Report on Forced and Damped Oscillations

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1- Introduction

Oscillatory systems play a fundamental role in understanding natural and engineered phenomena, ranging from seismic vibrations in structures to the acoustic properties of musical instruments. Among these systems, forced and damped oscillations stand out as critical models for examining how energy input, resistance, and inherent dynamics interact. This report investigates the behavior of such oscillatory systems, focusing on their theoretical and practical implications.

Forced oscillations arise when an external periodic force sustains motion within a system, counteracting resistive forces. Simultaneously, damping mechanisms dissipate energy, influencing the system's response and stability. These dynamics are often modeled using second-order differential equations, enabling predictions about resonance, amplitude, and phase behavior under varying conditions.

The primary objective of this report is to analyze the effects of external driving forces and damping coefficients on oscillatory behavior. Emphasis is placed on identifying resonance phenomena, evaluating critical damping conditions, and examining transient and steady-state responses through numerical simulations. Additionally, we explore scenarios with and without external forcing to compare their dynamic characteristics and highlight practical implications for engineering applications.

Through mathematical modeling and computational techniques, this study provides insights into oscillatory systems' behavior under different parameters, offering a foundation for understanding real-world systems where resonance and damping are key factors.

2- Problem Description and Mathematical Framework

2.1 Problem Description

Forced and Damped Oscillations (2 People): The aim of this project is the study of the forced mechanical oscillator with a damping term, obeying the following second order ODE:

$$m\frac{\mathrm{d}^2\mathrm{x}}{\mathrm{dt}^2} + b\frac{\mathrm{dx}}{\mathrm{dt}} + m\omega_o^2 x = F_o \cos\omega t$$

with initial conditions $x(t = 0\,\mathrm{s}) = 0$ m and $v(t = 0\,\mathrm{s}) = 1$ m/s. Assume, for instance, the following values for the constants: $b = 2$ kg / s, $\omega_o = 10\ \mathrm{s}^{-1}$, $m = 0.1$ kg; $F_o = 1$ N.

(a) Study the behaviour of the system for different values of ω (eg., $\omega = 0.2\omega_o$, $\omega = 0.4\omega_o$, $\omega = 0.6\omega_o$, $\ldots \omega = 2\omega_o$), for the the range $0 < t < 5T_o$, with T_o being the natural period of the oscillator ($T_o = 2\pi/\omega_o$).

(b) Also consider the cases above with $F_o = 0$ (damped oscillator).

Compare the results with the theoretical solutions.

2.2 Physical and Mathematical Approximations

$$m\,\frac{\mathrm{d}_x^2}{\mathrm{d}t^2} + b\,\frac{dx}{dt} + mw_0^2 x = F_0\cos(wt)$$

Where:

- m: Mass of the oscillator;
- b: Damping coefficient;
- ω_0: Natural angular frequency;
- F_0: Amplitude of the external driving force;
- ω: Driving angular frequency

This equation describes the interaction between inertia, damping, restoring forces, and external periodic forces. The model assumes linear damping, proportional to velocity, and a sinusoidal driving force, simplifying mathematical analysis while preserving essential dynamics. These simplifications are common in physical models to ensure mathematical tractability while capturing essential system behaviors.

A critical parameter in this model is the natural angular frequency, which defines the oscillator's intrinsic tendency to oscillate in the absence of damping and external forces. Additionally, the damping ratio determines the oscillatory regime (underdamped, critically damped, or overdamped). These parameters influence the system's transient and steady-state responses, enabling predictions of resonance, amplitude, and phase shifts under varying conditions.

3- Analytical and Numerical Solutions

3.1 Resolution of the Problem

The problem is analyzed under two distinct scenarios: forced oscillations with external periodic forces and free oscillations without external forces:

- Forced Oscillator: The system is subjected to an external periodic force with varying driving angular frequencies (ω). The critical damping coefficient ($bc = 2m\omega_0$) distinguishes three damping regimes:
    - $b < bc$: Underdamped oscillator
    - $b = bc$: Critically damped oscillator
    - $b > bc$: Overdamped oscillator

3.2 Splitting Method

To facilitate numerical computation, the second-order differential equation is transformed into a system of two first-order equations, separating position and velocity dynamics.:

$$\frac{dy_1}{dt} = y_2 \,;\; \frac{dy_2}{dt} = -\frac{b}{m}y_2 - w_0^2 y_1 + \frac{F_0}{m}\cos(\omega t)$$

where $y_1 = x$ (position) and $y_2 = v$ (velocity).

