DSC 190

DATA STRUCTURES & ALGORITHMS

Today's Lecture

Last Time

- Time needed for BST operations is proportional to height.
- If tree is balanced, $h = \Theta(\log n)$
- If tree is unbalanced, $h = O(n)$

Today

- How do we ensure that tree is balanced?
- Approach 1: Complicated rules, red-black trees.
- Approach 2: Randomization
- We'll introduce **treaps**.

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Red-Black Trees

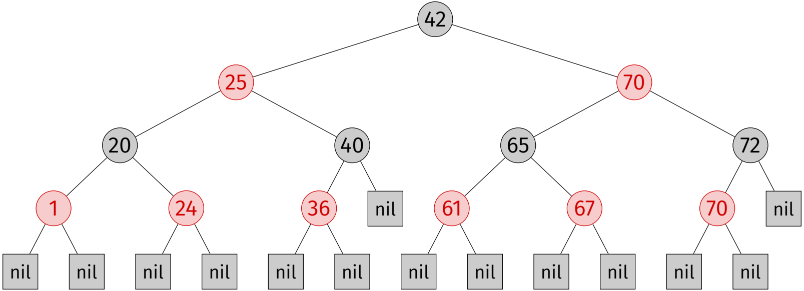
Self-Balancing BSTs

- We wish to ensure that the tree does not become unbalanced.
- Idea: If tree becoming unbalanced, it will balance itself.
- Several strategies, including **red-black** trees and **AVL** trees

Red-Black Trees

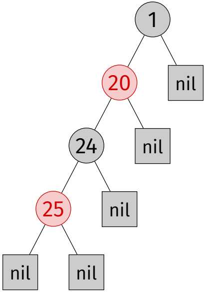
- A **red-black** tree is a BST whose nodes are colored **red** and **black**.
- Leaf nodes are “nil”.
- Must satisfy four additional properties:
  1. The root node is **black**.
  2. Every leaf node is **black**.
  3. If a node is **red**, both child nodes are **black**.
  4. For any node, all paths from the node to a leaf contain the same number of **black** nodes.

Example


42
25
70
20
40
65
72
1
24
36
nil
61
67
70
nil
nil
nil
nil
nil
nil
nil
nil
nil
nil
nil
nil
nil

Example

- This **not** a red-black tree.
  - Violates last property



Claim

If a red-black tree has n internal (non-nil) nodes, then the height is at most $2 \log(n + 1)$.

Proof Intuition[1]

- All paths from root to a leaf are about the same length ($\approx h$).
- Therefore, the tree is close to balanced.
- So height is proportional to $\log n$

[1]Formal proof proceeds by induction.

Non-Modifying Operations

- As a result, the non-modifying operations take $\Theta(\log n)$ time in red-black trees.
  - query
  - minimum/maximum
  - next smallest/largest

- Proof: these take $\Theta(h)$ time in any BST, and in a red-black tree $h = O(\log n)$.

Insertion and Deletion

- Standard BST `.insert` and `.delete` methods preserve BST, but **not** red-black properties.
- Insertion/deletion in a red-black tree is considerably more **complicated**.
- But both take $\Theta(\log n)$ time.

Implementing balanced trees is an exacting task and as a result balanced tree algorithms are rarely implemented except as part of a programming assignment in a data structures class[2].

Pugh, 1990

[2] For computer science majors.

Summary

- For red-black trees, worst cases:

query	$\Theta(\log n)$
minimum/maximum	$\Theta(\log n)$
next largest/smallest	$\Theta(\log n)$
insertion	$\Theta(\log n)$

- But they are **tricky** to implement.

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Randomization to the Rescue

Order Matters

- The structure of a BST depends on insertion order.

Example

- Insert 1,2,3,4,5,6 into BST, in that order.

Example

- Insert 3, 5, 1, 2, 4, 6 into BST, in that order.

Claim

The expected height of a BST built by inserting the keys in random order is $\Theta(\log n)$.

Idea

- To build a BST, take all n keys, shuffle them randomly, then insert.
- No need for Red-Black Trees, right?

Problem

- Usually don't have all the keys right now.
- This is a **dynamic set**, after all.
- The keys come to us in a stream, can't specify order.

Goal

- Design a data structure that **simulates** random insertion order without actually changing the order.

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Treaps

Randomization

- If insertions are in a random order, expected depth of a BST is $\Theta(\log n)$.
- But in **online** operation, we cannot randomize insertion order.
- Now: an elegant data structure simulating random insertion order in online operation.

