EP 307 Assignment 4

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Problem 1

$$\prod_{i\neq s}\frac{\hat{A}-a_i}{a_s-a_i}$$

If $|a\rangle = \sum c_i |a_i\rangle$, applying this operator we have

$$\begin{aligned}
\prod_{i\neq s}\frac{\hat{A}-a_i}{a_s-a_i}\left(\sum c_i |a_i\rangle\right) &= \sum_j \left(\prod_{i\neq s}\frac{\hat{A}-a_i}{a_s-a_i}\right) c_j |a_j\rangle \\
&= \left(\prod_{i\neq s}\frac{\hat{A}-a_i}{a_s-a_i}\right) c_s |a_s\rangle \\
&\quad \text{Since the other terms have numerator cancelling out for } i=j \\
&= \left(\prod_{i\neq s}\frac{a_s-a_i}{a_s-a_i}\right) c_s |a_s\rangle \\
&= c_s |a_s\rangle
\end{aligned}$$

Thus the operator is $\frac{|a_s\rangle\langle a_s|}{\langle a_s \mid a_s\rangle} = p_s$ (projection operator)

Problem 2

To prove:

$$\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$$

The trace of an operator can be denoted by $\sum_i \langle i \,|\, A \,|\, i \rangle$, where $\langle i|\,, |i\rangle$ are basis vectors corresponding to the ith element being 1 and the rest 0. We can also write $A_{ij} = \sum_{ij} A\,|j\rangle\,\langle i|$

Thus,

$$
\begin{aligned}
\mathrm{Tr}(AB) &= \sum \langle i \,|\, AB \,|\, i \rangle \\
&= \sum_i \sum_j \sum_k \sum_l \langle i|\, A_{jk}\,|j\rangle\,\langle k|\, B_{kl}\,|k\rangle\,\langle l|\,|i\rangle \\
&= \sum_i \sum_j \sum_k \sum_l \delta_{ij} A_{jk} B_{kl} \delta_{il} \\
&= \sum_i \sum_k A_{ik} B_{ki} \\
&= \sum_k \sum_i B_{ki} A_{ik} \qquad \text{(Interchanging the sums)} \\
&= \mathrm{Tr}(BA) \qquad \text{(Following the reverse steps)}
\end{aligned}
$$

Problem 7

Applying $\hat{A} = \hat{Q}\hat{C} + \hat{C}\hat{Q}$ to an eigenstate $|\psi_q\rangle$, we get:

$$
\begin{aligned}
\hat{A}\,|\psi_q\rangle &= (\hat{Q}\hat{C} + \hat{C}\hat{Q})\,|\psi_q\rangle \\
&= \hat{Q}\hat{C}\,|\psi_q\rangle + \hat{C}\hat{Q}\,|\psi_q\rangle \\
&= \hat{Q}\,|\psi_{-q}\rangle + q\hat{C}\,|\psi_q\rangle \\
&= -q\,|\psi_{-q}\rangle + q\,|\psi_{-q}\rangle \\
&= 0
\end{aligned}
$$

If $\hat{A}\,|\psi_q\rangle = 0$, then $\hat{A}\,(\sum a_q\,|\psi_q\rangle) = 0$. We can say that the eigenvalue of $\hat{A}$ is 0.

For a state $|\psi_q\rangle$ to be an eigenstate of $\hat{C}$, $c\,|\psi_q\rangle = \hat{C}\,|\psi_q\rangle = |\psi_{-q}\rangle$

Since $|\psi_{-q}\rangle\,, |\psi_q\rangle$ have different eigenvalues (except when $q = 0$), they are linearly independent, and thus the only possible value of c is 0, which isn't an eigenstate.

So the only common eigenstate is $|\psi_0\rangle$, provided that it is not a null vector.

Problem 9

$$
\begin{aligned}
[\hat{x}, \exp\left(\frac{i\hat{p}a}{\hbar}\right)]\,|\psi\rangle &= [\hat{x}, \exp\left(\frac{i(-i)\hbar\partial_x a}{\hbar}\right)]\,|\psi\rangle \\
&= [\hat{x}, \exp(a\partial_x)]\,|\psi\rangle \\
&= x\sum_i \frac{1}{i!}a^i\partial_x^i\,|\psi\rangle - \sum_i \frac{1}{i!}a^i\partial_x^i(x\,|\psi\rangle)
\end{aligned}
$$