This transformation enables the use of numerical methods like Euler-Centered schemes for computational analysis. The splitting method adapts well to oscillatory systems by separating velocity and position components, ensuring stability and accuracy in capturing dynamic behaviors. For the forced oscillator scenario, this approach effectively captures transient and steady-state responses across varying driving frequencies.

3.3 Second-Order Euler-Centered Method

The Euler-Centered method was chosen for its simplicity, stability, and suitability for oscillatory systems. While effective for small time steps, larger time steps may reduce accuracy, particularly near resonance conditions..

The Euler-Centered Method updates position and velocity iteratively by finite-difference approximations of derivatives:

$$\begin{aligned}
\bar{x} &= x^n + \frac{h}{2}, \\
\bar{y}_1 &= y_1^n + \frac{h}{2}y_2^n, \\
\bar{y}_2 &= y_2^n + \frac{h}{2}f(t^n, y_1^n, y_2^n), \\
y_1^{n+1} &= y_1^n + h\bar{y}_2, \\
y_2^{n+1} &= y_2^n + hf(t^{n+1}, \bar{y}_1, \bar{y}_2).
\end{aligned}$$

Intermediate values of position and velocity are used to enhance accuracy during updates, making this method suitable for small time steps in oscillatory systems. This method effectively captures resonance peaks and damping effects with high accuracy and computational efficiency, making it ideal for studying oscillatory behavior. For the current problem, the method reliably approximates the oscillator's behavior under varying driving frequencies, aligning well with theoretical expectations.

4 – QUESTION 1

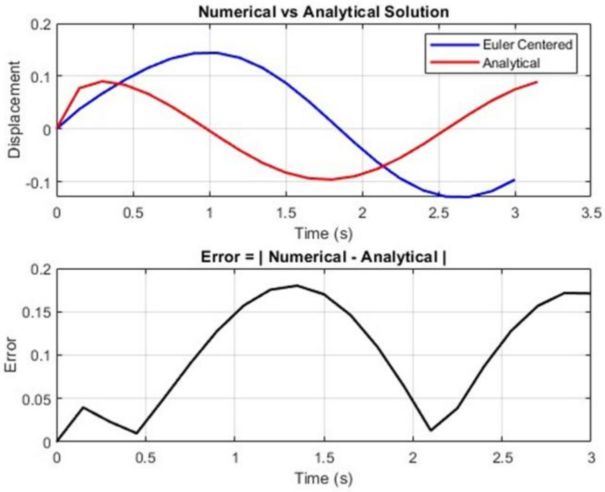
4.1 Euler Centered Code

In order to solve the problem we have developed a Matlab code that implements the second order euler centered method.

