EP 307 Assignment 4

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Problem 1

$$\prod_{i\neq s} \frac{\hat{A} - a_i}{a_s - a_i}$$

If $|a\rangle = \sum c_i |a_i\rangle$, applying this operator we have

$$\begin{aligned}
\prod_{i\neq s} \frac{\hat{A} - a_i}{a_s - a_i} \left(\sum c_i |a_i\rangle\right) &= \sum_j \left(\prod_{i\neq s} \frac{\hat{A} - a_i}{a_s - a_i}\right) c_j |a_j\rangle \\
&= \left(\prod_{i\neq s} \frac{\hat{A} - a_i}{a_s - a_i}\right) c_s |a_s\rangle \\
&\quad \text{Since the other terms have numerator cancelling out for } i = j \\
&= \left(\prod_{i\neq s} \frac{a_s - a_i}{a_s - a_i}\right) c_s |a_s\rangle \\
&= c_s |a_s\rangle
\end{aligned}$$

Thus the operator is $\frac{|a_s\rangle\langle a_s|}{\langle a_s \mid a_s\rangle} = p_s$ (projection operator)

Problem 2

To prove:

$$\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$$

The trace of an operator can be denoted by $\sum_i \langle i \mid A \mid i \rangle$, where $\langle i| \, , |i\rangle$ are basis vectors corresponding to the ith element being 1 and the rest 0.

Thus,

$$\begin{aligned}
\mathrm{Tr}(AB) &= \sum_i \langle i \mid AB \mid i \rangle \\
&= \sum_i \langle i \mid AIB \mid i \rangle \\
&= \sum_i \sum_j \langle i \mid A \, |j\rangle \, \langle j| \, B \mid i \rangle \\
&= \sum_i \sum_j \langle j \mid A \, |i\rangle \, \langle i| \, B \mid j \rangle \\
&= \sum_j \langle j \mid BIA \mid j \rangle \\
&= \mathrm{Tr}(BA)
\end{aligned}$$

Problem 7

Applying $\hat{A} = \hat{Q}\hat{C} + \hat{C}\hat{Q}$ to an eigenstate $|\psi_q\rangle$, we get:

$$\begin{aligned}
\hat{A} \, |\psi_q\rangle &= (\hat{Q}\hat{C} + \hat{C}\hat{Q}) \, |\psi_q\rangle \\
&= \hat{Q}\hat{C} \, |\psi_q\rangle + \hat{C}\hat{Q} \, |\psi_q\rangle \\
&= \hat{Q} \, |\psi_{-q}\rangle + q\hat{C} \, |\psi_q\rangle \\
&= -q \, |\psi_{-q}\rangle + q \, |\psi_{-q}\rangle \\
&= 0
\end{aligned}$$

If $\hat{A} \, |\psi_q\rangle = 0$, then $\hat{A} \left(\sum a_q \, |\psi_q\rangle\right) = 0$. We can say that the eigenvalue of $\hat{A}$ is 0.

For a state $|\psi_q\rangle$ to be an eigenstate of $\hat{C}$, $c \, |\psi_q\rangle = \hat{C} \, |\psi_q\rangle = |\psi_{-q}\rangle$

Since $|\psi_{-q}\rangle \, , |\psi_q\rangle$ have different eigenvalues (except when $q = 0$), they are linearly independent, and thus the only possible value of c is 0, which isn't an eigenstate.

So the only common eigenstate is $|\psi_0\rangle$, provided that it is not a null vector.

Problem 9

$$
\begin{aligned}
[\hat{x}, \exp\left(\frac{i\hat{p}a}{\hbar}\right)]\left|\psi\right\rangle &= [\hat{x}, \exp\left(\frac{i(-i)\hbar\partial_x a}{\hbar}\right)]\left|\psi\right\rangle \\
&= [\hat{x}, \exp(a\partial_x)]\left|\psi\right\rangle \\
&= x\sum_i \frac{1}{i!}a^i\partial_x^i\left|\psi\right\rangle - \sum_i \frac{1}{i!}a^i\partial_x^i(x\left|\psi\right\rangle) \\
&= x\sum_i \frac{1}{i!}a^i\partial_x^i\left|\psi\right\rangle - \sum_i \frac{1}{i!}a^i(x\partial_x^i(\left|\psi\right\rangle) + \partial_x^{i-1}i\left|\psi\right\rangle) \\
&= -\sum_i \frac{1}{i!}a^i\partial_x^{i-1}i\left|\psi\right\rangle \\
&= -\sum_i \frac{1}{(i-1)!}a^i\partial_x^{i-1}\left|\psi\right\rangle \\
&= -a\exp(a\partial_x)\left|\psi\right\rangle \\
&= -a\exp\left(ia\frac{\hat{p}}{\hbar}\right)\left|\psi\right\rangle \\
\therefore [\hat{x}, \exp\left(\frac{i\hat{p}a}{\hbar}\right)] &= -a\exp\left(ia\frac{\hat{p}}{\hbar}\right)
\end{aligned}
$$

Problem 11

The first allowed state (ground state) in the new system will be the first odd wavefunction, i.e. when $n = 1$. We need to calculate the probability that the current state ($n = 0$) becomes $n = 1$ in that region. Note that since the space is halved over a symmetric function, the wavefunctions will be normalized by an extra factor of $\frac{1}{\sqrt{2}}$

So, the overlap is

$$
\begin{aligned}
\int_0^\infty \frac{1}{\sqrt{2}}\psi_0(x)\frac{1}{\sqrt{2}}\psi_1(x) &= \int_0^\infty \frac{1}{2}\left(\frac{\alpha}{\sqrt{\pi}}\right)^{\frac{1}{2}} e^{-\alpha^2x^2/2}H_0(\alpha x)\left(\frac{\alpha}{\sqrt{\pi}2}\right)^{\frac{1}{2}} e^{-\alpha^2x^2/2}H_2(\alpha x) \\
&= \frac{1}{2}\frac{\alpha}{\sqrt{\pi}\sqrt{2}}\cdot\int_0^\infty e^{-\alpha^2x^2}\cdot 2\alpha x \\
&= \frac{\alpha^2}{\sqrt{2\pi}}\int_0^\infty e^{-\alpha^2x^2}\cdot x
\end{aligned}
$$

$$= \frac{\alpha^2}{\sqrt{2\pi}} \frac{1}{2\alpha^2} \qquad = \frac{1}{2\sqrt{\pi}}$$

Thus the probability is $\frac{1}{\sqrt{2\pi}}$