EP 307 Assignment 6

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Problem 2

To prove: $\langle n \,|\, T \,|\, n \rangle = \langle n \,|\, V \,|\, n \rangle$

Firstly, we note that $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})$

$$
\begin{aligned}
\therefore \langle n \,|\, T \,|\, n \rangle &= \left\langle n \,\middle|\, \frac{1}{2m}\hat{p}^2 \,\middle|\, n \right\rangle \\
&= -\frac{\omega\hbar}{4} \left\langle n \,\middle|\, (\hat{a}^\dagger - \hat{a})^2 \,\middle|\, n \right\rangle \\
\text{And} \quad \langle n \,|\, V \,|\, n \rangle &= \left\langle n \,\middle|\, \frac{1}{2}m\omega^2\hat{x}^2 \,\middle|\, n \right\rangle \\
&= \frac{\omega\hbar}{4} \left\langle n \,\middle|\, (\hat{a}^\dagger + \hat{a})^2 \,\middle|\, n \right\rangle \\
\therefore \langle n \,|\, V \,|\, n \rangle - \langle n \,|\, T \,|\, n \rangle &= \frac{\omega\hbar}{4} \left\langle n \,\middle|\, (\hat{a}^\dagger + \hat{a})^2 \,\middle|\, n \right\rangle + \frac{\omega\hbar}{4} \left\langle n \,\middle|\, (\hat{a}^\dagger - \hat{a})^2 \,\middle|\, n \right\rangle \\
&= \frac{\omega\hbar}{2} \left\langle n \,\middle|\, \hat{a}^{\dagger 2} + \hat{a}^2 \,\middle|\, n \right\rangle \\
&= 0
\end{aligned}
$$

since both $\hat{a}^\dagger$ will move the state to $|n+2\rangle$, which is orthogonal to $|n\rangle$, and $\hat{a}$ will move it to $|n-2\rangle$ or $0\,|0\rangle$ (if $n < 2$). In both cases the net result is zero

Thus $\langle n \,|\, T \,|\, n \rangle = \langle n \,|\, V \,|\, n \rangle$

Problem 3

Such a linear combination would be $\frac{1}{\sqrt{1+\eta^2}}(|1\rangle + \eta|0\rangle)$

$$\begin{aligned}
\langle x\rangle &= \frac{1}{1+\eta^2}(\langle 1| + \eta\langle 0|)\left(\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger)\right)(|1\rangle + \eta|0\rangle) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\frac{1}{1+\eta^2}(\langle 1| + \eta\langle 0|)\left(\hat{a}|1\rangle + \eta\hat{a}|0\rangle + \hat{a}^\dagger|1\rangle + \eta\hat{a}^\dagger|0\rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\frac{1}{1+\eta^2}(\langle 1| + \eta\langle 0|)\left(|0\rangle + 0 + \sqrt{2}|2\rangle + \eta|1\rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\frac{1}{1+\eta^2}(2\eta)
\end{aligned}$$

Maximizing $\frac{\eta}{1+\eta^2}$ we get $\eta = 1$, and thus the state is $\frac{|1\rangle+|0\rangle}{\sqrt{2}}$

Problem 4

We know that $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger), \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})$, and furthermore $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. Taking the conjugate, $\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|$

Now,

$$\begin{aligned}
\langle x\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left\langle \alpha \,\middle|\, \hat{a}+\hat{a}^\dagger \,\middle|\, \alpha\right\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}(\langle \alpha | \hat{a} | \alpha\rangle + \left\langle \alpha \,\middle|\, \hat{a}^\dagger \,\middle|\, \alpha\right\rangle) \\
&= \sqrt{\frac{\hbar}{2m\omega}}(\langle \alpha | \alpha | \alpha\rangle + \alpha^*\langle \alpha | \alpha\rangle) \\
&= \sqrt{\frac{\hbar}{2m\omega}}(\alpha + \alpha^*) \\
\left\langle x^2\right\rangle &= \frac{\hbar}{2m\omega}\left\langle \alpha \,\middle|\, (\hat{a}+\hat{a}^\dagger)^2 \,\middle|\, \alpha\right\rangle \\
&= \frac{\hbar}{2m\omega}(\left\langle \alpha \,\middle|\, \hat{a}^2 \,\middle|\, \alpha\right\rangle + \left\langle \alpha \,\middle|\, \hat{a}^{\dagger 2} \,\middle|\, \alpha\right\rangle + \left\langle \alpha \,\middle|\, \hat{a}\hat{a}^\dagger \,\middle|\, \alpha\right\rangle + \left\langle \alpha \,\middle|\, \hat{a}^\dagger\hat{a} \,\middle|\, \alpha\right\rangle) \\
&= \frac{\hbar}{2m\omega}(\left\langle \alpha \,\middle|\, \alpha^2 \,\middle|\, \alpha\right\rangle + (\alpha^*)^2\langle \alpha | \alpha\rangle + \left\langle \alpha \,\middle|\, \hat{a}^\dagger\hat{a} + [\hat{a},\hat{a}^\dagger] \,\middle|\, \alpha\right\rangle + \left\langle \alpha \,\middle|\, \hat{a}^\dagger\hat{a} \,\middle|\, \alpha\right\rangle)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\hbar}{2m\omega}(\alpha^2 * \alpha^{*2} + \left\langle \alpha \,\middle|\, \hat{a}^\dagger \hat{a} + 1 \,\middle|\, \alpha \right\rangle + \left\langle \alpha \,\middle|\, \hat{a}^\dagger \hat{a} \,\middle|\, \alpha \right\rangle) \\
&= \frac{\hbar}{2m\omega}(\alpha^2 + \alpha^{*2} + \langle \alpha \,|\, 1 \,|\, \alpha \rangle + 2 \left\langle \alpha \,\middle|\, \hat{a}^\dagger \hat{a} \,\middle|\, \alpha \right\rangle) \\
&= \frac{\hbar}{2m\omega}(\alpha^2 + \alpha^{*2} + 1 + 2\alpha^* \alpha) \\
\langle p \rangle &= i\sqrt{\frac{m\omega\hbar}{2}} \left\langle \alpha \,\middle|\, (\hat{a}^\dagger - \hat{a}) \,\middle|\, \alpha \right\rangle \\
&= i\sqrt{\frac{m\omega\hbar}{2}}(\alpha^* - \alpha) \\
\left\langle p^2 \right\rangle &= -\frac{m\omega\hbar}{2} \left\langle \alpha \,\middle|\, (\hat{a}^\dagger - \hat{a})^2 \,\middle|\, \alpha \right\rangle \\
&= -\frac{m\omega\hbar}{2}(\alpha^{*2} + \alpha^2 - 1 + 2\alpha^* \alpha) \\
\therefore \sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \\
&= \sqrt{\frac{\hbar}{2m\omega}\left(\alpha^2 + \alpha^{*2} + 1 + 2\alpha^*\alpha - (\alpha + \alpha^*)^2\right)} \\
&= \sqrt{\frac{\hbar}{2m\omega}} \\
\therefore \sigma_p &= \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \\
&= \sqrt{-\frac{m\omega\hbar}{2}\left(\alpha^2 + \alpha^{*2} - 1 - 2\alpha\alpha^* - (\alpha - \alpha^*)^2\right)} \\
&= \sqrt{-\frac{m\omega\hbar}{2}(-1)} \\
&= \sqrt{\frac{m\omega\hbar}{2}} \\
\therefore \sigma_x \sigma_p &= \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{m\omega\hbar}{2}}) \\
&= \frac{\hbar}{2}
\end{aligned}
$$

