1 Week 2

1.1 Recap Week 1

$$PV = \frac{FV}{(1+r)^n}$$

$$FV = P \cdot (1+r)^n$$

1.2 FV of Annuity: Concept

1.2.1 Multiple Payments: Annuities

Multiple payments over time.

- A special case of multiple payments:
  - annuities (C or PMT)

Year	Cash Flow	Years to End: n	Future Value
0	0	3	0
1	C	2	$C \cdot (1+r)^2$ [first payment at the end of first year]
2	C	1	$C \cdot (1+r)$
3	C	0	C

No cash flow occuring at time 0 by convention.

1.2.2 FV of an Annuity: Formula

Future Value of a Stream of Cash Flows as of n Periods from Now

$$\mathbf{FV = C_1 \cdot (1+r)^{n-1} + C_2 \cdot (1+r)^{n-2} + \ldots + C_{n-1} \cdot (1+r) + C_n}$$

Future Value of an Annuity Paying $C at the End of Each of n Periods.

$$\begin{aligned} \mathbf{FV_n} &= \sum_{\mathbf{i=0}}^{\mathbf{n-1}} \mathbf{C \cdot (1+r)^i} \\ &= \mathbf{C \cdot \frac{1}{r} \left[(1+r)^n - 1\right]} \end{aligned}$$

1.3 FV of Annuity: Example

What will be the value of your portfolio at retirement if you deposit $10,000 every year in a pension fund. You plan to retire in 40 years and expect to earn 8% on your portfolio?

Year	Cash Flow	Years to End: n	Future Value
0	0	40	0
1	$10,000	39	201152.9768
2	$10,000	38	186252.7563
3	$10,000	37	172456.2558
⋮	⋮	⋮	⋮
40	$10,000	0	$10,000
		Total:	**$2,590,565.1871**

Excel: =FV(0.08, 40, 10000)

Calculation in Google docs, Week2: 1.3 Ex1

The interest rate, 8%, wil not consistent over the 40 years.

1.4 FV of Annuity: Example 2

Suppose you want to guarantee yourself $500,000 when you reture 25 years from now. How much must you invest each year, starting at the end of this year, if the interst rate is 8%?

$$\mathbf{C} = \frac{\mathbf{FV} \cdot \mathbf{r}}{(\mathbf{r}+\mathbf{1})^{\mathbf{n}} - \mathbf{1}}$$

$$C = \frac{500000 \cdot 0.08}{(0.08+1)^{25} - 1} = 6839.3895$$

Excel: =PMT(0.08, 25, 0, 500000)

Suppose the interest rate was 0, what would you have? $\$7000 \cdot 7 = \$175,000$.

The 8% comes from the market. This rate will probably require a lot of risk.

1.5 PV of Annuity: Concept

Year	Cash Flow	Years to Discount ($y > 0$): n	Future Value
0	0	0	0
1	C	1	$\frac{C}{(1+r)}$
2	C	2	$\frac{C}{(1+r)^2}$
3	C	3	$\frac{C}{(1+r)^3}$

Present Value of a Stream of Cash Flows

$$PV_n = \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \ldots + \frac{C_n}{(1+r)^n}$$

Present Value of an Annuity

$$\mathbf{PV_C} = \frac{\mathbf{FV} \cdot \mathbf{A}}{(\mathbf{1}+\mathbf{r})^{\mathbf{n}}} = \frac{\mathbf{C}}{\mathbf{r}}\left[\mathbf{1} - \frac{\mathbf{1}}{(\mathbf{1}+\mathbf{r})^{\mathbf{n}}}\right]$$

1.6 PV of Annuity: Examples

Ex. 1:

How much money do you need in the bank today so that you can spend $10,000 every year for the next 25 years, starting at the end of this year. Suppose $r = 5\%$.

