Stability Analysis:

Forward Time Centered Space:

$\varphi_j^{n+1} = \varphi_j^n + \mathcal{J} f_j^n$ (1) $\quad f_j^{n+1} = f_j^n + \frac{\gamma}{2\Delta}(g_{j+1}^n - g_{j-1}^n) - \mathcal{J}\beta^2\varphi_j^n$ (2) $\quad g_j^{n+1} = g_j^n + \frac{\gamma}{2\Delta}(f_{j+1}^n - f_{j-1}^n)$ (3)

let $f^n = S^n e^{ik\Delta}$ and $g^n = \alpha S^n e^{ik\Delta}$ and $\lambda = \frac{\gamma}{\Delta}$ and $\varphi^n = S^n e^{ik\Delta}$

Substituting into (2) gives:

$$S^{n+1} e^{ik\Delta} = S^n e^{ik\Delta} + \frac{\lambda}{2}(\alpha S^n e^{2ik\Delta} - \alpha S^n) - \mathcal{J}\beta^2 S^n e^{ik\Delta}$$

$$S = 1 + \frac{\lambda\alpha}{2}(e^{ik\Delta} - e^{-ik\Delta}) - \mathcal{J}\beta^2$$

$$S = 1 + \lambda\alpha i \sin(k\Delta) - \lambda\Delta\beta^2 \qquad (4)$$

Substituting into (3) gives:

$$\alpha S^{n+1} e^{ikx} = \alpha S^n e^{ik\Delta} + \frac{\lambda}{2}(S^n e^{2ik\Delta} - S^n)$$

$$\alpha S = \alpha + \frac{\lambda}{2}(e^{ik\Delta} - e^{-ik\Delta})$$

$$\alpha S = \alpha + \lambda i \sin(k\Delta) \qquad (5)$$

Solving (5) for α gives:

$$\alpha = \frac{i\lambda \sin(k\Delta)}{S-1} \qquad (6)$$

Substituting (6) into (4) gives

$$S = 1 - \frac{\lambda^2 \sin^2(k\Delta)}{S-1} - \lambda\Delta\beta$$

$$S^2 - (2+\lambda\Delta\beta)S + (1 - \lambda\Delta\beta + \lambda^2\sin^2(k\Delta)) = 0$$

$$S = \frac{2+\lambda\Delta\beta \pm \sqrt{(2+\lambda\Delta\beta)^2 - 4(1-\lambda\Delta\beta+\lambda^2\sin^2(k\Delta))}}{2}$$

$$S = \frac{2+\lambda\Delta\beta \pm \sqrt{4+4\lambda\Delta\beta+\lambda^2\Delta^2\beta^2-4+4\lambda\Delta\beta-4\lambda^2\sin^2(k\Delta)}}{2}$$

$$S = \frac{2+\lambda\Delta\beta \pm \sqrt{8\lambda\Delta\beta+\lambda^2\Delta^2\beta^2-4\lambda^2\sin^2 k\Delta}}{2}$$

let $8\lambda\Delta\beta + \lambda^2\Delta^2\beta^2 - 4\lambda^2\sin^2 k\Delta = D$

If $D<0$, then $|S|^2 > 1$ since

$$|S|^2 = \left|\frac{2+\lambda\Delta\beta \pm i\sqrt{|D|}}{2}\right|^2 = \frac{4+4\lambda\Delta\beta+\lambda^2\Delta^2\beta^2+|D|}{4} = 1 + \frac{1}{4}(4\lambda\Delta\beta + \lambda^2\Delta^2\beta^2 + |D|)$$

If $D>0$ then $|S|^2>1$ since

$$|S|^2 = \left|\frac{2+\lambda\Delta\beta \pm \sqrt{D}}{2}\right| = \begin{cases} \frac{1}{4}(4+4\lambda\Delta\beta+4\sqrt{D}+\lambda^2\Delta^2\beta^2+2\lambda\Delta\beta\sqrt{D}+D) \\ \frac{1}{4}(4+4\lambda\Delta\beta-4\sqrt{D}+\lambda^2\Delta^2\beta^2-2\lambda\Delta\beta\sqrt{D}+D) \end{cases}$$

$$= \begin{cases} 1+\frac{1}{4}(4\lambda\Delta\beta+4\sqrt{D}+\lambda^2\Delta^2\beta^2+2\lambda\Delta\beta\sqrt{D}+D)>1 \\ 1+\frac{1}{4}(4\lambda\Delta\beta-4\sqrt{D}+\lambda^2\Delta^2\beta^2-2\lambda\Delta\beta\sqrt{D}+D) \ \ast \end{cases}$$

$\ast$ When the negative square root is taken, $|S|^2$ can be less than one. However, it is clear from the numerical results that this stability is overpowered by the instability of the positive square root.

If $D=0$ then $|S|^2>1$ since

$$|S|^2 = \left|\frac{2+\lambda\Delta\beta}{2}\right|^2 = \frac{1}{4}(4+4\lambda\Delta\beta+\lambda^2\Delta^2\beta^2) = 1+\frac{1}{4}(4\lambda\Delta\beta+\lambda^2\Delta^2\beta^2)>1$$

So $|S|^2>1 \ \forall \lambda$, and the FTCS scheme is unconditionally unstable.