Introductions

Review of functions

Ex 1

$f(x) \coloneqq x^2 + 6x - 10$

$f(0) = (0)^2 + 6(0) - 10 = -10$

$f(2) = (2)^2 + 6 \cdot (2) = 10 = 4 + 12 - 10 = 6$

$f(-3) = (-3)^2 + 6(-3) - 83 = 9 - 18 - 10 = +9$

$f(a) = (a)^2 + 6(a) - 10 = a^2 + 6a - 10$

$f(a+h) = (a+h)^2 + 6(a+h) - 10$
$= a^2 + 2ah + h^2 + 6a + 6h - 10$

$f(Will) = (Will)^2 + 6Will - 10$
$= Will^2 + 6Will - 10$

Ex 2 Ball dropped from drone
$y = 16x^2$ is distance ball has fallen x seconds

has fallen x seconds
after having been dropped.
How far has ball dropped
after 6 seconds?

WANT: distance fallen (y)
after 6 seconds (x)

$y(6)$ = dist fallen in 6 secs
$= 16(6)^2 = 16(36)$
$= 576$

So ball has fallen 576 feet in first 6 seconds.

Problem How fast is the ball traveling after 6 seconds?

(Reword) What is velocity of ball at 6 secs?

Attempt #1: velocity at 6 seconds

$= \dfrac{y(7) - y(6)}{\;}$ change in dist

$$= \frac{y(7) - y(6)}{7-6} \quad \frac{\text{change in dist}}{\text{change in time}}$$

NO!

This is the average velocity of the ball from 6-7 secs after being dropped

We want the velocity of the ball at exactly 6 seconds, and not around 6 seconds

Attempt #2: velocity $= \frac{\text{change in dist}}{\text{change in time}}$

$$\frac{y(6.5) - y(6)}{6.5 - 6} \quad \frac{\text{change in dist}}{\text{change in time}}$$

Again NO!

This again an average velocity over the interval of time from 6-6.5 seconds

6 0.5 seconds

We want an instantaneous velocity.

A velocity at the very instant of 6 seconds after the ball is released.

So... $\text{instantaneous velocity @ } 6s = \dfrac{y(6) - y(6)}{6 - 6}$

Trouble is this is $\frac{0}{0}$ an undefined quantity.

What is going wrong?

Plugging in a fixed quantity equal to or near 6 into

$$\frac{y(x) - y(6)}{x - 6}$$

doesn't get us the answer we want

What if we allow the value of

What if we allow the value of x to change and move close to 6 rather than be a single fixed value?

Let's write $x \to 6$ to say that the value of x gets as close to 6 as possible. What happens now?

$$\frac{y(x)-y(6)}{x-6} = \frac{16x^2 - 16(6)^2}{x-6}$$

$$= \frac{16x^2-576}{x-6} = \frac{16(x^2-36)}{x-6}$$

$$= \frac{16(x-6)(x+6)}{x-6} = 16(x+6)$$

provided $x \neq 6$

If $x=6$, we'd be back to our $\frac{0}{0}$ problem. So if x gets close to, but not equal to 0, we have

$y(x)-y(\ )$

but not equal to 6, we have

$$\frac{y(x)-y(6)}{x-6} = 16(x+6) \xrightarrow[x \to 6]{\text{as}} 16(6+6) = 192$$

This concept of looking at the value of a function as $x \to \#$ we will call a limit. It will be important to us only because we will be able to what's the instantaneous rate of change (velocity) of a function as x

Introduction to Limits

Monday, August 24, 2015 4:12 PM

Notation: $\lim\limits_{x \to c} f(x) = L$ or $f(x) \to L$ as $x \to c$

meaning

as x gets close to the value c

$f(x)$ gets close to the value L

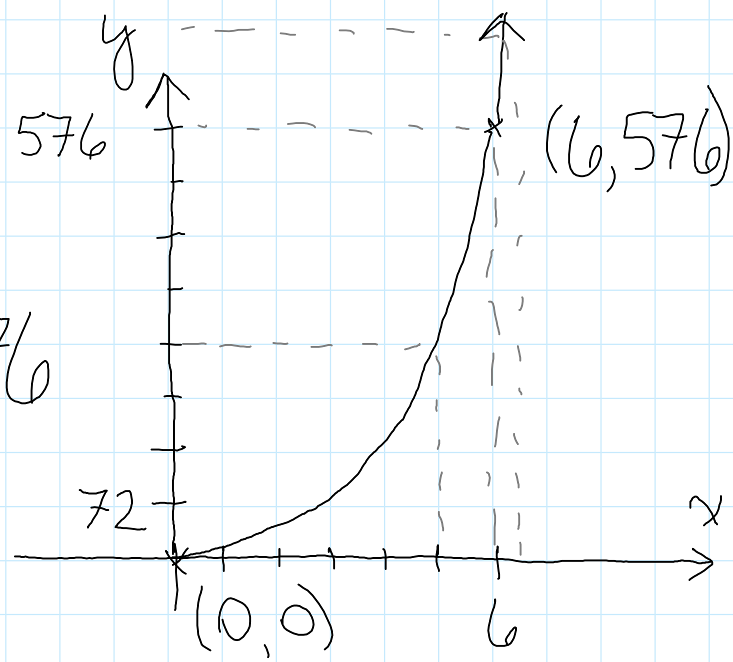
read

"limit of $f(x)$, as x approaches c, is L"

