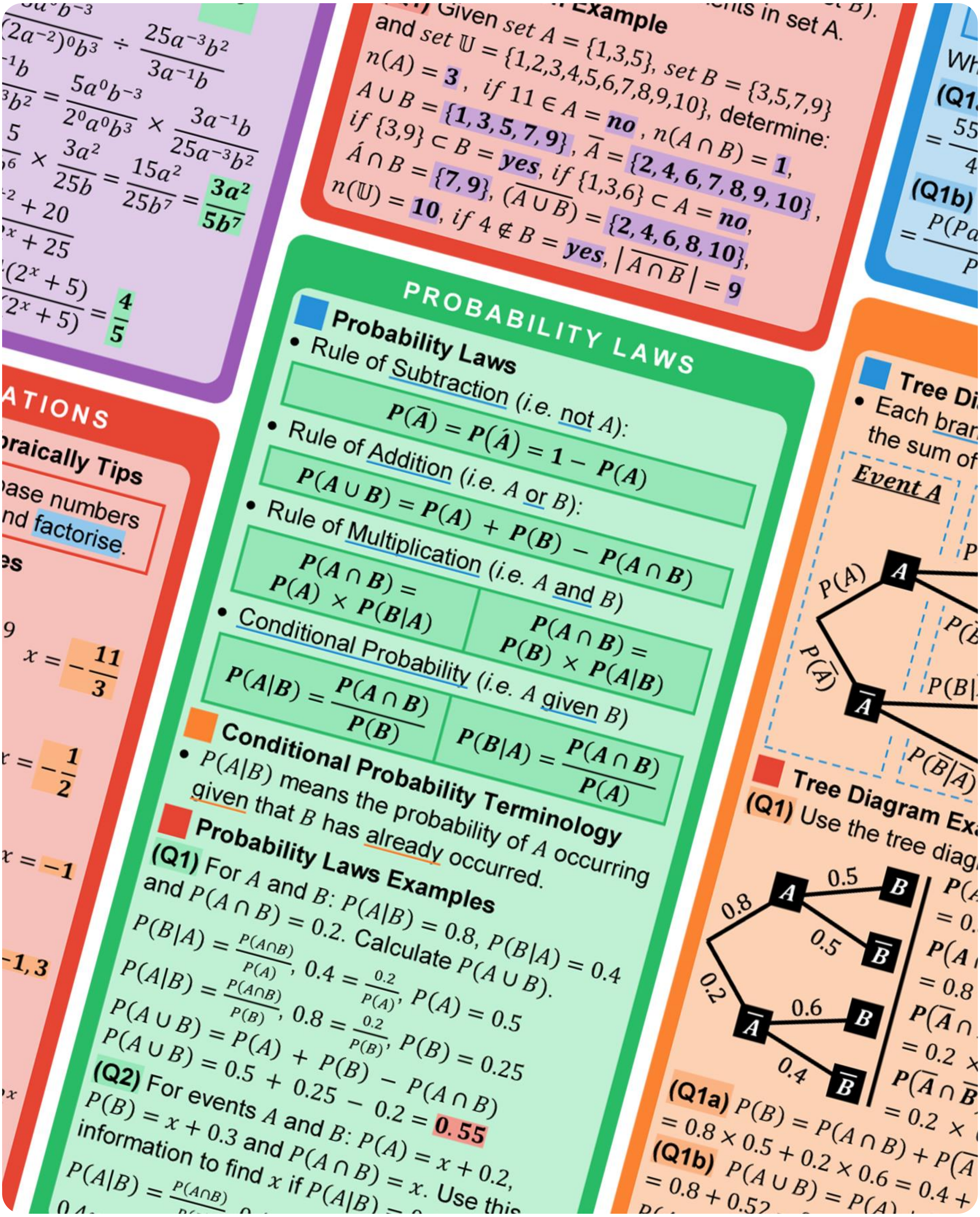

Given set $A = \{1,3,5\}$, set $B = \{3,5,7,9\}$ and set $\mathbb{U} = \{1,2,3,4,5,6,7,8,9,10\}$, determine:

$n(A) = \mathbf{3}$, $if\ 11 \in A = \mathbf{no}$, $n(A \cap B) = \mathbf{1}$,

$A \cup B = \mathbf{\{1,3,5,7,9\}}$, $\bar{A} = \mathbf{\{2,4,6,7,8,9,10\}}$,

$if\ \{3,9\} \subset B = \mathbf{yes}$, $if\ \{1,3,6\} \subset A = \mathbf{no}$,

$\acute{A} \cap B = \mathbf{\{7,9\}}$, $\overline{(A \cup B)} = \mathbf{\{2,4,6,8,10\}}$,

$n(\mathbb{U}) = \mathbf{10}$, $if\ 4 \notin B = \mathbf{yes}$, $|\overline{A \cap B}| = \mathbf{9}$

PROBABILITY LAWS

Probability Laws

- Rule of Subtraction (*i.e.* not *A*):

$$P(\bar{A}) = P(\acute{A}) = 1 - P(A)$$

- Rule of Addition (*i.e.* *A* or *B*):

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Rule of Multiplication (*i.e.* *A* and *B*)

$P(A \cap B) = P(A) \times P(B\|A)$	$P(A \cap B) = P(B) \times P(A\|B)$

- Conditional Probability (*i.e.* *A* given *B*)

$P(A\|B) = \frac{P(A \cap B)}{P(B)}$	$P(B\|A) = \frac{P(A \cap B)}{P(A)}$

Conditional Probability Terminology

- $P(A|B)$ means the probability of A occurring given that B has already occurred.

Probability Laws Examples

(Q1) For A and B: $P(A|B) = 0.8$, $P(B|A) = 0.4$ and $P(A \cap B) = 0.2$. Calculate $P(A \cup B)$.

$P(B|A) = \frac{P(A\cap B)}{P(A)}$, $0.4 = \frac{0.2}{P(A)}$, $P(A) = 0.5$

$P(A|B) = \frac{P(A\cap B)}{P(B)}$, $0.8 = \frac{0.2}{P(B)}$, $P(B) = 0.25$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = 0.5 + 0.25 - 0.2 = \mathbf{0.55}$

(Q2) For events A and B: $P(A) = x + 0.2$, $P(B) = x + 0.3$ and $P(A \cap B) = x$. Use this information to find x if $P(A|B) =$

ATAR Mathematics Methods Units 1 & 2

Exam Notes for Western Australian Year 11 Students

Created by Anthony Bochrinis
Version 3.0 (Updated 21/12/19)

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► About the Author - Anthony Bochrinis

Hello! My name is Anthony and I graduated from high school in 2012, completed a Bachelor of Actuarial Science in 2015, completed my Graduate Diploma in Secondary Education in 2017 and am now a secondary mathematics teacher!

My original exam notes (created in 2013) were inspired by Severus Snape's copy of Advanced Potion Making in Harry Potter and the Half-Blood Prince; a textbook filled with annotations containing all of the pro tips and secrets to help gain a clearer understanding.

Thank you for being a part of my journey in realising that teaching is my lifelong vocation!

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INDICES AND SURDS

INDEX AND SURD LAWS

Index Laws

$a^m \times a^n = a^{m+n}$	$a^m \div a^n = a^{m-n}$
$(a^m)^n = a^{m \times n}$	$a^0 = 1$
$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$	$(ab)^m = a^m \times b^m$
$a^{-m} = \frac{1}{a^m}$	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Surd Laws

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$	$\sqrt{a} \times \sqrt{a} = a$
$\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$	$m\sqrt{a} \pm n\sqrt{a} = (m \pm n)\sqrt{a}$

Rationalising a Surd

- Removes surd in denominator of a fraction.

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \times 1 = \frac{1}{\sqrt{a}} \times \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}$$

SIMPLIFYING EXPRESSIONS

Simplifying Expressions Tips

Tip 1	Remove all $\sqrt{\ }$ and replace with a power of ½ (*e.g.* $\sqrt{a} = a^{\frac{1}{2}}$).
Tip 2	To divide 2 fractions, flip second fraction upside down and change ÷ to a × (*e.g.* $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$).
Tip 3	Reverse multiplication of like terms index law (e.g. $2^{x+2} = 2^x \times 2^2$).

Simplifying Expressions Examples

(Q1) Simplify with positive indices: $\left(\frac{-2xy^7}{3wxy^{-2}z^3}\right)^{-2}$

$$= \frac{(-2)^{-2}x^{-2}y^{-14}}{3^{-2}w^{-2}x^{-2}y^4z^{-6}} = \frac{3^2w^2x^2z^6}{(-2)^2wx^2y^{18}} = \boldsymbol{\frac{9w^2z^6}{4y^{18}}}$$

(Q2) Simplify with positive indices: $\frac{\sqrt{25a^4b^2c}}{ab^{-1}\sqrt{c}}$

$$= \frac{(5^2a^4b^2c)^{\frac{1}{2}}}{ab^{-1}c^{\frac{1}{2}}} = \frac{5a^2bc^{\frac{1}{2}}}{ab^{-1}c^{\frac{1}{2}}} = \frac{5a^2b^2c^{\frac{1}{2}}}{ac^{\frac{1}{2}}} = \boldsymbol{5ab^2}$$

(Q3) Simplify with positive indices: $\frac{5a^0b^{-3}}{(2a^{-2})^0b^3} \div \frac{25a^{-3}b^2}{3a^{-1}b}$

$$= \frac{5a^0b^{-3}}{(2a^{-2})^0b^3} \times \frac{3a^{-1}b}{25a^{-3}b^2} = \frac{5a^0b^{-3}}{2^0a^0b^3} \times \frac{3a^{-1}b}{25a^{-3}b^2}$$

$$= \frac{5b^{-3}}{b^3} \times \frac{3a^{-1}b}{25a^{-3}b^2} = \frac{5}{b^6} \times \frac{3a^2}{25b} = \frac{15a^2}{25b^7} = \boldsymbol{\frac{3a^2}{5b^7}}$$

(Q4) Simplify with positive indices: $\frac{2^{x+2} + 20}{5 \times 2^x + 25}$

$$= \frac{(2^2 \times 2^x) + (4 \times 5)}{(5 \times 2^x) + (5 \times 5)} = \frac{4(2^x + 5)}{5(2^x + 5)} = \boldsymbol{\frac{4}{5}}$$

SOLVING EQUATIONS

Solving Equations Algebraically Tips

Tip	Convert to common base numbers (*e.g.* $4 = 2^2$, $8 = 2^3$) and factorise.

