EP 307 Assignment 4

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Problem 1

$$\prod_{i\neq s}\frac{\hat{A}-a_i}{a_s-a_i}$$

If $|a\rangle = \sum c_i |a_i\rangle$, applying this operator we have

$$\begin{aligned}
\prod_{i\neq s}\frac{\hat{A}-a_i}{a_s-a_i}\left(\sum c_i |a_i\rangle\right) &= \sum_j \left(\prod_{i\neq s}\frac{\hat{A}-a_i}{a_s-a_i}\right) c_j |a_j\rangle \\
&= \left(\prod_{i\neq s}\frac{\hat{A}-a_i}{a_s-a_i}\right) c_s |a_s\rangle \\
&\qquad \text{Since the other terms have numerator cancelling out for } i=j \\
&= \left(\prod_{i\neq s}\frac{a_s-a_i}{a_s-a_i}\right) c_s |a_s\rangle \\
&= c_s |a_s\rangle
\end{aligned}$$

Thus the operator is $\frac{|a_s\rangle\langle a_s|}{\langle a_s \mid a_s\rangle} = p_s$ (projection operator)

Problem 2

To prove:

$$\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$$

The trace of an operator can be denoted by $\sum_i \langle i \,|\, A \,|\, i \rangle$, where $\langle i| \,, |i\rangle$ are basis vectors corresponding to the ith element being 1 and the rest 0.

Thus,

$$\begin{aligned}
\mathrm{Tr}(AB) &= \sum_i \langle i \,|\, AB \,|\, i \rangle \\
&= \sum_i \langle i \,|\, AIB \,|\, i \rangle \\
&= \sum_i \sum_j \langle i \,|\, A \,|j\rangle \langle j|\, B \,|\, i \rangle \\
&= \sum_i \sum_j \langle j \,|\, A \,|i\rangle \langle i|\, B \,|\, j \rangle \\
&= \sum_j \langle j \,|\, BIA \,|\, j \rangle \\
&= \mathrm{Tr}(BA)
\end{aligned}$$

Problem 7

Applying $\hat{A} = \hat{Q}\hat{C} + \hat{C}\hat{Q}$ to an eigenstate $|\psi_q\rangle$, we get:

$$\begin{aligned}
\hat{A} \,|\psi_q\rangle &= (\hat{Q}\hat{C} + \hat{C}\hat{Q}) \,|\psi_q\rangle \\
&= \hat{Q}\hat{C} \,|\psi_q\rangle + \hat{C}\hat{Q} \,|\psi_q\rangle \\
&= \hat{Q} \,|\psi_{-q}\rangle + q\hat{C} \,|\psi_q\rangle \\
&= -q \,|\psi_{-q}\rangle + q \,|\psi_{-q}\rangle \\
&= 0
\end{aligned}$$

If $\hat{A} \,|\psi_q\rangle = 0$, then $\hat{A} \left(\sum a_q \,|\psi_q\rangle\right) = 0$. We can say that the eigenvalue of $\hat{A}$ is 0.

For a state $|\psi_q\rangle$ to be an eigenstate of $\hat{C}$, $c \,|\psi_q\rangle = \hat{C} \,|\psi_q\rangle = |\psi_{-q}\rangle$

Since $|\psi_{-q}\rangle \,, |\psi_q\rangle$ have different eigenvalues (except when $q = 0$), they are linearly independent, and thus the only possible value of c is 0, which isn't an eigenstate.

So the only common eigenstate is $|\psi_0\rangle$, provided that it is not a null vector.

Problem 9

$$
\begin{aligned}
[\hat{x}, \exp\left(\frac{i\hat{p}a}{\hbar}\right)] |\psi\rangle &= [\hat{x}, \exp\left(\frac{i(-i)\hbar\partial_x a}{\hbar}\right)] |\psi\rangle \\
&= [\hat{x}, \exp(a\partial_x)] |\psi\rangle \\
&= x\sum_i \frac{1}{i!} a^i \partial_x^i |\psi\rangle - \sum_i \frac{1}{i!} a^i \partial_x^i (x |\psi\rangle) \\
&= x\sum_i \frac{1}{i!} a^i \partial_x^i |\psi\rangle - \sum_i \frac{1}{i!} a^i (x\partial_x^i (|\psi\rangle) + \partial_x^{i-1} i |\psi\rangle) \\
&= -\sum_i \frac{1}{i!} a^i \partial_x^{i-1} i |\psi\rangle \\
&= -\sum_i \frac{1}{(i-1)!} a^i \partial_x^{i-1} |\psi\rangle \\
&= -a\exp(a\partial_x) |\psi\rangle \\
&= -a\exp\left(ia\frac{\hat{p}}{\hbar}\right) |\psi\rangle \\
\therefore [\hat{x}, \exp\left(\frac{i\hat{p}a}{\hbar}\right)] &= -a\exp\left(ia\frac{\hat{p}}{\hbar}\right)
\end{aligned}
$$