Thus this system follows Heisenberg's uncertainty relation. Ans (a)

$$\langle N \rangle = \left\langle \alpha \,\middle|\, \hat{a}^\dagger \hat{a} \,\middle|\, \alpha \right\rangle$$

$$
\begin{aligned}
&= \alpha \langle \alpha \,|\, \alpha \,|\, \alpha \rangle \\
&= \alpha^2
\end{aligned}
$$

The probability amplitude is $\langle n \,|\, \alpha \rangle$

$$
\begin{aligned}
\langle n \,|\, \alpha \rangle &= \left\langle n \,\middle|\, e^{-\frac{|\alpha|^2}{2}} e^{\alpha \hat{a}^\dagger} \,\middle|\, 0 \right\rangle \\
&= e^{-|\alpha|^2/2} \left\langle n \,\middle|\, 1 + \alpha \hat{a}^\dagger + \frac{1}{2} (\alpha \hat{a}^\dagger)^2 + ... \,\middle|\, 0 \right\rangle \\
&= e^{-|\alpha|^2/2} \langle n| \left(1 + \alpha \,|1\rangle + \frac{1}{2} (\alpha)^2 \sqrt{2} \,|2\rangle + ... \right) \\
&= e^{-|\alpha|^2/2} \frac{\alpha^n}{n!} \sqrt{n!}
\end{aligned}
$$

Thus the probability is $e^{-|\alpha|^2} \frac{\alpha^2 n}{n!}$. This is a Poisson distribution, due to the $\frac{\alpha^2 n}{n!}$ factor. Ans (b)

Since the displacement operator is equivalent to $e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}$, its conjugate is $e^{-\alpha \hat{a}^\dagger + \alpha^* \hat{a}}$ (conjugating all of the components).

Also, $e^{\hat{A}} e^{-\hat{A}} = e^{\hat{A} - \hat{A}} e^{[\hat{A}, -\hat{A}]} 1$.

Thus, $DD^\dagger = 1$, for $\hat{A} = \alpha \hat{a}^\dagger - \alpha^* \hat{a}$. Thus it is unitary

Problem 5

Problem 6

$$
\begin{aligned}
(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) &= \sigma_i a_i \sigma_j b_j \\
&= a_i b_j \sigma_i \sigma_j \\
&= a_i b_j (\delta_{ij} + \epsilon_{ijk} \sigma_k) \\
&= a_i b_i + \epsilon_{ijk} \sigma_k a_i b_j \\
&= \vec{a} \cdot \vec{b} + \vec{\sigma} \cdot (\vec{a} \times \vec{b}) \qquad \text{Ans. (a)}
\end{aligned}
$$

$$
\begin{aligned}
e^{i\vec{a}\cdot\sigma} &= e^{ia\hat{n}\cdot\sigma} \\
&= 1 + ia\hat{n}\cdot\hat{\sigma} + \frac{1}{2}i^2a^2(\hat{n}\cdot\hat{\sigma})^2 + ...
\end{aligned}
$$

For even terms, the $(\hat{n}\cdot\hat{\sigma})^{2n}$ is the identity, since

$$
\begin{aligned}
(\hat{n}\cdot\hat{\sigma})^2 &= (\hat{n}\cdot\hat{\sigma})(\hat{n}\cdot\hat{\sigma}) \\
&= \hat{n}\cdot\hat{n} + \sigma\cdot(\hat{n}\times\hat{n}) \\
&= I
\end{aligned}
$$

. For odd terms, a single $\hat{n}\cdot\hat{\sigma}$ remains.

Collecting the even terms we get $\cos(a)I$, and for the odd terms we get $i\sin(a)\hat{n}\cdot\hat{\sigma}$.

Thus, the final expression is $I\cos a + i\frac{\vec{a}\cdot\sigma}{a}\sin a$ Ans. (b)