EP 307 Assignment 6

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Problem 2

To prove: $\langle n \,|\, T \,|\, n \rangle = \langle n \,|\, V \,|\, n \rangle$

Firstly, we note that $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})$

$$\begin{aligned}
\therefore \langle n \,|\, T \,|\, n \rangle &= \left\langle n \,\middle|\, \frac{1}{2m}\hat{p}^2 \,\middle|\, n \right\rangle \\
&= -\frac{\omega\hbar}{4} \left\langle n \,\middle|\, (\hat{a}^\dagger - \hat{a})^2 \,\middle|\, n \right\rangle \\
\text{And} \quad \langle n \,|\, V \,|\, n \rangle &= \left\langle n \,\middle|\, \frac{1}{2}m\omega^2\hat{x}^2 \,\middle|\, n \right\rangle \\
&= \frac{\omega\hbar}{4} \left\langle n \,\middle|\, (\hat{a}^\dagger + \hat{a})^2 \,\middle|\, n \right\rangle \\
\therefore \langle n \,|\, V \,|\, n \rangle - \langle n \,|\, T \,|\, n \rangle &= \frac{\omega\hbar}{4} \left\langle n \,\middle|\, (\hat{a}^\dagger + \hat{a})^2 \,\middle|\, n \right\rangle + \frac{\omega\hbar}{4} \left\langle n \,\middle|\, (\hat{a}^\dagger - \hat{a})^2 \,\middle|\, n \right\rangle \\
&= \frac{\omega\hbar}{2} \left\langle n \,\middle|\, \hat{a}^{\dagger 2} + \hat{a}^2 \,\middle|\, n \right\rangle \\
&= 0
\end{aligned}$$

since both $\hat{a}^\dagger$ will move the state to $|n+2\rangle$, which is orthogonal to $|n\rangle$, and $\hat{a}$ will move it to $|n-2\rangle$ or $0\,|0\rangle$ (if $n < 2$). In both cases the net result is zero

Thus $\langle n \,|\, T \,|\, n \rangle = \langle n \,|\, V \,|\, n \rangle$

Problem 3

We know that $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger), \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})$, and furthermore $\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle$. Taking the conjugate, $\left\langle\alpha\right|\hat{a}^\dagger = \alpha * \left\langle\alpha\right|$

Now,

$$\begin{aligned}
\langle x\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left\langle\alpha\,\middle|\,\hat{a}+\hat{a}^\dagger\,\middle|\,\alpha\right\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}(\left\langle\alpha\,\middle|\,\hat{a}\,\middle|\,\alpha\right\rangle + \left\langle\alpha\,\middle|\,\hat{a}^\dagger\,\middle|\,\alpha\right\rangle) \\
&= \sqrt{\frac{\hbar}{2m\omega}}(\left\langle\alpha\,\middle|\,\alpha\,\middle|\,\alpha\right\rangle + \alpha^*\left\langle\alpha\,\middle|\,\alpha\right\rangle) \\
&= \sqrt{\frac{\hbar}{2m\omega}}(\alpha+\alpha^*) \\
\left\langle x^2\right\rangle &= \frac{\hbar}{2m\omega}\left\langle\alpha\,\middle|\,(\hat{a}+\hat{a}^\dagger)^2\,\middle|\,\alpha\right\rangle \\
&= \frac{\hbar}{2m\omega}(\left\langle\alpha\,\middle|\,\hat{a}^2\,\middle|\,\alpha\right\rangle + \left\langle\alpha\,\middle|\,\hat{a}^{\dagger 2}\,\middle|\,\alpha\right\rangle + \left\langle\alpha\,\middle|\,\hat{a}\hat{a}^\dagger\,\middle|\,\alpha\right\rangle + \left\langle\alpha\,\middle|\,\hat{a}^\dagger\hat{a}\,\middle|\,\alpha\right\rangle) \\
&= \frac{\hbar}{2m\omega}(\left\langle\alpha\,\middle|\,\alpha^2\,\middle|\,\alpha\right\rangle + (\alpha^*)^2\left\langle\alpha\,\middle|\,\alpha\right\rangle + \left\langle\alpha\,\middle|\,\hat{a}^\dagger\hat{a}-[\hat{a},\hat{a}^\dagger]\,\middle|\,\alpha\right\rangle + \left\langle\alpha\,\middle|\,\hat{a}^\dagger\hat{a}\,\middle|\,\alpha\right\rangle) \\
&= \frac{\hbar}{2m\omega}(\alpha^2 * \alpha^{*2} + \left\langle\alpha\,\middle|\,\hat{a}^\dagger\hat{a}-1\,\middle|\,\alpha\right\rangle + \left\langle\alpha\,\middle|\,\hat{a}^\dagger\hat{a}\,\middle|\,\alpha\right\rangle) \\
&= \frac{\hbar}{2m\omega}(\alpha^2 + \alpha^{*2} - \left\langle\alpha\,\middle|\,1\,\middle|\,\alpha\right\rangle + 2\left\langle\alpha\,\middle|\,\hat{a}^\dagger\hat{a}\,\middle|\,\alpha\right\rangle) \\
&= \frac{\hbar}{2m\omega}(\alpha^2 + \alpha^{*2} - 1 + 2\alpha^*\alpha) \\
\langle p\rangle &= i\sqrt{\frac{m\omega\hbar}{2}}\left\langle\alpha\,\middle|\,(\hat{a}^\dagger-\hat{a})\,\middle|\,\alpha\right\rangle \\
&= i\sqrt{\frac{m\omega\hbar}{2}}(\alpha^*-\alpha) \\
\left\langle p^2\right\rangle &= -\frac{m\omega\hbar}{2}\left\langle\alpha\,\middle|\,(\hat{a}^\dagger-\hat{a})^2\,\middle|\,\alpha\right\rangle \\
&= -\frac{m\omega\hbar}{2}(\alpha^{*2}+\alpha^2+1-+2\alpha^*\alpha) \\
\therefore \sigma_x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2}
\end{aligned}$$

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\begin{aligned}
&= \sqrt{\frac{\hbar}{2m\omega}\left(\alpha^2 + \alpha^{*2} - 1 + 2\alpha^*\alpha - (\alpha + \alpha^*)^2\right)} \\
&= \sqrt{\frac{\hbar}{2m\omega}\left(1 - 2|\alpha|^2\right)} \\
\therefore \sigma_p &= \sqrt{\langle p^2\rangle - \langle p\rangle^2} \\
&= \sqrt{-\frac{m\omega\hbar}{2}\left(\alpha^2 + \alpha^{*2} - 1 - (\alpha - \alpha^*)^2\right)} \\
&= \sqrt{-\frac{m\omega\hbar}{2}\left(-1 + 2|\alpha|^2\right)} \\
&= \sqrt{\frac{m\omega\hbar}{2}\left(1 - 2|\alpha|^2\right)} \\
\therefore \sigma_x\sigma_p &= \sqrt{\frac{\hbar}{2m\omega}\left(1 - 2|\alpha|^2\right)}\sqrt{\frac{m\omega\hbar}{2}\left(1 - 2|\alpha|^2\right)} \\
&= \frac{\hbar}{2}(1 - 2|\alpha|^2)
\end{aligned}
$$

From this we infer that $|\alpha|^2$

Problem 4

Problem 5

Problem 6