Chapter 1 – Reviewing linear equations

Solutions to Exercise 1A

1 a $x + 3 = 6$

$\therefore \quad x = 3$

b $x - 3 = 6$

$\therefore \quad x = 9$

c $3 - x = 2$

$-x = -1$

$\therefore \quad x = 1$

d $x + 6 = -2$

$x + 8 = 0$

$\therefore \quad x = -8$

e $2 - x = -3$

$-x = -5$

$\therefore \quad x = 5$

f $2x = 4$

$\therefore \quad x = 2$

g $3x = 5$

$\therefore \quad x = \frac{5}{3}$

h $-2x = 7$

$\therefore \quad x = -\frac{7}{2}$

i $-3x = -7$

$\therefore \quad x = \frac{7}{3}$

j $\frac{3x}{4} = 5$

$3x = 20$

$\therefore \quad x = \frac{20}{3}$

k $-\frac{3x}{5} = 2$

$-3x = 10$

$\therefore \quad x = -\frac{10}{3}$

l $-\frac{5x}{7} = -2$

$-5x = -14$

$\therefore \quad x = \frac{-14}{-5} = \frac{14}{5}$

2 a $x - b = a$

$\therefore \quad x = a + b$

b $x + b = a$

$\therefore \quad x = a - b$

c $ax = b$

$\therefore \quad x = \frac{b}{a}$

d $\frac{x}{a} = b$

$\therefore \quad x = ab$

e $\frac{ax}{b} = c$

$ax = bc$

$\therefore \quad x = \frac{bc}{a}$

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3 **a**

$$\begin{aligned} 2y - 4 &= 6 \\ 2y &= 10 \\ y &= 5 \end{aligned}$$

b

$$\begin{aligned} 3t + 2 &= 17 \\ 3t &= 15 \\ t &= 5 \end{aligned}$$

c

$$\begin{aligned} 2y + 5 &= 2 \\ 2y &= -3 \\ y &= -\frac{3}{2} \end{aligned}$$

d

$$\begin{aligned} 7x - 9 &= 5 \\ 7x &= 14 \\ x &= 2 \end{aligned}$$

e

$$\begin{aligned} 2a - 4 &= 7 \\ 2a &= 11 \\ a &= \frac{11}{2} \end{aligned}$$

f

$$\begin{aligned} 3a + 6 &= 14 \\ 3a &= 8 \\ a &= \frac{8}{3} \end{aligned}$$

g

$$\begin{aligned} \frac{y}{8} - 11 &= 6 \\ \frac{y}{8} &= 17 \\ y &= 136 \end{aligned}$$

h

$$\begin{aligned} \frac{t}{3} + \frac{1}{6} &= \frac{1}{2} \\ \frac{t}{3} &= \frac{1}{3} \\ t &= 1 \end{aligned}$$

i

$$\begin{aligned} \frac{x}{3} + 5 &= 9 \\ \frac{x}{3} &= 4 \\ x &= 12 \end{aligned}$$

j

$$\begin{aligned} 3 - 5y &= 12 \\ -5y &= 9 \\ y &= -\frac{9}{5} \end{aligned}$$

k

$$\begin{aligned} -3x - 7 &= 14 \\ -3x &= 21 \\ x &= -7 \end{aligned}$$

l

$$\begin{aligned} 14 - 3y &= 8 \\ -3y &= -6 \\ y &= 2 \end{aligned}$$

4 **a** $6x - 4 = 3x$

$$\begin{aligned} 3x &= 4 \\ \therefore \quad x &= \frac{4}{3} \end{aligned}$$

b $x - 5 = 4x + 10$

$$\begin{aligned} -3x &= 15 \\ \therefore \quad x &= \frac{15}{-3} = -5 \end{aligned}$$

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c $3x - 2 = 8 - 2x$

$5x = 10$

$\therefore \quad x = 2$

5 **a** $2(y + 6) = 10$

$y + 6 = 5$

$\therefore \quad y = 5 - 6 = -1$

b $2y + 6 = 3(y - 4)$

$2y + 6 = 3y - 12$

$-y = -18$

$\therefore \quad y = 18$

c $2(x + 4) = 7x + 2$

$2x + 8 = 7x + 2$

$-5x = -6$

$\therefore \quad x = \frac{6}{5}$

d $5(y - 3) = 2(2y + 4)$

$5y - 15 = 4y + 8$

$5y - 4y = 18 + 8$

$\therefore \quad y = 23$

e $x - 6 = 2(x - 3)$

$x - 6 = 2x - 6$

$-x = 0$

$\therefore \quad x = 0$

f $\frac{y + 2}{3} = 4$

$y + 2 = 12$

$\therefore \quad y = 10$

g $\frac{x}{2} + \frac{x}{3} = 10$

$\frac{5x}{6} = 10$

$5x = 60$

$\therefore \quad x = 12$

h $x + 4 = \frac{3x}{2}$

$-\frac{x}{2} = -4$

$-x = -8$

$\therefore \quad x = 8$

i $\frac{7x + 3}{2} = \frac{9x - 8}{4}$

$14x + 6 = 9x - 8$

$5x = -14$

$\therefore \quad x = -\frac{14}{5}$

j $\frac{2}{3}(1 - 2x) - 2x = -\frac{2}{5} + \frac{4}{3}(2 - 3x)$

$10(1 - 2x) - 30x = -6 + 20(2 - 3x)$

$10 - 20x - 30x = -6 + 40 - 60x$

$10x = 24$

$\therefore \quad x = \frac{12}{5}$

k $\frac{4y - 5}{2} - \frac{2y - 1}{6} = y$

$(12y - 15) - (2y - 1) = 6y$

$12y - 15 - 2y + 1 = 6y$

$4y = 14$

$\therefore \quad y = \frac{7}{2}$

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6 a $ax + b = 0$

$$ax = -b$$

$$\therefore \quad x = -\frac{b}{a}$$

b $cx + d = e$

$$cx = e - d$$

$$\therefore \quad x = \frac{e - d}{c}$$

c $a(x + b) = c$

$$x + b = \frac{c}{a}$$

$$\therefore \quad x = \frac{c}{a} - b$$

d $ax + b = cx$

$$ax - cx = -b$$

$$x(c - a) = b$$

$$\therefore \quad x = \frac{b}{c - a}$$

e $\frac{x}{a} + \frac{x}{b} = 1$

$$bx + ax = ab$$

$$x(a + b) = ab$$

$$\therefore \quad x = \frac{ab}{a + b}$$

f $\frac{a}{x} + \frac{b}{x} = 1$

$$\therefore \quad x = a + b$$

g $ax - b = cx - d$

$$ax - cx = b - d$$

$$x(a - c) = b - d$$

$$\therefore \quad x = \frac{b - d}{a - c}$$

h $\frac{ax + c}{b} = d$

$$ax + c = bd$$

$$ax = bd - c$$

$$\therefore \quad x = \frac{bd - c}{a}$$

7 a $0.2x + 6 = 2.4$

$$0.2x = -3.6$$

$$\therefore \quad x = -18$$

b $0.6(2.8 - x) = 48.6$

$$2.8 - x = 81$$

$$-x = 78.2$$

$$\therefore \quad x = -78.2$$

c $\frac{2x + 12}{7} = 6.5$

$$2x + 12 = 45.5$$

$$x + 6 = 22.75$$

$$\therefore \quad x = 16.75$$

d $0.5x - 4 = 10$

$$0.5x = 14$$

$$\therefore \quad x = 28$$

e $\frac{1}{4}(x - 10) = 6$

$$x - 10 = 24$$

$$\therefore \quad x = 34$$

f $6.4x + 2 = 3.2 - 4x$

$$10.4x = 1.2$$

$$\therefore \quad x = \frac{1.2}{10.4} = \frac{3}{26}$$

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8

$$\frac{b-cx}{a}+\frac{a-cx}{b}+2=0$$
$$b(b-cx)+a(a-cx)+2ab=0$$
$$b^2-bcx+a^2-acx+2ab=0$$
$$b^2+a^2+2ab=acx+bcx$$
$$(a+b)^2=cx(a+b)$$
$$\therefore \quad x=\frac{a+b}{c}$$

so long as $a+b\neq 0$

9

$$\frac{a}{x+a}+\frac{b}{x-b}=\frac{a+b}{x+c}$$
$$\frac{a(x-b)+b(x+a)}{(x+a)(x-b)}=\frac{a+b}{x+c}$$
$$\frac{ax-ab+bx+ab}{(x+a)(x-b)}=\frac{a+b}{x+c}$$
$$\frac{ax+bx}{(x+a)(x-b)}=\frac{a+b}{x+c}$$
$$\frac{x}{(x+a)(x-b)}=\frac{1}{x+c}$$
$$x(x+c)=(x+a)(x-b)$$
$$x^2+cx=x^2+ax-bx-ab$$
$$cx-ax+bx=-ab$$
$$x(a-b-c)=ab$$
$$\therefore \quad x=\frac{ab}{a-b-c}$$

so long as $a+b\neq 0$

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Solutions to Exercise 1B

1 **a** $x + 2 = 6$

$\therefore \quad x = 4$

b $3x = 10$

$\therefore \quad x = \frac{10}{3}$

c $3x + 6 = 22$

$3x = 16$

$\therefore \quad x = \frac{16}{3}$

d $3x - 5 = 15$

$3x = 20$

$\therefore \quad x = \frac{20}{3}$

e $6(x + 3) = 56$

$x + 3 = \frac{56}{6} = \frac{28}{3}$

$\therefore \quad x = \frac{19}{3}$

f $\frac{x + 5}{4} = 23$

$x + 5 = 92$

$\therefore \quad x = 87$

2 $A + 3A + 2A = 48$

$6A = 48$

$\therefore \quad A = 8$

A gets \$8, B \$24 and C \$16

3 $y = 2x; x + y = 42 = 3x$

$x = \frac{42}{3}$

$\therefore \quad x = 14, y = 28$

4 $\frac{x}{3} + \frac{1}{3} = 3$

$x + 1 = 9$

$\therefore \quad x = 8$ kg

5 $L = w + 0.5; A = Lw$

$P = 2(L + w)$

$= 2(2w + 0.5)$

$= 4w + 1$

$4w + 1 = 4.8$

$4w = 3.8$

$\therefore \quad w = 0.95$

$A = 0.95(0.95 + 0.5)$

$= 1.3775 \text{ m}^2$

6 $(n - 1) + n + (n + 1) = 150$

$3n = 150$

$\therefore \quad n = 50$

Sequence = 49, 50 & 51, assuming n is the middle number.

7 $n + (n + 2) + (n + 4) + (n + 6) = 80$

$4n + 12 = 80$

$4n = 68$

$\therefore \quad n = 17$

17, 19, 21 and 23 are the odd numbers.

8 $6(x - 3000) = x + 3000$

$6x - 18000 = x + 3000$

$5x = 21000$

$\therefore \quad x = 4200$ L

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9 $140(p-3) = 120p$

$140p - 420 = 120p$

$20p = 420$

$\therefore \quad p = 21$

10 $\dfrac{x}{6} + \dfrac{x}{10} = \dfrac{48}{60}$

$5x + 3x = 24$

$8x = 24$

$x = 3$ km

11 Profit = x for crate 1 and $0.5x$ for crate 2, where x = amount of dozen eggs in each crate.

$x + \dfrac{x+3}{2} = 15$

$2x + x + 3 = 30$

$3x = 27$

$\therefore \quad x = 9$

Crate 1 has 9 dozen, crate 2 has 12 dozen.

12 $3\left(\dfrac{45}{60}\right) + x\left(\dfrac{30}{60}\right) = 6$

$\dfrac{9}{4} + \dfrac{x}{2} = 6$

$\dfrac{x}{2} = \dfrac{15}{4}$

$\therefore \quad x = \dfrac{15}{2} = 7.5\text{km/hr}$

13

$t = \dfrac{x}{4} + \dfrac{x}{6} = \dfrac{45}{60}$

$60 \times \dfrac{x}{4} + 60 \times \dfrac{x}{6} = 45$

$15x + 10x = 45$

$25x = 45$

$x = \dfrac{45}{25}$

$= \dfrac{9}{5}$

$= 1.8$

Total $= 2 \times 1.8$

$= 3.6$ km (there and back)

Total $= 4 \times 0.9$

$= 3.6$ km there and back twice

14 $f = b + 24$

$(f+2) + (b+2) = 40$

$b + 26 + b + 2 = 40$

$2b = 12$

$\therefore \quad b = 6$

The boy is 6, the father 30.

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Solutions to Exercise 1C

1 a $y = 2x + 1 = 3x + 2$

$-x = 1, \therefore x = -1$

$\therefore y = 2(-1) + 1 = -1$

b $y = 5x - 4 = 3x + 6$

$2x = 10, \therefore x = 5$

$\therefore y = 5(5) - 4 = 21$

c $y = 2 - 3x = 5x + 10$

$-8x = 8, \therefore x = -1$

$\therefore y = 2 - 3(-1) = 5$

d $y - 4 = 3x \quad (1)$
$y - 5x + 6 = 0 \quad (2)$
From (1) $y = 3x + 4$
Subsitute in (2).

$$\begin{aligned} 3x + 4 - 5x + 6 &= 0 \\ -2x + 10 &= 0 \\ x &= 5 \end{aligned}$$

Substitute in (1). $y - 4 = 15$.
Therefore $x = 5$ and $y = 19$.

e $y - 4x = 3 \quad (1)$
$2y - 5x + 6 = 0 \quad (2)$
From (1) $y = 4x + 3$
Subsitute in (2).

$$\begin{aligned} 2(4x + 3) - 5x + 6 &= 0 \\ 3x + 12 &= 0 \\ x &= -4 \end{aligned}$$

Substitute in (1). $y + 16 = 3$.
Therefore $x = -4$ and $y = -13$.

f $y - 4x = 6 \quad (1)$
$2y - 3x = 4 \quad (2)$
From (1) $y = 4x + 6$
Subsitute in (2).

$$\begin{aligned} 2(4x + 6) - 3x &= 4 \\ 5x + 12 &= 4 \\ 5x &= -8 \\ x &= -\frac{8}{5} \end{aligned}$$

Substitute in (1). $y - 4 \times \left(-\frac{8}{5}\right) = 6$.
$y = \frac{50}{3}$
Therefore $x = -\frac{8}{5}$ and $y = -\frac{2}{5}$.

2 a $x + y = 6$

$$\begin{array}{rl} x - y &= 10 \\ \hline 2x &= 16 \end{array}$$

$\therefore \quad x = 8; y = 6 - 8 = -2$

b $y - x = 5$

$$\begin{array}{rl} y + x &= 3 \\ \hline 2y &= 8 \end{array}$$

$\therefore \quad y = 4; x = 3 - 4 = -1$

c $x - 2y = 6$

$$\begin{array}{rl} -(x + 6y &= 10) \\ \hline -8y &= -4 \end{array}$$

$\therefore \ y = \frac{1}{2}, \ x = 6 + \frac{2}{2} = 7$

3 a $2x - 3y = 7$

$$\begin{array}{rl} 9x + 3y &= 15 \\ \hline 11x &= 22 \end{array}$$

$\therefore \ px = 2$

$4 - 3y = 7, \therefore \ y = -1$

b $4x - 10y = 20$

$$\begin{array}{rl} -(4x + 3y &= 7) \\ \hline \end{array}$$

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$-13y = 13$
$\therefore\ y = -1$
$4x - 3 = 7, \therefore\ x = 2.5$

c $4m - 2n = 2$

$$\begin{array}{l} m + 2n = 8 \\ \hline 5m \qquad = 10 \end{array}$$

$\therefore\ m = 2$
$8 - 2n = 2, \therefore\ n = 3$

d $14x - 12y = 40$

$$\begin{array}{l} 9x + 12y = 6 \\ \hline 23x \qquad = 46 \end{array}$$

$\therefore\ x = 2$
$14 - 6y = 20, \therefore y = -1$

e $6s - 2t = 2$

$$\begin{array}{l} 5s + 2t = 20 \\ \hline 11s \qquad = 22 \end{array}$$

$\therefore\ s = 2$
$6 - t = 1, \therefore\ t = 5$

f $16x - 12y = 4$

$$\begin{array}{l} -15x + 12y = 6 \\ \hline x \qquad = 10 \end{array}$$

$\therefore\ 4y - 5(10) = 2$
$\therefore\ y = 13$

g $15x - 4y = 6$

$$\begin{array}{l} -(18x - 4y = 10) \\ \hline -3x \qquad = -4 \end{array}$$

$\therefore\ x = \frac{4}{3}$

$9\left(\frac{4}{3}\right) - 2y = 5$

$-2y = -7, \therefore\ y = \frac{7}{2}$

h $2p + 5q = -3$

$$\begin{array}{l} 7p - 2q = 9 \\ \hline \end{array}$$

$4p + 10q = -6\ 39p = 39$

$$\begin{array}{l} 35p - 10q = 45 \\ \hline p = 1 \end{array}$$

$\therefore\ q = -1$

i $2x - 4y = -12$

$$\begin{array}{l} 6x + 4y = 4 \\ \hline 8x \qquad = -8 \end{array}$$

$\therefore\ x = -1$

$2y - 3 - 2 = 0, \therefore\ y = \frac{5}{2}$

4 a $3x + y = 6 \qquad (1)$
$6x + 2y = 7 \qquad (2)$
Multiply (1) by 2.
$6x + 2y = 12 \qquad (3)$
Subtract (2) from (3)
$0 = 5$.
There are no solutions.
The graphs of the two straight lines are parallel.

b $3x + y = 6 \qquad (1)$
$6x + 2y = 12 \qquad (2)$
Multiply (1) by 2.
$6x + 2y = 12 \qquad (3)$
Subtract (2) from (3)
$0 = 0$.
There are infinitely many solutions.
The graphs of the two straight lines coincide.

c $3x + y = 6 \qquad (1)$
$6x - 2y = 7 \qquad (2)$
Multiply (1) by 2.
$6x + 2y = 12 \qquad (3)$
Add (2) and (3)
$12x = 19$.
$x = \frac{19}{12}$ and $y = \frac{5}{4}$. There is only one solution.

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The graphs intersect at the point$\left(\frac{19}{12}, \frac{5}{4}\right)$

d $3x - y = 6 \qquad (1)$

$6x + 2y = 7 \qquad (2)$

Multiply (1) by 2.

$6x - 2y = 12 \qquad (3)$

Add (2) and (3)

$12x = 19$.

$x = \frac{19}{12}$ and $y = -\frac{5}{4}$. There is only one solution.

The graphs intersect at the point$\left(\frac{19}{12}, -\frac{5}{4}\right)$

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Solutions to Exercise 1D

1 $x + y = 138$

$$\begin{array}{rl} x - y &= 88 \\ \hline 2x &= 226 \\ \therefore \quad x &= 113 \\ y &= 138 - 113 = 25 \end{array}$$

2 $x + y = 36$

$$\begin{array}{rl} x - y &= 9 \\ \hline 2x &= 45 \\ \therefore \quad x &= 22.5 \\ y &= 36 - 22.5 = 13.5 \end{array}$$

3 $6S + 4C = 58$

$5S + 2C = 35, \therefore\ 10S + 4C = 70$

$$\begin{array}{rl} 10S + 4C &= 70 \\ -(6S + 4C) &= 58 \\ \hline 4S &= 12 \end{array}$$

$\therefore\ S = \$3$

$2C = 35 - 35, \therefore\ C = \$10$

a $10S + 4C = 10 \times 3 + 4 \times 10$

$= 30 + 40 = \$70$

b $4S = 4 \times 3 = \$12$

c $S = \$3$

4 $7B + 4W = 213$

$B + W = 42, \therefore\ 4B + 4W = 168$

$$\begin{array}{rl} 7B + 4W &= 213 \\ -(4B + 4W &= 168) \\ \hline 3B &= 45 \end{array}$$

$\therefore\ B = 15$

$15 + W = 42, \therefore\ W = \$27$

a $4B + 4W = 4 \times 15 + 4 \times 27$

$= 60 + 108 = \$168$

b $3B = 3 \times 15 = \$45$

c $B = \$15$

5 $x + y = 45$

$$\begin{array}{rl} x - 7 &= 11 \\ \hline 2x &= 56 \end{array}$$

$\therefore\ x = 28;\ y = 17$

6 $m + 4 = 3(c + 4) \ldots (1)$

$m - 2 = 5(c - 4) \ldots (2)$

From (1), $m = 3c + 8$.

Substitute into (2):

$$\begin{aligned} 3c + 8 - 4 &= 5(c - 4) \\ 3c + 4 &= 5c - 20 \\ -2c &= -24, \therefore\ c = 12 \\ \therefore\ m - 4 &= 5(12 - 4) \\ m &= 44 \end{aligned}$$

7
$$\begin{aligned} h &= 5p \\ h + p &= 20 \\ \therefore\ 5p + p &= 30 \\ \therefore\ p &= 5;\ h = 25 \end{aligned}$$

8 Let one child have x marbles and the other y marbles.

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$$x + y = 110$$
$$\frac{x}{2} = y - 20$$
$$\therefore \quad x = 2y - 40$$
$$\therefore \; 2y - 40 + y = 110$$
$$3y = 150$$
$$\therefore \quad y = 50; \; x = 60$$
They started with 50 and 60 marbles, and finished with 30 each.

9 Let x be the number of adult tickets and y be the number of child tickets.
$x + y = 150$ (1)
$4x + 1.5y = 560$ (2)
Multiply (1) by 1.5.
$1.5x + 1.5y = 225$ (1')
Subtract (1') from (2)
$2.5x = 335$
$x = 134$
Substitute in (1). $y = 16$
There were 134 adult tickets and 16 child tickets sold.

10 Let a be the numerator and b be the denominator.
$a + b = 17$ (1)
$\frac{a+3}{b} = 1$ (2).
From (2), $a + 3 = b$ (1')
Substitute in (1)
$a + a + 3 = 17$
$2a = 14$
$a = 7$ and hence $b = 10$.
The fraction is $\frac{7}{10}$

11 Let the digits be m and n.
$m + n = 8$ (1)
$10n + m - (n + 10m) = 36$
$9n - 9m = 36$
$n - m=4$ (2)
Add (1) and (2)
$2n = 12$ implies $n = 6$.
Hence m = 2.
The initial number is 26 and the second number is 62.

12 Let x be the number of adult tickets and y be the numbr of child tickets.
$x + y = 960$ (1) $30x + 12y = 19\,080$ (2)
Multiply (1) by 12. $12x + 12y = 11\,520$ (1')
Subtract (1') from (2).
$18x = 7560$
$x = 420$.
There were 420 adults and 540 children.

13 $0.1x + 0.07y = 1400 \ldots (1)$
$0.07x + 0.1y = 1490 \ldots (2)$
From (1), $x = (14\,000 - 0.7y)$
From (2):
$$0.07(14\,000 - 0.7y) + 0.1y = 1490$$
$$\therefore \quad 980 - 0.049\,y + 0.1y = 1490$$
$$0.051y = 510$$
$$\therefore \quad y = \frac{510}{.051}$$
$$= 10\,000$$
From (1):
$$0.1x + 0.07 \times 10\,000 = 1400$$
$$0.1x = 1400 - 700$$
$$= 700$$
$$\therefore \quad x = 7000$$
So $x + y = \$17\,000$ invested.

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14 $\frac{100s}{3} + 20t = 10\,000 \ldots (1)$

$$\left(\frac{100}{3}\right)\left(\frac{s}{2}\right) + 20\left(\frac{2t}{3}\right) = 6000$$

$$\therefore \quad \left(\frac{50s}{3}\right) + \frac{40t}{3} = 6000 \ldots (2)$$

From (1):

$$20t = 10\,000 - \frac{100s}{3}$$

$$\therefore \ t = 500 - \frac{5s}{3} \ldots (3)$$

Substitute into (2):

$$\left(\frac{50s}{3}\right) + \left(\frac{40}{3}\right)\left(500 - \frac{5s}{3}\right) = 6000$$

$$150s + 120\left(500 - \frac{5s}{3}\right) = 54\,000$$

$$150s + 60\,000 - 200s = 54\,000$$

$$-50s = -6000$$

$$\therefore \quad s = 120$$

Substitute into (3):

$$t = 500 - \left(\frac{5}{3}\right) \times 120$$

$$= 500 - 200$$

$$\therefore \ t = 300$$

He sold 120 shirts and 300 ties.

15 Outback $= x$, BushWalker $= y$; $x = 1.2$y

$$200x + 350y = 177\,000$$

$$200(1.2y) + 350y = 177\,000$$

$$240y + 350y = 177\,000$$

$$\therefore \quad y = \frac{177\,000}{590} = 300$$

$$\therefore \quad x = 1.2 \times 300$$

$$= 360$$

16 Mydney $= x$ jeans; Selbourne $= y$ jeans

$$30x + 28\,000 = 24y + 35\,200 \ldots (1)$$

$$x + y = 6000 \ldots (2)$$

From (2): $y = 6000 - x$

Substitute in (1):

$$30x + 28\,000 = 24(6000 - x) + 35\,200$$

$$30x + 28\,000 = 144\,000 - 24x + 35\,200$$

$$54x = 151\,200$$

$$\therefore \quad x = 2800\,;\, y = 3200$$

17 Tea $A = \$10; B = \$11, C = \$12$ per kg

$B = C; C + B + A = 100$

$$10A + 11B + 12C = 1120$$

$$10A + 23B = 1120$$

$$\therefore \quad A = 100 - 2B$$

$$10(100 - 2B) + 23B = 1120$$

$$3B = 1120 - 1000$$

$$\therefore \quad B = 40$$

$A = 20$kg, $B = C = 40$ kg

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Solutions to Exercise 1E

1 a $x + 3 < 4$

$x < 4 - 3, \therefore\ x < 1$

b $x - 5 > 8$

$x > 8 + 5, \therefore\ x > 13$

c $2x \geq 6$

$\frac{2x}{2} \geq \frac{6}{2}, \therefore\ x \geq 3$

d $\frac{x}{3} \leq 4$

$3\left(\frac{x}{3}\right) \leq 12, \therefore\ x \leq 12$

e $-x \geq 6$

$0 \geq 6 + x$

$-6 \geq x, \therefore\ x \leq -6$

f $-2x < -6$

$-x < -3$

$0 < x - 3$

$3 < x, \therefore\ x > 3$

g $6 - 2x > 10$

$3 - x > 5$

$-x > 2$

$0 > x + 2$

$-2 > x, \therefore x < -2$

h $-\frac{3x}{4} \leq 6$

$-x \leq 8$

$0 \leq x + 8$

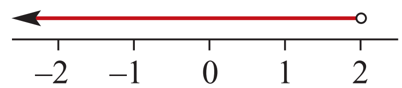
$-8 \leq x, \therefore\ x \geq -8$

i $4x - 4 \leq 2$

$x - 1 \leq \frac{1}{2}, \therefore\ x \leq \frac{3}{2}$

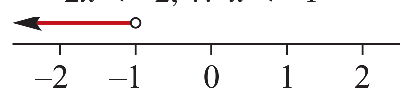
2 a $4x + 3 < 11$

$4x < 8, \therefore\ x < 2$



b $3x + 5 < x + 3$

$2x < -2, \therefore\ x < -1$

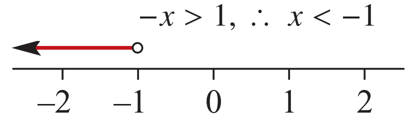


c $\frac{1}{2}(x + 1) - x > 1$

$\frac{x}{2} + \frac{1}{2} - x > 1$

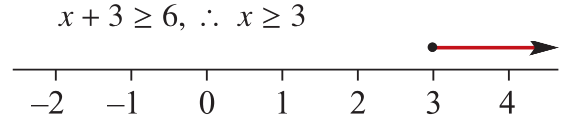
$-\frac{x}{2} > \frac{1}{2}$

$-x > 1, \therefore\ x < -1$



d $\frac{1}{6}(x + 3) \geq 1$

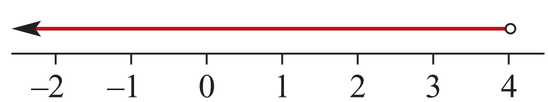
$x + 3 \geq 6, \therefore\ x \geq 3$



e $\frac{2}{3}(2x - 5) < 2$

$2x - 5 < 3$

$2x < 8, \therefore\ x < 4$



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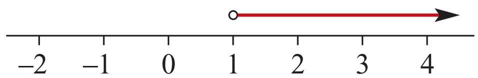
f $\dfrac{3x-1}{4} - \dfrac{2x+3}{2} < -2$

$(3x-1) - (4x+6) < -8$

$-x - 7 < -8$

$-x < -1, \therefore\ x > 1$



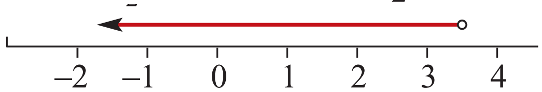
g $\dfrac{4x-3}{2} - \dfrac{3x-3}{3} < 3$

$\dfrac{4x-3}{2} - (x-1) < 3$

$4x - 3 - (2x-2) < 6$

$2x - 1 < 6$

$2x < 7, \therefore\ x < \dfrac{7}{2}$

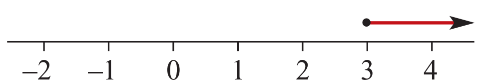


h $\dfrac{1-7x}{-2} \geq 10$

$\dfrac{7x-1}{2} \geq 10$

$7x - 1 \geq 20$

$7x \geq 21, \therefore x \geq 3$

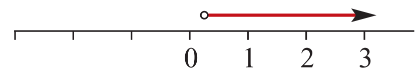


i $\dfrac{5x-2}{3} - \dfrac{2-x}{3} > -1$

$(5x - 2) - (2-x) > -3$

$6x - 4 > -3$

$6x > 1, \therefore x > \dfrac{1}{6}$



3 **a** $2x + 1 > 0$

$2x > -1, \therefore x > -\dfrac{1}{2}$

b $100 - 50x > 0$

$100 > 50x$

$2 > x, \therefore x < 2$

c $100 + 20x > 0$

$20x > -100, \therefore x > -5$

4 Let p be the number of sheets of paper.

$3p < 20$

$p < \dfrac{20}{3}$

$p \in \mathbb{Z}, \therefore\ p = 6$

5 $\dfrac{66 + 72 + x}{3} \geq 75$

$138 + x \geq 225$

$\therefore\ x \geq 87$

Lowest mark:87

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Solutions to Exercise 1F

1 a $c = ab$
$= 6 \times 3 = 18$

b $r = p + q$
$= 12 + -3 = 9$

c $c = ab$
$\therefore\ b = \dfrac{c}{a}$
$= \dfrac{18}{6} = 3$

d $r = p + q$
$\therefore\ q = r - p$
$= -15 - 3 = -18$

e $c = \sqrt{a}$
$= \sqrt{9} = 3$

f $c = \sqrt{a}$
$\therefore\ a = c^2$
$= 9^2 = 81$

g $p = \dfrac{u}{v}$
$= \dfrac{10}{2} = 5$

h $p = \dfrac{u}{v}$
$\therefore\ u = pv$
$= 2 \times 10 = 20$

2 a $S = a + b + c$

b $P = xy$

c $C = 5p$

d $T = dp + cq$

e $T = 60a + b$

3 a $E = IR$
$= 5 \times 3 = 15$

b $C = pd$
$= 3.14 \times 10 = 31.4$

c $P = R\left(\dfrac{T}{V}\right)$
$= 60 \times \dfrac{150}{9} = 1000$

d $I = \dfrac{E}{R}$
$= \dfrac{240}{20} = 12$

e $A = \pi rl$
$= 3.14 \times 5 \times 20 = 314$

f $S = 90(2n - 4)$
$= 90(6 \times 2 - 4) = 720$

4 a $PV = c, \therefore\ V = \dfrac{c}{P}$

b $F = ma, \therefore\ a = \dfrac{F}{m}$

c $I = Prt, \therefore\ P = \dfrac{I}{rt}$

d $w = H + Cr$
$\therefore Cr = w - H$
$\therefore\ r = \dfrac{w - H}{C}$

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e $S = P(1 + rt)$

$\therefore \dfrac{S}{P} = 1 + rt$

$\therefore rt = \dfrac{S}{P} - 1 = \dfrac{S - P}{P}$

$\therefore t = \dfrac{S - P}{rP}$

f $V = \dfrac{2R}{R - r}$

$\therefore (R - r)V = 2R$

$V - rV - 2R = 0$

$R(V - 2) = rV$

$\therefore \quad r = \dfrac{R(V - 2)}{V}$

5 a $D = \dfrac{T + 2}{P}$

$10 = \dfrac{T + 2}{5}$

$T + 2 = 50, \therefore T = 48$

b $A = \dfrac{1}{2}bh$

$40 = \dfrac{10h}{2}$

$10h = 80, \therefore h = 8$

c $V = \dfrac{1}{3}\pi hr^2$

$\therefore h = \dfrac{3V}{\pi r^2}$

$= \dfrac{300}{25 \times 3.14}$

$= \dfrac{12}{3.14} = 3.82$

d $A = \dfrac{1}{2}h(a + b)$

$50 = \dfrac{5}{2} \times (10 + b)$

$20 = 10 + b, \therefore b = 10$

6 a $l = 4a + 3w$

b $H = 2b + h$

c $A = 3 \times (h \times w) = 3hw$

d

$$\begin{aligned} Area &= H \times l - 3hw \\ &= (4a + 3w)(2b + h) - 3hw \\ &= 8ab + 6bw + 4ah + 3hw - 3hw \\ &= 8ab + 6bw + 4ah \end{aligned}$$

7 a i Circle circumferences $= 2\pi(p + q)$
Total wire length
$T = 2\pi(p + q) + 4h$

ii $T = 2\pi(20 + 24) + 4 \times 28$

$= 88\pi + 112$

b $A = \pi h(p + q)$

$\therefore p + q = \dfrac{A}{\pi h}$

$\therefore p = \dfrac{A}{\pi h} - q$

8 a $P = \dfrac{T - M}{D}$

$6 = \dfrac{8 - 4}{D}$

$6D = 4, \therefore D = \dfrac{2}{3}$

b $H = \dfrac{a}{3} + \dfrac{a}{b}$

$5 = \dfrac{6}{3} + \dfrac{6}{b}$

$\dfrac{6}{b} = 5 - 2 = 3$

$3b = 6, \therefore b = 2$

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c $a = \dfrac{90(2n-4)}{n}$

$6 = \dfrac{90(2n-4)}{n}$

$6n = 90(2n-4)$

$n = 15(2n-4)$

$n = 30n - 60$

$29n = 30, \therefore\ n = \dfrac{60}{29}$

d $R = \dfrac{r}{a} + \dfrac{r}{3}$

$4 = \dfrac{r}{2} + \dfrac{r}{3}$

$\dfrac{5r}{6} = 4$

$\therefore\ r = \dfrac{24}{5} = 4.8$

9 a a Big triangle area $= \dfrac{1}{2}bc$

Small triangle area $= \dfrac{1}{2}bk \times ck$

$= \dfrac{1}{2}bck^2$

Shaded area $D = \dfrac{1}{2}bc(1-k^2)$

b $D = \dfrac{1}{2}bc(1-k^2)$

$1 - k^2 = \dfrac{2D}{bc}$

$k^2 = 1 - \dfrac{2D}{bc}$

$\therefore\ k = \sqrt{1 - \dfrac{2D}{bc}}$

c $k = \sqrt{1 - \dfrac{2D}{bc}}$

$= \sqrt{1 - \dfrac{4}{12}}$

$= \sqrt{\dfrac{2}{3}} = \dfrac{\sqrt{6}}{3}$

10 a Width of each arm $= c$

Length of each ofthe 8 arms $= \dfrac{b-c}{2}$

$P = 8 \times \dfrac{b-c}{2} + 4c$

$= 4b - 4c + 4c = 4b$

b Area of each piece $= bc$, but the centre area (c^2) is counted twice

$\therefore\ A = 2bc - c^2$

c $2bc = A + c^2$

$\therefore\ b = \dfrac{A + c^2}{2c}$

11 a $a = \sqrt{a + 2b}$

$a^2 = a + 2b$

$2b = a(a-1)$

$\therefore\ b = \dfrac{a}{2}(a-1)$

b

$\dfrac{a+x}{a-x} = \dfrac{b-y}{b+y}$

$(a+x)(b+y) = (a-x)(b-y)$

$ab + bx + ay + xy = ab - bx - ay + xy$

$bx + ay = -bx - ay$

$2bx + 2ay = 0$

$2bx = -2ay$

$\therefore\ x = -\dfrac{ay}{b}$

c $px = \sqrt{3q - r^2}$

$p^2x^2 = 3q - r^2$

$r^2 = 3q - p^2x^2$

$\therefore\ r = \pm\sqrt{3q - p^2x^2}$

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d $$\frac{x}{y} = \sqrt{1 - \frac{v^2}{u^2}}$$

$$\frac{x^2}{y^2} = 1 - \frac{v^2}{u^2}$$

$$\frac{v^2}{u^2} = 1 - \frac{x^2}{y^2} = \frac{y^2 - x^2}{y^2}$$

$$v^2 = \frac{u^2}{y^2}(y^2 - x^2)$$

$$\therefore\ v = \pm\frac{u}{y}\sqrt{y^2 - x^2}$$

$$= \pm\sqrt{(u^2)\left(1 - \frac{x^2}{y^2}\right)}$$

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Solutions to Technology-free questions

1 a $2x + 6 = 8$

$2x = 2, \therefore x = 1$

b $3 - 2x = 6$

$-2x = 3, \therefore x = -\frac{3}{2}$

c $2x + 5 = 3 - x$

$\therefore \quad 3x = -2, \therefore x = -\frac{2}{3}$

d $\frac{3 - x}{5} = 6$

$3 - x = 30$

$-x = 27, \therefore x = -27$

e $\frac{x}{3} = 4, \therefore x = 12$

f $\frac{13x}{4} - 1 = 10$

$\frac{13x}{4} = 11$

$13x = 44, \therefore \; x = \frac{44}{13}$

g $3(2x + 1) = 5(1 - 2x)$

$6x + 3 = 5 - 10x$

$16x = 2, \therefore x = \frac{1}{8}$

h $\frac{3x + 2}{5} + \frac{3 - x}{2} = 5$

$2(3x + 2) + 5(3 - x) = 50$

$6x + 4 + 15 - 5x = 50$

$\therefore \quad x = 50 - 19 = 31$

2 a $a - t = b$

$a = t + b, \therefore t = a - b$

b $\frac{at + b}{c} = d$

$at + b = cd$

$at = cd - b$

$\therefore \quad t = \frac{cd - b}{a}$

c $a(t - c) = d$

$at - ac = d$

$at = d + ac$

$\therefore \quad t = \frac{d + ac}{a} = \frac{d}{a} + c$

d $\frac{a - t}{b - t} = c$

$a - t = c(b - t)$

$a - t = cb - ct$

$-t + ct = cb - a$

$t(c - 1) = cb - a$

$\therefore \quad t = \frac{cb - a}{c - 1}$

e $\frac{at + b}{ct - b} = 1$

$at + b = ct - b$

$at - ct = -2b$

$t(c - a) = 2b$

$\therefore \quad t = \frac{2b}{c - a}$

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f $\dfrac{1}{at+c} = d$

$dat + dc = 1$

$dat = 1 - dc$

$\therefore \quad t = \dfrac{1-dc}{ad}$

3 a $2 - 3x > 0$

$2 > 3x$

$\dfrac{2}{3} > x, \therefore x < \dfrac{2}{3}$

b $\dfrac{3-2x}{5} \geq 60$

$3 - 2x \geq 300$

$-2x \geq 297$

$-297 \geq 2x$

$-\dfrac{297}{2} \geq x$

$\therefore x \leq -148.5$

c $3(58x - 24) + 10 < 70$

$3(58\, x - 24) < 60$

$58x - 24 < 20$

$58x < 44, \ \therefore \ x < \dfrac{22}{29}$

d $\dfrac{3-2x}{5} - \dfrac{x-7}{6} \leq 2$

$6(3\ - 2x) - 5(x-7) \leq 60$

$18 - 12x - 5x - 35 \leq 60$

$53 - 17x \leq 60$

$-17x \leq 7$

$0 \leq\ 17x + 7$

$-\dfrac{7}{17} \leq x$

$\therefore \quad x \geq\ -\dfrac{7}{17}$

4 $z = \dfrac{x}{2} - 3t$

$\dfrac{1}{2}x = z + 3t$

$\therefore \quad x = 2z + 6t$

When $z = 4$ and $t = -3$:

$x = 2 \times 4 + 6 \times -3$

$= 8 - 18 = -10$

5 a $d = e^2 + 2f$

b $d - e^2 = 2f$

$\therefore \quad f = \dfrac{1}{2}(d - e^2)$

c If $d = 10$ and $e = 3$,

$f = \dfrac{1}{2}(10 - 3^2) = \dfrac{1}{2}$

6 $A = 400\pi$ cm^3

7 The volume of metal in a tube is given by the formula $V = \pi\ell[r^2 - (r-t)^2]$, where ℓ, is the length of the tube, r is the radius of the outside surface and t is the thickness of the material.

a $\ell = 100, r = 5$ and $t = 0.2$

$V = \pi \times 100[5^2 - (5 - 0.2)^2]$

$= \pi \times 100(5 - 4.8)(5 + 4.8)$

$= \pi \times 100 \times 0.2 \times 9.8$

$= \pi \times 20 \times 9.8$

$= 196\pi$

b $\ell = 50, r = 10$ and $t = 0.5$

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$$\begin{aligned} V &= \pi \times 50[10^2 - (10 - 0.5)^2] \\ &= \pi \times 50(10 - 9.5)(10 + 9.5) \\ &= \pi \times 50 \times 0.5 \times 19.5 \\ &= \pi \times 25 \times 19.5 \\ &= \frac{975\pi}{2} \end{aligned}$$

8 **a** $A = \pi r s \qquad (r)$

$$\begin{aligned} A &= \pi r s \\ r &= \frac{A}{\pi s} \end{aligned}$$

b $T = P(1 + rw) \qquad (w)$

$$\begin{aligned} T &= P(1 + rw) \\ T &= P + Prw \\ T - P &= Prw \\ w &= \frac{T - P}{Pr} \end{aligned}$$

c $v = \sqrt{\dfrac{n - p}{r}} \qquad (r)$

$$\begin{aligned} v^2 &= \frac{n - p}{r} \\ r \times v^2 &= n - p \\ r &= \frac{n - p}{v^2} \end{aligned}$$

d $ac = b^2 + bx \qquad (x)$

$$\begin{aligned} ac &= b^2 + bx \\ ac - b^2 &= bx \\ x &= \frac{ac - b^2}{b} \end{aligned}$$

9 $s = \left(\dfrac{u + v}{2}\right)t.$

a $u = 10, v = 20$ and $t = 5$.

$$\begin{aligned} s &= \left(\frac{10 + 20}{2}\right) \times 5 \\ &= 75 \end{aligned}$$

b $u = 10, v = 20$ and $s = 120$.

$$\begin{aligned} 120 &= \left(\frac{10 + 20}{2}\right)t \\ 120 &= 15t \\ t &= 8 \end{aligned}$$

10 $V = \pi r^2 h$ where r cm is the radius and h cm is the height

$V = 500\pi$ and $h = 10$.

$$500\pi = \pi r^2 \times 10$$

$r^2 = 50$ and therefore $r = 5\sqrt{2}$

The radius is $r = 5\sqrt{2}$ cm.

11 Let the lengths be x m and y m.

$10x + 5y = 205 \qquad (1)$

$3x - 2y = 2 \qquad (2)$

Multiply (1) by 2 and (2) by 5.

$20x + 10y = 410 \qquad (3)$

$15x - 10y = 10 \qquad (4)$

Add (3) and (4)

$35x = 420$

$x = 12$ and $y = 17$.

The lengths are 12 m and 17 m.

12 $\dfrac{m + 1}{n} = \dfrac{1}{5} \qquad (1).$

$\dfrac{m}{n - 1} = \dfrac{1}{7} \qquad (2).$

They become:

$5m + 5 = n \qquad (1)$ and $7m = n - 1$ (2)

Substitute from (1) in (2).

$7m = 5m + 5 - 1$

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$m = 2$ and $n = 15$.

13 ■ Mr Adonis earns \$7200 more than Mr Apollo

■ Ms Aphrodite earns \$4000 less than Mr Apollo.

■ If the total of the three incomes is \$303 200, find the income of each person.

Let Mr Apollo earn $\$x$.
Mr Adonis earns $\$(x + 7200)$
Ms Aphrodite earns $\$(x - 4000)$
We have

$$\begin{aligned} x + x + 7200 + x - 4000 &= 303\,200 \\ 3x + 3200 &= 303\,200 \\ 3x &= 300\,000 \\ x &= 100\,000 \end{aligned}$$

Mr Apollo earns \$100 000 ; Mr Adonis earns \$107 200 and Ms Aphrodite earns \$96 000.