Google docs: Week2: 1.6 Ex1

$$PV_{10000,25} = 140939.45$$

Excel: =PV(0.05, 25, 10000)

Ex. 2:

You plan to attend a business school and you will be forced to take out \$100,000 in a loan at 10%. You want to figure out your yearly payments, given that you will have 5 years to pay back the loan.

Periodic Payment of Annuity with fixed interest rates

$$\mathbf{PMT} = \frac{\mathbf{PV \cdot r \cdot (r+1)^n}}{\mathbf{(r+1)^n - 1}}$$

$$PMT = 26379.75 \text{ per year}$$

Excel: =PMT(0.1, 5, 100000)

What should be the present value of 26379.75 with $m = 5$, $r = 5\%$? Should be \$100000.

1.7 A Loan: The Power of Finance

1.7.1 Example II: Loan Amortization

Loan Amortization Table

Take the problem from 1.6

Interest is 10% of what you borrowed, so first year it's \$10,000

Year	Beginning Balance	Yearly Payment	Interest (10%)	Principal Payment
1	\$100,000	\$26,380	\$10,000	$26,380 - 10,000 = \$16,380$
2	$100,000 - 16,380 = \$83,620$	\$26,380	$\$83,620 \cdot 10\% = \$8,362$	\$18,018
3	\$65,602	\$26,380	\$6,560.20	\$19,819.80
4	\$45,782.20	\$26,380	\$4,578.22	\$21,801.78
5	\$23,980.42	\$26,380	\$2,398.04	\$23,981.96

Google docs: Week 2: 1.7 Ex2

Ex. 3:

How much would you owe at the beginning of year 3?

Year	Beginning Balance	Yearly Payment	Interest (10%)	Principal Payment
1	\$100000	\$26,380	\$10,000	\$16,380
2	\$83,620 ($n = 4$)	\$26,380	\$8,362	\$18,018
3	\$65,602 ($n = 3$)	\$26,380	\$6,560.20	\$19,819.80
4	\$45,782.20 ($n = 2$)	\$26,380	\$4,578.22	\$21,801.78
5	\$23,980.42 ($n = 1$)	\$26,380	\$2,398.04	\$23,981.96

How much do you owe the bank the moment you walk out of the bank. You can't take 26380 and multiply by 5 because of compounding.

$n = 5$

$C = 26380$

$r = 10\%$

$PV_0 = 100000$

All value is determined by standing at a point in time and looking forward. The bank makes money by charging you a bit more on borrowing/lending rates difference.

1.8 Compounding

Simple Interest

1.8.1 Compound Intereset

You plan to attend a business school and you will be forced to take out \$100,000 in a loan at 10%.

What are your monthly payments, given thta you will have 5 years to pay back the loan?

What is your "real" interest rate?

Timeline Changed

$r = 5\%$ annual, 5 years →60 months

$\frac{r}{mo} = \frac{0.1}{12}$

$PV = 100000$ →?

$PMT = 60$

Periodicity changes.

so take 60 periods

0—-1—–2—–...——————60

divide interest rate by 12

0–$r = \frac{0.10}{12}$–1–$r = \frac{0.10}{12}$–2—–...——————60

Excel: =PMT(0.1/12, 60, 100000)

$$\begin{aligned} PMT &= \frac{PV \cdot r \cdot (r+1)^n}{(r+1)^n - 1} \\ &= \frac{100000 \cdot \frac{0.1}{12} \cdot \left(1 + \frac{0.1}{12}\right)^{60}}{\left(1 + \frac{0.1}{12}\right)^{60} - 1} \\ &= 2124.7 \end{aligned}$$

How much will you owe after 30 months?

