Tutorial 2

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Problem 1

(i) $\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ is a tensor for all kinds of transformations. $\frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(x^\nu \frac{\mathrm{d}x'^\mu}{\mathrm{d}x^\nu}\right) = \frac{\mathrm{d}x'^\mu}{\mathrm{d}x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$, and thus we get a contravariant tensor

(ii) $\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}$ is also a tensor because it is the derivative of a tensor (Use the property proved in (i) by replacing x^μ with $\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$). This tensor is also contravariant.

(iii) $\partial_\mu\phi$ is a tensor. $\partial'_\mu\phi = \frac{\partial\phi}{\partial x_\nu \frac{\mathrm{d}x'_\mu}{\mathrm{d}x_\nu}} = \frac{\mathrm{d}x_\nu}{\mathrm{d}x'_\mu}\frac{\partial\phi}{\partial x_\nu} = \frac{\mathrm{d}x_\nu}{\mathrm{d}x'_\mu}\partial_\mu\phi$, making it a covariant tensor.

(iv) This is only a tensor when special relativistic constraints are present. We know that the covariant derivative of the tensor F, $F_{\mu\nu;\rho} = \partial_\rho F_{\mu\nu} - \Gamma^\sigma_{\nu\rho}A_{\mu\sigma} - \Gamma^\sigma_{\mu\rho}A_{\nu\sigma}$ is a tensor itself. However, $\Gamma^\sigma_{\nu\rho}F_{\mu\sigma}, \Gamma^\sigma_{\mu\rho}F_{\nu\sigma}$ are not tensors in the general case as proved in class, so $\partial_\rho F_{\mu\nu}$ is not. However, in special relativity the Christoffel symbols are zero, so the remaining portion of the equation must be a tensor too.

Problem 2

We first calculate the velocity and then γ_v of the particle in the frame:

$$
\begin{aligned}
v^2 &= \dot{x}^2 + \dot{y}^2 + \dot{x}^2 \\
&= (a + b\omega\cos\omega t)^2 + (-b\omega\sin\omega t)^2 \\
&= a^2 + b^2\omega^2 + 2ab\omega\cos\omega t \\
\therefore \gamma_v &= \frac{1}{\sqrt{1 - \frac{a^2+b^2\omega^2+2ab\omega\cos\omega t}{c^2}}}
\end{aligned}
$$

From this, we get $w^\mu = \frac{1}{\sqrt{1-\frac{a^2+b^2\omega^2+2ab\omega\cos\omega t}{c^2}}} \begin{bmatrix} a + b\omega\cos\omega t \\ -b\omega\sin\omega t \\ 0 \\ c \end{bmatrix}$

The acceleration four vector can be calculated by differentiating this.

Problem 3

$$
t = \frac{X}{a}\sinh aT, x\frac{X}{a}\cosh aT
$$

$$
\begin{aligned}
ds^2 &= dx^2 - dt^2 \\
&= (\sinh aT dX + X\cosh aT dT)^2 - (X\sinh aT dT + \cosh aT dX)^2 \\
&= -dX^2 + dT^2X^2
\end{aligned}
$$

This gives us the metric

$$
g_{\mu\nu} \equiv \begin{pmatrix} X^2 & 0 \\ 0 & -1 \end{pmatrix}
$$

The inverse is

$$
g^{\mu\nu} \equiv \begin{pmatrix} \frac{1}{X^2} & 0 \\ 0 & -1 \end{pmatrix}
$$

Now, $\Gamma^\mu_{\rho\sigma} = g^{\mu\nu}\frac{1}{2}(\partial_\sigma g_{\rho\nu} + \partial_\rho g_{\sigma\nu} - \partial_\nu g_{\rho\sigma})$

$$
\begin{aligned}
\Gamma^{\mu}_{11} &= g^{\mu\nu}\frac{1}{2}(\partial_1 g_{1\nu} + \partial_1 g_{1\nu} - \partial_\nu g_{11}) \\
&= g^{\mu\nu}\frac{1}{2}(0 + 0 - \partial_\nu g_{11}) && \because \text{ g is independant of } T \\
&= 0 \\
\Gamma^{\mu}_{10} = \Gamma^{\mu}_{01} &= g^{\mu\nu}\frac{1}{2}(\partial_0 g_{1\nu} + \partial_1 g_{0\nu} - \partial_\nu g_{10}) \\
&= g^{\mu\nu}\frac{1}{2}(\partial_1 g_{0\nu} - \partial_\nu g_{10}) \\
&= g^{\mu 0}\frac{1}{2}(\partial_1 g_{00} - \partial_0 g_{10}) + g^{\mu 1}\frac{1}{2}(\partial_1 g_{01} - \partial_1 g_{10}) \\
&= -g^{\mu 0}\frac{1}{2X} \\
\Gamma^{\mu}_{00} &= g^{\mu\nu}\frac{1}{2}(\partial_0 g_{0\nu} + \partial_0 g_{0\nu} - \partial_\nu g_{00}) \\
&= g^{\mu\nu}\frac{1}{2}(-\partial_\nu g_{00}) \\
&= g^{\mu 1}\frac{1}{2}(-\partial_1 g_{00}) \\
&= g^{\mu 1}\frac{1}{2X}
\end{aligned}
$$

$$
\therefore \Gamma^{\mu}_{\rho\sigma} = \begin{pmatrix} 0 & -g^{\mu 0}\frac{1}{2X} \\ -g^{\mu 0}\frac{1}{2X} & g^{\mu 1}\frac{1}{2X} \end{pmatrix}
$$