14 **a** $4a - b = 11$ (1)
$3a + 2b = 6$ (2)
Multiply (1) by 2.
$8a - 2b = 22$ (3)
Add (3) and (2).
$11a = 28$ which implies $a = \dfrac{28}{11}$.
From(1), $b = -\dfrac{9}{11}$

b $a = 2b + 11$ (1)
$4a - 3b = 11$ (2)
Substitute from (1) in (2).
$4(2b + 11) - 3b = 11$
$5b = -33$
$b = -\dfrac{33}{5}$
From (1), $a = 2 \times \left(-\dfrac{33}{5}\right) + 11 = -\dfrac{11}{5}$.

15 Let t_1 hours be the time spent on hig-ways and t_2 hours be the time travelling through towns.
$t_1 + t_2 = 6$ (1)
$80t_1 + 24t_2 = 424$ (2)
From (1) $t_2 = 6 - t_1$
Substitute in (2).
$80t_1 + 24(6 - t_1) = 424$
$56t_1 = 424 - 6 \times 24$
$t_1 = 5$ and $t_2 = 1$.
The car travelled for 5 hours on highways and 1 hour through towns.

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Solutions to multiple-choice questions

1 D $3x - 7 = 11$

$$3x = 18$$

$$x = 6$$

2 D $\frac{x}{3} + \frac{1}{3} = 2$

$$x + 1 = 6$$

$$x = 5$$

3 C $x - 8 = 3x - 16$

$$-2x = -8$$

$$x = 4$$

4 A $7 = 11(x - 2)$

5 C $2(2x - y) = 10$

$\therefore\ 4x - 2y = 20$

$$\begin{array}{r} x + 2y = 0 \\ \hline 5x \quad\quad = 20 \end{array}$$

$\therefore x = 4; y = -2$

6 C Average cost $=$ total \$/total items

$$= \frac{ax + by}{x + y}$$

7 B $\frac{x+1}{4} - \frac{2x-1}{6} = x$

$$3(x + 1) - 2(2x - 1) = 12x$$

$$3x + 3 - 4x + 2 = 12x$$

$$-13x = -5$$

$$\therefore x = \frac{5}{13}$$

8 B $\frac{72 + 15z}{3} > 4$

$$72 + 15z > 12$$

$$15z > -60$$

$$\therefore z > -4$$

9 A $A = \frac{hw + k}{w}$

$$Aw = hw + k$$

$$w(A - h) = k$$

$$\therefore\ w = \frac{k}{A - h}$$

10 B Total time taken (hrs)

$$= \frac{x}{2.5} + \frac{8x}{5} = \frac{1}{2}$$

$$\frac{2x}{5} + \frac{8x}{5} = \frac{1}{2}$$

$$\frac{10x}{5} = \frac{1}{2},\ \therefore\ x = \frac{1}{4}$$

$x = \frac{1}{4}$ km $= 250$ m

11 E The lines $y = 2x + 4$ and $y = 2x + 6$ are parallel but have different y-axis intercepts.
Alternatively if $2x + 4 = 2x + 6$ then $4 = 6$ which is impossible.

12 B $5(x + 3) = 5x + 15$ for all x.

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Solutions to extended-response questions

1 a $$F = \frac{9}{5}C + 32$$

If $F = 30$, then $$30 = \frac{9}{5}C + 32$$

and $$\frac{9}{5}C = -2$$

which implies $$C = -\frac{10}{9}$$

A temperature of 30°F corresponds to $\left(-\frac{10}{9}\right)^{\circ}$C.

b If $C = 30$, then
$$\begin{aligned} F &= \frac{9}{5} \times 30 + 32 \\ &= 54 + 32 = 86 \end{aligned}$$

A temperature of 30°C corresponds to a temperature of 86°F.

c $x°$C = $x°$ F when
$$\begin{aligned} x &= \frac{9}{5}x + 32 \\ -\frac{4}{5}x &= 32 \end{aligned}$$

$\therefore$ $$x = -40$$

Hence $$-40°\text{F} = -40°\text{C}.$$

d
$$\begin{aligned} x &= \frac{9}{5}(x + 10) + 32 \\ 5x &= 9x + 90 + 160 \\ -4x &= 250 \end{aligned}$$

$\therefore$ $$x = -62.5$$

e
$$\begin{aligned} x &= \frac{9}{5}(2x) + 32 \\ \frac{-13x}{5} &= 32 \end{aligned}$$

$\therefore$ $$x = \frac{-160}{13}$$

f
$$\begin{aligned} k &= \frac{9}{5}(-3k) + 32 \\ 5k &= -27k + 160 \\ 32k &= 160 \end{aligned}$$

$\therefore$ $$k = 5$$

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2 a $$\frac{1}{v}+\frac{1}{u}=\frac{2}{r}$$

Obtain the common denominator $$\frac{u+v}{vu}=\frac{2}{r}$$

Take the reciprocal of both sides $$\frac{vu}{u+v}=\frac{r}{2}$$

Make r the subject $$r=\frac{2vu}{u+v}$$

b $$m=\left(v-\frac{2vu}{u+v}\right)\div\left(\frac{2vu}{u+v}-u\right)$$
$$=\frac{v^2-vu}{u+v}\div\frac{uv-u^2}{u+v}$$
$$=\frac{v^2-vu}{u+v}\times\frac{u+v}{uv-u^2}$$
$$=\frac{v(v-u)}{u(v-u)}=\frac{v}{u}$$

3 a

The total length of wire is given by $T=6w+6l$.

b i If $w=3l$, then $$T=6w+6\left(\frac{w}{3}\right)$$
$$=8w$$

ii If $T=100$, then $$8w=100$$

Hence $$w=\frac{25}{2}$$
$$l=\frac{w}{3}$$
$$=\frac{25}{6}$$

c i $$L=6x+8y$$

Make y the subject $$8y=L-6x$$

and $$y=\frac{L-6x}{8}$$

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ii When $L = 200$ and $x = 4$,

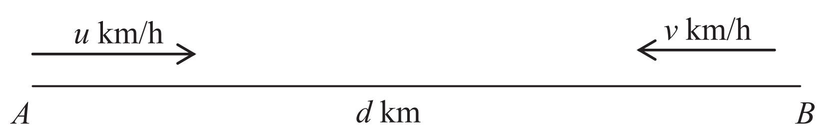
$$y = \frac{200 - 6 \times 4}{8}$$

$$= \frac{176}{8} = 22$$

d The two types of mesh give

	$6x + 8y = 100$	(1)
and	$3x + 2y = 40$	(2)
Multiply (2) by 2	$6x + 4y = 80$	(3)
Subtract (3) from (1) to give	$4y = 20$	
Hence	$y = 5$	
Substitute in (1)	$6x + 40 = 100$	
Hence	$x = 10$	

4



a At time t hours, Tom has travelled ut km and Julie has travelled vt km.

b **i** The sum ofthe two distances must be d when they meet.

Therefore $ut + vt = d$

and $t = \dfrac{d}{u + v}$

They meet after $\dfrac{d}{u+v}$ hours.

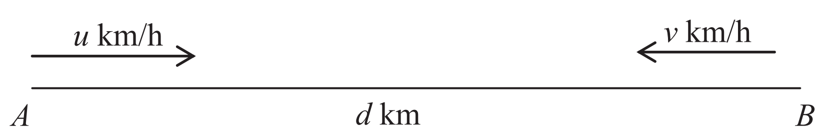
ii The distance from A is $u \times \dfrac{d}{u+v} = \dfrac{ud}{u+v}$ km.

c If $u = 30$, $v = 50$ and $d = 100$, the distance from $A = \dfrac{30 \times 100}{30 + 50}$

$= 37.5$ km

The time it takes to meet is $\dfrac{100}{30 + 50} = 1.25$ hours.

5



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a The time taken to go from A to B is $\frac{d}{u}$ hours. The time taken to go from B to A is $\frac{d}{v}$ hours.

$$\text{The total time taken} = \frac{d}{u} + \frac{d}{v}$$

$$\begin{aligned}\text{Therefore, average speed} &= 2d \div \left(\frac{d}{u} + \frac{d}{v}\right)\\ &= 2d \div \frac{dv + du}{uv}\\ &= 2d \times \frac{uv}{d(u + v)}\\ &= \frac{2uv}{u + v} \text{ km/h}\end{aligned}$$

b **i** The time to go from A to B is T hours.

Therefore $\quad T = \frac{d}{u} \quad (1)$

The time for the return trip $= \frac{d}{v} \quad (2)$

From (1) $\quad d = uT$

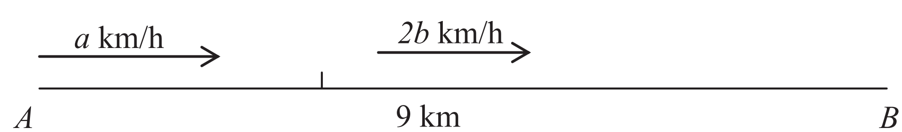
and substituting in (2) gives

the time for the return trip $= \frac{uT}{v}$.

ii The time for the entire trip $= T + \frac{uT}{v}$

$$= \frac{vT + uT}{v} \text{ hours.}$$

6 **a**



One-third of the way is 3 km.

$$\begin{aligned}\text{The time taken} &= \frac{3}{a} + \frac{6}{2b}\\ &= \frac{3}{a} + \frac{3}{b}\end{aligned}$$

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b The return journey is 18 km and therefore, if the man is riding at $3c$ km/h,

$$\text{the time taken} = \frac{18}{3c} = \frac{6}{c}$$

Therefore, if the time taken to go from A to B at the initial speeds is equal to the time taken for the return trip travelling at $3c$ km/h,

then $\qquad \dfrac{6}{c} = \dfrac{3}{a} + \dfrac{3}{b}$

and hence $\qquad \dfrac{2}{c} = \dfrac{1}{a} + \dfrac{1}{b}$

c **i** $$\frac{2}{c} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$$

To make c the subject, take the reciprocal of both sides.

$$\frac{c}{2} = \frac{ab}{a+b}$$

and $\qquad c = \dfrac{2ab}{a+b}$

ii If $a = 10$ and $b = 20$, $c = 400 \div 30 = \dfrac{40}{3}$

7 **a** $\dfrac{x}{8}$ hours at 8 km/h

$\dfrac{y}{10}$ hours at 10 km/h

b

$$\begin{aligned}\text{Average speed} &= (x+y) \div \left(\frac{x}{8} + \frac{y}{10}\right) \\ &= (x+y) \div \frac{10x+8y}{80} \\ &= (x+y) \times \frac{80}{10x+8y} \\ &= \frac{80(x+y)}{10x+8y}\end{aligned}$$

c $$10 \times \frac{x}{8} + 8 \times \frac{y}{10} = 72$$

and, from the statement of the problem,

$$x + y = 70 \qquad (1)$$

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Therefore simultaneous equations in x and y

$$\frac{5x}{4} + \frac{4y}{5} = 72 \qquad (2)$$

Multiply (2) by 20 $\quad 25x + 16y = 1440 \qquad (3)$

Multiply (1) by 16 $\quad 16x + 16y = 1120 \qquad (4)$

Subtract (4) from (3)

$$9x = 320$$

which gives $\quad x = \frac{320}{9}$ and $y = \frac{310}{9}$.

8 First solve the simultaneous equations:

$2y - x = 2 \qquad (1)$

$y + x = 7. \qquad (2)$

Add (1) and (2).

$3y = 9$

$y = 3$ and from (2) $x = 4$.

Now check in

$y - 2x = -5 \qquad (3)$

LHS $= 3 - 8 = -5 =$ RHS.

The three lines intersect at $(4, 3)$.

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Chapter 2 – Coordinate geometry and linear relations

Solutions to Exercise 2A

1 a $A(2, 12), B(8, 4)$

$x = \frac{1}{2}(2 + 8) = 5$

$y = \frac{1}{2}(12 + 4) = 8$

M is at (5, 8).

b $A(-3, 5), B(4, -4)$

$x = \frac{1}{2}(-3 + 4) = 0.5$

$y = \frac{1}{2}(5 + -4) = 0.5$

M is at (0.5, 0.5).

c $A(-1.6, 3.4), B(4.8, -2)$

$x = \frac{1}{2}(-1.6 + 4.8) = 1.6$

$y = \frac{1}{2}(3.4 + -2) = 0.7$

M is at (1.6, 0.7).

d $A(3.6, -2.8), B(-5, 4.5)$

$x = \frac{1}{2}(3.6 + -5) = -0.7$

$y = \frac{1}{2}(-2.8 + 4.5) = 0.85$

M is at $(-0.7, 0.85)$

2 A is (1,1), B is (5,5) and C is (11, 2).

AB: $x = y = \frac{1}{2}(5 - 1) = 3$

Midpoint is at (3, 3).

BC: $x = \frac{1}{2}(5 + 11) = 8$

$y = \frac{1}{2}(5 + 2) = 3.5$

Midpoint is at (8, 3.5).

AC: $x = \frac{1}{2}(1 + 11) = 6$

$y = \frac{1}{2}(1 + 2) = 1.5$

Midpoint is at (6, 1.5).

3 A (3.1, 7.1), B (8.9,10.5)

$x = \frac{1}{2}(3.1 + 8.9) = 6$

$y = \frac{1}{2}(7.1 + 10.5) = 8.8$

C is at (6, 8.8).

4 a $X(-4, 2), M(0, 3)$

For midpt x: $0 = \frac{1}{2}(-4 + x)$

$\therefore \quad x = 4$

For midpt y: $3 = \frac{1}{2}(2 + y)$

$1 + \frac{y}{2} = 3$

$\frac{y}{2} = 2, \therefore \; y = 4$

Point Y is at (4,4).

b $X(-1, -3), M(0.5, -1.6)$

For midpt x: $0.5 = \frac{1}{2}(-1 + x)$

$1 = -1 + x, \therefore x = 2$

For midpt y: $-1.6 = \frac{1}{2}(-3 + y)$

$-3.2 = -3 + 2, \therefore y = -0.2$

Point Y is at $(2, -0.2)$.

c $X(6, 3), M(2, 1)$

For midpt x: $2 = \frac{1}{2}(6 + x)$

$4 = 6 + x, \therefore \; x = -2$

For midpt y: $1 = \frac{1}{2}(-3 + y)$

$2 = -3. + y, \therefore y = 5$

Point Y is at $(-2, 5)$.

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d $X(4,-3), M(0,-3)$

For midpt x: $\quad 0 = \frac{1}{2}(4+x)$

$\therefore x = 4$

For midpt y: does not change so $y=-3$ Point Y is at $(-4,-3)$

5 At midpoint: $x = \frac{1}{2}(1+a)$; $y = \frac{1}{2}(4+b)$

$x = \frac{1}{2}(1+a) = 5$

$1 + a = 10, \therefore a = 9$

$y = \frac{1}{2}(4+b) = -1$

$4 + b = -2, \therefore b = -6$

6 a Distance between $(3,6)$ and $(-4,5)$

$= \sqrt{(6-5)^2 + (3--4)^2}$

$= \sqrt{1^2+7^2}$

$= \sqrt{50} = 5\sqrt{2} \approx 7.07$

b Distance between $(4,1)$ and $(5,-3)$

$= \sqrt{(4-5)^2 + (1--3)^2}$

$= \sqrt{(-1)^2+4^2}$

$= \sqrt{17} \approx 4.12$

c Distance between $(-2,-3)$ and $(-5,-8)$

$= \sqrt{(-2--5)^2 + (-3--8)^2}$

$= \sqrt{3^2+5^2}$

$= \sqrt{34} \approx 5.83$

d Distance between $(6,4)$ and $(-7,4)$

$= \sqrt{(6--7)^2 + (4-4)^2}$

$= \sqrt{13^2+0^2}$

$= 13.00$

7 $A=(-3,-4), B=(1,5), C=(7,-2)$

$AB = \sqrt{(1--3)^2+(5--4)^2}$

$= \sqrt{4^2+9^2}$

$= \sqrt{97}$

$BC = \sqrt{(7-1)^2+(-2-5)^2}$

$= \sqrt{6^2+(-7)^2}$

$= \sqrt{85}$

$AC = \sqrt{(7--3)^2+(-2--4)^2}$

$= \sqrt{10^2+2^2}$

$= \sqrt{104}$

$P = \sqrt{97} + \sqrt{85} + \sqrt{104} \approx 29.27$

8 $A(6,6), B(10,2), C(-1,5), D(-7,1)$

For P: $x = \frac{1}{2}(6+10) = 8$

$y = \frac{1}{2}(6+2) = 4$

P is at $(8,4)$.

For M: $x = \frac{1}{2}(-1+-7) = -4$

$y = \frac{1}{2}(5+1) = 3$

M is at $(-4,3)$.

$\therefore PM = \sqrt{(-4-8)^2+(3-4)^2}$

$= \sqrt{(-12)^2+(-1)^2}$

$= \sqrt{145} \approx 12.04$

9 $DM = \sqrt{(-6-0)^2+(1-6)^2}$

$= \sqrt{(-6)^2+(-5)^2}$

$= \sqrt{61}$

$DN = \sqrt{(3-0)^2+(-1-6)^2}$

$= \sqrt{3^2+7^2}$

$= \sqrt{58}$

DN is shorter.

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Solutions to Exercise 2B

1 **a** $m = \dfrac{4-0}{0-(-1)} = 4$

b $m = \dfrac{6-0}{3-0} = 2$

c $m = \dfrac{1-0}{4-0} = \dfrac{1}{4}$

d $m = \dfrac{4-0}{0-1} = -4$

e $m = \dfrac{3-0}{3-0} = 1$

f $m = \dfrac{3-0}{-3-0} = -1$

g $m = \dfrac{10-0}{6-(-2)} = \dfrac{5}{4}$

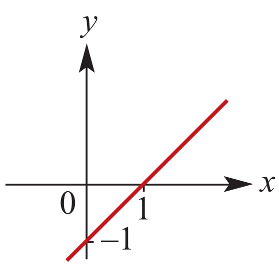
h $m = \dfrac{8-2}{0-3} = \dfrac{6}{-3} = -2$

i $m = \dfrac{5-0}{0-4} = \dfrac{5}{-4} = -\dfrac{5}{4}$

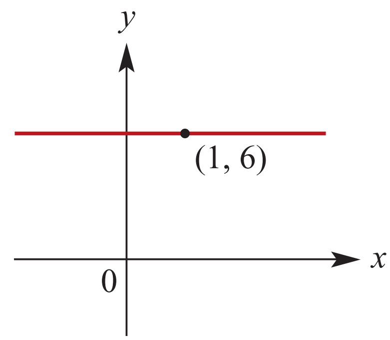
j $m = \dfrac{4-0}{0-(-3)} = \dfrac{4}{3}$

k Rise = zero so $m = 0$

2 Line with gradient 1:



3 $m = 0$ so $y = mx + c$ means $y = c$.
Here $c = 6$:



4 **a** $m = \dfrac{4-3}{2-6} = -\dfrac{1}{4}$

b $m = \dfrac{-6-4}{1-(-3)} = -\dfrac{5}{2}$

c $m = \dfrac{-3-7}{11-6} = -2$

d $m = \dfrac{0-8}{6-5} = -8$

e Rise = zero so $m = 0$

f $m = \dfrac{0-(-6)}{-6-0} = -1$

g $m = \dfrac{16-9}{4-3} = 7$

h $m = \dfrac{36-25}{6-5} = 11$

i $m = \dfrac{64-25}{-8-(-5)} = \dfrac{39}{-3} = -13$

j $m = \dfrac{100-1}{10-1} = \dfrac{99}{9} = 11$

k $m = \dfrac{1000-1}{10-1} = \dfrac{999}{9} = 111$

l $m = \dfrac{64-125}{4-5} = \dfrac{-61}{-1} = 61$

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5 **a** $m = \dfrac{6a - 2a}{3a - 5a}$

$= \dfrac{4a}{-2a} = -2$

b $m = \dfrac{2b - 2a}{5b - 5a}$

$= \dfrac{2}{5}\left(\dfrac{b - a}{b - a}\right) = \dfrac{2}{5}$

6 **a** $m = \dfrac{a - 6}{7 - (-1)}$

$= \dfrac{a - 6}{8} = 6$

$a - 6 = 48, \therefore a = 54$

b $m = \dfrac{7 - 6}{b - 1}$

$= \dfrac{1}{b - 1} = -6$

$1 = 6(1 - b) = 6 - 6b$

$6b = 5, \ \therefore b = \dfrac{5}{6}$

7 We only need positive angles, so negative ones have 180° added.

a $(0, 3), (-3, 0);\ m = \dfrac{0 - (-3)}{3 - 0} = 1$
Angle $= \tan^{-1}(1) = 45°$

b $(0, -4), (4, 0);\ m = \dfrac{0 - (-4)}{4 - 0} = 1$
Angle $= \tan^{-1}(1) = 45°$

c $(0, 2), (-4, 0);\ m = \dfrac{0 - 2}{-4 - 0} = \dfrac{1}{2}$
Angle $= \tan^{-1}\left(\dfrac{1}{2}\right) = 26.57°$

d $(0, -5), (-5, 0);\ m = \dfrac{0 - -5}{-5 - 0} = -1$
Angle $= \tan^{-1}(-1) + 180° = 135°$

8 **a** $(-4, -2), (6,8);\ m = \dfrac{8 - -2}{6 - -4} = 1$
Angle $= \tan^{-1}(1) = 45°$

b $(2,6), (-2, 4);\ m = \dfrac{4 - 6}{-2 - 2} = \dfrac{1}{2}$
Angle $= \tan^{-1}\left(\dfrac{1}{2}\right) = 26.57°$

c $(-3, 4), (6, 1);\ m = \dfrac{1 - 4}{6 - -3} = -\dfrac{1}{3}$
Angle $= \tan^{-1}\left(-\dfrac{1}{3}\right) = 161.57°$

d $(-4, -3), (2, 4);\ m = \dfrac{4 - -3}{2 - -4} = \dfrac{7}{6}$
Angle $= \tan^{-1}\left(\dfrac{7}{6}\right) = 49.4°$

e $(3b, a), (3a, b);\ m = \dfrac{b - a}{3a - 3b}$
$= (b - a)\left(\dfrac{1}{-3}\right) = -\dfrac{1}{3}$
Angle $= \tan^{-1}\left(-\dfrac{1}{3}\right) = 161.57°$

f $(c, b), (b, c);\ m = \dfrac{c - b}{b - c} = -1$
Angle $= \tan^{-1}(-1) + 180° = 135°$

9 **a** $\tan 45° = 1$

b $\tan 135° = -1$

c $\tan 60° = \sqrt{3}$

d $\tan 120° = -\sqrt{3}$

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Solutions to Exercise 2C

1 **a** $m = 3, c = 6$

b $m = -6, c = 7$

c $m = 3, c = -6$

d $m = -1, c = -4$

2 **a** $y = mx + c; m = 3, c = 5$
so $y = 3x + 5$

b $y = mx + c; m = -4, c = 6$
so $y = -4x + 6$

c $y = mx + c; m = 3, c = -4$
so $y = 3x - 4$

3 **a** $y = 3x - 6$; Gradient $= 3$; y-axis intercept $= -6$

b $y = 2x - 4$; Gradient $= 2$; y-axis intercept $= -4$

c $y = \frac{1}{2}x - 2$; Gradient $= \dfrac{1}{2}$; y-axis intercept $= -2$

d $y = \dfrac{1}{3}x - \dfrac{5}{3}$; Gradient $= \dfrac{1}{3}$; y-axis intercept $= -\dfrac{5}{3}$

4 **a** $2x - y = 9$

$$-y = -2x + 9$$

$$y = 2x - 9, \therefore m = 2, c = -9$$

b

$3x + 4y = 10$

$$4y = -3x + 10$$

$$y = -\frac{3}{4}x + \frac{5}{2}, \therefore m = -\frac{3}{4}, c = \frac{5}{2}$$

c

$-x - 3y = 6$

$$-3y = x + 6$$

$$y = -\frac{1}{3}x - 2, \therefore m = -\frac{1}{3}, c = -2$$

d $5x - 2y = 4$

$$-2y = -5x + 4$$

$$y = \frac{5}{2}x - 2, \therefore m = \frac{5}{2}, c = -2$$

5 **a** The equation is of the form
$y = 3x + c$;
When $x = 6, y = 7$
$\therefore 7 = 3 \times 6 + c$
$\therefore c = -11$
The equation is $y = 3x - 11$

b The equation is of the form
$y = -2x + c$;
When $x = 1, y = 7$
$\therefore 7 = -2 \times 1 + c$
$\therefore c = 9$
The equation is $y = -2x + 9$

6 **a** $(-1, 4), (2, 3)$

$m = \dfrac{3 - 4}{2 - -1} = -\dfrac{1}{3}$

Using $(2, 3)$: $y = -\dfrac{2}{3} + c = 3$

$$c = \frac{11}{3}$$

$\therefore y = -\dfrac{1}{3}x + \dfrac{11}{3}$

$3y = -x + 11$

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$\therefore x + 3y = 11$

b $(0, 4), (5, -3)$

$m = \dfrac{-3 - 4}{5 - 0} = -\dfrac{7}{5}$

Using (0,4): $y = c = 4$

$\therefore y = -\dfrac{7}{5}x + 4$

$5y = -7x + 20$

$\therefore 7x + 5y = 20$

c $(3, -2), (4, -4)$

$\therefore m = \dfrac{-4 - -2}{4 - 3} = -2$

Using $(3, -2)$: $y = -2 \times 3 + c = -2$

$c = 4$

$\therefore y = -2x + 4$

$\therefore 2x + y = 4$

d $(5, -2), (8, 9)$

$\therefore m = \dfrac{9 - -2}{8 - 5} = \dfrac{11}{3}$

Using $(5, -2)$: $y = \dfrac{11}{3} \times 5 + c = -2$

$c + \dfrac{55}{3} = -2$

$c = -\dfrac{61}{3}$

$\therefore\ y = \dfrac{11}{3}x - \dfrac{61}{3}$

$3y = 11x - 61$

$\therefore\ -11x + 3y = -61$

7 **a** The line passses through the point $(0, 6)$ and $(1, 8)$.

Therefore gradient $= \dfrac{8 - 6}{1 - 0} = 2$

b The equation is $y = 2x + 6$

8 **a** The equation is of the form $y = 2x + c$;

When $x = 1, y = 6$

$\therefore 6 = 2 \times 1 + c$

$\therefore c = 4$

The equation is $y = 2x + 4$

b The equation is of the form $y = -2x + c$;

When $x = 1, y = 6$

$\therefore 6 = -2 \times 1 + c$

$\therefore c = 8$

The equation is $y = -2x + 8$

9 **a** The equation is of the form $y = 2x + c$;

When $x = -1, y = 4$

$\therefore 4 = 2 \times (-1) + c$

$\therefore c = 6$

The equation is $y = 2x + 6$

b The equation is of the form $y = -2x + c$;

When $x = 0, y = 4$

$\therefore c = 4$

The equation is $y = -2x + 4$

c The equation is of the form $y = -5x + c$;

When $x = 3, y = 0$

$\therefore 0 = -5 \times 3 + c$

$\therefore c = 15$

The equation is $y = -5x + 15$

10 **a** $y = mx + c$; $m = \dfrac{0 - 4}{6 - 0} = -\dfrac{2}{3}$

Using (0,4), $c = 4$

$y = -\dfrac{2x}{3} + 4$

b $y = mx + c$; $m = \dfrac{-6 - 0}{0 - -3} = -\dfrac{6}{3} = -2$

Using $(0, -6)$, $c = -6$

$y = -2x - 6$

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c $y = mx + c$; $m = \dfrac{0-4}{4-0} = -\dfrac{4}{4} = -1$
Using (0,4), $c = 4$
$y = -x + 4$

d $y = mx + c$; $m = \dfrac{3-0}{0-2} = -\dfrac{3}{2}$
Using (0,3):
$y = -\dfrac{3}{2}x + 3$
$\therefore y = -x + 8$

11 a Gradient $= \dfrac{6-4}{3-0} = \dfrac{2}{3}$
Passes through $(0, 4)$, $\therefore c = 4$
Therefore equation is $y = \dfrac{2}{3}x + 4$

b Gradient $= \dfrac{2-0}{4-1} = \dfrac{2}{3}$
When $x = 1, y = 0$
$\therefore 0 = \dfrac{2}{3} \times 1 + c$
$\therefore c = -\dfrac{2}{3}$
Therefore equation is $y = \dfrac{2}{3}x - \dfrac{2}{3}$

c Gradient $= \dfrac{3-0)}{3-(-3)} = \dfrac{1}{2}$
When $x = -3, y = 0$
$\therefore 0 = \dfrac{1}{2} \times (-3) + c$
$\therefore c = \dfrac{3}{2}$
Therefore equation is $y = \dfrac{1}{2}x + \dfrac{3}{2}$

d Gradient $= \dfrac{0-3}{4-(-2)} = -\dfrac{1}{2}$
When $x = 4, y = 0$
$\therefore 0 = -\dfrac{1}{2} \times 4 + c$
$\therefore c = 2$
Therefore equation is $y = -\dfrac{1}{2}x + 2$

e Gradient $= \dfrac{8-2}{4.5-(-1.5)} = 1$
When $x = -1.5, y = 2$
$\therefore 2 = 1 \times (-1.5) + c$
$\therefore c = 3.5$
Therefore equation is $y = x + 3.5$

f Gradient $= \dfrac{-2-1.75}{4.5-(-3)} = -0.5$
When $x = -3, y = 1.75$
$\therefore 1.75 = -0.5 \times (-3) + 0.25$
$\therefore c = 0.25$
Therefore equation is
$y = -0.5x + 0.25$

12 a Axis intercepts: (0,4) and $(-1, 0)$
$m = \dfrac{4-0}{0--1} = 4$,
$c = 4$ so $y = 4x + 4$

b Specified points: $(-3, 2)$ and (0,0)
$m = \dfrac{2-0}{-3-0} = -\dfrac{2}{3}$
$c = 0$ so $y = -\dfrac{2x}{3}$

c Axis intercepts: $(-2, 0)$ and $(0, -2)$
$m = \dfrac{0--2}{-2-0} = 1$
$c = -2$ so $y = -x - 2$

d Axis intercepts: (2,0) and $(0, -1)$
$m = \dfrac{0--1}{2-0} = \dfrac{1}{2}$,
$c = -1$ so $y = \dfrac{x}{2} - 1$

e $m = 0$, $c = 3.5$ so $y = 3.5$

f m undefined. Vertical line is $x = k$
so $x = -2$

13 P and Q are on the line $y = mx + c$;
$m = \dfrac{1--3}{2-1} = 4$
Using Q at (2,1):
$y = 4 \times 2 + c = 1$ so $c = -7$
Line PQ has equation $y = 4x - 7$
Q and R are on the line $y = ax + b$:

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$a = \dfrac{3-1}{2.5-2} = \dfrac{2}{0.5} = 4$

Using Q at $(2, 1)$:

$y = 4 \times 2 + b = 1$ so $b = -7$

Line QR also has equation $y = 4x - 7$

P, Q and R are collinear.

14 **a** $y + x = 1$

Does not pass through $(0, 0)$ because $y = 1 - x$ has $c = 1$

b $y + 2x = 2(x + 1)$

Does not pass through (0,0): this equation simplifies to $y = 2$, so y is never 0.

c $x + y = 0$

Passes through (0,0) because $c = 0$

d $x - y = 1$

Does not pass through (0,0) because $y = x + 1$ has $c = 1$

15 **a** $x = 4$

b $y = 11$

c $x = 11$

d $y = -1$

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Solutions to Exercise 2D

1 **a** $x + y = 4$

If $x = 0, y = 4$; if $y = 0, x = 4$

Axis intercepts are at (0,4) and (4,0)

b $x - y = 4$

If $x = 0, y = -4$; if $y = 0, x = 4$

Axis intercepts are at $(0, -4)$ and (4,0)

c $-x - y = 6$

If $x = 0, y = -6$; if $y = 0, x = -6$

Axis intercepts are at $(0, -6)$ and $(-6, 0)$

d $y - x = 8$

If $x = 0, y = 8$; if $y = 0, x = -8$

Axis intercepts are at (0,8) and $(-8, 0)$

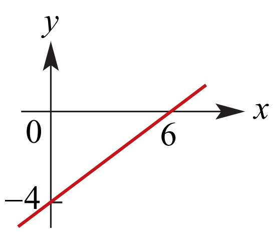
2 **a** $2x - 3y = 12$

If $x = 0, -3y = 12$

$\therefore y = \dfrac{12}{-3} = -4$

If $y = 0, 2x = 12$

$\therefore x = \dfrac{12}{2} = 6$

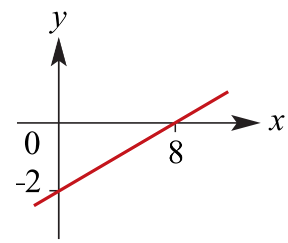


b $x - 4y = 8$:

If $x = 0, -4y = 8$

$\therefore y = \dfrac{8}{-4} = -2$

If $y = 0, x = 8$



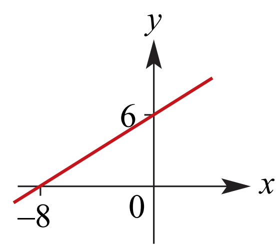
c $-3x + 4y = 24$

If $x = 0$, $4y = 24$

$\therefore y = \dfrac{24}{4} = 6$

If $y = 0, -3x = 24$

$\therefore x = \dfrac{24}{-3} = -8$



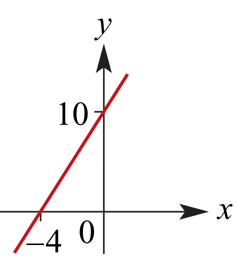
d $-5x + 2y = 20$

If $x = 0, 2y = 20$

$\therefore y = \dfrac{20}{2} = 10$

If $y = 0, -5x = 20$

$\therefore x = \dfrac{20}{-5} = -4$



e $4x - 3y = 15$

If $x = 0, -3y = 15$

$\therefore y = \dfrac{15}{-3} = -5$

If $y = 0, 4x = 15$

$\therefore x = \dfrac{15}{4} = 3.75$

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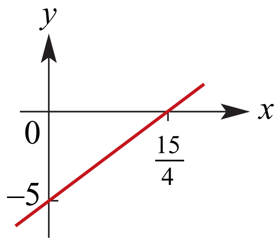


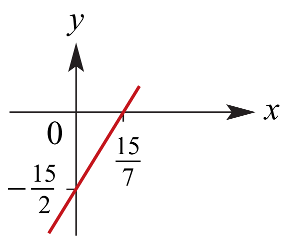
f $7x - 2y = 15$

If $x = 0, -2y = 15$

$\therefore y = \dfrac{15}{-2} = -7.5$

If $y = 0, 7x = 15$

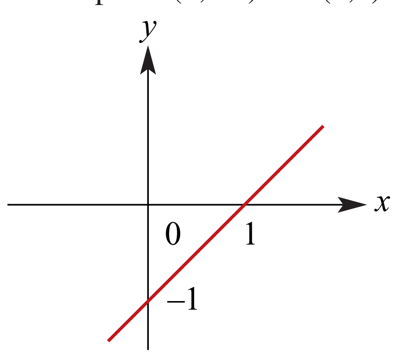
$\therefore x = \dfrac{15}{7}$



3 a $y = x - 1$

If $x = 0, y = -1$; if $y = 0, x = 1$

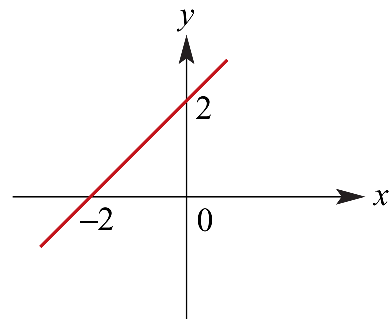
Intercepts at $(0, -1)$ and $(1,0)$



b $y = x + 2$

If $x = 0, y = 2$; if $y = 0, x = -2$

Intercepts at $(0, 2)$ and $(-2, 0)$

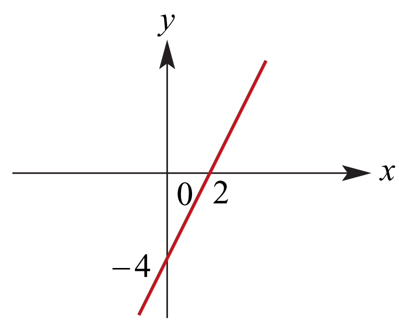


c $y = 2x - 4$

If $x = 0, y = -4$;

if $y = 0$, $2x - 4 = 0$, so $x = 2$

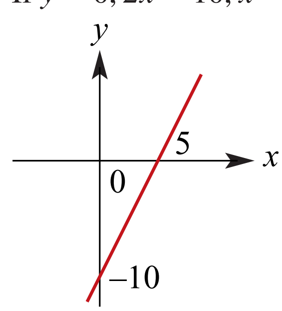
Intercepts at $(0, -4)$ and $(2,0)$



4 a $y = 2x - 10$

If $x = 0, y = -10$ so $(0, -10)$

If $y = 0, 2x = 10, x = 5$ so $(5,0)$



b $y = 3x - 9$

If $x = 0, y = -9$ so $(0, -9)$

If $y = 0, 3x = 9, x = 3$ so $(3,0)$

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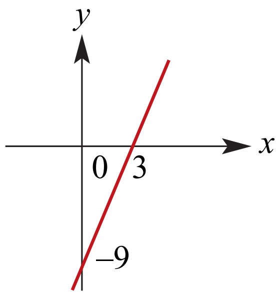
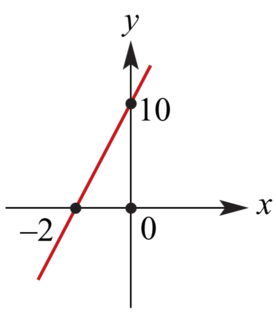


c $y = 5x + 10$

If $x = 0, y = 10$ so $(0,10)$

If $y = 0, 5x + 10 = 0,$

so $5x = -10$ and $x = -2$ so $(-2, 0)$

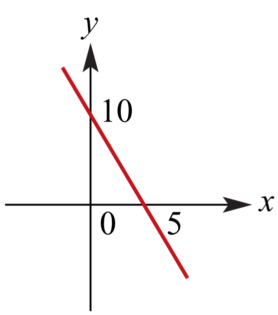


d $y = -2x + 10$

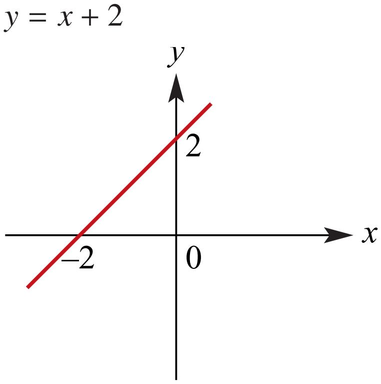
If $x = 0, y = 10$ so $(0,10)$

If $y = 0, -2x + 10 = 0,$

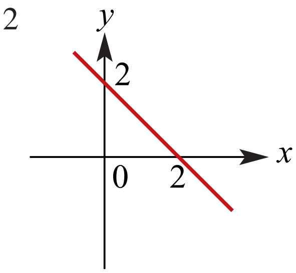
so $2x = 10$ and $x = 5$ so $(5,0)$



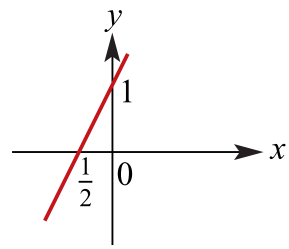
5 a $y = x + 2$



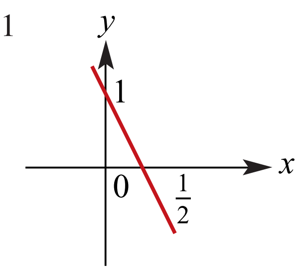
b $y = -x + 2$



c $y = 2x + 1$

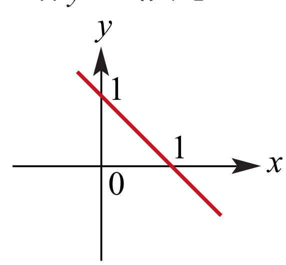


d $y = -2x + 1$



6 a $x + y = 1$

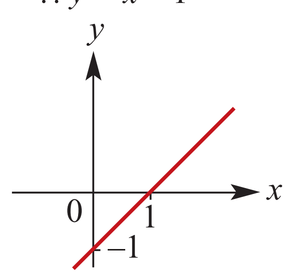
$\therefore y = -x + 1$



b $x - y = 1$

$x - 1 = y$

$\therefore y = x - 1$



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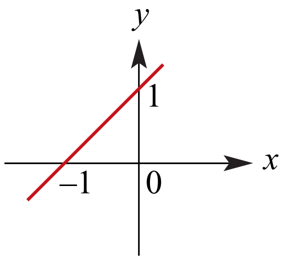
c $y - x = 1$

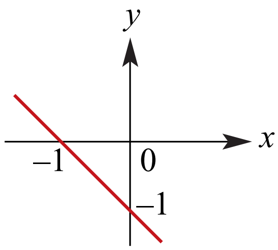
$\therefore y = x + 1$



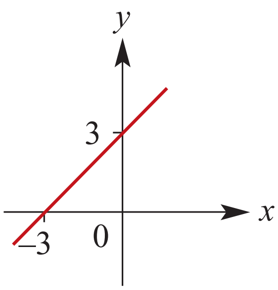
d $-x - y = 1$

$-y = x + 1$

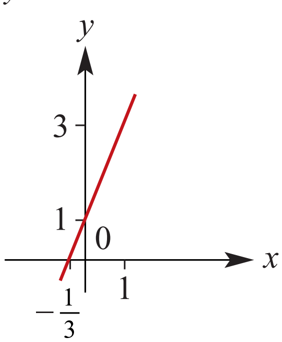
$\therefore y = -x - 1$



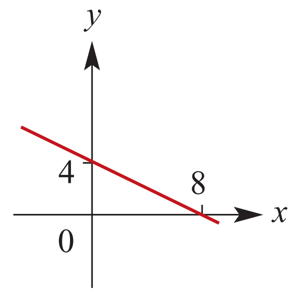
7 a $y = x + 3$



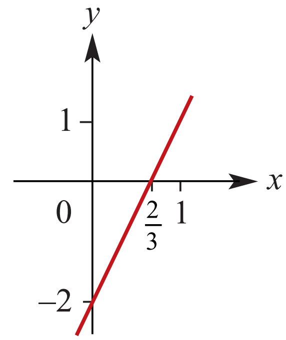
b $y = 3x + 1$



c $y = 4 - \frac{1}{2}x$



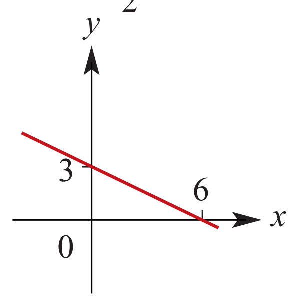
d $y = 3x - 2$



e $4y + 2x = 12$

$4y = 12 - 2x$

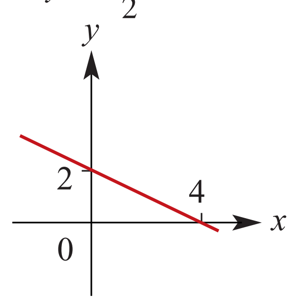
$\therefore y = -\frac{x}{2} + 3$



f $3x + 6y = 12$

$6y = 12 - 3x$

$\therefore y = -\frac{x}{2} + 2$



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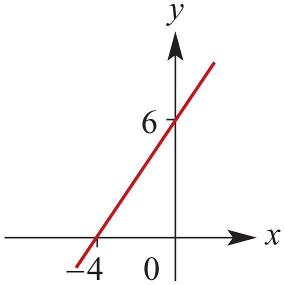
g $4y - 6x = 24$

$4y = 24 + 6x$

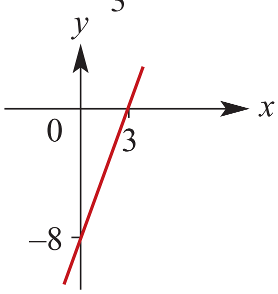
$\therefore y = \dfrac{3x}{2} + 6$



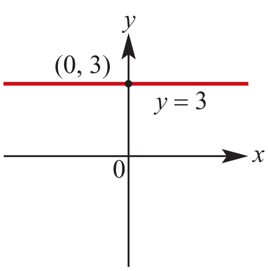
h $8x - 3y = 24$

$-3y = 24 - 8x$

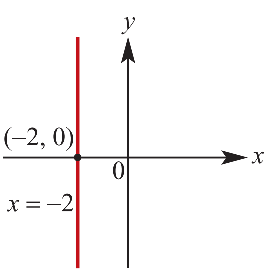
$\therefore y = \dfrac{8x}{3} - 8$



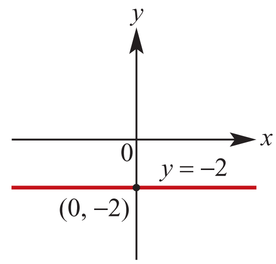
8 a



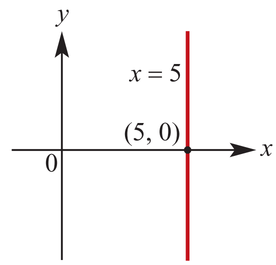
b



c



d



9 a $y = x$ so $m = 1; 45°$

b $y = -x$ so $m = -1; 135°$

c $y = x + 1$ so $m = 1; 45°$

d $x + y = 1$
$y = -x + 1$ so $m = -1; 135°$

e $y = 2x$ so $m = 2;$
$\tan^{-1}(2) = 63.43°$

f $y = -2x; m = -2;$
$\tan^{-1}(-2) + 180° = 116.57°$

10 a $y = 3x + 2; m = 3$
$\tan^{-1}(3) = 71.57°$

b $2y = -2x + 1$
$\therefore y = -x + \dfrac{1}{2}; m = -1$
$\tan^{-1}(-1) = 135°$

c $2y - 2x = 6$

$y - x = 3$
$\therefore y = x + 3; m = 1$
$\tan^{-1}(1) = 45°$

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d $3y + x = 7$

$3y = -x + 7$

$\therefore y = -\dfrac{x}{3} + \dfrac{7}{3}; m = -\dfrac{1}{3}$

$\tan^{-1}\left(-\dfrac{1}{3}\right) + 180^\circ = 161.57^\circ$

11 A straight line has equation $y = 3x - 4$

$(0, a)$: $a = -4$

$(b, 0)$: $0 = 3b - 4$

$3b = 4, \therefore b = \dfrac{4}{3}$

$(1, d)$: $d = 3 - 4 = -1$

$(e, 10)$: $10 = 3e - 4$

$3e = 14, \therefore e = \dfrac{14}{3}$

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Solutions to Exercise 2E

1 **a** Gradient = 2

Equation is of the form $y = 2x + c$

When $x = 4, y = -2$

$\therefore -2 = 2 \times 4 + c$

$\therefore c = -10$

The equation is $y = 2x - 10$

b Gradient $= -\dfrac{1}{2}$

Equation is of the form $y = -\dfrac{1}{2}x + c$

When $x = 4, y = -2$

$\therefore -2 = -\dfrac{1}{2} \times 4 + c$

$\therefore c = 0$

The equation is $y = -\dfrac{1}{2}x$

c Gradient $= -\dfrac{1}{2}$

Equation is of the form $y = -2x + c$

When $x = 4, y = -2$

$\therefore -2 = -2 \times 4 + c$

$\therefore c = 6$

The equation is $y = -2x + 6$

d Gradient $= -\dfrac{1}{2}$

Equation is of the form $y = \dfrac{1}{2}x + c$

When $x = 4, y = -2$

$\therefore -2 = \dfrac{1}{2} \times 4 + c$

$\therefore c = -4$

The equation is $y = \dfrac{1}{2}x - 4$

e $2x - 3y = 4$

$$-3y = -2x + 4$$

$$\therefore \quad y = \frac{2}{3}x - \frac{4}{3}$$

So the gradient we want is $\dfrac{2}{3}$.

