All of Statistics

My proposed solutions to the book:

All of Statistics - A Concise Course in Statistical Inference.

I really needed a proper refresher on statistics.

Let me know if you find any mistakes!

Contents

1 Chapter 2 - Random Variables

Exercises

2.1
Claim: $\mathbb{P}(X = x) = F(x^+) - F(x^-)$. (Discrete)
PROOF. By definition of the CDF:

$$F(x^+) = \lim_{z \downarrow x} F(z) = \lim_{z \downarrow x} \mathbb{P}(X \leqslant z), \qquad F(x^-) = \lim_{y \uparrow x} F(y) = \lim_{y \uparrow x} \mathbb{P}(X \leqslant y)$$

(so $y < x$ and $y \to x$, and $x < z$ and $x \leftarrow z$). By the right continuous property, we can deduce that $z > y$ and we can set $z = x$ and $y = x - 1$.

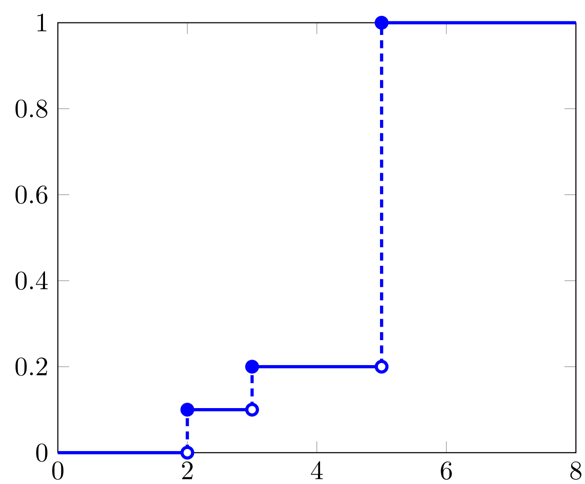
$$\mathbb{P}(X \leqslant x^+) = \mathbb{P}(X \leqslant x) = \mathbb{P}(X = x) + \mathbb{P}(X \leqslant x - 1), \quad \mathbb{P}(X \leqslant x^-) = \mathbb{P}(X \leqslant x - 1)$$

So:

$$\begin{aligned} \mathbb{P}(X = x) &= \mathbb{P}(X = x) + \mathbb{P}(X \leqslant x - 1) - \mathbb{P}(X \leqslant x - 1) \\ &= \mathbb{P}(X \leqslant x) - \mathbb{P}(X \leqslant x - 1) \\ &= \mathbb{P}(X \leqslant x^+) - \mathbb{P}(X \leqslant x^-) \\ &= F(x^+) - F(x^-) \end{aligned}$$

□

2.2
Let X be such that $\mathbb{P}(X = 2) = \mathbb{P}(X = 3) = 1/10$ and $\mathbb{P}(X = 5) = 8/10$. Here is a plot of the CDF.



By reading the plot, we can see that:

$$\mathbb{P}(2 < X \leqslant 4.8) = F(4.8) - F(2) = 2/10 - 1/10 = 1/10$$

$$\mathbb{P}(2 \leqslant X \leqslant 4.8) = F(4.8) = 2/10$$

2.3

Lemma 2.15 Let F be the CDF for a random variable X. Then:

1. $\mathbb{P}(X = x) = F(x) - F(x^-)$
2. $\mathbb{P}(x < X \leqslant y) = F(y) - F(x)$
3. $\mathbb{P}(X > x) = 1 - F(x)$
4. If X is continuous, then
$$F(b) - F(a) = \mathbb{P}(a < X < b) = \mathbb{P}(a \leqslant X < b) = \mathbb{P}(a < X \leqslant b) = \mathbb{P}(a \leqslant X \leqslant b)$$

Proof. We will prove each statement in turn. (1.) was proved in exercise **2.1**. Doing (3) first, since we need it to prove (2).