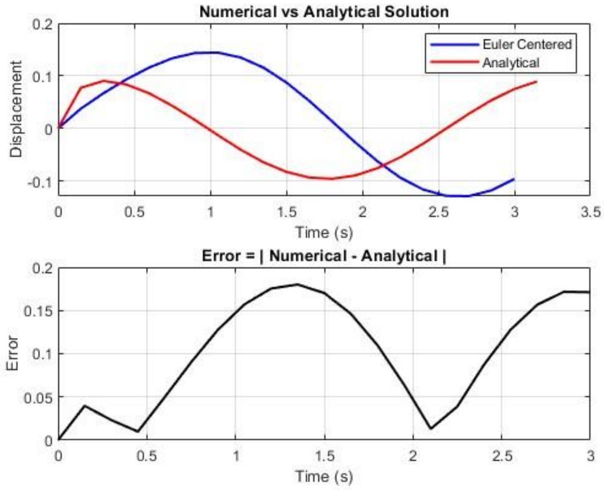
```
%% EULER SECOND ORDER CENTERED


% m*x"=-b*x'-m*w0^2*x+F*cos*w*t
% x(t=0)=0   x'(t=0)=1  b=2   m=0.1   F=1   w0=10   0<t<pi

b=2;
m=0.1;
F=1;
w0=10;
w=0.2*w0;

h=0.001;

f=@(x2,x1,t) (-b / m) * x2 - w0^2 * x1 + (F / m) * cos(w * t);

x1(1)=0;
x2(1)=1;
t(1)=0;
t(2)=h;
i=2;
while t(i)<=pi
  tbar= t(i)+h/2;
  x1_bar=x1(i-1)+h/2*x2(i-1);
  x2_bar=x2(i-1)+h/2*(f(x2(i-1),x1(i-1),t(i-1)));
  x1(i)=x1(i-1)+h*x2_bar;
  x2(i)=x2(i-1)+h*(f(x2_bar,x1_bar,tbar));
  i=i+1;
  t(i)=t(i-1)+h;
 end

plot(t(1:end-1),x1)
```

4.2 Step size and accuracy

Initially, we tested the code with different step-time values and we observed that the accuracy improves significantly with a small decrease in step size. The results are presented here below for ω=0.2*ω0, where the corresponding error decreases from an initial peak value of approximately 0.2 to a peak value on the order of 10^{-4}.

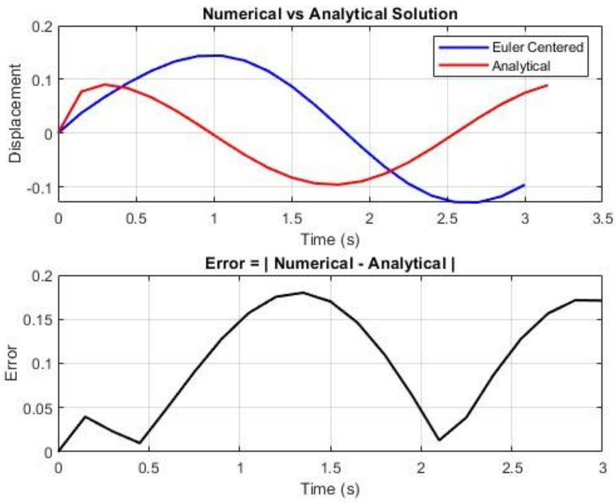


Time step = 0.15

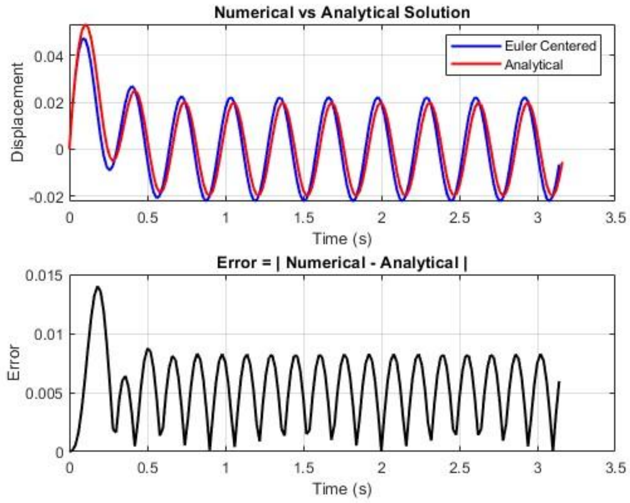


Time step = 0.1

To investigate the impact of different ω values, we also analyzed the behavior with ω=2*ω0. The results showed that for larger ω, the system requires a smaller step size to achieve higher precision. Specifically, with a large step size, the resulting curve is not smooth because only a few points are evaluated to represent a graph with several oscillations.