Using the point $(4, -2)$:

$$y - -2 = \frac{2}{3}(x - 4)$$

$$y = \frac{2}{3}(x - 4) - 2$$

$$y = \frac{2}{3}x - \frac{8}{3} - 2$$

$$y = \frac{2}{3}x - \frac{14}{3}$$

$$3y = 2x - 14$$

$$\therefore 2x - 3y = 14$$

f $2x - 3y = 4$

$$\therefore 3y = 2x - 4$$

$$\therefore y = \frac{2}{3}x - \frac{4}{3}$$

Gradient $= -\dfrac{3}{2}$

Equation is of the form $y = -\dfrac{3}{2}x + c$

When $x = 4, y = -2$

$\therefore -2 = -\dfrac{3}{2} \times 4 + c$

$\therefore c = 4$

The equation is $y = -\dfrac{3}{2}x + 4$

g $x + 3y = 5$

$$\therefore 3y = -x + 5$$

$$\therefore y = -\frac{1}{3}x + \frac{5}{3}$$

Gradient $= -\dfrac{1}{3}$

Equation is of the form $y = -\dfrac{1}{3}x + c$

When $x = 4, y = -2$

$\therefore -2 = -\dfrac{1}{3} \times 4 + c$

$\therefore c = -\dfrac{2}{3}$

The equation is $y = -\dfrac{1}{3}x - \dfrac{2}{3}$

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h $x + 3y = -4$

$\therefore 3y = -x - 4$

$\therefore y = -\frac{1}{3}x - \frac{4}{3}$

Gradient = 3

Equation is of the form $y = 3x + c$

When $x = 4, y = -2$

$\therefore -2 = 3 \times 4 + c$

$\therefore c = -14$

The equation is $y = 3x - 14$

2 a $2y = 6x + 4; y = 3x + 4$

Parallel: $m = 3$ for both

b $x = 4 - y; 2x + 2y = 6$

Parallel: $m = -1$ for both

c $3y - 2x = 12; y + \frac{1}{3} = \frac{2}{3}x$

Parallel: $m = \frac{2}{3}$ for both

d $4y - 3x = 4; 3y = 4x - 3$

Not parallel:

$4y - 3x = 4$

$4y = 3x + 4$

$\therefore \quad y = \frac{3x}{4} + 1$

$3y = 4x - 3$

$\therefore y = \frac{4x}{3} - 1$

3 a $y = 4$ (The y-coordinate)

b $x = 2$ (The x-coordinate)

c $y = 4$ (The y-coordinate)

d $x = 3$ (The x-coordinate)

4 Gradient of $y = -\frac{1}{2}x + 6$ is $-\frac{1}{2}$.

So perpendicular gradient is

$-1 \div -\frac{1}{2} = 2$

Using the point (1,4):

$y - 4 = 2(x - 1)$

$y = 2(x - 1) + 4$

$\therefore \ y = 2x + 2$

5 $A(1, 5)$ and $B(-3, 7)$

Midpoint

$M(\frac{1 + (-3)}{2}, \frac{7 + 5}{2}) = M(-1, 6)$

Gradient $AB = \frac{7 - 5}{-3 - 1} = -\frac{1}{2}$

$\therefore$ gradient of line perpendicular to $AB = 2$. The equation of the line is of the form $y = 2x + c$

When $x = -1, y = 6$

$\therefore 6 = 2 \times (-1) + c$

$\therefore c = 8$

Equation of line is $y = 2x + 8$

6 Gradient of $AB = \frac{-3 - 2}{2 - 5} = \frac{5}{3}$

Gradient of $BC = \frac{3 - -3}{-8 - 2} = -\frac{3}{5}$

Product of these gradients

$= -\frac{3}{5} \times \frac{5}{3} = -1$

AB and BC are perpendicular, so ABC is a right–angled triangle.

7 $A(3, 7), B(6, 1), C(-8, 3)$

Gradient $AB = \frac{7 - 1}{3 - 6} = -2$

Gradient $BC = \frac{8 - 1}{20 - 6} = \frac{1}{2}$

$\therefore AB \perp BC$

8 Gradient of $RS = \frac{4 - 6}{6 - 2} = -\frac{1}{2}$

Gradient of $ST = \frac{-4 - 4}{2 - 6} = 2$

Product of these gradients $= -1$, so RS

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and ST are perpendicular.

Gradient of $TU = \frac{-2 - -4}{-2 - 2} = -\frac{1}{2}$

Gradient of $UR = \frac{6 - -2}{2 - -2} = 2$

Similarly, TU and UR are perpendicular, as are ST and TU, and RS and UR.

So $RSTU$ must be a rectangle.

9 $4x - 3y = 10$

$$-3y = 10 - 4x$$

$$3y = 4x - 10$$

$$\therefore \quad y = \frac{4}{3}x - \frac{10}{3}$$

Gradient $= \frac{4}{3}$

$$4x - ly = m$$

$$-ly = m - 4x$$

$$ly = 4x - m$$

$$\therefore \quad y = \frac{4}{l}x - \frac{m}{l}$$

Gradient $= \frac{4}{l}$

These lines are perpendicular, so their gradients multiplied equal -1:

$$\frac{4}{3} \times \frac{4}{l} = -1$$

$$\frac{16}{3} = -l$$

$$\therefore \quad l = -\frac{16}{3}$$

At intersection (4, 2) the y and x values are equal. From $4x - ly = m$:

$$m = 16 - 2\left(-\frac{16}{3}\right)$$

$$= 16 + \frac{32}{3} = \frac{80}{3}$$

10 **a** The line perpendicular to AB through B has gradient $-\frac{1}{2}$ and passes through $(-1, 6)$.

The equation of this line is $y = -\frac{1}{2}x + \frac{11}{2}$.

b Intersects AB when $2x + 3 = -\frac{1}{2}x + \frac{11}{2}$.

$\therefore x = 1, y = 5$ are the coordinates of point B.

c The coordinates of A and B are $(0, 3)$ and $(1, 5)$ respectively.

$\therefore$ the coordinates of C are $(2, 7)$.

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Solutions to Exercise 2F

1 The point $(2, 7)$ is on the line $y = mx - 3$.
Hence $7 = 2m - 3$
That is, $m = 5$

2 The point $(3, 11)$ is on the line $y = 2x + c$.
Hence $11 = 2 \times 3 + c$ That is, $c = 5$

3 **a** Gradient of line perpendicular to the line $y = mx + 3$ is $-\frac{1}{m}$ The y-intercept is 3.
The equation of the second line is $y = -\frac{x}{m} + 3$.

b If $(1, -4)$ is on the line, $-4 = -\frac{1}{m} + 3$.
Hence $-\frac{1}{m} = -7$.
That is, $m = \frac{1}{7}$

4 $8 = m \times 3 + 2$
$m = 2$

5 $f: R \to R, f(x) = mx - 3, m \in R/\{0\}$

a x-axis intercept: $mx - 3 = 0, \therefore x = \frac{3}{m}$

b $6 = 5m - 3$
$5m = 9$
$m = \frac{9}{5}$

c x-axis intercept ≤ 1 for $\frac{3}{m} \leq 1$,
$\therefore m \geq 3$

d $y = f(x)$ has gradient $= m$, so a perpendicular line has gradient $= -\frac{1}{m}$.
Using the straight line formula for the point $(0, -3)$:
$y - (-3) = -\frac{1}{m}(x - 0)$
$\therefore y = -\frac{1}{m}x - 3$
OR $my + x = -3m$

6 $f: R \to R, f(x) = 2x + c$, where $c \in R$

a x-axis intercept: $2x + c = 0, \therefore x = -\frac{c}{2}$

b $6 = 5 \times 2 + c$
$c = -4$

c $-\frac{c}{2} \leq 1$
$c \geq -2$

d $y = f(x)$ has gradient $= 2$, so a perpendicular line has gradient $= -\frac{1}{2}$.
Using the straight line formula for the point $(0, c)$:
$y - c = -\frac{1}{2}(x - 0)$
$\therefore y = -\frac{1}{2}x + c$

7 $\frac{x}{a} - \frac{y}{12} = 4$

a When $y = 0$, $\frac{x}{a} = 4, \therefore x = 4a$
The coordinates of the x-axis intercept are $(4a, 0)$.

b Rearranging to make y the subject.
$y = \frac{12x}{a} - 48$
The gradient of the line is $\frac{12}{a}$

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c **i** When the gradient is $2, \frac{12}{a} = 2, \therefore a = 6$

ii When the gradient is $-2, \frac{12}{a} = -2, \therefore a = -6$

iii $OM = \sqrt{\left(\frac{c}{4}\right)^2 + \left(\frac{c}{2}\right)^2}$

$= \sqrt{\frac{5c^2}{16}}$

If $OM = 2\sqrt{5}$ then $\sqrt{\frac{5c^2}{16}} = 2\sqrt{5}$

$\therefore c = 8$

8 **a** When $y = 0, x = \frac{c}{2}$

b $y = -2x + c$. When $x = 1, y = 7$

$\therefore 7 = -2 + c$

$\therefore c = 9.$

c $\frac{c}{2} \le 1 \Leftrightarrow c \le 2$

d Line perpendicular to $y = -2x + c$ has gradient $\frac{1}{2}$

Therefore $y = \frac{1}{2}x + c$

e $A(\frac{c}{2}, 0)$ and $B(0, c)$

i The midpoint of line segment AB has coordinates $\left(\frac{c}{4}, \frac{c}{2}\right)$

If $\left(\frac{c}{4}, \frac{c}{2}\right) = (3, 6)$ then $c = 12$

ii The area of the triangle $AOB = \frac{1}{2} \times c \times \frac{c}{2} = \frac{c^2}{4}$

If the area is 4, $\frac{c^2}{4} = 4$ which implies $c^2 = 16$. Therefore $c = 4$ since ($c > 0$

9 $3x + by = 12$

a $3x + by = 12$

$by = -3x + 12$

$y = -\frac{3}{b}x + \frac{12}{b}$

$\therefore$ y-axis intercept is $\frac{12}{b}$.

b $\therefore$ gradient $= -\frac{3}{b}$

c **i** $-\frac{3}{b} = 1$

$b = -3$

ii $-\frac{3}{b} = -2$

$b = \frac{3}{2}$

d Gradient of perpendicular line is $\frac{b}{3}$

The line is of the form $y = \frac{b}{3}x + c$

When $x = 4, y = 0$

$0 = \frac{b}{3} \times 4 + c$

$c = -\frac{4b}{3}$

$\therefore y = \frac{b}{3}x - \frac{4b}{3}$ or $3y = bx - 4b$

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Solutions to Exercise 2G

1 At $n = 0, w = \$350,$ paid at \$20 per n

$\therefore w = 20n + 350; n \in N \cup \{0\}$

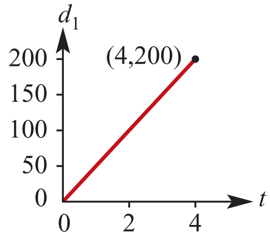
2 a At $t = 0, d_1 = 0$ and $v = 50$ km/h

$\therefore d_1 = vt = 50\,t$

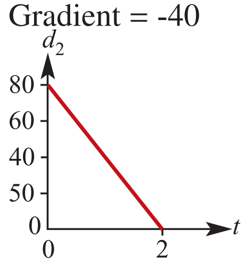
b At $t = 0, d_2 = 80$ and $v = -40$ km/h

$\therefore d_2 = 80 - 40\,t$

c Gradient = 50



Gradient = -40



3 a At $t = 0, V = 0,$ fills at 5 L/min

$\therefore V = 5t$

b At $t = 0, V = 10,$ fills at 5L/min

$\therefore V = 5t + 10$

4 a At $t = 0, v = 500,$ empties at 2.5 L/min

$\therefore v = -2.5t + 500$

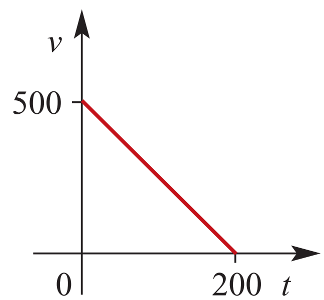
b Since the bag is emptying, $v \leq 500$
The bag cannot contain a negative volume so $v \geq 0$

$\therefore 0 \leq v \leq 500$
The bag does not go back in time so $t \geq 0$
The bag empties when

$$-2.5t + 500 = 0$$
$$2.5t = 500$$
$$t = 200$$

After that the function no longer holds true,
$\therefore 0 \leq t \leq 200$

c



5 At $n = 0, C = 2.6, C$ per km = 1.5
$\therefore C = 1.5n + 2.6$

6 a At $x = 0, C = 85, C$ per km = 0.24
$\therefore C = 0.24x + 85$

b When $x = 250,$

$$C = 0.24\,(250) + 85$$
$$= 60 + 85 = \$145$$

7 At $t = 0, d = 200$ km,
$v = -5$ km/h from B
$\therefore d = -5t + 200$

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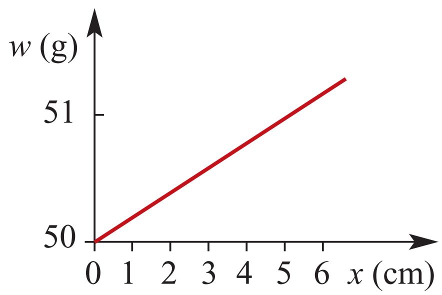
8 **a**



b $w = 50 + 0.2x$

c If $w = 52.5$ g,

$$x = 5 \times (52.5 - 50)$$
$$= 5 \times 2.5 = 12.5 \text{ cm}$$

9 **a** $C = an + b$

If $n = 800, C = 47$; if $n = 600, C = 35$

$$\begin{array}{rl} 800a + b &= 47 \\ 600a + b &= 35 \\ \hline 200a &= 12 \end{array}$$

$$\therefore a = \frac{12}{200} = \frac{3}{50} = 0.06$$

Substitute into 2nd equation:

$$600 \times \frac{3}{50} + b = 35$$
$$36 + b = 35$$
$$b = -1$$
$$\therefore C = 0.06n - 1$$

b If $n = 1000$,

$$c = 0.06(1000) - 1$$
$$= 60 - 1 = \$59$$

10 **a** $C = an + b$

If $n = 160, C = 975$; if $n = 120, C = 775$

$$\begin{array}{rl} 160a + b &= 975 \\ 120a + b &= 775 \\ \hline 40a &= 200 \end{array}$$

$$\therefore a = \frac{200}{40} = 5$$

Substitute into 2nd equation:

$$600 + b = 775$$
$$b = 175$$
$$\therefore C = 5n + 175$$

b Yes, because $b \neq 0$

c When $n = 0$, $C = \$\,175$

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Solutions to Exercise 2H

1 The lines $x + y = 6$ and $2x + 2y = 13$ both have gradient -1 but different y-intercepts.

2 Let $x = \lambda$. Then solution is $\{(\lambda, 6 - \lambda) : \lambda \in R\}$

3 **a** $m = 4$. The line $y = 4x + 6$ is parallel to the line $y = 4x - 5$

b $m \neq 4$

c $15 = 5m + 6$
$\therefore m = \dfrac{9}{5}$
Check: (5,15) lies on the line $y = 4x - 5$

4 $6 = 4 + k$ and $6 = 2m - 4$

$\therefore k = 2$ and $m = 5$

5
$$2(m - 2) + 8 = 4 \ldots (1)$$
$$2m + 24 = k \ldots (2)$$
From (1) $2m - 4 + 8 = 4$
$$m = 0$$
From (2) $k = 24$

6 The simultaneous equations have no solution when the cooresponding lines have the same gradient and no point in common.
Gradient of $mx - y = 5$ is m.

Gradient of $3x + y = 6$ is -3.
$\therefore$ lines are parallel when $m = -3$

7
Gradient of $3x + my = 5$ is $-\dfrac{3}{m}$.

Gradient of $(m + 2)x + 5y = m$ is $-\dfrac{m + 2}{5}$.
If the gradients are equal
$$-\frac{3}{m} = -\frac{m + 2}{5}$$
$15 = m^2 + 2m$

$m^2 + 2m - 15 = 0$

$(m + 5)(m - 3) = 0$

$m = -5$ or $m = 3$

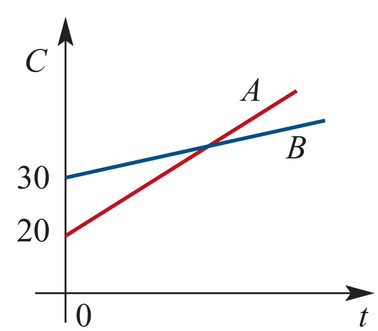
a When $m = -5$ the equations become
$3x - 5y = 5$

$-3x + 5y = -5$
They are equations of the same line.
There are infinitely many solutions.

b When $m = 3$ the equations become
$3x + 3y = 5$

$5x + 5y = 3$
They are the equations of parallel lines with no common point.
No solutions

8 A: $C = 10t + 20$
B: $C = 8t + 30$

a


b Costs are equal when

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$$10t + 20 = 8t + 30$$

$$2t = 10, \therefore t = 5$$

9 Day 1:

John: $v = \dfrac{1}{a}$ m/s

Michael: $v = \dfrac{1}{b}$ m/s

$d = vt = 50$ m, so Michael's time is:

$t = 50\dfrac{1}{v} = 50\,b$

Similarly, John's time is:

$t = 50\dfrac{1}{v} = 50\,a$

Michael wins by 1 second

$\therefore 50\,a = 50\,b + 1$

Day 2:

John runs only 47 m:

$t = 47\,a$

Michael runs the same time:

$t = 50\,b$

Michael wins by 0.1 seconds

$\therefore 47\,a = 50\,b + 0.1$

From day 1 : $50\,b = 50\,a - 1$

$\therefore 47\,a = 50\,a - 1 + 0.1$

$3\,a = 0.9, \therefore a = 0.3$

$50\,b = 50 \times 0.3 - 1 = 14$

$\therefore b = 0.28$

Michael's speed:

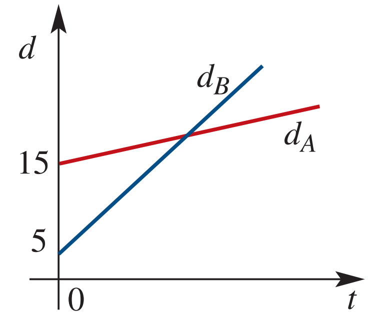
$v = \dfrac{1}{b} = \dfrac{1}{0.28} = \dfrac{25}{7}$ m/s

10 $d_A = 10t + 15$

$d_B = 20t + 5$

t is the time in hours after 1.00 p.m.

a



b $d_A = d_B$ when

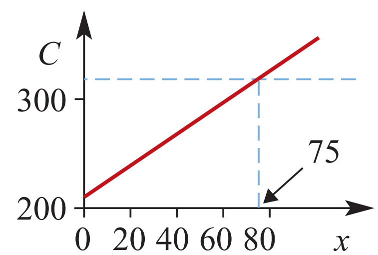
$20t + 5 = 10t + 15$

$10t = 10, \therefore t = 1$

11 **a** A: $C = 1.6x + 210$

B: $C = 330$

b



c Costs are equa1 when

$l.6x + 210 = 330$

$1.6x = 120$

$x = \dfrac{120}{1.6} = 75$

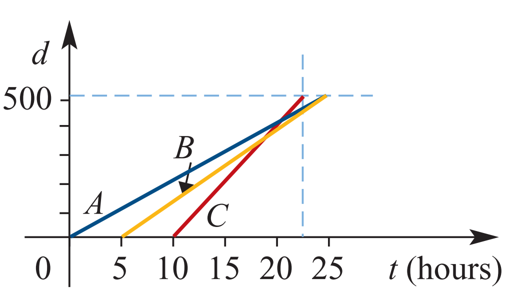
When $x < 75$, method A is cheaper;

when $x > 75$, method B is cheaper.

So the fixed charge method is cheaper

when $x > 75$.

12 **a**



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b C wins the race

c $A(t) = 20t$

$B(t) = 25(t - 5)$

$C(t) = 40(t - 10)$

After 25 hours,

$A(25) = 500$

$B(25) = 25(25 - 5) = 500$

$C(25) = 40(25 - 10) = 600$

C completes the course in t hours where:

$40(t - 10) = 500$

$$t - 10 = \frac{500}{40} = 12.5$$

$$t = 12.5 + 10 = 22.5$$

d C, leaving 5 hours after B, overtakes B $13\frac{1}{2}$ hours after B had started and then overtakes A 20 hours after A had started. C wins the race with a total handicap time of $22\frac{1}{2}$ hours ($12\frac{1}{2}$ hours for journey +10 hours handicap) with A and B deadheating for 2nd, each with a total handicap time of 25 hours.

13 $y = -\frac{3}{4}x$ meets $y = \frac{3}{2}x - 12$ when

$$-\frac{3x}{4} = \frac{3x}{2} - 12$$

$$-3x = 6x - 48$$

$$9x = 48$$

$$x = \frac{48}{9} = \frac{16}{3} = 5\tfrac{1}{3}$$

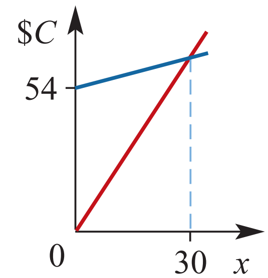
$$\therefore y = -\frac{3}{4} \times \frac{16}{3} = -4$$

The paths cross at $\left(5\frac{1}{3}, -4\right)$

14 **a** A: $\$C = 2.8x$

B: $\$C = x + 54$

b



c Costs are equal when $2.8x = x + 54$

$$1.8x = 54$$

$$x = 30$$

It is more economical if there are more than 30 students.

15 **a** Anne: when $t = 0, d = 0$;

$v = 20$ km/h $= \frac{1}{3}$km/min

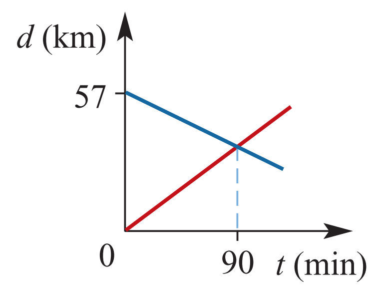
$\therefore d = \frac{t}{3}$

Maureen: when $t = 0, d = 57$;

$v = -18$ km/h $= -\frac{3}{10}$km/min

$\therefore d = 57 - \frac{3}{10}t$

b



c They meet when

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$$\frac{t}{3} = 57 - \frac{3t}{10}$$

$$\frac{t}{3} + \frac{3t}{10} = 37$$

$$10t + 9t = 1710$$

$$19t = 1710$$

$$t = \frac{1710}{19} = 90$$

They meet after 90 min, i.e. 10.30 a.m.

d Substitute into either equation, but Anne's is easier:

$d = \frac{t}{3} = \frac{90}{3} = 30$

Anne has traveled 30 km, so Maureen must have traveled $57 - 30 = 27$ km.

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Solutions to Technology-free questions

1 **a** A$(1, 2)$ and $B(5, 2)$: y does not change.

Length $AB = 5 - 1 = 4$

Midpoint $x = \dfrac{1+5}{2} = 3$

$\therefore$ midpoint is at $(3, 2)$.

b A$(-4, -2)$ and $B(3, -7)$

Length $AB = \sqrt{(-4-3)^2 + (-2 - -7)^2}$

$= \sqrt{(-7)^2 + 5^2} = \sqrt{74}$

Midpoint $x = \dfrac{-4+3}{2} = -\dfrac{1}{2}$

$y = \dfrac{-2 + -7}{2} = -\dfrac{9}{2}$

$\therefore$ midpoint is at $\left(-\dfrac{1}{2}, -\dfrac{9}{2}\right)$.

c A$(3, 4)$ and $B(7, 1)$

Length $AB = \sqrt{(7-3)^2 + (1-4)^2}$

$= \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$

Midpoint $x = \dfrac{3+7}{2} = 5$

$y = \dfrac{4+1}{2} = \dfrac{5}{2}$

$\therefore$ midpoint is at $\left(5, \dfrac{5}{2}\right)$.

2 **a** $m = \dfrac{12-3}{8-4} = \dfrac{9}{4}$

b $m = \dfrac{-6-4}{8 - -3} = -\dfrac{10}{11}$

c x does not change so gradient is undefined.

d $m = \dfrac{0-a}{a-0} = -1$

e $m = \dfrac{b-0}{a-0} = \dfrac{b}{a}$

f $m = \dfrac{0-b}{a-0} = -\dfrac{b}{a}$

3 If $m = 4$ then $y = 4x + c$

a Passing though $(0, 0)$; $y = 4x$

b Passing though $(0, 5)$; $y = 4x + 5$

c Passing though $(1, 6)$;

$y = 4 + c = 6, \therefore c = 2$

$y = 4x + 2$

d Passing though $(3, 7)$;

$y = 12 + c = 7, \therefore c = -5$

$y = 4x - 5$

4 **a** $y = 3x - 5$

Using $(1, a)$, $a = 3 - 5 = -2$

b $y = 3x - 5$

Using $(b, 15)$, $3b - 5 = 15$

$3b = 20, \therefore b = \dfrac{20}{3}$

5 $y = mx + c$; $m = \dfrac{-4-2}{3 - -5} = -\dfrac{3}{4}$

Using $(3, -4)$:

$-4 = \left(-\dfrac{3}{4}\right)3 + c$

$\dfrac{9}{4} - 4 = c = -\dfrac{7}{4}$

$y = -\dfrac{3}{4}x - \dfrac{7}{4}$

$4y = -3x - 7$

$\therefore 3x + 4y = -7$

6 $y = mx + c : m = -\dfrac{2}{3}$

Using $(-4, 1)$:

$y = -\dfrac{2}{3}(-4) + c = 1$

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$$\frac{8}{3} + c = 1 \therefore c = -\frac{5}{3}$$

$$y = -\frac{2}{3x} - \frac{5}{3}$$

$$3y = -2x - 5$$

$$\therefore 2x + 3y = -5$$

7 **a** Lines parallel to the x-axis are $y = c$.
Using (5, 11), $y = 11$

b Parallel to $y = 6x + 3$ so gradient
$m = 6$
When $x = 0, y = -10$, so $c = -10$
$y = 6x - 10$

c

$$3x - 2y + 5 = 0$$

$$-2y = -3x - 5$$

$$\therefore \quad y = \frac{3}{2}x + \frac{5}{2}$$

$m = \frac{3}{2}$ so perpendicular gradient
$= -\frac{2}{3}$
Using (0,1), $c = 1$

$$y = -\frac{2}{3}x + 1$$

$$3y = -2x - 3$$

$$\therefore \quad 2x + 3y = -3$$

8 $y = mx + c$:
Line at 30° to x-axis, $m = \tan 30° = \frac{1}{\sqrt{3}}$

Using (2, 3) : $\quad 3 = \frac{2}{\sqrt{3}} + c$

$$\therefore \quad c = 3 - \frac{2}{\sqrt{3}}$$

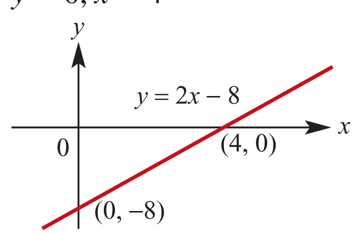
$$\therefore \sqrt{3}y - x = 3\sqrt{3} - 2$$

9 $y = mx + c$:
Line at 135° to x-axis,
$m = \tan 135° = -1$
Using $(-2, 3) : 3 = -1 \times -2 + c$
So $c = 1$ and $y = -x + 1$
$\therefore x + y = 1$

10 Gradient of a line perpendicular to
$y = -3x + 2$ is $\frac{1}{3}$.
Therefore required line is of the form
$y = \frac{1}{3}x + c$.
When $x = 4, y = 8$
$\therefore 8 = \frac{1}{3} \times 4 + c$
Hence $c = 8 - \frac{4}{3} = \frac{20}{3}$
$y = \frac{1}{3}x + \frac{20}{3}$

11 $y = 2x + 1$
When $x = 0, y = 1. \therefore a = 1$
When $y = 0, 2x + 1 = 0. \therefore b = -\frac{1}{2}$
When $x = 2, y = 5. \therefore d = 5$
When $y = 7, 2x + 1 = 7. \therefore e = 3$

12 **a** $y = 2x - 8$
When $x = 0, y = -8$ and when
$y = 0, x = 4$



b $3x + y = 6$
When $x = 0, y = 6$ and when
$y = 0, x = 2$

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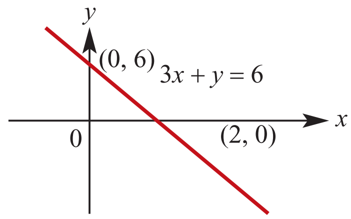
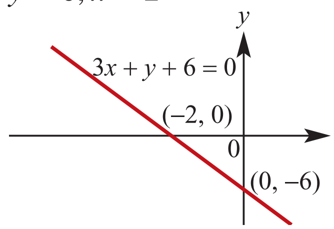


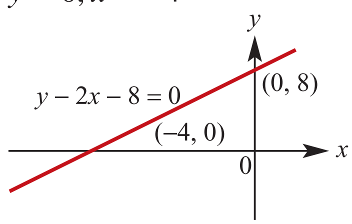
c $3x + y + 6 = 0$

When $x = 0, y = -6$ and when $y = 0, x = 2$



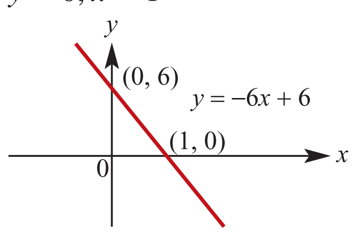
d $y - 2x - 8 = 0$

When $x = 0, y = 8$ and when $y = 0, x = -4$



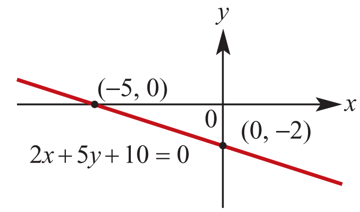
e $y = -6x + 6$

When $x = 0, y = 6$ and when $y = 0, x = 1$



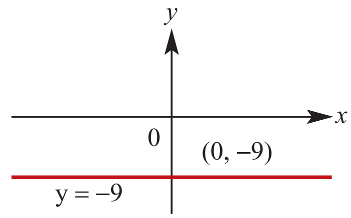
f $2x + 5y + 10 = 0$

When $x = 0, y = -2$ and when $y = 0, x = -5$



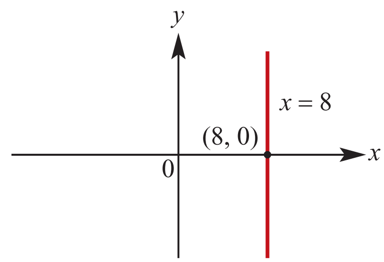
13 a Line is of the form $y = c$.

$\therefore y = -9$ is the equation

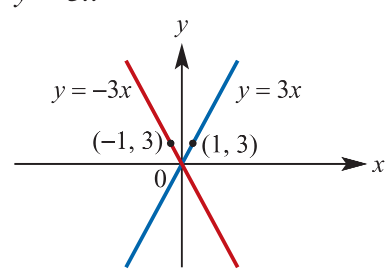


b Line is of the form $x = a$.

$\therefore x = 8$ is the equation.



c **i** $y = 3x$



ii $y = -3x$

14 a $d = 60t$

b gradient is 60

15 $S = 800 + 500n$

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16 a $y = ax + 2$

When $x = 2, y = 6$

$\therefore 6 = 2a + 2$

$\therefore a = 2$

The equation is $y = 2x + 2$

b i When $y = 0, ax + 2 = 0$

$\therefore x = \dfrac{-2}{a}$

ii

$$\frac{-2}{a} > 1$$

$$-2 < a \quad \text{(Since } a < 0\text{)}$$

$$\therefore 0 > a > -2$$

Equivalently $-2 < a < 0$

c $x + 3 = ax + 2$ Substitute in

$x - ax = 2 - 3$

$x(1 - a) = -1$

$x = \dfrac{1}{a - 1}$

$y = x + 3$ to find the y-coordinate.

$y = \dfrac{1}{a-1} + 3$

$y = \dfrac{3a - 2}{a - 1}$

.

The coordinates of the point of intersection are $\left(\dfrac{1}{a-1}, \dfrac{3a-2}{a-1}\right)$

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Solutions to multiple-choice questions

1 D Midpoint $x = \dfrac{4+6}{2} = 5$

Midpoint $y = \dfrac{12+2}{2} = 7$

Midpt is at (5, 7)

2 E Midpoint x-coordinate

$6 = \dfrac{-4+x}{2}, \therefore x = 16$

Midpoint y-coordinate

$3 = \dfrac{-6+y}{2}, \therefore y = 12$

$\therefore x + y = 28$

3 A Gradient $= \dfrac{-10-(-8)}{6-5} = -2$

4 E Gradient $= \dfrac{2a-(-3a)}{4a-9a} = -1$

5 C $y = mx + c; m = 3$

Using $(1, 9)$:

$9 = 3 + c$ so $c = 6$

$y = 3x + 6$

6 D $y = mx + c; m = \dfrac{-14 - -6}{-2-2} = 2$

Using $(2, -6)$:

$y = 4 + c = -6; c = -10$

So $y = 2x - 10$

7 B $y = 2x - 6$

Using $(a, 2)$:

$y = 2a - 6 = 2, \therefore a = 4$

8 E Axis intercepts at $(1, 0)$ and $(0, -3)$:

$y = mx + c : c = -3$

Using (1, 0): $0 = m - 3$ so $m = 3$

$y = 3x - 3$

9 C $5x - y + 7 = 0$

$$-y = -5x - 7$$

$$\therefore \quad y = 5x + 7$$

Gradient = 5

$$ax + 2y - 11 = 0$$

$$2y = -ax + 11$$

$$\therefore \quad y = -\frac{a}{2}x + \frac{11}{2}$$

Parallel lines mean gradients are equal:

$-\dfrac{a}{2} = 5, \therefore a = -10$

10 E

$C = 2.5x + 65 = 750$

$2.5x = 685, \therefore x = 274$

11 C

$$2ax + 2by = 3 \ldots (1)$$

$$3ax - 2by = 7 \ldots (2)$$

Add (1) and (2)

$$5ax = 10$$

$$\therefore x = \frac{2}{a}$$

Substitute in (1)

$$y = -\frac{1}{2b}$$

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Solutions to extended-response questions

1 a $C = 100n + 27.5n + 50 + 62.5n = 550 + 190n$

b

$$\begin{aligned} C \le 3000 \quad \therefore\ 550 + 190n &\le 3000 \\ \therefore\quad 190n &\le 2450 \\ \therefore\quad n &\le \frac{2450}{190} \\ \therefore\quad n &< 12.9 \end{aligned}$$

The cruiser can be hired for up to and including 12 days by someone wanting to spend no more than $3000.

c $300n < 550 + 190n$

$110n < 550$

$n < 5$

It's cheaper to hire from the rival company for cruises less than 5 days.

2 a It is the cost of the plug.

b It is the cost per metre of the cable.

c 1.8

d

$$\begin{aligned} 24.5 &= 4.5 + 1.8x \\ \therefore\quad 20 &= 1.8x \\ \therefore\quad x &= \frac{20}{1.8} \\ &= \frac{100}{9} \\ &= 11\frac{1}{9} \end{aligned}$$

$11\frac{1}{9}$ metres of cable would give a total cost of \$ 24.50.

3 a It is the maximum profit when the bus has no empty seats, i.e. $x = 0$.

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b
$$\begin{aligned} P &< 0 \\ 1020 - 24x &< 0 \\ -24x &< -1020 \\ x &> \frac{-1020}{-24} \\ x &> 42.5 \end{aligned}$$

43 empty seats is the least number to cause a loss on a single journey.

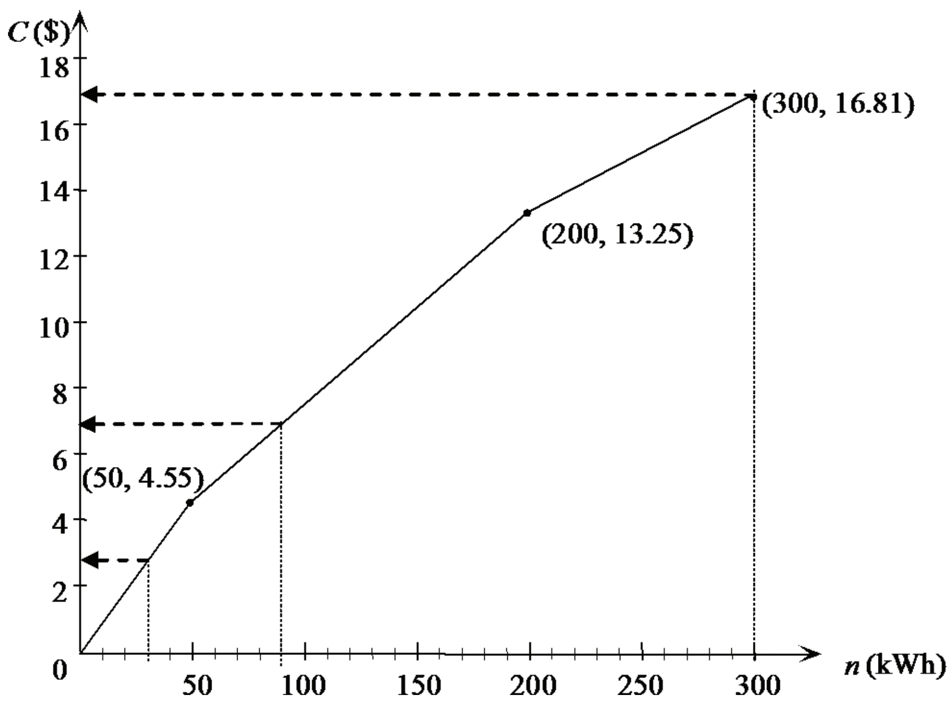
c The profit reduces by \$24 for each empty seat.

4 a i $C = 0.091n, \qquad 0 < n \le 50$

ii
$$\begin{aligned} C &= 0.058(n - 50) + 0.091 \times 50 \\ &= 0.058n - 2.9 + 4.55 \\ &= 0.058n + 1.65, \qquad 50 < n \le 200 \end{aligned}$$

iii
$$\begin{aligned} C &= 0.0356(n - 200) + 0.058 \times 200 + 1.65 \\ &= 0.0356n - 7.12 + 11.6 + 1.65 \\ &= 0.0356n + 6.13, \qquad n > 200 \end{aligned}$$

b



i When $n = 30$ kWh,

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from the graph $C \approx \$3$

from the formula $C = 0.091 \times 30$

$= \$2.73$

ii When $n = 90$ kWh,

from the graph $C \approx \$7$

from the formula $C = 0.058 \times 90 + 1.65$

$= \$6.87$

iii When $n = 300$ kWh,

from the graph $C \approx 17$

from the formula $C = 0.0356 \times 300 + 6.13$

$= \$16.81$

c When $C = 20$,

$$\begin{aligned} 20 &= 0.0356n + 6.13 \\ \therefore \quad 13.87 &= 0.0356n \\ \therefore \quad n &= 389.60\ldots \end{aligned}$$

Approximately 390 kWh of electricity could be used for \$20.

5 a Let $(x_1, y_1) = (2, 10)$ and $(x_2, y_2) = (8, -4)$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 10}{8 - 2} \\ &= \frac{-14}{6} \\ &= \frac{-7}{3} \end{aligned}$$

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ \therefore \quad y - 10 &= \frac{-7}{3}(x - 2) \\ \therefore \quad y - 10 &= \frac{-7}{3}x + \frac{14}{3} \\ \therefore \quad y + \frac{7}{3}x &= 10 + \frac{14}{3} \\ \therefore \quad y + \frac{7}{3}x &= \frac{44}{3} \\ \therefore \quad 3y + 7x &= 44 \end{aligned}$$

$\therefore y = -\frac{7}{3}x + 14\frac{2}{3}$

The equation describing the aircraft's flight path is $7x + 3y = 44$.

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b When $x = 15$, $\quad 7 \times 15 + 3y = 44$

$$\therefore \quad 105 + 3y = 44$$

$$\therefore \quad 3y = -61$$

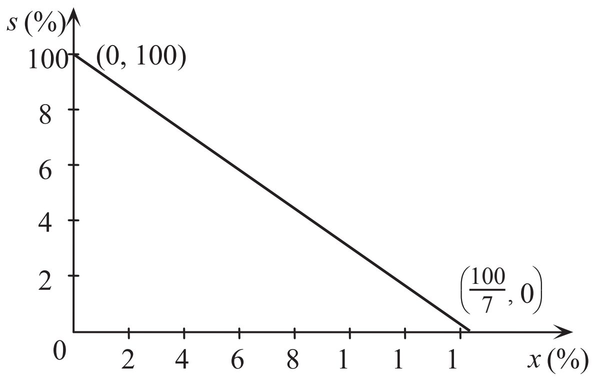
$$\therefore \quad y = \frac{-61}{3}$$

$$= -20\frac{1}{3}$$

When $x = 15$, the aircraft is $20\frac{1}{3}$ km south of O.

6 a $s = 100 - 7x$

b



c $\quad 100 - 7x \geq 95$

$$\therefore \quad -7x \geq -5$$

$$\therefore \quad x \leq \frac{5}{7}$$

i.e. $\frac{5}{7}$% air can be left in the concrete for at least 95% strength.

d $\quad 100 - 7x = 0$

$$\therefore \quad -7x = -100$$

$$\therefore \quad x = \frac{100}{7}$$

$$= 14\frac{2}{7}$$

i.e. the concrete contains $14\frac{2}{7}$% air when at 0% strength.

e Concrete at 0% strength would not be useful, therefore not a sensible model.

f $\{x : 0 \leq x \leq 14\frac{2}{7}\}$

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7 a For line AB, let $(x_1, y_1) = (0, 2)$ and $(x_2, y_2) = (4, 6)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-2}{4-0} = 1$$

(Alternatively, $y = mx + c$, where $m = 1$ as shown and $c = 2, \therefore\ y = x + 2$.)

