EP 307 Assignment 5

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Problem 1

$$\hat{H} = \frac{\hat{p}^2}{2m} + \mathcal{E}\hat{x}$$

In the momentum space,

$$\hat{H}\tilde{\psi}(p) = \frac{p^2}{2m}\psi + \mathcal{E}i\hbar\frac{\partial\psi(p)}{\partial p} = E\psi(p)$$

This gives us

$$\frac{\partial\tilde{\psi}}{\partial p} = \frac{1}{\mathcal{E}i\hbar}\left(E - \frac{p^2}{2m}\right)\psi$$

$$\boxed{\therefore \tilde{\psi}(p) = \exp\left(\frac{Ep - \frac{p^3}{6m}}{\mathcal{E}i\hbar}\right)}$$

Thus the position space wavefunction is

$$\psi(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(i\frac{p}{\hbar}x + \frac{Ep - \frac{p^3}{6m}}{\mathcal{E}i\hbar}\right) dp$$

This integral can only be partially evaluated. One can see by looking at the original equation that the solution involved Airy functions (since the position space equation is of the form $y'' + xy = \lambda y$ with suitable normalization, and these cannot be written out in closed form.

Problem 2

1.

$$\begin{aligned}[\hat{r}_i, \hat{L}_j] &= [\hat{r}_i, \epsilon_{jkl}\hat{r}_k\hat{p}_l] \\ &= \epsilon_{jkl}([\hat{r}_i, \hat{r}_k]\hat{p}_l + \hat{r}_k[\hat{r}_i, \hat{p}_l]) \\ &= \epsilon_{jkl}(\hat{r}_k\delta_{il}i\hbar) \\ &= \epsilon_{jki}i\hbar\hat{r}_k \\ &= \epsilon_{ijk}i\hbar\hat{r}_k\end{aligned}$$

2.

$$\begin{aligned}[\hat{p}_i, \hat{L}_j] &= [\hat{p}_i, \epsilon_{jkl}\hat{r}_k\hat{p}_l] \\ &= \epsilon_{jkl}([\hat{p}_i, \hat{r}_k]\hat{p}_l + \hat{r}_k[\hat{p}_i, \hat{p}_l]) \\ &= \epsilon_{jkl}(-i\hbar\delta_{ik}p_l) \\ &= \epsilon_{jil}(-i\hbar p_l) \\ &= \epsilon_{ijl}i\hbar p_l\end{aligned}$$

3.

$$\begin{aligned}[\mathbf{r}\cdot\mathbf{r}, \hat{L}_i] &= [\hat{r}_j\hat{r}_j, \hat{L}_i] \\ &= \hat{r}_j[\hat{r}_j, L_i] + [\hat{r}_j, \hat{L}_i]\hat{r}_j \\ &= \hat{r}_i\epsilon_{jil}(i\hbar\hat{r}_l) + \epsilon_{jil}(i\hbar\hat{r}_l)\hat{r}_i \\ &= 2\epsilon_{ijk}i\hbar\hat{r}_i\hat{r}_j\end{aligned}$$

4.

$$\begin{aligned}[\hat{x}^2, \hat{p}_x^2] &= [\hat{x}\hat{x}, \hat{p}_x\hat{p}_x] \\ &= \hat{x}[\hat{x}, \hat{p}_x\hat{p}_x] + [\hat{x}, \hat{p}_x\hat{p}_x]\hat{x} \\ &= \{\hat{x}, [\hat{x}, \hat{p}_x\hat{p}_x]\} \\ &= \{\hat{x}, ([\hat{x}, \hat{p}_x]\hat{p}_x + \hat{p}_x[\hat{x}, \hat{p}_x])\} \\ &= \{\hat{x}, 2i\hbar\hat{p}_x\} \\ &= 2i\hbar\{\hat{x}, \hat{p}_x\} \\ &= 2i\hbar(\hat{x}\hat{p}_x + \hat{p}_x\hat{x})\end{aligned}$$

Problem 6

$$
\begin{aligned}
\frac{\mathrm{d}^2 \langle \hat{x}(t) \rangle}{\mathrm{d}t^2} &= \frac{\mathrm{d}^2 \langle \psi \,|\, \hat{x}(t) \,|\, \psi \rangle}{\mathrm{d}t^2} \\
&= \langle \psi | \frac{\mathrm{d}^2 x \, |\psi\rangle}{\mathrm{d}t^2} \\
&\text{Since an operator can be written as } U^\dagger(t)\hat{x}U(t) \\
&= \langle \psi | \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial x}{\partial t} + x\frac{1}{i\hbar}\hat{H} - \frac{1}{i\hbar}\hat{H}x \right) |\psi\rangle \\
&= \langle \psi | \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial x}{\partial t} + \frac{1}{i\hbar}[x, \hat{H}] \right) |\psi\rangle \\
&= \langle \psi | \left(\frac{\partial^2 x}{\partial t^2} + \frac{1}{i\hbar}[\frac{\partial x}{\partial t}, \hat{H}] + \frac{1}{i\hbar}\frac{\partial [x, \hat{H}]}{\partial t} + \frac{1}{-\hbar^2}[[x, \hat{H}], \hat{H}] \right) |\psi\rangle \\
&= \left\langle \psi \,\middle|\, \frac{1}{-\hbar^2}[[x, \hat{H}], \hat{H}] \,\middle|\, \psi \right\rangle \quad \text{(No explicit time dependence)} \\
&= \left\langle \psi \,\middle|\, \frac{1}{-2m\hbar^2}[[x, \hat{p}^2], \hat{H}] \,\middle|\, \psi \right\rangle \quad \text{(No explicit time dependence)} \\
&= \left\langle \psi \,\middle|\, \frac{1}{-\hbar^2}[i\hbar p, \hat{H}] \,\middle|\, \psi \right\rangle \quad \text{(No explicit time dependence)} \\
&= \langle \psi \,|\, 0 \,|\, \psi \rangle \\
&= 0
\end{aligned}
$$

Since for a free particle, acceleration is 0, Ehrenfest theorem is proved.