(3) By definition of complements of sets $A = \{X > x\}$ means $A^c = \{X \leqslant x\}$, and it follows that:
$$\mathbb{P}(X > x) = \mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - \mathbb{P}(X \leqslant x) = 1 - F(x).$$

(2) Assume $x < y$. We will need that $\{X > x\} \cup \{X \leqslant y\} = \Omega$, and we will also use Lemma 1.6 (in reverse).
$$\begin{aligned}\mathbb{P}(x < X \leqslant y) &= \mathbb{P}(\{X > x\} \cap \{X \leqslant y\}) \\ &= \mathbb{P}(X > x) + \mathbb{P}(X \leqslant y) - \mathbb{P}(\{X > x\} \cup \{X \leqslant y\}) \\ &= 1 - F(x) + F(y) - 1 \\ &= F(y) - F(x)\end{aligned}$$

(4) Similar argument for all cases, so will just do one. We just need to turn the inequalities into strict inequalities. For continuous random variables, pointwise probabilities are 0. Again, we will need to use $\{X > a\} \cup \{X < b\} = \Omega$.

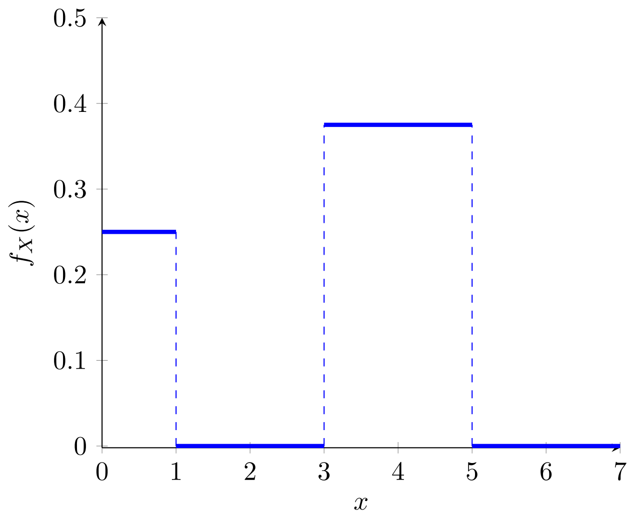
Define $A := \{a \leqslant X\}$ and $B := \{X < b\}$. First, we make the following observation:
$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(\{a \leqslant X\}) \\ &= \mathbb{P}(\{a = X\} \cup \{a < X\}) \\ &= \mathbb{P}(\{a = X\}) + \mathbb{P}(\{a < X\}) + \mathbb{P}(\{a = X\} \cap \{a < X\}) \\ &= 0 + \mathbb{P}(A') + 0 \\ &= \mathbb{P}(A')\end{aligned}$$

where $A' = \{a < X\}$. We get 0 for the pointwise probability, since this is continuous, and we get 0 because the sets are disjoint. We have shown that $\mathbb{P}(A) = \mathbb{P}(A')$ and can use this to conclude the proof.
$$\begin{aligned}\mathbb{P}(a \leqslant X < b) &= \mathbb{P}(\{a \leqslant X\} \cap \{X < b\}) \\ &= \mathbb{P}(A \cap B) \\ &= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(A \cup B) \\ &= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(\Omega) \\ &= \mathbb{P}(A') + \mathbb{P}(B) + \mathbb{P}(A' \cup B) \\ &= \mathbb{P}(A' \cap B) \\ &= \mathbb{P}(a < X < b)\end{aligned}$$
□

2.4
X has the probability density (PDF):

$$f_X(x) = \begin{cases} 1/4 & 0 < x < 1 \\ 3/8 & 3 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

Plot of the PDF:



From the relatively simple structure, we can easily determine the area under the graph:

$$A = (1)\left(\frac{1}{4}\right) + (2)\left(\frac{3}{8}\right) = \frac{2}{8} + \frac{6}{8} = 1$$