Now $\quad y - y_1 = m(x - x_1)$

$\therefore \quad y - 2 = 1(x - 0)$

$\therefore \quad y = x + 2$ is the equation of line AB.

For line CD, let $(x_1, y_1) = (3, 0)$ and $(x_2, y_2) = (5, 4)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-0}{5-3} = 2$$

Now $\quad y - y_1 = m(x - x_1)$

$\therefore \quad y - 0 = 2(x - 3)$

$\therefore \quad y = 2x - 6$

The equation of line CD is $y = 2x - 6$.

b The lines intersect where $\quad x + 2 = 2x - 6$

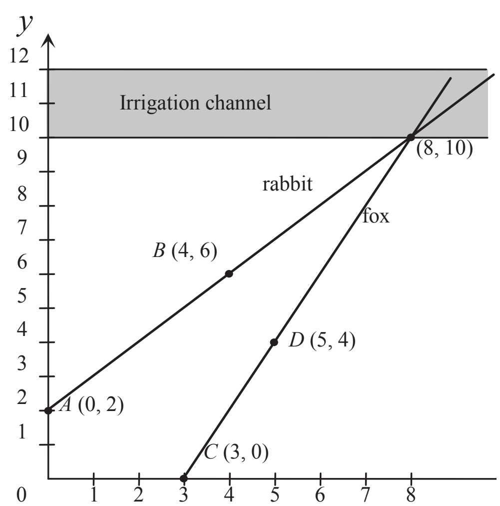
$\therefore \quad 2 = x - 6$

$\therefore \quad x = 8$

When $x = 8$, $\quad y = 2 \times 8 - 6$

$\therefore \quad y = 10$

i.e. the coordinates of the point of intersection are (8, 10), and the paths of the rabbit and the fox meet on the near bank of the irrigation channel.



8 a For the equation of line AB, let $(x_1, y_1) = (-4.5, 2)$ and $(x_2, y_2) = (0.25, 7)$.

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$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{7 - 2}{0.25 - (-4.5)} \\ &= \frac{5}{4.75} \\ &= \frac{20}{19} \end{aligned}$$

Now $\quad y - y_1 = m(x - x_1)$

$\therefore \quad y - 2 = \frac{20}{19}(x - (-4.5))$

$\therefore \quad y = \frac{20}{19}x + \frac{90}{19} + 2$

$\therefore \quad y = \frac{20}{19}x + \frac{128}{19}$

The equation of line AB is $y = \frac{20}{19}x + \frac{128}{19}$.

$\therefore$ the y-coordinate of the point V is the y-axis intercept of $\frac{128}{19}$.

b For the equation of line VC, let $(x_1, y_1) = (0, \frac{128}{19})$ and $(x_2, y_2) = (5, 1.5)$.

$C = \frac{128}{19}$ and $\quad m = \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{1.5 - \frac{128}{19}}{5 - 0} -$$

$$= \frac{\frac{57}{38} - \frac{256}{38}}{5} = \frac{1}{5} \times -\frac{199}{38} = -\frac{199}{190}$$

Now $\quad y = mx + c$

$y = -\frac{199}{190}x + \frac{128}{19}$ is the equation of the line VC.

c Cuts AB and VC are not equally inclined to the vertical axis because the gradients of AB and VC (although opposite in sign) are not the same size, (gradient $AB \approx 1.053$, gradient $VC \approx -1.047$).

9 **a** For the equation of line PQ, let $(x_1, y_1) = (4, -75)$ and $(x_2, y_2) = (36, -4)$.

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$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-4 - (-75)}{36 - 4}$$
$$= \frac{71}{32}$$

Now $\quad y - y_1 = m(x - x_1)$

$\therefore \quad y - (-75) = \frac{71}{32}(x - 4)$

$\therefore \quad y + 75 = \frac{71}{32}x - \frac{71}{8}$

$\therefore \quad y = \frac{71}{32}x - \frac{671}{8}$ is the equation of line PQ.

When $7x = 20$, $\quad y = \frac{71}{32} \times 20 - \frac{671}{8}$
$$= \frac{355}{8} - \frac{671}{8}$$
$$= \frac{-316}{8}$$
$$= -39\frac{1}{2}$$

i.e. line PQ does not pass directly over a hospital located at $H(20, -36)$.

b When $y = -36$, $\quad -36 = \frac{71}{32}x - \frac{671}{8}$

$\therefore \quad \frac{383}{8} = \frac{71}{32}x$

$\therefore \quad x = \frac{383}{8} \times \frac{32}{71} = 21\frac{41}{71}$

i.e. when $y = -36$, the aircraft is $1\frac{41}{71}$ km east of H.

10 a 5 km due south of E is $D(68, 30)$.

For the equation of line AD, let $(x_1, y_1) = (48, 10)$ and $(x_2, y_2) = (68, 30)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{30 - 10}{68 - 48} = 1$$

Now $\quad y - y_1 = m(x - x_1)$

$\therefore \quad y - 10 = 1(x - 48)$

$\therefore \quad y = x - 38$ is the equation ofthe new runway.

b $A(48, 10)$

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$\therefore \quad B((48+8), y) = (56, y)$

When $x = 56$,

$$\begin{aligned} y &= x - 38 \\ &= 56 - 38 \\ &= 18 \end{aligned}$$

$\therefore$ the coordinates of B are (56, 18).

c Consider the line connecting $C(88, -10)$ and $D(68, 30)$.

Let $(x_1, y_1) = (88, -10)$ and $(x_2, y_2) = (68, 30)$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{30 - (-10)}{68 - 88} \\ &= \frac{40}{-20} = -2 \end{aligned}$$

Now $\quad y - y_1 = m(x - x_1)$

$\therefore \quad y - (-10) = -2(x - 88)$

$\therefore \quad y + 10 = -2x + 176$

$\therefore \quad y = -2x + 166$ is the equation defining the line CD.

d $y = 10$ for an auxiliary beacon due east of A.

$\therefore \quad 2x + 10 = 166$

$\therefore \quad 2x = 156$

$\therefore \quad x = 78$

i.e. the coordinates of the auxiliary beacon are (78, 10).

11 a Gradient of $AB = \dfrac{6 - 8}{8 - 2}$

$$\begin{aligned} &= -\frac{2}{6} \\ &= -\frac{1}{3} \end{aligned}$$

As $AD \perp AB$, gradient of $AD = 3$.

Equation of the line through A and D is

$$\begin{aligned} y - 8 &= 3(x - 2) \\ &= 3x - 6 \end{aligned}$$

$\therefore \quad y = 3x + 2$

b D lies on the line with equation $y = 3x + 2$, and has an x-coordinate of 0.

When $x = 0, y = 2 \therefore D$ is the point $(0, 2)$.

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c Let M be the midpoint of AB, with coordinates (x, y).

$$x = \frac{2+8}{2} = 5$$

and $$y = \frac{8+6}{2} = 7$$

$\therefore$ M is the point (5, 7).

The point M lies on the perpendicular bisector of AB that has a gradient 3.

The equation of the perpendicular bisector of AB is

$$y - 7 = 3(x - 5)$$

$$= 3x - 15$$

$$\therefore \quad y = 3x - 8$$

d Let C be the point (a, b).

C lies on the lines with equation $y = 3x - 8$ and $3y = 4x - 14$

$$\therefore \quad b = 3a - 8 \qquad (1)$$

and $$3b = 4a - 14 \qquad (2)$$

Substituting (1) into (2) yields

$$3(3a - 8) = 4a - 14$$

$$\therefore \quad 9a - 24 = 4a - 14$$

$$\therefore \quad 5a = 10$$

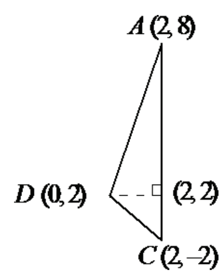
$$\therefore \quad a = 2$$

$$\therefore \quad b = 3(2) - 8 = -2$$

$\therefore$ C is the point (2, −2).

e Area of triangle ADC

Using $\frac{1}{2}$ (base)(height), area ΔADC is simply $\frac{1}{2}(2)(10) = 10$ square units [since AC = base length = 10 units]



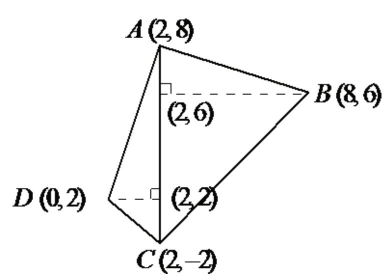
f Area ΔABC is simply $\frac{1}{2}(6)(10) = 30$ square units

Area of quadrilateral $ABCD$

= area of ΔADC + area of ΔABC

= 10 + 30

= 40 square units



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12 **a** $C = 40x + 30\,000$

b When $x = 6000$,

$$\begin{aligned} C &= 40 \times 6000 + 30\,000 \\ &= 270\,000 \\ \text{Cost per wheelbarrow} &= \frac{270\,000}{6000} \\ &= 45 \end{aligned}$$

i.e. overall cost per wheelbarrow is \$45.

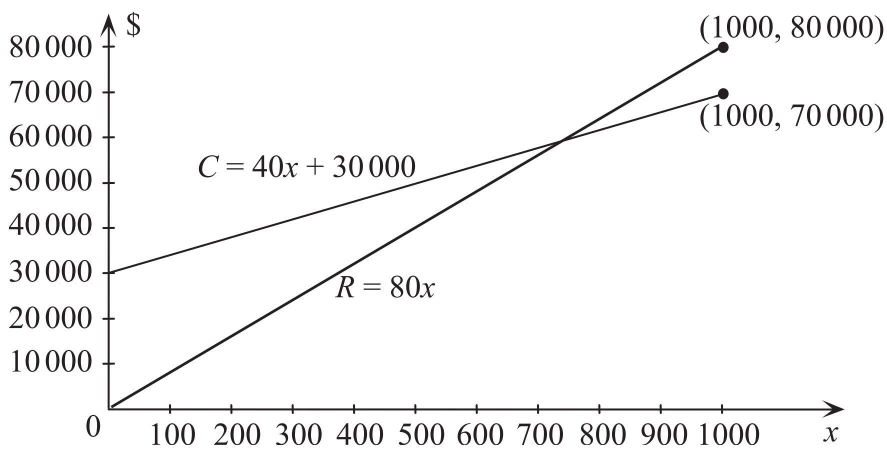
c Cost per wheelbarrow = \$46

$$\begin{aligned} \therefore \quad \frac{40x + 30\,000}{x} &= 46 \\ \therefore \quad 40x + 30\,000 &= 46x \\ \therefore \quad 30\,000 &= 6x \\ \therefore \quad x &= 5000 \end{aligned}$$

i.e. 5000 wheelbarrows must be made for an overall cost of \$46 each.

d $R = 80x$

e



f $R > C$, $\quad \therefore 80x > 40x + 30\,000$

$$\begin{aligned} \therefore \quad 40x &> 30\,000 \\ \therefore \quad x &> 750 \end{aligned}$$

i.e. minimum number of wheelbarrows to make a profit is 751.

g $P = R - C$

$$\begin{aligned} &= 80x - (40x + 30\,000) \\ &= 40x - 30\,000 \end{aligned}$$

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13 a Cost of Method 1 $= 100 + 0.08125 \times 1560$

$= 226.75$

Cost of Method 2 $= 4 \times 27.5 + 0.075 \times 1560$

$= 227$

i.e. Method 1 is cheaper for 1560 units.

b

	Number of units of electricity			
	0	1000	2000	3000
Cost ($) by Method 1	100	181.25	262.50	343.75
Cost ($) by Method 2	110	185	260	335

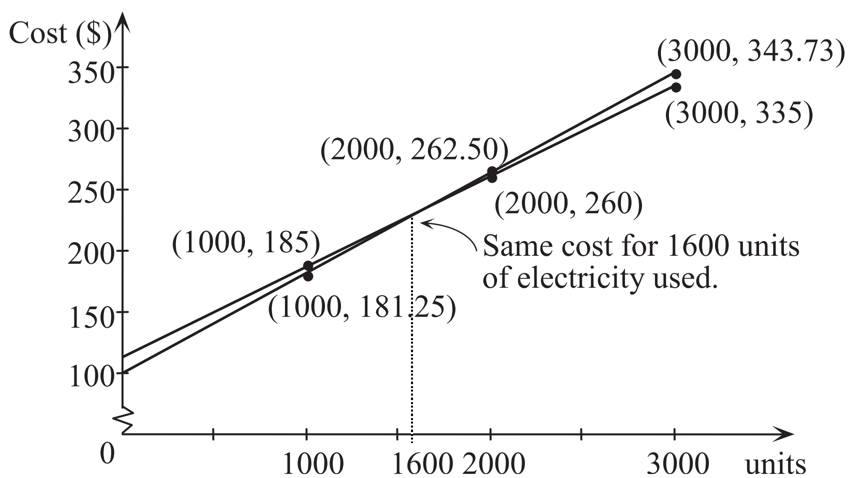
Calculations for Method 1:

$$100 + 0.08125 \times 0 = 100$$
$$100 + 0.08125 \times 1000 = 181.25$$
$$100 + 0.08125 \times 2000 = 262.50$$
$$100 + 0.08125 \times 3000 = 343.75$$

Calculations for Method 2:

$$4 \times 27.5 + 0.075 \times 0 = 110$$
$$4 \times 27.5 + 0.075 \times 1000 = 185$$
$$4 \times 27.5 + 0.075 \times 2000 = 260$$
$$4 \times 27.5 + 0.075 \times 3000 = 335$$

c



d $C_1 = 100 + 0.08125x$

$C_2 = 4 \times 27.5 + 0.075 \times x$

$= 110 + 0.075x$

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When $C_1 = C_2$, $\quad 100 + 0.08125x = 110 + 0.075x$

$$0.00625x = 10$$

$$x = 1600$$

14 a M is the point directly below the intersection of lines AC and BD.
Find the equations of the lines AC and BD.
M is the midpoint of both AC and BD.
Use the midpoint formula to find the coordinates of M.
Using line AC:

$$M = \left(\frac{10+24}{2}, \frac{16+8}{2}\right)$$
$$= (17, 12)$$

i.e. the point to which the hook must be moved has coordinates (17, 12).

b (17, 12) is a point on the line, and m = gradient of AB

where $\quad (x_1, y_1) = (10, 16)$

and $\quad (x_2, y_2) = (16, 20)$

$$\therefore \quad m = \frac{20-16}{16-10} = \frac{2}{3}$$

$\therefore$ equation of line parallel to AB is

$$y - 12 = \frac{2}{3}(x - 17)$$

$$\therefore \quad y = \frac{2}{3}x - \frac{34}{3} + 12$$

$$\therefore \quad 3y = 2x + 2$$

15 a Find equations of lines PA, AB, BC, CD and DP using the formula $y - y_1 = m(x - x_1)$, where $m = -\dfrac{y_2 - y_1}{x_2 - x_1}$.

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For line PA
$$\begin{aligned} m &= \frac{60-120}{100-0} \\ &= -\frac{3}{5} \\ y-120 &= -\frac{3}{5}(x-0) \\ \therefore \quad y &= -\frac{3}{5}x+120 \end{aligned}$$

For line AB
$$\begin{aligned} m &= \frac{100-60}{200-100} \\ &= \frac{2}{5} \\ y-60 &= \frac{2}{5}(x-100) \\ \therefore \quad y &= \frac{2}{5}x+20 \end{aligned}$$

For line BC
$$\begin{aligned} m &= \frac{200-100}{160-200} \\ &= -\frac{5}{2} \\ y-100 &= -\frac{5}{2}(x-200) \\ y &= -\frac{5}{2}x+600 \end{aligned}$$

For line CD
$$\begin{aligned} m &= \frac{160-200}{60-160} \\ &= \frac{2}{5} \\ y-200 &= \frac{2}{5}(x-160) \\ \therefore \quad y &= \frac{2}{5}x+136 \end{aligned}$$

For line DP
$$\begin{aligned} m &= \frac{120-160}{0-60} \\ &= \frac{2}{3} \\ y-120 &= \frac{2}{3}(x-0) \\ \therefore \quad y &= \frac{2}{3}x+120 \end{aligned}$$

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b

$$m_{PA} = -\frac{3}{5}$$

$$m_{AB} = \frac{2}{5}$$

$$m_{BC} = -\frac{5}{2}$$

$$m_{CD} = \frac{2}{5}$$

and $$m_{DP} = \frac{2}{3}$$

Now $$m_{AB} = m_{CD} = \frac{2}{5}$$

and $$m_{AB} \times m_{BC} = \frac{2}{5} \times -\frac{5}{2} = -1$$

$\therefore$ line BC is perpendicular to lines AB and CD, which are parallel.

Hence $\angle ABC$ and $\angle BCD$ are right angles.

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Chapter 3 – Quadratics

Solutions to Exercise 3A

1 a $2(x-4) = 2x-8$

b $-2(x-4) = -2x+8$

c $3(2x-4) = 6x-12$

d $-3(4-2x) = 6x-12$

e $x(x-1) = x^2-x$

f $2x(x-5) = 2x^2-10x$

2 a $(2x+4x)+1 = 6x+1$

b $(2x+x)-6 = 3x-6$

c $(3x-2x)+1 = x+1$

d $(-x+2x+4x)-3 = 5x-3$

3 a $8(2x-3)-2(x+4)$

$= 16x-24-2x-8$

$= 14x-32$

b $2x(x-4)-3x$

$= 2x^2-8x-3x$

$= 2x^2-11x$

c $4(2-3x)+4(6-x)$

$= 8-12x+24-4x$

$= 32-16x$

d $4-3(5-2x)$

$= 4-15+6x$

$= 6x-11$

4 a $2x(x-4)-3x$

$= 2x^2-8x-3x$

$= 2x^2-11x$

b $2x(x-5)+x(x-5)$

$= 2x^2-10x+x^2-5x$

$= 3x^2-15x$

c $2x(-10-3x)$

$= -20x-6x^2$

d $3x(2-3x+2x^2)$

$= 6x-9x^2+6x^3$

e $3x-2x(2-x)$

$= 3x-4x+2x^2$

$= 2x^2-x$

f $3(4x-2)-6x$

$= 12x-6-6x$

$= 6x-6$

5 a $(3x-7)(2x+4)$

$= 6x^2+12x-14x-28$

$= 6x^2-2x-28$

b $(x-10)(x-12)$

$= x^2-10x-12x+120$

$= x^2-22x+120$

c $(3x-1)(12x+4)$

$= 36x^2+12x-12x-4$

$= 36x^2-4$

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d $(4x-5)(2x-3)$

$= 8x^2 - 12x - 10x + 15$

$= 8x^2 - 22x + 15$

e $(x-\sqrt{3})(x-2)$

$= x^2 - 2x - \sqrt{3}x + 2\sqrt{3}$

$= x^2 - (2+\sqrt{3})x + 2\sqrt{3}$

f $(2x-\sqrt{5})(x+\sqrt{5})$

$= 2x^2 + 2\sqrt{5}x - \sqrt{5}x - 5$

$= 2x^2 + \sqrt{5}x - 5$

g $(3x-\sqrt{7})(x+\sqrt{7})$

$= 3x^2 + 3\sqrt{7}x - 2\sqrt{7}x - 14$

$= 3x^2 + \sqrt{7}x - 14$

h $(5x-3)(x+2\sqrt{2})$

$= 5x^2 + 10\sqrt{2}x - 3x - 6\sqrt{2}$

$= 5x^2 + (10\sqrt{2}-3)x - 6\sqrt{2}$

i $(\sqrt{5}x-3)(\sqrt{5}x-32\sqrt{2})$

$= 5x^2 - 32\sqrt{10}x - 3\sqrt{5}x + 96\sqrt{2}$

$= 5x^2 - (32\sqrt{10}+3\sqrt{5})x + 96\sqrt{2}$

6 a $(2x-3)(3x^2+2x-4)$

$= 6x^3 + 4x^2 - 8x - 9x^2 - 6x + 12$

$= 6x^3 - 5x^2 - 14x + 12$

b $(x-1)(x^2+x+1)$

$= x^3 + x^2 + x - x^2 - x - 1$

$= x^3 - 1$

c $(6-2x-3x^2)(4-2x)$

$= 24 - 12x - 8x + 4x^2 - 12x^2 + 6x^3$

$= 24 - 20x - 8x^2 + 6x^3$

d $(5x-3)(x+2)-(2x-3)(x+3)$

$= (5x^2+10x-3x-6)$

$\quad - (2x^2+6x-3x-9)$

$= (5x^2+7x-6)-(2x^2+3x-9)$

$= 3x^2 + 4x + 3$

e $(2x+3)(3x-2)-(4x+2)(4x-2)$

$= (6x^2-4x+9x-6)$

$\quad - (16x^2-8x+8x-4)$

$= (6x^2+5x-6)-(16x^2-4)$

$= -10x^2 + 5x - 2$

7 a $(x-4)^2$

$= x^2 - 4x - 4x + 16$

$= x^2 - 8x + 16$

b $(2x-3)^2$

$= 4x^2 - 6x - 6x + 9$

$= 4x^2 - 12x + 9$

c $(6-2x)^2$

$= 36 - 12x - 12x + 4x^2$

$= 36 - 24x + 4x^2$

d $\left(x-\frac{1}{2}\right)^2$

$= x^2 - \frac{x}{2} - \frac{x}{2} + \frac{1}{4}$

$= x^2 - x + \frac{1}{4}$

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e $(x-\sqrt{5})^2$

$= x^2 - \sqrt{5}x - \sqrt{5}x + 5$

$= x^2 - 2\sqrt{5}x + 5$

f $(x-2\sqrt{3})^2$

$= x^2 - 2\sqrt{3}x - 2\sqrt{3}x + 4(3)$

$= x^2 - 4\sqrt{3}x + 12$

8 a $(x-3)(x+3)$

$= x^2 - 3x + 3x - 92$

$= x^2 - 9$

b $(2x-4)(2x+4)$

$= 4x^2 + 8x - 8x - 16$

$= 4x^2 - 16$

c $(9x-11)(9x+11)$

$= 81x^2 + 99x - 99x + 121$

$= 81x^2 - 121$

d $(2x-3)(2x+3)$

$= 4x^2 - 9$

e $(2x+5)(2x-5)$

$= 4x^2 - 25$

f $(x-\sqrt{5})(x+\sqrt{5})$

$= x^2 - 5$

g $(2x+3\sqrt{3})(2x+3\sqrt{3})$

$= 4x^2 - 27$

h $(\sqrt{3}-\sqrt{7})(\sqrt{3}+\sqrt{7})$

$= 3x^2 - 7$

9 a $(x-y+z)(x-y-z)$

$= ((x-y)+z)((x-y)-z)$

$= (x-y)^2 - z^2$

$= x^2 - 2xy + y^2 - z^2$

b $(2a-b+c)(2a-b-c)$

$= ((2a-b)+c)((2a-b)-c)$

$= (2a-b)^2 - c^2$

$= 4a^2 - 4ab + b^2 - c^2$

c $(3w-4z+u)(3w+4z-u)$

$= (3w-(4z-u))((3w+(4z-u))$

$= (3w)^2 - (4z-u)^2$

$= 9w^2 - 16z^2 + 8zu - u^2$

d $(2a-\sqrt{5}b+c)(2a+\sqrt{5}b+c)$

$= (2a+c-\sqrt{5}b)(2a+c-\sqrt{5}b)$

$= (2a+c)^2 - 5b^2$

$= 4a^2 + 4ac + c^2 - 5b^2$

10 a **i** $A = x^2 + 2x + 1$

ii $A = (x+1)^2$

b **i** $A = (x-1)^2 + 2(x-1) + 1$

ii $A = x^2$

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Solutions to Exercise 3B

1 a $2x + 4 = 2(x + 2)$

b $4a - 8 = 4(a - 2)$

c $6 - 3x = 3(2 - x)$

d $2x - 10 = 2(x - 5)$

e $18x + 12 = 6(3x + 2)$

f $24 - 16x = 8(3 - 2x)$

2 a $4x^2 - 2xy = 2x(2x - y)$

b $8ax + 32xy = 8x(a + 4y)$

c $6ab - 12b = 6b(a - 2)$

d $6xy + 14x^2y = 2xy(3 + 7x)$

e $x^2 + 2x = x(x + 2)$

f $5x^2 - 15x = 5x(x - 3)$

g $-4x^2 - 16x = -4x(x + 4)$

h $7x + 49x^2 = 7x(1 + 7x)$

i $2x - x^2 = x(2 - x)$

3 a $6x^3y^2 + 12y^2x2 = 6x^2y^2(x + 2)$

b $7x^2y - 6y^2x = xy(7x - 6y)$

c $8x^2y^2 + 6y^2x = 2xy^2(4x + 3)$

4 a $x^3 + 5x^2 + x + 5$

$= x^2(x + 5) + (x + 5)$

$= (x + 5)(x^2 + 1)$

b $xy + 2x + 3y + 6$

$= x(y + 2) + 3(y + 2)$

$= (x + 3)(y + 2)$

c $x^2y^2 - x^2 - y^2 + 1$

$= x^2(y^2 - 1) - (y^2 - 1)$

$= (x^2 - 1)(y^2 - 1)$

$= (x - 1)(x + 1)(y - 1)(y + 1)$

d $ax + ay + bx + by$

$= a(x + y) + b(x + y)$

$= (a + b)(x + y)$

e $a^3 - 3a^2 + a - 3$

$= a^2(a - 3) + (a - 3)$

$= (a^2 + 1)(a - 3)$

f $2ab - 12a - 5b + 30$

$= 2a(b - 6) - 5(b - 6)$

$= (b - 6)(2a - 5)$

g $2x^2 - 2x + 5x - 5$

$= 2x(x - 1) + 5(x - 1)$

$= (x - 1)(2x + 5)$

h $x^3 - 4x + 2x^2 - 8$

$= x(x^2 - 4) + 2(x^2 - 4)$

$= (x^2 - 4)(x + 2)$

$= (x - 2)(x + 2)(x + 2)$

i $x^3 - bx^2 - a^2x + a^2b$

$= x^2(x - b) - a^2(x - b)$

$= (x^2 - a^2)(x - b)$

$= (x - a)(x + a)(x - b)$

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5 a $x^2 - 36 = (x - 6)(x + 6)$

b $x^2 - 81 = (x - 9)(x + 9)$

c $x^2 - a^2 = (x - a)(x + a)$

d $4x^2 - 81 = (2x - 9)(2x + 9)$

e $9x^2 - 16 = (3x - 4)(3x + 4)$

f $25x^2 - y^2 = (5x - y)(5x + y)$

g $3x^2 - 48 = 3(x^2 - 16)$

$= 4(x - 4)(x + 4)$

h $2x^2 - 98 = 2(x^2 - 49)$

$= 2(x - 7)(x + 7)$

i $3ax^2 - 27a = 3a(x^2 - 9)$

$= 3a(x - 3)(x + 3)$

j $a^2 - 7 = (a - \sqrt{7})(x + \sqrt{7})$

k $2a^2 - 5 = (\sqrt{2}a - \sqrt{5})(\sqrt{2}a + \sqrt{5})$

l $x^2 - 12 = (x - \sqrt{12})(x + \sqrt{12}) =$
$(x - 2\sqrt{3})(x + 2\sqrt{3})$

6 a $(x - 2)^2 - 16$

$= (x - 2 - 4)(x - 2 + 4)$

$= (x - 6)(x + 2)$

b $25 - (2 + x)^2$

$= (5 - (2 + x))(5 + (2 + x))$

$= (3 - x)(7 + x)$

c

$3(x + 1)^2 - 12 = 3((x + 1)^2 - 4)$

$= 3(x + 1 - 2)(x + 1 + 2)$

$= 3(x - 1)(x + 3)$

d

$(x - 2)^2 - (x + 3)^2$

$= ((x - 2) - (x + 3))((x - 2) + (x + 3))$

$= (x - 2 - x - 3)(x - 2 + x + 3)$

$= -5(2x + 1)$

e

$(2x - 3)^2 - (2x + 3)^2$

$= ((2x - 3) - (2x + 3))((2x - 3) + (2x + 3))$

$= (-6)(4x)$

$= -24x$

f

$(2x - 1)^2 - (3x + 6)^2$

$= ((2x - 1) - (3x + 6))((2x - 1) + (3x + 6))$

$= (-x - 7)(5x + 5)$

$= -5(x + 7)(x + 1)$

7 a Check signs: must be + and −
$x^2 - 7x - 18 = (x - 9)(x + 2)$

b Check signs: must be − and −
$y^2 - 19y + 48 = (y - 16)(y - 3)$

c $a^2 - 14a + 24 = (a - 12)(a - 2)$

d $a^2 + 18a + 81 = (a + 9)(a + 9) =$
$(a + 9)^2$

e $x^2 - 5x - 24 = (x - 8)(x + 3)$

f $x^2 - 2x - 120 = (x - 12)(x + 10)$

8 a Check signs: must be − and −
$3x^2 - 7x + 2 = (3x - a)(x - b)$
$a + 3b = 7; ab = 2$
$b = 2, a = 1:$
$3x^2 - 7x + 2 = (3x - 1)(x - 2)$

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b Check signs: must be + and +
$6x^2 + 7x + 2 = (6x + a)(x + b)$
$a + 6b = 7,\ ab = 2$; no solution.

Try:
$6x^2 + 7x + 2 = (3x + a)(2x + b)$
$2a + 3b = 7,\ ab = 2$
$a = 2,\ b = 1$
$6x^2 + 7x + 2 = (3x + 2)(2x + 1)$

c $5x^2 + 23x + 12 = (5x + a)(x + b)$
$a + 5b = 23; ab = 12$
$\therefore b = 4,\ a = 3$
$5x^2 + 23x + 12 = (5x + 3)(x + 4)$

d $2x^2 + 9x + 4$
$= 2x^2 + x + 8x + 4$
$= x(2x + 1) + 4(2x + 1)$
$= (2x + 1)(x + 4)$

e $6x^2 - 19x + 10$
$= 6x^2 - 15x - 4x + 10$
$= 3x(2x - 5) - 2(2x - 5)$
$= (2x - 5)(3x - 2)$

f $6x^2 - 7x - 3$
$= 6x^2 - 9x + (2x - 3)$
$= 3x(2x - 3) + (2x - 3)$
$= (2x - 3)(3x + 1)$

g $12x^2 - 17x + 6$
$= 12x^2 - 9x - 8x + 6)$
$= 3x(4x - 3) - 2(4x - 3)$
$= (4x - 3)(3x - 2)$

h $5x^2 - 4x - 12$
$= 5x^2 - 10x + 6x - 12)$
$= 5x(x - 2) + 6(x - 2)$
$= (x - 2)(5x + 6)$

i $5x^3 - 16x^2 + 12x$
$= x(5x^2 - 16x + 12)$
$= x(5x^2 - 10x - 6x + 12)$
$= x(5x(x - 2) - 6(x - 2))$
$= x(x - 2)(5x - 6)$

9 **a** Check signs: must be + and −
$3y^2 - 12y - 36 = 3(y^2 - 4y - 12)$
$= 3(y^2 - 4y - 12)$
$= 3(y + a)(y - b)$
$a - b = -4; ab = 12$
$\therefore a = 2,\ b = 6$
$3y^2 - 12y - 36 = 3(y + 2)(y - 6)$

b $2x^2 - 18x + 28 = 2(x^2 - 9x + 14)$
$= 2(x - 2)(x - 7)$

c $4x^2 - 36x + 72 = 4(x^2 - 9x + 18)$
$= 4(x - 6)(x - 3)$

d $3x^2 + 15x + 18 = 3(x^2 + 5x + 6)$
$= 3(x + 3)(x + 2)$

e $ax^2 + 7ax + 12a = a(x^2 + 7x + 12)$
$= a(x + 3)(x + 4)$

f
$48x - 24x^2 + 3x^3 = 3x(16 - 8x + x^2)$
$= 3x(4 - x)^2$ or $3x(x - 4)^2$

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10 **a** $(x-1)^2+4(x-1)+3$

Put $y=x-1$:

$=y^2+4y+3$

$=(y+3)(y+1)$

$=(x-1+3)(x-1+1)$

$=x(x+2)$

b $2(x-1)^2+5(x-1)-3$

Put $a=x-1$:

$=2a^2+5a-3$

$=(2a-1)(a+3)$

$=(2(x-1)-1)(x-1+3)$

$=(2x-3)(x+2)$

c $(2x+1)^2+7(2x+1)+12$

Put $a=2x+1$:

$=a^2+7a+12$

$=(a+3)(a+4)$

$=(2x+1+3)(2x+1+4)$

$=(2x+4)(2x+5)$

$=2(x+2)(2x+5)$

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Solutions to Exercise 3C

1 **a** $(x-2)(x-3)=0, \therefore\ x=2,3$

b $x(2x-4)=0, \therefore\ 2x(x-2)=0$

$\therefore\ x=0,2$

c $(x-4)(2x-6)=0$

$\therefore\ 2(x-4)(x-3)=0$

$\therefore\ x=3,4$

d $(3-x)(x-4)=0$

$\therefore\ x=3,4$

e $(2x-6)(x+4)=0$

$\therefore\ 2(x-3)(x+4)=0$

$\therefore\ x=3,-4$

f $2x(x-1)=0, \therefore\ x=0,1$

g $(5-2x)(6-x)=0$

$\therefore\ 2\left(\frac{5}{2}-x\right)(6-x)=0$

$\therefore\ x=\frac{5}{2},6$

h $x^2=16, \therefore\ x^2-16=0$

$\therefore\ (x-4)(x+4)=0$

$\therefore\ x=4,-4$

2 **a** $x^2-4x-3=0$

$\therefore\ x=-0.65,\ 4.65$

b $2x^2-4x-3=0$

$\therefore\ x=-0.58, 2.58$

c $-2x^2-4x+3=0$

$\therefore\ x=-2.58, 0.58$

3 **a** $x^2-x-72=0$

$\therefore\ (x-9)(x+8)=0$

$\therefore\ x=9,-8$

b $x^2-6x+8=0$

$\therefore\ (x-2)(x-4)=0$

$\therefore\ x=2,4$

c Check signs: must be + and −

$x^2-8x-33=0$

$\therefore\ (x-a)(x+b)=0$

$a-b=8; ab=33$

$a=11; b=3$

$(x-11)(x+3)=0$

$\therefore\ x=11,-3$

d $x(x+12)=64$

$x^2+12x-64=0$

Check signs: must be + and −

$\therefore\ (x-a)(x+b)=0$

$b-a=12; ab=64;$

$b=16; a=4$

$(x-4)(x+16)=0$

$\therefore\ x=4,-16$

e Check signs: must be + and −

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$x^2 + 5x - 14 = 0$

$(x - a)(x + b) = 0$

$b - a = 5; ab = 14;$

$b = 7; a = 2$

$(x - 2)(x + 7) = 0$

$\therefore\ x = 2, -7$

f $x^2 = 5x + 24,\ \therefore\ x^2 - 5x - 24 = 0$

Check signs: must be + and −

$\therefore\ (x - a)(x + b) = 0$

$a - b = 5; ab = 24$

$a = 8; b = 3$

$(x - 8)(x + 3) = 0$

$\therefore x = 8, -3$

4 a $2x^2 + 5x + 3 = 0$

$\therefore\ (2x + a)(x + b) = 0$

$a + 2b = 5; ab = 3$

$a = 3; b = 2$

$(2x + 3)(x + 1) = 0$

$\therefore\ x = -\frac{3}{2}, -1$

b $4x^2 - 8x + 3 = 0$

$\therefore\ (2x - a)(2x - b) = 0$

$2a + 2b = 8; ab = 3$

$a = 3; b = 1$

$(2x - 3)(2x - 1) = 0$

$\therefore x = \frac{3}{2}, \frac{1}{2}$

c $6x^2 + 13x + 6 = 0$

$\therefore\ (3x + a)(2x + b) = 0$

$2a + 3b = 13; ab = 6$

$a = 2; b = 3$

$(3x + 2)(2x + 3) = 0$

$\therefore\ x = -\frac{2}{3}, -\frac{3}{2}$

d $2x^2 - x = 6$

$\therefore\ 2x^2 - x - 6 = 0$

$\therefore\ x = -\frac{3}{2}, 2$

e $6x^2 + 15 = 23x$

$\therefore\ 6x^2 - 23x + 15 = 0$

$\therefore\ (6x - a)(x - b) = 0$

$a + 6b = 23; ab = 15$

$b = 3; a = 5$

$(6x - 5)(x - 3) = 0$

$\therefore\ x = \frac{5}{6}, 3$

f Check signs: must be + and −

$2x^2 - 3x - 9 = 0$

$\therefore\ (2x - a)(x + b) = 0$

$2b - a = -3; ab = 9$

$b = -3; a = -3$

$(2x + 3)(x - 3) = 0$

$\therefore\ x = -\frac{3}{2}, 3$

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g $10x^2 - 11x + 3 = 0$

$\therefore\ (5x - a)(2x - b) = 0$

$2a + 5b = 11; ab = 3$

$a = 3; b = 1$

$(5x - 3)(2x - 1) = 0$

$\therefore\ x = \dfrac{3}{5}, \dfrac{1}{2}$

h $12x^2 + x = 6$

$\therefore\ 12x^2 + x - 6 = 0$

Check signs: must be + and −

$\therefore\ (6x - a)(2x + b) = 0$

$6b - 2a = 1; ab = 6;$ no solution

$\therefore\ (4x - a)(3x + b) = 0$

$4b - 3a = 1; ab = 6$

$a = -3; b = -2$

$(4x + 3)(3x - 2) = 0$

$\therefore\ x = -\dfrac{3}{4}, \dfrac{2}{3}$

i $4x^2 + 1 = 4x$

$\therefore\ 4x^2 - 4x + 1 = 0$

$\therefore\ (2x - 1)^2 = 0, \therefore\ x = \dfrac{1}{2}$

j $x(x + 4) = 5$

$x^2 + 4x - 5 = 0$

Check signs: must be + and −

$\therefore\ (x - a)(x + b) = 0$

$b - a = 4; ab = 5$

$b = 5; a = 1$

$(x - 1)(x + 5) = 0$

$\therefore x = 1, -5$

k $\dfrac{1}{7}x^2 = \dfrac{3}{7}x$

$\therefore\ x^2 = 3x, \therefore\ x^2 - 3x = 0$

$\therefore\ x(x - 3) = 0, \therefore\ x = 0, 3$

l $x^2 + 8x = -15$

$x + 8x + 15 = 0$

$(x + 5)(x + 3) = 0$

$\therefore\ x = -5, -3$

m $5x^2 = 11x - 2$

$\therefore\ 5x^2 - 11x + 2 = 0$

$\therefore\ (5x - a)(x - b) = 0$

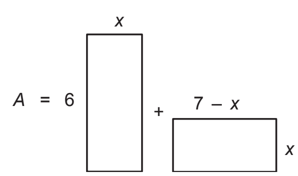
$a + 5b = 11; ab = 2$

$a = 1; b = 2$

$(5x - 1)(x - 2) = 0$

$\therefore\ x = \dfrac{1}{5}, 2$

5 Cut vertically down middle:



$\therefore\ A = 6x + x(7 - x) = 30$

$\therefore\ 6x + 7x - x^2 = 30$

$\therefore\ x^2 - 13x - 30 = 0$

$\therefore\ (x - 3)(x - 10) = 0$

$\therefore\ x = 3, 10$

However, $0 < x < 7$ so $x = 3$

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6 $M = \frac{wl}{2}x - \frac{w}{2}x^2$

$\therefore 104x - 8x^2 = 288$

$\therefore\ x^2 - 13x + 36 = 0$

$\therefore\ (x-4)(x-9) = 0$

$\therefore\ x = 4, 9$

7 $h = 70t - 16t^2 = 76$

$\therefore 16t^2 - 70t + 76 = 0$

$\therefore 8t^2 - 35t + 38 = 0$

$\therefore\ (8t - 19)(t - 2) = 0$

$\therefore\ t = 2,\ \frac{19}{8}$ seconds

8 $D = \frac{n}{2}(n-3) = 65$

$\therefore\ n^2 - 3n - 130 = 0$

$\therefore\ (n-a)(n+b) = 0$

$b - a = -3; ab = 130$

$b = 10; a = 13$

$(n-10)(n+13) = 0$

$\therefore\ n = -10, 13$

Since $n > 0$, the polygon has 13 sides.

9 $R = 1.6 + 0.03v + 0.003v^2 = 10.6$

$\therefore 3v^2 + 1600 + 30v = 10600$

$\therefore 3v^2 + 30v - 9000 = 0$

$\therefore v^2 + 10v - 3000 = 0$

$\therefore (v-a)(v+b) = 0$

$b - a = 10; ab = 3000$

$b = 60, a = 50$

$(v-50)(v+60) = 0$

$\therefore\ v = 50, -60$

$v \geq 0, \therefore\ v = 50$ km/h

10 $P = 2L + 2W = 16$

$\therefore\ L = 8 - W$

$A = LW = W(8-W) = 12$

$\therefore 8W - W^2 = 12$

$\therefore\ W^2 - 8W + 12 = 0$

$\therefore\ (W-2)(W-6) = 0$

$\therefore\ w = 2, 6$

Length = 6 cm, width = 2 cm

11 $A = \frac{bh}{2} = 15$

$h = b - 1, \therefore\ A = \frac{b}{2}(b-1)$

$\frac{b}{2}(b-1) = 15$

$b^2 - b = 30, \therefore\ b^2 - b - 30 = 0$

$\therefore\ (b+5)(b-6) = 0$

$b = 6, -5$

$b \geq 0, \therefore\ b = 6$ cm

Therefore height (altitude) = 5 cm

12 $e = c + 30 \ldots (1)$

$\frac{1800}{e} + 10 = \frac{1800}{c} \ldots (2)$

Substitute (1) into (2):

$\frac{1800}{c+30} + 10 = \frac{1800}{c}$

$\therefore 1800c + 10c(c+30) = 1800(c+30)$

$\therefore 1800c + 10c^2 + 300c = 1800c + 54000$

$\therefore 10c^2 + 300c = 54000$

$\therefore c^2 + 30c - 5400 = 0$

$\therefore (c-a)\,(c+b) = 0$

$b - a = 30;$

$ab = 5400$

$b = 90, a = 60$

$(c-60)(c+90) = 0$

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$\therefore\ c = \$60$

Cheap seats are \$60, expensive \$90

13 Original cost per person = x

Original members = N where

$Nx = 2100$

$\therefore x = \dfrac{2100}{N}$

Later: $(N - 7)(x + 10) = 2100$

$\therefore\ (N - 7)(\dfrac{2100}{N} + 10) = 2100$

$\therefore\ (N - 7)(2100 + 10N) = 2100N$

$\therefore 2100N - 14700 + 10N^2 - 70N$

$= 2100N$

$\therefore -14700 + 10N^2 - 70N = 0$

$\therefore\ N^2 - 7N - 1470 = 0$

$\therefore\ (N - a)(N + b) = 0$

$a - b = 7; ab = 1470$

$a = 42; b = 35$

$\therefore\ (N - 42)(N + 35) = 0$

Since $N > 7,\ N = 42$

So 42 members originally agreed to go on the bus.

