EP 222 Assigment 2

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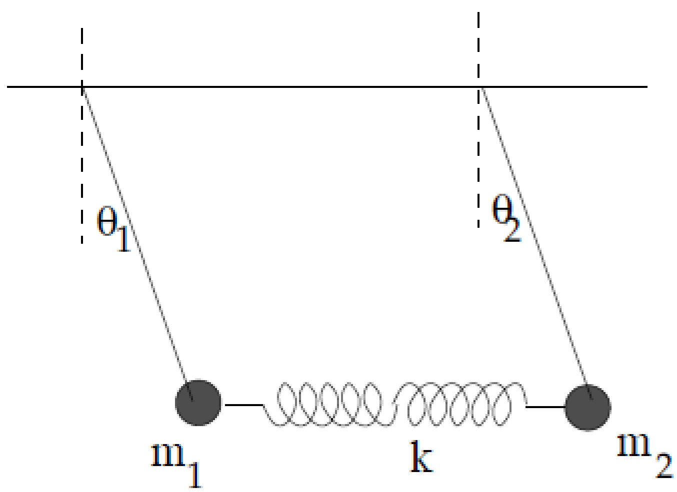
1 Question 1



We shall take θ_1, θ_2 as our coordinates. Let the rigid supports be of length l, and let the distance between the supports be equal to the equilibrium length of the spring, d. The kinetic energy of this system is $T = \frac{l^2}{2}(m_1\dot{\theta}_1^2 + m_2\dot{\theta}_2^2)$ Rewriting T as a matrix, $\mathbf{T} = [t_{ij}] = \left[\frac{\partial^2 T}{\partial\dot{\theta}_i\partial\dot{\theta}_j}\right] = l^2 \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$

The potential energy is

$$\begin{aligned} V = &\quad m_1 gl(1 - \cos\theta_1) + m_2 gl(1 - \cos\theta_2) + \frac{1}{2}k(l(\theta_2 - \theta_1))^2 \\ \approx &\quad m_1 gl\theta_1^2 + m_2 gl\theta_2^2 + \frac{1}{2}kl^2(\theta_1^2 + \theta_2^2 + 2\theta_1\theta_2) \end{aligned}$$

Rewriting V as a matrix,

$$\begin{aligned}\mathbf{V} = &\quad [V_{ij}] \\ = &\quad \left[\frac{\partial^2 V}{\partial\theta_i \partial\theta_j}\right] \\ = &\quad \begin{pmatrix} 2m_1gl + kl^2 & -kl^2 \\ -kl^2 & 2m_2gl + kl^2 \end{pmatrix}\end{aligned}$$

Now, the normal mode frequencies are given by the eigenvalue equation

$$|\mathbf{V} - \omega^2\mathbf{T}| = 0$$

$$\therefore \left| \begin{pmatrix} 2m_1gl + kl^2 & -kl^2 \\ -kl^2 & 2m_2gl + kl^2 \end{pmatrix} - \omega^2 l^2 \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \right| = 0$$

Rewriting $\lambda = \omega^2 l^2$

$$\begin{aligned}\therefore \begin{vmatrix} kl^2 + 2gm_1l - \lambda m_1 & kl^2 \\ kl^2 & kl^2 + 2gm_2l - \lambda m_2 \end{vmatrix} &= 0 \\ \implies 4g^2l^2m_1m_2 + 2gkl^3m_1 + 2gkl^3m_2 & \\ - 4g\lambda lm_1m_2 - k\lambda l^2m_1 - k\lambda l^2m_2 + \lambda^2m_1m_2 &= 0\end{aligned}$$

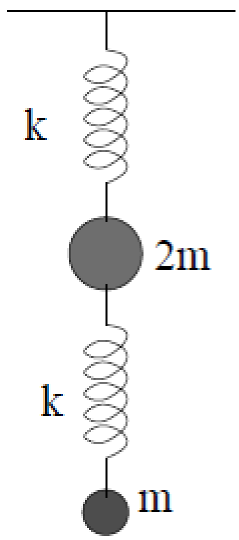
From this we get roots for λ as $2gl$ and $\frac{kl^2}{\mu} + 2gl$, so the normal modes are $\boxed{\omega = \sqrt{2\frac{g}{l}}}$ anand $\boxed{\omega = \sqrt{2\frac{g}{l} + \frac{k}{\mu}}}$ where μ is the reduced mass.