Solving Equations Examples

(Q1) Solve for x: $4^{3x+1} = 8^{x-3}$

$(2^2)^{3x+1} = (2^3)^{x-3}$ $\quad 6x + 2 = 3x - 9$
$2^{6x+2} = 2^{3x-9}$ $\quad 3x = -11$ $\quad x = \boldsymbol{-\frac{11}{3}}$

(Q2) Solve for x: $25(5^{-2x}) = 125$

$(5^2)(5^{-2x}) = 5^3$ $\quad -2x + 2 = 3$
$5^{-2x+2} = 5^3$ $\quad 2x = -1$ $\quad x = \boldsymbol{-\frac{1}{2}}$

(Q3) Solve for x: $3^{2x+1} = 27 \times 81^x$

$3^{2x+1} = (3^3) \times (3^4)^x$ $\quad 2x + 1 = 4x + 3$
$3^{2x+1} = 3^{3+4x}$ $\quad -2x = 2$ $\quad x = \boldsymbol{-1}$

(Q4) Solve for x: $(x^2 - 2x)^4 = 81$

$(x^2 - 2x)^4 = 3^4$ $\quad x^2 - 2x - 3 = 0$
$x^2 - 2x = 3$ $\quad (x-3)(x+1) = 0$ $\quad x = \boldsymbol{-1, 3}$

(Q5) Solve for x: $2^{2x} - 10 \times 2^x + 16 = 0$

$(2^x)^2 - 10(2^x) + 16 = 0$
Substitute $y = 2^x$ $\quad y = 2, 8$
$y^2 - 10y + 16 = 0$ $\quad \therefore 2 = 2^x$ and $8 = 2^x$
$(y-2)(y-8) = 0$ $\quad x = \boldsymbol{1, 3}$

(Q6) Solve for x: $3^x = 15$

Graph $y = 3^x$ and $y = 15$ on calculator, finding the intersection gives $x = \boldsymbol{1.97}$ $(2dp)$

SCIENTIFIC NOTATION

Scientific Notation (Standard Form)

- Expresses any number as a product of a number between 0 and 10 exclusive and a power of 10 (*e.g.* $712 = 7.12 \times 10^2$).
  - Positive indices move decimal point right and represent numbers larger than 1.
  - Negative indices move decimal point left and represent numbers between 0 and 1.

Scientific Notation Examples

(Q1) $385{,}000$ in standard form $= \boldsymbol{3.85 \times 10^5}$

(Q2) 0.0039 in standard form $= \boldsymbol{3.9 \times 10^{-3}}$

(Q3) 3.06×10^4 as a basic numeral $= \boldsymbol{30{,}600}$

(Q4) 2.5×10^{-2} as a basic numeral $= \boldsymbol{0.025}$

SIGNIFICANT FIGURES

Significant Figures *(sig. fig.)*

- Significant figures are numbers that are correct within a stated degree of accuracy.
- 3 rules of determining significant figures:

Rule 1	All non-zero digits are significant (*e.g.* 1234 has 4 sig. fig.).
Rule 2	All zeroes that appear between any non-zero digits are significant (*e.g.* 1014 has 4 sig. fig.).
Rule 3	All zeroes that are both to the right of a decimal point and to the right of the first non-zero digit after the decimal point are significant (*e.g.* 0.00040650 has 5 sig. fig.).

Significant Figure Examples

(Q1) Write 47.502 with 4 sig. fig. $= \boldsymbol{47.50}$

(Q2) Write 780,582 with 4 sig. fig. $= \boldsymbol{780{,}600}$

(Q3) Write 0.050899 with 3 sig. fig. $= \boldsymbol{0.0509}$

(Q4) Write 29.86 with 1 sig. fig. $= \boldsymbol{30}$

PROBABILITY

SET NOTATION

Logic Functions and Symbols

- $\overline{A}$ or $\acute{A}$: complement of an event (not A).
- $A \cup B$: union of two events (A or B).
- $A \cap B$: intersection of two events (A and B).

Set Notation and Symbols

- $\in$: element (found in a given set).
- $\notin$: not an element (not found in a given set).
- $\emptyset$ or $\{\ \}$: empty set (contains no elements).
- $\mathbb{U}$: universal set (contains all elements).
- $\subset$: subset ($A \subset B$ means that all elements of set A is found within the elements of set B).
- $n(A)$ or $|A|$: number of elements in set A.

Set Notation Example

(Q1) Given $set\ A = \{1,3,5\}$, $set\ B = \{3,5,7,9\}$ and $set\ \mathbb{U} = \{1,2,3,4,5,6,7,8,9,10\}$, determine:

$n(A) = \boldsymbol{3}$, $if\ 11 \in A = \boldsymbol{no}$, $n(A \cap B) = \boldsymbol{1}$, $A \cup B = \boldsymbol{\{1,3,5,7,9\}}$, $\overline{A} = \boldsymbol{\{2,4,6,7,8,9,10\}}$, $if\ \{3,9\} \subset B = \boldsymbol{yes}$, $if\ \{1,3,6\} \subset A = \boldsymbol{no}$, $\acute{A} \cap B = \boldsymbol{\{7,9\}}$, $(\overline{A \cup B}) = \boldsymbol{\{2,4,6,8,10\}}$, $n(\mathbb{U}) = \boldsymbol{10}$, $if\ 4 \notin B = \boldsymbol{yes}$, $|\overline{A \cap B}| = \boldsymbol{9}$

PROBABILITY LAWS

Probability Laws

- Rule of Subtraction (*i.e.* not A):

$$\boldsymbol{P(\overline{A}) = P(\acute{A}) = 1 - P(A)}$$

- Rule of Addition (*i.e.* A or B):

$$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

- Rule of Multiplication (*i.e.* A and B)

$\boldsymbol{P(A \cap B) = P(A) \times P(B \mid A)}$	$\boldsymbol{P(A \cap B) = P(B) \times P(A \mid B)}$

- Conditional Probability (*i.e.* A given B)

$\boldsymbol{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$	$\boldsymbol{P(B \mid A) = \frac{P(A \cap B)}{P(A)}}$

Conditional Probability Terminology

- $P(A|B)$ means the probability of A occurring given that B has already occurred.

Probability Laws Examples

(Q1) For A and B: $P(A|B) = 0.8$, $P(B|A) = 0.4$ and $P(A \cap B) = 0.2$. Calculate $P(A \cup B)$.

$P(B|A) = \frac{P(A \cap B)}{P(A)}$, $0.4 = \frac{0.2}{P(A)}$, $P(A) = 0.5$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$, $0.8 = \frac{0.2}{P(B)}$, $P(B) = 0.25$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = 0.5 + 0.25 - 0.2 = \boldsymbol{0.55}$

(Q2) For events A and B: $P(A) = x + 0.2$, $P(B) = x + 0.3$ and $P(A \cap B) = x$. Use this information to find x if $P(A|B) = 0.4$.

$P(A|B) = \frac{P(A \cap B)}{P(B)}$, $0.4 = \frac{x}{x+0.3}$, $0.4(x + 0.3) = x$

$0.4x + 0.12 = x$, $0.6x = 0.12$, $x = \boldsymbol{0.2}$

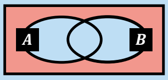
SHADING VENN DIAGRAMS

Tips for Shading Venn Diagrams

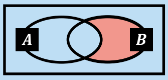
- Not ($\overline{A}$ or $\acute{A}$): shade outer region.
- Or ($A \cup B$): shade region A & B and overlap.
- And ($A \cap B$): shade overlapping region.

Shading Venn Diagrams Examples

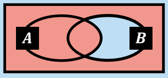
(Q1) Shade $\overline{A \cup B}$



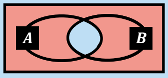
(Q2) Shade $\overline{A} \cap B$



(Q3) Shade $A \cup \overline{B}$

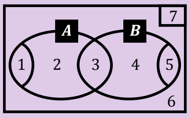


(Q4) Shade $\overline{A} \cup \overline{B}$



VENN DIAGRAMS

Different Venn Diagram Regions



- Different region definitions:

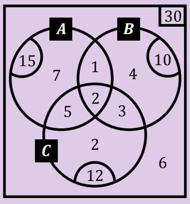
1: $\boldsymbol{A}$, 2: $\boldsymbol{A \cap \overline{B}}$, 3: $\boldsymbol{A \cap B}$, 4: $\boldsymbol{\overline{A} \cap B}$, 5: $\boldsymbol{B}$, 6: $\boldsymbol{\overline{A \cup B}}$, 7: $\mathbb{U}$

- 3 rules of adding Venn Diagram regions:

Rule 1	Regions $2 + 3 + 4 + 6 = 7$		
Rule 2	$2 + 3 = 1$	**Rule 3**	$3 + 4 = 5$

Triple Venn Diagram Example

(Q1) Three events A, B and C are such that: $n(\mathbb{U}) = 30$, $n(A) = 15$, $n(B) = 10$, $n(C) = 12$, $n(A \cap B) = 3$, $n(B \cap C) = 5$, $n(A \cap C) = 7$, $n(A \cap B \cap C) = 2$, $n(\overline{A \cup B \cup C}) = 6$. Find:



(Q1a) $P(A \cup B \cup C)$ $= 1 - \frac{6}{30} = \frac{24}{30} = \boldsymbol{\frac{5}{6}}$

(Q1b) $P(A \cap B \cap C)$ $= \frac{2}{30} = \boldsymbol{\frac{1}{15}}$

(Q1c) $P(A \cap \overline{B} \cap C)$ $= \frac{5}{30} = \boldsymbol{\frac{1}{6}}$

(Q1d) Probability of A or B, given C occurs.

$$P(A \cup B | C) = \frac{P((A \cup B) \cap C)}{P(C)} = \frac{5}{30} \div \frac{12}{30} = \boldsymbol{\frac{5}{12}}$$

TWO-WAY TABLES

Two-Way Table Example

(Q1) A clothes shop has 400 items in stock:

Type/Colour	Red	Blue	Yellow
Shirt	55	70	40
Pants	45	67	24
Shoes	0	50	49

What is probability of randomly selecting:

(Q1a) Red item or a shirt $= P(R \cup Shirt)$

$$= \frac{55 + 45}{400} + \frac{55 + 70 + 40}{400} - \frac{55}{400} = \frac{200}{400} = \boldsymbol{\frac{1}{2}}$$

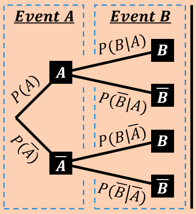
(Q1b) Pants given its blue $= P(Pants \mid Blue)$

$$= \frac{P(Pants \cap Blue)}{P(Blue)} = \frac{67}{400} \div \frac{187}{400} = \boldsymbol{\frac{67}{187}}$$

TREE DIAGRAMS

Tree Diagrams

- Each branch of the tree diagram as well as the sum of the final outcomes adds to 1.