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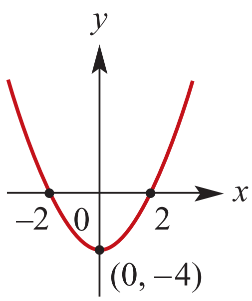
Solutions to Exercise 3D

1 a $y = x^2 - 4$

i turning point at $(0, -4)$

ii the axis of symmetry $x = 0$

iii the x-axis intercepts $(-2, 0)$ and $(2,0)$

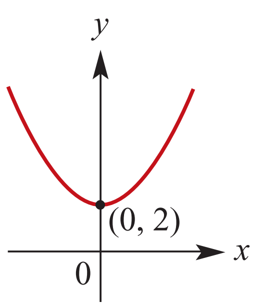


b $y = x^2 + 2$

i turning point at $(0,2)$

ii the axis of symmetry $x = 0$

iii No x-axis intercepts: $y(\text{min}) = 2$

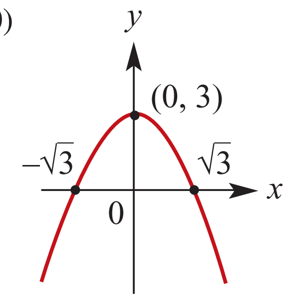


c $y = -x^2 + 3$

i turning point at $(0, 3)$

ii the axis of symmetry $x = 0$

iii the x-axis intercepts $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$

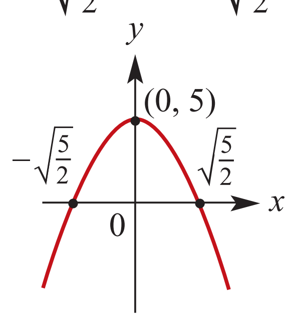


d $y = -2x^2 + 5$

i turning point at $(0, 5)$

ii the axis of symmetry $x = 0$

iii the x-axis intercepts $\left(-\sqrt{\frac{5}{2}}, 0\right)$ and $\left(\sqrt{\frac{5}{2}}, 0\right)$



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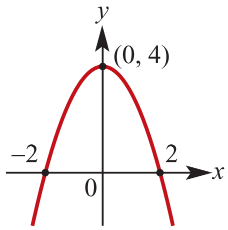
e $y = -x^2 + 4$

i turning point at $(0, 4)$

ii the axis of symmetry $x = 0$

iii the x-axis intercepts $(-2, 0)$ and $(2, 0)$

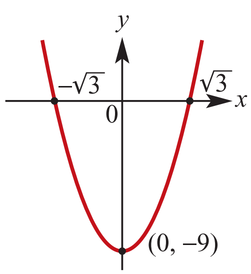


f $y = 3x^2 - 9$

i turning point at $(0, -9)$

ii the axis of symmetry $x = 0$

iii the x-axis intercepts $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$

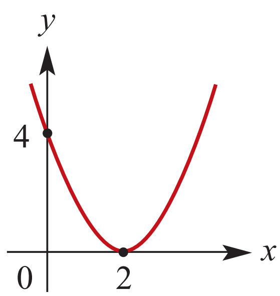


2 **a** $y = (x - 2)^2$

i turning point at $(2, 0)$

ii the axis of symmetry $x = 2$

iii the x-axis intercept $(2, 0)$ (turning pt)

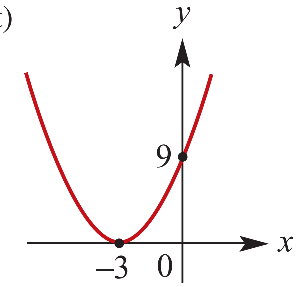


b $y = (x + 3)^2$

i turning point at $(-3, 0)$

ii the axis of symmetry $x = -3$

iii the x-axis intercept $(-3, 0)$ (= turning pt)

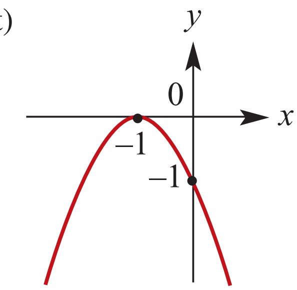


c $y = -(x + 1)^2$

i turning point at $(-1, 0)$

ii the axis of symmetry $x = -1$

iii the x-axis intercept $(-1, 0)$ (= turning pt)



d $y = -\dfrac{1}{2}(x - 4)^2$

i turning point at $(4, 0)$

ii the axis of symmetry $x = 4$

iii the x-axis intercept $(4, 0)$ (= turning pt)

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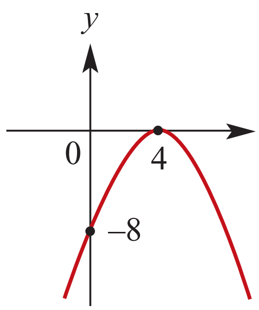
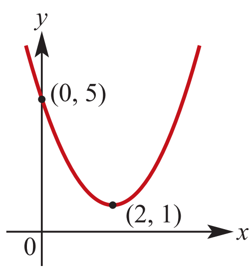


3 **a** $y = (x-2)^2 + 1$

i turning point at $(2, 1)$

ii the axis of symmetry $x = 2$

iii no x-axis intercepts.

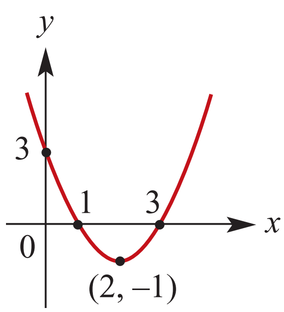


b $y = (x-2)^2 - 1$

i turning point at $(2, -1)$

ii the axis of symmetry $x = 2$

iii the x-axis intercepts $(1, 0)$ and $(3, 0)$

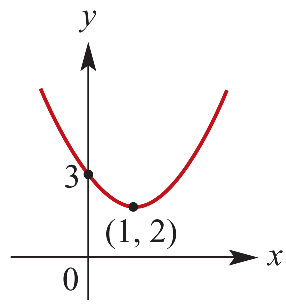


c $y = (x-1)^2 + 2$

i turning point at $(1, 2)$

ii the axis of symmetry $x = 1$

iii no x-axis intercepts

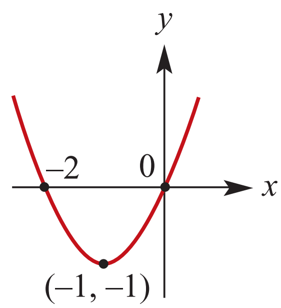


d $y = (x+1)^2 - 1$

i turning point at $(-1, -1)$

ii the axis of symmetry $x = -1$

iii the x-axis intercepts $(0, 0)$ and $(-2, 0)$



e $y = -(x-3)^2 + 1$

i turning point at $(3, 1)$

ii the axis of symmetry $x = 3$

iii the x-axis intercepts:

$$y = -(x-3)^2 + 1 = 0$$

$$\therefore (x-3)^2 = 1$$

$$\therefore x - 3 = \pm 1$$

$$\therefore x = 3 \pm 1$$

$(2,0)$ and $(4,0)$

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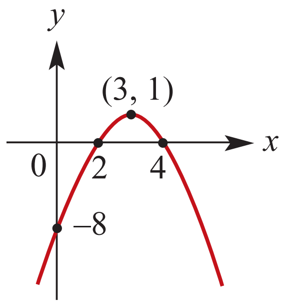
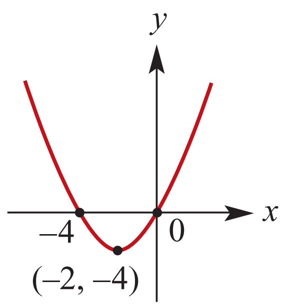


f $y = (x+2)^2 - 4$

i turning point at $(-2, -4)$

ii the axis of symmetry $x = -2$

iii the x-axis intercepts $(0, 0)$ and $(-4, 0)$



g $y = 2(x+2)^2 - 18$

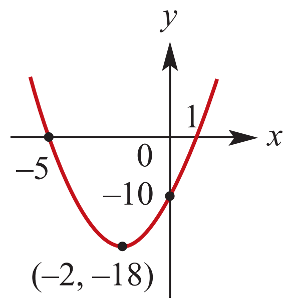
i turning point at $(-2, -18)$

ii the axis of symmetry $x = -2$

iii the x-axis intercepts:

$$y = y = 2(x+2)^2 - 18 = 0$$
$$\therefore 2(x+2)^2 = 18$$
$$\therefore (x+2)^2 = 9$$
$$\therefore x + 2 = \pm 3$$
$$\therefore x = -2 \pm 3$$

$(-5, 0)$ and $(1, 0)$



h $y = -3(x-4)^2 + 3$

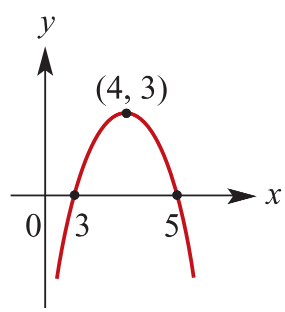
i turning point at $(4,3)$

ii the axis of symmetry $x = 4$

iii the x-axis intercepts:

$$y = -3(x-4)^2 + 3 = 0$$
$$\therefore 3(x-4)^2 = 3$$
$$\therefore (x-4)^2 = 1$$
$$\therefore x - 4 = \pm 1$$
$$\therefore x = 4 \pm 1$$

$(5,0)$ and $(3,0)$



i $y = -\frac{1}{2}(x+5)^2 - 2$

i turning point at $(-5, -2)$

ii the axis of symmetry $x = -5$

iii the x-axis intercepts:

$$y = -\frac{1}{2}(x+5)^2 - 2 = 0$$
$$\therefore -\frac{1}{2}(x+5)^2 = 2$$
$$\therefore (x+5)^2 = -4$$

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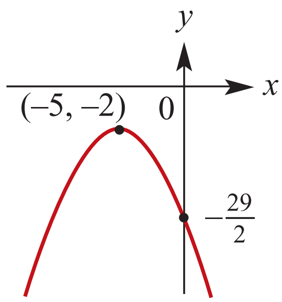
No x-axis intercepts because no real roots.



j $y = 3(x + 2)^2 - 12$

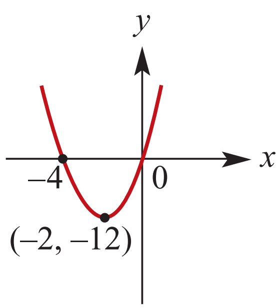
i turning point at $(-2, -12)$

ii the axis of symmetry $x = -2$

iii the x-axis intercepts:

$$y = 3(x + 2)^2 - 12 = 0$$
$$\therefore 3(x + 2)^2 = 12$$
$$\therefore (x + 2)^2 = 4$$
$$\therefore x + 2 = \pm 2$$
$$\therefore x = -2 \pm 2$$

(0,0) and $(-4, 0)$



k $y = -4(x - 2)^2 + 8$

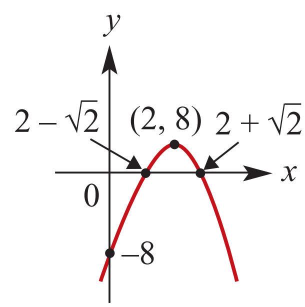
i turning point at $(2, 8)$

ii the axis of symmetry $x = 2$

iii the x-axis intercepts:

$$y = -4(x - 2)^2 + 8 = 0$$
$$\therefore 4(x - 2)^2 = 8$$
$$\therefore (x - 2)^2 = 2$$
$$\therefore x - 2 = \pm\sqrt{2}$$
$$\therefore x = 2 \pm \sqrt{2}$$

$(2 - \sqrt{2}, 0)$ and $(2 + \sqrt{2}, 0)$

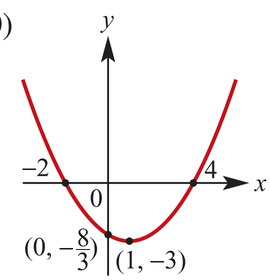


l $y = \frac{1}{3}(x - 1)^2 - 3$

i turning point at $(1, -3)$

ii the axis of symmetry $x = 1$

iii the x-axis intercepts $(-2, 0)$ and $(4, 0)$



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Solutions to Exercise 3E

1 **a** $(x-1)^2 = x^2 - 2x + 1$

b $(x+2)^2 = x^2 + 4x + 4$

c $(x-3)^2 = x^2 - 6x + 9$

d $(-x+3)^2 = x^2 - 6x + 9$

e $$\begin{aligned}(-x-2)^2 &= (-1)^2(x+2)^2\\ &= x^2 + 4x + 4\end{aligned}$$

f $(x-5)^2 = x^2 - 10x + 25$

g $\left(x - \frac{1}{2}\right)^2 = x^2 - x + \frac{1}{4}$

h $\left(x - \frac{3}{2}\right)^2 = x^2 - 3x + \frac{9}{4}$

2 **a** $x^2 - 4x + 4 = (x-2)^2$

b $x^2 - 12x + 36 = (x-6)^2$

c $$\begin{aligned}-x^2 + 4x - 4 &= -(x^2 - 4x + 4)\\ &= -(x-2)^2\end{aligned}$$

d $$\begin{aligned}2x^2 - 8x + 8 &= 2(x^2 - 4x + 4)\\ &= 2(x-2)^2\end{aligned}$$

e $$\begin{aligned}&-2x^2 + 12x - 18\\ &= -2(x^2 - 6x + 9)\\ &= -2(x-3)^2\end{aligned}$$

f $x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$

g $x^2 - 3x + \frac{9}{4} = \left(x - \frac{3}{2}\right)^2$

h $x^2 + 5x + \frac{25}{4} = \left(x + \frac{5}{2}\right)^2$

3 **a** $x^2 - 2x - 1 = 0$

$$\begin{aligned}\therefore\ x^2 - 2x + 1 - 2 &= 0\\ \therefore\ (x-1)^2 - 2 &= 0\\ \therefore\ (x-1)^2 &= 2\\ \therefore\ x - 1 &= \pm\sqrt{2}\\ \therefore\ x &= 1 \pm \sqrt{2}\end{aligned}$$

b $x - 4x - 2 = 0$

$$\begin{aligned}\therefore\ x^2 - 4x + 4 - 6 &= 0\\ \therefore\ (x-2)^2 - 6 &= 0\\ \therefore\ (x-2)^2 &= 6\\ \therefore\ x - 2 &= \pm\sqrt{6}\\ \therefore\ x &= 2 \pm \sqrt{6}\end{aligned}$$

c $x^2 - 6x + 2 = 0$

$$\begin{aligned}\therefore\ x^2 - 6x + 9 - 7 &= 0\\ \therefore\ (x-3)^2 - 7 &= 0\\ \therefore\ (x-3)^2 &= 7\\ \therefore\ x - 3 &= \pm\sqrt{7}\\ \therefore\ x &= 3 \pm \sqrt{7}\end{aligned}$$

d $x^2 - 5x + 2 = 0$

$$\begin{aligned}\therefore\ x^2 - 5x + \frac{25}{4} + 2 - \frac{25}{4} &= 0\\ \therefore\ \left(x - \frac{5}{2}\right)^2 - \frac{17}{4} &= 0\\ \therefore\ \left(x - \frac{5}{2}\right)^2 &= \frac{17}{4}\\ \therefore\ x - \frac{5}{2} &= \pm\frac{1}{2}\sqrt{17}\\ \therefore\ x &= \frac{5 \pm \sqrt{17}}{2}\end{aligned}$$

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e $2x^2 - 4x + 1 = 0$

$$\therefore\ 2\left(x^2 - 2x + \frac{1}{2}\right) = 0$$
$$\therefore\ x^2 - 2x + 1 - \frac{1}{2} = 0$$
$$\therefore\ (x-1)^2 = \frac{1}{2}$$
$$\therefore\ x - 1 = \pm\frac{1}{\sqrt{2}}$$
$$\therefore\ x = \frac{2 \pm \sqrt{2}}{2}$$

f $3x^2 - 5x - 2 = 0$

$$\therefore\ 3\left(x^2 - \frac{5x}{3} - \frac{2}{3}\right) = 0$$
$$\therefore\ x^2 - \frac{5x}{3} - \frac{2}{3} = 0$$
$$\therefore\ x^2 - \frac{5x}{3} + \frac{25}{36} - \frac{2}{3} - \frac{25}{36} = 0$$
$$\therefore\ \left(x - \frac{5}{6}\right)^2 - \frac{49}{36} = 0$$
$$\therefore\ \left(x - \frac{5}{6}\right)^2 = \frac{49}{36}$$
$$\therefore\ x - \frac{5}{6} = \pm\frac{7}{6}$$
$$\therefore\ x = \frac{5}{6} \pm \frac{7}{6}$$
$$= 2, -\frac{1}{3}$$

g $x^2 + 2x + k = 0$

$$\therefore\ x^2 + 2x + 1 - (1-k) = 0$$
$$\therefore\ (x+1)^2 - (1-k) = 0$$
$$\therefore\ x + 1 = \pm\sqrt{1-k}$$

h $kx^2 + 2x + k = 0$

$$\therefore\ x^2 + \frac{2x}{k} + 1 = 0$$
$$\therefore\ x^2 + \frac{2x}{k} + \frac{1}{k^2} - \frac{1}{k^2+1} = 0$$
$$\therefore\ \left(x + \frac{1}{k}\right)^2 - \left(\frac{1}{k^2} - 1\right) = 0$$
$$\therefore\ \left(x + \frac{1}{k}\right)^2 = \frac{1-k^2}{k^2}$$
$$\therefore\ x + \frac{1}{k} = \pm\frac{1}{k}\sqrt{1-k^2}$$
$$\therefore\ x = \frac{-1 \pm \sqrt{1-k^2}}{k}$$

i $x^2 - 3kx + 1 = 0$

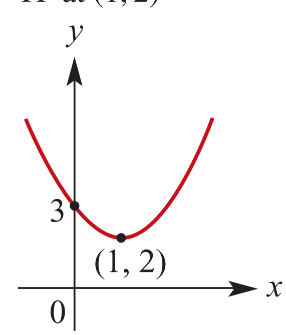
$$\therefore\ x^2 - 3kx + \frac{9}{4}k^2 - \left(\frac{9}{4}k^2 - 1\right) = 0$$
$$\therefore\ \left(x - \frac{3k}{2}\right)^2 - \left(\frac{9}{4}k^2 - 1\right) = 0$$
$$\therefore\ \left(x - \frac{3k}{2}\right)^2 = \left(\frac{9}{4}k^2 - 1\right)$$
$$\therefore\ x - \frac{3k}{2} = \pm\sqrt{\frac{9}{4}k^2 - 1}$$
$$\therefore\ x = \frac{3k \pm \sqrt{9k^2-4}}{2}$$

4 **a** $x^2 - 2x + 3$

$= x^2 - 2x + 1 + 2$

$= (x-1)^2 + 2$

TP at $(1, 2)$



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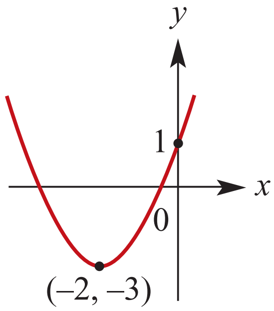
b $x^2 + 4x + 1$

$= x^2 + 4x + 4 - 3$

$= (x + 2)^2 - 3$

TP at $(-2, -3)$

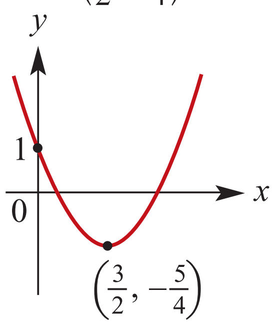


c $x^2 - 3x + 1$

$= x^2 - 3x + \frac{9}{4} - \frac{5}{4}$

$= \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$

TP at $\left(\frac{3}{2}, -\frac{5}{4}\right)$



5 a $y = 2x^2 - 2x - 5$

$= 2\left(x^2 - x - \frac{5}{2}\right)$

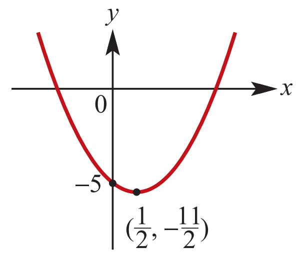
$= 2\left(x^2 - x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - \frac{5}{2}\right)$

$= 2\left((x - \frac{1}{2})^2 - \left(\frac{1}{2}\right)^2 - \frac{5}{2}\right)$

$= 2\left((x - \frac{1}{2})^2 - \frac{11}{4}\right)$

$= 2\left((x - \frac{1}{2})^2\right) - \frac{11}{2}$

TP at $\left(\frac{1}{2}, -\frac{11}{2}\right)$



b $y = 4x^2 + 8x + 8$

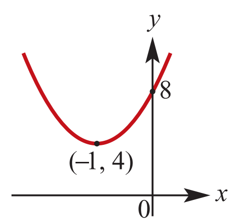
$= 4\left(x^2 + 2x + 2\right)$

$= 4\left(x^2 + 2x + 1 - 1 + 2\right)$

$= 4\left((x + 1)^2 + 1\right)$

$= 4(x + 1)^2 + 4$

TP at $(-1, 4)$



c $y = 3x^2 - 6x - 4$

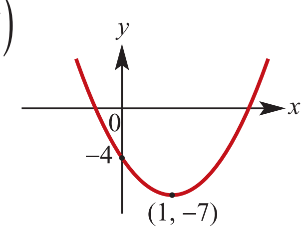
$= 3\left(x^2 - 2x - \frac{4}{3}\right)$

$= 3\left(x^2 - 2x + 1 - 1 - \frac{4}{3}\right)$

$= 3\left((x - 1)^2 - \left(\sqrt{\frac{7}{3}}\right)^2\right)$

$= 3(x - 1)^2 - 7$

TP at $\left(1, -7\right)$



6 a $x^2 - 8x + 12$

$= x^2 - 8x + 16 - 4$

$= (x - 4)^2 - 4$

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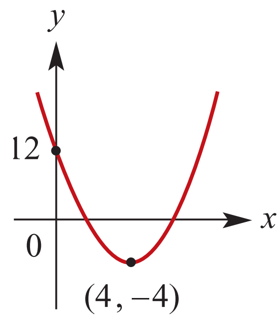
TP at $(4, -4)$

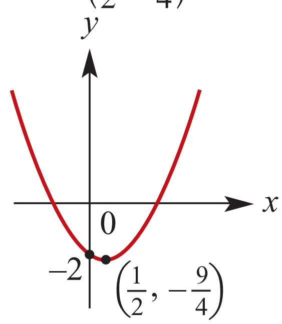


b $x^2 - x - 2$

$= x^2 - x + \frac{1}{4} - \frac{9}{4}$

$= \left(x - \frac{1}{2}\right)^2 - \frac{9}{4}$

TP at $\left(\frac{1}{2}, -\frac{9}{4}\right)$



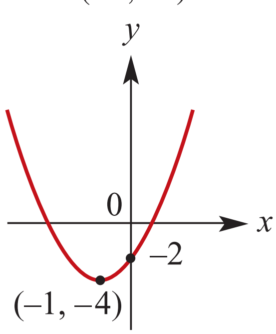
c $2x^2 + 4x - 2$

$= 2(x^2 + 2x - 1)$

$= 2(x^2 + 2x + 1 - 2)$

$= 2(x + 1)^2 - 4$

TP at $(-1, -4)$



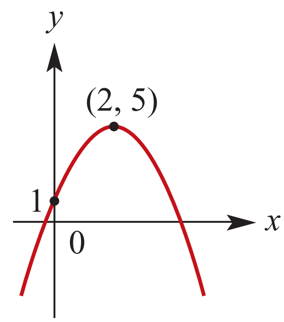
d $-x^2 + 4x + 1$

$= -(x^2 - 4x - 1)$

$= -(x^2 - 4x + 4 - 5)$

$= -(x - 2)^2 + 5$

TP at $(2, 5)$



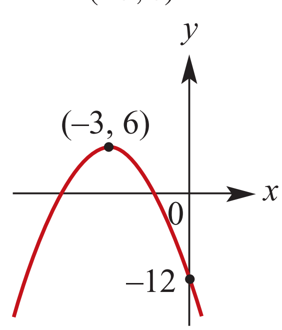
e $-2x^2 - 12x - 12$

$= -2(x^2 + 6x + 6)$

$= -2(x^2 + 6x + 9 - 3)$

$= -2(x + 3)^2 + 6$

TP at $(-3, 6)$



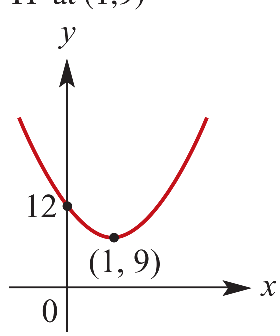
f $3x^2 - 6x + 12$

$= 3(x^2 - 2x + 4)$

$= 3(x^2 - 2x + 1 + 3)$

$= 3(x - 1)^2 + 9$

TP at $(1, 9)$



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Solutions to Exercise 3F

1 **a** x-axis intercepts 4 and 10;
x-coordinate of vertex
$= \frac{1}{2}(4 + 10) = 7$

b x-axis intercepts 6 and 8;
x-coordinate of vertex $= \frac{1}{2}(6 + 8) = 7$

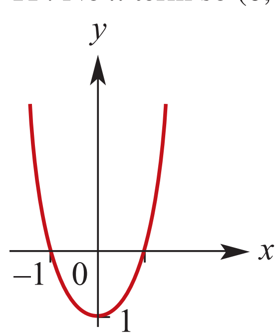
c x-axis intercepts -6 and 8;
x-coordinate of vertex
$= \frac{1}{2}(-6 + 8) = 1$

2 **a** x-axis intercepts a and 6;
x-coordinate of vertex $= \frac{1}{2}(a + 6) = 2$
$\therefore\ a + 6 = 4, \therefore\ a = -2$

b x-axis intercepts a and -4;
x-coordinate of vertex $= \frac{1}{2}(a - 4) = 2$
$\therefore\ a - 4 = 4, \therefore\ a = 8$

c x-axis intercepts a and 0;
x-coordinate of vertex $\frac{1}{2}(a + 0) = 2$
$\therefore\ a = 4$

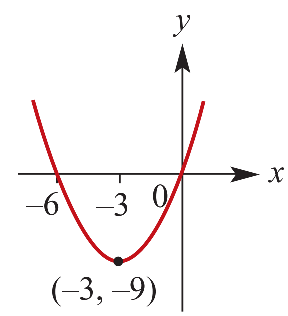
3 **a** $y = x^2 - 1$
x-intercepts: $y = x^2 - 1 = 0$
$\therefore\ (x - 1)(x + 1) = 0$
$\therefore x = 1, -1$
x-int: $(-1, 0)$ and $(1, 0)$
TP: No x term so $(0, -1)$



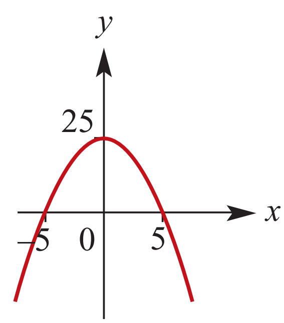
b $y = x^2 + 6x$
x-intercepts: $y = x^2 + 6x = 0$
$\therefore x(x + 6) = 0$
$\therefore x = 0, -6$
x-int: $(-6, 0)$ and $(0, 0)$
TP: $y = x^2 + 6x$

$$= x^2 + 6x + 9 - 9$$
$$= (x + 3)^2 - 9$$

TP at $(-3, -9)$



c $y = 25 - x^2$
x-intercepts: $y = 25 - x^2 = 0$
$\therefore\ (5 - x)(5 + x) = 0$
$\therefore x = 5, -5$
TP: No x term so (0,25)



d $y = x^2 - 4$
x-intercepts: $y = x^2 - 4 = 0$
$\therefore\ (x - 2)(x + 2) = 0$
$\therefore\ x = 2, -2$
x-int: $(-2, 0)$ and $(2, 0)$
TP: No x term so $(0, -4)$

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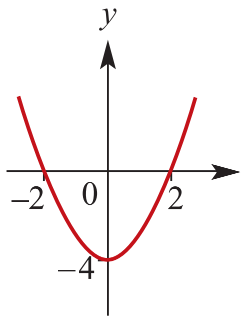


e $y = 2x^2 + 3x$

x-intercepts: $y = 2x^2 + 3x = 0$

$\therefore\ x(2x + 3) = 0$

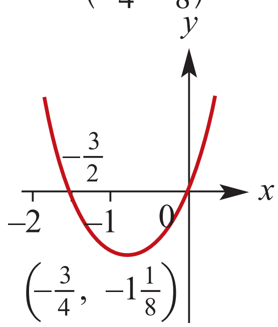
x-int: $\left(-\dfrac{3}{2}, 0\right)$ and $(0, 0)$

TP: $y = 2x^2 + 3x$

$$= 2\left(x^2 + \frac{3}{2}x + \frac{9}{16} - \frac{9}{16}\right)$$

$$= 2\left(x + \frac{3}{4}\right)^2 - \frac{9}{8}$$

TP at $\left(-\dfrac{3}{4}, -\dfrac{9}{8}\right)$



f $y = 2x^2 - 4x$

x-intercepts: $y = 2x^2 - 4x = 0$

$\therefore\ 2x(x - 2) = 0$

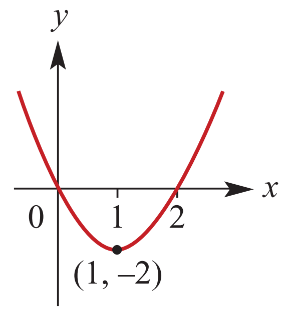
x-int: $(2, 0)$ and $(0, 0)$

TP: $y = 2x^2 - 4x$

$$= 2(x^2 - 2x + 1 - 1)$$

$$= 2(x - 1)^2 - 2$$

TP at $(1, -2)$



g $y = -2x^2 - 3x$

x-intercepts: $y = -2x^2 - 3x = 0$

$\therefore\ -x(2x + 3) = 0$

x-int: $\left(-\dfrac{3}{2}, 0\right)$ and $(0, 0)$

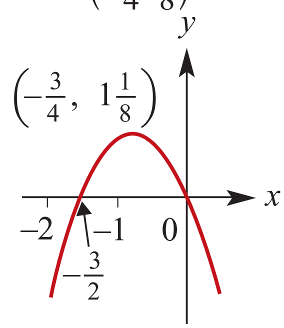
TP: $y = -2x^2 - 3x$

$$= -2\left(x^2 + \frac{3}{2}x + \frac{9}{16} - \frac{9}{16}\right)$$

$$= -2\left((x + \frac{3}{4}\right)^2 + \frac{9}{8}$$

$$= -2\left((x + \frac{3}{4}\right)^2 + \frac{9}{8}$$

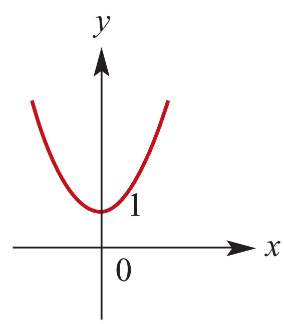
TP at $\left(-\dfrac{3}{4}, \dfrac{9}{8}\right)$



h $y = x^2 + 1$

No x-intercepts since $y > 0$ for all x

TP: No x term so $(0, 1)$



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4 **a** $y = x^2 + 3x - 10$

x-intercepts: $y = x^2 + 3x - 10 = 0$

$\therefore (x + 5)(x - 2) = 0$

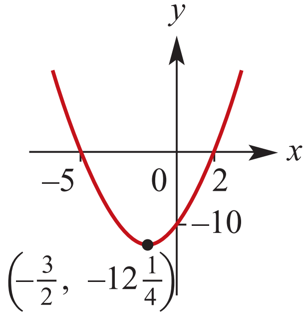
x-int: $(-5, 0)$ and $(2, 0)$

TP: $y = x^2 + 3x - 10$

$$y = x^2 + 3x + \frac{9}{4} - \frac{9}{4} - 10$$

$$y = \left(x + \frac{3}{2}\right)^2 - \frac{49}{4}$$

TP at $\left(-\frac{3}{2}, -\frac{49}{4}\right)$



b $y = x^2 - 5x + 4$

x-intercepts: $y = x^2 - 5x + 4 = 0$

$\therefore (x - 1)(x - 4) = 0$

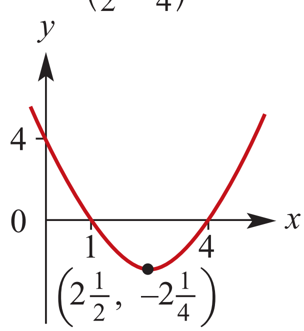
x-int: $(1, 0)$ and $(4, 0)$

TP: $y = x^2 - 5x + 4$

$$y = x^2 - 5x + \frac{25}{4} - \frac{25}{4} + 4$$

$$y = \left(x - \frac{5}{2}\right)^2 - \frac{9}{4}$$

TP at $\left(\frac{5}{2}, -\frac{9}{4}\right)$



c $y = x^2 + 2x - 3$

x-intercepts: $y = x^2 + 2x - 3$

$\therefore (x - 1)(x + 3) = 0$

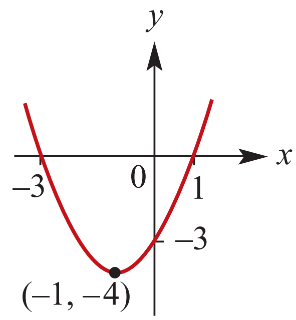
x-int: $(1, 0)$ and $(-3, 0)$

TP: $y = x^2 + 2x - 3$

$$y = x^2 + 2x + 1 - 4$$

$$y = (x + 1)^2 - 4$$

TP at $(-1, -4)$



d $y = x^2 + 4x + 3$

x-intercepts: $y = x^2 + 4x + 3 = 0$

$\therefore (x + 1)(x + 3) = 0$

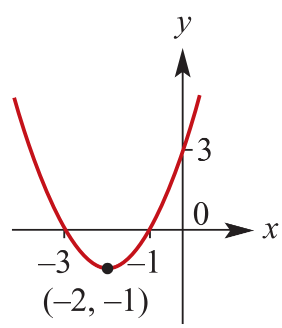
x-int: $(-1, 0)$ and $(-3, 0)$

TP: $y = x + 4x + 3$

$$y = x^2 + 4x + 4 - 1$$

$$y = (x + 2)^2 - 1$$

TP at $(-2, -1)$



e $y = 2x^2 - x - 1$

x-intercepts: $y = 2x^2 - x - 1 = 0$

$\therefore (2x + 1)(x - 1) = 0$

x-int: $\left(-\frac{1}{2}, 0\right)$ and $(1, 0)$

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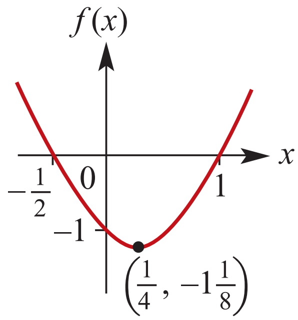
TP: $y = 2x^2 - x - 1$

$$y = 2\left(x^2 - \frac{x}{2} + \frac{1}{2}\right)$$

$$y = 2\left(x^2 - \frac{x}{2} + \frac{1}{16} - \frac{9}{16}\right)$$

$$y = 2\left(x - \frac{1}{4}\right)^2 - \frac{9}{8}$$

TP at $\left(\frac{1}{4}, -\frac{9}{8}\right)$



f $y = 6 - x - x^2$

x-intercepts: $y = -(x^2 + x - 6) = 0$

$\therefore -(x + 3)(x - 2) = 0$

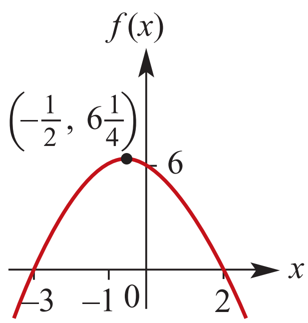
x-int: $(-3, 0)$ and $(2,0)$

TP: $y = -(x^2 + x - 6)$

$$y = -\left(x^2 + x + \frac{1}{4} - 6 - \frac{1}{4}\right)$$

$$y = -\left(x + \frac{1}{2}\right)^2 + \frac{25}{4}$$

TP at $\left(-\frac{1}{2}, \frac{25}{4}\right)$



g $y = -x^2 - 5x - 6$

x-intercepts: $y = -(x^2 + 5x + 6) = 0$

$\therefore -(x + 3)(x + 2) = 0$

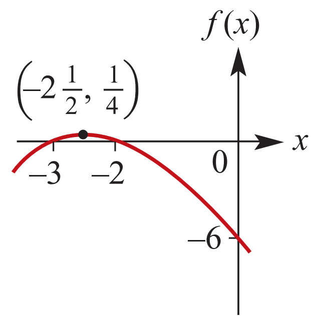
x-int: $(-3, 0)$ and $(-2, 0)$

TP: $y = -(x^2 + 5x + 6)$

$$y = -\left(x^2 + 5x + \frac{25}{4} + 6 - \frac{25}{4}\right)$$

$$y = -\left(x + \frac{5}{2}\right)^2 + \frac{1}{4}$$

TP at $\left(-\frac{5}{2}, \frac{1}{4}\right)$



h $y = x^2 - 5x - 24$

x-intercepts: $y = x^2 - 5x - 24 = 0$

$\therefore (x + 3)(x - 8) = 0$

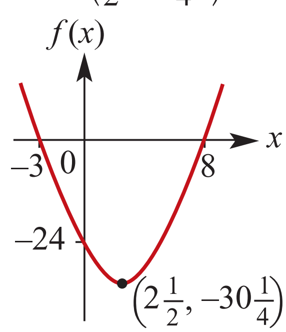
x-int: $(-3, 0)$ and $(8,0)$

TP: $y = x^2 - 5x - 24$

$$y = x^2 - 5x + \frac{25}{4} - 24 - \frac{25}{4}$$

$$y = \left(x - \frac{5}{2}\right)^2 - \frac{121}{4}$$

TP at $\left(\frac{5}{2} - \frac{121}{4}\right)$



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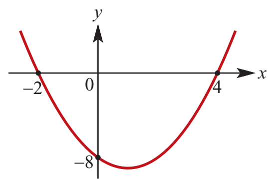
Solutions to Exercise 3G

1 **a** $x^2 + 2x - 8 = 0$
$\therefore (x + 2)(x - 4) = 0$
$\therefore x = -2, 4$
'Positive coefficient of x^2':

b



c $x^2 - 2x - 8 \le 0 \Leftrightarrow -2 \le x \le 4$

d $x^2 - 2x - 8 > 0 \Leftrightarrow x > 4$ or $x < -2$

2 **a** Positive coefficient of x^2
$x \le -2$ or $x \ge 3$

b Positive coefficient of x^2
$-4 < x < 3$

c Positive coefficient of x^2
$-4 \le x \le \frac{1}{2}$

d Positive coefficient of x^2
$x < 2$ or $x > 6$

e Positive coefficient of x^2
$2 < x < 3$

f Negative coefficient of x^2
$\frac{3}{2} \le x \le \frac{7}{2}$

g Positive coefficient of x^2
$-\frac{7}{2} < x < 2$

h Positive coefficient of x^2
$-2 \le x \le \frac{5}{2}$

i Negative coefficient of x^2
$x < -5$ or $x > \frac{5}{2}$

j Negative coefficient of x^2
$-2 \le x \le \frac{7}{2}$

k Negative coefficient of x^2
$x < \frac{2}{5}$ or $x > \frac{7}{2}$

l Positive coefficient of x^2
$x \le \frac{5}{2}$ or $x \ge \frac{11}{2}$

3 **a** Negative coefficient of x^2
$x < -5$ or $x > 5$

b Negative coefficient of y^2
$-\frac{2}{3} \le y \le \frac{2}{3}$

c Negative coefficient of y^2
$y > 4$ or $y < -4$

d Negative coefficient of x^2
$-\frac{6}{5} \le x \le \frac{6}{5}$

e Negative coefficient of y^2
$y \le -\frac{1}{4}$ or $y \ge \frac{1}{4}$

f Negative coefficient of y^2
$y < -\frac{5}{6}$ or $y > \frac{5}{6}$

4 **a** $x^2 + 2x - 8 = 0$
$\therefore (x + 4)(x - 2) = 0$
$\therefore x = 2, -4$
'Positive coefficient of x^2':
$x \le -2$ or $x \ \ge 4$

b $x^2 - 5x - 24 = 0$

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$\therefore (x+3)(x-8)=0$

$\therefore x=-3, 8$

'Positive coefficient of x^2':

$\{x: \ -3 < x < 8\}$

c $x - 4x - 12 = 0$

$\therefore (x+2)(x-6)=0$

$\therefore x=-2, 6$

'Positive coefficient of x^2':

$\{x: \ -2 \le x \le 6\}$

d $2x^2 - 3x - 9 = 0$

$\therefore (2x+3)(x-3)=0$

$\therefore x=-\frac{3}{2}, 3$

'Positive coefficient of x^2':

$\{x: x < -\frac{3}{2}\} \cup \{x: x > 3\}$

e $6x^2 + 13x < -6$

$\therefore 6x^2 + 13x + 6 < 0$

$6x^2 + 13x + 6 = 0$

$\therefore (3x+2)(2x+3)=0$

$x=-\frac{2}{3}, -\frac{3}{2}$

'Positive coefficient of x^2':

$\left\{x: -\frac{3}{2} < x < -\frac{2}{3}\right\}$

f $-x^2 - 5x - 6 = 0$

$\therefore -(x+2)(x+3)=0$

$\therefore x=-2, -3$

'Negative coefficient of x^2':

$\{x: \ -3 \le x \le -2\}$

g $12x^2 + x > 6$

$\therefore 12x^2 + x - 6 > 0$

$12x^2 + x - 6 = 0$

$\therefore (4x+3)(3x-2)=0$

$\therefore x=-\frac{3}{4}, \frac{3}{2}$

'Positive coefficient of x^2':

$\left\{x: x < -\frac{3}{4}\right\} \cup \left\{x: x > \frac{3}{2}\right\}$

h $10x^2 - 11x \le -3$

$\therefore 10x^2 - 11x + 3 \le 0$

$10x^2 - 11x + 3 = 0$

$\therefore (5x-3)(2x-1)=0$

$\therefore x=\frac{1}{2}, \frac{3}{5}$

'Positive coefficient of x^2':

$\left\{x: \frac{1}{2} \le x \le \frac{3}{5}\right\}$

i $x(x-1) \le 20$

$\therefore x^2 - x - 20 \le 0$

$x^2 - x - 20 = 0$

$\therefore (x-5)(x+4)=0$

$x=-4, 5$

'Positive coefficient of x^2':

$\{x: \ -4 \le x \le 5\}$

j $4 + 5p - p^2 = 0$

$\therefore p = \frac{-5 \pm \sqrt{41}}{-2}$

'Negative coefficient of x^2':

$\left\{p: \frac{5-\sqrt{41}}{2} \le p \le \frac{5+\sqrt{41}}{2}\right\}$

k $3 + 2y - y^2 = 0$

$\therefore (1+y)(3-y)=0$

$\therefore y=-1, 3$

'Negative coefficient of x^2':

$\{y: y < -1\} \cup \{y: y > 3\}$

l $x^2 + 3x \ge -2$

$\therefore x^2 + 3x + 2 \ge 0$

$x^2 + 3x + 2 = 0$

$\therefore (x+2)(x+1)=0$

$\therefore x=-2; -1$

'Positive coefficient of x^2':

$\{x: x \le -2\} \cup \{x: x \ge -1\}$

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5 **a**

$x^2 + 3x - 5 \geq 0$

$\Leftrightarrow \left(x + \frac{3}{2}\right)^2 - \frac{29}{4} \geq 0$

$\Leftrightarrow \left(x + \frac{3}{2}\right)^2 \geq \frac{29}{4}$

$\Leftrightarrow x \leq -\frac{3}{2} - \frac{\sqrt{29}}{2}$ or $x \geq -\frac{3}{2} + \frac{\sqrt{29}}{2}$

b $x^2 - 5x + 2 < 0$

$\Leftrightarrow 2\left(x - \frac{5}{2}\right)^2 - \frac{17}{4} < 0$

$\Leftrightarrow \left(x - \frac{5}{2}\right)^2 < \frac{17}{4}$

$\Leftrightarrow \frac{5}{2} - \frac{\sqrt{17}}{2} < x < \frac{5}{2} + \frac{\sqrt{17}}{2}$

c $2x^2 - 3x - 1 \leq 0$

$\Leftrightarrow 2\left(x - \frac{3}{4}\right)^2 - \frac{17}{8} \leq 0$

$\Leftrightarrow 2\left(x - \frac{3}{4}\right)^2 \leq \frac{17}{8}$

$\Leftrightarrow \left(x - \frac{3}{4}\right)^2 \leq \frac{17}{16}$

$\Leftrightarrow \frac{3}{4} - \frac{\sqrt{17}}{4} < x < \frac{3}{4} + \frac{\sqrt{17}}{4}$

d $2x^2 - 3x - 1 \leq 0$

$\Leftrightarrow -\left(x + \frac{3}{2}\right)^2 + \frac{41}{4} > 0$

$\Leftrightarrow \left(x - \frac{3}{4}\right)^2 < \frac{41}{4}$

$\Leftrightarrow \left(x - \frac{3}{4}\right)^2 < \frac{41}{16}$

$\Leftrightarrow \frac{3}{4} - \frac{\sqrt{41}}{4} < x < \frac{3}{4} + \frac{\sqrt{41}}{4}$

e $2x^2 + 7x + 1 \leq 0$

$\Leftrightarrow 2\left(x^2 + \frac{7}{2} + \frac{49}{16} - \frac{49}{16} + \frac{1}{2}\right) < 0$

$\Leftrightarrow 2\left(x + \frac{7}{4}\right)^2 - \frac{41}{16} < 0$

$\Leftrightarrow \left(x + \frac{7}{4}\right)^2 < \frac{41}{16}$

$\Leftrightarrow \frac{-7 - \sqrt{41}}{4} < x < \frac{-7 + \sqrt{41}}{4}$

f $2x^2 - 8x + 5 \geq 0$

$\Leftrightarrow 2\left(x^2 - 4 + 4 - 4 + \frac{5}{2}\right) \geq 0$

$\Leftrightarrow 2\,(x - 2)^2 - \frac{3}{2} \geq 0$

$\Leftrightarrow (x - 2)^2 \geq \frac{3}{2}$

$\Leftrightarrow x \leq \frac{4 - \sqrt{6}}{2}$ or $x \geq \frac{4 + \sqrt{6}}{2}$

