x] < array_y[index_y]:
            array_sorted.append(array_x[index_x])
            index_x += 1
        else:
            array_sorted.append(array_y[index_y])
            merge_inversion_count += size_x - index_x
            index_y += 1

    return merge_inversion_count, array_sorted


# Read file and return an array
def read_file(file_path):
    f = open(file_path, 'r')
    content = f.readlines()
    array = map(int, content)
    return array
```

```
# Run the algorithm
arr = read_file('./Q8.txt')
print sort_and_count(arr)[0]
```

2. 运行结果

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3. 无法使用快速排序的思想解决该问题

因为在对元素进行比较分组时，改变了元素的位置，而因改变位置的同时，逆序数数量发生了变化，变化的大小无法获取。所以无法使用快速排序的思想解决问题。

问题五 （11）

> Implement the Karatsuba algorithm for Multiplication problem in your favourite language, and compare the performance with quadratic grade-school method.

1. python实现Karatsuba算法

Python

```python
def karatsuba(x, y):
    """ Multiply two numbers with more efficient way than grade school's

    :param x: number one to multiply
    :param y: number two to multiply
    :return: product of x and y
    """

    # maximum length of x and y
    max_len = max(len(str(x)), len(str(y)))

    # stop condition of recursive call
    # Stop when one of the value equals 0 or their length equal 1
    if x == 0 or y == 0:
        return 0
    if max_len == 1:
        return x * y

    # Split x and y into high part and low part
    # It's easier for a human
    splitter = 10 ** (max_len / 2)  # Set the split location
    x_high = x // splitter
    x_low = x - x_high * splitter
    y_high = y // splitter
    y_low = y - y_high * splitter

    # parts will be used in product computing
    high_high = karatsuba(x_high, y_high)
    low_low = karatsuba(x_low, y_low)

    # The trick: grade school multiply twice. Only once in this way
    # One time multiply to get the result of x_high * y_low + x_low * y_high
    high_lows = (karatsuba(x_high + x_low, y_high + y_low)
                 - high_high - low_low)

    # Calculate the product of x and y by using computed parts
    result = (high_high * (splitter ** 2)
              + high_lows * splitter + low_low)

    return result


print "Input two number:"
num_a, num_b = map(int, raw_input().split())
print "Multiple Result: ", karatsuba(num_a, num_b)
```

2. 与小学计算方法的性能比较

- 小学计算方法：使用了四次乘法

$$T(n) = 4T(n/2) + cn \Rightarrow T(n) = O(n^{log_2 4}) = O(n^2)$$

- Karatsuba算法：使用了三次乘法

$$T(n) = 3T(n/2) + cn \Rightarrow T(n) = O(n^{log_2 3}) = O(n^{1.585})$$