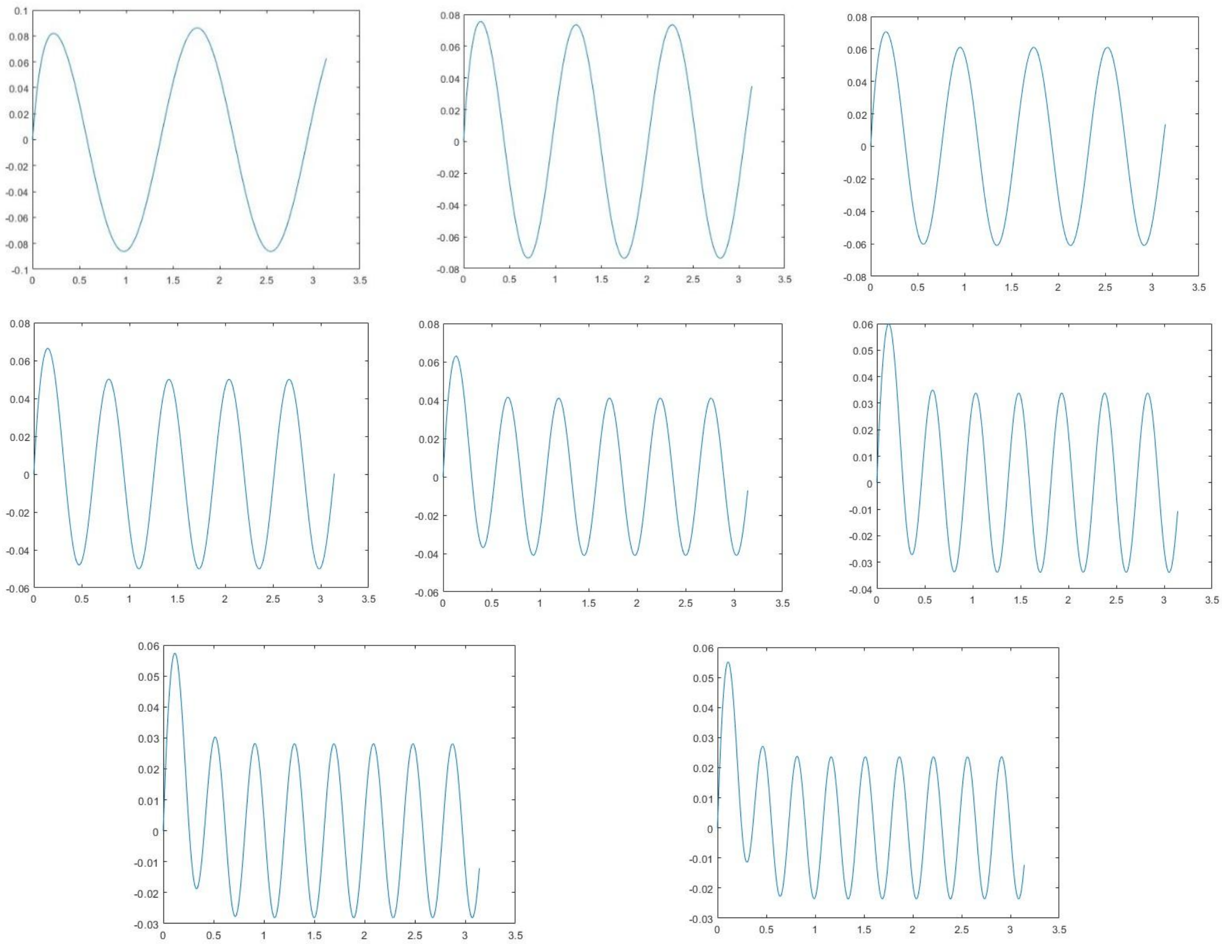
Time step = 0.1



Time step = 0.02

4.3 Oscillatory behavior with varying 'ω'

The graphs below display the system's behavior for all the ω values :



