Introduction to Mathematical Thinking

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1

It's false. Since when $n \geq 3$, $5n \geq 15 > 12$, we only need to consider when $n = 1$ or $n = 2$, because of $n \in N$. If $n = 2$, then $3m + 10 = 12$, or $3m = 2$, $m = \frac{2}{3}$, which is not a nature number. If $n = 1$, then $3m + 5 = 12$, or $3m = 7$, $m = \frac{7}{3}$ which is not a nature number, too. Therefore, it's false.

2

It's true. Let arbitrary five consecutive integers be $n-2, n-1, n, n+1, n+2$, we have $(n-2) + (n-1) + n + (n+1) + (n+2) = 5n$ by algebra, which is divisible by 5. So it's true.

3

It's true. $n^2 + n + 1 = n(n+1) + 1$ by algebra. Noticed that $n(n+1)$ is always even, for if n is odd, then $n+1$ is even, and if n is even, then $n+1$ is odd, so $n(n+1)$ is even. Therefore, $n(n+1)+1$ is odd, and n^2+n+1 is odd as well. It's true.

4

A arbitrary nature number is one of the forms of $4n, 4n+1, 4n+2$ or $4n+3$ by the Division Theorem. Since $4n+2$ and $4n$ are always even. So odd nature number can only belong to $4n+1$ of $4n+3$. And $4n+1$ and $4n+3$ are also odd number. These prove the result.

5

The integer n can be expressed as one of $3k$, $3k+1$, $3k+2$ by the Division Theorem, where k is an integer. If $n = 3k$, n is divisible by 3. If $n = 3k+1$, $n+2 = 3k+3 = 3(k+1)$, which is divisible by 3. If $n = 3k+2$, $n+4 = 3k+6 = 3(k+2)$, which is divisible by 3. These prove the result.

6

Consider a possible triple in the form of $n, n+2, n+4$. We require either of them to be a prime, so we have to let $n \geq 2$. From the above question (Question 5), we know at least one element in the triple is divisible by 3. So if $n \geq 4$, the triple is not a prime triple. If $n = 3$, 3,5,7 is a valid prime triple. If $n = 2$, 2,4,6 is not valid, as 4 and 6 are not primes. In conclusion, the only prime triple is 3,5,7.

7

If n $= 1$, the left side is 2, and the right side is $2^{1+1}-2 = 2^2-2 = 4-2 = 2$, which is equal. Assume it's true for $n = k$, which means $2+2^2+2^3+\cdots+2^k = 2^{k+1}-2$, we next prove it's true for $n = k+1$, that is,$2+2^2+2^3+\cdots+2^{k+1} = 2^{k+2}-2$
Since $2+2^2+2^3+\cdots+2^{k+1}$
$= 2+2^2+2^3+\cdots+2^k+2^{k+1}$ (seperate the last term from the formula)
$= 2^{k+1}-2+2^{k+1}$ (use the assumption that it's true for $n = k$)
$= 2^{k+2}-2$ (by algebra)
so it is true for $n = k+1$. These above prove the result.

8

Let $\epsilon > 0$ be given. Since $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, there exists an N s.t. if $n \geq N$, then $|a_n - L| \leq \frac{\epsilon}{M}$. So we have $|Ma_n - ML| \leq M \cdot \frac{\epsilon}{M} = \epsilon$, which means that $\{Ma_n\}_{n=1}^{\infty}$ tends to limit ML as $n \to \infty$, which proves the result.

9

Let $A_n = (0, \frac{1}{n})$, which satisfies that $A_{n+1} \subset A_n$. As $\cap_{n=1}^{\infty} A_n \subset (0,1)$, so if there are elements in the intersect, then they must belong to (0,1). However, for all $x \in (0,1)$, we can find a natural number n s.t. $\frac{1}{n} < x$, which means $x \notin A_n$, and not in the intersect as well. Therefore, $\cap_{n=1}^{\infty} A_n = \emptyset$

10

Let $A_n = [0, \frac{1}{n})$, which satisfies that $A_{n+1} \subset A_n$. As $\cap_{n=1}^{\infty} A_n \subset [0,1)$, so if there are elements in the intersect, then they must belong to [0,1). Obviously, the element 0 is in the intersect. However, for all $x \in (0,1)$, we can find a natural number n s.t. $\frac{1}{n} < x$, which means $x \notin A_n$, and not in the intersect as well. Therefore, $\cap_{n=1}^{\infty} A_n = \{0\}$, which consists of a single real number 0.