Outcomes

$\boldsymbol{P(A \cap B)}$ $= P(A) \times P(B|A)$

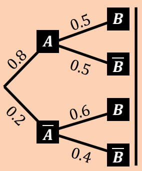
$\boldsymbol{P(A \cap \overline{B})}$ $= P(A) \times P(\overline{B}|A)$

$\boldsymbol{P(\overline{A} \cap B)}$ $= P(\overline{A}) \times P(B|\overline{A})$

$\boldsymbol{P(\overline{A} \cap \overline{B})}$ $= P(\overline{A}) \times P(\overline{B}|\overline{A})$

Tree Diagram Examples

(Q1) Use the tree diagram below to find:



$\boldsymbol{P(A \cap B)} = P(A) \times P(B|A)$ $= 0.8 \times 0.5 = 0.4$

$\boldsymbol{P(A \cap \overline{B})} = P(A) \times P(\overline{B}|A)$ $= 0.8 \times 0.5 = 0.4$

$\boldsymbol{P(\overline{A} \cap B)} = P(\overline{A}) \times P(B|\overline{A})$ $= 0.2 \times 0.6 = 0.12$

$\boldsymbol{P(\overline{A} \cap \overline{B})} = P(\overline{A}) \times P(\overline{B}|\overline{A})$ $= 0.2 \times 0.4 = 0.08$

(Q1a) $P(B) = P(A \cap B) + P(\overline{A} \cap B)$ $= 0.8 \times 0.5 + 0.2 \times 0.6 = 0.4 + 0.12 = \boldsymbol{0.52}$

(Q1b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $= 0.8 + 0.52 - 0.4 = \boldsymbol{0.92}$ or alternatively, $P(A \cup B) = 1 - P(\overline{A} \cap \overline{B}) = 1 - 0.08 = \boldsymbol{0.92}$

(Q1d) $P(\overline{A}|B) = \frac{P(\overline{A} \cap B)}{P(B)} = \frac{0.15}{0.4+0.12} = \frac{0.15}{0.54} = \boldsymbol{0.28}$

EVENT RELATIONSHIPS

Mutually Exclusive Events

- Events cannot co-occur and one event does influence the outcome of the other event (e.g. you can't roll a 3 and 5 on the same die at the same time as rolling a 3 prevents rolling a 5).
- 2 rules of Mutually Exclusive events:

Rule 1	$P(A \cap B) = 0$
Rule 2	$P(A \cup B) = P(A) + P(B)$

Independent Events

- Events can co-occur and one event does not influence the outcome of the other event (e.g. rolling a dice and then flipping a coin).
- 2 rules of Independent events:

Rule 1	$P(A \cap B) = P(A) \times P(B)$	
Rule 2	$P(A \mid B) = P(A)$	$P(B \mid A) = P(B)$

► **Topic Is Continued In Next Column** ◄

EVENT RELATIONSHIPS

Event Relationships Examples

(Q1) For independent events A and B: $P(A) = 0.2$ and $P(B) = 0.15$. Calculate the following: $P(A|B) = \boldsymbol{0.2}$, $P(A \cap B) = 0.2 \times 0.15 = \boldsymbol{0.03}$, $P(A \cup B) = 0.2 + 0.15 - (0.2 \times 0.15) = \boldsymbol{0.32}$

(Q2) For events A and B: $P(A) = x + 0.1$, $P(B) = x + 0.4$ and $P(A \cap B) = x$. Find x if A and B are:

(Q2a) Mutually Exclusive: $P(A \cap B) = 0$, $x = \boldsymbol{0}$

(Q2b) Independent: $P(A \cap B) = P(A) \times P(B)$ $x = (x + 0.1)(x + 0.4)$, expand and rearrange: $0 = x^2 - 0.5x + 0.04$, solving gives $x = \boldsymbol{\frac{1}{10}}$ and $\boldsymbol{\frac{2}{5}}$

TESTING EVENT RELATIONSHIPS

Testing for Event Relationship Types

Test 1	Use mutual exclusivity rules to test if the events are mutually exclusive.

If test 1 works, events are mutually exclusive ↓ If test 1 fails, go to next test

Test 2	Use independence rules to then test if events are independent.

If test 2 works, events are independent ↓ If test 2 fails, go to result

Result	Both events are neither mutually exclusive nor independent.

Event Relationship Test Examples

(Q1) Find relationship between A and B if: $P(A) = 0.4$, $P(B) = 0.3$ and $P(\overline{A \cup B}) = 0.3$

- First Test: if A and B are Mutually Exclusive

Testing using the rule: $P(A \cup B) = P(A) + P(B)$ $0.7 = 0.4 + 0.3$, $0.7 = 0.7$ which is true. Test 1 passes, $\therefore$ ***A and B are mutually exclusive***

(Q2) Find relationship between A and B if: $P(A \cup B) = 0.9$, $P(A \cap B) = 0.4$, $P(A|B) = 0.5$.

From $P(A|B) = \frac{P(A \cap B)}{P(B)}$, $0.5 = \frac{0.4}{P(B)}$, $\therefore P(B) = 0.8$

Also, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$0.9 = P(A) + 0.8 - 0.4$, $\therefore P(A) = 0.5$

- First Test: A and B are Mutually Exclusive

Testing using the rule: $P(A \cap B) = 0$
$0.4 \neq 0$ which is false. Test 1 fails; try Test 2.

- Second Test: A and B are Independent

Testing using the rule: $P(A \cap B) = P(A) \times P(B)$
$0.4 = 0.8 \times 0.5$, $0.4 = 0.4$ which is true.
Test 2 passes, $\therefore$ ***A and B are independent***

COMBINATORICS

FACTORIALS

Factorial *($n!$)*

- The product of all positive integers less than or equal to a number n (e.g. $3! = 3 \times 2 \times 1$).

$n!$ is pronounced "n factorial", for $n > 0$:
$$\boldsymbol{n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1}$$

- Finding $n!$ shows the number of ways that n distinct objects can be arranged in a line.

Factorial rule exception: $0! = 1$
As there is 1 way to arrange 0 objects

Factorial Example

(Q1) Determine the value of $5! \div 3!$

$$\frac{5!}{3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{5 \times 4 \times \cancel{3 \times 2 \times 1}}{\cancel{3 \times 2 \times 1}} = \boldsymbol{20}$$

(Q2) Simplify the expression $(n+2)!/n!$

$$= \frac{(n+2) \times (n+1) \times n!}{n!} = \boldsymbol{(n+2)(n+1)}$$

COMBINATIONS

Combination Notation

- A combination is number of ways of choosing r items from a collection of n items.

$$\boldsymbol{{}^nC_r = n\ choose\ r = \binom{n}{r} = \frac{n!}{(n-r)! \times r!}}$$

Combination Examples

(Q1) How many selections of 3 chocolates can be made from 7 chocolates?

$$\binom{7}{3} = \frac{7!}{(7-3)! \times 3!} = \frac{7!}{4! \times 3!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = \boldsymbol{35}$$

(Q2) Out of 20 candidates (11 males and 9 females), only 5 will be chosen to form a council. How many combinations of different councils can be formed if there needs to be atleast 3 males?

$$= (3M\ and\ 2F) + (4M\ and\ 1F) + (5M\ and\ 1F)$$

$$= \left[\binom{11}{3} \times \binom{9}{2}\right] + \left[\binom{11}{4} \times \binom{9}{1}\right] + \left[\binom{11}{5} \times \binom{9}{0}\right]$$

$$= 5{,}940 + 2{,}970 + 462 = \boldsymbol{9{,}372\ councils}$$

ATAR Math Methods
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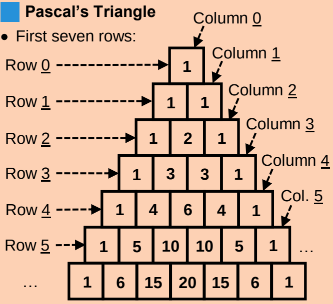
PASCAL'S TRIANGLE

Pascal's Triangle

- First seven rows:



How to Create Pascal's Triangle

- Each number in Pascal's Triangle is the sum of the two numbers directly above it.

Combinations and Pascal's Triangle

$$\binom{n}{r} = \left(\begin{matrix}\textit{Row}\ \#\textit{of Pascal's Tri.}\\ \textit{Column}\ \#\textit{of Pascal's Tri.}\end{matrix}\right)$$

Pascal's Triangle Example

(Q1) $\binom{4}{2} = \left(\begin{matrix}4^{th}\ \textit{row of P.T.}\\ 2^{nd}\ \textit{column of P.T.}\end{matrix}\right) = \mathbf{6}$

BINOMIAL EXPANSION

Expanding Brackets with Two Terms

- Expanding binomials to large powers of n:

Expanding $(ax + by)^n$

$$= \binom{n}{0}(ax)^n(by)^0 + \binom{n}{1}(ax)^{n-1}(by)^1 + \cdots + \binom{n}{n-1}(ax)^1(by)^{n-1} + \binom{n}{n}(ax)^0(by)^n$$

Expanding Brackets Tips

- Expanding binomials to large powers of n:

Tip 1	If there is an addition between the 2 terms, each term of the answer is added together.
Tip 2	If there is a subtraction between the 2 terms, each term of the answer follows the pattern: $-, +, -, +, -, \ldots$ starting with $-$.

Expanding Brackets Examples

(Q1) Expand $(x + 3y)^4$

$= \binom{4}{0}(x)^4(3y)^0 + \binom{4}{1}(x)^3(3y)^1 + \binom{4}{2}(x)^2(3y)^2 + \binom{4}{3}(x)^1(3y)^3 + \binom{4}{4}(x)^0(3y)^4$

$= (1)(x^4)(1) + (4)(x^3)(3y) + (6)(x^2)(9y^2) + (4)(x^1)(27y^3) + (1)(x^0)(81y^4)$

$= \mathbf{x^4 + 12x^3y + 54x^2y^2 + 108xy^3 + 81y^4}$

(Q2) Expand $(2x - y)^4$

$= \binom{4}{0}(2x)^4(y)^0 - \binom{4}{1}(2x)^3(y)^1 + \binom{4}{2}(2x)^2(y)^2 - \binom{4}{3}(2x)^1(y)^3 + \binom{4}{4}(2x)^0(y)^4$

$= (1)(16x^4)(1) - (4)(8x^3)(y) + (6)(4x^2)(y^2) -(4)(2x)(y^3) + (1)(1)(y^4)$

$= \mathbf{16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4}$

(Q3) Expand $(x^2 - 2)^5$

$= \binom{5}{0}(x^2)^5(2)^0 - \binom{5}{1}(x^2)^4(2)^1 + \binom{5}{2}(x^2)^3(2)^2 - \binom{5}{3}(x^2)^2(2)^3 + \binom{5}{4}(x^2)^1(2)^4 - \binom{5}{5}(x^2)^0(2)^5$

$= (1)(x^{10})(1) - (5)(x^8)(2) + (10)(x^6)(4) -(10)(x^4)(8) + (5)(x^2)(16) - (1)(1)(32)$

$= \mathbf{x^{10} - 10x^8 + 40x^6 - 80x^4 + 80x^2 - 32}$

SEQUENCES

TYPES OF SEQUENCES

Sequence Notation

- $\boldsymbol{T_n}$: the value of the n^{th} term in the sequence.
- $\boldsymbol{a}$: the value of the first (initial) term in the sequence (i.e. the value of T_1).
- $\boldsymbol{d}$: common difference between each term (note: arithmetic sequences only).
- $\boldsymbol{r}$: common ratio between each term (note: geometric sequences only).
- $\boldsymbol{S_n}$: the sum of the first n terms in the sequence (i.e. $S_n = T_1 + T_2 + T_3 + \cdots + T_n$).
- $\boldsymbol{S_\infty}$: the sum of all terms (to infinity) in the sequence (i.e. $S_\infty = T_1 + T_2 + T_3 + \cdots$).