$PMT = 2125$

$r = \frac{0.1}{12}$

$m = 30$

Take the Present Value

$$\begin{aligned} P &= \frac{C}{r}\left[1 - \frac{1}{(1+r)^n}\right] \\ &= \frac{2125}{\frac{0.1}{12}}\left[1 - \frac{1}{\left(1 + \frac{0.1}{12}\right)^{30}}\right] \\ &= 56199.97 \end{aligned}$$

How much is the Annual Interest Rate (more)

Effective Annual Interest Rate

$$\mathbf{EAR} = \left(\mathbf{1} + \frac{\mathbf{r}}{\mathbf{k}}\right)^{\mathbf{k}} - \mathbf{1}$$

$k = 12$, monthly

r is annual

$$
\begin{aligned}
EAR &= \left(1 + \frac{0.1}{12}\right) - 1 \\
&= 10.47\%
\end{aligned}
$$

So 10% is a stated interest rate, 10.47% is the "real" interest rate.

1.9 Valuing Perpetuities

A perpetuity is simply a set of equal payments that are paid forever, with or without growth.

Ex.:

- Something that pays 1 poud over time
- Stock (goes on forever) vs Bonds (limited in maturity)

Forever doesn't mean forever.

1.9.1 Power of Perpetuities

0—-$10—–...—–

$$\mathbf{PV} = \frac{\mathbf{C}}{\mathbf{r}}$$

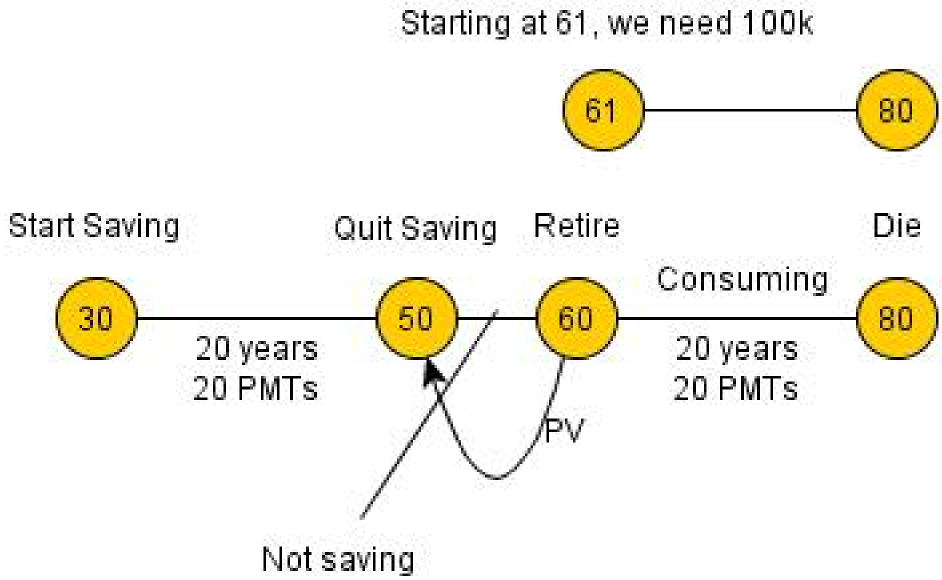
If the growth rate is r,

$$\mathbf{PV} = \frac{\mathbf{C}}{\mathbf{r - g}}$$

(Growth stocks)

1.10 Mega Example: Putting it All Together

Suppose you are exactly 30 years old. You believe that you will be able to save for the next 20 years, until you are 50. For 10 years following that, and till your retirement at age 60, you will have a spike in your expenses due to your kids' college expenses, weddings, etc., and you will not be able to save. If you want to guarantee yourself $100,000 per year starting on your 61st birthday, how much should you save every year, for the next 20 years, starting at the end of this year. Assume that your investments are expected to yield 8% and you are likely to live till 80.