6 The square of any real number is zero or positive.

7 The negative of the square of any real number is zero or negative.

8 $x^2 + 2x + 7$

$= x^2 + 2x + 1 - 1 + 7$

$= (x + 1)^2 + 6$

Since $(x + 1)^2 \geq 0$ for all x

$(x + 1)^2 + 6 \geq 6$ for all x

9 $- x^2 - 2x - 7$

$= -(x^2 + 2x + 1 - 1 + 7$

$= -((x + 1)^2 - 6$

Since $-(x + 1)^2 \leq 0$ for all x

$-(x + 1)^2 - 6 \leq -6$ for all x

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Solutions to Exercise 3H

1 **a** $a = 2, b = 4$ and $c = -3$

i $b^2 - 4ac = 4^2 - 4(-3)2 = 40$

ii $\sqrt{b^2 - 4ac} = \sqrt{40} = 2\sqrt{10}$

b $a = 1, b = 10$ and $c = 18$

i $b^2 - 4ac = 10^2 - 4(18)1 = 28$

ii $\sqrt{b^2 - 4ac} = \sqrt{28} = 2\sqrt{7}$

c $a = 1, b = 10$ and $c = -18$

i $b^2 - 4ac = 10^2 - 4(-18)1 = 172$

ii $\sqrt{b^2 - 4ac} = \sqrt{172} = 2\sqrt{43}$

d $a = -1, b = 6$ and $c = 15$

i $b^2 - 4ac = 6^2 - 4(15)(-1) = 96$

ii $\sqrt{b^2 - 4ac} = \sqrt{96} = 4\sqrt{6}$

e $a = 1, b = 9$ and $c = -27$

i $b^2 - 4ac = 9^2 - 4(-27)1 = 189$

ii $\sqrt{b^2 - 4ac} = \sqrt{189} = 3\sqrt{21}$

2 **a** $\dfrac{2 + 2\sqrt{5}}{2} = 1 + \sqrt{5}$

b $\dfrac{9 - 3\sqrt{5}}{6} = \dfrac{3 - \sqrt{5}}{2}$

c $\dfrac{5 + 5\sqrt{5}}{10} = \dfrac{1 + \sqrt{5}}{2}$

d $\dfrac{6 + 12\sqrt{2}}{6} = 1 + 2\sqrt{2}$

3 **a** $x^2 + 6x = 4$

$\therefore x^2 + 6x - 4 = 0$

$\therefore x = \dfrac{-6 \pm \sqrt{6^2 - 4(-4)1}}{2}$

$\therefore x = \dfrac{-6 \pm \sqrt{52}}{2}$

$\therefore x = -3 \pm \sqrt{13}$

b $x^2 - 7x - 3 = 0$

$\therefore x = \dfrac{7 \pm \sqrt{7^2 - 4(-3)1}}{2}$

$\therefore x = \dfrac{7 \pm \sqrt{61}}{2}$

c $2x^2 - 5x + 2 = 0$

$\therefore x = \dfrac{5 \pm \sqrt{5^2 - 4(2)2}}{4}$

$\therefore x = \dfrac{5 \pm \sqrt{9}}{4}$

$\therefore x = \dfrac{5 \pm 3}{4} = \dfrac{1}{2}, 2$

d $2x^2 + 4x - 7 = 0$

$\therefore x = \dfrac{-4 \pm \sqrt{4^2 - 4(-7)2}}{4}$

$\therefore x = \dfrac{-4 \pm \sqrt{72}}{4}$

$\therefore x = -1 \pm \dfrac{6}{4}\sqrt{2}$

$\therefore x = -1 \pm \dfrac{3}{2}\sqrt{2}$

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e $2x^2 + 8x = 1$

$\therefore 2x^2 + 8x - 1 = 0$

$\therefore x = \dfrac{-8 \pm \sqrt{8^2 - 4(-1)2}}{4}$

$\therefore x = -2 \pm \dfrac{\sqrt{72}}{4}$

$\therefore x = -2 \pm \dfrac{3}{2}\sqrt{2}$

f $5x^2 - 10x = 1$

$\therefore 5x^2 - 10x - 1 = 0$

$\therefore x = \dfrac{10 \pm \sqrt{10^2 - 4(-1)5}}{10}$

$\therefore x = 1 \pm \dfrac{\sqrt{120}}{10}$

$\therefore x = 1 \pm \dfrac{\sqrt{30}}{5}$

g $-2x^2 + 4x - 1 = 0$

$\therefore x = \dfrac{-4 \pm \sqrt{4^2 - 4(-1)(-2)}}{-4}$

$\therefore x = 1 \pm \dfrac{\sqrt{8}}{4}$

$\therefore x = 1 \pm \dfrac{\sqrt{2}}{2}$

h $2x^2 + x = 3$

$\therefore 2x^2 + x - 3 = 0$

$\therefore x = \dfrac{-1 \pm \sqrt{1^2 - 4(-3)2}}{4}$

$\therefore x = \dfrac{-1 \pm \sqrt{25}}{4}$

$\therefore x = \dfrac{-1 \pm 5}{4} \; \therefore x = 1, -\dfrac{3}{2}$

i $2.5x^2 + 3x + 0.3 = 0$

$\therefore x = \dfrac{-3 \pm \sqrt{3^2 - 4(0.3)2.5}}{5}$

$\therefore x = \dfrac{-3 \pm \sqrt{6}}{5}$

j $-0.6x^2 - 1.3x = 0.1$

$\therefore -6x^2 - 13x - 1 = 0$

$\therefore 6x^2 + 13x + 1 = 0$

$\therefore x = \dfrac{-13 \pm \sqrt{13^2 - 4(1)6}}{12}$

$\therefore x = \dfrac{-13 \pm \sqrt{145}}{12}$

k $2kx^2 - 4x + k = 0$

$\therefore x = \dfrac{4 \pm \sqrt{4^2 - 4(2k)k}}{4k}$

$\therefore x = 1 \pm \dfrac{\sqrt{16 - 8k^2}}{4k}$

$\therefore x = \dfrac{2 \pm \sqrt{4 - 2k^2}}{2k}$

l $2(1 - k)x^2 - 4kx + k = 0$

$\therefore x = \dfrac{4k \pm \sqrt{16k^2 - 8k(1 - k)}}{4(1 - k)}$

$\therefore x = \dfrac{4k \pm \sqrt{24k^2 - 8k}}{4(1 - k)}$

$\therefore x = \dfrac{2k \pm \sqrt{6k^2 - 2k}}{2(1 - k)}$

4 a $y = x^2 + 5x - 1$

x-axis intercepts:

$x = \dfrac{-5 \pm \sqrt{29}}{2}$

$x = -\dfrac{5}{2};$

$y = \dfrac{25}{4} - \dfrac{25}{2} - 1 = -\dfrac{29}{4}$

TP at $(-2.5, -7.25)$

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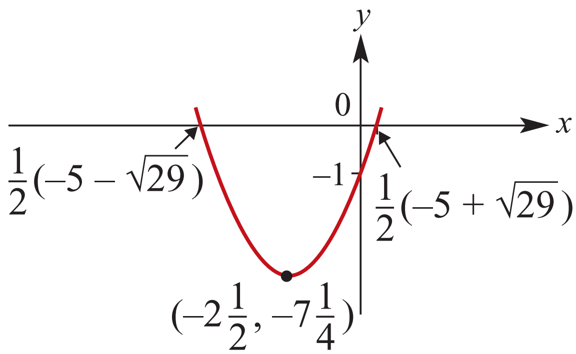


b $y = 2x^2 - 3x - 1$

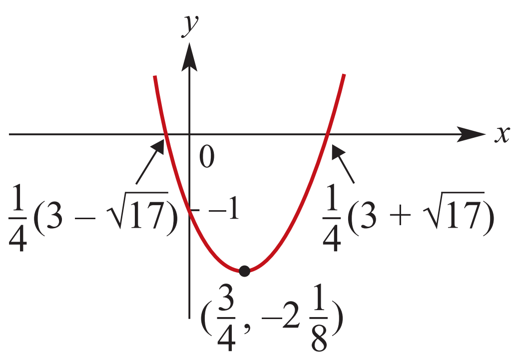
x-axis intercepts:

$\therefore x = \dfrac{3 \pm \sqrt{17}}{4}$

$x = \dfrac{3}{4}$;

$y = \dfrac{9}{8} - \dfrac{9}{4} - 1 = -\dfrac{17}{8}$

TP at $(0.75, -2.125)$



c $y = -x^2 - 3x + 1$

x-axis intercepts:

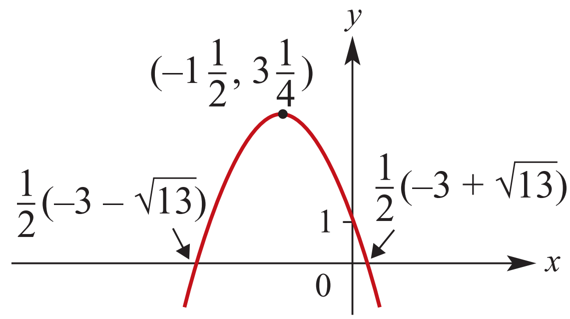
$\therefore x = \dfrac{3 \pm \sqrt{13}}{-2}$

$\therefore x = \dfrac{-3 \pm \sqrt{13}}{2}$

$x = -\dfrac{3}{2}$;

$y = -\dfrac{9}{4} + \dfrac{9}{2} + 1 = \dfrac{13}{4}$

TP at $(-1.5, 3.25)$



d $y + 4 = x^2 + 2x$

$\therefore y = x^2 + 2x - 4$

x-axis intercepts:

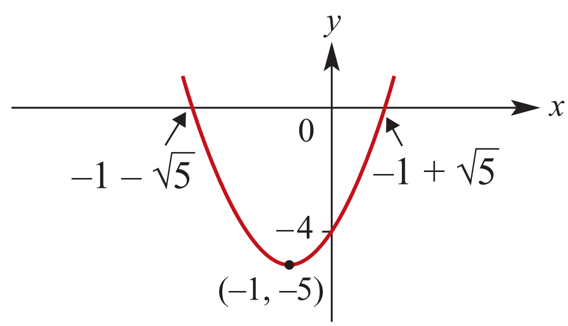
$\therefore x = \dfrac{-2 \pm \sqrt{20}}{2}$

$\therefore x = -1 \pm \sqrt{5}$

$x = -1$;

$y = 1 - 2 - 4 = -5$

TP at $(-1, -5)$



e $y = 4x^2 + 5x + 1$

x-axis intercepts:

$\therefore x = \dfrac{-5 \pm \sqrt{25-16}}{8}$

$\therefore x = \dfrac{-5 \pm 3}{8}$

$\therefore x = -1, -\dfrac{1}{4}$

$x = -\dfrac{5}{8}$;

$y = \dfrac{100}{64} - \dfrac{25}{8} + 1 = -\dfrac{9}{16}$

TP at $(-0.625, -0.5625)$

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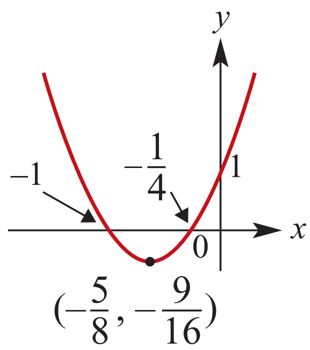


f $y = -3x^2 + 4x - 2$

x-axis intercepts:

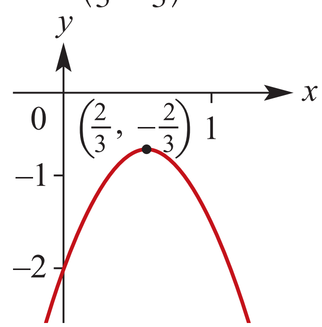
$\therefore x = \dfrac{-4 \pm \sqrt{16 - 24}}{-6}$

This is not defined, so no x-intercepts.

$x = \dfrac{2}{3}$;

$y = -\dfrac{4}{3} + \dfrac{8}{3} - 2 = -\dfrac{2}{3}$

TP at $\left(\dfrac{2}{3}, -\dfrac{2}{3}\right)$



g $y = -x^2 + 5x + 6$ When $y = 0$

$$-x^2 + 5x + 6 = 0$$

$$x^2 - 5x - 6 = 0$$

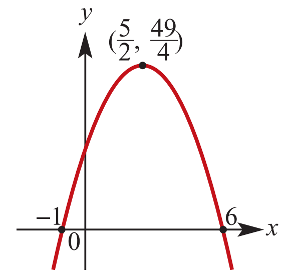
$$(x - 6)(x + 1) = 0$$

$$x = 6 \text{ or } x = -1$$

When $x = 0, y = 6$

Axis of symmetry: $x = \dfrac{5}{2}$

Coordinates of turning point $\left(\dfrac{5}{2}, \dfrac{49}{4}\right)$

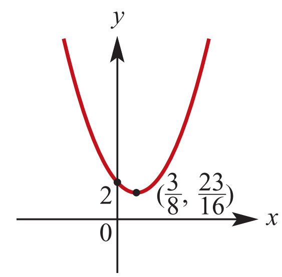


h $y = 4x^2 - 3x + 2$

$\Delta = 9 - 4 \times 4 \times 2 < 0$

Therefore no x-axis intercepts. Axis of symmetry: $x = \dfrac{3}{8}$

Coordinates of turning point $\left(\dfrac{3}{8}, \dfrac{23}{16}\right)$



i $y = 3x^2 - x - 4$

When $y = 0$,

$x = \dfrac{1 \pm \sqrt{1 - 4 \times 3 \times (-4)}}{6}$

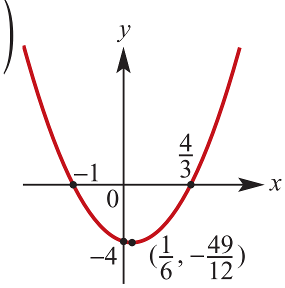
That is $x = \dfrac{1 \pm 7}{6}$

$x = -1$ or $x = \dfrac{4}{3}$

Axis of symmetry: $x = \dfrac{1}{6}$

Coordinates of turning point

$\left(\dfrac{1}{6}, -\dfrac{49}{12}\right)$



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Solutions to Exercise 3I

1 a $x^2 + 2x - 4;$
$\Delta = 2^2 - 4(-4) = 20$

b $x^2 + 2x + 4;$
$\Delta = 2^2 - 4(4) = -12$

c $x^2 + 3x - 4;$
$\Delta = 3^2 - 4(-4) = 25$

d $2x^2 + 3x - 4;$
$\Delta = 3^2 - 8(-4) = 41$

e $-2x^2 + 3x + 4;$
$\Delta = 3^2 - 4(-2)(4) = 41$

2 a $x^2 - 5x + 2;$
$\Delta = 5^2 - 4(2) = 17$
$\Delta > 0$ so graph crosses the x-axis

b $-4x^2 + 2x - 1;$
$\Delta = 2^2 - 4(-4)(-1) = -12$
$\Delta < 0$ so graph does not cross the x-axis

c $x^2 - 6x + 9;$
$\Delta = 6^2 - 4(9) = 10$
$\Delta = 0$ so graph touches the x-axis

d $-2x^2 - 3x + 8;$
$\Delta = 3^2 - 4(-2)8 = 73$
$\Delta > 0$ so graph crosses the x-axis

e $3x^2 + 2x + 5;$
$\Delta = 2^2 - 4(5)(3) = -56$
$\Delta < 0$ so graph does not cross the x-axis

f $-x^2 - x - 1;$
$\Delta = 1^2 - 4(-1)(-1) = -3$
$\Delta < 0$ so graph does not cross the x axis

3 a $x^2 + 8x + 7;$
$\Delta = 8^2 - 4(7) = 36$
$\Delta > 0$ so the equation has 2 real roots

b $3x^2 + 8x + 7;$
$\Delta = 8^2 - 4(7)(3) = -20$
$\Delta < 0$ so no real roots

c $10x^2 - x - 3;$
$\Delta = 1^2 - 4(-3)(10) = 121$
$\Delta > 0$ so the equation has 2 real roots

d $2x^2 + 8x - 7;$
$\Delta = 8^2 - 4(-7)2 = 120$
$\Delta > 0$ so the equation has 2 real roots

e $3x^2 - 8x - 7;$
$\Delta = 8^2 - 4(-7)3 = 148$
$\Delta > 0$ so the equation has 2 real roots

f $10x^2 - x + 3;$
$\Delta = 1^2 - 4(10)(3) = -119$
$\Delta < 0$ so the equation has no real roots

4 a $9x^2 - 24x + 16;$
$\Delta = 24^2 - 4(9)(16) = 0$
$\Delta = 0$ so the equation has 1 rational root

b $-x^2 - 5x - 6;$
$\Delta = 5^2 - 4(-6)(-1) = 1$
$\Delta > 0$ so the equation has 2 rational roots.

c $x^2 - x - 4;$

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$\Delta = 1^2 - 4(-4) = 17$
$\Delta > 0$ so the equation has 2 irrational roots, and is not a perfect square

d $25x^2 - 20x + 4;$
$\Delta = 20^2 - 4(25)(4) = 0$
$\Delta = 0$ so the equation has 1 rational root and is a perfect square.

e $6x^2 - 3x - 2;$
$\Delta = 3^2 - 4(6)(-2) = 57$
$\Delta > 0$ so the equation has 2 irrational roots and is not a perfect square

f $x^2 + 3x + 2;$
$\Delta = 3^2 - 4(2) = 1$
$\Delta > 0$ so the equation has 2 rational roots and is not a perfect square

5 **a** $x^2 - 4mx + 20 = 0$
$\Delta = 16m^2 - 80 = 16(m^2 - 5)$

i If $(m^2 - 5) < 0$, no real solutions:
$\{m\colon -\sqrt{5} < m < \sqrt{5}\}$

ii If $(m^2 - 5) = 0$, one real solution:
$\{m\colon m = \pm\sqrt{5}\}$

iii If $(m^2 - 5) > 0$, 2 distinct solutions:
$\{m\colon m < -\sqrt{5}\} \cup \{m\colon m > \sqrt{5}\}$

b $mx^2 - 3mx + 3 = 0$
$\Delta = 9m^2 - 12m = 3m(3m - 4)$

i If $\Delta < 0$, no real solutions:
$\Delta = 0$ at $m = 0, \frac{4}{3}$
Upright parabola, so
$\{m\colon 0 < m < \frac{4}{3}\}$

ii If $\Delta = 0$, one real solution;
$m = 0, \frac{4}{3}$ satisfies this, but there is no solution to the equation if $m = 0$, so $\{m\colon m = \frac{4}{3}\}$

iii If $(3m^2 - 4) > 0$, 2 distinct solutions:
$\{m\colon m < 0\} \cup \{m\colon m > \frac{4}{3}\}$

c $5x^2 - 5mx - m = 0$
$\Delta = 25m^2 + 20m = 5m(5m + 4)$

i If $5m(5m + 4) < 0$, no real solutions
$\Delta = 0$ at $m = 0, -\frac{4}{5}$
Quadratic in m is upright:
$\{m\colon -\frac{4}{5} < m < 0\}$

ii If $5m(5m + 4) = 0$, one real solution:
$\{m\colon m = 0, -\frac{4}{5}\}$

iii If $5m(5m + 4) > 0$, 2 distinct solutions:
$\{m\colon m < -\frac{4}{5}\} \cup \{m\colon m > 0\}$

d $x^2 + 4mx - 4(m - 2) = 0$
$\Delta = 16m + 16(m - 2)$
$= 16(m^2 + m - 2)$

i If $m^2 + m - 2 < 0$, no real solutions:
$m^2 + m - 2 = (m + 2)(m - 1)$
Quadratic in m is upright, so
$\{m\colon -2 < m < 1\}$

ii If $m^2 + m - 2 = 0$, one real solution:
$\{m\colon m = -2, 1\}$

iii If $m^2 + m - 2 > 0$, 2 distinct

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solutions:
$\{m: m < -2\} \cup \{m: m > 1\}$

6 $mx^2 + (2m+n)x + 2n = 0$

$$\begin{aligned}\Delta &= (2m+n)^2 - 8mn\\ &= 4m^2 + 4mn + n^2 - 8mn\\ &= 4m^2 - 4mn + n^2\\ &= (2m-n)^2\end{aligned}$$

This is a perfect square for all rational m and n, so the solution is rational also.

7 $px^2 + 2(p+2)x + p + 7 = 0$

$$\begin{aligned}\Delta &= 4(p+2)^2 - 4p(p+7)\\ &= 4p^2 + 16p + 16 - 4p - 28p\\ &= 16 - 12p = 4(4-3p)\end{aligned}$$

This equation has no real solution if $\Delta < 0$, i.e. if $p > \dfrac{4}{3}$

8 $(1-2p)x^2 + 8px - (2+8p) = 0$

$$\begin{aligned}\Delta &= 64p^2 + 4(1-2p)(2+8p)\\ &= 64p^2 - 8(2p-1)(4p+1)\\ &= 64p^2 - 8(8p^2 - 2p - 1)\\ &= 8(2p+1)\end{aligned}$$

This equation has one real solution if $\Delta = 0$;
$2p + 1 = 0$ or $p = -\dfrac{1}{2}$

9 **a** $px^2 - 6x + 9 = 0$
$\Delta = 36 - 4p^2$
One solution $\Leftrightarrow \Delta = 0$

$$\begin{aligned}36 - 4p^2 &= 0\\ 36 &= 4p^2\\ 9 &= p^2\\ p &= \pm 3\end{aligned}$$

b $2x^2 - 4x + 3 - p = 0$
$\Delta = 16 - 4 \times 2(3-p)$
Two solution $\Leftrightarrow \Delta > 0$

$$\begin{aligned}16 - 4 \times 2(3-p) &> 0\\ 8p - 8 &> 0\\ p &> 1\end{aligned}$$

c $3x^2 - 2x - p + 1 = 0$
$\Delta = 4 - 4 \times 3(1-p)$
Two solution $\Leftrightarrow \Delta > 0$
$12p - 8 > 0$

$p > \dfrac{2}{3}$

d $x^2 - 2x + 2 - p = 0$
$\Delta = 4 - 4 \times (2-p)$
Two solution $\Leftrightarrow \Delta > 0$
$4p - 4 > 0$

$p > 1$

10 $y = px^2 + 8x + p - 6$

$$\begin{aligned}\Delta &= 64 - 4p(p-6)\\ &= 4(-p + 6p + 16)\end{aligned}$$

If the graph crosses the x-axis, $\Delta > 0$:
$\Delta = 0$ when $p = \dfrac{-6 \pm \sqrt{100}}{-2}$
$\therefore p = 3 \pm 5 = -2, 8$
Inverted quadratic, so $\Delta > 0$ when:
$\{p: -2 < p < 8\}$

11 $(p^2+1)x^2 + 2pqx + q^2 = 0$

$$\begin{aligned}\Delta &= 4p^2q^2 - 4q^2(p^2+1)\\ &= 4q^2(p^2 - p^2 - 1)\\ &= -4q^2\end{aligned}$$

This is negative for all values of p, and for all non-zero q, so there are no real solutions.

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12 **a** For $x^2 + 4mx + 24m - 44$

$\Delta = (4m)^2 - 4(24m - 44)$

$= 16m^2 - 96m + 176$

b $4mx^2 + 4(m - 1)x + m - 2 = 0$ has a solution for all values of m if and only if $\Delta \geq 0$ for all m.

$16m^2 - 96m + 176$

$= 16(m^2 - 6m + 11)$

$= 16(m^2 - 6m + 9 + 2)$

$= 16(m - 3)^2 + 32 \geq 0 \quad$ for all m

13 $4mx^2 + 4(m - 1)x + m - 2$

a $\Delta = 16(m - 1)^2 - 4(4m)(m - 2)$

$= 16m^2 - 32m + 16 - 16m^2 + 32m$

$= 16$

b Δ is a perfect square and thus the solutions are rational for all m.

14 $4x^2 + (m - 4)x - m = 0$

$\Delta = (m - 4)^2 - 4(4)(-m)$

$= m^2 - 8m + 16 + 16m$

$= m^2 + 8m + 16$

$= (m + 4)^2$

$\therefore \Delta$ is a perfect square for all m

15 $x^2 - (m + 2n)x + 2mn = 0$

$\Delta = (m + 2n)^2 - 4 \times 2mn$

$= m^2 + 4mn + 4n^2 - 8mn$

$= m^2 - 4mn + 4n^2$

$= (m - 2n)^2$

Therefore Δ is a perfect square. The roots of the equation are rational.

16 $\Delta = b^2 - 4(a)(-c) = b^2 + 4ac > 0 \therefore$ the graph of $y = x^2 + bx - c$ where a and c are positive always intersects with the x-axis.

17 $\Delta = b^2 - 4(a)(c) = b^2 - 4ac > 0$ if .
$\therefore$ the graph of $y = x^2 + bx + c$ where a is negative and c is positive always intersects with the x-axis.

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Solutions to Exercise 3J

1 a

$$y = x - 2 \ldots (1)$$
$$y = x^2 - x - 6 \ldots (2)$$
$$\therefore x^2 - x - 6 = x - 2$$
$$x^2 - 2x - 4 = 0$$
$$\therefore x = \frac{2 \pm \sqrt{20}}{2}$$
$$= \frac{2 \pm 2\sqrt{5}}{2}$$
$$= 1 \pm \sqrt{5}$$

Therefore points of intersection are $(1 - \sqrt{5}, -1 - \sqrt{5})$ and $(1 + \sqrt{5}, -1 + \sqrt{5})$

b

$$x + y = 6 \ldots (1)$$
$$y = x^2 \ldots (2)$$

From (1), $y = 6 - x$

$$\therefore x^2 = 6 - x$$
$$x^2 + x - 6 = 0$$
$$(x + 3)(x - 2) = 0$$
$$\therefore x = 2 \text{ or } x = -3$$

Therefore points of intersection are $(2, 4)$ and $(-3, 9)$

c

$$5x + 4y = 21 \ldots (1)$$
$$y = x^2 \ldots (2)$$

Substitute from (2) in (1),

$$5x + 4x^2 = 21$$
$$4x^2 + 5x - 21 = 0$$
$$(4x - 7)(x + 3) = 0$$
$$\therefore x = \frac{7}{4} \text{ or } x = -3$$

Therefore points of intersection are $(-3, 9)$ and $\left(\frac{7}{4}, \frac{49}{16}\right)$

d

$$y = 2x + 1 \ldots (1)$$
$$y = x^2 - x + 3 \ldots (2)$$

Substitute from (1) in (2),

$$x^2 - x + 3 = 2x + 1$$
$$x^2 - 3x + 2 = 0$$
$$(x - 2)(x - 1) = 0$$
$$\therefore x = 2 \text{ or } x = 1$$

Therefore points of intersection are $(2, 5)$ and $(1, 3)$

2 a $y = x^2 + 2x - 8$ and $y = 2 - x$ meet where $x^2 + 2x - 8 = 2 - x$

$\therefore x^2 + 3x - 10 = 0$

$\therefore (x + 5)(x - 2) = 0$

$\therefore x = -5, 2$

When $x = -5, y = 2 - (-5) = 7$

When $x = 2, y = 2 - 2 = 0$

Curves meet at $(-5, 7)$ and $(2,0)$.

b $y = x^2 - x - 3$ and $y = 4x - 7$ meet where $x^2 - x - 3 = 4x - 7$

$\therefore x^2 - 5x + 4 = 0$

$\therefore (x - 4)(x - 1) = 0$

$\therefore x = 4, 1$

When $x = 1, y = 4 - 7 = -3$

When $x = 4, y = 16 - 7 = 9$

Curves meet at $(1, -3)$ and $(4,9)$.

c $y = x^2 + x - 5$ and $y = -x - 2$ meet where $x^2 + x - 5 = -x - 2$

$\therefore x^2 + 2x - 3 = 0$

$\therefore (x + 3)(x - 1) = 0$

$\therefore x = -3, 1$

When $x = -3, y = 3 - 2 = 1$

When $x = 1, y = -1 - 2 = -3$

Curves meet at $(-3, 1)$ and $(1, -3)$.

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d $y = x^2 + 6x + 6$ and $y = 2x + 3$ meet where $x^2 + 6x + 6 = 2x + 3$
$\therefore x^2 + 4x + 3 = 0$
$\therefore (x + 3)(x + 1) = 0$
$x = -3, -1$
When $x = -3, y = -6 + 3 = -3$
When $x = -1, y = -2 + 3 = 1$
Curves meet at $(-3, -3)$ and $(-1, 1)$.

e $y = -x^2 - x + 6$ and $y = -2x - 2$ meet where $-x^2 - x + 6 = -2x - 2$
$\therefore -x^2 + x + 8 = 0$
$\therefore x^2 - x - 8 = 0$
$\therefore x = \dfrac{1 \pm \sqrt{1 - 4(8)}}{2}$
$\therefore x = \dfrac{1 \pm \sqrt{33}}{2}$
When $x = \dfrac{1 - \sqrt{33}}{2}$, $y = -3 + \sqrt{33}$
When $x = \dfrac{1 + \sqrt{33}}{2}$, $y = -3 - \sqrt{33}$
Curves meet at $\left(\dfrac{1 - \sqrt{33}}{2}, -3 + \sqrt{33}\right)$
and $\left(\dfrac{1 + \sqrt{33}}{2}, -3 - \sqrt{33}\right)$.

f $y = x^2 + x + 6$ and $y = 6x + 8$ meet where $x^2 + x + 6 = 6x + 8$
$\therefore x^2 - 5x - 2 = 0$
$\therefore x = \dfrac{5 \pm \sqrt{25 - 4(-2)}}{2}$
$\therefore x = \dfrac{5 \pm \sqrt{33}}{2}$
When $x = \dfrac{5 - \sqrt{33}}{2}$, $y = 23 - 3\sqrt{33}$
When $x = \dfrac{5 + \sqrt{33}}{2}$, $y = 23 + \sqrt{33}$
Curves meet at
$\left(\dfrac{5 - \sqrt{33}}{2}, 23 - 3\sqrt{33}\right)$ and
$\left(\dfrac{5 + \sqrt{33}}{2}, 23 + 3\sqrt{33}\right)$.

3 If the straight line meets the parabola only once, then the $y_1 = y_2$ quadratic will produce a perfect square.

a $x - 6x + 8 = -2x + 4$
$\therefore x^2 - 4x + 4 = 0$
$\therefore (x - 2)^2 = 0, \therefore x = 2$
Touches at (2,0).

b $x^2 - 2x + 6 = 4x - 3$
$\therefore x^2 - 6x + 9 = 0$
$\therefore (x - 3)^2 = 0, \therefore x = 3$
Touches at (3,9).

c $2x^2 + 11x + 10 = 3x + 2$
$\therefore 2x^2 + 8x + 8 = 0$
$\therefore 2(x + 2)^2 = 0, \therefore x = -2$
Touches at $(-2, -4)$.

d $x^2 + 7x + 4 = -x - 12$
$\therefore x^2 + 8x + 16 = 0$
$\therefore (x + 4)^2 = 0, \therefore x = -4$
Touches at $(-4, -8)$.

4 **a** $y = x^2 - 6x; y = 8 + x$
$\therefore \quad x^2 - 6x = 8 + x$
$x^2 - 7x - 9 = 0$
$(x - 8)(x + 1) = 0$
$\therefore \quad x = 8, -1$
$x = 8; y = 8 + 8 = 16$
$x = -1; y = 8 + 1 = 7$

b $y = 3x^2 + 9x; y = 32 - x$
$\therefore \quad 3x^2 + 9x = 32 - x$
$3x^2 + 10x - 32 = 0$
$(3x + 16)(x - 2) = 0$
$\therefore \quad x = -\dfrac{16}{3}, 2$
$x = -\dfrac{16}{3}; y = 32 + \dfrac{16}{3} = \dfrac{112}{3}$
$x = 2; y = 32 - 2 = 30$

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c $y = 5x^2 + 9x; y = 12 - 2x$

$$\therefore \quad 5x^2 + 9x = 12 - 2x$$

$$5x^2 + 11x - 12 = 0$$

$$(5x - 4)(x + 3) = 0$$

$$\therefore \quad x = \frac{4}{5}, -3$$

$x = \frac{4}{5}; y = 12 - \frac{8}{5} = \frac{52}{5}$

$x = -3; y = 12 - (-6) = 18$

d $y = -3x^2 + 32x; y = 32 - 3x$

$$\therefore \quad -3x^2 + 32x = 32 - 3x$$

$$-3x^2 + 35x - 32 = 0$$

$$3x^2 - 35x + 32 = 0$$

$$(x - 1)(3x - 32) = 0$$

$$x = 1, \frac{32}{3}$$

$x = 1; y = 32 - 3 = 29$

$x = \frac{32}{3}; y = 32 - 32 = 0$

e $y = 2x^2 - 12; y = 3(x - 4)$

$\therefore 2x^2 - 12 = 3x - 12$

$$2x^2 - 3x = 0$$

$$x(2x - 3) = 0$$

$$x = 0, \frac{3}{2}$$

$x = 0; y = 3(-4) = -12$

$x = \frac{3}{2}; y = 3\left(\frac{3}{2} - 4\right) = -\frac{15}{2}$

f $y = 11x^2; y = 21 - 6x$

$\therefore 11x^2 + 6x - 21 = 0$

$$\therefore x = \frac{-6 \pm \sqrt{6^2 - 4(-21)(11)}}{22}$$

$$= \frac{-3 \pm \sqrt{240}}{11} = -3 \pm \frac{4}{11}\sqrt{15}$$

$x = \frac{-3 - 4\sqrt{15}}{11}$;

$y = 21 + \frac{6}{11}(3 + 4\sqrt{15}) = \frac{249 + 24\sqrt{15}}{11}$

$x = \frac{-3 + 4\sqrt{15}}{11}$;

$y = 21 + \frac{6}{11}(3 - 4\sqrt{15}) = \frac{249 - 24\sqrt{15}}{11}$

Using a calculator: $x = 1.14, y = 14.19$;

$x = -1.68, y = 31.09$

5 a If $y = x + c$ is a tangent to the parabola $y = x^2 - x - 12$, then $x^2 - x - 12 = x + c$ must reduce to a quadratic with zero discriminant.

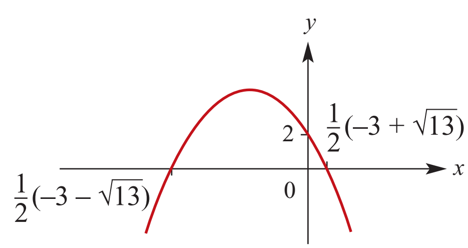
$x^2 - x - 12 = x + c$

$\therefore x^2 - 2x - (12 + c) = 0$

$\therefore \Delta = 4 + 4(12 + c)$

$= 4c + 52 = 0, \therefore c = -13$

b i $y = -2x^2 - 6x + 2$



ii If $y = mx + 6$ is a tangent to the parabola,

$-2x^2 - 6x + 2 = mx + 6$

$\therefore -2x^2 - (6 + m)x - 4 = 0$

$\therefore 2x^2 + (6 + m)x + 4 = 0$

For a tangent, $\Delta = 0$:

$\therefore \Delta = (6 + m)^2 - 4(4)(2) = 0$

$\therefore (6 + m)^2 = 32$

$\therefore 6 + m = \pm\sqrt{32} = \pm 4\sqrt{2}$

$m = -6 \pm 4\sqrt{2}$

6 a $y = x^2 + 3x$ has as a tangent

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$y = 2x + c$
$\Delta = 0$ for $x^2 + 3x = 2x + c$
$\therefore x^2 + x - c = 0$
$\therefore \Delta = 1 + 4c = 0, \therefore c = -\frac{1}{4}$

b For two intersections, $\Delta > 0$ so
$c > -\frac{1}{4}$

7 $y = x$ is a tangent to the parabola
$y = x^2 + ax + 1$
$\therefore x^2 + ax + 1 = x$
$\therefore x^2 + (a - 1)x + 1 = 0$
$\Delta = (a - 1)^2 - 4 = 0$
$\therefore a - 1 = \pm 2$
$\therefore a = 1 \pm 2 = -1, 3$

8 $y = -x$ is a tangent to the parabola
$y = x^2 + x + b$
$\therefore x^2 + x + b = -x$
$\therefore x^2 + 2x + b = 0$
$\Delta = 4 - 4b = 0$
$\therefore b = 1$

9 A straight line passing through the point $(1, -2)$ has the form $y - (-2) = m(x - 1)$
$\therefore y = m(x - 1) - 2$
If this line is a tangent to $y = x^2$ then
$x^2 = m(x - 1) - 2$
$\therefore x^2 - m(x - 1) + 2 = 0$
$\therefore x^2 - mx + m + 2 = 0$
$\Delta = 0$ for a tangent here:
$\Delta = m^2 - 4(m + 2)$
$= m^2 - 4m - 8 = 0$
$m^2 - 4m - 8 = 0$
$\therefore m = \frac{4 \pm \sqrt{16 + 32}}{2}$
$\therefore m = 2 \pm \sqrt{12} = 2 \pm 2\sqrt{3}$
$\therefore y = (2 \pm 2\sqrt{3})(x - 1) - 2$
$y = 2(1 + \sqrt{3})x - 4 - 2\sqrt{3}$ and
$y = 2(1 - \sqrt{3})x - 4 + 2\sqrt{3}$

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Solutions to Exercise 3K

1 $y = ax^2 + c$ passes through $(0, 6)$ and $(-1, 2)$.
$\therefore a(0)^2 + c = 6, \therefore c = 6$
$a(-1)^2 + 6 = 2, \therefore a = -4$

2 $y = ax^2 + bx + 4$

a $\Delta = b^2 - 16a$

b If the turning point lies on the x axis, $\Delta = 0$.
$\therefore b^2 - 16a = 0$
This implies, $a = \dfrac{b^2}{16}$.

c Turning point when $x = -\dfrac{b}{2a}$
Therefore,

$$-4 = -\frac{b}{2a} \ldots (1)$$

$$a = \frac{b^2}{16} \ldots (2)$$

Rearranging(1)

$$a = \frac{b}{8}$$

$$\therefore \frac{b}{8} = \frac{b^2}{16}$$

$$\therefore b = 2 \qquad \text{(If } b = 0 \text{ then } a = 0\text{)}$$

$$\therefore a = \frac{1}{4}$$

3 a $y = k(x + 2)(x - 6)$
When $x = 1, y = -30$
$-30 = k(3)(-5)$

$k = 2$

$\therefore y = 2(x + 2)(x - 6)$

b $y = a(x - h)^2 + k$
Turning point $(-2, 4)$

$$\therefore y = a(x + 2)^2 + 4$$

Passes through $(3, -46)$

$$\therefore -46 = a(25) + 4$$

$$\therefore -50 = a(25)$$

$$\therefore a = -2$$

$$\therefore y = -2(x + 2)^2 + 4$$

c Passes through the points $(1, -2), (0, -3), (-1, -6)$
Use $y = ax^2 + bx + c$
Passes through $(0, -3)$,
$\therefore c = -3$

$$y = ax^2 + bx - 3$$

When$x = 1, y = -2$

$$\therefore -2 = a + b - 3$$

$$\therefore a + b = 1 \ldots (1)$$

When$x = -1, y = -6$

$$\therefore -6 = a + b - 3$$

$$\therefore a - b = -3 \ldots (2)$$

Add (1) and (2)

$$2a = -2$$

$$a = -1$$

$$\therefore b = 2$$

$$\therefore y = -x^2 + 2x - 3$$

4 $y = ax^2$ passes through (2,8).
$\therefore 8 = a(2)^2, \therefore a = 2$

5 $y = ax^2 + bx$ passes through (6,0) and $(-1, 4)$.
$\therefore a(6)^2 + 6b = 0$
$\therefore 36a + 6b = 0, \therefore b = -6a$
$a(-1)^2 - 6a(-1) = 4$

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$\therefore 7a = 4$

$\therefore a = \frac{4}{7}; b = -\frac{24}{7}$

6 $y = a(x - b)^2 + c$

The vertex is at (1,6) so $y = a(x - 1)^2 + 6$

$y = a(x - 1)^2 + 6$ passes through (2,4)

$\therefore a(2 - 1)^2 + 6 = 4$

$\therefore a = -2; b = 1; c = 6$

7 **a** $y = a(x - b)^2 + c$

The vertex is at (0,5) so

$y = (x - 0)^2 + 5$

$y = ax^2 + 5$

$y = ax^2 + 5$ passes through (0,4)

$\therefore a(4)^2 + 5 = 0$

$\therefore a = -\frac{5}{16}$

$y = -\frac{5x^2}{16} + 5$

b $y = a(x - b)^2 + c$

The vertex is at (0,0) so $y = ax^2$

$y = ax^2$ passes through $(-3, 9)$

$\therefore a(-3)^2 = 9$

$\therefore a = 1$

$y = x^2$

c $y = ax^2 + bx + c$

This is of the form $y = ax(x + 7)$

For (4,4)

$4 = a(4)(4 + 7)$

$4 = 44a$

Therefore $a = \frac{1}{11}$

And the rule is $y = \frac{x^2}{11} + \frac{7x}{11}$

d $y = a(x + b)(x + c)$

From x-intercepts, a and b are -1 and -3:

$y = a(x - 1)(x - 3)$

From y-intercept,

$a(-1)(-3) = 3, \therefore a = 1$

$\therefore y = (x - 1)(x - 3)$

e $y = a(x - b)^2 + c$

The vertex is at $(-1, 5)$ so

$y = a(x + 1)^2 + 5$

$y = a(x + 1)^2 + 5$ passes through $(1, 0)$

$\therefore a(2)^2 + 5 = 0$

$\therefore a = -\frac{5}{4}$

$y = -\frac{5}{4}(x + 1)^2 + 5$

OR $y = -\frac{5}{4}x^2 - \frac{5}{2}x + \frac{15}{4}$

Check with 3rd pt: $y = 0$ at $x = -3$

f $y = a(x - b)^2 + c$

The vertex is at (2, 2) so

$y = a(x + 2)^2 + 2$

$y = a(x - 2)^2 + 2$ passes through (0,6)

$\therefore a(-2)^2 + 2 = 6$

$\therefore a = 1$

$y = (x - 2)^2 + 2$

OR $y = x^2 - 4x + 6$

Check with 3rd pt: $y = 6$ at $x = 4$

8 $y = a(x - b)^2 + c$

The vertex is at $(-1, 3)$ so

$y = a(x + 1)^2 + 3$

$y = a(x + 1)^2 + 3$ passes through (3,8)

$\therefore a(4)^2 + 3 = 8$

$\therefore 16a = 5, \therefore a = \frac{5}{16}$

$y = \frac{5}{16}(x + 1)^2 + 3$

9 $y = a(x + b)(x + c)$

From x intercepts, a and b are 6 and -3:

$y = a(x - 6)(x + 3)$

Using (1, 10):

$a(1 - 6)(1 + 3) = 10$

$\therefore -20a = 10, \therefore a = -\frac{1}{2}$

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$\therefore y = -\frac{1}{2}(x - 6)(x + 3)$

OR $y = -\frac{1}{2}(x^2 - 3x - 18)$

10 $y = a(x - b)^2 + c$

The vertex is at $(-1, 3)$ so

$y = a(x + 1)^2 + 3$

$y = a(x + 1)^2 + 3$ passes through (0,4)

$\therefore a + 3 = 4, \therefore a = 1$

$y = (x + 1)^2 + 3$

OR $y = x^2 + 2x + 4$

11 The suspension cable forms a parabola:

$y = a(x - b)^2 + c$

The vertex is at (90, 30) so

$y = a(x - 90)^2 + 30$

When $x = 0, y = 75$, so:

$y = a(-90)^2 + 30 = 75$

$\therefore 8100a = 45, \therefore a = \frac{1}{180}$

$y = \frac{1}{180}(x - 90)^2 + 30$

$\therefore y = \frac{1}{180}x^2 - x + 75$

12 $y = 2(x - b)^2 + c$

$(1, -2) = \text{TP (vertex)} = (b, c)$

$\therefore y = 2(x - 1)^2 - 2$

OR $y = 2x^2 - 4x$

13 $y = a(x - b)^2 + c$

$(1, -2) = \text{TP (vertex)} = (b, c)$

$\therefore y = a(x - 1)^2 - 2$

Using the point (3,2),

$a(3 - 1)^2 - 2 = 2$

$\therefore 4a - 2 = 2, \therefore a = 1$

$\therefore y = (x - 1)^2 - 2$

OR $y = x^2 - 2x - 1$

14 a $y = \frac{1}{3}(x + 4)(8 - x)$

Squared term is negative, so inverted parabola; must be **A** or **C**.

The x-intercepts must be at 8 and -4, so **C**.

b $y = x^2 - x + 2$

Positive squared term gives an upright parabola; must be **B** or **D**.