Substituting Terms into Sequences

(Q1) Let $T_{n+1} = T_n + T_{n-1} + T_{n-2}$, calculate the value of T_4 if $T_1 = 4$, $T_2 = 7$ and $T_3 = 10$.

- $T_{3+1} = T_3 + T_{3-1} + T_{3-2}$, $T_4 = T_3 + T_2 + T_1$

From substitution $\therefore T_4 = 10 + 7 + 4 = \mathbf{21}$

Arithmetic Sequences *(+ or −)*

- Each term is found by adding or subtracting a constant to or from the previous term.
- a.k.a. arithmetic progression (AP).

$$\mathbf{4} \xrightarrow[+3]{} \mathbf{7} \xrightarrow[+3]{} \mathbf{10} \xrightarrow[+3]{} \mathbf{13} \xrightarrow[+3]{} \mathbf{17}$$

$$\boldsymbol{d = T_{n+1} - T_n} \ \text{or}\ \boldsymbol{d = T_2 - T_1}$$

► Topic Is Continued In Next Column ◄

TYPES OF SEQUENCES

Geometric Sequences *(× or ÷)*

- Each term is found by multiplying or dividing a constant to or from the previous term.
- a.k.a. geometric progression (GP).

$$\mathbf{0.5} \xrightarrow[\times 4]{} \mathbf{2} \xrightarrow[\times 4]{} \mathbf{8} \xrightarrow[\times 4]{} \mathbf{32} \xrightarrow[\times 4]{} \mathbf{128}$$

$$\boldsymbol{r = T_{n+1} \div T_n} \ \text{or}\ \boldsymbol{r = T_2 \div T_1}$$

Explicit and Recursive Formulae

- Explicit: finds the value of any term.
- Recursive: finds the next term in the sequence if the previous term is known.
  - Note: T_1 must also be stated with the rule.

Arithmetic Sequences (AP)	
Explicit	$\boldsymbol{T_n = a + (n - 1) \times d}$
Recursive	$\boldsymbol{T_{n+1} = T_n + d,\ T_1 = a}$
Geometric Sequences (GP)	
Explicit	$\boldsymbol{T_n = a \times r^{n-1}}$
Recursive	$\boldsymbol{T_{n+1} = T_n \times r,\ T_1 = a}$

Sum of Series and to Infinity Formulae

Arithmetic Sequences (AP)	
Series Sum	$\boldsymbol{S_n = \frac{n}{2}(2a + (n - 1) \times d)}$
Infinity Sum	$\boldsymbol{S_\infty = \infty\ or\ -\infty}$
Geometric Sequences (GP)	
Series Sum*	$\boldsymbol{S_n = \frac{a(1 - r^n)}{1 - r}\ or\ \frac{a(r^n - 1)}{r - 1}}$
*if $-1 < \mathrm{r} < 1$, use the first formula or if $\mathrm{r} > 1$, $\mathrm{r} < -1$, use the second formula.	
Infinity Sum	$\boldsymbol{S_\infty = \frac{a}{1 - r}}$

ARITHMETIC SEQUENCES

Arithmetic Sequence Examples

(Q1) Which term of the following arithmetic sequence 2, 6, 10, … is equal to 110?

- $a = 2$, $d = 4$, using explicit AP formula:

$T_n = a + (n - 1) \times d = 2 + 4(n - 1) = 4n - 2$

$110 = 4n - 2$, $112 = 4n$, $n = 28$, $\therefore \mathbf{28^{th}\ term}$

(Q2) An arithmetic progression has a first term of 8 and a common difference of 3. Find the recursive AP rule and the next 2 terms.

- $a = 8$, $d = 3$, using recursive AP formula:

$\boldsymbol{T_{n+1} = T_n + 3, T_1 = 8}$, $T_2 = \mathbf{11}$, $T_3 = \mathbf{14}$

(Q3) Calculate $9 + 12 + 15 + 18 + \cdots + 138$.

- $a = 9$, $d = 3$, using explicit AP formula:

$T_n = a + (n - 1) \times d = 9 + 3(n - 1) = 3n + 6$

$\therefore 138 = 3n + 6$, $132 = 3n$, $n = 44$, using AP sum of series, find $S_{44} = \frac{n}{2}(2a + (n - 1) \times d) = (44/2)(2(9) + 3(44 - 1)) = 22 \times 147 = \mathbf{3234}$

(Q4) The 3^{rd} term of an AP sequence is 19 and the 20^{th} term is 121. Determine the values of the first 5 terms in this sequence.

- $a =?$, $d =?$, $T_3 = 19$, $T_{20} = 121$
- There are 2 missing variables (a and d)
- $\therefore$ need to solve simultaneously.

Use explicit AP formula: $T_n = a + (n - 1) \times d$

$T_3 = a + d(3 - 1)\ \therefore 19 = a + 2d \rightarrow Eq.\boxed{1}$

$T_{20} = a + d(20 - 1) \therefore 121 = a + 19d \rightarrow Eq.\boxed{2}$

Subtracting $\boxed{2} - \boxed{1}$: $102 = 17d \therefore d = \frac{102}{17} = 6$

$19 = a + 2(6)$, $a = 7 \therefore T_n = 7 + 6(n - 1)$:

$T_1 = \mathbf{7}$, $T_2 = \mathbf{13}$, $T_3 = \mathbf{19}$, $T_4 = \mathbf{25}$, $T_5 = \mathbf{31}$

GEOMETRIC SEQUENCES

Geometric Sequence Examples

(Q1) Find the 10^{th} term of the GP sequence: 2, 2.4, 2.88, 3.456, 4.1472 … to 2 d.p.

- $a = 2$, $r = T_{n+1}/T_n = T_2/T_1 = 2.4/2 = 1.2$
- explicit GP rule: $T_n = a \times r^{n-1} = 2 \times 1.2^{n-1}$

$T_{10} = 2 \times 1.2^{10-1} = 2 \times 1.2^9 = 10.319 \approx \mathbf{10.32}$

(Q2) $x - 1$, x, $x + 2$ are 3 consecutive terms of a GP sequence. Find the value of x.

$r = \frac{T_{n+1}}{T_n} = \frac{T_2}{T_1} = \frac{T_3}{T_2}$, hence $\frac{x}{x-1} = \frac{x+2}{x}$, rearranging this equation and solving: $x^2 = (x - 1)(x + 2)$, $x^2 = x^2 + x - 2$, $0 = x - 2$, $x = \mathbf{2}$

(Q3) Find S_9 for the GP: 3, 6, 12, 24, 48 …

- $a = 3, r = T_{n+1}/T_n = T_2/T_1 = 6/3 = 2$

$\therefore S_9 = \frac{3(2^9 - 1)}{2 - 1} = 3 \times 511 = \mathbf{1533}$

(Q4) Find r if a GP has $a = 40$ and $S_\infty = 400$

$S_\infty = \frac{a}{1 - r}, 400 = \frac{40}{1 - r}, 1 - r = \frac{400}{40}, r = \mathbf{0.9}$

(Q5) Determine S_∞ for the GP $T_n = 5(0.8)^{n-1}$

$a = 5, r = 0.8, S_\infty = \frac{a}{1 - r} = \frac{5}{1 - 0.8} = \frac{5}{0.2} = \mathbf{25}$

(Q6) Determine x if … $8, x, 18,$ … is part of a GP: $8 \times r = x$ and $x \times r = 18$, rearranging 2nd eq. gives $r = \frac{18}{x}$ which substitutes into 1st eq.

$8 \times r = x$ $\qquad x^2 = 18 \times 8 = 144$

$8 \times 18/x = 18$ $\qquad x = \mathbf{\pm 12}$

GROWTH AND DECAY

Exponential Growth and Decay

- Exponential growth/decay is a type of GP.

Growth $(+)$ Formulae	
Explicit	$\boldsymbol{T_n = a \times (1 + r)^t}$
Recursive	$\boldsymbol{T_{n+1} = (1 + r) \times T_n,\ T_1 = a}$
Decay $(-)$ Formulae	
Explicit	$\boldsymbol{T_n = a \times (1 - r)^t}$
Recursive	$\boldsymbol{T_{n+1} = (1 - r) \times T_n,\ T_1 = a}$

Exponential Growth/Decay Examples

(Q1) Write a recursive rule to model 8 rabbits growing in population at a rate of 40% per year.

- Recursive, $a = 8$, $r = 40\% = 0.4$

$T_{n+1} = (1 + 0.4) \times T_n$, $\boldsymbol{T_{n+1} = 1.04T_n, T_1 = 8}$

(Q2) Write a rule to show the area of a $350m^2$ oil slick that reduces by 6% every hour.

- Explicit, $a = 350$, $r = 6\% = 0.06$

$T_n = 350 \times (1 - 0.06)^n$, $\boldsymbol{T_n = 350 \times 0.94^n}$

FUNCTIONS

LINEAR RELATIONS

Linear Equations

$\boldsymbol{y = mx + c}$	$\boldsymbol{m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}}$
• $\boldsymbol{m}$: gradient (*i.e.* steepness of the line). • $\boldsymbol{c}$: $y-$intercept (*i.e.* where the equation crosses the $y-$*axis at the point* $(0, c)$).	

Parallel and Perpendicular Gradients

- Parallel Lines ($m_1 \parallel m_2$) are two lines that never meet (i.e. they have equal gradient).

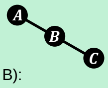
$\boldsymbol{m_1 = m_2}$
• $\boldsymbol{m_1}$ **and** $\boldsymbol{m_2}$ **:** gradient of line 1 and line 2

- Perpendicular Lines ($m_1 \perp m_2$) are two lines that are at 90º to each other.

$\boldsymbol{m_1 = -\frac{1}{m_2}}$	$\boldsymbol{m_1 \times m_2 = -1}$
• $\boldsymbol{m_1}$ **and** $\boldsymbol{m_2}$ **:** gradient of line 1 and line 2	

Midpoint and Endpoint Co-ordinates

A, B & C are three points on a line, point B is the midpoint, points A and C are endpoints.



- Midpoint Co-ordinates (Point B):

$\boldsymbol{\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)}$
• $\boldsymbol{(x_1, y_1)}$ **&** $\boldsymbol{(x_2, y_2)}$**:** both endpoint co-ords

- Endpoint Co-ordinates (Point A or C):

$\boldsymbol{(2x_1 - x_2, 2y_1 - y_2)}$
• $\boldsymbol{(x_1, y_1)}$**:** midpoint co-ords • $\boldsymbol{(x_2, y_2)}$**:** other unused endpoint co-ords

Solving Linear Equations Examples

(Q1) Solve for x: $3(1 - 2x) - 2(x - 1) = 9$

$3 - 6x - 2x + 2 = 9$ $\qquad 8x = -4$

$5 - 8x = 9$ $\qquad x = \mathbf{-0.5}$

(Q2) Solve the following for x: $\frac{4 - 2x}{4} - \frac{x - 2}{5} = \frac{x + 4}{10}$

$\frac{5(4 - 2x) - 4(x - 2)}{20} = \frac{2(x + 4)}{20}$ $\qquad x = \frac{20}{16}$

$\therefore 5(4 - 2x) - 4(x - 2) = 2(x + 4)$ $\qquad x = \frac{5}{4}$

$20 - 10x - 4x + 8 = 2x + 8$

$28 - 14x = 2x + 8 \rightarrow 20 = 16x$ $\qquad x = \mathbf{1.25}$

FINDING LINEAR EQUATIONS

Methods of Finding Linear Equation

- Determine formula given two random co-ordinates (x_1, y_1) and (x_2, y_2) on the line.

