EP 307 Assignment 2

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January 29, 2014

Problem 1

$$
\begin{aligned}
\langle e_1 \,|\, e_1 \rangle = \langle e_2 \,|\, e_2 \rangle &= 2 \\
\langle e_3 \,|\, e_3 \rangle &= 4 \\
\langle e_1 \,|\, e_2 \rangle &= i\sqrt{2} \\
\langle e_1 \,|\, e_3 \rangle &= 1 + i \\
\langle e_2 \,|\, e_3 \rangle &= 2
\end{aligned}
$$

First, we normalize the states to get:

$$
\begin{aligned}
\langle a_1 \,|\, a_1 \rangle = \langle a_2 \,|\, a_2 \rangle &= 1 \\
\langle e_3 \,|\, a_3 \rangle &= 1 \\
\langle a_1 \,|\, a_2 \rangle &= i\frac{1}{\sqrt{2}} \\
\langle a_1 \,|\, a_3 \rangle &= \frac{1+i}{2\sqrt{2}} \\
\langle a_2 \,|\, a_3 \rangle &= \frac{1}{\sqrt{2}}
\end{aligned}
$$

Taking $\hat{p}_{a_i} = |a_i\rangle \langle a_i|$ to be the projection operator with respect to a_i, by Gram-Schmidt orthogonalization procedure,

$$
\begin{aligned}
|\phi_1\rangle &= |a_1\rangle \\
|\phi_2\rangle &= |a_2\rangle - \hat{p}_{a_1}|a_2\rangle \\
|\phi_3\rangle &= |a_3\rangle - \hat{p}_{a_1}|a_3\rangle - \hat{p}_{a_2}|a_3\rangle
\end{aligned}
$$

Thus

$$
\begin{aligned}
|\phi_1\rangle &= |a_1\rangle \\
|\phi_2\rangle &= |a_2\rangle - |a_1\rangle\langle a_1 \mid a_2\rangle \\
|\phi_3\rangle &= |a_3\rangle - |a_1\rangle\langle a_1 \mid a_3\rangle - |a_2\rangle\langle a_2 \mid a_3\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
|\phi_1\rangle &= |a_1\rangle \\
|\phi_2\rangle &= |a_2\rangle + \frac{1}{\sqrt{2}i}|a_1\rangle \\
|\phi_3\rangle &= |a_3\rangle - \frac{1+i}{2\sqrt{2}}|a_1\rangle - \frac{1}{\sqrt{2}}|a_2\rangle
\end{aligned}
$$

We can calculate

$$
\begin{aligned}
\langle\phi_1 \mid \phi_1\rangle &= 1 \\
\langle\phi_2 \mid \phi_2\rangle &= 1 - \frac{1}{2} = \frac{1}{2} \\
\langle\phi_3 \mid \phi_3\rangle &= 1 + \frac{(1-i)(1+i)}{8} + \frac{1}{2} = \frac{7}{4}
\end{aligned}
$$

and from here we get an orthonormal basis

$$
\begin{aligned}
|\psi_1\rangle &= |a_1\rangle \\
|\psi_2\rangle &= \sqrt{2}\left(|a_2\rangle + \frac{1}{\sqrt{2}i}|a_1\rangle\right) \\
|\psi_3\rangle &= \frac{2}{\sqrt{7}}\left(|a_3\rangle - \frac{1+i}{2\sqrt{2}}|a_1\rangle - \frac{1}{\sqrt{2}}|a_2\rangle\right)
\end{aligned}
$$

Rewriting in terms of the original vectors

$$
\begin{aligned}
|\psi_1\rangle &= \frac{1}{\sqrt{2}} |e_1\rangle \\
|\psi_2\rangle &= |e_2\rangle + \frac{1}{\sqrt{2}i} |e_1\rangle \\
|\psi_3\rangle &= \frac{4}{\sqrt{7}} \left(|e_3\rangle - \frac{1+i}{2\sqrt{2}} |e_1\rangle - \frac{1}{\sqrt{2}} |e_2\rangle \right)
\end{aligned}
$$

$|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$ are our orthonormal basis vectors.

Problem 2

First, we normalize them by taking integrals in $[-1, 1]$, and get $c_0 = \frac{1}{\sqrt{2}}, c_1 = \sqrt{\frac{3}{2}}, c_2 = \sqrt{\frac{5}{2}}$

Applying the orthogonalization process,

$$
\begin{aligned}
|\phi_1\rangle &= |\psi_1\rangle \\
|\phi_2\rangle &= |\psi_2\rangle - |\psi_1\rangle \langle \psi_1 \mid \psi_2\rangle \\
|\phi_3\rangle &= |\psi_3\rangle - |\psi_1\rangle \langle \psi_1 \mid \psi_3\rangle - |\psi_2\rangle \langle \psi_2 \mid \psi_3\rangle
\end{aligned}
$$

calculating the inner products, we get

$$
\begin{aligned}
|\phi_1\rangle &= |\psi_1\rangle \\
|\phi_2\rangle &= |\psi_2\rangle - 0 |\psi_1\rangle \\
|\phi_3\rangle &= |\psi_3\rangle - \frac{\sqrt{5}}{2} \frac{2}{3} |\psi_1\rangle - 0 |\psi_2\rangle
\end{aligned}
$$

which evaluates to

$$
\begin{aligned}
\phi_1 &= \frac{1}{\sqrt{2}} \\
\phi_2 &= \sqrt{\frac{3}{2}}x \\
\phi_3 &= \sqrt{\frac{5}{2}}\left(x^2 - \frac{1}{3}\right)
\end{aligned}
$$