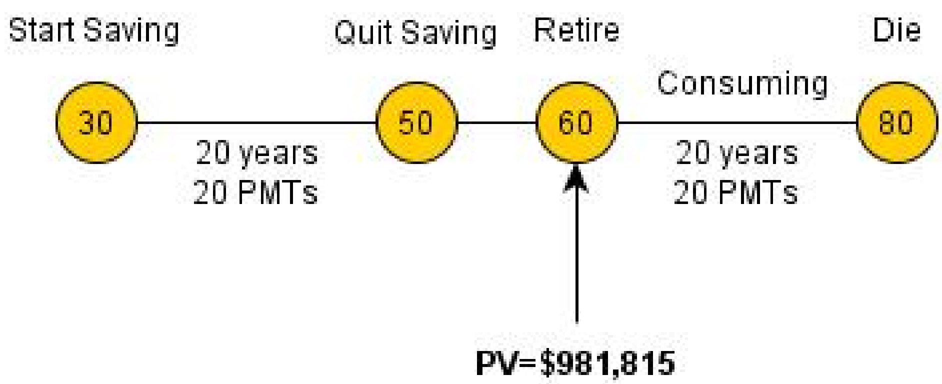
Problem is called **PMT Present Value Problem**

We solve for PV at point 60, we'll know PV at point 50 as well.

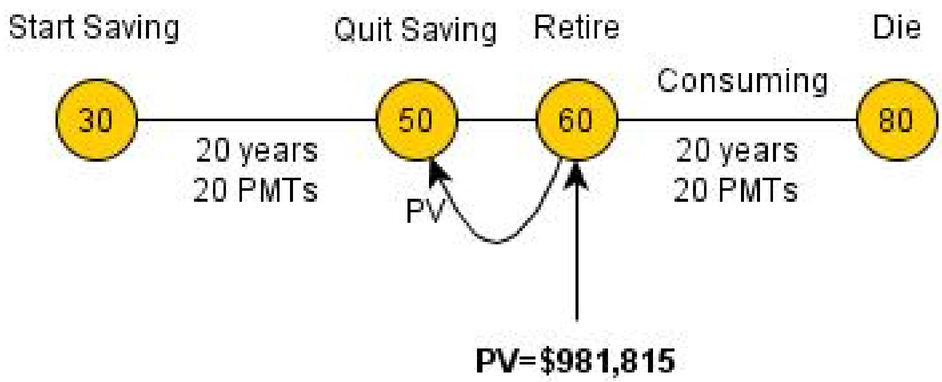
Excel:

=PV(0.08, 20, 100000)

$$PV_{60} = \$981,814.74$$



The problem is the gap between 50 and 60. The PV as in year 60. We need to bring it back to year 50.

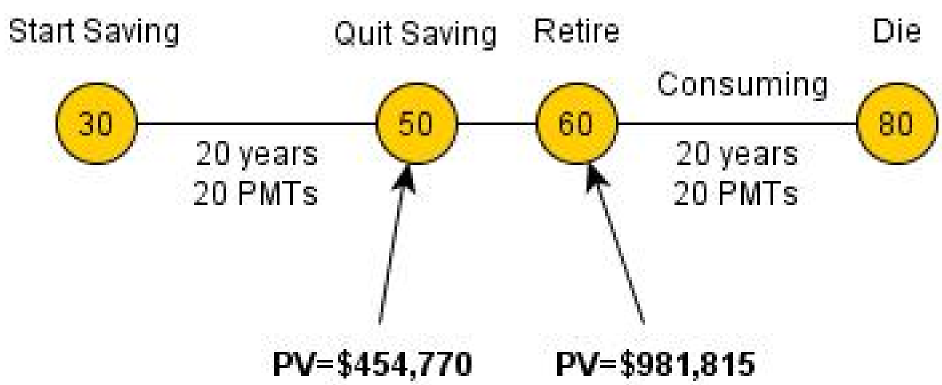


PV_{50} will become the FV of the saving annuity we're trying to solve.

Excel:

=PV(0.08, 10, 0, 981814.74)

=$454,770



If we know PV_{50}, it can become the FV of the problem.

So how much do we put in the bank, so it becomes $454,770.

Excel:

=PMT(0.08, 20, 0, 454770)

=**$9,937.73** every year