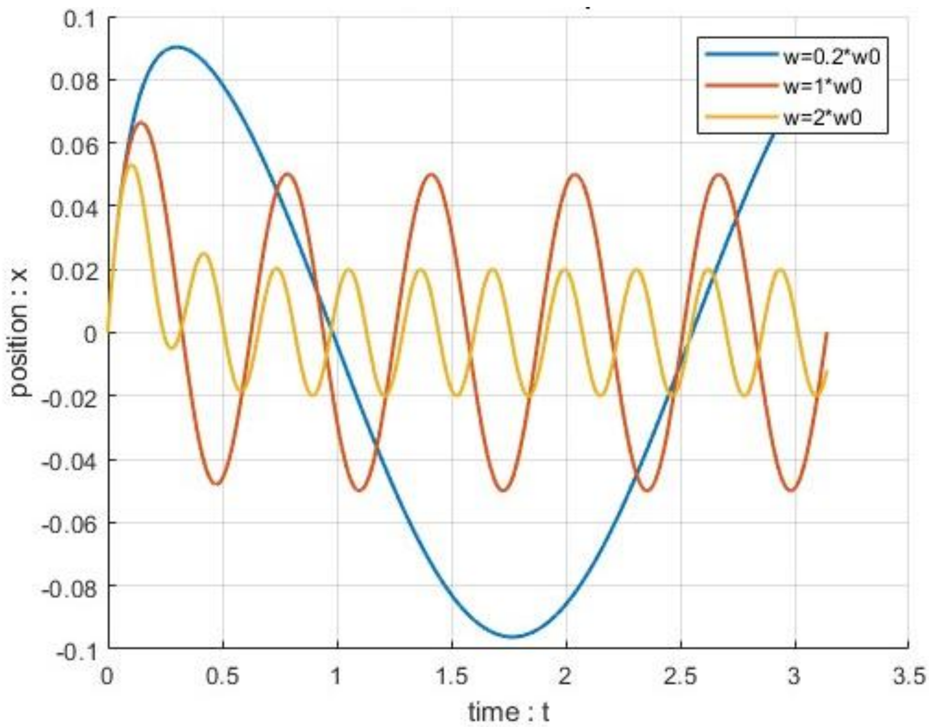
Graphs in the corresponding order : ω=0.4ω0, ω=0.6ω0, ω=0.8ω0, ω=ω0, ω=1.2ω0, ω=1.4ω0, ω=1.6ω0, ω=1.8ω0

From the graphs we can see that:

- ***ω≪ω0****: The system oscillates slowly with a long period and is able to closely follow the external force, resulting in a high amplitude response.*
- ***ω=ω0****: Resonance would be expected at this frequency (as will be explained), but due to the high damping coefficient, the amplitude is limited.*

- ***ω≫ω0****: The system oscillates rapidly with a short period but cannot keep pace with the external force, as it changes too quickly. Consequently, the amplitude is significantly reduced.*

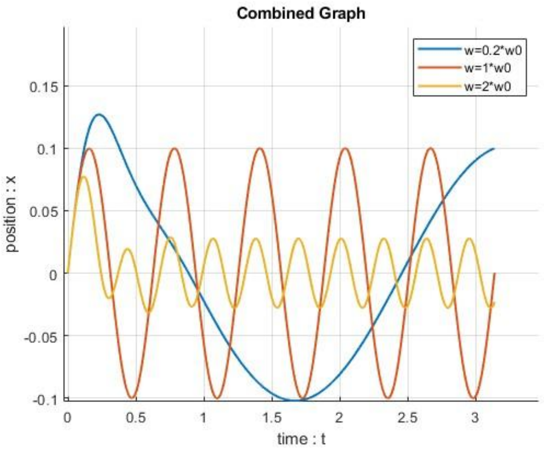
In summary, as ω increases, the response amplitude decreases, the period shortens (corresponding to a higher frequency), and the system oscillates more rapidly. For each increment in ω, the number of oscillations within a fixed time interval increases.