(a) Finding the CDF by integrating the PDF. We will split up the integral in several parts. First for the case when $y \in (0, 1)$:

$$F_X(y) = \int_{-\infty}^{y} f_X(t)dt = \frac{1}{4}\int_0^y 1dt = \frac{1}{4}\Big[t\Big]_0^y = \frac{y}{4}$$

When $y = 1$ we have $F_X(1) = 1/4$. Next, we must consider the case $y \in (1, 3)$. Here the PDF is 0, so it doesn't increase. It remains constant at 1/4 (since the CDF doesn't decrease).

$$F_X(y) = \frac{1}{4}$$

Next is the case $y \in (3, 5)$. Consider the intermediary integral:

$$I_1 = \int_3^y \frac{3}{8}dt = \frac{3}{8}\Big[t\Big]_3^y = \frac{3y-9}{8}$$

For values $y \in (3, 5)$ we start on 1/4, so the CDF in this region becomes:

$$F_X(y) = \frac{3y-9}{8} + \frac{1}{4}$$

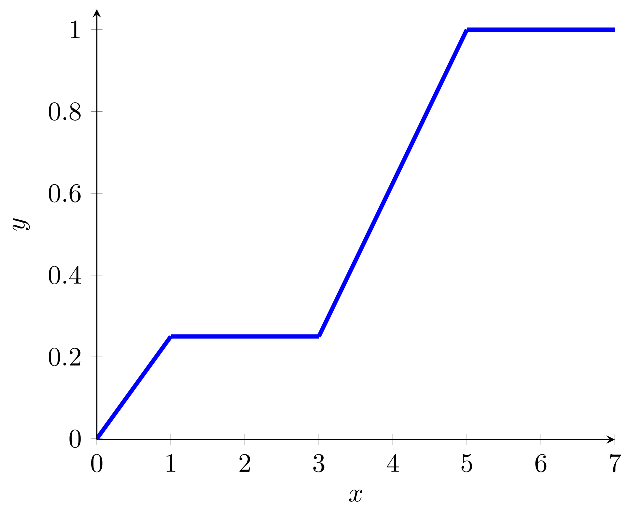
So, the full expression for the CDF becomes:

$$F_X(y) = \begin{cases} y/4 & y \in (0,1) \\ 1/4 & y \in (1,3) \\ \dfrac{3y-9}{8} + \dfrac{1}{4} & y \in (3,5) \\ 1 & y \geqslant 5 \end{cases}$$

Note that when $y = 5$ we get:

$$F_X(5) = \frac{3(5)-9}{8} + \frac{1}{4} = \frac{6}{8} + \frac{2}{8} = 1$$

Plot of the CDF:



(b) Defining $Y = 1/X$ and finding the PDF of Y. Following the hint we are given, we will consider the following three sets:

$$A_1 = \frac{1}{5} \leqslant y \leqslant \frac{1}{3}, \quad A_2 = \frac{1}{3} \leqslant y \leqslant 1, \quad A_3 = y \geqslant 1$$

Where A_1 corresponds to $(3,5)$, A_2 to $(1,3)$ and A_3 to $(0,1)$. We can express the CDF for $F_Y(y)$ in terms of $F_X(x)$:

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leqslant y) = \mathbb{P}(\frac{1}{X} \leqslant y) \\ &= \mathbb{P}(X \geqslant \frac{1}{y}) \\ &= 1 - \mathbb{P}(X \leqslant \frac{1}{y}) \\ &= 1 - F_X(\frac{1}{y}) \end{aligned}$$

First, we consider $A_1 : y \in [1/5, 1/3]$, and when we input $1/y$ to $F_X(\cdot)$, it will be in $(3, 5)$. So:

$$\begin{aligned}
F_Y(y) &= 1 - F_X(1/y) \\
&= 1 - \left(\frac{3(\frac{1}{y}) - 9}{8} + \frac{1}{4} \right) \\
&= 1 - \frac{3 - 9y}{8y} - \frac{1}{4} \\
&= \frac{3}{4} + \frac{9y - 3}{8y} \\
&= \frac{15y - 3}{8y}
\end{aligned}$$

