EP 222: Assignment 1

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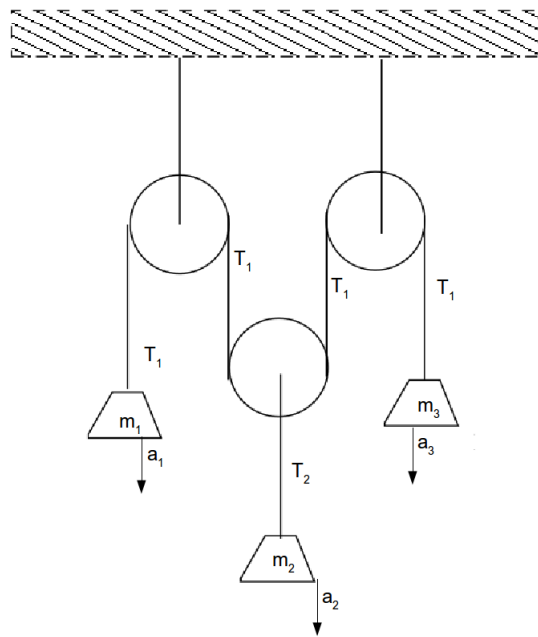July 28, 2013

1 Atwood's Machine

Figure 1: Diagram for Problem 1

I have defined the variables in Figure 1.
Now, assuming that the pulleys are massless, and writing Newton's laws of motion for the central pulley, with FBD in Figure 2 (d), we get:

$$2T_1 - T_2 = 0 \times a_2 = 0$$

$$\therefore T_2 = 2T_1$$

Now, if we assume that each block m_i moves x_i (downward positive) in a given time, the net work done by the strings will be $-(T_1 x_1 + T_2 x_2 + T_1 x_3)$. This must be 0, as strings as a whole cannot do work. Differentiating twice with respect to time,
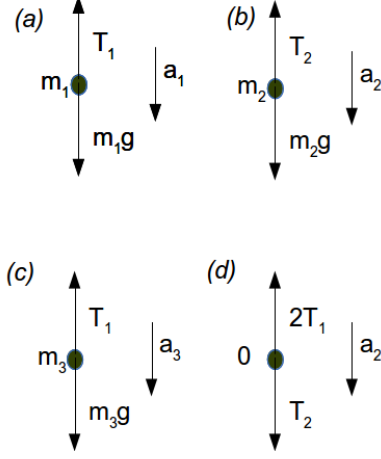

Figure 2: Free body diagrams: (a) For leftmost weight. (b) for middle weight. (c) For rightmost weight. (d) For middle pulley.

$$0 = T_1 \ddot{x}_1 + T_2 \ddot{x}_2 + T_1 \ddot{x}_3$$

$$\therefore T_1 a_1 + 2T_1 a_2 + T_1 a_3 = 0$$

$$\therefore a_1 + 2a_2 + a_3 = 0$$

Now, applying Newton's second law on the other three FBDs, we get:

$$T_1 - m_1 g = m_1 a_1$$
$$2T_1 - m_2 g = m_2 a_2$$
$$T_1 - m_3 g = m_3 a_3$$

Firstly, let us substitute $a_i' = a_i + g$
This reduces our equations to

$$4g = a_1' + 2a_2' + a_3'$$
$$T_1 = m_1 a_1'$$
$$2T_1 = m_2 a_2'$$
$$T_1 = m_3 a_3'$$

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Substituting values for a_1', a_2', a_3' into the first equation, we get:

$$4g = \frac{T_1}{m_1} + 4\frac{T_1}{m_2} + \frac{T_1}{m_3}$$

$$\therefore T_1 = \frac{4g}{\frac{1}{m_1} + \frac{4}{m_2} + \frac{1}{m_3}} = \frac{4gm_1m_2m_3}{m_1m_2 + m_2m_3 + 4m_1m_3}$$

Substituting these values for T_1 back, we get:

$$a_1 = \frac{4gm_2m_3}{m_1m_2 + m_2m_3 + 4m_1m_3} - g$$

$$a_2 = 2\frac{4gm_1m_3}{m_1m_2 + m_2m_3 + 4m_1m_3} - g$$

$$a_3 = \frac{4gm_1m_2}{m_1m_2 + m_2m_3 + 4m_1m_3} - g$$
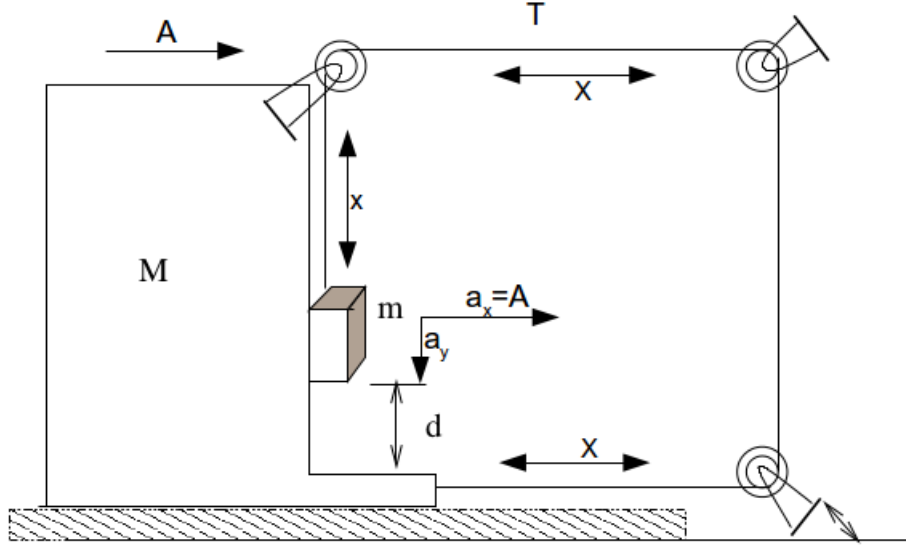
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2 Block system

Figure 3: Diagram for Problem 2

Let a_x, a_y be the components of acceleration of the small block, and let A be the acceleration of the large block.

Let x denote the extension of the vertical string and X denote the contraction of the horizontal string where marked.

Now, if the block M moves forward by X, the horizontal motion will "contract" these portions of the string by $2X$ (net change in length). This must be compensated for with the expansion of the vertical portion of the string x, so $x = 2X$ (as the total length of the string does not change, it is just redistributed). Differentiating, $\ddot{x} = 2\ddot{X} \implies a_y = 2A$.
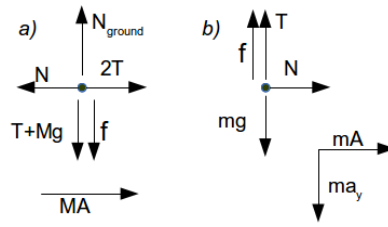

Figure 4: Free body diagrams: (a) For large block M (b) For small block m

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Now, as the system is in motion, $f = \mu N$. Analyzing the horizontal direction with Newton's laws in Figure 4 (a), we get:

$$2T - N = MA$$

Analyzing Figure 4 (b), we get:

$$N = mA$$

$$mg - f - T = ma_y \implies mg - \mu N - T = ma_y$$

Substituting $N = mA$ and $a_y = 2A$, we get:

$$2T = (m + M)A$$

$$mg - \mu mA - T = 2mA$$

Solving, we get

$$A = \frac{2g}{5 + 2\mu + \frac{M}{m}}$$

This implies

$$a_y = \frac{4g}{5 + 2\mu + \frac{M}{m}}$$

Substituting $\frac{M}{m} = 10, \mu = 0.5$, we get:

$$A = a_x = \frac{1}{4}g, a_y = \frac{1}{2}g$$

Thus vector acceleration of M is $\boxed{\frac{1}{4}g\hat{i}}$, acceleration of m is $\boxed{\frac{1}{4}g(\hat{i} - 2\hat{j})}$

As initual downward velocity of m is 0, time taken to cover a distance of $d = 2.45$ m is $\sqrt{\frac{2d}{a_y}}$ (from $s = ut + \frac{1}{2}at^2$), which is $\sqrt{\frac{4.9}{9.8 \times \frac{1}{4}}} = \sqrt{2}$ s

Thus the block takes $\boxed{1.414 \text{ seconds}}$ to reach the bottom.
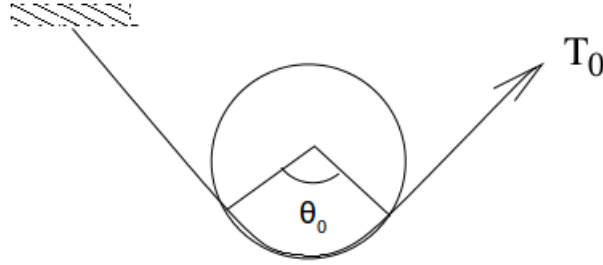
3   Rod on belt

Figure 5: Diagram for Problem 3

Let us analyze a small portion of the belt, as tension on that belt is changing. We assume that friction acts to the right.

Now, the horizontal component of tension is $(T + dT - T)\cos\frac{1}{2}d\theta = dT$. The vertical component is $(T + T + dT)\sin\frac{1}{2}d\theta = Td\theta$.

Balancing forces, we get $N = Td\theta$, and $f = dT$. Now, $f \le \mu N \implies dT \le \mu Td\theta$.