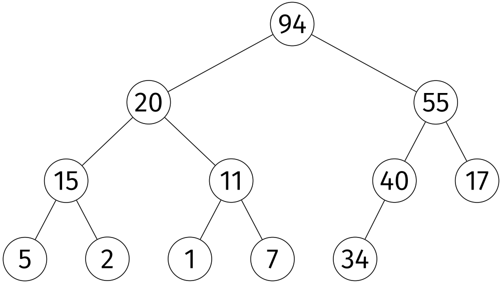
First: Recall Heaps

- A **max heap** is a **binary tree** where:
  - each node has a priority.
  - if y is a child of node x, then

$$\texttt{y.priority} \leq \texttt{x.priority}$$

Example

- This is a max heap:

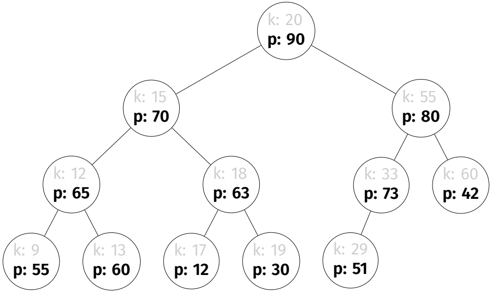


Treaps

- A **treap** is a binary tree in which each node has both a **key** and a **priority**.
- It is a **max heap** w.r.t. its priorities.
- It is a **binary search tree** w.r.t. its keys.

Example

- This is a treap:



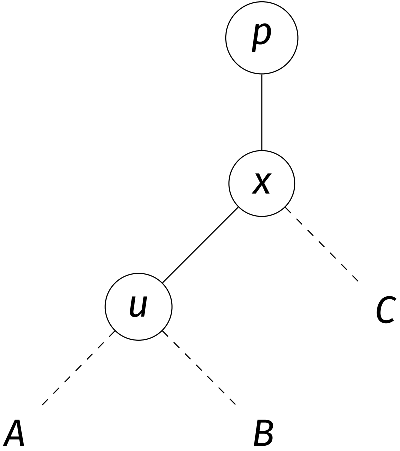
BST Operations

- Because a treap is a BST, querying, finding max/min by key is done the same.
- Insertion and deletion require care to preserve **heap** property.

Insertion

- Find place to insert node as usual.
- While priority of new node is > than parent's:
  - Left rotate new node if it is the right child.
  - Right rotate new node if it is the left child.
- Rotate preserves BST, repeat until heap property satisfied.

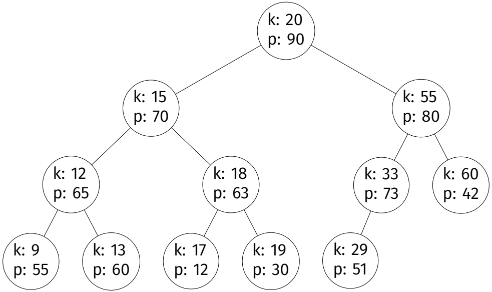
(Right) Rotation



→

Example: Insertion

- Insert key: 16, priority: 65.



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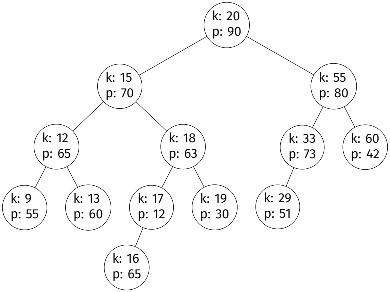

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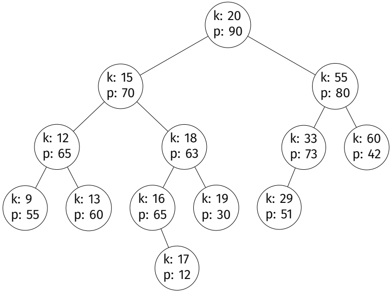

Example: Insertion

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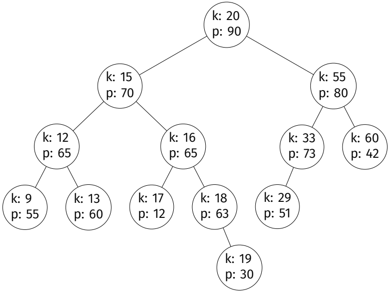

Deletion

- While node is not a leaf:
  - Rotate it with child of highest priority.
- Once it is a leaf, delete it.

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Treap Properties

Good Question

- Is it always possible to build a treap?

Claim

Given any set of (key, priority) pairs, inserting them one-by-one into a treap always results in a valid treap (no matter the insertion order).