Step 1	Calculate gradient $m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$
Step 2	Using $y = mx + c$, substitute either (x_1, y_1) or (x_2, y_2) into x and y, sub in m and rearrange to solve for c.

- Find rule given (x_1, y_1) and line gradient m.

Step 1	Using $y = mx + c$, sub (x_1, y_1) into x and y, sub in m and then rearrange to solve for c.

- Determine formula given one co-ordinate (x_1, y_1) and perpendicular gradient $m_\perp$.

Step 1	Calculate gradient $m = -1 \div m_\perp$
Step 2	Using $y = mx + c$, sub (x_1, y_1) into x and y, sub in m and then rearrange to solve for c.

Linear Equations Example

(Q1) Find equation of line that is perpendicular to $y = 6 - x/3$ and passes through (2,1).

- $Gradient = -1 \div (-1/3) = 3$
- $y = mx + c \rightarrow 1 = 3(2) + c$

$\rightarrow 1 = 6 + c \rightarrow c = -5\ \therefore \boldsymbol{y = -3x - 5}$

QUADRATIC RELATIONS

Types of Quadratic Functions

General Form: $\boldsymbol{y = ax^2 + bx + c}$

• $\boldsymbol{a}$: concavity (*i.e.* orientation of curve).
  ▪ $a > 0$: ∪-shape (*i.e.* concave up/min).
  ▪ $a < 0$: ∩-shape (*i.e.* concave down/max).
• $\boldsymbol{c}$: $y-$intercept (*i.e. at the point* $(0, c)$).
• **Quadratic formula:** x-intercept(s)

Factored Form: $\boldsymbol{y = s(x - t)(x - u)}$
• $\boldsymbol{(t, 0)\ and\ (u, 0)}$**:** x-intercept(s) • $\boldsymbol{a}$: concavity (*i.e.* orientation of curve).

Turning Point Form: $\boldsymbol{y = a(x - h)^2 + k}$
• $\boldsymbol{(h, k)}$ **:** turning point/vertex co-ordinates • $\boldsymbol{x = h}$ **:** line of symmetry (*i.e.* vertical line). • $\boldsymbol{a}$: concavity (*i.e.* orientation of curve).

Quadratic Formula

- Uses general form of a quadratic to find the x co-ordinate(s) of the roots of the parabola.

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Discriminant

- Uses general form to find the number of roots.

$$\boldsymbol{\Delta = b^2 - 4ac}$$

  - If discriminant $\Delta > 0$, parabola has 2 roots.
  - If discriminant $\Delta = 0$, function has 1 root.
  - If discriminant $\Delta < 0$, function has no roots.

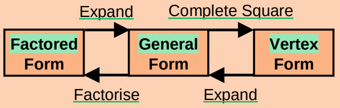
Line of Symmetry *(a.k.a. LOS)*

- Uses general form to find equation of vertical line of symmetry (*i.e.* splits parabola in two).

$$\boldsymbol{Line\ of\ Symmetry\ (LOS) \rightarrow x = \frac{-b}{2a}}$$

QUADRATIC CONVERSIONS

Converting between Quadratic Forms



Method of Completing the Square

Step 1	Factor out a (if any) and determine: $\boldsymbol{y = a\left[\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c\right]}$
Step 2	Simplify equation to vertex form: $\boldsymbol{y = a(x - h)^2 + k}$

Completing the Square Example

(Q1) Complete the square: $y = 2x^2 - 20x - 42$

- Factor out a: $y = 2(x^2 - 10x - 21)$

$y = 2\left[\left(x - \frac{10}{2}\right)^2 - \left(-\frac{10}{2}\right)^2 - 21\right]$

$y = 2[(x - 5)^2 - 25 - 21] \rightarrow \boldsymbol{y = 2(x - 5)^2 - 92}$

CUBIC RELATIONS

Features of a Cubic Function

- **General form**: $y = ax^2 + bx^2 + cx + d$
  - $\boldsymbol{a > 0}$: $x \rightarrow \infty, y \rightarrow \infty$ and $x \rightarrow -\infty, y \rightarrow -\infty$
  - $\boldsymbol{a < 0}$: $x \rightarrow \infty, y \rightarrow -\infty$ and $x \rightarrow -\infty, y \rightarrow \infty$

Factorising a Cubic Function

Step 1	Use guess and check to find one root of the equation (that when you sub into x it equals 0). Test integers starting with $0, 1, -1, 2, -2, \ldots$
Step 2	Write $y = (x - m)(ax^2 + bx + c)$ where m is the root from step 1.
Step 3	By inspection, evaluate a, b and c and factorise the resulting quadratic into two more brackets.
Step 4	Using $y = k(x - m)(x - n)(x - o)$, the roots are $x = m, n, o$

Function Notation and Cubic Functions

Tip 1	$f(a) = 0$ means that $x = a$ is a root of the cubic function (*i.e.* x-intercept)
Tip 2	$f(a) = b$ means that when you sub $x = a$ into the function you get $y = b$

Factorising a Cubic Function Example

(Q1) Find roots of $f(x) = x^2 + 2x^2 - 21x + 18$

- Find first root: $f(0) = 20, f(1) = 0$ is a root

$\therefore (x - 1)(ax^2 + bx + c) = x^2 + 2x^2 - 21x + 18$

$\therefore a = 2, b = -6, c = -20 \rightarrow 2x^2 - 6x - 20$

- Factorise: $2x^2 - 6x - 20 = (2x + 4)(x - 5)$
- Cubic: $f(x) = (x - 1)(2x + 4)(x - 5)$

Pull out factor: $f(x) = 2(x - 1)(x + 2)(x - 5)$

$\therefore$ roots of the cubic function are $x = \mathbf{1, -2, 5}$

ATAR Math Methods
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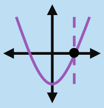
Version: **3.0**

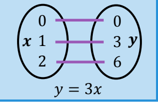
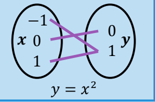
TYPES OF FUNCTIONS

Definition of a Function

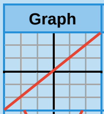
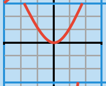
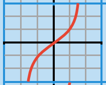
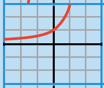
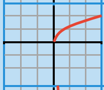
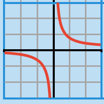
- A function satisfies any of the following:

Passes Vertical Line Test
If all possible vertical lines drawn at all points along the curve cut the curve once, it passes the vertical line test.



One-to-One	Many-to-One
 $y = 3x$	 $y = x^2$

Types of Functions

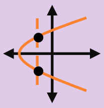
Graph	Type and Description
	**Linear (Direct)** $\boldsymbol{y = mx + c}$ Domain $= \{x \in \mathbb{R}\}$ Range $= \{y \in \mathbb{R}\}$
	**Quadratic** $\boldsymbol{y = a[b(x+c)]^2 + d}$ Domain $= \{x \in \mathbb{R}\}$ Range $= \{y \in \mathbb{R}: y \geq 0\}$
	**Cubic** $\boldsymbol{y = a[b(x+c)]^3 + d}$ Domain $= \{x \in \mathbb{R}\}$ Range $= \{y \in \mathbb{R}\}$
	**Exponential** $\boldsymbol{y = a^{b(x+c)} + d}$ Domain $= \{x \in \mathbb{R}\}$ Range $= \{y \in \mathbb{R}: y > 0\}$
	**Square Root** $\boldsymbol{y = a\sqrt{b(x+c)} + d}$ Domain $= \{x \in \mathbb{R}: x \geq 0\}$ Range $= \{y \in \mathbb{R}: y \geq 0\}$
	**Reciprocal (Inverse)** $\boldsymbol{y = \frac{a}{b(x+c)} + d}$ Domain $= \{x \in \mathbb{R}: x \neq 0\}$ Range $= \{y \in \mathbb{R}: y \neq 0\}$

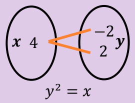
TYPES OF NON-FUNCTIONS

Definition of a Non-Function

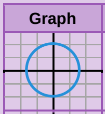
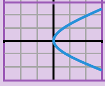
- A non-function (*a.k.a.* a relation) satisfies:

Fails Vertical Line Test
If all vertical lines drawn at all points along the curve cut the curve more than once, it fails the vertical line test.



Many-to-One	
 $y^2 = x$	$y^2 = x \rightarrow y = \pm\sqrt{x}$ $y = \sqrt{x}$ and $y = -\sqrt{x}$ *i.e.* two functions plotted together

Types of Non-Functions

Graph	Type and Description
	**Circle** $\boldsymbol{(x-a)^2 + (y-b)^2 = r^2}$ Domain$=\{x \in \mathbb{R}: -r \leq x \leq r\}$ Range$=\{y \in \mathbb{R}: -r \leq y \leq r\}$
	**Pos/Neg Square Root** $\boldsymbol{y^2 = x}$ Domain $= \{x \in \mathbb{R}: x \geq 0\}$ Range $= \{y \in \mathbb{R}\}$

FUNCTION TRANSFORMATIONS

Function Transformations

- Impact of changing the values in any function in the form: $\boldsymbol{a \times f(b(x+c)) + d}$

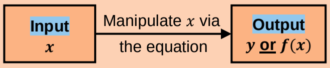
Variable	Condition and Description	
a Multiplies all y-values by **a**	**a** > 0	Dilation in the direction of the y-axis by scale factor **a**
	a < 0	Dilation in the direction of the y-axis by scale factor |**a**| and reflection in the x-axis
b Multiplies all x-values by 1/**b**	**b** > 0	Dilation in the direction of the x-axis by scale factor 1/**b**
	b < 0	Dilation in the direction of the x-axis by scale factor |1/**b**| and reflection in the y-axis
c Adds **c** to all x-values	**c** > 0	Translate horizontally **c** units to the left
	c < 0	Translate horizontally **c** units to the right
d Adds **d** to all y-values	**d** > 0	Translate vertically **d** units upwards
	d < 0	Translate vertically **d** units downwards

- **|a| and |1/b|:** represent absolute values. This means to change the number inside the lines from negative to positive.
  - *e.g.* $|-1| = 1$ or $|-1/2| = 1/2$

FUNCTION NOTATION

Function Notation

- $f(x)$ and y both mean the equation output.