Integrating, $\int_{T_0}^{T} \frac{dT}{T} = \int_{0}^{\theta} \mu d\theta$. From here, we get that

$$T(\theta) \le T_0 e^{\mu\theta} \implies T_{wall} \le T_0 e^{\mu\theta_0}$$

If we assume that friction acts to the left, we get $T_{wall} \ge T_0 e^{-\mu\theta_0}$

Now, the macroscopic variable T_{wall} depends on the microscopic distribution of elongation in the string, which in turn depends on the process used to create this system. This is an unknown, which gives us a range (instead of an absolute answer) for T_{wall} However, Newton's laws allow all values of T_{wall} be-
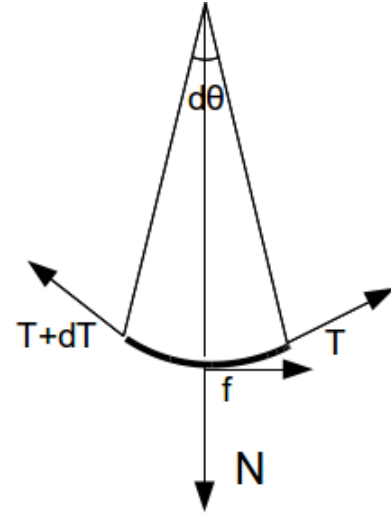

Figure 6:   Close up of small element of rope

tween $T_0 e^{-\mu\theta_0}$ and $T_0 e^{\mu\theta_0}$. Thus, we can claim that a state can be constructed where $\boxed{T_{wall} = T_0 e^{\mu\theta_0}}$, and this is the maximum tension possible.
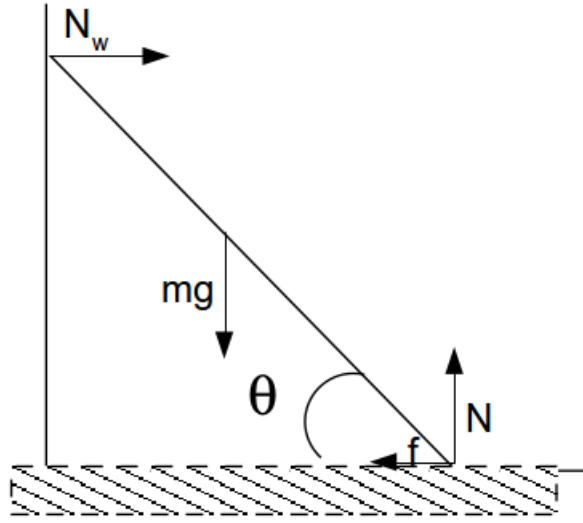
4   Leaning ladder

Figure 7: Diagram for Problem 4

Firstly, let us assume that the ladder has a length $2L$ and mass m.

Now, note that the smallest angle in equilibrium will be when the static friction attains its maximum value (μN). So, $f = \mu N$.

Now, first writing translational equilibrium equations:

$$mg = N$$

$$\mu N = f = N_w$$

The net torque on the ladder is $\tau = N_w L \sin\theta + fL \sin\theta - NL\cos\theta$. The ladder is in rotational equilibrium, so $\tau = 0$. So,

$$N_w L \sin\theta + fL \sin\theta = NL\cos\theta$$

Making substitutions from previously derived equations, we get

$$2\mu NL \sin\theta = \cos\theta$$

or

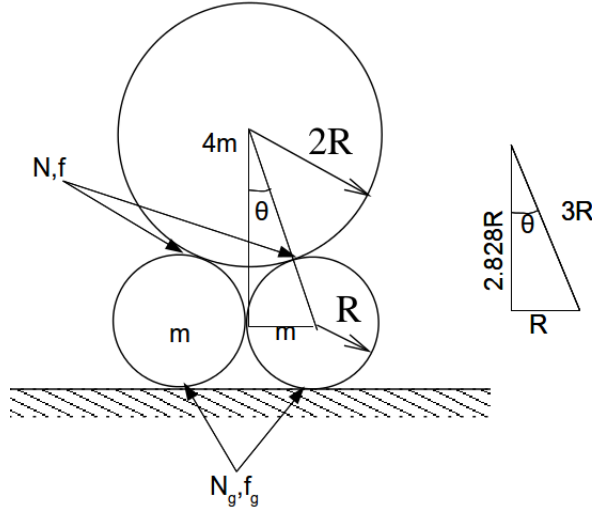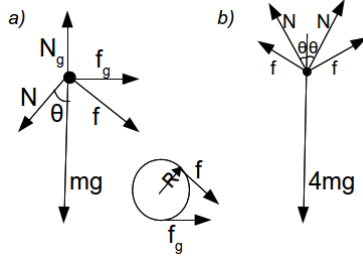$$\boxed{\theta = \cot^{-1} 2\mu}$$

5 Cylinders

Figure 8: Diagram for Problem 5

Here, we assume that there is friction between the small cylinders and the large one, as otherwise we get that the system does not stay in rotational equilibrium. This is easily verified, as the only forces providing torque to the smaller cylinders are the friction from the ground and the friction between them and the larger cylinder.