Proof Idea

- Start with a treap (possibly empty).
- Inserting new (key, priority) preserves treap:
  - **BST**: rotation preserves BST property
  - **heap**: initially violated, but rotation repeated until it is satisfied

Claim

Given any set of (key, priority) pairs, if both keys and priorities are unique, then the treap is **unique**.

Claim

Corollary: Given any set of (key, priority) pairs, if both keys and priorities are unique, inserting them one-by-one into a treap results in the same treap, no matter the insertion order.

Example

- Insert (3, 40), (1, 20), (10, 50), (6, 30), (5, 100), in that order

Example

- Insert (5, 100), (10, 50), (3, 40), (6, 30), (1, 20), in that order

Proof Idea

- Root node must be node w/ highest priority.
- Root's left (right) child must have highest priority among nodes with key < (>) root key.
- Apply recursively.

Claim

Given any set of (key, priority) pairs, if both keys and priorities are unique, then inserting them one-by-one into a treap (in any order) results in the **same** BST one would obtain by inserting into a BST in decreasing order of priority.

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Randomized Binary Search Trees

Claim

Given any set of keys, if they are inserted into a BST in random order, the result is (almost surely) balanced. The expected height is $\Theta(\log n)$.

Claim

Given any set of (key, priority) pairs, if both keys and priorities are unique, then inserting them one-by-one into a treap (in any order) results in the **same** BST one would obtain by inserting into a BST in decreasing order of priority.

The Idea

- When inserting a node into a treap, generate priority **randomly**.
- The resulting treap will be the same tree as a BST built with nodes randomly ordered according to these priorities.
- It will almost surely be balanced.
- This is called a **randomized binary search tree**[3].

[3] Sometimes people call these treaps

Example

- Insert 1, 2, 3, 4, 5, 6 into a treap, generating priorities randomly.

Time Complexities

- For randomized BSTs, expected times:

query	$\Theta(\log n)$
minimum/maximum	$\Theta(\log n)$
next largest/smallest	$\Theta(\log n)$
insertion	$\Theta(\log n)$

- Worst case times are $\Theta(n)$, but very rare

Comparison to Red-Black Trees

- When compared to red-black trees, randomized BSTs are:
  - same in terms of expected time;
  - perhaps slightly slower in practice;
  - **much** easier to implement/modify.

- Good trade-off for a data scientist!

Priority Hacks

- Several interesting strategies for generating a new node’s priority, beyond simply generating a random number.

Idea #1: Hashing

- Instead of randomly generating a number, hash the key to get priority.
- Works, provided hash function looks random.
- **Careful!** In python, `hash(300) == 300`

Idea #2: "Learning"

- Idea: Frequently-queried items should be near top of tree.

- When an item is queried, update its priority:

  new priority = *max*(old priority, random number)

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Order Statistic Trees

Modifying BSTs

- More than most other data structures, BSTs must be modified to solve unique problems.
- Red-black trees are a pain to modify.
- Treaps/randomized BSTs are easy!

Order Statistics

- Given n numbers, the **kth order statistic** is the kth smallest number in the collection.

Example

```
[99, 42, -77, -12, 101]
```

- 1st order statistic:
- 2nd order statistic:
- 4th order statistic:

Exercise

Some special cases of order statistics go by different names. Can you think of some?

Special Cases

- **Minimum**: 1st order statistic.
- **Maximum**: nth order statistic.
- **Median**: $\lceil n/2 \rceil$th order statistic[4].
- p**th Percentile**: $\lceil \frac{p}{100} \cdot n \rceil$th order statistic.

[4] What if n is even?

Computing Order Statistics

- Quickselect finds any order statistic in linear expected time.
- This is efficient for a static set.
- Inefficient if set is dynamic.

Goal

- Create a dynamic set data structure that supports fast computation of **any** order statistic.

Exercise

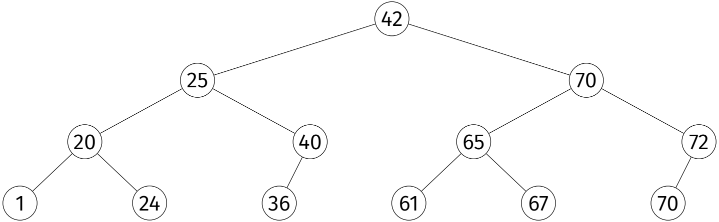
Does the “two heaps” trick from before work?

BST Solution

- For each node, keep attribute `.number_lt`, containing the number of nodes < current node
- Example: for median[5], find node where `.number_lt` $\approx n/2$

[5] Remember, we're assuming keys are unique.

Example: Insert/Delete



Challenge

- `.number_lt` changes when nodes are inserted/deleted
- We must **modify** the code for insertion/deletion
- A pain with R-B tree; easy with treap!