Next, we consider $A_2 : y \in [1/3, 1]$. The input to $F_X(\cdot)$ will be in $(1, 3)$:

$$\begin{aligned}
F_Y(y) &= 1 - F_X(1/y) \\
&= 1 - \frac{1}{4} \\
&= \frac{3}{4}
\end{aligned}$$

Next, we consider $A_3 : y \geqslant 1$. The input to $F_X(\cdot)$ will be in $(0, 1)$:

$$\begin{aligned}
F_Y(y) &= 1 - F_X(1/y) \\
&= 1 - \frac{\frac{1}{y}}{4} \\
&= 1 - \frac{1}{4y}
\end{aligned}$$

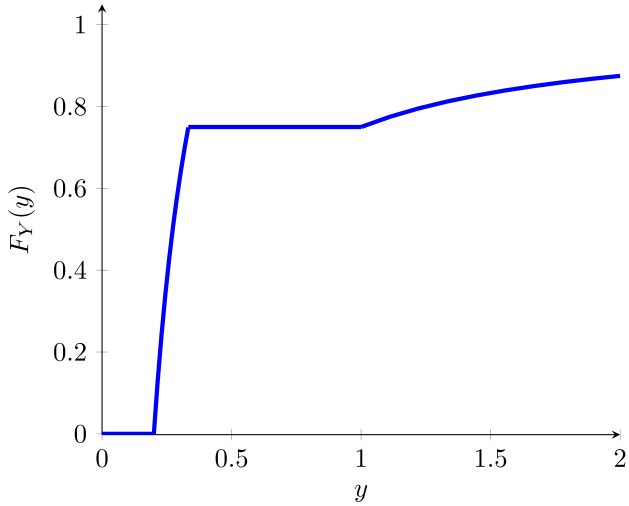
Also, whenever $y < 1/5$, then $1/y > 5$ which means $F_X(\cdot) = 1$, and so:

$$F_Y(y) = 1 - F_X(1/y) = 1 - 1 = 0.$$

This gives a full description of the CDF for $F_Y(y)$.

$$F_Y(y) = \begin{cases} 0 & y < 1/5 \\ \dfrac{15y - 3}{8y} & 1/5 \leqslant y \leqslant 1/3 \\ \dfrac{3}{4} & 1/3 \leqslant y \leqslant 1 \\ 1 - \dfrac{1}{4y} & y \geqslant 1 \end{cases}$$

Plot of CDF:



Finally, we can find the PDF of Y. We differentiate each of the parts in the CDF. When $y \in (1/5, 1/3)$:

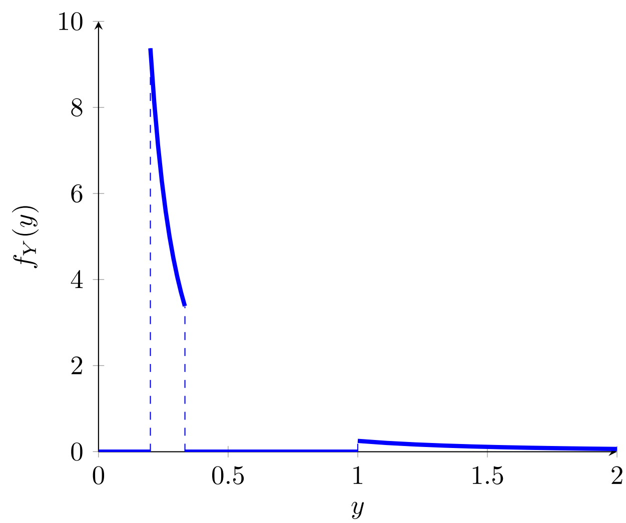
$$\frac{d}{dy}\left(\frac{15y-3}{8y}\right) = \frac{3}{8y^2}$$