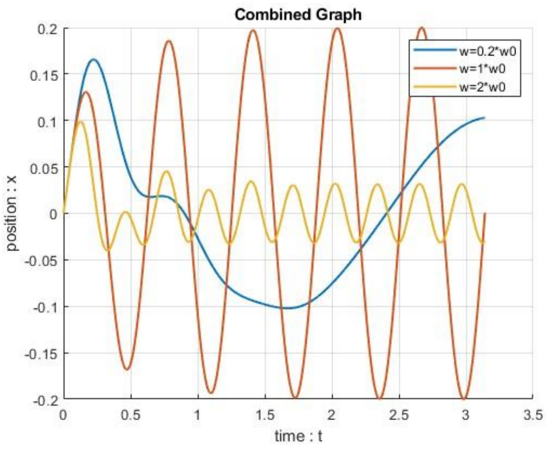
*Combined graph with only the values ω=0.2ω0 (blue), ω=ω0 (red), ω=2*ω0 (yellow)*

4.4 Effect of varying ‘b’

Next, we examined the effect of different b (damping coefficient) values on the system's behavior.



b=1

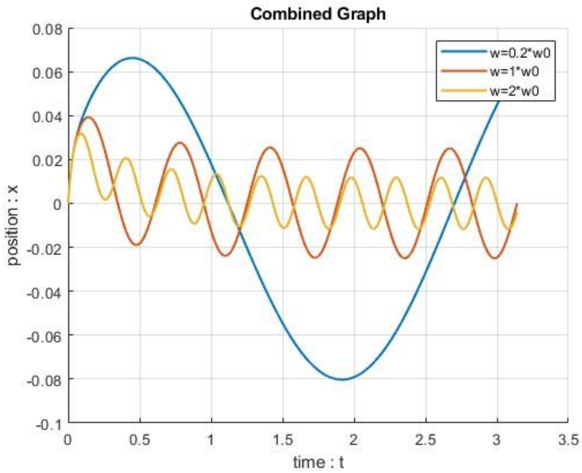


b=0.5

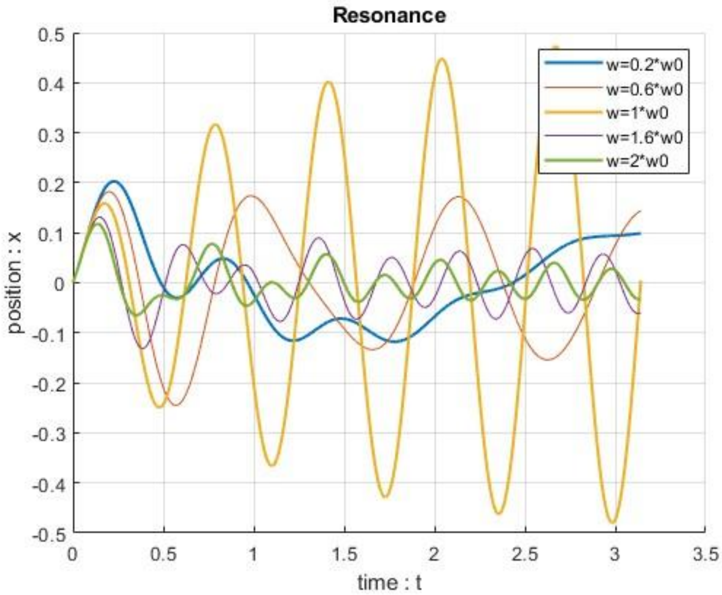
As we mentioned before, for b<<2 (critical damping coefficient), the oscillator is underdamped. In an underdamped system, the resonance effect becomes evident, where the curve corresponding to ω=ω0 exhibits the largest amplitude due to the forcing frequency being in sync with the system's natural frequency.

For b=1, the period remains almost unchanged, but the amplitude increases, particularly in the resonance case (red curve). For b=0.5, the resonance effect becomes more pronounced, with the red curve showing the largest amplitude. This occurs because the resonance effect is most significant when b is substantially below the critical damping coefficient.

b=4



Conversely, as b increases, the amplitude decreases for all ω values, and the red curve no longer exhibits the largest amplitude. With b=4 we can see that the strong damping coefficient dissipates the energy very quickly.



With b=0.2, the resonance effect is clearly visible, with significantly smaller amplitudes for other ω values.

4.5 Resonance effect

Resonance occurs when an oscillating system is subjected to an external force that has a frequency equal to the natural frequency of the system. In this case, the energy transferred to the system adds up with each oscillation, leading to increasingly larger oscillations.

