

# EP 307 Assignment 4

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## Problem 1

$$\prod_{i \neq s} \frac{\hat{A} - a_i}{a_s - a_i}$$

If  $|a\rangle = \sum c_i |a_i\rangle$ , applying this operator we have

$$\begin{aligned} \prod_{i \neq s} \frac{\hat{A} - a_i}{a_s - a_i} \left( \sum c_i |a_i\rangle \right) &= \sum_j \left( \prod_{i \neq s} \frac{\hat{A} - a_i}{a_s - a_i} \right) c_j |a_j\rangle \\ &= \left( \prod_{i \neq s} \frac{\hat{A} - a_i}{a_s - a_i} \right) c_s |a_s\rangle \end{aligned}$$

Since the other terms have numerator cancelling out for  $i = j$

$$\begin{aligned} &= \left( \prod_{i \neq s} \frac{a_s - a_i}{a_s - a_i} \right) c_s |a_s\rangle \\ &= c_s |a_s\rangle \end{aligned}$$

Thus the operator is  $\frac{|a_s\rangle\langle a_s|}{\langle a_s|a_s\rangle} = p_s$  (projection operator)

## Problem 2

To prove:

$$\text{Tr}(XY) = \text{Tr}(YX)$$

The trace of an operator can be denoted by  $\sum_i \langle i | A | i \rangle$ , where  $\langle i |, | i \rangle$  are basis vectors corresponding to the  $i$ th element being 1 and the rest 0.

Thus,

$$\begin{aligned}
\text{Tr}(AB) &= \sum_i \langle i | AB | i \rangle \\
&= \sum_i \langle i | AIB | i \rangle \\
&= \sum_i \sum_j \langle i | A | j \rangle \langle j | B | i \rangle \\
&= \sum_i \sum_j \langle j | A | i \rangle \langle i | B | j \rangle \\
&= \sum_j \langle j | BIA | j \rangle \\
&= \text{Tr}(BA)
\end{aligned}$$

## Problem 7

Applying  $\hat{A} = \hat{Q}\hat{C} + \hat{C}\hat{Q}$  to an eigenstate  $|\psi_q\rangle$ , we get:

$$\begin{aligned}
\hat{A}|\psi_q\rangle &= (\hat{Q}\hat{C} + \hat{C}\hat{Q})|\psi_q\rangle \\
&= \hat{Q}\hat{C}|\psi_q\rangle + \hat{C}\hat{Q}|\psi_q\rangle \\
&= \hat{Q}|\psi_{-q}\rangle + q\hat{C}|\psi_q\rangle \\
&= -q|\psi_{-q}\rangle + q|\psi_{-q}\rangle \\
&= 0
\end{aligned}$$

If  $\hat{A}|\psi_q\rangle = 0$ , then  $\hat{A}(\sum a_q |\psi_q\rangle) = 0$ . We can say that the eigenvalue of  $\hat{A}$  is 0.

For a state  $|\psi_q\rangle$  to be an eigenstate of  $\hat{C}$ ,  $c|\psi_q\rangle = \hat{C}|\psi_q\rangle = |\psi_{-q}\rangle$

Since  $|\psi_{-q}\rangle, |\psi_q\rangle$  have different eigenvalues (except when  $q = 0$ ), they are linearly independent, and thus the only possible value of  $c$  is 0, which isn't an eigenstate.

So the only common eigenstate is  $|\psi_0\rangle$ , provided that it is not a null vector.

