Date ______ Prepared By Subodh Mishra

Explanation

①

parameter θ follows a $\beta(2,5)$ pdf

$$f(\theta) = \frac{\theta^{2-1}(1-\theta)^{5-1}}{B(2,5)} = \frac{\theta(1-\theta)^4}{B(2,5)} \longrightarrow \text{prior}$$

Sampleds follow a binomial $(40,\theta)$ distribution

$X = (X_1, X_2, \ldots X_8)$:

$$f_X(\theta) = \prod_{i=1}^{8}\binom{40}{x_i}\theta^{x_i}(1-\theta)^{40-x_i} = \left[\prod_{i=1}^{8}\binom{40}{x_i}\right]\theta^{\Sigma x_i}(1-\theta)^{320-\Sigma x_i}$$

$$\Rightarrow f_X(\theta) = \left[\prod_{i=1}^{8}\binom{40}{x_i}\right]\theta^{\Sigma x_i}(1-\theta)^{320-\Sigma x_i} \longrightarrow \text{Liklihood}$$

The posterior is

$$f_{\theta|X} = \eta \text{ liklihood} \times \text{prior}$$

$$= \eta\left[\prod_{i=1}^{8}\binom{40}{x_i}\right]\theta^{\Sigma x_i}(1-\theta)^{320-\Sigma x_i} \quad \frac{\theta(1-\theta)^4}{B(2,5)}$$

$$= \eta\prod_{i=1}^{8}\binom{40}{x_i}\frac{\theta^{1+\Sigma x_i}(1-\theta)^{324-\Sigma x_i}}{B(2,5)}$$

$$= \eta\frac{\prod_{i=1}^{8}\binom{40}{x_i}}{B(2,5)}\theta^{(1+\Sigma x_i)}(1-\theta)^{(324-\Sigma x_i)}$$

$$\eta = \int_0^1 \frac{\prod_{i=1}^{8} \binom{40}{x_i}}{B(2,5)} \theta^{(1+\Sigma x_i)} (1-\theta)^{(324-\Sigma x_i)}$$

$$f_{\theta|x} = \frac{\theta^{(1+\Sigma x_i)} (1-\theta)^{(324-\Sigma x_i)}}{\int \theta^{(1+\Sigma x_i)} (1-\theta)^{(324-\Sigma x_i)} d\theta}$$

Now: let $\alpha - 1 = 1 + \Sigma x_i \Rightarrow \alpha = 2 + \Sigma x_i$

$\beta - 1 = 324 - \Sigma x_i \Rightarrow \beta = 325 - \Sigma x_i$

Now

$$\int \theta^{\alpha-1} (1-\theta)^{\beta-1} d\theta = \Gamma(\alpha)\Gamma(\beta) / \Gamma(\alpha+\beta)$$

| by definition.

$$f_{\theta|x} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

$$\hat{\theta} = E[\theta|x] = \int_0^1 f_{\theta|x}\, \theta\, d\theta$$

$$\cancel{\Rightarrow \hat{\theta} = \int_0^1 \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \theta^{\alpha} (1-\theta)^{\beta-1} d\theta}$$

$$\hat{\theta} = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha} (1-\theta)^{\beta-1} d\theta$$

$$\hat{\theta} = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{(\alpha-1)+1} (1-\theta)^{\beta-1} \, d\theta$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \theta^{(\alpha+1)-1} (1-\theta)^{\beta-1} \, d\theta$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cancel{\Gamma(\beta)}} \frac{\Gamma(\alpha+1)\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+1)}$$

$$\Rightarrow \hat{\theta} = \frac{\alpha\cancel{\Gamma(\alpha+\beta)}}{(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}} = \frac{\alpha}{\alpha+\beta}$$

$$\Rightarrow \hat{\theta} = \frac{2+\sum x_i}{2+\cancel{\sum x_i}+325-\cancel{\sum x_i}} = \frac{2+\sum x_i}{327}$$

$$\Rightarrow \boxed{\hat{\theta} = \frac{2+\sum x_i}{327}}$$