Input x → Manipulate x via the equation → **Output** y or $f(x)$

Function Notation Examples

(Q1) $f(x) = 2x + 1,\ g(x) = x^2 + 3,\ h(x) = \frac{4}{x}$

(Q1a) Determine the value of $f(2)$

$f(x) = 2x + 1 \rightarrow f(2) = 2(2) + 1 \rightarrow f(2) = \mathbf{5}$

(Q1b) Determine the equation of $g(a)$

$g(x) = x^2 + 3 \rightarrow \therefore g(a) = \boldsymbol{a^2 + 3}$

(Q1c) Determine the value of $h(-1) + g(1)$

- $h(-1) + g(1)$ means to sub -1 into $h(x)$ and 1 into $g(x)$ and add the two answers:

$h(-1) + g(1) = \left(\frac{4}{-1}\right) + 1^2 + 3 = -4 + 4 = \mathbf{0}$

(Q1d) Determine the value of a if $f(a) = 1$

- $f(a) = 1$ means to sub a into the function and find what makes it equal to 1.
- $f(a) = 1 \rightarrow 2a + 1 = 1$ then solve for a:

$2a + 1 = 1 \rightarrow 2a = 0 \rightarrow \therefore a = \mathbf{0}$

(Q1e) Determine x that satisfies $f(x) = h(x)$

- $f(x) = h(x)$ means to solve for the value of x that makes both equations equal.

$2x + 1 = \frac{4}{x}$ $\quad 2x^2 + x = 4$ $\quad$ Solve for x

$x(2x+1) = 4$ $\quad 0 = 2x^2 + x - 4$ $\quad = \mathbf{-1.69, 1.19}$

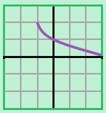
FINDING FUNCTION EQUATIONS

Determining Functions from Graphs

Tip 1	Identify any horizontal or vertical translation (affecting c & d values)
Tip 2	Identify reflection in x-axis or y-axis (making a or b values pos/neg).
Tip 3	Don't choose a root to sub into the function to solve for a variable.
Tip 4	Locate asymptotes (*i.e.* horizontal or vertical lines that the graph doesn't cross) to identify shifts.

Determining Functions Examples

(Q1) Determine the equation of the curve:

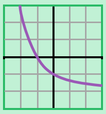


- Type: square root function
- Upward shift of 2: $d = 2$
- Leftward shift of 1: $c = 1$
- Reflect x-axis: $a = negative$
- Use point (0,1) to sub/solve

$y = a\sqrt{x+1} + 2$ $\quad 1 = a + 2 \rightarrow a = -1$

$1 = a\sqrt{0+1} + 2$ $\quad \therefore \boldsymbol{y = -\sqrt{x+1} + 2}$

(Q2) Determine the equation of the curve:

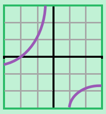


- Type: exponential function
- Down shift of 2: $d = -2$
- No horizontal shift: $c = 0$
- Reflect y-axis: $b = negative$
- Use point $(-2,2)$ to sub/solve

$y = a^{-x} - 2$ $\quad 4 = a^2 \rightarrow a = 2$

$2 = a^{-(-2)} - 2$ $\quad \therefore \boldsymbol{y = 2^{-x} - 2}$

(Q3) Determine the equation of the curve:



- Type: reciprocal function
- Down shift of 1: $d = -1$
- No horizontal shift: $c = 0$
- Reflect y-axis: $a = negative$
- Use point $(-1,1)$ to sub/solve

$y = \frac{a}{x} - 1$ $\quad 1 = \frac{a}{-1} - 1$ $\quad a = -2$ $\quad \therefore \boldsymbol{y = -\frac{2}{x} - 2}$

DOMAIN AND RANGE

Natural Domain and Range

- Natural Domain: what values of x can be inputted into a function (*e.g.* $y = \sqrt{x}$ can only have x values inputted that are ≥ 0).
- Natural Domain: what values of y can be outputted from a function (*e.g.* $y = \sqrt{x}$ can only have y values outputted that are ≥ 0).

Natural Domain	Natural Range
$\{x \in \mathbb{R}: restriction\}$	$\{y \in \mathbb{R}: restriction\}$

Given Domain and Range Examples

(Q1) Find the range for the given domain of $x \geq 2$ for the function $f(x) = (x+3)(x-2)^2$

- Equation represents a cubic function
- Minimum turning point at co-ord (2,0)

Range $= \boldsymbol{\{y \in \mathbb{R}: y \geq 0\}}$

(Q2) Find the range for the given domain of $-3 \leq x < 2$ for the function $f(x) = -3x - 1$

- Equation represents a linear function
- $f(-3) = 8$ and $f(2) = -7$ are the bounds

Range $= \boldsymbol{\{y \in \mathbb{R}: -7 < y \leq 8\}}$

(Q3) Find the domain for the given range of $1 < y \leq 4$ for the function $(x+1)^2 + y^2 = 16$

- Function represents a circle relation that has a radius of 4 and shifted 1 to the left.
- $f(-1) = 4$ (*i.e.* the top of the circle).
- Solving for $f(a) = 1$ gives $a = -1 \pm \sqrt{15}$

Range $= \boldsymbol{\{y \in \mathbb{R}: -1 - \sqrt{15} < y \leq -1 + \sqrt{15}\}}$

DIRECT/INVERSE PROPORTION

Direct Proportion *(Linear)*

- Function where if the input (x) increases, the output (y) increases as well and vice versa.

$\boldsymbol{y = kx}$	• $\boldsymbol{k}$**:** constant of proportionality

Inverse Proportion *(Reciprocal)*

- Function where if the input (x) increases, the output (y) decreases and vice versa.

$\boldsymbol{y = \frac{k}{x}}$	• $\boldsymbol{k}$**:** constant of proportionality

Direct/Inverse Proportion Examples

(Q1) y is directly proportional to x. If $x = 3$ when $y = 15$, what is y when $x = 1$?

$y = kx$ $\quad 15 = 3k$ $\quad k = 5$ $\quad \therefore y = 5(1) = \mathbf{5}$

(Q2) y is inversely proportional to x. If $x = 3$ when $y = 2$, what is y when $x = 2$?

$y = k/x$ $\quad 2 = k/3$ $\quad k = 6$ $\quad \therefore y = 6/2 = \mathbf{3}$

EQUATIONS OF RELATIONS

Circle Relations

- Converting expanded form to completed square form: $\boldsymbol{x^2 + y^2 + ax + by + c = 0}$

Step 1	Using variables above, calculate: $\left(x + \frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 = -c + \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2$
Step 2	Simplify $(x-m)^2 + (x-n)^2 = r^2$ where (m,n) is circle centre and r is the radius of the circle in units.

Positive/Negative Square Root Relation

- Converting to two square root functions:

Step 1	Square root both sides of equation: $y^2 = x \rightarrow \sqrt{y^2} = \pm\sqrt{x} \rightarrow y = \pm\sqrt{x}$

TRIGONOMETRY

RIGHT ANGLE TRIANGLES

Pythagoras' Theorem

2-D Pythagoras	3-D Pythagoras
$c^2 = a^2 + b^2$	$d^2 = a^2 + b^2 + c^2$

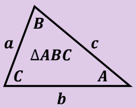
Trigonometric Ratios

Sine	Cosine	Tangent
$sin\theta = \frac{O}{H}$	$cos\theta = \frac{A}{H}$	$tan\theta = \frac{O}{A}$
$\theta = \sin^{-1}\left(\frac{O}{H}\right)$	$\theta = \cos^{-1}\left(\frac{A}{H}\right)$	$\theta = \tan^{-1}\left(\frac{O}{A}\right)$

NON-RIGHT ANGLE TRIANGLES

Triangle Notation

- Angles are capitalized.
- Sides are in lower case.
- Opposing angles and sides have same letter.



Sine Rule and Ambiguous Case

- Used when two pairs of opposing angles and sides are given and one element is missing.

$\boldsymbol{\frac{a}{sinA} = \frac{b}{sinB}}$	$\boldsymbol{\frac{sinA}{a} = \frac{sinB}{b}}$

- Ambiguous Case: Using sine rule to find an angle gives two answers: A and $180 - A$; check question to select which one to use.

Cosine Rule

- Used when three sides and one angle is given and one element is missing.

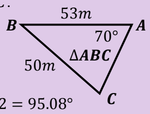
$\boldsymbol{c^2 = a^2 + b^2 - 2 \times a \times b \times cos(C)}$
$\boldsymbol{Angle\ C = cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2 \times a \times b}\right)}$

Area of Non-Right Triangle

$\boldsymbol{Area\ \Delta ABC = \frac{1}{2} \times a \times b \times sin(C)}$

Using Sine and Cosine Rule Examples

(Q1) Find angle B that gives the smallest possible side length of AC:



- Two pairs of angle and side: sine rule

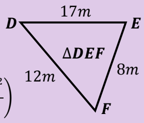
$\frac{\sin(C)}{53} = \frac{\sin(70)}{50}$

$C = 84.92°\ or\ 180 - 84.92 = 95.08°$

- Smallest AC requires smallest angle B

$\therefore Angle\ B = 180 - 70 - 95.08 = \mathbf{14.92°}$

(Q2) Find angle D in the following triangle:



- Three sides and one angle: cosine rule

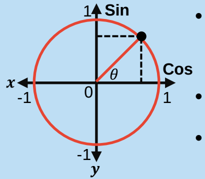
$D = cos^{-1}\left(\frac{e^2 + f^2 - d^2}{2 \times e \times f}\right)$

$D = cos^{-1}\left(\frac{12^2 + 17^2 - 8^2}{2 \times 12 \times 17}\right)$

$D = cos^{-1}\left(\frac{369}{408}\right) = \cos^{-1}(0.9044) = \mathbf{25.26°}$

UNIT CIRCLE

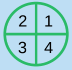
Unit Circle Definitions



- The unit circle is plotted on a set of cartesian axes with a radius of 1.
- Values on x-axis measure $cos(\theta)$.
- Values on y-axis measure $sin(\theta)$.

Pos/Neg of Trigonometric Ratios

- Positive trig ratios: All Stations To Central

Quad.	Q.1	Q.2	Q.3	Q.4	Unit Circle
Sin	+	+	−	−	
Cos	+	−	−	+	
Tan	+	−	+	−	

Range of Trigonometric Ratios

Sin	Cos	Tan
$-1\ to\ 1$	$-1\ to\ 1$	$-\infty\ to\ \infty$

RADIAN MEASURE

Common Angles in Degrees & Radians

Deg.	30	45	60	90	120	135	150	180
Rad.	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Deg.	210	225	240	270	300	315	330	360
Rad.	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π

Converting between Degrees & Radians

Degrees to Radians	Radians to Degrees
$multiply\ by\ \boldsymbol{\frac{\pi}{180}}$	$multiply\ by\ \boldsymbol{\frac{180}{\pi}}$

Reference Angles *(β)*

- Calculates the size that any angle has with the x-axis (used in conjunction with trigonometry).