The y-intercept is at (0,2) so only **B** is possible

c $y = -10 + 2(x - 1)^2$

Positive squared term gives an upright parabola; must be **B** or **D**.

Vertex is at $(1, -10)$ so **D**.

d $y = \frac{1}{2}(9 - x^2)$

Squared term is negative so inverted parabola; must be **A** or **C**.

Vertex at $\left(0, \frac{9}{2}\right)$ so **A**..

15 a $ax^2 + 2x + a$

$$= a\left(x^2 + \frac{2}{a}x + 1\right)$$

$$= a\left(x^2 + \frac{2}{a}x + \frac{1}{a^2} - \frac{1}{a^2} + 1\right)$$

$$= a\left(\left(x + \frac{1}{a}\right)^2 - \frac{1}{a^2} + 1\right)$$

$$= a\left(x + \frac{1}{a}\right)^2 - \frac{1}{a} + a$$

b Turning point: $\left(-\frac{1}{a}, a - \frac{1}{a}\right)$

c Perfect square when $a - \frac{1}{a} = 0$

That is, when $a^2 = 1$

$\therefore a = \pm 1$

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d Two solutions when $1 - a^2 > 0$, That is, $-1 < a < 1$

16 $y = a(x - b)^2 + c$
$(2, 2) = \text{TP (vertex)} = (b, c)$
$\therefore y = a(x - 2)^2 + 2$
Using the point $(4, -6)$,
$a(4 - 2)^2 + 2 = -6$
$\therefore 4a + 2 = -6 \therefore, a = -2$
$\therefore y = -2(x - 2)^2 + 2$
OR $y = -2x^2 + 8x - 6$

17 **(a)** has x-intercepts at 0 and 10, so
$y = a(x - b)(x - c)$
$b = 0, c = 10$
$\therefore y = ax(x - 10)$
$a > 0$ because upright parabola

(b) has x-intercepts at -4 and 10, so
$y = a(x - b)(x - c)$
$b = -4, c = 10$
$\therefore y = a(x + 4)(x - 10)$
$a < 0$ because upright parabola

(c) has no x intercepts, so
$y = a(x - b)^2 + c$
Vertex is at (6,6) so $b = c = 6$
$y = a(x - 6)^2 + 6$
y-intercept is at (0,8):
$a(0 - 6)^2 + 6 = 8$
$\therefore 36a = 2, \therefore a = \dfrac{1}{18}$
$y = \dfrac{1}{18}(x - 6)^2 + 6$

(d) $y = a(x - b)^2 + c$
Vertex is at (8,0) so $b = 8; c = 0$
$y = a(x - 8)^2$
$a < 0$ because inverted parabola

18 **(a)** $y = ax^2 + x + b$
Using $D = (0, 2), b = 2$
Using $A = (2, 3)$,

$$4a + 2 + 2' = 3$$
$$\therefore \qquad a = -\frac{1}{4}$$

$y = -\dfrac{1}{4}x^2 + x + 2$

(b) $y = ax^2 + x + b$
Using $C = (0, -5), b = -5$
Using $B = (2, 1)$,

$$4a + 2 - 5 = 1$$
$$\therefore \qquad a = 1$$

$y = x^2 + x - 5$

19 $r = at^2 + bt + c$
(1) $t = 5, r = 3 : 25a + 5b + c = 3$
(2) $t = 9, r = 6 : 81a + 9b + c = 6$
(3) $t = 13, r = 5 : 169a + 13b + c = 5$
(2) − **(1)** gives $56a + 4b = 3$
(3) − **(2)** gives $88a + 4b = -1$
From these 2 equations, $32a = -4$
so $a = -\dfrac{1}{8}$
Substitute into $56a + 4b = 3$:
$-\dfrac{56}{8} + 4b = 3$
$\therefore 4b = 10, \therefore b = \dfrac{5}{2}$
Substitute into **(1)**:
$-\dfrac{25}{8} + \dfrac{25}{2} + c = 3$
$\therefore c = -\dfrac{51}{8}$
$r = -\dfrac{1}{8}(t^2 - 20t + 51)$
$\therefore r = -\dfrac{1}{8}t^2 + \dfrac{5}{2}t - \dfrac{51}{8}$

20 **a** $y = (x - 4)^2 - 3$
Upright parabola, vertex $(4, -3)$ so **B**

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b $y = 3 - (x - 4)^2$

Inverted parabola, vertex (4,3) so **D**

21 a $y = ax^2 + bx + c$

$(-2, -1): 4a - 2b + c = -1 \ldots$ **(1)**

$(1, 2) : a + b + c = 2 \ldots$ **(2)**

$(3, -16)\ 9a + 3b + c = -16 \ldots$ **(3)**

(2) − **(1)** gives $3b - 3a = 3$ or

$b = a + 1$

(3) − **(2)** gives $8a + 2b = -18$ or

$b = -9 - 4a$

$b = a + 1 = -9 - 4a$

$\therefore 5a = -10, \therefore a = -2; b = -1$

Substitute into **(1)**:

$-8 + 2 + c = -1 \therefore c = 5$

$y = -2x^2 - x + 5$

b $y = ax^2 + bx + c$

$(-1, -2) : a - b + c = -2 \ldots$ **(1)**

$(1, -4) : a + b + c = -4 \ldots$ **(2)**

$(3, 10): 9a + 3b + c = 10 \ldots$ **(3)**

(2) − **(1)** gives $2b = -2$ or $b = -1$

(3) − **(2)** gives $8a + 2b = 14$

$\therefore 8a = 16 \therefore a = 2$

Substitute into **(2)**:

$2 - 1 + c = -4, \therefore c = -5$

$y = 2x^2 - x - 5$

c $y = ax^2 + bx + c$

$(-3, 5) : 9a - 3b + c = 5 \ldots$ **(1)**

$(3, 20): 9a + 3b + c = 20 \ldots$ **(2)**

$(5, 57): 25a + 5b + c = 57 \ldots$ **(3)**

(2) − **(1)** gives $6b = 15$ or $b = \frac{5}{2}$

(3) − **(2)** gives $16a + 2b = 37$

$\therefore 16a + 5 = 37, \therefore a = 2$

Substitute into **(2)**:

$18 + \frac{15}{2} + c = 20, \therefore c = -\frac{11}{2}$

$y = 2x^2 + \frac{5}{2}x - \frac{11}{2}$

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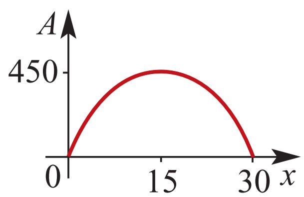
Solutions to Exercise 3L

1 **a** Width of paddock = x;
length = $60 - 2x$
$\therefore A = x(60 - 2x) = 60x - 2x^2$

b

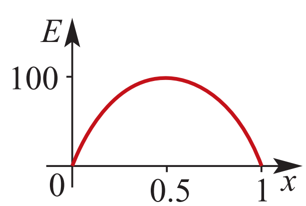


c Maximum area is at the vertex,
i.e. when $x = 15$ (halfway between the two x-intercepts).
When $x = 15$,
$A = 15(60 - 30) = 450\ \text{m}^2$

2 $A = x(10 - x)$; Maximum area = 25 m^2

3 $E = 400(x - x^2)$

a



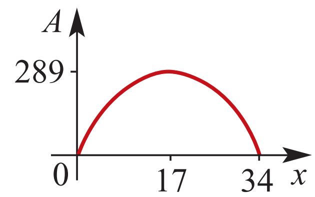
b Zero efficiency rating when $x = 0$ and 1

c Maximum efficiency rating is at the vertex where $x = 0.5$

d $E \geq 70$ when $400x - 400x^2 - 70 \geq 0$
i.e. $\{x: 0.23 < x < 0.77\}$

4 **a** If x cm = length of the rectangle, then
$2x + 2w = 68, \therefore w = 34 - x$
$A = lw = x(34 - x) = 34x - x^2$

b



c Maximum area formed is at the vertex where $x = 17$:
$A = 17(34 - 17) = 172 = 289\ \text{cm}^2$

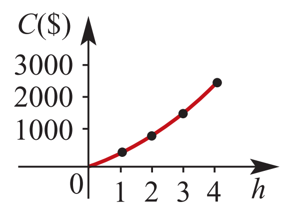
5 **a** $4x + 10y = 80$

b **i** $A = 1.64x^2 - 25.6x + 256$

ii 31.22 and 48.78

6 $C = 240h + 100h^2$

a



p

h is most unlikely to be less than zero in an alpine area, and will be less than 10, since the highest mountain on Earth is less high than 10 km above sea level.

b C's maximum value is at the top of the highest peak in the mountains (8.848 km for Mt Everest).

c For $h = 2.5$ km,
$C = 240(2.5) + 100(2.5)^2$
$= 600 + 625 = \$1225$

7 $T = 290(8t - 0.5t^2 - 1.4)$

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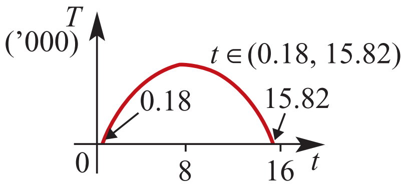
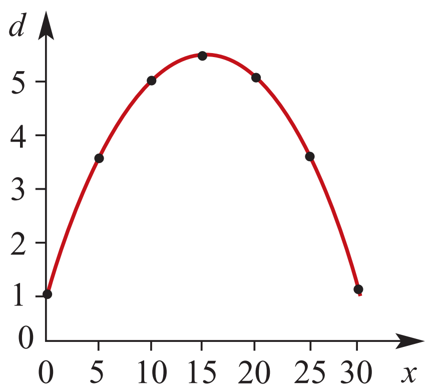
a 

Solving $8t - 0.5t^2 - 1.4 = 0$ with a CAS gives $t = 0.18, 15.82$. So $t \in (0.18, 15.82)$

b At the vertex $t = 8$, $T = 8874$ units

8 **a** $d = 1 + \frac{3}{5}x - \frac{1}{50}x^2, x \geq 0$

b 

c **i** Maximum height = 5.5 m

ii When $y = 2$, $x = 15 \pm 5\sqrt{7}$
($x = 1.9$ m or 28.1 m)

iii y-intercept = 1, so it was struck 1 metre above the ground.

9 The x-intercepts are 0 and 1.5

So $y = ax(x - 1.5)$

A is the point $(0.75, 0.6)$ so:

$0.6 = \text{a}(0.75)(0.75 - 1.5)$

$\frac{3}{5} = -\frac{9}{16}a$

So $a = -16$

$y = -\frac{16}{15}x^2 + \frac{8}{5}x$

$a = -\frac{16}{15},\ b = \frac{8}{5}, c = 0$

10 **a** $s = at^2 + bt + c$

$900a + 30b + c = 7.2 \ldots \mathbf{(1)}$

$22\,500a + 150b + c = 12.5 \ldots \mathbf{(2)}$

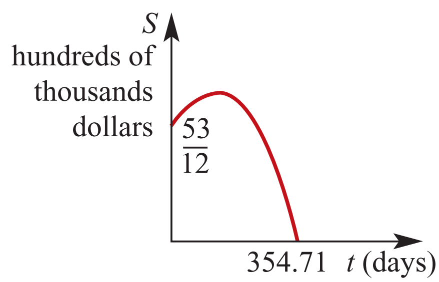
$90\,000a + 300b + c = 6 \ldots \mathbf{(3)}$

$\mathbf{(2)} - \mathbf{(1)}$ gives $21600a + 120b = 5.3$

$\mathbf{(3)} - \mathbf{(2)}$ gives $67500a + 150b = -6.5$

Using a CAS, the solution is:

$a = -\frac{7}{21\,600}; b = \frac{41}{400}; c = \frac{53}{12}$

b 

c **i** $t = 180$, $s = 12.36$, so spending is estimated at \$1 236 666.

ii $t = 350$, $s = 0.59259$, so spending is estimated at \$59259

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Solutions to Technology-free questions

1 a $x^2 + 9x + \frac{81}{4} = \left(x + \frac{9}{2}\right)^2$

b $x^2 + 18x + 81 = (x + 9)^2$

c $x^2 - \frac{4}{5}x + \frac{4}{25} = \left(x - \frac{2}{5}\right)^2$

d $x^2 + 2bx + b^2 = (x + b)^2$

e $9x^2 - 6x + 1 = (3x - 1)^2$

f $25x^2 + 20x + 4 = (5x + 2)^2$

2 a $-3(x - 2) = -3x + 6$

b $-a(x - a) = -ax + a^2$

c $(7a - b)(7a + b) = 49a^2 - b^2$

d $(x + 3)(x - 4) = x^2 + 3x - 4x - 12$
$= x^2 - x - 12$

e $(2x + 3)(x - 4) = 2x^2 + 3x - 8x - 12$
$= 2x^2 - 5x - 12$

f $(x + y)(x - y) = x^2 - y^2$

g $(a - b)(a^2 + ab + b^2)$
$= a^3 - a^2b + a^2b - ab^2 + ab^2 - b^3$
$= a^3 - b^3$

h
$(2x + 2y)(3x + y) = 6x^2 + 6xy + 2xy + 2y^2$
$= 6x^2 + 8xy + 2y^2$

i $(3a + 1)(a - 2) = 3a^2 + a - 6a - 2$
$= 3a^2 - 5a - 2$

j
$(x + y)^2 - (x - y)^2$
$= ((x + y) - (x - y))((x + y) + (x - y))$
$= (2y)(2x) = 4xy$

k $u(v + 2) + 2v(1 - u)$
$= uv + 2u + 2v - 2uv$
$= 2u + 2v - uv$

l $(3x + 2)(x - 4) + (4 - x)(6x - 1)$
$= (3x + 2)(x - 4) + (x - 4)(1 - 6x)$
$= (x - 4)(3x + 2 + 1 - 6x)$
$= (x - 4)(3 - 3x)$
$= -3x^2 + 15x - 12$

3 a $4x - 8 = 4(x - 2)$

b $3x^2 + 8x = x(3x + 8)$

c $24ax - 3x = 3x(8a - 1)$

d $4 - x^2 = (2 - x)(2 + x)$

e $au + 2av + 3aw = a(u + 2v + 3w)$

f $4a^2b^2 - 9a^4 = a^2(4b^2 - 9a^2)$
$= a^2(2b - 3a)(2b + 3a)$

g $1 - 36x^2a^2 = (1 - 6ax)(1 + 6ax)$

h $x^2 + x - 12 = (x + 4)(x - 3)$

i $x^2 + x - 2 = (x + 2)(x - 1)$

j $2x^2 + 3x - 2 = (2x - 1)(x + 2)$

k $6x^2 + 7x + 2 = (3x + 2)(2x + 1)$

l $3x^2 - 8x - 3 = (3x + 1)(x - 3)$

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m $3x^2 + x - 2 = (3x - 2)(x + 1)$

n $6a^2 - a - 2 = (3a - 2)(2a + 1)$

o $6x^2 - 7x + 2 = (3x - 2)(2x - 1)$

4 a
$$\begin{aligned} x^2 - 2x - 15 &= 0 \\ (x - 5)(x + 3) &= 0 \\ x = 5 \text{ or } x &= -3 \end{aligned}$$

b
$$\begin{aligned} x^2 - 9x &= 0 \\ x(x - 9) &= 0 \\ x = 0 \text{ or } x &= 9 \end{aligned}$$

c
$$\begin{aligned} 2x^2 - 10x + 12 &= 0 \\ 2(x^2 - 5x + 6) &= 0 \\ (x - 3)(x - 2) &= 0 \\ x = 3 \text{ or } x &= 2 \end{aligned}$$

d
$$\begin{aligned} x^2 - 24x - 25 &= 0 \\ (x - 25)(x + 1) &= 0 \\ x = 25 \text{ or } x &= -1 \end{aligned}$$

e
$$\begin{aligned} 3x^2 + 15x + 18 &= 0 \\ 3(x^2 + 5x + 6) &= 0 \\ (x + 3)(x + 2) &= 0 \\ x = -3 \text{ or } x &= 2 \end{aligned}$$

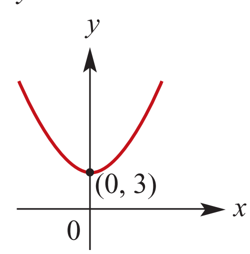
f
$$\begin{aligned} x^2 - 12x + 36 &= 0 \\ (x - 6)(x - 6) &= 0 \\ x &= 6 \end{aligned}$$

g
$$\begin{aligned} 2x^2 - 5x - 3 &= 0 \\ 2x^2 - 6x + x - 3 &= 0 \\ 2x(x - 3) + (x - 3) &= 0 \\ (x - 3)(2x + 1) &= 0 \\ x = 3 \text{ or } x &= -\frac{1}{2} \end{aligned}$$

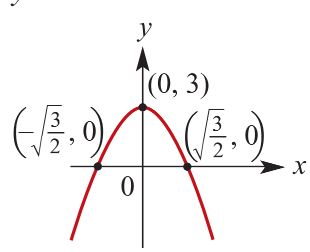
h
$$\begin{aligned} 12x^2 - 8x - 15 &= 0 \\ 12x^2 - 18x + 10x - 15 &= 0 \\ 6x(2x - 3) + 5(2x - 3) &= 0 \\ (6x + 5)(2x - 3) &= 0 \\ x = -\frac{5}{6} \text{ or } x &= \frac{3}{2} \end{aligned}$$

i
$$\begin{aligned} 5x^2 + 7x - 12 &= 0 \\ 5x^2 + 12x - 5x - 12 &= 0 \\ x(5x + 12) - (5x + 12) &= 0 \\ (5x + 12)(x - 1) &= 0 \\ x = 1 \text{ or } x &= -\frac{12}{5} \end{aligned}$$

5 a $y = 2x^2 + 3$



b $y = -2x^2 + 3$



c $y = 2(x - 2)^2 + 3$

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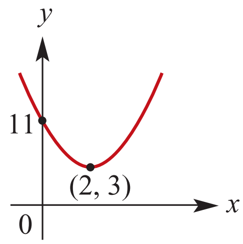
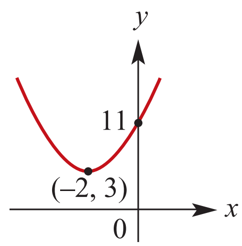
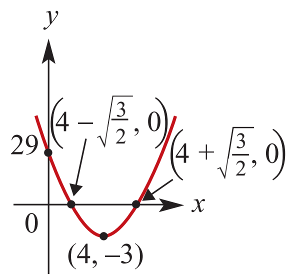


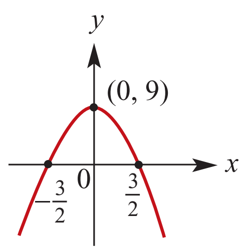
d $y = 2(x + 2)^2 + 3$



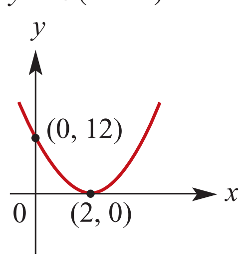
e $y = 2(x - 4)^2 - 3$



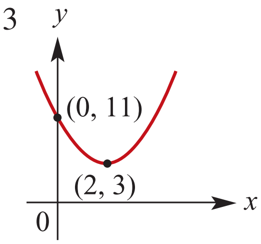
f $y = 9 - 4x^2$



g $y = 3(x - 2)^2$



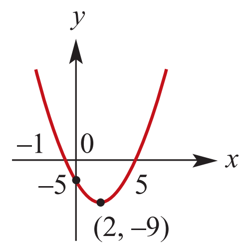
h $y = 2(2 - x)^2 + 3$



6 a $y = x^2 - 4x - 5$

$= x^2 - 4x + 4 - 9$

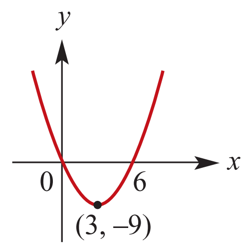
$\therefore y = (x - 2)^2 - 9$



b $y = x^2 - 6x$

$= x^2 - 6x + 9 - 9$

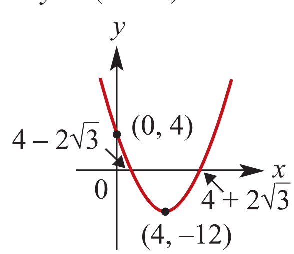
$\therefore y = (x - 3)^2 - 9$



c $y = x^2 - 8x + 4$

$= x^2 - 8x + 16 - 12$

$\therefore y = (x - 4)^2 - 12$



d $y = 2x^2 + 8x - 4$

$= 2(x^2 + 4x - 2)$

$\therefore y = 2(x^2 + 4x + 4 - 6)$

$\therefore y = 2(x + 2)^2 - 12$

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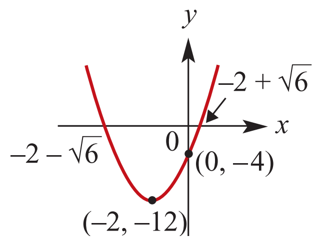
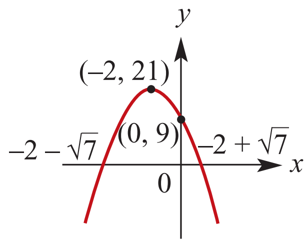


e $y = -3x^2 - 12x + 9$

$= -3(x^2 + 4x - 3)$

$= -3(x^2 + 4x + 4 - 7)$

$\therefore y = -3(x + 2)^2 + 21$

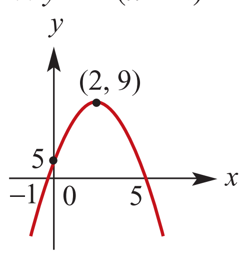


f $y = -x^2 + 4x + 5$

$\therefore y = -(x^2 - 4x - 5)$

$\therefore y = -(x^2 - 4x + 4 - 9)$

$\therefore y = -(x - 2)^2 + 9$



7 **i** y-intercepts are at $(0, c)$ in each case; x-intercepts are where the factors equal zero.

ii The axis of the symmetry is at $x = -\dfrac{b}{2a}$

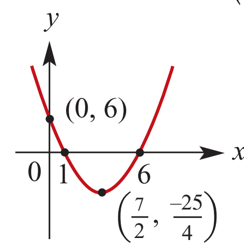
iii The turning point is on the axis of symmetry with the y-value for that point.

a $y = x^2 - 7x + 6 = (x - 6)(x - 1)$

i $(0, 6)$, $(6,0)$ and $(1,0)$

ii $x = -\dfrac{b}{2a} = \dfrac{7}{2}$

iii Turning point at $\left(\dfrac{7}{2}, -\dfrac{25}{4}\right)$



b $y = -x^2 - x + 12$

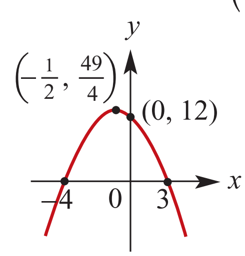
$= -(x^2 + x - 12)$

$= -(x + 4)(x - 3)$

i $(0, 12)$, $(-4, 0)$ and $(3, 0)$

ii $x = -\dfrac{b}{2a} = -\dfrac{1}{2}$

iii Turning point at $\left(-\dfrac{1}{2}, \dfrac{49}{4}\right)$



c $y = -x^2 + 5x + 14$

$= -(x^2 - 5x - 14)$

$= -(x - 7)(x + 2)$

i $(0, 14)$, $(-2, 0)$ and $(7, 0)$

ii $x = -\dfrac{b}{2a} = \dfrac{5}{2}$

iii turning point at $\left(\dfrac{5}{2}, \dfrac{81}{4}\right)$

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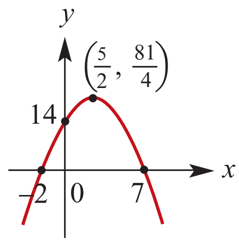
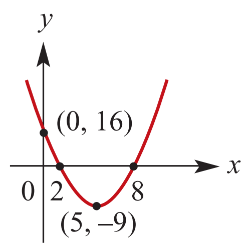


d $y = x^2 - 10x + 16 = (x - 8)(x - 2)$

i (0,16), (2,0) and (8,0)

ii $x = -\dfrac{b}{2a} = 5$

iii turning point at $(5, -9)$

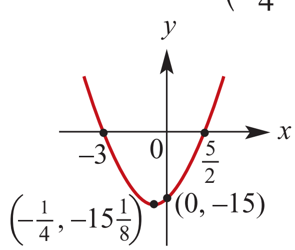


e $y = 2x^2 + x - 15 = (2x - 5)(x + 3)$

i $(0, -15), \left(\dfrac{5}{2}, 0\right)$ and $(-3, 0)$

ii $x = -\dfrac{b}{2a} = -\dfrac{1}{4}$

iii Turning point at $\left(-\dfrac{1}{4}, -\dfrac{121}{8}\right)$

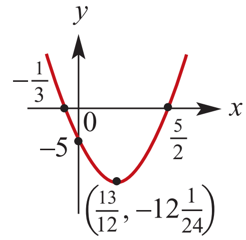


f $y = 6x^2 - 13x - 5 = (3x + 1)(2x - 5)$

i $(0, -5), \left(\dfrac{5}{2}, 0\right)$ and $\left(-\dfrac{1}{3}, 0\right)$

ii $x = -\dfrac{b}{2a} = \dfrac{13}{12}$

iii Turning point at $\left(\dfrac{13}{12}, -\dfrac{289}{24}\right)$

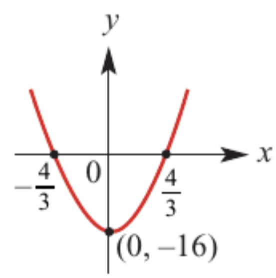


g $y = 9x^2 - 16 = (3x - 4)(3x + 4)$

i $(0, -16), \left(\dfrac{4}{3}, 0\right)$ and $\left(-\dfrac{4}{3}, 0\right)$

ii $x = -\dfrac{b}{2a} = 0$

iii Turning point at $(0, -16)$

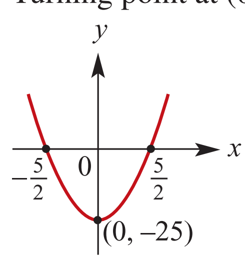


h $y = 4x^2 - 25 = (2x - 5)(2x + 5)$

i $(0, -25), \left(\dfrac{5}{2}, 0\right)$ and $\left(-\dfrac{5}{2}, 0\right)$

ii $x = -\dfrac{b}{2a} = 0$

iii Turning point at $(0, -25)$



8 $(5p - 1)x^2 - 4x + 2p - 1$

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$$\begin{aligned}\Delta &= 16 - 4(2p-1)(5p-1)\\ &= 16 - 4(10p^2 - 7p + 1)\\ &= 16 - 40p^2 + 28p - 4\\ &= 12 - 40p^2 + 28p\\ &= -4(10p^2 - 7p - 3)\\ &= -4(10p-3)(p-1)\end{aligned}$$

$\Delta = 0 \Rightarrow p = -\dfrac{3}{10}$ or $p = 1$

9 **a** $x^2 > x$

$\Leftrightarrow x^2 - x > 0$

$\Leftrightarrow x(x-1) > 0$

$\Leftrightarrow x < 0$ or $x > 1$

b $(x+2)^2 \le 34$

$\Leftrightarrow (x+2)^2 - 34 \le 0$

$\Leftrightarrow (x + 2 - \sqrt{34})(x + 2 + \sqrt{34}) \le 0$

$\Leftrightarrow -2 - \sqrt{34} \le x \le -2 + \sqrt{34}$

c $3x^2 + 5x - 2 \le 0$

$\Leftrightarrow 3x^2 + 6x - x - 2 \le 0$

$\Leftrightarrow 3x(x+2) - (x+2) \le 0$

$\Leftrightarrow (x+2)(3x-1) \le 0$

$\Leftrightarrow -2 \le x \le \dfrac{1}{3}$

d $-2x^2 + 13x \ge 15$

$\Leftrightarrow -2x^2 + 13x - 15 \ge 0$

$\Leftrightarrow -(2x^2 - 13x + 15) \ge 0$

$\Leftrightarrow 2x^2 - 13x + 15 \le 0$

$\Leftrightarrow (2x-3)(x-5) \le 0$

$\Leftrightarrow \dfrac{3}{2} \le x \le 5$

10 $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

a $x^2 + 6x + 3 = 0$

$\therefore x = \dfrac{-6 \pm \sqrt{36 - 12}}{2}$

$= \dfrac{-6 \pm 2\sqrt{6}}{2} = -3 \pm \sqrt{6}$

$x = -0.55, -5.45$ from calculator

b $x^2 + 9x + 12 = 0$

$\therefore x = \dfrac{-9 \pm \sqrt{81 - 48}}{2}$

$= \dfrac{-9 \pm \sqrt{33}}{2}$

$x = -1.63, -7.37$ from calculator

c $x^2 - 4x + 2 = 0$

$\therefore x = \dfrac{4 \pm \sqrt{16 - 8}}{2}$

$= 2 \pm \sqrt{2}$

$x = 3.414, 0.586$ from calculator

d $2x^2 + 7x + 2 = 0$

$\therefore x = \dfrac{-7 \pm \sqrt{49 - 16}}{4}$

$= \dfrac{-7 \pm \sqrt{33}}{2}$

$x = -0.314, -3.186$ from calculator

e $2x^2 + 7x + 4 = 0$

$\therefore x = -7 \pm \sqrt{49 - 32}$

$= \dfrac{-7 \pm \sqrt{17}}{e}$

$x = -0.719, -2.7816$ from calculator

f $3x^2 + 9x - 1 = 0$

$\therefore x = \dfrac{-9 \pm \sqrt{81 + 12}}{6}$

$= \dfrac{-9 \pm \sqrt{93}}{6}$

$x = -0.107, 3.107$ from calculator

11 $y = a(x-b)(x-c)$

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Assume the graph cuts the axis at (0,0) and (5,0), $b = 0$ and $c = 5$
Using (6, 10): $y = ax(x - 5) = 10$

$$\therefore ax^2 - 5ax - 10 = 0$$
$$36a - 30a - 10 = 0$$
$$6a - 10 = 0$$
$$\therefore \quad a = \frac{5}{3}$$

$$y = \frac{5}{3}x(x - 5)$$

12 A parabola has the same shape as $y = 3x^2$, but its vertex is at (5,2).
$y = 3(x - 5)^2 + 2$

13 $(2m - 3)x^2 + (5m - 1)x + (3m - 2) = 0$

$$\begin{aligned}\Delta &= (5m - 1)^2 - 4 \times (2m - 3)(3m - 2)\\ &= 25m^2 - 10m + 1 - 4(6m^2 - 13m + 6)\\ &= m^2 + 42m - 23\\ &= m^2 + 42m + 441 - 441 - 23\\ &= (m + 21)^2 - 464\end{aligned}$$

$$\begin{aligned}\Delta > 0 &\Leftrightarrow (m - 21)^2 - 464 > 0\\ &\Leftrightarrow (m + 21 - 4\sqrt{29})(m + 21 + 4\sqrt{29} > 0\\ &\Leftrightarrow x < -21 - 4\sqrt{29} \text{ or } m > -21 + 4\sqrt{29}\end{aligned}$$

14 Let a and b be the numbers.
$a + b = 30 \therefore b = 30 - a$
$P = ab = a(30 - a)$
Maximum occurs when $a = 15$.
Maximum product is 225

15 The vertex is at (1,5).
$\therefore y = a(x - 1)^2 + 5$
Using (2, 10):
$y = a(2 - 1)^2 + 5 = 10$
$\therefore a = 5$
$\therefore y = 5(x - 1)^2 + 5$
OR $y = 5x^2 - 10x + 10$

16 a $y = 2x + 3$ and $y = x^2$ meet where:
$x^2 = 2x + 3, \therefore x^2 - 2x - 3 = 0$
$\therefore (x - 3)(x + 1) = 0$
Where $x = 3, y = 9$; where $x = -1, y = 1$
Curves meet at (3,9) and $(-1, 1)$.

b $y = 8x + 11$ and $y = 2x^2$ meet where:
$2x^2 = 8x + 11$

$$\therefore x = 2x^2 - 8x - 11 = 0$$
$$\therefore x = \frac{8 \pm \sqrt{64 + 88}}{4}$$
$$\therefore x = 2 \pm \frac{\sqrt{38}}{2}$$

Where $x = 2 - \dfrac{\sqrt{38}}{2}$, $y = 27 - 4\sqrt{38}$
Where $x = 2 + \dfrac{\sqrt{38}}{2}$, $y = 27 + 4\sqrt{38}$
From calculator: curves meet at $(-1.08, 2.34)$ and (5.08,51.66).

c $y = 3x^2 + 7x$ and $y = 2$ meet where:
$3x^2 + 7x = 2$
$\therefore 3x^2 + 7x - 2 = 0$

$$\therefore x = \frac{-7 \pm \sqrt{49 - 24}}{6}$$
$$\therefore x = \frac{-7 \pm \sqrt{73}}{6}$$

Curves meet at $\left(\dfrac{-7 \pm \sqrt{73}}{6}, 2\right)$.
From calculator: (0.26, 2) and $(-2.62, 2)$

d $y = 2x^2$ and $y = 2 - 3x$ meet where
$2x^2 = 2 - 3x$
$\therefore 2x^2 + 3x - 2 = 0$
$\therefore (2x - 1)(x + 2) = 0, \therefore x = \dfrac{1}{2}, -2$

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Where $x = \frac{1}{2}, y = \frac{1}{2}$; where $x = -2, y = 8$

Curves meet at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $(-2, 8)$.

17 **a** Equation is of the form
$y = k(x + 4)(x - 1)$
When $x = -1, y = -12$
Hence, $-12 = k(3)(-2)$
$\therefore k = 2$
$\therefore y = 2(x + 4)(x - 1)$

b Equation is of the form
$y = a(x + 1)^2 + 3$
When $x = 1, y = -5$
Hence, $-5 = a(4) + 3$
$\therefore a = -2$
$\therefore y = -2(x + 1)^2 + 3$

c Equation is of the form
$y = ax^2 + bx - 3$
When $x = 1, y = -3$
$\therefore -3 = a + b - 3 \ldots (1)$
When $x = -1, y = 1$
$\therefore 1 = a - b - 3 \ldots (2)$
Simplifying the equations
$a + b = 0 \ldots (1')$
$a - b = 4 \ldots (2')$
Add$(1')$ and $(2')$
$2a = 4$
$a = 2, b = -2$
$\therefore y = 2x^2 - 2x - 3$

18

$S = 9.42r^2 + 6(6.28)r = 125.6$
$\therefore 9.42r^2 + 37.68r - 125.6 = 0$

$$\therefore r = \frac{-37.68 \pm \sqrt{37.68^2 + 4(125.6)9.42}}{2(9.42)}$$

$$\therefore r = -2 \pm \frac{\sqrt{6152.4}}{18.84}$$

Since r > 0,

$$r = -2 + \frac{\sqrt{6152.4}}{18.84} = 2.16 \text{ m}$$

19 **a** $2x^2 + mx + 1 = 0$ has exactly one solution where $\Delta = 0$:
$\Delta = m^2 - 8 = 0, \therefore m^2 = 8$
$\therefore m = \pm 2\sqrt{2}$

b $x^2 - 4mx + 20 = 0$ has real solutions where $\Delta \geq 0$:
$\Delta = 16m^2 - 80 \geq 0$
$\therefore m^2 \geq 5$
Solution set:
$\{m : m \leq -\sqrt{5}\} \cap \{m : m \geq \sqrt{5}\}$

20 $y = x^2 + bx$

a When $y = 0, x(x + b) = 0$
$x = 0$ or $x = -b$

b Completing the square
$y = x^2 + bx + \frac{b^2}{4} - \frac{b^2}{4}$
$\therefore y = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4}$
The vertex is at $\left(-\frac{b}{2}, -\frac{b^2}{4}\right)$

c $x^2 + bx = x$

i $x^2 + (b - 1)x = 0$
$\therefore x(x + (b - 1)) = 0$
$\therefore x = 0$ or $x = 1 - b$

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The coordinates of the points of intersection are
$(0, 0)$ and $(1 - b, 1 - b)$

ii There is one point of intersection when $b = 1$.

iii There are two points of intersection when $b \neq 1$.

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Solutions to multiple-choice questions

1 A $12x^2 + 7x - 12 = (3x + 4)(4x - 3)$

2 C $x^2 - 5x - 14 = 0$
$\therefore (x - 7)(x + 2) = 0$
$\therefore x = -2, 7$

3 C $y = 8 + 2x - x^2$
$= 9 - (x^2 - 2x + 1)$
$= 9 - (x - 1)^2$
Maximum value of y is 9 when $x = 1$

4 E $y = 2x^2 - kx + 3$
If the graph of touches the x-axis then $\Delta = 0$:
$\Delta = (-k)^2 - 24 = 0$
$\therefore k^2 = 24$
$\therefore k = \pm\sqrt{24} = \pm 2\sqrt{6}$

5 B $x^2 - 56 = x$
$\therefore x^2 - x - 56 = 0$
$\therefore (x - 8)(x + 7) = 0$
$\therefore x = -7, 8$

6 C $x + 3x - 10$
$\Delta = 3^2 + 40 = 49$

7 E $y = 3x^2 + 6x - 1$
$= 3x + 6x + 3 - 4$
$= 3(x + 1)^2 - 4$
TP is at $(-1, -4)$.

8 E $5x^2 - 10x - 2$
$= 5(x^2 - 2x + 1) - 7$
$= 5(x - 1)^2 - 7$

9 D If two real roots of $mx^2 + 6x - 3 = 0$ exist, then $\Delta > 0$:
$\Delta = 6^2 + 12m = 12(m + 3)$
$m > -3$

10 A $6x^2 - 8xy - 8y^2$
$= (3x + 2y)(2x - 4y)$

11 B $y = x^2 - ax + \dfrac{a^2}{4} - \dfrac{a^2}{4}$
$y = \left(x - \dfrac{a}{2}\right) - \dfrac{a^2}{4}$
Therefore vertex $\left(\dfrac{a}{2}, -\dfrac{a^2}{4}\right)$

12 E $x^2 > b^2$
$(x - b)(x + b) > 0$
But $b < 0$ and therefore $-b > 0$
$(x - b)(x + b) > 0 \Leftrightarrow x > -b$ or $x < b$

13 D $\Delta = 4a^2 - 4b$
One solution when $\Delta = 0$
$\therefore a^2 = b$
$\therefore a = \pm\sqrt{b}$

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Solutions to extended-response questions

1 a The turning point (h, k) is $\left(25, \frac{9}{2}\right)$

$\therefore \quad y = a(x - 25)^2 + \frac{9}{2}$

When $x = 0$, $\quad y = 0$

$\therefore \quad 0 = a(0 - 25)^2 + \frac{9}{2}$

$\therefore \quad 0 = 625a + \frac{9}{2}$

$\therefore \quad 625a = \frac{-9}{2}$

$\therefore \quad a = \frac{-9}{1250}$

Hence the equation for the parabola is $y = \frac{-9}{1250}(x - 25)^2 + \frac{9}{2}$, for $0 \le x \le 50$.
This can also be written as $y = -0.0072x(x - 50)$ [the intercept form].

b

x	0	5	10	15	20	25	30	35	40	45	50
y	0	1.62	2.88	3.78	4.32	4.5	4.32	3.78	2.88	1.62	0

You can find these values using a CAS calculator, or:

When $x = 10$, $y = \frac{-9}{1250}(10 - 25)^2 + \frac{9}{2}$

$= \frac{-9}{1250} \times 225 + \frac{9}{2}$

$= \frac{-81}{50} + \frac{225}{50}$

$= \frac{144}{50}$

$= \frac{72}{25}$

$= 2.88$

When $x = 20$, $y = \frac{-9}{1250}(20 - 25)^2 + \frac{9}{2}$

$= \frac{-9}{1250} \times 25 + \frac{9}{2}$

$= 4.32$

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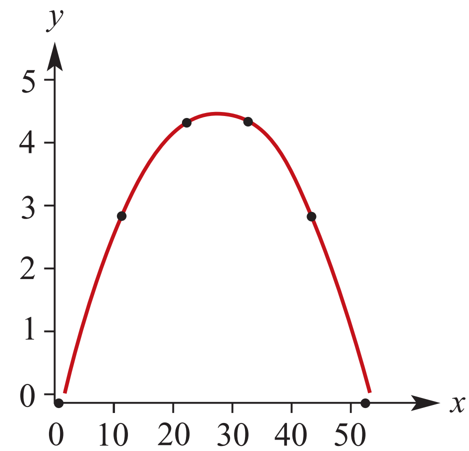
$$\text{When } x = 30,\ y = \frac{-9}{1250}(30-25)^2 + \frac{9}{2}$$
$$= \frac{-9}{1250} \times 25 + \frac{9}{2}$$
$$= 4.32$$

$$\text{When } x = 40,\ y = \frac{-9}{1250}(40-25)^2 + \frac{9}{2}$$
$$= \frac{-9}{1250} \times 225 + \frac{9}{2}$$
$$= 2.88$$



c

$$\text{When } y = 3, \quad \frac{-9}{1250}(x-25)^2 + \frac{9}{2} = 3$$
$$\therefore \quad \frac{-9}{1250}(x-25)^2 = \frac{-3}{2}$$
$$\therefore \quad (x-25)^2 = \frac{-3}{2} \times \frac{-1250}{9} = \frac{625}{3}$$
$$\therefore \quad x - 25 = \pm\sqrt{\frac{625}{3}}$$
$$\therefore \quad x = 25 \pm \frac{25\sqrt{3}}{3}$$
$$\therefore \quad x \approx 10.57 \text{ or } x \approx 39.43$$

Hence the height of the arch is 3 m above water level approximately 10.57 m and 39.43 m horizontally from A. This can also be solved using a CAS calculator.

d When $x = 12$, $y = \frac{-9}{1250}(12-25)^2 + \frac{9}{2}$

$$= \frac{-9}{1250} \times 169 + \frac{9}{2} = 3.2832$$

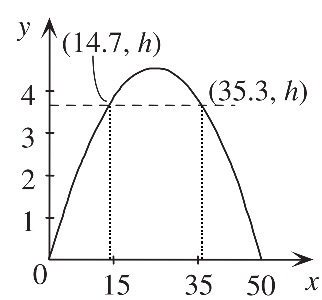
The height of the arch is 3.2832 m at a horizontal distance of 12 m from A.

e The greatest height of the deck above water level, h m, is when

$x + 0.3 = 15$ and $x - 0.3 = 35$

i.e. when $x = 14.7$ and $x = 35.3$

$$\therefore h = \frac{-9}{1250}(14.7-25)^2 + \frac{9}{2}$$
$$= 3.736152$$



Hence the greatest height of the deck above water level is approximately 3.736 m.

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2 **a** If x cm is the side length of the square then $4x$ cm has been used to form the square, so the perimeter of the rectangle is $P = 12 - 4x$.

Let a cm be the width of the rectangle and $2a$ cm be the length of the rectangle,

SO $P = a + a + 2a + 2a = 6a$

$\therefore \quad 6a = 12 - 4x$

$\therefore \quad a = 2 - \frac{2}{3}x \qquad \text{and} \qquad 2a = 4 - \frac{4}{3}x$

Hence the dimensions of the rectangle are $\left(2 - \frac{2}{3}x\right)$ cm $\times \left(4 - \frac{4}{3}x\right)$ cm.

b Let A_1 be the area of the square and A_2 be the area of the rectangle.

$$\begin{aligned}\therefore A = A_1 + A_2 &= x^2 + \left(2 - \frac{2}{3}x\right)\left(4 - \frac{4}{3}x\right)\\ &= x^2 + 8 - \frac{8}{3}x - \frac{8}{3}x + \frac{8}{9}x^2\\ &= \frac{17}{9}x^2 - \frac{16}{3}x + 8\end{aligned}$$

Hence the combined area of the square and the rectangle in cm^2 is defined by the rule $A = \frac{17}{9}x^2 - \frac{16}{3}x + 8$.

c TP occurs when $x = -\frac{b}{2a}$

$$= \frac{16}{3} \div \frac{34}{9}$$

$$= \frac{24}{17}$$

Minimum occurs when $x = \frac{24}{17}$.

When $x = \frac{24}{17}$, $\quad 4x = \frac{96}{17} \approx 5.65$

and $\qquad 12 - 4x = \frac{108}{17} \approx 6.35$

Hence, the wire needs to be cut into lengths of 5.65 cm and 6.35 cm (correct to 2 decimal places) for the sum of the areas to be a minimum.