When $y \geqslant 1$:

$$\frac{d}{dy}\left(1 - \frac{1}{4y}\right) = \frac{1}{4y^2}$$

(All other parts are constant, so they become 0). This gives us the PDF and its plot:

$$f_Y(y) = \begin{cases} 0 & y < 1/5 \\ \dfrac{3}{8y^2} & 1/5 \leqslant y \leqslant 1/3 \\ 0 & 1/3 < y < 1 \\ \dfrac{1}{4y^2} & y \geqslant 1 \end{cases}$$



2.5
Let X and Y be discrete RV. X and Y are independent if and only if $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all x and y.
PROOF.
$\Rightarrow$) Assume that X and Y are independent. That means that for any x, y, we have

$$\mathbb{P}(X = x \cap Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$$

Starting with the definition of the joint pdf:

$$\begin{aligned}
f_{X,Y}(x,y) &= \mathbb{P}(X = x, Y = y) \\
&= \mathbb{P}(X = x \cap Y = y) \\
&= \mathbb{P}(X = x)\mathbb{P}(Y = y) \\
&= f_X(x)f_Y(y)
\end{aligned}$$

Which shows that $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all x and y.

$\Leftarrow$) Assume that $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all x and y. By definition:

$$\begin{aligned}
f_{X,Y}(x,y) &= \mathbb{P}(X = x, Y = y) \\
&= \mathbb{P}(X = x \cap Y = y)
\end{aligned}$$

And,

$$f_X(x)f_Y(y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$$

From our assumption, these are equal, so $\mathbb{P}(X = x \cap Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$ which shows that X and Y are independent.
By implication both ways, the statement is proved. $\square$

2.6
Let X have distribution F and density f, and let A be a subset of the real line, e.g. $A = (a, b)$ for some $a, b \in \mathbb{R}$ and $a < b$. We have the indicator function

$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

We will set $Y = I_A(X)$ and find the PDF and CDF of Y.

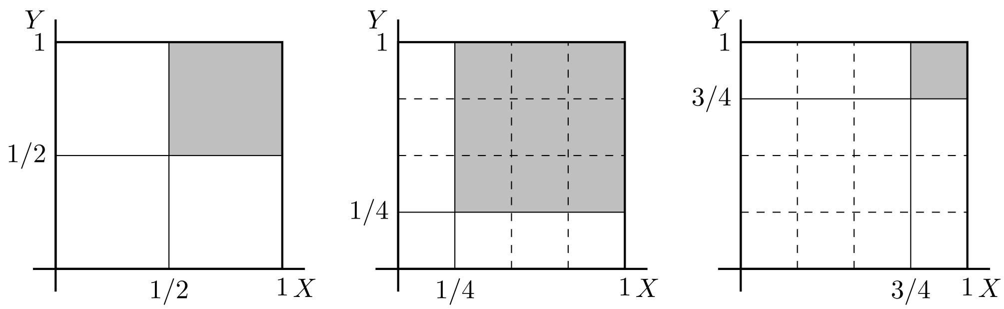
The exercise asks for a probability mass function, but that cannot be correct. Since X has a density f, it is a continuous RV. If $X \sim U(0,1)$ and $A = (0,1)$, then $Y = X$ and it will be a uniform variable with a continuous distribution, so not necessarily continuous.

And if it can be a continuous distribution, what happens if we define $A = \mathbb{Q} \subset \mathbb{R}$? There will be an infinite number of points in any interval with measure 0. Then we cannot define a PDF at all... Poorly formulated exercise in my opinion! Need to fill up with extra assumptions?

Skipping for now.

2.7

Let X and Y be independent and suppose that $X, Y \sim U(0,1)$. For $Z = \min(X, Y)$ we will find the density $f_Z(z)$ for Z. Following the hint, we will first find $\mathbb{P}(Z > z)$. Since any observations of $x, y \in (0,1)$, then we can immediately see that $\mathbb{P}(Z > 0) = 1$ and $\mathbb{P}(Z > 1) = 0$. But what happens for other values? Best way to find out is with some illustrations. Here are plots of the cases $\mathbb{P}(Z > 1/2)$, $\mathbb{P}(Z > 1/4)$ and $\mathbb{P}(Z > 3/4)$.