Problem 13

When we calculate $\langle \hat{x} \rangle$, we get some time-independant function into $\sin(\frac{E_2 - E_1}{\hbar} t)$. To reach the other side, it needs to undergo a rotation of $\frac{\pi}{2}$, so the time taken is $\frac{\pi\hbar}{2(E_2 - E_1)} = \frac{\pi\hbar 2mL}{2(4-1)\hbar^2\pi^2}$

Thus, the time taken is $\frac{mL}{3\pi\hbar}$

Problem 16

Since the wavefunction must be continuous, but zero everywhere except Ω_1, Ω_2, it cannot continuously move from one region to another in its entirety.

$\psi_i(r,t) = \phi_i(r)e^{-iEt/\hbar}$ in region Ω_i.

$|\psi(r,t)|^2 = \frac{1}{\sqrt{2}}|\psi_1 + \psi_2|^2 = \frac{1}{\sqrt{2}}\left(\psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_1^*\psi_2 + \psi_2^*\psi_1\right)$

The first two terms are time independant, as the $e^{-iEt/\hbar}$ and $\left(e^{-iEt/\hbar}\right)^*$ term cancel out. The last term is zero as the two functions are never simultaneously nonzero.

If they do overlap, the last term is no longer zero.

$$\frac{\psi_1^*\psi_2 + \psi_2^*\psi_1}{\sqrt{2}} = \frac{\phi_1^*\phi_2 e^{-i(E_2-E_1)t/\hbar} + \phi_2^*\phi_1 e^{-i(E_1-E_2)t/\hbar}}{\sqrt{2}}$$

which is the sum of two periodic functions, and periodic or quasiperiodic itself.

Problem 19

The wavefunction is separable into cartesian components:

$\psi(x,y,z) = X(x)Y(y)Z(z)$, where $X(x) = \sin(k_1 x), Y(y) = \cos(k_2 y), Z(z) = e^{ik_3 z}$. These are all free particles, with momentum $\hbar k_i$. Thus, the net momentum is $\hbar\left(k_1 x + k_2 y + k_3 z\right)$. However, the sin and cos wavefunctions have half the probability as the exponential one, since it when you integrate the former you get only $\int \sin^2$ or $\int \cos^2$, but for the latter you have $\int sin^2 + \cos^2$. Thus the relative probabilities are 1:1:2.

Problem 22

$$\begin{aligned}
\left\langle \vec{p}_1 \left| \frac{e^{-\mu r}}{r} \right| \vec{p}_2 \right\rangle &= \left\langle \vec{p}_1 \left| I\frac{e^{-\mu r}}{r} I \right| \vec{p}_2 \right\rangle \\
&= \iiint \iiint \langle \vec{p}_1 \,|\, \vec{r} \rangle \langle \vec{r} | \frac{e^{-\mu r}}{r} | \vec{r}' \rangle \langle \vec{r}' \,|\, \vec{p}_2 \rangle \, \mathrm{d}V \mathrm{d}V' \\
&= \iiint \iiint e^{-i\vec{p}_1 \cdot \vec{r}} e^{i\vec{p}_2 \cdot \vec{r}'} \delta(r - r') \frac{e^{-\mu r}}{r} \mathrm{d}V \mathrm{d}V'
\end{aligned}$$

$$
\begin{aligned}
&= \iiint \iiint e^{-i\overrightarrow{p_1}\cdot\hat{e}_r r} e^{i\overrightarrow{p_1}\cdot\hat{e}_r r'} \delta(r-r') \frac{e^{-\mu r}}{r} \mathrm{d}V \mathrm{d}V' \\
&= (4\pi)^2 \int \int e^{-ip_{1,r}r} e^{ip_{2,r}r'} \delta(r-r') \frac{e^{-\mu r}}{r} r^2 \mathrm{d}r r'^2 \mathrm{d}r' \\
&= (4\pi)^2 \int e^{-ip_{1,r}r} e^{ip_{2,r}r} \frac{e^{-\mu r}}{r} r^4 \mathrm{d}r \\
&= (4\pi)^2 \int e^{ir(p_{2,r}-p_{1,r})} \frac{e^{-\mu r}}{r} r^4 \mathrm{d}r \\
&= (4\pi)^2 \int \frac{e^{r(i(p_{2,r}-p_{1,r})-\mu)}}{r} r^4 \mathrm{d}r \\
&= \frac{6 \cdot (4\pi)^2}{(\mu + ip_{1,r} - ip_{2,r})^4}
\end{aligned}
$$