The resonance effect can be illustrated with the example of a swing. When a person on the swing oscillates and another person pushes to increase its velocity, the timing of the push is critical. If the push occurs before the swing starts to slow down and change direction ($\omega \gg \omega_0$) or after the swing changes direction ($\omega \gg \omega_0$), the frequencies are not synchronized and the system cannot efficiently transfer energy from the external force to the swing.

On the other hand, if the push occurs when the swing is about to reverse direction ($\omega = \omega_0$, resonance) system's energy increases progressively with each oscillation.

This principle applies to many physical systems, including bridges, buildings, and musical instruments. While resonance can be beneficial in some cases, it can also pose significant dangers.

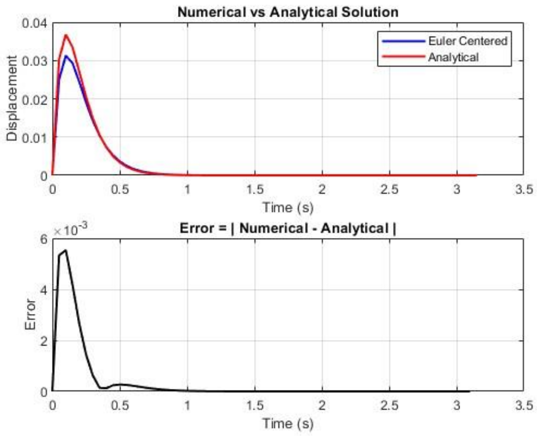
5 – QUESTION 2

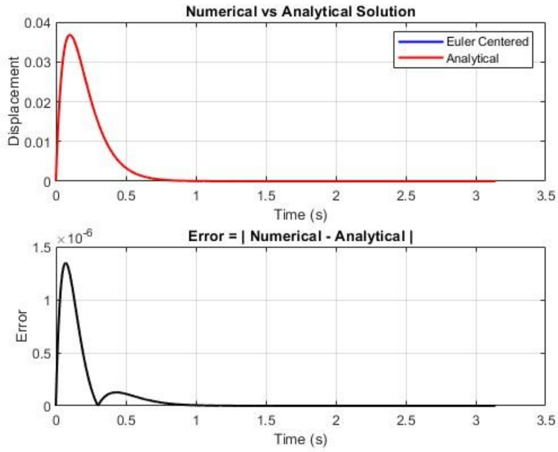
5.1 Physical case

The second question requires analyzing the system in the absence of an external force (F_0=0). How does this scenario differ from the previous case? The elimination of the external force results in a system that naturally converges to the steady state (x=0), with its behavior governed solely by the parameters b, ω_0, and m, independent of ω. Moreover, the absence of the external force implies the elimination of any resonance effects.

5.2 Step time and accuracy

The code employed to analyze the system remains unchanged from the previous case, except for the removal of F_0. By comparing the analytical solution with the generated graphs, it becomes evident that with a step time of 0.05, the resulting curve lacks smoothness, exhibiting an error on the order of 10^{-3}. Reducing the step time to 0.001 produces a smooth curve with a significantly reduced error on the order of10^{-6}.