Quad.	Angle $\boldsymbol{\beta}$	Quad.	Angle $\boldsymbol{\beta}$
1	$\beta = \theta$	3	$\beta = \theta - 180$
2	$\beta = 180 - \theta$	4	$\beta = 360 - \theta$

Exact Values of Trigonometric Ratios

Deg.	$\mathbf{0°}$	$\mathbf{30°}$	$\mathbf{45°}$	$\mathbf{60°}$	$\mathbf{90°}$
Rad.	$\mathbf{0}$	$\boldsymbol{\frac{\pi}{6}}$	$\boldsymbol{\frac{\pi}{4}}$	$\boldsymbol{\frac{\pi}{3}}$	$\boldsymbol{\frac{\pi}{2}}$
Sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	N/A

TRIGONOMETRIC ALGEBRA

Unit Circle Formulae

$\boldsymbol{sin(-x) = -sin(x)}$	$\boldsymbol{cos(-x) = cos(x)}$	$\boldsymbol{tan(-x) = -tan(x)}$

$\boldsymbol{tan(x) = \frac{sin(x)}{cos(x)}}$	$\boldsymbol{sin^2(x) + cos^2(x) = 1}$

Trigonometric Identities

- Sine Angle Sum/Difference Formulae:

$$\mathbf{\sin(a \pm b) = \sin(a)\cos(b) \pm \sin(b)\cos(a)}$$

- Cosine Angle Sum/Difference Formulae:

$$\mathbf{\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)}$$

- Tangent Angle Sum/Difference Formulae:

$$\mathbf{\tan(a \pm b) = \frac{\tan(a) \pm \tan(b)}{1 \mp \boldsymbol{tan(a)tan(b)}}}$$

Trigonometric Algebra Examples

(Q1) Solve $\sqrt{2}sinx + 1 = 0$ for $0° \leq x \leq 360°$

$sinx = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2} \rightarrow x = \mathbf{225°, 315°}$

(Q2) Solve $2(cosx + 1) = 1$ for $0 \leq x \leq 2\pi$

$2cosx + 2 = 1 \rightarrow cosx = -\frac{1}{2} \rightarrow x = \boldsymbol{\frac{2\pi}{3}, \frac{4\pi}{3}}$

(Q3) Solve $4sin^2x - 1 = 0$ for $-\pi \leq x \leq \pi$

$sin^2x = \frac{1}{4} \rightarrow sinx = \pm\frac{1}{2} \rightarrow x = \boldsymbol{-\frac{\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}}$

(Q4) Determine the exact value of $sin(345)$

- Convert using reference angle & unit circle:

$sin(345) = -sin(15) = -sin(60 - 45)$

$= -[sin(60)\,cos(45) - sin(45)\,cos(60)]$

$= sin(45)\,cos(60) - sin(60)\,cos(45)$

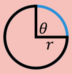
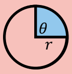
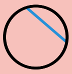
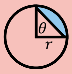
$= \left(\frac{\sqrt{2}}{2} \times \frac{1}{2}\right) - \left(\frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \boldsymbol{\frac{\sqrt{2} - \sqrt{6}}{4}}$

ATAR Math Methods
Units 1 & 2 Exam Notes

Created by **Anthony Bochrinis**
More resources at **reviseonline.com**

CIRCLE MEASURE

Circle Measure Terms

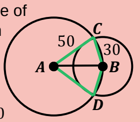
Arc	Sector	Chord	Segment
			

Circle Measure Formulae

Length of an Arc		Area of a Sector	
Radians	Degrees	Radians	Degrees
$r\theta$	$r\left(\frac{\pi\theta}{180}\right)$	$\frac{1}{2}r^2\theta$	$\frac{1}{2}r^2\left(\frac{\pi\theta}{180}\right)$

Area of a Segment (Sector – Triangle)	
Radians	Degrees
$\frac{1}{2}r^2(\theta - sin\theta)$	$\frac{1}{2}r^2\left[\left(\frac{\pi\theta}{180}\right) - sin\left(\frac{\pi\theta}{180}\right)\right]$

Circle Measure Examples



(Q1) Circumference of circle of radius 50m passes through the centre of a circle of radius 30m. Find the area of the overlapping region.

- $AC = AB = 50, CB = 30$
- Cosine rule: $\angle CAB = 34.92°, \angle CBA = 72.54°$

$\therefore \angle CAD = 2\angle CAB = 69.84° = 1.22\, radians$

$\therefore \angle CBD = 2\angle CBA = 145.08° = 2.53\, radians$

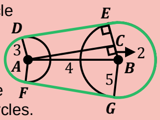
- Calculate area of segment CAD:

$CAD = \frac{1}{2}(50^2)[1.22 - sin(1.22)] = 351.13m^2$

- Calculate area of segment CBD:

$CBD = \frac{1}{2}(30^2)[2.53 - sin(2.53)] = 880.12m^2$

$\therefore Overlap = 351.13 + 880.12 = \mathbf{1231.25m^2}$



(Q2) The centre of a circle of radius 5m is 4m away from the centre of a second circle of radius 3m, find the length of the belt that connects the circles.

- Side $AC = DE = \sqrt{4^2 - 2^2} = \sqrt{12} = 3.46m$
- Calculate arc length for small circle:

$\angle CAB = \sin^{-1}\left(\frac{2}{4}\right) = 30, \angle DAB = 90 + 30$

$= 120°, \angle DAF = 120 \times 2 = 240°$

$\therefore 360 - 240 = 120°$ other side $= 2.09\, radians$

$\therefore Arc\ Length\ DF = 3 \times 2.09 = 6.28m$

- Calculate arc length for large circle:

$\angle CBA = \cos^{-1}\left(\frac{2}{4}\right) = 60, \angle EBG = 2(60) = 120°$

$\therefore 360 - 120 = 240°$ other side $= 4.19\, radians$

$\therefore Arc\ Length\ DF = 5 \times 4.19 = 20.94m$

- Calculate total length of the belt:

$\therefore Belt = 2 \times 3.46 + 12.57 + 20.94 = \mathbf{40.43m}$

TRIGONOMETRIC FUNCTIONS

Period, Amplitude and Phase

- Period: how long it takes for a trigonometric function to complete 1 full cycle.
  - Period relates to 'b' in each equation:

Ratio	Sine	Cosine	Tangent
Period	2π	2π	π
b	$2\pi/Period$	$2\pi/Period$	$\pi/Period$

- Amplitude: maximum vertical distance in units from the x-axis to max/min points.
  - Amplitude relates to 'a' in each equation:

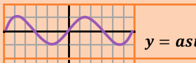
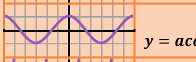
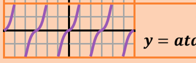
$$\boldsymbol{a = \frac{max\ y_{value} - min\ y_{value}}{2}}$$

- Phase: refers to any left or rightward shifts.
  - Phase relates to 'c' in each equation.
  - Sine and Cosine have a phase shift of $\pi/2$:

$\boldsymbol{sin\theta = cos\left(\theta - \frac{\pi}{2}\right)}$	$\boldsymbol{cos\theta = sin\left(\theta + \frac{\pi}{2}\right)}$

- Vertical Shift: relates to 'd' in each equation.

Sine, Cosine and Tangent Functions

	**Sine** $\boldsymbol{y = asin[b(x+c)] + d}$
	**Cosine** $\boldsymbol{y = acos[b(x+c)] + d}$
	**Tangent** $\boldsymbol{y = atan[b(x+c)] + d}$

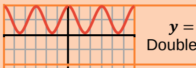
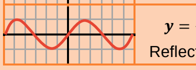
Trigonometric Function Examples

(Q1) The height of a chair on a Ferris wheel is given by $y = 5\cos(\pi t/25) + 7$ where t is time in seconds. Describe the features of this graph.

- Cosine curve starts at maximum @ $\mathbf{(0, 12)}$.
- Amplitude is **5** and is shifted upwards by **7**.
- Graph completes 1 full cycle (period) every:

$b = \frac{2\pi}{Period} \rightarrow \frac{\pi}{25} = \frac{2\pi}{Period} \rightarrow Period = \mathbf{50\ s}$

(Q2) Sketch the following cosine functions:

	$\boldsymbol{y = cos(2x) + 1}$ Double period, up by 1
	$\boldsymbol{y = -cos\left(x + \frac{\pi}{2}\right)}$ Reflect y-axis, left 90º

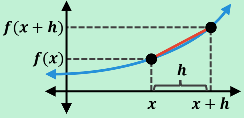
CALCULUS

FIRST PRINCIPLES

Concept of the Derivative

- Derivatives find the gradient at any point of any function with a given equation.
  - Linear functions always have the same gradient at every point along the line.
  - All non-linear functions (such as quadratic and cubic functions) have different values of gradients at different points (i.e. some parts of the curve are steeper/shallower than other parts of the curve).

Derivation of First Principles Formula



- Note: $f(x)$ can be any function.

Step	
Step 1	Find the gradient of red line via the method used for linear equations.

$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$ (x_1, y_1) is $(x, f(x))$ and (x_2, y_2) is $(x + h, f(x + h))$

$\therefore m = \frac{f(x+h) - f(x)}{(x+h) - (x)} = \frac{f(x+h) - f(x)}{h}$

Step 2	Reduce horizontal distance (*i.e.* h) to 0 to find gradient at the point x.

$$\boldsymbol{f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)}$$

First Principle Derivatives Examples

(Q1) Determine the derivative of $f(x) = 2x^2$

- Determine $f(x)$ and $f(x+h)$:

$f(x) = 2x^2$ and $f(x+h) = 2(x+h)^2$

- Use First Principles Derivative formula:

$f'(x) = \lim_{h\to 0}\left(\frac{2(x+h)^2 - 2x^2}{h}\right)$ *Substitute into formula

$f'(x) = \lim_{h\to 0}\left(\frac{2(x^2 + 2xh + h^2) - 2x^2}{h}\right)$ *Expand

$f'(x) = \lim_{h\to 0}\left(\frac{2x^2 + 4xh + 2h^2 - 2x^2}{h}\right)$ *Simplify

$f'(x) = \lim_{h\to 0}\left(\frac{4xh + 2h^2}{h}\right) = \lim_{h\to 0}(4x + 2h)$

- Remove limit by substituting $h = 0$:

$f'(x) = 4x + (2 \times 0) = \mathbf{4x}$

(Q2) Use first principles to find the gradient at the point (2,2) of the function $f(x) = \frac{x}{x-1}$

$f'(2) = \lim_{h\to 0}\frac{\frac{2+h}{2-1+h} - \frac{2}{2-1}}{h} = \lim_{h\to 0}\frac{\frac{2+h}{1+h} - \frac{2}{1}}{h}$

$f'(2) = \lim_{h\to 0}\frac{\frac{2+h-2-2h}{1+h}}{h} = \lim_{h\to 0}\frac{\frac{-h}{1+h}}{h}$

$f'(2) = \lim_{h\to 0}\frac{-1}{1+h} \rightarrow f'(2) = -\frac{1}{1} = \mathbf{-1}$

DIFFERENTIATION

Differentiation Notation

- Lagrange and Leibniz: writing derivatives of functions can be denoted in two ways:

Lagrange Notation	Leibniz Notation
$y \rightarrow \frac{dy}{dx}$	$f(x) \rightarrow f'(x)$

Differentiating Polynomials

- Power Rule: instead of using first principles on polynomials to differentiate, use the rule:

$$\boldsymbol{If\ f(x) = x^n,\ then\ f'(x) = nx^{n-1}}$$

- Each time a polynomial is differentiated, its power is reduced by 1 and follows the pattern shown on the right:

Cubic (x^3)
Quadratic $(3x^2)$
Linear $(6x)$
Constant (6)
Zero (0)

Tip 1	All constants (*i.e.* any number such as 6 or π) differentiate to 0.
Tip 2	Expand and collect like terms of the equation before differentiating it.