The eigenvectors can be found by solving $\mathbf{VA} = \omega^2\mathbf{TA}$. This expands to

$$\begin{pmatrix} 2glm_1 + kl^2 & -kl^2 \\ -kl^2 & 2glm_2 + kl^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \lambda \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

which gives rise to eigenvectors $\boxed{\mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} m_2 \\ -m_1 \end{pmatrix}}$, or relative amplitudes $(\theta_2 : \theta_1)$ 1 and $-\frac{m_1}{m_2}$.

2 Question 2



Our generalized coordinates shall be the elongations in each spring, x_1, x_2 (x_1 is for the upper spring).
The velocities of the masses are $\dot{x}_1$ for $2m$ and $\dot{x}_1 + \dot{x}_2$ for m.
$\therefore T = \frac{1}{2}2m\dot{x}_1^2 + \frac{1}{2}m(\dot{x}_1 + \dot{x}_2)^2 = \frac{1}{2}(3m\dot{x}_1^2 + m\dot{x}_2^2 + 2m\dot{x}_1\dot{x}_2)$
Rewriting T as a matrix, $\mathbf{T} = [t_{ij}] = \left[\frac{\partial^2 T}{\partial \dot{x}_i \partial \dot{x}_j}\right] = \frac{m}{2}\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}$
The potential energy is simply $\frac{1}{2}(kx_1^2+kx_2^2)-2mg(l_1+x_1)-mg(l_1+x_1+l_2+x_2)$ (l_1, l_2 are natural lengths of springs), giving $\mathbf{V} = [V_{ij}] = \left[\frac{\partial^2 V}{\partial x_i \partial x_j}\right] = \frac{k}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
Now, the normal mode frequencies are given by the eigenvalue equation

$$|\mathbf{V} - \omega^2\mathbf{T}| = 0$$

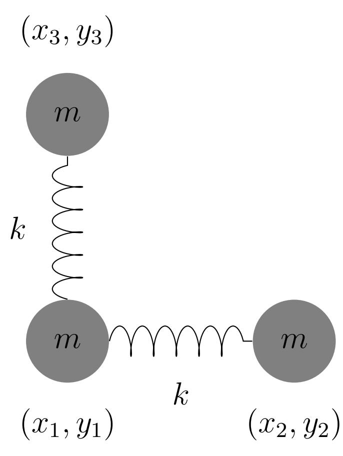
$$\therefore \left| \frac{k}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \omega^2 \frac{m}{2}\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix} \right| = 0$$

Rewriting $\lambda = \omega^2 \frac{m}{k}$

$$
\begin{aligned}
\therefore \begin{vmatrix} 2-6\lambda & 0-2\lambda \\ 0-2\lambda & 2-2\lambda \end{vmatrix} &= 0 \\
\implies (2-6\lambda)(2-2\lambda) - (-2\lambda)^2 &= 0 \\
\implies 8\lambda^2 - 16\lambda + 4 &= 0 \\
\implies 2\lambda^2 - 4\lambda + 1 &= 0 \\
\implies \lambda &= \frac{4 \pm \sqrt{16-8}}{4} \\
&= 1 \pm \frac{1}{\sqrt{2}} \\
\implies \omega &= \sqrt{\frac{k}{m}\left(1 \pm \frac{1}{\sqrt{2}}\right)}
\end{aligned}
$$

Thus, the normal modes are $\boxed{\omega_1 = \sqrt{\frac{k}{m}\left(1 + \frac{1}{\sqrt{2}}\right)}, \quad \omega_2 = \sqrt{\frac{k}{m}\left(1 - \frac{1}{\sqrt{2}}\right)}}$

3 Question 3



Let q_1, q_2, q_3 be the x coordinates, and q_4, q_5, q_6 be the y coordinates.