As the grund friction is nonzero, there must be friction between the smaller cylinders and the larger one otherwise the system does not stay at rest. Note that the friction between the cylinders must be zero due to the symmetry of the system.

The free body diagrams have been labelled taking into account the symmetry of the system. Figure 8 indicates the contact forces at various points (direction is not shown as it depends on which body we are considering, refer Figure 9 for details on the direction of the forces). Now, we first write the equation for torational equilibrium for the small cylinders, $\tau = rf_g - rf = 0 \implies f_g = f$. We can thus conveniently replace all f_g with f.

Writing the equations for translational equilibrium of the small cylinders, we get

$$mg + N\cos\theta + f\sin\theta = N_g$$

Figure 9: Free body diagrams: (a) For small cylinders (m, R) (b) For large cylinder $(4m, 2R)$

$$N\sin\theta = f + f\cos\theta$$

Writing equilibrium equations in the vertical direction for the large cylinder:

$$2N\cos\theta + 2f\sin\theta = 4mg \implies N\cos\theta + f\sin\theta = 2mg$$

Note that θ is known, with $\sin\theta = \frac{1}{3}, \cos\theta = \frac{\sqrt{8}}{3}$

$$\therefore N = f(1 + \sqrt{8})$$

$$N\sqrt{8} + f = 6mg$$

$$\therefore f(1 + \sqrt{8}) + f = 6mg \implies f = \frac{6mg}{2 + \sqrt{8}}$$

$$\therefore N = 6mg\frac{1 + \sqrt{8}}{2 + \sqrt{8}}$$

As

$$Ng = mg + N\cos\theta + f\sin\theta$$

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$$N_g = mg + 6mg\cos\theta\frac{1+\sqrt{8}}{2+\sqrt{8}} + \sin\theta\frac{6mg}{2+\sqrt{8}}$$

$$= mg\left(1 + \frac{6}{3(2+\sqrt{8})}\left(\sqrt{8}(1+\sqrt{8}) + (2+\sqrt{8})\right)\right)$$

$$= mg\left(1 + \frac{1}{1+\sqrt{2}}\left(10 + 4\sqrt{2}\right)\right)$$

$$= mg\frac{11 + 5\sqrt{2}}{1+\sqrt{2}}$$

$$\therefore \frac{N_g}{f_g} = \frac{\left(mg\frac{11+5\sqrt{2}}{1+\sqrt{2}}\right)}{\left(\frac{3mg}{1+\sqrt{2}}\right)} = \frac{11+5\sqrt{2}}{3}$$

Now, $\mu N_g \geq f_g$. In this case, the minimum μ is during equality, so

$$\boxed{\mu = \frac{N_g}{f_g} = \frac{11+5\sqrt{2}}{3}}$$
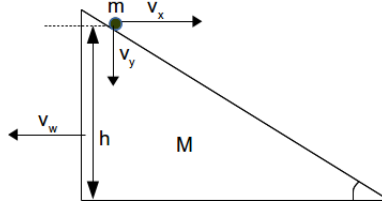
6   Mass on wedge

Figure 10: Diagram for Problem 6

Here, I am assuming that the values of final velocity are required just before the mass m reaches the table (as to calculate this value at the exact time will require knowledge of the nature of the collision). For the purposes of this problem I shall take the symbols v_w, v_x, v_y to mean the velocities at the final stage. Initial velocities are zero.

Now, we can conserve momentum in the horizontal direction here and overall energy as well. Conserving momentum in the horizontal direction, we get $mv_x = Mv_w \implies v_x = \frac{M}{m}v_w$.

Conserving energy, $\frac{1}{2}(Mv_w^2 + mv_x^2 + mv_y^2) = mgh$

Finally, in the wedge frame, the mass must move along the wedge surface. Thus, $\frac{v_y}{v_x + v_w} = \tan\theta \implies v_y = \tan\theta(v_x + v_w)$

Substituting the values for v_x and v_y in the energy conservation equation, we get

$$2mgh = Mv_w^2 + m\left(\frac{M}{m}v_w\right)^2 + m\tan^2\theta(v_x + v_w)^2 = Mv_w^2 + m\left(\frac{M}{m}v_w\right)^2 + m\tan^2\theta v_w^2\left(1 + \frac{M}{m}\right)^2$$

$$\therefore v_w^2 = \frac{2mgh}{M + \frac{M^2}{m} + m\tan^2\theta\left(1 + \frac{M}{m}\right)^2}$$

$$\boxed{\therefore v_w = \sqrt{\frac{2mgh}{M + \frac{M^2}{m} + m\tan^2\theta\left(1 + \frac{M}{m}\right)^2}}}$$