Limits Graphically

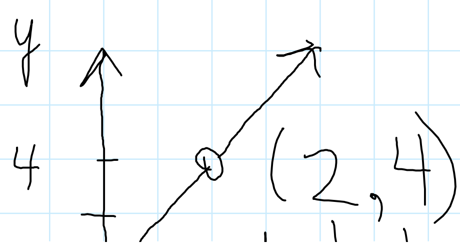
Ex limit exists

$$\lim_{x \to 6} 16x^2 = 576$$

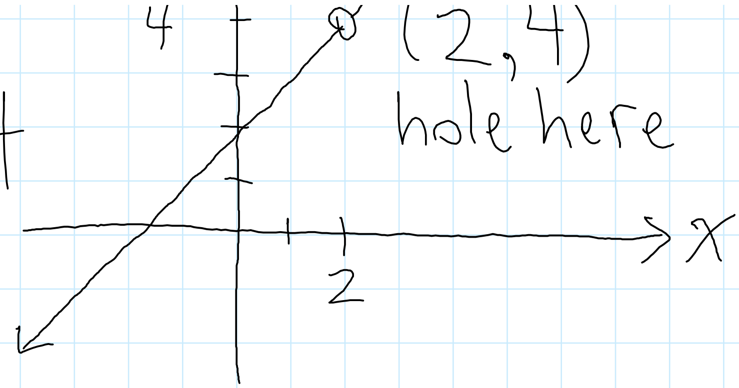


Ex limit exists

lim x^2-4



$$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$$

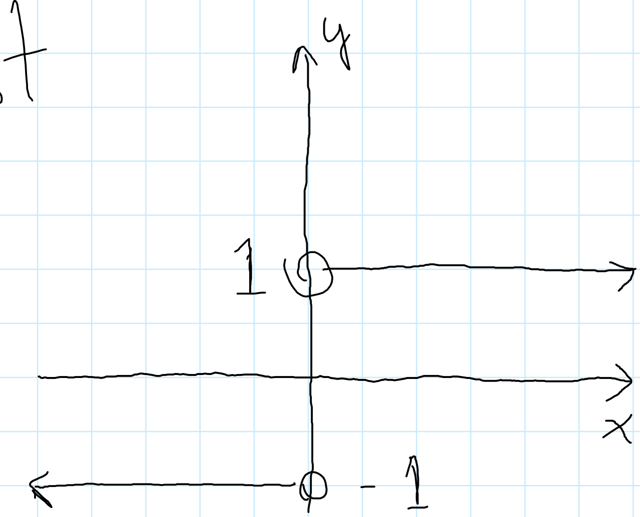


$\frac{x^2-4}{x-2}$ doesn't exist for $x=2$

but limit exists as x gets close to 2

Ex limit does not exist

$\lim_{x\to 0} \frac{|x|}{x}$ does not exist



Left handed (sided) limit

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = -1$$

indicates left hand limit

Right handed limit

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{+x}{x} = \lim_{x \to 0^+} +1 = 1$$

right hand side

So as x approaches 0 from the right hand side, $|x|/x$ approaches 1

But as x approaches 0 from the left hand side, $|x|/x$ approaches -1

Now what is the (two sided) limit supposed to be? $+1$? -1?

Neither <u>Limit is undefined when left and right sides disagree</u>

Limits Algebraically

1) $\lim_{x \to c} k = k$

2) $\lim_{x \to c} x = c$

3&4) $\lim_{x \to c} (f(x) \pm g(x)) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$

$x \to c$ $\quad x \to c \quad x \to c$

one rule for $+$ & one for $-$

5) $\lim\limits_{x \to c} k f(x) = k \lim\limits_{x \to c} f(x)$

6) $\lim\limits_{x \to c} \left(f(x)\, g(x) \right) = \left(\lim\limits_{x \to c} f(x) \right) \left(\lim\limits_{x \to c} g(x) \right)$

7) $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$ $\left(\text{assuming } \lim\limits_{x \to c} g(x) \neq 0 \right)$

8) $\lim\limits_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to c} f(x)}$

$\left(\text{assuming } \lim\limits_{x \to c} f(x) \geq 0 \right)$

Ex ∘ $\lim\limits_{x \to 2} (x^3 - 5x - 1)$

$= \lim\limits_{x \to 2} x^3 - \lim\limits_{x \to 2} 5x - \lim\limits_{x \to 2} 1$