The kinetic energy is $T = \frac{m}{2}(\sum \dot{q}_i^2)$, giving $\mathbf{T} = [t_{ij}] = \left[\frac{\partial^2 T}{\partial \dot{q}_i \dot{q}_j}\right] = m\mathbf{I}$, where $\mathbf{I}$ is the 6×6 identity matrix.

The potential energy is

$$V = \frac{k}{2}\left(\left(\sqrt{(x_1 - x_2)^{\,2} + (y_1 - y_2)^{\,2}} - a\right)^{\,2} + \left(\sqrt{(x_3 - x_1)^{\,2} + (y_3 - y_1)^{\,2}} - a\right)^{\,2}\right)$$

. This gives us a matrix $\mathbf{V} = [V_{ij}] = \left[\frac{\partial^2 V}{\partial q_i q_j}\right]$, which when expanded gives us an unprintably large matrix. However, on substituting the equilibrium values of the coordinates, we get the much more manageable

$$\frac{k}{2}\begin{pmatrix} 2 & -2 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \end{pmatrix}$$

Now, the eigenvalue equation is $|\mathbf{V} - \omega^2 \mathbf{T}| = 0$, giving us

$$\left|\frac{k}{2}\begin{pmatrix} 2 & -2 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \end{pmatrix} - \omega^2 m \mathbf{I}\right| = 0$$

Writing $\lambda = \frac{2m\omega^2}{k}$

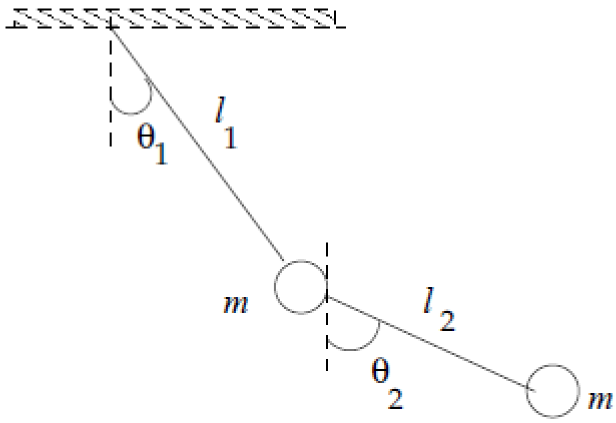
$$\therefore \begin{vmatrix} 2-\lambda & -2 & 0 & 0 & 0 & 0 \\ -2 & 2-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 2-\lambda & 0 & -2 \\ 0 & 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -2 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^2\left(\lambda^4 - 8\lambda^3 + 16\lambda^2\right) = 0$$

$$\therefore (\lambda - 4)^2 \lambda^4 = 0$$

This gives us eigenvalues $\lambda = 0, 0, 0, 0, 4, 4$. Thus, the normal modes are $\boxed{\sqrt{2\frac{k}{m}}}$ with multiplicity 2.

4 Question 4



We shall take our coordinates as θ_1, θ_2. The velocity of the upper bob is $\frac{1}{2}ml_1^2\dot{\theta}_1^2$. For the second bob, its position is $\mathbf{r}_2 = l_1(\cos\theta_1\hat{\imath} + \sin\theta_1\hat{\jmath}) + l_2(\cos\theta_2\hat{\imath} + \sin\theta_2\hat{\jmath})$. This gives us velocity

$$\dot{\mathbf{r_2}} = l_1\dot{\theta}_1(-\sin\theta_1\hat{\imath} + \cos\theta_1\hat{\jmath}) + l_2\dot{\theta}_2(-\sin\theta_2\hat{\imath} + \cos\theta_2\hat{\jmath})$$

, which can be rewritten as

$$\dot{\mathbf{r_2}} = \hat{\imath}(-l_1\dot{\theta}_1\sin\theta_1 - l_2\dot{\theta}_2\sin\theta_2) + \hat{\jmath}(l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2)$$

. The kinetic energy

$$T = \frac{1}{2}ml_1^2\dot{\theta}_1^2 + \frac{1}{2}m|\dot{\mathbf{r_2}}|^2 = \frac{1}{2}m\left(2\dot{\theta}_1^2l_1^2 + \dot{\theta}_2^2l_2^2 + 2\dot{\theta}_2\dot{\theta}_1l_1l_2\cos(\theta_1 - \theta_2)\right)$$

