1 Week 1

1.1 Time Value of Money

1.1.1 Future Value, Present Value and Compounding

- \$V invested for n years at simple interest rate R per year
- Compounding of interest occurs at end of year

$$FV_n = V \cdot (1+R)^n$$

where FV_n is future value after n years.

So after, say, 1 year, the initial investment grows to $V \cdot (1+R)$, after 2 years $V \cdot (1+R)(1+R)$

Example: Consider putting \$1000 in an interest checking account that pays a simple annual percentage rate of 3%. The future value after n = 1, 5 and 10 years is, respectively,

$FV_1 = \$1000 \cdot (1.03)\ 1 = \$1030$,

$FV_5 = \$1000 \cdot (1.03)\ 5 = \$1159.27$,

$FV_{10} = \$1000 \cdot (1.03)\ 10 = \$1343.92$.

FV function is a relationship between four variables FV_n, V, R, n. Given three variables, you can solve for the fourth:

Present Value

$$V = \frac{FV_n}{(1+R)^n}$$

Compound annual return - average annual interest rate that you get every year that compounds such that the investment grows to the future value

$$R = \left(\frac{FV_n}{V}\right)^{\frac{1}{n}} - 1$$

Investment horizon

$$n = \frac{ln\left(\frac{FV_n}{V}\right)}{ln(1+R)}$$

Investment horizon answers a question "how long does it take the money to double". Related to the "rule of 70"

$$\begin{aligned} 2 &= V(1+R)^n \\ \ln 2 &= n \ln(1+R) \\ n &= \frac{\ln(2)}{\ln(1+R)} \approx \frac{0.7}{R} \end{aligned}$$

Ex.: If $R = 0.01$, then $n = \frac{0.7}{0.01} = 70$ years for the money to double.

Compounding can occur m times per year

$$FV_n^m = V \cdot (1 + \frac{R}{m})^{m \cdot n}$$

$$\frac{R}{m} = \text{periodic interest rate}$$

Continuous compounding - if we're moving compounding from daily to hourly to minutely to secondly...$m \to \infty$

$$FV_n^{\infty} \lim_{m \to \infty} V \cdot \left(1 + \frac{R}{m}\right)^{m \cdot n} = Ve^{R \cdot n}$$

Example: If the simple annual percentage rate is 10% then the value of \$1000 at the end of one year ($n = 1$) for different values of m is given in the table below.

Compounding Frequency	Value of \$1000 at end of 1 year ($R = 10\%$)
Annually ($m = 1$)	1100.00
Quarterly ($m = 4$)	1103.81
Weekly ($m = 52$)	1105.06
Daily ($m = 365$)	1105.16
Continuously ($m = \infty$)	1105.17

For historical reasons, compounding interesting more than once a year, let banks get around certain kinds of regulations that capped interest rates.

Because we can have compounding periods different from a year, there's a concept of **Effective Annual Rate**. For example, we effectively have a higher interest rate when we compoud daily than annually, like in the above example. So for the above example the effective annual rate for daily compounding frequency is 10.5%.

Effective Annual Rate

Annual rate R_A that equates FV_n^m with FV_n i.e.

$$\begin{aligned} V \cdot \left(1 + \frac{R}{m}\right)^{m \cdot n} &= V \cdot (1 + R_A)^n \\ (1 + \frac{R}{m})^m &= 1 + R_A \\ \mathbf{R_A} &= \mathbf{\left(1 + \frac{R}{m}\right)^m - 1} \end{aligned}$$

$V \cdot \left(1 + \frac{R}{m}\right)^{m \cdot n}$- future value with compounding m per year

$V \cdot (1 + R_A)^n$ - future value compounding once per year at R_A, effective annual rate.