## Problem 9

$$\begin{aligned}
[\hat{x}, \exp\left(\frac{i\hat{p}a}{\hbar}\right)]|\psi\rangle &= [\hat{x}, \exp\left(\frac{i(-i)\hbar\partial_x a}{\hbar}\right)]|\psi\rangle \\
&= [\hat{x}, \exp(a\partial_x)]|\psi\rangle \\
&= x \sum_i \frac{1}{i!} a^i \partial_x^i |\psi\rangle - \sum_i \frac{1}{i!} a^i \partial_x^i (x |\psi\rangle) \\
&= x \sum_i \frac{1}{i!} a^i \partial_x^i |\psi\rangle - \sum_i \frac{1}{i!} a^i (x \partial_x^i (|\psi\rangle) + \partial_x^{i-1} |\psi\rangle) \\
&= - \sum_i \frac{1}{i!} a^i \partial_x^{i-1} |\psi\rangle \\
&= - \sum_i \frac{1}{(i-1)!} a^i \partial_x^{i-1} |\psi\rangle \\
&= -a \exp(a\partial_x) |\psi\rangle \\
&= -a \exp\left(ia\frac{\hat{p}}{\hbar}\right) |\psi\rangle \\
\therefore [\hat{x}, \exp\left(\frac{i\hat{p}a}{\hbar}\right)] &= -a \exp\left(ia\frac{\hat{p}}{\hbar}\right)
\end{aligned}$$

## Problem 11

The first allowed state (ground state) in the new system will be the first odd wavefunction, i.e. when  $n = 1$ . We need to calculate the probability that the current state ( $n = 0$ ) becomes  $n = 1$  in that region. Note that since the space is halved over a symmetric function, the wavefunctions will be normalized by an extra factor of  $\frac{1}{\sqrt{2}}$

So, the overlap is

$$\begin{aligned}
\int_0^\infty \psi_0(x) \frac{1}{\sqrt{2}} \psi_1(x) &= \int_0^\infty \left(\frac{\alpha}{\sqrt{\pi}}\right)^{\frac{1}{2}} e^{-\alpha^2 x^2/2} H_0(\alpha x) \left(\frac{\alpha}{\sqrt{\pi}2}\right)^{\frac{1}{2}} e^{-\alpha^2 x^2/2} H_2(\alpha x) \\
&= \frac{\alpha}{\sqrt{\pi}\sqrt{2}} \cdot \int_0^\infty e^{-\alpha^2 x^2} \cdot 2\alpha x \\
&= \frac{2\alpha^2}{\sqrt{2\pi}} \int_0^\infty e^{-\alpha^2 x^2} \cdot x
\end{aligned}$$

$$= \frac{2\alpha^2}{\sqrt{2\pi}} \frac{1}{2\alpha^2} = \frac{1}{\sqrt{2\pi}}$$

Thus the probability is  $\left(\frac{1}{\sqrt{2\pi}}\right)^2 = \boxed{\frac{1}{2\pi}}$

### Problem 13

$$\hat{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & ib \\ 0 & ib & 0 \end{pmatrix}$$

To find eigenvectors of  $\hat{B}$ , we can multiply it with the vector  $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ .

We get  $\hat{B}X = \begin{pmatrix} bx \\ ibz \\ iby \end{pmatrix} = cX$ , so for an eigenvector we either have  $c = b, ibz =$

$y, iby = z$  (giving  $y = z = 0$  for a general  $b$ , and eigenvector  $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ). If  $c \neq b$

we have  $x = 0$ , and  $y = \pm z$  with eigenvalues  $\pm ib$

Therefore the eigenvectors are:

1.  $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  or  $|1\rangle$  with eigenvalue  $b$
2.  $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$  or  $|2\rangle + |3\rangle$  with eigenvalue  $ib$
3.  $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$  or  $|2\rangle - |3\rangle$  with eigenvalue  $-ib$

Ans. (a)

Now,

$$\hat{A}\hat{B} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & ib \\ 0 & ib & 0 \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\hat{B}\hat{A} = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & ib \\ 0 & ib & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

As  $\hat{A}\hat{B} = \hat{B}\hat{A}$  in matrix form, the operators commute. Ans. (b)

The eigenkets of  $\hat{A}$  can be easily seen to be  $|1\rangle$  with eigenvalue  $a$ , and  $l|2\rangle + m|3\rangle$  with eigenvalue  $-a$  ( $\forall l, m$ )