Rewriting as a matrix,

$$\mathbf{T} = [t_{ij}] = \left[\frac{\partial^2 T}{\partial\dot{\theta}_i\dot{\theta}_j}\right] = m\begin{pmatrix} 2l_1^2 & l_1l_2\cos(\theta_1 - \theta_2) \\ l_1l_2\cos(\theta_1 - \theta_2) & l_2^2 \end{pmatrix}$$

Near equilibrium $(\theta_1, \theta_2 = 0)$, $\mathbf{T} = \begin{pmatrix} 2l_1^2 & l_1l_2 \\ l_1l_2 & l_2^2 \end{pmatrix}$

The potential energy $V = -mgl_1\cos\theta_1 - (mgl_1\cos\theta_1 + mgl_2\cos\theta_2)$. This gives us a potential matrix

$$\mathbf{V} = \left[\frac{\partial^2 V}{\partial\theta_i\theta_j}\right] = \begin{pmatrix} 2mgl_1\cos\theta_1 & 0 \\ 0 & mgl_2\cos\theta_2 \end{pmatrix}$$

which, at equilibrium, is $mg\begin{pmatrix} 2l_1 & 0 \\ 0 & l_2 \end{pmatrix}$

The eigenvalue equation is $|\mathbf{V} - \omega^2\mathbf{T}| = 0$, which comes out to be

$$\left| mg\begin{pmatrix} 2l_1 & 0 \\ 0 & l_2 \end{pmatrix} - \omega^2\begin{pmatrix} 2l_1^2 & l_1l_2 \\ l_1l_2 & l_2^2 \end{pmatrix} \right| = 0$$

With $\lambda = \frac{\omega^2}{mg}$, we get

$$\begin{aligned}
\left| \begin{pmatrix} 2l_1 & 0 \\ 0 & l_2 \end{pmatrix} - \lambda\begin{pmatrix} 2l_1^2 & l_1l_2 \\ l_1l_2 & l_2^2 \end{pmatrix} \right| &= 0 \\
\therefore \lambda^2 l_2^2 l_1^2 - 2\lambda l_2 l_1^2 - 2\lambda l_2^2 l_1 + 2l_2l_1 &= 0 \\
\therefore \lambda &= \frac{l_1 + l_2 \pm \sqrt{l_1^2 + l_2^2}}{l_1l_2} \\
\therefore \omega &= \sqrt{mg\frac{l_1 + l_2 \pm \sqrt{l_1^2 + l_2^2}}{l_1l_2}}
\end{aligned}$$

To find the eigenvectors, we solve $\mathbf{VA} = \omega^2\mathbf{TA}$, which expands to

$$\begin{pmatrix} 2l_1 & 0 \\ 0 & l_2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \lambda\begin{pmatrix} 2l_1^2 & l_1l_2 \\ l_1l_2 & l_2^2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

This gives us eigenvectors

$$\mathbf{A} = \begin{pmatrix} l_2\left(\pm(l_1 + l_2) + \sqrt{l_1^2 + l_2^2}\right) \\ -2l_1\left(\sqrt{l_1^2 + l_2^2} \pm l_1\right) \end{pmatrix}$$

These are already orthogonal ($\mathbf{A}_1^T\mathbf{T}\mathbf{A_2} = 0$). On normalizing them ($\mathbf{A}_i^T\mathbf{T}\mathbf{A_i} = 1$), we get

$$\boxed{\mathbf{A} = \frac{1}{2l_1l_2\sqrt{l_1^2 \pm l_1\sqrt{l_1^2 + l_2^2} + l_2^2}}\begin{pmatrix} l_2\left(\pm(l_1 + l_2) + \sqrt{l_1^2 + l_2^2}\right) \\ -2l_1\left(\sqrt{l_1^2 + l_2^2} \pm l_1\right) \end{pmatrix}}$$

for corresponding normal modes $\boxed{\omega = \sqrt{mg\frac{l_1 + l_2 \pm \sqrt{l_1^2 + l_2^2}}{l_1 l_2}}}$