MA 207 scribbles

Manish

November 10, 2013

1 Series

- If A series covnerges then terms tend to zero
- $\sum f(n)$ converges iff $\int\limits_1^\infty$ converges
- abs convergent $\implies$ convergent.
- Alternating ($\pm$) the terms of a monotone decr series is convergent
- (Cond convgt = convegent but not absolutely). Such series can be rearranged to any preassigned value. Absolutely convergent series have no change on rearangement.
- Cauchy product converges if both converge and atleast one is absolutely convergent

2 Polynomial-type ODEs

- Legendre: $(1-x^2)y'' - 2xy' + p(p+1)y = 0$
- Tchebychev: $(1-x^2)y'' - xy' + p^2y = 0$ (orthogonal on $(-1,1)$ with wieght $(1-x^2)^{-0.5}$)
- Bessel: $x^2y'' + xy' + (x^2 - p^2)y = 0$
- Laguerre: $xy'' + (1-x)y' + py = 0$

- Hypergeometric: $x(1-x)y'' + (c-(a+b+1)x)y' - aby = 0$
- (others)
- Hermite: $y'' - 2xy' + 2ny = 0$ (Orthogonal on $\mathbb{R}$ wrt e^{-x^2})
- Bernoulli: $B_n' = nB_{n-1}$, $\int\limits_0^1 B_n = 0, n \geq 1$

If f_n is a seq of orth pol, then zeroes are real+distinct (and are in the interval of orthogonality); it satisfies a 3term recursion formula, and the zeroes interlace.

2.1 Legendre

Gen soln:

$$a_0 \left(1 - \frac{p(p+1)x^2}{2!} + \frac{p(p-2)(p+1)(p+3)x^4}{4!} - ...\right) +$$
$$a_1 \left(x - \frac{(p-1)(p+2)x^3}{2!} + \frac{(p-1)(p-3)(p+2)(p+4)x^5}{4!} - ...\right)$$

Self adjoint: $D((1-x^2)P_k') + k(k+1)P_k$
Rodrigues' formula: $P_n(x) = \frac{1}{2^n n!} D^n (x^2-1)^n$
Recursion: $(n+1)P_{n+1} - x(2n+1)P_n + nP_{n-1} = 0$
Generating func: $\sum\limits_{n=0}^{\infty} t^n P_n(x) = \frac{1}{\sqrt{1-2xt+t^2}}$
Laplace's formula: $\frac{1}{\pi} \int\limits_0^{\pi} (x + \sqrt{x^2-1}\cos\phi)^n d\phi$

2.2 Gamma/etc

$$\Gamma(p) = \int\limits_0^{\infty} e^{-t} t^{p-1} dt$$

$\Gamma(p+1) = p\Gamma(p); \Gamma \sim (p-1)!$
Stirling's approx: $n! \sim n^n e^{-n} \sqrt{2n\pi}$; extends to Γ

(F-F thm) If an ODE has a regular sing pt, then $\exists$ at least one soln of the form $(x-x_0)^p \sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$, +ve ratio of convergence. If the roots of ρ do not differ by an integer, the solutions are linearly independant. Otherwise, they may still be LD, or the recursion formula may break down, or we get a multiple.

2.3 Bessel

$$J_p = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!\Gamma(n+p+1)}$$

- $J_{\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} \sin x$, for -0.5 cos
- $a = x^{-p} D x^p, a^\dagger = x^p D x^{-p}$; $(D(x^p J_p) = x^p J_{p-1})$
- $xJ_p' \pm pJ_p = \pm x J_{p\mp 1}$, $2J_p' = J_{p-1} - J_{p+1}$, $2nJ_n = x(J_{n-1} + J_{n+1})$
- Bilateral sum of J_n is 1. Schlömilch's formula: $\sum\limits_{-\infty}^{\infty} J_n(x) t^n = \exp \frac{x}{2}\left(t - \frac{1}{t}\right)$
- $J_m(x) = \frac{1}{\pi} \int\limits_0^{\pi} \cos(x \sin\theta - m\theta) d\theta$
- Has a zero every interval of π
- Lommel's formula: For $f(r) = \sum\limits_1^{\infty} A_j J_0(\zeta_j r)$, $A_j = \frac{2}{J_1^2(\zeta_j)} \int\limits_0^1 r f(r) J_0(\zeta_j r) dr$ (Orthogonal on $[0,1]$ with weight x)
- $J_0(x) = \int\limits_1^{\infty} \frac{\sin(tx)}{\sqrt{t^2-1}} dt$

3 Sturm-Liouville

$$\int_\Omega u \frac{\partial v}{\partial x} d\tau = -\int_\Omega u \frac{partialv}{\partial x} d\tau + \int_{\partial\Omega} uv\hat{i} \cdot dS$$

- Energy method: Write $F = f_2 - f_1$, now substitute in initial equation and boundary eqn. Multiply DE with $\partial_t F$ and integrate over disc covered by boundary (this is energy). Prove that this is constantly 0 (or something manageable) via derivatives. The boundary term in the integral (by parts) will vanish.

- Generalized: $y'' + \lambda\rho(x)y = 0$ with boundary conditions (Dirichlet BC: $y(x_0)$ given, Neumann: $y'(x_0)$). ρ is continuous and positive.

- The eigenfunctions for such a problem on $[0, l]$ are orthogonal on the same interval with weight ρ

- For dirichlet, there is 1 EF per EV. For any SL problem, there is an infinite sequence of EVs tending to infinity. The EFs cover the space of all Lipschitz functions.