Instantaneous Rate of Change *(IROC)*

- The rate that the function changes at a point (*i.e.* the gradient of a function at a point).

$$\boldsymbol{IROC\ @\ time = t: f'(t)}$$

Differentiation Examples

(Q1) Find the derivative of $f(x) = 10x + \pi - x^3$

$\therefore f'(x) = 10 + 0 + 3 \times x^{3-1} = \mathbf{10 - 3x^2}$

(Q2) Find the derivative of $f(x) = (x-2)^3 + x^2$

$f(x) = (x-2)(x-2)(x-2) + x^2$

$f(x) = x^3 - 5x^2 + 12x - 8$ *Expand and simplify first

$\therefore f'(x) = \mathbf{3x^2 - 10x + 12}$

(Q3) Find the derivative of $f(x) = \frac{8}{x^4}$

$f(x) = 8 \times x^{-4} \therefore f'(x) = -32 \times x^{-3} = \mathbf{-32/x^3}$

(Q4) Find the instantaneous rate of change when $x = 1$ of the function $f(x) = -x^2 + 5x$

$f'(x) = -2x + 5, \therefore f'(1) = -2(1) + 5 = \mathbf{3}$

DERIVATIVE APPLICATIONS

Finding Gradient at a Point

Step 1	Determine the derivative of the function $f'(x)$ using the power rule.
Step 2	Sub the x co-ord of the point into the derivative, this is the gradient.

Finding Co-ords with a given Gradient

Step 1	Determine the derivative of the function $f'(x)$ using the power rule.
Step 2	Make the given gradient equal to the derivative and solve for x.
Step 3	Sub the x co-ord found in step 2 into the original equation to find the y co-ord, present answer as (x, y).

Co-ords of a Stationary Point

Step 1	Determine the derivative of the function $f'(x)$ using the power rule.
Step 2	Make derivative equal to 0 and solve for x (*note*: can be more than one answer when solving).
Step 3	Sub the x co-ord found in step 2 into the original equation to find the y co-ord, present answer as (x, y).

Equation of the Tangent at a Point

Step 1	Determine the derivative of the function $f'(x)$ using the power rule.
Step 2	Sub x co-ord of the point into the derivative, this is m in $y = mx + c$.
Step 3	Sub m found in step 2 and x/y co-ord into $y = mx + c$ and solve for c.
Step 4	Write $y = mx + c$ using m from step 2 and c from step 3.

Derivative Application Examples

(Q1) Find gradient of $f(x) = 2x^3 - x$ at $x = 2$

$f'(x) = 6x^2 - 1 \rightarrow f'(2) = 6(2)^2 - 1 = \mathbf{23}$

(Q2) Find all co-ordinates of the function $f(x) = x^2 - x$ that has a gradient of -1.

$f'(x) = 2x - 1 \rightarrow -1 = 2x - 1 \rightarrow 2x = -2$

$\therefore x = -1 \rightarrow f(-1) = (-1)^2 - (-1) = 2$

$\therefore$ The co-ords $\mathbf{(-1, 2)}$ has a gradient of -1.

(Q3) Find the equation to the tangent when $x = 3$ of the function $y = 2x^3 - 5x + 9$

- Substitute x co-ord into to $f(x)$ find y:

$f(3) = 2(3)^3 - 5(3) + 9 = 54 - 15 + 9 = 48$

- Differentiate and sub in x to find gradient:

$f'(x) = 6x - 5, f'(3) = 6(3) - 5 = 13 = m$

- Substitute co-ords and m into $y = mx + c$

$y = mx + c \rightarrow 48 = 13(3) + c \rightarrow 48 = 39 + c$

$c = 48 - 39 = 9, \therefore y = \mathbf{13x + 9}$

OPTIMISATION

Optimising Dimensions of a Scenario

Step 1	Draw a diagram of the scenario and define all variables.
Step 2	If there are more than 2 variables, reduce the number of variables to 2 by substitution and simplification.
Step 3	Determine the derivative of the function, $f'(x)$, using power rule.
Step 4	Make derivative equal to 0 and solve for x to find turning points.
Step 5	Find nature of all turning points by subbing in x co-ord found in step 4 as well as two arbitrary values (above and below) into derivative and find if they are positive (*i.e.* /) or negative (*i.e.* \) to find max/min.

x		TP		x		TP	
dy/dx	\	–	/	dy/dx	/	–	\
∴ Minimum TP				∴ Maximum TP			

Step 6	Find optimal dimensions and maximum or minimum value required according to question.

Optimisation Example

(Q1) A cylinder has the sum of two times its radius and its height equal to 10cm. Find the dimensions that maximises its volume.

- Identify all equations relevant to question:

$2r + h = 10$ and Cylinder Volume: $V = \pi r^2 h$

- Reduce to two variables by substitution:

Rearranging: $2r + h = 10 \rightarrow h = 10 - 2r$, sub:
$V = \pi r^2 h = \pi r^2(10 - 2r) = 10\pi r^2 - 2\pi r^3$

- Find derivative and test all turning points:

$\frac{dV}{dr} = 20\pi r - 6\pi r^2 \rightarrow 0 = 20\pi r - 6\pi r^2$

$\therefore 0 = \pi r(20 - 6r) \rightarrow r = 0cm, 20/6 = \mathbf{3.33cm}$

x	3	3.33	4	*Note*: ignore $r = 0cm$ as it's an impossible result in context of the question.
dy/dx	/	–	\	
∴ Maximum TP				

- Find dimensions and maximum volume:

$h = 10 - 2r = 10 - 2(3.33) = \mathbf{3.33cm}$ as well

$\therefore Max\ V = \pi(3.33)^2(3.333) = \mathbf{116.36cm^3}$

With radius of $\mathbf{3.33cm}$ and height of $\mathbf{3.33cm}$

SKETCHING FUNCTIONS

Sketching Complex Polynomials

Step 1	Determine co-ords of the y intercept by subbing $x = 0$ into the equation.
Step 2	Determine co-ords of all x intercepts by factorising and solving.
Step 3	Find co-ords of stationary points by finding the derivative of the function and solving for when it equals 0.
Step 4	Find the nature of each turning point by running the max/min test.
Step 5	Find long term behaviour for y values as x tends toward $\pm\infty$.

Sketching Functions Example

(Q1) Sketch the polynomial $y = 3x^2 - 2x^3$

- Finding all x and y intercept co-ords:

$y = 3(0)^2 - 2(0)^3 = 0, \therefore y - int\ (0,0)$

$0 = x^2(3 - 2x), x = 0, 1.5 \therefore x - int\ (0,0), (1.5,0)$

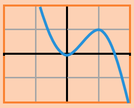
- Finding location and nature of turning points:

$\frac{dy}{dx} = 6x - 6x^2 \rightarrow 0 = 6x - 6x^2 \rightarrow 0 = 6x(1 - x)$

$\therefore$ Turning Point when $x = 0$ and $x = 1$:

TP nature at $x = 0$				TP nature at $x = 1$			
x	-0.5	0	0.5	x	0.5	1	1.5
dy/dx	\	–	/	dy/dx	/	–	\
∴ Min at (0,0)				∴ Max at (1,1)			

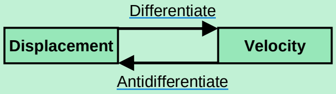
- Long term behaviour as x tends toward $\pm\infty$:
  $x \rightarrow +\infty, y \rightarrow -\infty$
  $x \rightarrow -\infty, y \rightarrow +\infty$
- Sketching polynomial:



RECTILINEAR MOTION

Displacement and Velocity

- Displacement (s): distance from origin.
- Velocity (v): speed toward/away from origin.



Differentiate

Displacement | Velocity

Antidifferentiate

Var.	Displacement	Velocity
+ **Pos.**	Positioned in front of origin	Moving towards the origin
– **Neg.**	Positioned behind origin	Moving away from origin

Rectilinear Motion Example

(Q1) Position of particle in metres and seconds according to $s = (t - 1)(t - 4)$ over $0 \leq t \leq 7$.

(Q1a) What is the initial displacement at $t = 0$?

$s = (0 - 1)(0 - 4) = -1 \times -4 = \mathbf{4\ metres}$

(Q1b) What is the velocity of particle at $t = 0$?

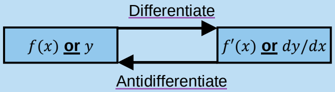
$s = (t - 1)(t - 4) = t^2 - 5t + 4$ *Expanding

$\therefore v = \frac{ds}{dt} = 2t - 5 \rightarrow v = 2(0) - 5 = \mathbf{-5m/s}$

ANTIDIFFERENTIATION

Antiderivative Notation

- Finds original function from derivative function.



Differentiate

$f(x)$ **or** y | $f'(x)$ **or** dy/dx

Antidifferentiate

Antidifferentiating Polynomials

$\boldsymbol{f(x) = \int f'(x)\, dx}$	$\boldsymbol{\int x^n\, dx = \frac{x^{n+1}}{n+1} + c}$

- Antiderivatives are written as $\int f'(x)\, dx$ which means "integrate function with respect to x".
- All antiderivatives produce a constant $(+c)$ as it caters for possibility of there being a constant in the original function $f(x)$, which disappears (*i.e.* becomes 0) after being differentiated.

Step 1	Use the integral rule for polynomials on the derivative function.
Step 2	Sub x & y co-ord into antiderivative function and solve for c.

Antidifferentiation Examples

(Q1) Find the integral $\int 6x^2 + 2x\, dx$

$\therefore f(x) = 6x^3/3 + 2x^2/2 + c = \mathbf{2x^3 + x^2 + c}$

(Q2) Find the integral $\int \sqrt[3]{x^2}\, dx$

$f'(x) = x^{\frac{2}{3}} \therefore f(x) = x^{\frac{5}{3}}/(5/3) + c = \mathbf{3x^{\frac{5}{3}}/5 + c}$

(Q3) Find $f(1)$ of $f'(x) = 3 - 2x$ if $f(2) = 3$

$f(x) = \int 3 - 2x\, dx = 3x - \frac{2x^2}{2} + c = 3x - x^2 + c$

$\therefore 3 = 3(2) - (2)^2 + c \rightarrow 3 = 2 + c \rightarrow c = 1$

$\therefore f(x) = 3x - x^2 + 1 \rightarrow f(1) = 3 - 1 + 1 = \mathbf{3}$

ATAR Math Methods
Units 1 & 2 Exam Notes

Created by **Anthony Bochrinis**
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