For continuous compouding we do the same computations

$$\begin{aligned} Ve^{R \cdot n} &= V \cdot (1 + R_A)^n \\ e^R &= (1 + R_A) \\ R_A &= e^R - 1 \end{aligned}$$

$Ve^{R \cdot n}$ - future value of continuous compounding

$V \cdot (1 + R_A)^n$ - future value with compounding once per year

Example. Compute effective annual rate with semi-annual compounding The effective annual rate associated with an investment with a simple annual rate $R = 10\%$ and semi-annual compounding ($m = 2$) is determined by solving

$$
\begin{aligned}
(1+R_A) &= \left(1+\frac{0.10}{2}\right)^2 \\
R_A &= \left(1+\frac{0.10}{2}\right)^2 - 1 \\
&= 0.1025
\end{aligned}
$$

Compounding Frequency	Value of \$1000 at end of 1 year ($R = 10\%$)	R_A
Annually ($m = 1$)	1100.00	10%
Quarterly ($m = 4$)	1103.81	10.38
Weekly ($m = 52$)	1105.06	10.51%
Daily ($m = 365$)	1105.16	10.52%
Continuously ($m = \infty$)	1105.17	10.52%

We get a slightly higher rate when we compound more often, reason for CCs quoting an annual interest rate but compounding daily, so the effective rate is higher.

1.2 Simple Returns

1.2.1 Asset Returns

Simple Returns Buy stock, hold it for a period of time and sell, want to know whta the return on that will be.

Holding Period Return The idea is that we have a price 0, P_0, of some asset. At some t in the future we're going to sell the asset at P_t. **Holding Period Return:**

$$
\underset{\textbf{over holding period}}{\%\Delta P} = \frac{P_t - P_0}{P_0}
$$

Holding periods are often standardized to 1 month, they could be 2.34 days.

P_t - price at the end of month t on an asset that pays no dividends

P_{t-1}- price at the end of month $t-1$

Net return over month t:

$$
R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \%\Delta P_t
$$

Gross return over month t:

$$
1 + R_t = \frac{P_t}{P_{t-1}}
$$

can think of this as: if we have \$1, invested at $t-1$, how much does that dollar grow after 1 month.

Example: One month investment in Microsoft stock. Buy stock at end of month $t-1$ at $P_t - 1 = \$85$ and sell stock at end of next month for $P_t = \$90$. Assuming that Microsoft does not pay a dividend between months $t-1$ and t, the one-month simple net and gross returns are:

$$
\begin{aligned}
R_t &= \frac{90-85}{85} = \frac{90}{85} - 1 = 1.0588 - 1 \\
&= 0.588
\end{aligned}
$$

$$
1 + R_t = 1.0588
$$

The one month investment in Microsoft yielded a 5.88% per month return. (unusual, maybe in the 90s)

Multi-period Returns

- **Simple two-month return**:
  Formula: What's the percentage rate of return over 2 months.

- **Relationship to one month returns**:
  What's the relationship between 2 month return and two one month returns? Geometric average of two one period returns.

Simple k-month Return We can do the above if our investment horizon is k months.

$$1 + R_t(k) = \prod_{j=0}^{k-1} (1 + R_{t-j})$$

1.2.2 Portfolio Returns

Portfolio - a clutch of assets that you want to invest in, you spread spread welath across those assets.

Case with 2 assets. We define our portfolio by how much of our wealth we invest in each of the securities.

When we study the portfolios, we generally assume that we exhaust all of our wealth amongst the assets that we have.

Errata:

x_B= share of \$V invested in B; \$V $\times$ x_B = \$ amount

Now we think about what is the rate of return on this portfolio over a given holding period and how does that rate of return relate to the rates of return on the individual securities A and B.

We start with the initial investment \$V and then the portfolio grows to $(1 + R_{p,t})$, $R_{p,t}$ being the portfolio's rate of return.

Portfolio shares add up to 100%, hence the reduction to 1 (line 3).

Rates of return on portfolios are linear combination of returns on securities.

msft/sbux Example:

We made money on msft, but lost on sbux

In general For n assets, weighted average across all the portfolio weights times the rates of return on individual securities.

We can simplify this using linear algebra, portfolio weights and returns, x_i and $R_{i,t}$ can be vectors and $x_i \times R_{i,t}$ for weighted average.

1.2.3 Dividends

Adjusting for Dividends If we buy stock in a publicly held company, it decides to share some of its profits during a meeting held every 3 months. Some types of stocks pay dividends regularly, public utilities, established companies. Stocks like Google typically don't share dividends, just reinvest into growth. You buy shares for their price appreciation, while with established companies the price may stay stationary and you make money from the dividend payment.

D_t =divident payment between months $t-1$ and t.

