College Algebra Lecture Notes

2016 Spring

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Contents

List of Figures

List of Tables

Part I.

Equations and Inequalities

1. Solutions Bound in Sets

New Things

Definitions

Set a collection of items

Element of set A
An item inside the set A. If x is an element of the set A, then we write $x \in A$. Otherwise, we write $x \notin A$ to say that the item x is not an element of A.

Empty Set
the unique set which has nothing inside it, i.e. no elements. We write $\emptyset$ to mean the empty set.

Real Numbers
the set which contains all integers, fractions, and irrational numbers. We write $\mathbb{R}$ to mean the real numbers, or more properly, the set of real numbers.

Notation

- Roster Method for describing a set
- Set-Builder Method for describing a set

Sets Contain Solutions

Definition

A **set** is simply a container, albeit an abstract container, of various items. When we talk about a specific set, lets just use an arbitrary set A as an example, the items contained inside this set A are called the **elements** of this set. When talking about a specific item, say x for example, we will write $x \in A$ to mean that the item x is an element of the set A. Likewise, we write $x \notin A$ to mean that x is not an element of A.

Naming Sets

We need to have some examples of the kind of sets that we'll be working with in this class. There are two ways of naming, or describing, a set.

Roster Method

Here is one such example.

$$\{2, 3, 5, 7\}$$

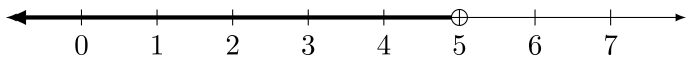
This is the set that contains the numbers 2, 3, 5, and 7, and no other elements. The method of writing a set in this way is known as the **roster method**. It's most convenient when dealing with sets that have a small number of elements.

Set-Builder Method

But we will be dealing with sets that have infinitely many members. For these sets, we describe them using **Set-Builder method**.

$$\{x \mid x < 5\}\,\text{read: the set of all element } x \text{ such that } x \text{ is less than 5}$$

It starts, and ends, with the same curly braces $\{\}$, describes the type of elements it contains, e.g. a single number represented by x, and gives a condition which determine whether or not an item is a member in the set, e.g. $x < 5$. For our set above, if any single number, represented by x, is less than 5, then x is an element of this. And every element of our set must be numbers less than 5. Another way to represent this set is with a number line.



Sets in College Algebra

The other common types of sets we will encounter are the set of all real numbers $\mathbb{R}$, and the empty set $\emptyset$, which is the set that contains nothing at all.

2. Linear Equations and Inequalities

New Things

Definitions

Linear Equation
An equation which can be put in the form $ax + b = 0$.

Contradiction
An equation or inequality which has no solution, like $x + 1 = x$, or is outright false, like $0 = 1$.

Identity
An equation or inequality which is true regardless of the value of the variable(s), for example $(x + 1)^2 - (x - 1)^2 = 4x$.

Conditional
An equation or inequality which is true when the variables have certain values, and is false otherwise.

Notations

Interval Notation
A method of denoting an interval using '(', '[', ')', and ']' to indicate the inclusion '[]' or exclusion '()' of endpoints. '$\cup$' is used to join together intervals when needed.

Linear Equations

Examples

Example 2-1

The cost, in dollars, to rent a storage unit for t months is given by $C := 150 + 52.50t$. If you have \$1,200 budgeted for storage, how long can you rent such a unit?

WANT Time t when cost is \$1,200.

KNOW $C = \$1,200$.

$$
\begin{aligned}
C &= 1200 \\
150 + 52.50t &= 1200 \\
52.50t &= 1050 \\
t &= 20
\end{aligned}
$$

So you can rent the unit for up to 20 months while staying within budget.

Example 2-2

How much of a 4% acid solution should be mixed with 200 mL of a 12% to make a 9% solution?

WANT Amount of 4% solution, x, to add.

	4% Solution	**12% Solution**	**9% Solution**
Total Amount	x	200	x + 200
Amount of Acid	$0.04x$	$0.12 \cdot 200$	$0.09\,(x+200)$

$$
\begin{aligned}
0.04x + 0.12 \cdot 200 &= 0.09\,(x + 200) \\
0.04x + 24 &= 0.09x + 18 \\
6 &= 0.05x \\
x &= 120
\end{aligned}
$$

Mix 120 mL of 4% solution with 200 mL of 12% solution to get 320 mL of 9%.