If we simulate lots of X and Y values, we see that about 1/4th of them will have both X and Y values larger than 0.5, so $\mathbb{P}(Z > 0.5) = 1/4$. Similarly, we get $\mathbb{P}(Z > 1/4) = 9/16$ and $\mathbb{P}(Z > 3/4) = 1/16$. Confirming this with a simulation.

```
# 2.7 - Simulating U(0,1)
N = 100000;X = runif(N);Y = runif(N)

Z = pmin(X, Y) # This is: Z = min{X, Y}

# Comparing simualted vs. theoretical results
sum(Z > 0.5)/N
1/4
sum(Z > 0.25)/N
9/16
sum(Z > 0.75)/N
1/16
```

```
> # Comparing simualted vs. theoretical results
> sum(Z > 0.5)/N
[1] 0.24888
> 1/4
[1] 0.25

> sum(Z > 0.25)/N
[1] 0.56156
> 9/16
[1] 0.5625

> sum(Z > 0.75)/N
[1] 0.06205
> 1/16
[1] 0.0625
```

By inspecting the images on the previous page, we can determine the 'shape' of the probabilities. For $Z > 1/4$ we remove the union of $X \leqslant 1/4$ and $Y \leqslant 1/4$. We define $A = \{X \leqslant z\}$ and $B = \{Y \leqslant z\}$, and can write the general case as:

$$\begin{aligned}
\mathbb{P}(Z > z) &= 1 - \mathbb{P}(\{X \leqslant z\} \cup \{Y \leqslant z\}) \\
&= 1 - \mathbb{P}(A \cup B) \\
&= 1 - \big[\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)\big] \\
&= 1 - \big[\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B)\big]
\end{aligned}$$

Where we used Lemma 1.6, and the fact that X and Y are independent. By using the probability law of complements, we can find the expression for $\mathbb{P}(Z \leqslant z)$.

$$\mathbb{P}(Z \leqslant z) = \mathbb{P}(X \leqslant z) + \mathbb{P}(Y \leqslant z) - \mathbb{P}(X \leqslant z)\mathbb{P}(Y \leqslant z)$$

The CDF for a uniform distribution on $U(a, b)$ is:

$$F(z) = \frac{z - a}{b - a} \implies F_X(z) = F_Y(z) = \frac{z - 0}{1 - 0} = z$$

Which means:

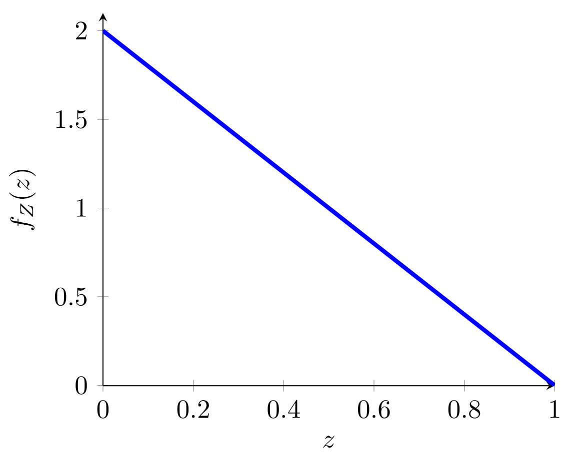
$$F_Z(z) = F_X(z) + F_Y(z) - F_X(z)F_Y(z) = 2z - z^2$$

