

EP 307 Assignment 4

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Problem 1

$$\prod_{i \neq s} \frac{\hat{A} - a_i}{a_s - a_i}$$

If $|a\rangle = \sum c_i |a_i\rangle$, applying this operator we have

$$\begin{aligned} \prod_{i \neq s} \frac{\hat{A} - a_i}{a_s - a_i} \left(\sum c_i |a_i\rangle \right) &= \sum_j \left(\prod_{i \neq s} \frac{\hat{A} - a_i}{a_s - a_i} \right) c_j |a_j\rangle \\ &= \left(\prod_{i \neq s} \frac{\hat{A} - a_i}{a_s - a_i} \right) c_s |a_s\rangle \end{aligned}$$

Since the other terms have numerator cancelling out for $i = j$

$$\begin{aligned} &= \left(\prod_{i \neq s} \frac{a_s - a_i}{a_s - a_i} \right) c_s |a_s\rangle \\ &= c_s |a_s\rangle \end{aligned}$$

Thus the operator is $\frac{|a_s\rangle\langle a_s|}{\langle a_s|a_s\rangle} = p_s$ (projection operator)

Problem 2

To prove:

$$\text{Tr}(XY) = \text{Tr}(YX)$$

The trace of an operator can be denoted by $\sum_i \langle i | A | i \rangle$, where $\langle i |, | i \rangle$ are basis vectors corresponding to the i th element being 1 and the rest 0.

Thus,

$$\begin{aligned}
\text{Tr}(AB) &= \sum_i \langle i | AB | i \rangle \\
&= \sum_i \langle i | AIB | i \rangle \\
&= \sum_i \sum_j \langle i | A | j \rangle \langle j | B | i \rangle \\
&= \sum_i \sum_j \langle j | A | i \rangle \langle i | B | j \rangle \\
&= \sum_j \langle j | BIA | j \rangle \\
&= \text{Tr}(BA)
\end{aligned}$$

Problem 7

Applying $\hat{A} = \hat{Q}\hat{C} + \hat{C}\hat{Q}$ to an eigenstate $|\psi_q\rangle$, we get:

$$\begin{aligned}
\hat{A}|\psi_q\rangle &= (\hat{Q}\hat{C} + \hat{C}\hat{Q})|\psi_q\rangle \\
&= \hat{Q}\hat{C}|\psi_q\rangle + \hat{C}\hat{Q}|\psi_q\rangle \\
&= \hat{Q}|\psi_{-q}\rangle + q\hat{C}|\psi_q\rangle \\
&= -q|\psi_{-q}\rangle + q|\psi_{-q}\rangle \\
&= 0
\end{aligned}$$

If $\hat{A}|\psi_q\rangle = 0$, then $\hat{A}(\sum a_q |\psi_q\rangle) = 0$. We can say that the eigenvalue of \hat{A} is 0.

For a state $|\psi_q\rangle$ to be an eigenstate of \hat{C} , $c|\psi_q\rangle = \hat{C}|\psi_q\rangle = |\psi_{-q}\rangle$

Since $|\psi_{-q}\rangle, |\psi_q\rangle$ have different eigenvalues (except when $q = 0$), they are linearly independent, and thus the only possible value of c is 0, which isn't an eigenstate.

So the only common eigenstate is $|\psi_0\rangle$, provided that it is not a null vector.

Problem 9

$$\begin{aligned}
[\hat{x}, \exp\left(\frac{i\hat{p}a}{\hbar}\right)] |\psi\rangle &= [\hat{x}, \exp\left(\frac{i(-i)\hbar\partial_x a}{\hbar}\right)] |\psi\rangle \\
&= [\hat{x}, \exp(a\partial_x)] |\psi\rangle \\
&= x \sum_i \frac{1}{i!} a^i \partial_x^i |\psi\rangle - \sum_i \frac{1}{i!} a^i \partial_x^i (x |\psi\rangle) \\
&= x \sum_i \frac{1}{i!} a^i \partial_x^i |\psi\rangle - \sum_i \frac{1}{i!} a^i (x \partial_x^i (|\psi\rangle) + \partial_x^{i-1} |\psi\rangle) \\
&= - \sum_i \frac{1}{i!} a^i \partial_x^{i-1} |\psi\rangle \\
&= - \sum_i \frac{1}{(i-1)!} a^i \partial_x^{i-1} |\psi\rangle \\
&= -a \exp(a\partial_x) |\psi\rangle \\
&= -a \exp\left(ia\frac{\hat{p}}{\hbar}\right) |\psi\rangle \\
\therefore [\hat{x}, \exp\left(\frac{i\hat{p}a}{\hbar}\right)] &= -a \exp\left(ia\frac{\hat{p}}{\hbar}\right)
\end{aligned}$$