Example 2-3

Two companies, company A and company B, both make yard signs among other things. Company A charges \$1.20 per sign, while B charges a flat fee of \$15.90 for the design on top of \$1.10 per sign. First write out a model for the cost of ordering x signs from both companies. Obviously, Company A is cheaper if you're only going to make a few signs; but if you're going to make a lot of signs then eventually Company B becomes cheaper. How many signs need to be ordered so that the costs from both companies become equal?

$$
\begin{aligned}
A &= 1.2x \\
B &= 1.1x + 15.9
\end{aligned}
$$

WANT Number of signs needed, x, for $A = B$.

$$\begin{aligned}
A &= B \\
1.2x &= 1.1x + 15.9 \\
0.1x &= 15.9 \\
x &= 159
\end{aligned}$$

So ordering up to 159 signs, company A is cheaper. But when ordering over 159 signs, company B is cheaper.

Contradictions and Identities

An equation involving the variable x that is true when x is one value but false when x is another value is called a **conditional equation**. There are equations which are true regardless of what the value of x is. We call these kinds of equations **identities** or tautologies. Finally, the third type of equation is one which is never true for any value of x. These are **contradictions**.

Example 2-4

Identify each equation as either as being conditional, a contradiction, or an identity. Also find the solution set of each.

1. $4x + 1 - x = 6x - 2$
2. $2(-5x - 1) = 2x - 12x + 6$
3. $2(3x - 1) = 6(x + 1) - 8$

1.

$$\begin{aligned}
4x + 1 - x &= 6x - 2 \\
3x + 1 &= 6x - 2 \\
3 &= 3x \\
1 &= x \\
x &= 1
\end{aligned}$$

If $x = 1$, then the original equation will be true. So this is a conditional equation whose solution set is $\{1\}$.

2.

$$\begin{aligned}
2(-5x - 1) &= 2x - 12x + 6 \\
-10x - 2 &= -10x + 6 \\
0 &= 8
\end{aligned}$$

This is clearly false. So the original equation must always be false, regardless of the value of x. This is a contraction whose solution set is the empty set $\emptyset$.

3.

$$\begin{aligned} 2\,(3x-1) &= 6\,(x+1)-8 \\ 6x-2 &= 6x+6-8 \\ 6x-2 &= 6x-2 \\ 0 &= 0 \end{aligned}$$

This is obviously true. The original equation must always be true. So we have an identity with the solution set being the set of all real numbers $\mathbb{R}$.

Linear Inequalties

Interval Notation

Most of the sets that we work with in this class will be written in a style known as interval notations. As the name suggests, this notation represents intervals on the number line. There are 9 basic types of intervals which are listed in table 2.1 along with their meaning.

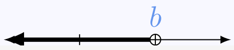
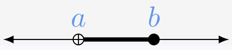
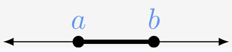
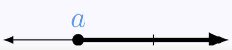
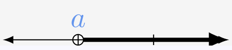
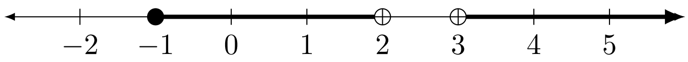
Interval Notation	Set Notation	Number Line	Meaning
$(-\infty, b)$	$\{x \mid x < b\}$		all #s $< b$
$(-\infty, b]$	$\{x \mid x \leq b\}$		all #s $\leq b$
(a, b)	$\{x \mid a < x < b\}$		all #s btwn a & b, excl. both endpoints
$(a, b]$	$\{x \mid a < x \leq b\}$		all #s btwn a & b, excl. a
$[a, b)$	$\{x \mid a \leq x < b\}$		all #s btwn a & b excl. b
$[a, b]$	$\{x \mid a \leq x \leq b\}$		all #s btwn a & b
$[a, +\infty)$	$\{x \mid x \geq a\}$		all numbers $\geq a$
$(a, +\infty)$	$\{x \mid x > a\}$		all numbers $> a$
$(-\infty, +\infty)$	$\mathbb{R}$		all real numbers

Table 2.1.: Table of Interval Notation

Joining Intervals Together

We will also want to join two intervals together. To join two intervals, say for example $[-1, 2)$ and $(3, +\infty)$, we write $[-1, 2) \cup (3, +\infty)$.



Examples of Linear Inequalities

Example 2-5

Donovan has offers for two different sales jobs. Job A pays a base salary of \$25,000 plus 10% commission on all sales. Job B pays a base salary of \$30,000 plus 8% commission. How much would Donovan have to sell for the salary from Job A to exceed that of Job B?

$$\begin{aligned} A &:= 25000 + 0.10x \\ B &:= 30000 + 0.08x \\ A &> B \\ 25000 + 0.10x &> 30000 + 0.08x \\ 0.02x &> 5000 \\ x &> 250000 \end{aligned}$$

So the solution set is the interval $(250,000, +\infty)$. This means that Donovan must sell at least \$250,000 in order for Job A to pay better than Job B.