We can see that the eigenkets of  $B$  are also eigenkets of  $A$ . They are also orthogonal, after normalizing we have:

Orthonormal Eigenket	Eigenvalue with $\hat{A}$	Eigenvalue with $\hat{B}$
$ 1\rangle$	$b$	$a$
$\frac{ 2\rangle+ 3\rangle}{\sqrt{2}}$	$ib$	$-a$
$\frac{ 2\rangle- 3\rangle}{\sqrt{2}}$	$-ib$	$-a$

The eigenkets are not completely determined from the eigenvalues from individual eigenvalues, as the eigenvalue  $-a$  has multiplicity 2 and thus has an entire space of eigenkets.

However, knowing the eigenvalues from both operators completely specifies the eigenket, barring a constant. Ans.

(c)

## Problem 16

$\psi(x) = \frac{1}{\sqrt{2a}}$  in  $[-a, a]$ . The momentum space function can be found via the fourier transform,

$$\begin{aligned} \tilde{\psi}(p) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \psi(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^a \frac{1}{\sqrt{2a}} e^{-ikx} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{i}{2k\sqrt{a\pi}} e^{-ikx} \Big|_{-a}^a \\
&= \frac{i}{2k\sqrt{a\pi}} (e^{-ika} - e^{ika}) \\
&= \frac{i}{2k\sqrt{a\pi}} \cdot -2i \sin ka \\
\therefore \tilde{\psi}(p) &= \frac{\sin\left(\frac{ap}{\hbar}\right)}{p\sqrt{a\pi}} \quad (\text{Already normalized})
\end{aligned}$$

Ans.

Now,

$$\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\int \psi^*(x) x^2 \psi(x) dx - 0} \\
&= \sqrt{\int_{-a}^a \left(\frac{1}{\sqrt{2a}}\right)^2 x^2 dx} \\
&= \frac{a}{\sqrt{3}}
\end{aligned}$$

And

$$\begin{aligned}
\sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \sqrt{\int \tilde{\psi}^*(p) p^2 \tilde{\psi}(p) dp - 0} \\
&= \sqrt{\int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{ap}{\hbar}\right)}{\sqrt{\pi}\sqrt{ap}}\right)^2 p^2 dp - 0} \quad (\text{The wavefunction is odd so the second term vanishes}) \\
&= \sqrt{\infty}
\end{aligned}$$

The product of the two uncertainties is greater than  $\frac{\hbar}{2}$

## Problem 17

$$\psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p) e^{-\frac{t(ip^2)}{2m} + ipx} dp$$

$$\begin{aligned} \langle p \rangle &= \int \psi^*(x, t) (-i\hbar) \frac{\partial}{\partial x} \psi(x, t) dx \\ &= \int \psi(x, t) \hbar p \psi^*(x, t) dx \quad \because \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \phi(p) e^{-\frac{t(ip^2)}{2m} + ipx} = ip \int_{-\infty}^{\infty} \phi(p) e^{-\frac{t(ip^2)}{2m} + ipx} \\ &= \int \hbar p \psi^*(x, t) \psi(x, t) dx \\ \frac{d \langle p \rangle}{dt} &= \int \hbar p \frac{\partial \psi^*(x, t) \psi(x, t)}{\partial t} dx \\ &= \int \hbar p \frac{\partial \psi^*(x, t)}{\partial t} \psi(x, t) + \frac{\partial \psi(x, t)}{\partial t} \psi^*(x, t) dx \\ &= \int \hbar p \psi^*(x, t) \psi(x, t) \left( \frac{ip^2}{2m} - \frac{ip^2}{2m} \right) \\ &\quad \because \frac{\partial \psi(x, t)}{\partial t} = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \phi(p) e^{-\frac{t(ip^2)}{2m} + ipx} = \frac{-ip^2}{2m} \psi(x, t) \\ &= 0 \end{aligned}$$

Thus,  $\langle \hat{p} \rangle$  is independent of time

Ans. (a)