Problem 11

The first allowed state (ground state) in the new system will be the first odd wavefunction, i.e. when $n = 1$. We need to calculate the probability that the current state ($n = 0$) becomes $n = 1$ in that region. Note that since the space is halved over a symmetric function, the wavefunctions will be normalized by an extra factor of $\frac{1}{\sqrt{2}}$

So, the overlap is

$$
\begin{aligned}
\int_0^\infty \psi_0(x)\frac{1}{\sqrt{2}}\psi_1(x) &= \int_0^\infty \left(\frac{\alpha}{\sqrt{\pi}}\right)^{\frac{1}{2}} e^{-\alpha^2x^2/2} H_0(\alpha x) \left(\frac{\alpha}{\sqrt{\pi}2}\right)^{\frac{1}{2}} e^{-\alpha^2x^2/2} H_2(\alpha x) \\
&= \frac{\alpha}{\sqrt{\pi}\sqrt{2}} \cdot \int_0^\infty e^{-\alpha^2x^2} \cdot 2\alpha x \\
&= \frac{2\alpha^2}{\sqrt{2\pi}} \int_0^\infty e^{-\alpha^2x^2} \cdot x
\end{aligned}
$$

$$= \frac{2\alpha^2}{\sqrt{2\pi}} \frac{1}{2\alpha^2} \qquad = \frac{1}{\sqrt{2\pi}}$$

Thus the probability is $\left(\frac{1}{\sqrt{2\pi}}\right)^2 = \boxed{\frac{1}{2\pi}}$

Problem 13

$$\hat{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & ib \\ 0 & ib & 0 \end{pmatrix}$$

To find eigenvectors of $\hat{B}$, we can multiply it with the vector $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

We get $\hat{B}X = \begin{pmatrix} bx \\ ibz \\ iby \end{pmatrix} = cX$, so for an eigenvector we either have $c = b, ibz = y, iby = z$ (giving $y = z = 0$ for a general b, and eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. If $c \neq b$ we have $x = 0$, and $y = \pm z$ with eigenvalues $\pm ib$

Therefore the eigenvectors are:

1. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ or $|1\rangle$ with eigenvalue b
2. $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ or $|2\rangle + |3\rangle$ with eigenvalue ib
3. $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ or $|2\rangle - |3\rangle$ with eigenvalue $-ib$

Ans. (a)

Now,

$$\hat{A}\hat{B} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & ib \\ 0 & ib & 0 \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\hat{B}\hat{A} = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & ib \\ 0 & ib & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

As $\hat{A}\hat{B} = \hat{B}\hat{A}$ in matrix form, the operators commute. Ans. (b)

The eigenkets of $\hat{A}$ can be easily seen to be $|1\rangle$ with eigenvalue a, and $l\,|2\rangle + m\,|3\rangle$ with eigenvalue $-a$ $(\forall\ l, m)$

We can see that the eigenkets of B are also eigenkets of A. They are also orthogonal, after normalizing we have:

Orthonormal Eigenket	Eigenvalue with $\hat{A}$	Eigenvalue with $\hat{B}$
$\lvert 1\rangle$	b	a
$\frac{\lvert 2\rangle+\lvert 3\rangle}{\sqrt{2}}$	ib	$-a$
$\frac{\lvert 2\rangle-\lvert 3\rangle}{\sqrt{2}}$	$-ib$	$-a$

The eigenkets are not completely determined from the eigenvalues from individual eigenvalues, as the eigenvalue $-a$ has multiplicity 2 and thus has an entire space of eigenkets.