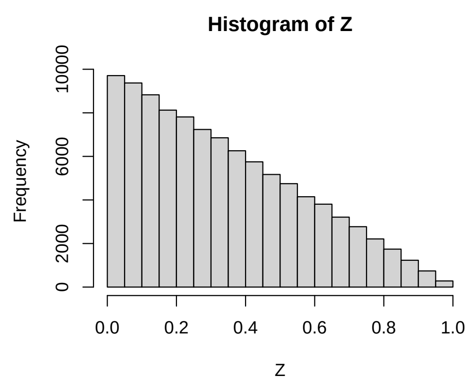
We can confirm our illustrations and simulated examples again by noting that:

$$\begin{aligned}
F_Z(1/2) &= \frac{1}{2} + \frac{1}{2} - \left(\frac{1}{2} \cdot \frac{1}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4} \\
F_Z(1/4) &= \frac{1}{4} + \frac{1}{4} - \left(\frac{1}{4} \cdot \frac{1}{4}\right) = \frac{8}{16} - \frac{1}{16} = \frac{7}{16} \\
F_Z(3/4) &= \frac{3}{4} + \frac{3}{4} - \left(\frac{3}{4} \cdot \frac{3}{4}\right) = \frac{24}{16} - \frac{9}{16} = \frac{15}{16}
\end{aligned}$$

which gives us the opposite results as expected (since we simulated and illustrated $\mathbb{P}(Z > z)$). The PDF $f_Z(z)$ is the derivative of $F_Z(z)$. Including PDF-plot and histogram of the simulated Zs.

$$f_Z(z) = \frac{d}{dz}\Big(F_Z(z)\Big) = 2 - 2z.$$





2.8

The RV X has CDF F. Finding the CDF of $X^+ = \max\{0, X\}$.

From the definition of CDF:

$$\begin{aligned} F_{X^+}(u) &= \mathbb{P}\big(\max(0, X) \leqslant u\big) \\ &= \mathbb{P}\Big(\{\omega \in \Omega \;:\; X(\omega) \leqslant u \text{ and } u \geqslant 0\}\Big) \end{aligned}$$

We must consider two cases. When $u < 0$:

$$\begin{aligned} F_{X^+}(u) &= \mathbb{P}\Big(\{\omega \in \Omega \;:\; X(\omega) \leqslant u \text{ and } u \geqslant 0\}\Big) \\ &= \mathbb{P}(\emptyset) \\ &= 0 \end{aligned}$$

When $u \geqslant 0$:

$$\begin{aligned} F_{X^+}(u) &= \mathbb{P}\Big(\{\omega \in \Omega \;:\; X(\omega) \leqslant u \text{ and } u \geqslant 0\}\Big) \\ &= \mathbb{P}\Big(\{\omega \in \Omega \;:\; X(\omega) \leqslant u\}\Big) \\ &= \mathbb{P}(X \leqslant u) \\ &= F_X(u) \end{aligned}$$

So in summary:

$$F_{X^+}(u) = \begin{cases} 0 & u < 0 \\ F_X(u) & u \geqslant 0 \end{cases}$$

2.9

We have $X \sim \text{Exp}(\beta)$. The PDF is given by:

$$f(x) = \frac{1}{\beta} e^{-x/\beta}, \quad x > 0$$

Finding the CDF by integrating the PDF.

$$\begin{aligned} F(y) &= \int_{-\infty}^{y} f(x) dx \\ &= \frac{1}{\beta} \int_{0}^{y} e^{-x/\beta} dx \\ &= \frac{1}{\beta} \big[-\beta e^{-x/\beta} \big]_0^y \\ &= \big[-e^{-x/\beta} \big]_0^y \\ &= 1 - e^{-y/\beta} \end{aligned}$$

To find the inverse $F^{-1}(q)$ we set $q = F(y)$ and solve for y.

$$q = 1 - e^{-y/\beta} \quad \implies \quad y = -\beta \log(1 - q) \quad \implies \quad F^{-1}(q) = -\beta \log(1 - q)$$

2.10
If X and Y are independent, then $g(X)$ and $h(Y)$ are independent for some functions g and h.

PROOF.