The equation of motion is $\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^{\mu}_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} = 0$
For $\mu = 0$, we have

$$
\begin{aligned}
0 &= \frac{\mathrm{d}^2 T}{\mathrm{d}\tau^2} + \Gamma^{0}_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \\
&= \frac{\mathrm{d}^2 T}{\mathrm{d}\tau^2} - 2g^{00}\frac{1}{2X}\frac{\mathrm{d}X}{\mathrm{d}\tau}\frac{\mathrm{d}T}{\mathrm{d}\tau} \\
&= \frac{\mathrm{d}^2 T}{\mathrm{d}\tau^2} + X\frac{\mathrm{d}X}{\mathrm{d}\tau}\frac{\mathrm{d}T}{\mathrm{d}\tau}
\end{aligned}
$$

For $\mu = 1$, we have

$$\begin{aligned} 0 &= \frac{\mathrm{d}^2 X}{\mathrm{d}\tau^2} + \Gamma^1_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} & &= \frac{\mathrm{d}^2 X}{\mathrm{d}\tau^2} + g^{11}\frac{1}{2X}\left(\frac{\mathrm{d}X}{\mathrm{d}\tau}\right)^2 \\ &= \frac{\mathrm{d}^2 X}{\mathrm{d}\tau^2} - \frac{1}{2X}\left(\frac{\mathrm{d}X}{\mathrm{d}\tau}\right)^2 \end{aligned}$$

Thus the geodesic trajectory is

$$\begin{aligned} \frac{\mathrm{d}^2 X}{\mathrm{d}\tau^2} &= \frac{1}{2X}\left(\frac{\mathrm{d}X}{\mathrm{d}\tau}\right)^2 \\ \frac{\mathrm{d}^2 T}{\mathrm{d}\tau^2} &= -X\frac{\mathrm{d}X}{\mathrm{d}\tau}\frac{\mathrm{d}T}{\mathrm{d}\tau} \end{aligned}$$

Problem 4

$$x = \mu\nu \qquad y = \frac{1}{2}(\mu^2 - \nu^2)$$

We have

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 \\ &= (\mu\mathrm{d}\nu + \nu\mathrm{d}\mu)^2 + (\mu\mathrm{d}\mu - \nu\mathrm{d}\nu)^2 \\ &= \mathrm{d}\mu^2\nu^2 + 2\mathrm{d}\mu\mathrm{d}\nu\mu\nu + \mathrm{d}\nu^2\mu^2 + \mathrm{d}\mu^2\mu^2 - 2\mathrm{d}\mu\mathrm{d}\nu\mu\nu + \mathrm{d}\nu^2\nu^2 \\ &= (\mu^2 + \nu^2)(\mathrm{d}\mu^2 + \mathrm{d}\nu^2) \end{aligned}$$

giving us the new metric

$$g_{\mu\nu} \equiv \begin{pmatrix} \mu^2 + \nu^2 & 0 \\ 0 & \mu^2 + \nu^2 \end{pmatrix}$$

Problem 5

(Summation notation not used)

$$x^{\mu'} = \frac{1}{\epsilon_\mu + \frac{1}{x^\mu}}$$

$$
\begin{aligned}
x^{\mu'} - x^{\mu} &= \frac{1}{\epsilon_\mu + \frac{1}{x^\mu}} - x^\mu \\
&= \frac{1 - \epsilon_\mu x^\mu - 1}{\epsilon_\mu + \frac{1}{x^\mu}}
\end{aligned}
$$

Taking limits, we get $x^{\mu'} - x^{\mu} = -\epsilon_\mu (x^\mu)^2$ (not expected answer)

Problem 6

(Some simplification done with the help of *Mathematica*)

We have the metric as

$$
g_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b^2 + r^2 & 0 \\ 0 & 0 & 0 & (b^2 + r^2)\sin^2\theta \end{pmatrix}
$$

The inverse is

$$
g^{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{b^2+r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{(b^2+r^2)\sin^2\theta} \end{pmatrix}
$$

We can calculate the Christoffel tensor via $\Gamma^\mu_{\rho\sigma} = g^{\mu\nu}\frac{1}{2}(\partial_\sigma g_{\rho\nu} + \partial_\rho g_{\sigma\nu} - \partial_\nu g_{\rho\sigma})$

We get

$$
\Gamma = \begin{pmatrix} (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) \\ (0,0,0,0) & (0,0,0,0) & \left(0,0,\frac{2r}{b^2+r^2},0\right) & \left(0,0,0,\frac{2r}{b^2+r^2}\right) \\ (0,0,0,0) & \left(0,0,\frac{2r}{b^2+r^2},0\right) & (0,-2r,0,0) & (0,0,0,2\cot(\theta)) \\ (0,0,0,0) & \left(0,0,0,\frac{2r}{b^2+r^2}\right) & (0,0,0,2\cot(\theta)) & \left(0,-2r\sin^2(\theta),-2\cos(\theta)\sin(\theta),0\right) \end{pmatrix}
$$

where the row and column of the outer matrix correspond to ρ,σ, and μ differentiates the components of the inner matrix.

The geodesic equation is $\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} = 0$

Solving it, we get

$$
\begin{aligned}
0 &= \frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} \\
0 &= -2r\sin^2(\theta)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 - 2r\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} \\
0 &= \frac{4r\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau}}{b^2+r^2} + \frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} - 2\sin(\theta)\cos(\theta)\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 \\
0 &= \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\left(\frac{2r\frac{\mathrm{d}r}{\mathrm{d}\tau}}{b^2+r^2} + 2\cot(\theta)\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right) + \frac{2r\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}}{b^2+r^2} + 2\cot(\theta)\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}
\end{aligned}
$$