Problem 3

$$\hat{B}g(x) = g(-x)$$

The operator is linear, since $\hat{B}(\alpha f(x) + \beta g(x)) = \alpha f(-x) + \beta g(-x) = \alpha\hat{B}f(x) + \beta\hat{B}g(x)$

It also is Hermitian, as

$$
\begin{aligned}
\langle f|\,(\hat{B}\,|g\rangle) &= \int_{-\infty}^{\infty} \bar{f}(x)\hat{B}g(x)dx \\
&= \int_{-\infty}^{\infty} \bar{f}(x)g(-x)dx \\
&= \int_{-\infty}^{\infty} \bar{f}(-x)g(x)dx \\
&= \int_{-\infty}^{\infty} \bar{f}(-x)g(x)dx \\
&= \int_{-\infty}^{\infty} \overline{\hat{B}f(x)}g(x) \\
&= \left\langle \overline{\hat{B}}f \,\middle|\, g \right\rangle \\
\therefore \left\langle f \,\middle|\, \hat{B}g \right\rangle &= \left\langle \overline{\hat{B}}f \,\middle|\, g \right\rangle
\end{aligned}
$$

To find eigenvalues, $\hat{B}g(x) = bg(x) = g(-x)$

Since $bg(x) = g(-x)$ and $bg(-x) = g(x)$, $b = \pm 1$ (neglecting the trivial $b = 0$ solution).

Thus, we have eigenvectors:

$$\sum_n a_n x^{2n} \qquad \text{(even function)} \qquad \text{for eigenvalue } b = 1$$
$$\sum_n b_n x^{2n+1} \qquad \text{(odd function)} \qquad \text{for eigenvalue } b = -1$$

To calculate the commutator,

$$\begin{aligned}
[B, \hat{x}^n] f &= \hat{B}\hat{x}^n f(x) - \hat{x}n\hat{B} f(x) \\
&= \hat{B}x^n f(x) - \hat{x}^n f(-x) \\
&= (-x)^n f(-x) - x^n f(-x) \\
&= f(-x)((-x)^n - x^n)
\end{aligned}$$

For this to be zero for all states f, n must be even. Thus the commutator is zero only for even n

Problem 4

$$\hat{\Omega} = |\psi\rangle \langle\phi|$$

For it to be Hermitian, $\hat{\Omega}^\dagger = \Omega$. Applying it to the state $|\Psi\rangle = |\psi\rangle + |\phi\rangle$,

$$\begin{aligned}
|\psi\rangle \langle \phi \,|\, \Psi\rangle &= |\phi\rangle \langle \psi \,|\, \Psi\rangle \\
|\psi\rangle \langle \phi \,|\, \psi\rangle + |\phi\rangle \langle \psi \,|\, \psi\rangle &= |\psi\rangle \langle \phi \,|\, \phi\rangle + |\phi\rangle \langle \psi \,|\, \phi\rangle \\
(\langle \phi \,|\, \psi\rangle - \langle \phi \,|\, \phi\rangle) |\psi\rangle &= (\langle \psi \,|\, \phi\rangle - \langle \psi \,|\, \psi\rangle) |\phi\rangle
\end{aligned}$$

For this to hold, either $|\psi\rangle = c\,|\phi\rangle$ or $(\langle \phi \,|\, \psi\rangle - \langle \phi \,|\, \phi\rangle) = 0 = (\langle \psi \,|\, \phi\rangle - \langle \psi \,|\, \psi\rangle)$

In the latter case, we get $\langle \psi \,|\, \psi\rangle = \langle \phi \,|\, \psi\rangle = \langle \phi \,|\, \phi\rangle$, which can only happen if the two states are equivalent.

Thus condition for the operator to be hermitian is that $|\psi\rangle = c\,|\phi\rangle$.

For it to be a projection operator, $c = \frac{1}{\langle \phi \,|\, \phi\rangle}$

1 Problem 5

$$\hat{B}\psi(x) = \int_{-\infty}^{x} x'\psi(x')dx'$$

The eigenvalue problem $\hat{B}\psi = \lambda\psi$ thus becomes $\int_{-\infty}^{x} x'\psi(x')dx' = \lambda\psi(x)$. Differentiating with respect to x, this is

$$\begin{aligned}
x\psi(x) &= \lambda\psi'(x) \\
\implies \frac{\psi'(x)}{\psi(x)} &= \frac{x}{\lambda} \\
\implies \int_{\psi(0)}^{\psi} \frac{\psi'(x)}{\psi(x)} &= \int_{0}^{x} \frac{x}{\lambda} \\
\implies \log\frac{\psi(x)}{\psi(0)} &= \frac{1}{\lambda}\frac{x^2}{2} \\
\implies \psi(x) = \psi(0)e^{\frac{x^2}{2\lambda}}
\end{aligned}$$

This is an acceptable (square-integrable) wavefunction only when λ is negative.