Example 2-6

Body temperature usually between 36.5 °Cand 37.5 °C. Given that $C = \frac{5}{9}(F - 32)$ converts from F °Fto C °C, find the typical range for body temperature in °F.

$$36.5 < C < 37.5 \tag{2.1}$$

$$36.5 < \frac{5}{9}(F - 32) < 37.5 \tag{2.2}$$

$$4.5 < \frac{5}{9}F - \frac{160}{9} < 5.5 \tag{2.3}$$

$$\frac{977}{18} < \frac{5}{9}F < \frac{995}{18} \tag{2.4}$$

$$\frac{977}{10} < F < \frac{199}{2} \tag{2.5}$$

$$97.7 < F < 99.5 \tag{2.6}$$

$$\tag{2.7}$$

In Fahrenheit, typical body temperature ranges between 97.5–99.5 °F.

Example 2-7

For what values of x will the following expressions avoid taking square roots of negative numbers.

1. $\sqrt{x-6}$
2. $\sqrt{6-x}$

1.

$$\begin{aligned} x-6 &\geq 0 \\ x &\geq 6 \end{aligned}$$

The values of x that avoid square roots of negative numbers are the values of the interval $[6,+\infty)$.

2.

$$\begin{aligned} 6-x &\geq 0 \\ 6 &\geq x \\ x &\leq 6 \end{aligned}$$

The desired values of x are exactly those in the interval $(-\infty,6]$.

Example 2-8

Solve the following inequalities.

1. $3\,(2x+1)+4 \leq 6x+2$
2. $9+4c > 3\,(c+1)+c$

1.

$$\begin{aligned} 3\,(2x+1)+4 &\leq 6x+2 \\ 6x+3+4 &\leq 6x+2 \\ 7 &\leq 2 \end{aligned}$$

This is a contradiction. So the original inequality is a contradiction and has no solution. The solution set is empty $\emptyset$.

2.

$$\begin{aligned}
9 + 4c &> 3\,(c + 1) + c \\
9 + 4c &> 3c + 3 + c \\
9 + 4c &> 4c + 3 \\
6 &> 0
\end{aligned}$$

This is always true. So the original inequality is an identity and every x-value is a solution. The solution set is all reals $\mathbb{R}$.

3. Complex Numbers

Important Things

Definitions

Real Numbers
All the numbers you know and love/hate like 0, 1, 7/11, $\sqrt{2}$, and π.

i The square root of -1, meaning $i^2 = -1$.

Complex Numbers
All "numbers" of the form $a + bi$ where both a and b are real numbers.

Conjugate of a Complex Number
The conjugate of an arbitrary complex number, like $a + bi$, is $a - bi$. Simply put, you change the sign on the number in front of i.

Procedures

Arithmetic of Complex Numbers
Including when two complex numbers equal each other as well as how to add, subtract, multiply, and divide complex numbers.

Notations

Standard Form of a Complex Number
It is writing a complex number as $a + bi$.

4. Quadratic Equations

New Things

Definitions

Quadratic Equation
An equation which can be put in the form $ax^2 + bx + c = 0$.

Procedures

Factoring a Quadratic

Completing the Square
A process that allows us to rewrite a quadratic from standard form into vertex form.

Quadratic Formula

Notation

Standard Form of a Quadratic $ax^2 + bx + c$

Vertex Form of a Quadratic $a\,(x - h)^2 + k$

Factoring

Completing the Square

Quadratic Formula

5. Polynomial and Rational Equations

New Things

Definitions

Polynomial Expression

Polynomial Equation

Rational Expression

Rational Equation

Procedures

Factoring a Polynomial

Solving a Rational Equation

6. Polynomial and Rational Inequalities

Important Things

Definitions

Polynomial Expression

Rational Expression

Partition Numbers

Polynomial Inequality

Rational Inequalities

Part II.

Functions and Relations

7. Graphs and Circles

New Things

Definitions
Procedures
Notations

8. Functions and Their Graphs

New Things

Definitions
Procedures
Notations

9. Transformations of Functions

New Things

Definitions
Procedures
Notations

10. Arithmetic of Functions

New Things

Definitions
Procedures
Notations

11. Inverse Functions

New Things

Definitions
Procedures
Notations

Part III.

Library of Functions

12. Linear Functions

13. Quadratic Functions

14. Polynomial Functions

15. Dividing Polynomials

16. Zeros of Polynomial Functions

17. Rational Functions

18. Exponential Functions

19. Logarithmic Functions

Part IV.

Linear Systems

20. System of Linear Equations

21. Introduction to Matrices

22. Inconsistent and Dependent Systems

23. Arithmetic with Matrices

24. Inverse Matrices and Matrix Equations