$= \left(\lim\limits_{x \to 2} x \right)^3 - 5 \lim\limits_{x \to 2} x - 1$

$= (2)^3 - 5(2) - 1$

$$= \frac{8 - 10 - 1}{1}$$

$$= -3$$

- $\lim_{x \to -1} \sqrt{2x^2+3} = \sqrt{\lim_{x \to -1} (2x^2+3)}$

$$= \sqrt{2\left(\lim_{x \to -1} x\right)^2 + 3} = \sqrt{2(-1)^2+3}$$

$$= \sqrt{5}$$

Ex Limits of piecewise functions

$$f(x) := \begin{cases} \dfrac{x}{x+3} & \text{if } x \geq 0 \\ \dfrac{x}{x-3} & \text{if } x < 0 \end{cases}$$

a) $\lim_{x \to 0^-} f(x)$

$x \to 0^-$ means $x < 0$

for $x < 0$, $f(x) = \dfrac{x}{x-3}$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x}{x-3}$$

$$= \lim_{x \to 0^-} x$$

$$= \frac{\quad}{\lim_{x \to 0^-} (x-3)}$$

$$= \frac{0}{0-3}$$

$$= 0$$

b) $\lim_{x \to 0^+} f(x)$

$x \to 0^+$ means $x > 0$

for $x > 0$ $f(x) = \frac{x}{x+3}$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{x+3}$$

$$= \frac{0}{0+3}$$

$$= 0$$

c) $\lim_{x \to 0} f(x)$

both $\lim_{x \to 0^-} f(x)$ & $\lim_{x \to 0^+} f(x)$

agree (both $= 0$)

So $\lim_{x \to 0} f(x) = 0$

Practically how to evaluate limits

$\lim_{x \to c} f(x)$

1. plug-in $x = c$
   find $f(c)$
2. if you get a # without running into 0's in denominators or -#s under $\sqrt{}$'s, that # you got from $f(c)$ is the value of the limit
3. if you get $\frac{0}{0}$, you have more work to do (examples follow).
   otherwise there does not exist a value for the limit.

Ex $\frac{0}{0}$ in limits

- $\lim \dfrac{x+3}{x^2+3x}$

$x \to -3$

try plugging in $x = -3$

$$\frac{(-3)+3}{(-3)^2+3(-3)} = \frac{0}{9-9} = \frac{0}{0}$$

problem & must work harder

$$\lim_{x \to -3} \frac{x+3}{x^2+3x} = \lim_{x \to -3} \frac{\cancel{x+3}}{x\cancel{(x+3)}}$$ we try factoring

$$= \lim_{x \to -3} \frac{1}{x}$$ cancel

$$\boxed{= \frac{1}{-3}}$$ try plugging in again

∘ $$\lim_{x \to 2} \frac{x^2+x-6}{x-2}$$

try plugging in $$\frac{(2)^2+(2)-6}{2-2}$$

$$= \frac{4+2-6}{0} = \frac{0}{0}$$

nope

$$\lim_{x \to 2} \frac{x^2+x-6}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+3)}{\cancel{x-2}}$$ factor

$$= \lim_{x \to 2} x+3$$ cancel

$$= (2)+3$$ try plugging in again

$$= 5$$

- $$\lim_{x \to -1} \frac{3x^2+2x-1}{x^2+3x+2}$$

trying $$\left\{ \frac{3(-1)^2+2(-1)-1}{(-1)^2+3(-1)+2} \right\}$$

$$= \left\{ \frac{3-2-1}{1-3+2} \right\} = \left\{ \frac{0}{0} \right\}$$

$$\lim_{x \to -1} \frac{3x^2+2x-1}{x^2+3x+2} = \lim_{x \to -1} \frac{(3x-1)\cancel{(x+1)}}{(x+2)\cancel{(x+1)}}$$

$$= \lim_{x \to -1} \frac{3x-1}{x+2}$$

$$= \left\{ \frac{3(-1)-1}{(-1)+2} \right\}$$ try again

$$= \left\{ \frac{-3-1}{1} \right\} = \left\{ \frac{-4}{1} \right\}$$

$$\boxed{= -4}$$ no problems our answer

- $$\lim \frac{x^2+25}{\ldots} = \left\{ \frac{(-5)^2+25}{\ldots} \right\}$$

$$\lim_{x \to -5} \frac{x^2+25}{x+5} = \left\{ \frac{(-5)^2+25}{-5+5} \right\}$$

$$= \left\{ \frac{25+25}{0} \right\} = \left\{ \frac{50}{0} \right\}$$

only $\frac{0}{0}$ means more work $50/0$ means

$$\lim_{x \to -5} \frac{x^2+25}{x+5}$$ Does not exist