However, knowing the eigenvalues from both operators completely specifies the eigenket, barring a constant. Ans. (c)

Problem 16

$\psi(x) = \frac{1}{\sqrt{2a}}$ in $[-a, a]$. The momentum space function can be found via the fourier transform,

$$\begin{aligned} \tilde{\psi}(p) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \psi(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \frac{1}{\sqrt{2a}} e^{-ikx} dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{i}{2k\sqrt{a\pi}}\ e^{-ikx}\Big|_{-a}^{a} \\
&= \frac{i}{2k\sqrt{a\pi}}\left(e^{-ika} - e^{ika}\right) \\
&= \frac{i}{2k\sqrt{a\pi}} \cdot -2i\sin ka \\
\therefore \tilde{\psi}(p) &= \frac{\sin\left(\frac{ap}{\hbar}\right)}{p\sqrt{a\pi}} \qquad \text{(Already normalized)}
\end{aligned}
$$

Ans.

Now,

$$
\begin{aligned}
\sigma_x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\
&= \sqrt{\int \psi^*(x)x^2\psi(x)dx - 0} \\
&= \sqrt{\int_{-a}^{a}\left(\frac{1}{\sqrt{2a}}\right)^2 x^2 dx} \\
&= \frac{a}{\sqrt{3}}
\end{aligned}
$$

And

$$
\begin{aligned}
\sigma_p &= \sqrt{\langle p^2\rangle - \langle p\rangle^2} \\
&= \sqrt{\int \tilde{\psi}^*(p)p^2\tilde{\psi}(p)dp - 0} \\
&= \sqrt{\int_{-\infty}^{\infty}\left(\frac{\sin\left(\frac{ap}{\hbar}\right)}{\sqrt{\pi}\sqrt{a}p}\right)^2 p^2 dp - 0} \qquad \text{(The wavefunction is odd so the second term vanishes)} \\
&= \sqrt{\infty}
\end{aligned}
$$

The product of the two uncertantainties is greater than $\frac{\hbar}{2}$

Problem 17

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p) e^{-\frac{t\left(ip^2\right)}{2m} + ipx} \, dp$$

Now, we can rewrite this as $\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx} \left(\phi(p) e^{-\frac{t\left(ip^2\right)}{2m}} \right) dp$

which is an inverse fourier transform.

Thus, $\tilde{\psi}(p,t) = \mathcal{F}(\psi(x,t)) = \phi(p) e^{-\frac{t\left(ip^2\right)}{2m}}$

Now,

$$\begin{aligned}
\langle p \rangle &= \int \tilde{\psi}^*(p,t) p \tilde{\psi}(p,t) dp \\
&= \int \phi^*(p) e^{\frac{t\left(ip^2\right)}{2m}} p \phi(p) e^{-\frac{t\left(ip^2\right)}{2m}} dp \\
&= \int \phi^*(p) p \phi(p) dp
\end{aligned}$$

Thus, $\langle \hat{p} \rangle$ is independant of time Ans. (a)

Note that $\phi(p) = \tilde{\psi}(p, t=0)$ Now,

$$\begin{aligned}
\langle x \rangle = \int \tilde{\psi}^*(p,t) i\hbar \frac{\mathrm{d}}{\mathrm{d}p} \tilde{\psi}(p,t) dp \\
&= \int \phi^*(p) e^{\frac{t\left(ip^2\right)}{2m}} i\hbar \left(\frac{\partial \phi}{\partial p} e^{-\frac{t\left(ip^2\right)}{2m}} + \phi(p) \frac{2pt}{2m} e^{-\frac{t\left(ip^2\right)}{2m}} \right) dp \\
&= \int \left(\phi^*(p) i\hbar \frac{\partial \phi}{\partial p} + \frac{t}{m} \phi^*(p) p \phi(p) \right) dp \\
&= \left\langle i\hbar \frac{\partial}{\partial p} \right\rangle_{t=t_0} + \frac{t}{m} \langle p \rangle_{t=t_0} \\
\therefore \langle x \rangle = \langle x \rangle_{t=0} + \frac{t}{m} \langle p \rangle_{t=0}
\end{aligned}$$

Since this is linear, we can shift by t_0 to get $\boxed{\langle x \rangle = \langle x \rangle_{t=t_0} + \frac{t-t_0}{m} \langle p \rangle_{t=t_0}}$

Ans. (b)

1 Problem 19

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$$

$$\begin{aligned}
[H, \hat{x}] &= \frac{\hat{p}^2}{2m}x + V(x)x - x\frac{\hat{p}^2}{2m} + xV(x) \\
&= -x\frac{\hat{p}^2}{2m} \\
[[H, \hat{x}], x] &= -x\frac{\hat{p}^2}{2m}x - -x^2\frac{\hat{p}^2}{2m} \\
&= \frac{x^2\hat{p}^2}{2m}
\end{aligned}$$