Capital gain return + divident yield (gross)

$$R_t^{total} = \frac{P_t + D_t - P_{t-1}}{P_{t-1}} = \frac{P_t - P_{t-1}}{P_{t-1}} + \frac{D_t}{P_{t-1}}$$

$$1 + R_t^{total} = \frac{P_t + D_t}{P_{t-1}}$$

We can brake the total return, R_t^{total} into:

- Pure percentage change in price called **capital gains**

$$\frac{P_t - P_{t-1}}{P_{t-1}}$$

- Divident payment divided by initial price called **dividend yield**

$$\frac{D_t}{P_{t-1}}$$

- The total return is: capital gain return + dividend yield

When you have a company like Google which doesn't have dividend, the return is essentially pure capital gain

1.2.4 Inflation

Adjusting for Inflation Accounting for growth in the general level of prices. We're computing the real rage of return as opposed to the nominal rate of return.

CPI_t - consumer price index, aggregated index of a typical basket of goods a household purchases. Published every month.

Inflation π_t - can be defined as percentage change in the CPI, $\%\Delta CPI_t$

Real rate of return, R_t^{Real} is related to the nominal rate R_t and inflation π_t.

$$R_t^{Real} = \frac{1 + R_t}{1 + \pi_t} - 1$$

Example:

From a practical point of view if the rate of return is not too big and the inflation rate is not too big,

$$R_t^{Real} \approx R_t - \pi_t$$

1.2.5 Annualizing Returns

Rates of returns are often expressed as an annual rate of return.

Problem: how do you convert a monthly rate of return to an annual rate of return, if say, you have an investment only for one month.

1.3 Continuously Compounded Returns

1.3.1 Continuously Compounded Returns

Simple returns are based on the percentage change in price. One of their features is that if we look at the relationship between single-period returns and multi-period returns, it was multiplicative. So one year return was e.g. a geometric average of one month returns. From the point of view of modeling, it's better if things are additive. Continuously compounded returns give us this property.

r_t- Continuously Compounded Return, continuously compounded growth rate of prices between two periods.

Computational Trick When you see macro data, like GDP plotted over time, it's often a logarithm instead of the price. That's done, because the slope of the logarithm represents the growth rate from one period to the next. So the elasticity in this case would be the same as the continuously compounded growth rate between two time periods. Same for prices, slope of the log-price graph would be the continuously compounded return between two time periods.

Example:

Why is $r_t < R_t$? With R_t we're compounding once over a period, with r_t we have continuously compounding returns and rate of return may be smaller and we're getting interest on the interest.

Multi-period Returns One of the features of r_t is the relationship between single-period continuously compounding returns and multi-period continuously compounding returns.

We can ask what's the relation between two-month CCR and two one-month CCRs. Two-month CCR is the sum of two one-month CCRs.

1.3.2 CC Portfolio Returns and Inflation

Portfolio Returns Portfolio return is not a weighted average of CCR of each of the securities, not additive in continuously compounding space.

Adjusting for inflation Becomes easier.

1.4 Excel Examples

1.4.1 Getting Financial Data from Yahoo!

Bid and Ask - dealers hold and inventory in stock, when you buy from a dealer you buy at the ask price, when you sell to the dealer you sell at the bid price.

Bid-Ask spread - represents profits to the dealer on a round-trip transaction.

Average volume - number of shares that were transacted

Adjusted close - closing price adjusted for dividends and stock splits. If msft paid a dividend today, it would get added to the adjusted close $P_t + D_t$, whereas the Closing Price is just P_t.

Stock split - that means every share holder of a company receives an additional share. Now there's twice as many shares of stock outstanding. Correspondingly, the price of that stock on that day gets cut in half. So if a stock is trading at $100 and then suddenly at $50, that may mean there was a 2 for 1 stock split. There's not going to be a jump in the adjusted closing price because of that, they're split-adjusted.

1.4.2 Return Calculations

1.4.3 Growth of $1

Start with $1

in month:

$1 \to \$1 \cdot (1 + R_1)$

$2 \to (1 + R_1) \cdot (1 + R_2)$

$t \to (1 + R_1) \cdot (1 + R_2) \cdot \cdots \cdot (1 + R_t)$

in excel, data sheet

FV: I3=(1+D3)=93.4%

I4=I3*(1+D4)=1.138554

etc

This plots the **equity curve** that shows what the dollar grew to every month.