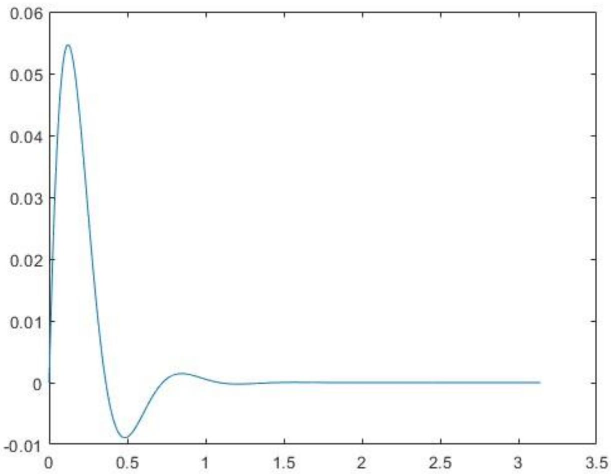


Time step=0.05 ***Time step=0.001***

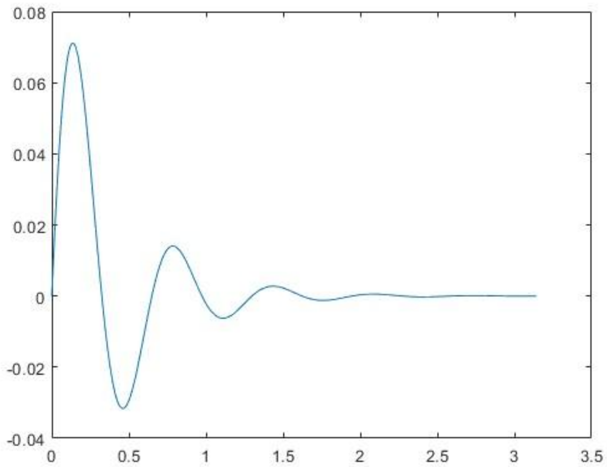
(b=2)

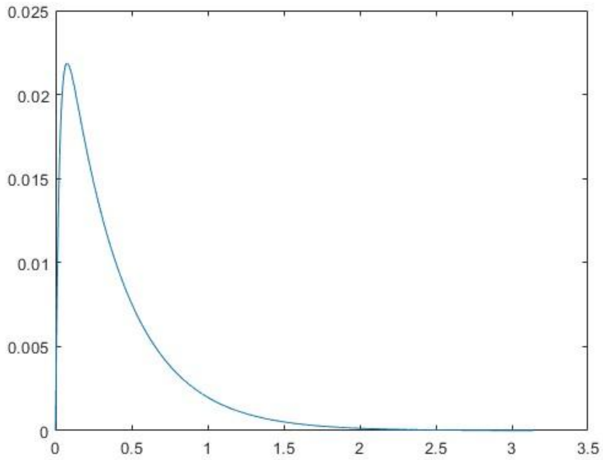
5.2 Comparison with different damping coefficients

b=1



b=0.5





b=4

The graphs indicate that when b=2 (critically damped oscillator), the system transitions to the steady state without oscillating. Comparing this scenario to the graphs obtained for other values of b, the following observations can be made:

- For b=1 and b=0.5 (underdamped oscillators), the system exhibits oscillatory behavior around the equilibrium position before gradually settling over time. Furthermore, as bbb decreases, the amplitude and number of oscillations increase.
- For b=4 (overdamped oscillator), the system returns to equilibrium without oscillations but at a slower rate than in the critically damped case.

Among these scenarios, the critically damped oscillator (b=2) achieves the fastest convergence to equilibrium. Deviations from the critical damping coefficient, whether bbb is lower or higher, result in longer convergence times.

6- Conclusion

This report successfully analyzed the behavior of forced and damped oscillatory systems, focusing on resonance effects, damping regimes, and transient and steady-state responses. Through mathematical modeling and numerical simulations, we examined the influence of external driving forces and damping coefficients on oscillatory behavior.

The results confirmed the theoretical predictions, demonstrating that:

- For frequencies near the natural frequency ($\omega = \omega_0$), resonance effects were observed, with amplitude peaks significantly influenced by the damping coefficient.
- Systems with lower damping coefficients exhibited pronounced resonance, while higher damping coefficients suppressed oscillations, aligning with expected behavior for underdamped, critically damped, and overdamped regimes.
- The Euler-Centered method proved effective in capturing dynamic responses, providing accurate approximations for small time steps.

By comparing scenarios with and without external forcing, we highlighted the transition from oscillatory to stable equilibrium states in the absence of external energy input. These findings emphasize the importance of damping control in engineering applications, such as vibration mitigation in structures.

In conclusion, the study achieved its objectives, providing insights into oscillatory systems and validating theoretical predictions with numerical simulations. Future work may explore higher-order numerical methods to further improve accuracy and efficiency in modeling complex oscillatory behaviors.