- To minimize $\int y'(t)^2 dt$ ($\langle y, \hat{A}y\rangle$) subject to $\int y(t)^2\rho(t)dt = 1$ ($\langle y, y\rangle = 1$), we solve the corresp S-L problem, and the EF corresp to smallest EV is the solution.

- Sturm's comparison: For weights $\rho(x) > \sigma(x)$, between two zeroes of $y_\sigma(x)$ there is at least one zero of $y_\rho(x)$

- $\int\limits_{\Omega} u\frac{\partial v}{\partial x}dxdy = \int\limits_{\Omega} uv\hat{i}dS - \int\limits_{\partial Omega} v\frac{\partial u}{\partial x}dxdy$

4 Fourier

Discrete

- Dirichlet theorem: Coefficients are $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx$. At a point of discontinuity, it will converge to the mean of the two sides.

- Riemann-Lebesgue: The fourier coefficients of a cont fn tend to zero as $n \to \infty$

- Weierstraß approx: If f is continuous then given any $\epsilon > 0$, $\exists P(x)$ (polynomial) such that $|f(x) - P(x)| < \epsilon$ for all x in interval.

- If s_ns are the successive partial fourier series as trig polynomials, and P_n is an arbit trig polynomial of degree at least n, $||f - x_n||_2 \leq ||f - P||_2$, where the L^2 norm is $||f||_2 = \sqrt{\frac{1}{b-a}\int_a^b |f(t)|^2 dt}$

- Parseval: For a pair of Riemann integrable functions, then inner product of two ($\frac{1}{2\pi}\int f\bar{g}$) is equal to sum of products of coefficients ($a_0\bar{a_0'} + \frac{1}{2}\sum a_n\bar{a_n'} + b_n\bar{b_n'}$).

- Poisson kernel: $\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}$. For a 3D system $\Delta u = 0$, and the value of u on the boundary is f, $f(\mathbf{y}) = \int_{\partial B} \mathrm{ker}(||\mathbf{x}||, \alpha) f(\mathbf{x}) dS$, where α is the angle between the two vectors. Works for 2D with norms replaced by r, and α with $\theta - t$ (t is the variable of integration)

Continuous

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt; \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega$$

$f(t-t_0)$	$\hat{f}(\omega)e^{-i\omega t_0}$	$f(at)$	$\frac{1}{\lvert a\rvert}\hat{f}\left(\frac{\omega}{a}\right)$
$(-it)^n f(t)$	$\hat{f}^{(n)}(\omega)$	e^{-t^2}	$\sqrt{\pi}e^{-\frac{\omega^2}{4}}$
$\mathrm{rect}(t)$	$\mathrm{sinc}(\frac{\omega}{2})$		

- If the integral over space is finite, $\hat{f(\xi)} = \int_{-\infty}^{\infty} f(t)e^{-it\xi}dt$

- Schwartz space: Space for which $t^n f^{(m)}(x)$ stays bounded for all integers. Integral over space is finite, so fourier transform exists.

- $\widehat{f'(t)} = i\xi\hat{f}(\xi)$, and $\widehat{tf(t)} = i\frac{d}{d\xi}\hat{f}(\xi)$

- $\int\limits_{-\infty}^{\infty} \hat{f}(\xi)d\xi = 2\pi f(0)$

- $\lim\limits_{k\to\infty} \int\limits_{-\infty}^{\infty} \frac{\sin kt}{\sin t} f(t)dt = \pi\left(f(0) + 2f(\pi) + 2f(2\pi)\cdots\right)$

- Theta fn identity: $\pi\left(1 + 2e^{-a^2\pi^2} + 2e^{-4a^2\pi^2} + \cdots\right) \frac{\sqrt{\pi}}{a}\left(1 + 2e^{-1/a^2} + 2e^{-4/a^2} + \cdots\right)$

- Poisson summation: $\sum\limits_{-\infty}^{\infty} f(n) = \sum\limits_{-\infty}^{\infty} \hat{f}(2n\pi)$
- Parseval: $\int\limits_{-\infty}^{\infty} f\bar{g} = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \hat{f}\bar{\hat{g}}$
- Convolution: $\widehat{f * g} = \hat{f}\hat{g}$
- Heat kernel: $\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(s) e^{-\frac{(x-s)^2}{4t}} ds$

5 Useful stuff

Laplace transforms

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t) dt$$

1	$\frac{1}{s}$	x^n	$\frac{n!}{s^{n+1}}$
$\sin(ax), \cos(ax)$	$\frac{a}{s^2+a^2}, \frac{s}{s^2+a^2}$	s^{ax}	$\frac{1}{s-a}$
$x\text{cis}(ax)$	$\frac{s^2-a^2+2ais}{(s^2+a^2)^2}$	$\sinh(ax)$	$\frac{a}{s^2-a^2}$
$u_c(x)$	$\frac{e^{-cs}}{s}$	$e^{ct} f(x)$	$\mathcal{L}(f)(s-c)$
$f'(x)$	$s\mathcal{L}(f) - f(0)$	$\frac{f(x)}{x}$	$\int_s^\infty \mathcal{L}(f)$
$\int_0^x f$	$\frac{\mathcal{L}(f)}{s}$	$f(cx)$	$\frac{1}{c}\mathcal{L}(f)(\frac{s}{c})$
$f^{(n)}(x)$	$s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - f^{(n-1)}(0)$		