3 **a**

$V = \text{rate} \times \text{time}$

When $x = 5$, $\quad V = 0.2 \times 60 = 12$

When $x = 10$, $V = 0.2 \times 60 \times 5 = 60$

When $x = 0$, $\quad V = 0$

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$\therefore\ c = 0$ (y-axis intercept is 0)

$\therefore \qquad V = ax^2 + bx$

When $x = 5,\ V = 12,\qquad 12 = 25a + 5b \qquad (1)$

When $x = 10, V = 60,\qquad 60 = 100a + 10b \qquad (2)$

$2 \times (1) \qquad 24 = 50a + 10b \qquad (3)$

$(2) - (3) \qquad 36 = 50a$

$\therefore \qquad a = \dfrac{36}{50} = \dfrac{18}{25}$

Substitute $a = \dfrac{18}{25}$ in (1) $\quad 12 = 25 \times \dfrac{18}{25} + 5b$

$\therefore \qquad 12 = 18 + 5b$

$\therefore \qquad 5b = -6$

$\therefore \qquad b = -\dfrac{6}{5}$

Hence, the rule for V in terms of x is $V = \dfrac{18}{25}x^2 - \dfrac{6}{5}x,\ x \geq 0$, or $v = 0.72x^2 - 1.2x$

b When $x = 20$ (i.e. a depth of 20 cm),

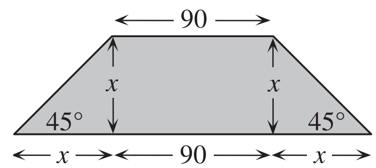
$$\begin{aligned} V &= \frac{18}{25}(20)^2 - \frac{6}{5}(20) \\ &= \frac{18 \times 400}{25} - 24 \\ &= 18 \times 16 - 24 = 264 \end{aligned}$$

Now $\qquad V = \text{rate} \times \text{time}$

$$\begin{aligned} \therefore \qquad \text{time} &= \frac{V}{\text{rate}} \\ &= \frac{264}{0.2} = 1320 \text{ minutes} \\ &= 22 \text{ hours} \end{aligned}$$

Water can be pumped into the tank for 22 hours before overflowing.

4 a Let V_E m^3 be the volume of the embankment.



$$\begin{aligned} V_E &= 120 \times \text{ shaded area} \\ &= 120\left(90x + \frac{1}{2}x^2 + \frac{1}{2}x^2\right) \\ &= 120x^2 + 10\,800x, x > 0 \end{aligned}$$

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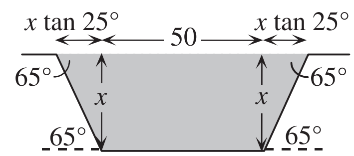
b Let V_C m^3 be the volume of the cutting.



$$\begin{aligned} V_C &= 100 \times \text{shaded area} \\ &= 100(50x + x^2 \tan 25°) \\ &\approx 100(50x + 0.466\,308x^2) \\ &\approx 46.63x^2 + 5000x, \; x > 0 \end{aligned}$$

c When $x = 4$, $\quad V_C \approx 46.63 \times 4^2 + 5000 \times 4$

$$\approx 20746.08$$

Now $V_E = L \times (x^2 + 90x)$, where L m is the length of the embankment.

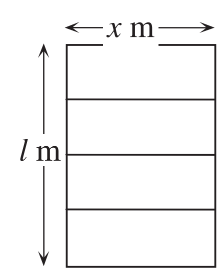
$$\therefore \quad L = \frac{V_E}{x^2 + 90x}$$

If using soil from the cutting, $V_C = V_E$

$$\begin{aligned} \therefore \quad L &= \frac{V_C}{x^2 + 90x} \\ &= \frac{20746.08}{4^2 + 90 \times 4} \approx 55.18 \end{aligned}$$

Hence, when $x = 4$ m, an embankment 55.18 m long could be constructed from the soil taken from the cutting.

5 a $5x + 2l = 100$



$$\therefore \quad 2l = 100 - 5x$$

$$\therefore \quad l = 50 - \frac{5}{2}x$$

b $A = x \times l$

$$= 50x - \frac{5}{2}x^2$$

c When $A = 0$, $\quad -\frac{5}{2}x^2 + 50x = 0$

$$\therefore \quad x\left(-\frac{5}{2}x + 50\right) = 0$$

$\therefore$ either $x = 0$ or $\quad -\frac{5}{2}x + 50 = 0$

If $-\frac{5}{2}x + 50 = 0$, $\quad \frac{5}{2}x = 50$

$$\begin{aligned} \therefore \quad x &= \frac{2 \times 50}{5} \\ &= 20 \end{aligned}$$

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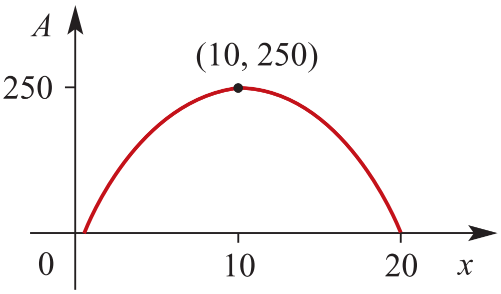
Turning point is halfway between the x-intercepts, i.e. at $x = 10$.

When $x = 10$,

$$A = -\frac{5}{2} \times 10^2 + 50 \times 10$$
$$= -250 + 500 = 250$$



Completing the square may also be used to find the vertex.

d The maximum area is 250 m^2 when x is 10 metres.

6 Given $AP = 1$, $AB = 1 - x$, $AD = x$ and $\dfrac{AP}{AD} = \dfrac{AD}{AB}$

then $\quad \dfrac{1}{x} = \dfrac{x}{1-x}$

$\therefore \quad 1 - x = x^2$

$\therefore\ x^2 + x - 1 = 0$

Using the general quadratic formula:

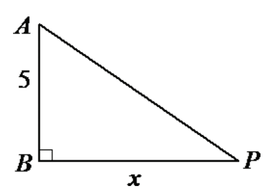
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{where } a = 1, b = 1, c = -1$$

$$\therefore \quad x = \frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

$$= \frac{-1 - \sqrt{5}}{2} \text{ or } \frac{-1 + \sqrt{5}}{2}$$

but $x > 0$, so $\quad x = \dfrac{-1 + \sqrt{5}}{2}$

7 **a** Using Pythagoras' theorem



$$PA^2 = 5^2 + x^2$$
$$= x^2 + 25$$
$$\therefore \quad PA = \sqrt{x^2 + 25}$$

b **i** $PC = BC - BP$

$= 16 - x$

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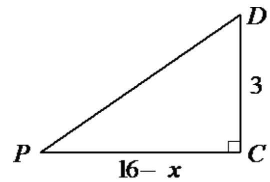
ii Using Pythagoras' theorem



$$PD^2 = (16 - x)^2 + 3^2$$
$$= x^2 - 32x + 256 + 9$$
$$\therefore \quad PD = \sqrt{x^2 - 32x + 265}$$

c If $PA = PD$, $\sqrt{x^2 + 25} = \sqrt{x^2 - 32x + 265}$

$\therefore \quad x^2 + 25 = x^2 - 32x + 265$

$\therefore \quad 25 = -32x + 265$

$\therefore \quad 32x = 240$

$\therefore \quad x = 7.5$

d If $PA = 2PD$, $\sqrt{x^2 + 25} = 2\sqrt{x^2 - 32x + 265}$

$\therefore \quad x^2 + 25 = 4(x^2 - 32x + 265)$

$\therefore \quad = 4x^2 - 128x + 1060$

$\therefore \quad 3x^2 - 128x + 1035 = 0$

Using the general quadratic formula,

$$x = \frac{128 \pm \sqrt{(-128)^2 - 4(3)(1035)}}{2(3)}$$
$$= \frac{128 \pm \sqrt{3964}}{6}$$
$$= \frac{128 \pm 2\sqrt{991}}{6}$$
$$= \frac{64 \pm \sqrt{991}}{3}$$
$$= 31.826\,71\ldots \text{ or } 10.839\,94\ldots$$
$$\approx 10.840 \text{ (as } 0 \le x \le 16)$$

e If $PA = 3PD$, $\sqrt{x^2 + 25} = 3\sqrt{x^2 - 32x + 265}$

$\therefore \quad x^2 + 25 = 9(x^2 - 32x + 265)$

$= 9x^2 - 288x + 2385$

$\therefore \quad 8x^2 - 288x + 2360 = 0$

$\therefore \quad 8(x^2 - 36x + 295) = 0$

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Using the general quadratic formula,

$$\begin{aligned} x &= \frac{36 \pm \sqrt{(-36)^2 - 4(1)(295)}}{2(1)} \\ &= \frac{36 \pm \sqrt{116}}{2} \\ &= \frac{36 \pm 2\sqrt{29}}{2} = 18 \pm \sqrt{29} \\ &= 23.385\,16\ldots \text{ or } 12.615\,83\ldots \end{aligned}$$

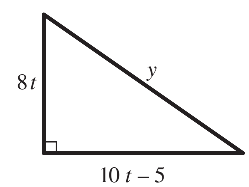
≈ 12.615 (as $0 \le x \le 16$)

Note: Parts **c**, **d** and **e** can be solved using the CAS calculator. Plot the graphs of **$f1 = \sqrt{(}x$^2 + 25)**, **$f2 = \sqrt{(}x$^2 − 32x + 265)**, **$f3 = 2 \times f2(x)$** and **$f4 = 3 \times f2(x)$** for

$x \in [0, 16]$. The points of intersection of ***f1*** with each of the other graphs provide the solutions for x.

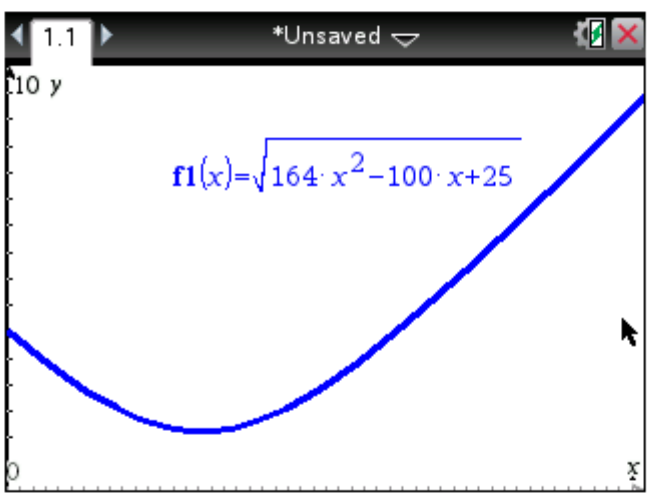
8 a i Consider AB and CD to be a pair of Cartesian axes with O at the point $(0, 0)$. The first jogger is at the point $(8t, 0)$ at time t. The second jogger is at the point $(0, 10t - 5)$ at time t.

Using Pythagoras' theorem



$$\begin{aligned} y^2 &= (8t)^2 + (10t - 5)^2 \\ &= 64t^2 + 100t^2 - 100t + 25 \\ \therefore\ y &= \sqrt{164t^2 - 100t + 25} \end{aligned}$$

ii



iii On a CAS calculator, enter **solve(√(164x^2 − 100x + 25) = 4, x)**.

The points of intersection are $\left(\frac{9}{82}, 4\right)$ and $\left(\frac{1}{2}, 4\right)$.

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Therefore joggers are 4 km apart after 0.11 hours (1.07 pm), correct to 2 decimal places, and after 0.5 hours (1.30 pm).

Or consider $\sqrt{164t^2 - 100t + 25} = 4$

$\therefore \quad 164t^2 - 100t + 9 = 0$

$$\therefore \quad t = \frac{100 \pm \sqrt{(-100)^2 - 4(9)(164)}}{2(164)}$$

$$= \frac{100 \pm \sqrt{4096}}{328}$$

$$= \frac{100 \pm 64}{328}$$

$$= \frac{1}{2} \text{ or } \frac{9}{82}$$

iv With the graph from part **ii** on screen

TI: Press **Menu→6:Analyze Graph→2:Minimum**

CP: Tap **Analysis→ G-Solve→Min**

to yield (0.30487837, 3.12 4752).

Therefore joggers are closest when they are 3.12 km apart after 0.30 hours, correct to 2 decimal places.

Alternatively, the minimum of $\sqrt{164t^2 - 100t + 25}$ occurs when $164t^2 - 100t + 25$ is a minimum.

This occurs when $t = \frac{100}{2 \times 164}$

$$= \frac{25}{82} \text{(1.18 pm)}$$

$$\therefore \text{ minimum distance apart} = \frac{20}{\sqrt{41}}$$

$$= \frac{20\sqrt{41}}{41}$$

$$\approx 3.123 \text{ km}$$

b i When $y = 5$, $\quad 5 = \sqrt{164t^2 - 100t + 25}$

$\therefore \quad 25 = 164t^2 - 100t + 25$

$\therefore \quad 164t^2 - 100t = 0$

$\therefore \quad 4t(41t^2 - 25t) = 0$

$\therefore \quad t = 0 \text{ or } t = \frac{25}{41}$

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ii When $y = 6$, $\quad 6 = \sqrt{164t^2 - 100t + 25}$

$\therefore \quad 36 = 164t^2 - 100t + 25$

$\therefore \quad 164t^2 - 100t - 11 = 0$

Using the general quadratic formula,

$$t = \frac{100 \pm \sqrt{(-100)^2 - 4(164)(-11)}}{2(164)}$$

$$= \frac{100 \pm \sqrt{17216}}{328} = \frac{25 \pm 2\sqrt{269}}{82}$$

9 **a** $BC = x, CD = y, BD =$ diameter of circle $= 2a$

Using Pythagoras' theorem,

$$BC^2 + CD^2 = BD^2$$

$\therefore \quad x^2 + y^2 = 4a^2$, as required.

b Perimeter $= b$, $\quad$ but perimeter $= 2(x + y)$

$\therefore \quad 2(x + y) = b$

c $2(x + y) = b \quad \therefore 2x + 2y = b$

$\therefore \quad 2y = b - 2x$

$\therefore \quad y = \frac{1}{2}b - x \qquad (1)$

Substituting (1) into $\quad x^2 + y^2 = 4a^2$ gives

$$x^2 + \left(\frac{1}{2}b - x\right)^2 = 4a^2$$

$\therefore \quad 2x^2 - bx + \frac{1}{4}b^2 - 4a^2 = 0 \qquad (2)$

$\therefore \quad 8x^2 - 4bx + b^2 - 16a^2 = 0$

d Now $x + y > 2a$ $\therefore$ using (1), $\quad x + \left(\frac{1}{2}b - x\right) > 2a$

$\therefore \quad \frac{1}{2}b > 2a$

$\therefore \quad b > 4a$

Considering the discriminant, Δ, of (2)

$$\Delta = (-b)^2 - 4(2)\left(\frac{1}{4}b^2 - 4a^2\right)$$

$$= b^2 - 8\left(\frac{1}{4}b^2 - 4a^2\right)$$

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$= b^2 - 2b^2 + 32a^2$

$= 32a^2 - b^2$

For the inscribed rectangle to exist, $\Delta \geq 0$

$\therefore\ 32a^2 - b^2 \geq 0$

$\therefore\quad b^2 \leq 32a^2$

$\therefore\quad b \leq 4\sqrt{2}a$

$\therefore\ 4a < b \leq 4\sqrt{2}a$, as required.

e **i** Substituting $a = 5$ and $b = 24$ into (2) gives

$$2x^2 - 24x + \left(\frac{1}{4}(24)^2 - 4(5)^2\right) = 0$$

$$\therefore\quad 2x^2 - 24x + 44 = 0$$

$$\therefore\quad x^2 - 12x + 22 = 0$$

Using the general quadratic formula,

$$x = \frac{12 \pm \sqrt{(-12)^2 - 4(1)(22)}}{2(1)}$$

$$= \frac{12 \pm \sqrt{56}}{2}$$

$$= 6 \pm \sqrt{14}$$

Now $\quad y = \frac{1}{2}b - x$

$$= \frac{1}{2}(24) - x = 12 - x$$

When $x = 6 \pm \sqrt{14}$, $\quad y = 12 - (6 \pm \sqrt{14})$

When $x = 6 + \sqrt{14}$, $\quad y = 6 - \sqrt{14}$

When $x = 6 - \sqrt{14}$, $\quad y = 6 + \sqrt{14}$

ii If $b = 4\sqrt{2}a$, then (2) gives

$$2x^2 - 4\sqrt{2}ax + \left(\frac{1}{4}(4\sqrt{2}a\right)^2 - 4a^2) = 0$$

$$\therefore\quad 2x^2 - 4\sqrt{2}ax + 8a^2 - 4a^2 = 0$$

$$\therefore\quad 2x^2 - 4\sqrt{2}ax + 4a^2 = 0$$

$$\therefore\quad x^2 - 2\sqrt{2}ax + 2a^2 = 0$$

$$\therefore\quad (x - \sqrt{2}a)^2 = 0$$

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$\therefore \quad x = \sqrt{2}a$

$\therefore \quad y = \frac{1}{2}b - x$

$$= 2\sqrt{2}a - \sqrt{2}a$$

$$= \sqrt{2}a$$

f If $\frac{b}{a} = 5$, then $b = 5a$ and, from (2):

$$2x^2 - (5a)x + \left(\frac{1}{4}(5a)^2 - 4a^2\right) = 0$$

$$\therefore \quad 2x^2 - 5ax + \left(\frac{25}{4}a^2 - 4a^2\right) = 0$$

$$\therefore \quad 2x^2 - 5ax + \frac{9}{4}a^2 = 0$$

Using the general quadratic formula,

$$x = \frac{5a \pm \sqrt{(-5a)^2 - 4 \times 2 \times \frac{9}{4}a^2}}{2(2)}$$

$$= \frac{5a \pm \sqrt{25a^2 = 18a^2}}{4}$$

$$= \frac{5a \pm \sqrt{7}a}{4}$$

Now $\quad y = \frac{1}{2}b - x$

$$= \frac{1}{2}(5a) - x$$

$$= \frac{5}{2}a - x$$

When $x = \frac{5a \pm \sqrt{7}a}{4}$, $y = \frac{5}{2}a - \frac{5a \pm \sqrt{7}a}{4}$

When $x = \frac{5a + \sqrt{7}a}{4}$, $y = \frac{5a - \sqrt{7}a}{4}$

When $x = \frac{5a - \sqrt{7}a}{4}$, $y = \frac{5a + \sqrt{7}a}{4}$

g The following program can be input into a CAS calculator to solve equation (2) in part **c** for x and y, given a and $b (a, b \in \mathbb{R})$, correct to 2 decimal places.

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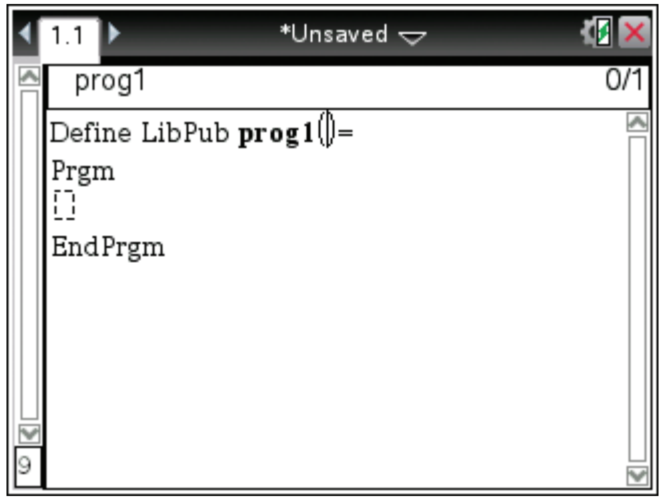
TI: In the calculator application press **menu→9:Functions & Programs→ 1:Program Editor1:New.** Name the program progl. The following information is shown automatically. Complete the screen as follows: Complete the screen as follows:




```
Define LibPub progl () =
Prgm
setMode(5,2)
setMode(1,16)
Local a,b,w,x,y,z
Request "a = ",a
Request "b = ",b
(b + √(32a^2 – b^2))/4 → x
b/2 – x → y
(b – √(32a^2 – b^2))/4 → w
b/2 – w → z
Disp "x= ",x
Disp "and y= ",y
Disp "OR"
Disp "x=",w
Disp "and y =", z
EndPrgm
```

10 **a** Equation of curve A is

$y = (x - h)^2 + 3$

$(0, 4): 4 = (0 - h)^2 + 3$

$h^2 = 1$

$h = 1$ (since $h > 0$)

So $y = (x - 1)^2 + 3$

$= x^2 - 2x + 4$

Giving $b = -2, c = 4$ and $h = 1$

b **i** The coordinates of P' are $(x, -6 + 4x - x^2)$

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ii Let (m, n) be the coordinates of M.

$\therefore \quad m = x$

and $\quad n = \dfrac{(x^2 - 2x + 4) + (-6 + 4x - x^2)}{2}$

$= \dfrac{2x - 2}{2} = x - 1$

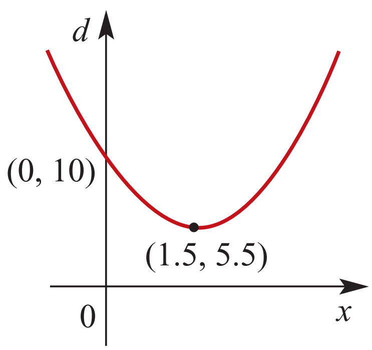
$\therefore$ the coordinates of M are $(x, x - 1)$.

iii The coordinates of M for $x = 0, 1, 2, 3, 4$ are $(0, -1), (1, 0), (2, 1), (3, 2)$ and $(4, 3)$ respectively.

iv $y = x - 1$ is the equation of the straight line on which the points (0, −1), (1, 0), (2, 1), (3, 2) and (4, 3) all lie.

c **i** $d = (x^2 - 2x + 4) - (-6 + 4x - x^2) = 2x^2 - 6x + 10$

ii



iii

Consider $\quad 2(x^2 - 3x + 5) = 2\left(x^2 - 3x + \dfrac{9}{4} + 5 - \dfrac{9}{4}\right)$

$= 2\left(\left(x - \dfrac{3}{2}\right)^2 + \dfrac{11}{4}\right) = 2\left(x - \dfrac{3}{2}\right)^2 + \dfrac{11}{2}$

$\therefore$ minimum value of d is $\dfrac{11}{2}$ and occurs for $x = \dfrac{3}{2}$.

11 **a** Length of path $= \sqrt{(60 + 30)^2 + (30 + 15)^2}$

$= \sqrt{10\,125}$

$= 45\sqrt{5}$

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b **i** $y = ax^2 + bx + c$

At $(-20, 45)$,	$45 = 400a - 20b + c$	(1)
At $(40, 40)$,	$40 = 1600a + 40b + c$	(2)
At $(30, 35)$,	$35 = 900a + 30b + c$	(3)
(2) − (1) gives	$-5 = 1200a + 60b$	(4)
(2) − (3) gives	$5 = 700a + 10b$	(5)
6 × (5) − (4) gives	$35 = 3000a$	
$\therefore$	$a = \dfrac{35}{3000} = \dfrac{7}{600}$	

Substituting $a = \dfrac{7}{600}$ into (5) gives:

$$\begin{aligned} 5 &= 700\left(\frac{7}{600}\right) + 10b \\ &= \frac{49}{6} + 10b \end{aligned}$$

$$\therefore \quad 10b = \frac{-19}{6}$$

$$\therefore \quad b = \frac{-19}{60}$$

Substituting $a = \dfrac{7}{600}$ and $b = \dfrac{-19}{60}$ into (1) gives:

$$\begin{aligned} 45 &= 400\left(\frac{7}{600}\right) - 20\left(\frac{-19}{60}\right) + C \\ &= \frac{14}{3} + \frac{19}{3} + C \end{aligned}$$

$$\therefore \quad c = 34$$

$$\therefore \quad y = \frac{7}{600}x^2 - \frac{19}{60}x + 34$$

TI: Press **Menu→6:Analyze Graph→ 2:Minimum** with ***f*1=7/600*x*^2-19/60*x*+34**,
CP: Tap **Analysis→G-Solve→Min**
to yield (13.571521, 31.85, 19). Therefore the vertex of the parabola has coordinates (13.57, 31.85), correct to 2 decimal places.

Or consider $\dfrac{7}{600}x^2 \dfrac{19}{60}x + 34 = \dfrac{7}{600}\left(x^2 - \dfrac{190}{7}x + \dfrac{20400}{7}\right)$

$$\begin{aligned} &= \frac{7}{600}\left(\left(x - \frac{95}{7}\right)^2 + \frac{133775}{49}\right) \\ &= \frac{7}{600}\left(x - \frac{95}{7}\right)^2 + \frac{5351}{168} \end{aligned}$$

$\therefore$ minimum value is $\dfrac{5351}{168}$ and this occurs when $x = \dfrac{95}{7}$.

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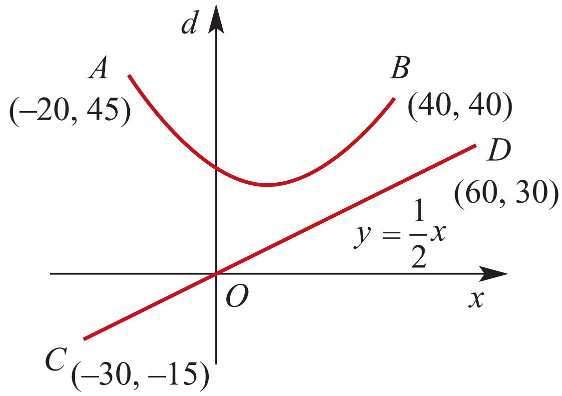
c



d **i** The expression $y = (ax^2 + bx + c) - \frac{1}{2}x$ determines the distance, perpendicular to the x-axis, between $y = ax^2 + bx + c$ and $y = \frac{1}{2}x$ at the point x. In this question, it is the distance between the path and the pond.

ii With ***f1* = 7*x*2/600 – 19*x*/60+34-*x*/2**
TI: Press **Menu→6:Analyze Graph→2:Minimum**
CP: Tap **Analysis→G-Solve→ Min**
to yield (35.000 004, 19.708 333). Therefore the minimum value is 19.71 which occurs when $x = 35.00$, correct to 2 decimal places.

Or consider
$$\begin{aligned}\frac{7x^2}{600} - \frac{19x}{60} + 34 - \frac{x}{2} &= \frac{7x^2}{600} - \frac{49x}{60} + 34\\
&= \frac{7}{600}\left(x^2 - 70x + \frac{20400}{7}\right)\\
&= \frac{7}{600}\left(x^2 - 70x + 1225 + \frac{11825}{7}\right)\\
&= \frac{7}{600}\left((x-35)^2 + \frac{11825}{7}\right)\\
&= \frac{7}{600}(x-35)^2 + \frac{473}{24}\end{aligned}$$

$\therefore$ minimum value is $\frac{473}{24}$ which occurs when $x = 35$.

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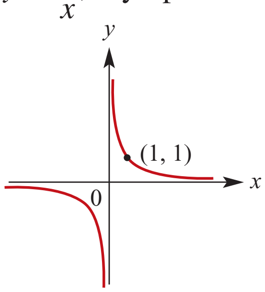
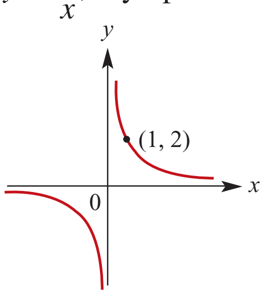
Chapter 4 – A gallery of graphs

Solutions to Exercise 4A

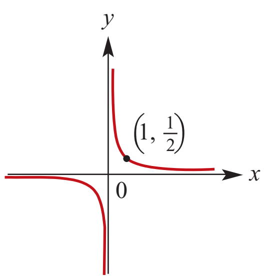
1 **a** $y = \dfrac{1}{x}$; asymptotes at $x = 0$ and $y = 0$



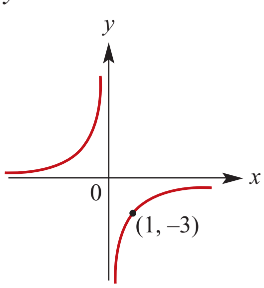
b $y = \dfrac{2}{x}$; asymptotes at $x = 0$ and $y = 0$



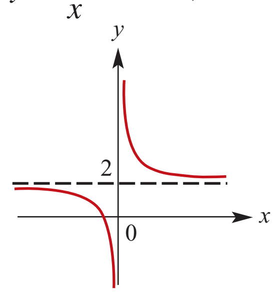
c $y = \dfrac{1}{2x}$; asymptotes at $x = 0$ and $y = 0$



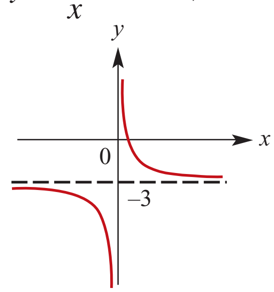
d $y = -\dfrac{3}{x}$; asymptotes at $x = 0$ and $y = 0$



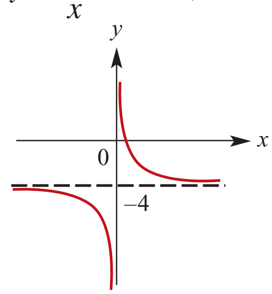
e $y = \dfrac{1}{x} + 2$; asymptotes at $x = 0$ and $y = 2$ x-intercept where $y = \dfrac{1}{x} + 2 = 0, \therefore x = -\dfrac{1}{2}$



f $y = \dfrac{1}{x} - 3$; asymptotes at $x = 0$ and $y = -3$ x-intercept where $y = \dfrac{1}{x} - 2 = 0, \therefore x = \dfrac{1}{3}$



g $y = \dfrac{2}{x} - 4$; asymptotes at $x = 0$ and $y = -4$ x-intercept where $y = \dfrac{2}{x} - 4 = 0, \therefore x = \dfrac{1}{2}$



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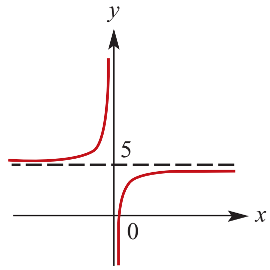
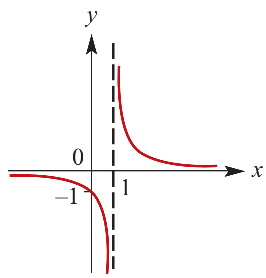
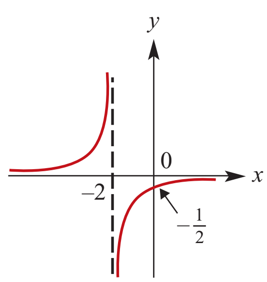
h $y = -\dfrac{1}{2x} + 5$;
asymptotes at $x = 0$ and $y = 5$
x-intercept where
$y = -\dfrac{1}{2x} + 5 = 0. \therefore x = 0.1$



i $y = \dfrac{1}{x - 1}$; asymptotes at $x = 1$ and $y = 0$
y-intercept where $y = \dfrac{1}{0 - 1} = -1$



j $y = -\dfrac{1}{x + 2}$;
Asymptotes at $x = -2$ and $y = 0$
y-intercept where $y = -\dfrac{1}{0 + 2} = -\dfrac{1}{2}$



k $y = \dfrac{1}{x + 1} + 3$;
asymptotes at $x = -1$ and $y = 3$
x-intercept where

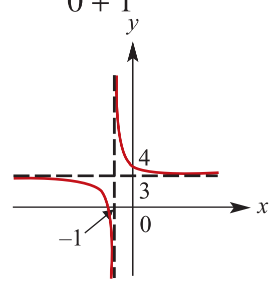
$$y = \frac{1}{x + 1} + 3 = 0$$

$$\therefore \frac{1}{x + 1} = -3$$

$$\therefore x + 1 = -\frac{1}{3}$$

$$\therefore x = -\frac{4}{3}$$

y-intercept where
$y = \dfrac{1}{0 + 1} + 3 = 4$



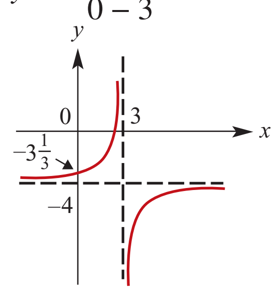
l $y = -\dfrac{2}{x - 3} - 4$;
asymptotes at $x = 3$ and $y = -4$
x-intercept where $y = -\dfrac{2}{x - 3} - 4 = 0$

$$\therefore \frac{2}{x - 3} = -4$$

$$\therefore x - 3 = -\frac{1}{2}$$

$$\therefore x = \frac{5}{2}$$

y-intercept where
$y = -\dfrac{2}{0 - 3} - 4 = -\dfrac{10}{3}$



2 Answers given in question 1

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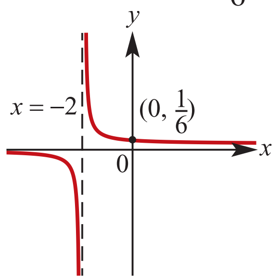
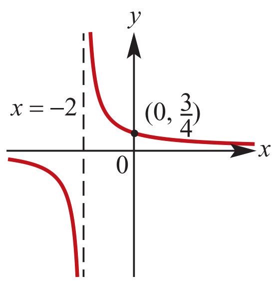
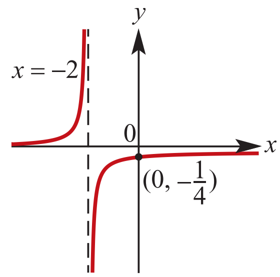
3 a The graph of $y = \dfrac{1}{3x+6}$ can be obtained by translating the graph of $y = \dfrac{1}{3x}$ two units to the left.
The equations of the asymptotes are $x = -2$ and $y = 0$.
When $x = 0, y = \dfrac{1}{6}$



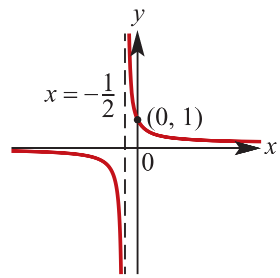
b The graph of $y = \dfrac{3}{2x+4}$ can be obtained by translating the graph of $y = \dfrac{3}{2x}$ two units to the left.
The equations of the asymptotes are $x = -2$ and $y = 0$.
When $x = 0, y = \dfrac{3}{4}$



c The graph of $y = \dfrac{-1}{2x+4}$ can be obtained by translating the graph of $y = \dfrac{-1}{2x}$ two units to the left.
The equations of the asymptotes are $x = -2$ and $y = 0$.
When $x = 0, y = -\dfrac{1}{4}$

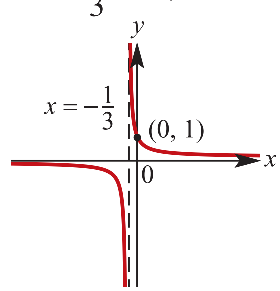


d The graph of $y = \dfrac{1}{2x+1}$ can be obtained by translating the graph of $y = \dfrac{1}{2x}$ half a unit to the left.
The equations of the asymptotes are $x = -\dfrac{1}{2}$ and $y = 0$.
When $x = 0, y = 1$



4 a $\dfrac{1}{3x+1} = \dfrac{1}{3(x+\frac{1}{3})}$.

Translate the graph of $y = \dfrac{1}{3x}$ one third of a unit to the left.
When $x = 0, y = 1$
Theequations of the asymptotes are $x = -\dfrac{1}{3}$ and $y = 0$



b Translate the graph of $y = \dfrac{1}{3x+1}$ one unit in the negative direcion of the y-axis.

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When $x = 0, y = 0$

When $y = 0, \dfrac{1}{3x+1} - 1 = 0$

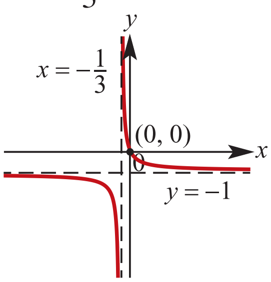
$$\frac{1}{3x+1} - 1 = 0$$

$$\frac{1}{3x+1} = 1$$

$$3x + 1 = 1$$

$$x = 0$$

The equations of the asymptotes are $x = -\dfrac{1}{3}$ and $y = -1$.



c Reflect the graph of $y = \dfrac{1}{3x+1}$ in the x axis and translate the image, $y = -\dfrac{1}{3x+1}$ one unit in the negative direcion of the y-axis.

When $x = 0, y = -2$

When $y = 0, -\dfrac{1}{3x+1} - 1 = 0$

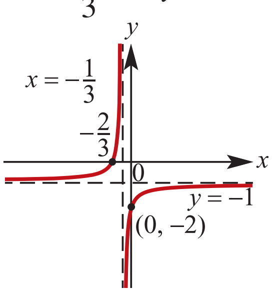
$$-\frac{1}{3x+1} - 1 = 0$$

$$-\frac{1}{3x+1} = 1$$

$$3x + 1 = -1$$

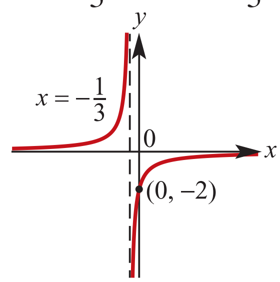
$$x = -\frac{2}{3}$$

The equation of the asymptotes are $x = -\dfrac{1}{3}$ and $y = -1$.



d Reflect the graph of $y = \dfrac{2}{3x+1}$ in the x axis. When $x = 0, y = -2$

The equation of the asymptotes are $x = -\dfrac{1}{3}$ and $y = -\dfrac{1}{3}$.



e Translate the graph of $y = \dfrac{-2}{3x+1}$ four units in the negative direcion of the y-axis.

When $x = 0, y = -6$

When $y = 0, -\dfrac{2}{3x+1} - 4 = 0$

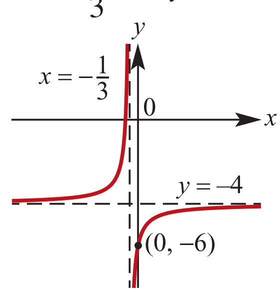
$$-\frac{2}{3x+1} - 4 = 0$$

$$-\frac{1}{3x+1} = 2$$

$$6x + 2 = -1$$

$$x = -\frac{1}{2}$$

The equation of the asymptotes are $x = -\dfrac{1}{3}$ and $y = -4$.



f Translate the graph of $y = \dfrac{-2}{3x+1}$ three units in the positive direcion of the y-axis.

When $x = 0, y = 1$

When $y = 0, -\dfrac{2}{3x+1} + 3 = 0$

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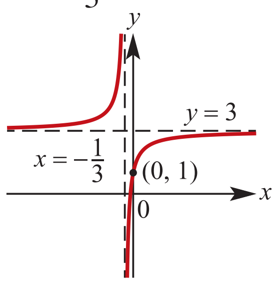
$$-\frac{2}{3x+1}+3=0$$
$$\frac{1}{3x+1}=3$$
$$9x+3=1$$
$$x=-\frac{2}{9}$$
The equation of the asymptotes are $x=-\frac{1}{3}$ and $y=3$.

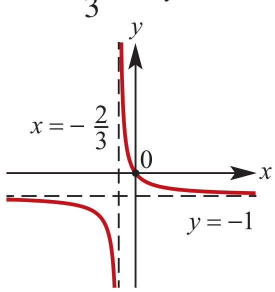


g $\dfrac{2}{3x+2}=\dfrac{2}{3(x+\frac{2}{3})}$

Translate the graph of $y=\dfrac{2}{3x}$ two thirds units in the negative direcion of the x-axis and one unit in the negative direction of the y-axis.

When $x=0, y=0$

When $y=0, \dfrac{2}{3x+2}-1=0$
$$\frac{2}{3x+2}-1=0$$
$$\frac{2}{3x+2}=1$$
$$3x+2=2$$
$$x=0$$
The equation of the asymptotes are $x=-\frac{2}{3}$ and $y=-1$.

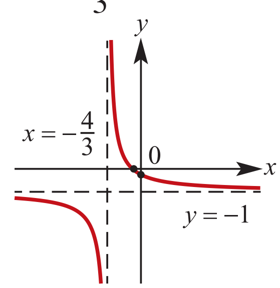


h $\dfrac{3}{3x+4}=\dfrac{3}{3(x+\frac{4}{3})}=\dfrac{1}{3x+\frac{4}{3}}$

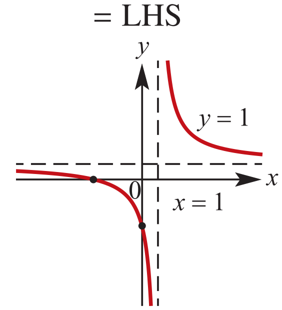
Translate the graph of $y=\dfrac{1}{x}$ four thirds units in the negative direcion of the x-axis and one unit in the negative direction of the y-axis.

When $x=0, y=-\dfrac{3}{4}$

When $y=0, \dfrac{3}{3x+4}-1=0$
$$\frac{3}{3x+4}-1=0$$
$$\frac{3}{3x+4}=1$$
$$3x+4=3$$
$$x=-\frac{1}{3}$$
The equation of the asymptotes are $x=-\frac{4}{3}$ and $y=-1$.



5
$$\begin{aligned}\text{RHS} &= \frac{4}{x-1}+1\\ &= \frac{4+x-1}{x-1}\\ &= \frac{x+3}{x-1}\\ &= \text{LHS}\end{aligned}$$



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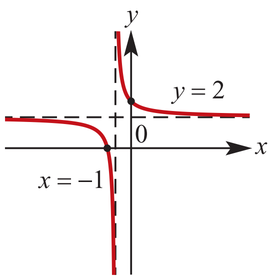
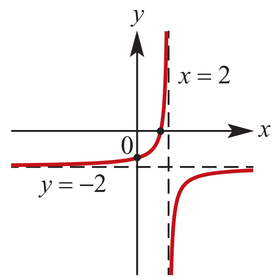
6 $$\text{RHS} = \frac{1}{x+1} + 2 = \frac{1 + 2(x+1)}{x+1} = \frac{2x+3}{x+1} = \text{LHS}$$



7 $$\text{RHS} = -\frac{1}{x-2} - 2 = \frac{-1 - 2(x-2)}{x-2} = \frac{-1 - 2x + 4}{x-2} = \frac{3 - 2x}{x-2} = \text{LHS}$$



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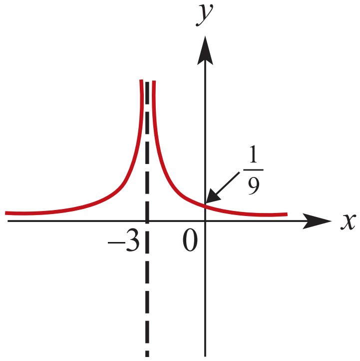
Solutions to Exercise 4B

1 a $y = \dfrac{1}{(x+3)^2}$

Asymptotes at $x = -3$ and $y = 0$

y-intercept where $y = \dfrac{1}{(0+3)^2} = \dfrac{1}{9}$

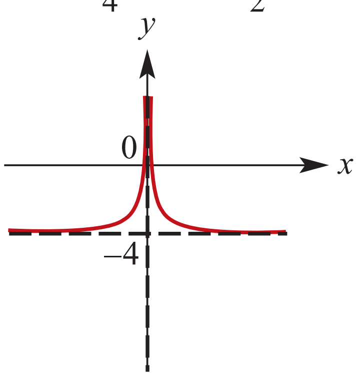


b $y = \dfrac{1}{x^2} - 4$

Asymptotes at $x = -3$ and $y = 0$

x-intercept where $y = \dfrac{1}{x^2} - 4 = 0$

$\therefore x^2 = \dfrac{1}{4}, \therefore x = \pm\dfrac{1}{2}$

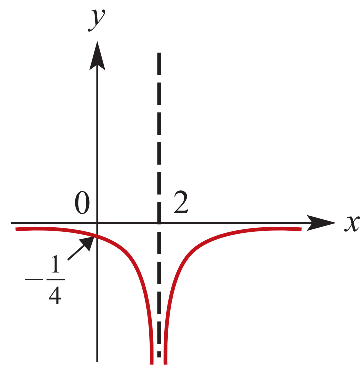


c $y = -\dfrac{1}{(x-2)^2}$

Asymptotes at $x = 2$ and $y = 0$

y-intercept where $y = -\dfrac{1}{(0-2)^2} =$

$-\dfrac{1}{4}$

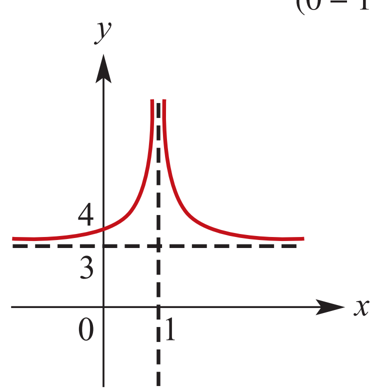


d $y = \dfrac{1}{(x-1)^2} + 3$

Asymptotes at $x = 1$ and $y = 3$

No x-intercepts: $\dfrac{1}{(x-1)^2} + 3 > 0$ for all x

y-intercept where $y = \dfrac{1}{(0-1)^2} + 3 = 4$



e $y = \dfrac{1}{2(x+3)^2} - 4$

Asymptotes at $x = -3$ and $y = -4$

x-intercepts: $y = \dfrac{1}{2(x+3)^2} - 4 = 0$

$\therefore \dfrac{1}{(x+3)^2} = 8$

$\therefore x + 3 = \pm\dfrac{1}{4}\sqrt{2}$

$\therefore x = -3 \pm \dfrac{1}{4}\sqrt{2}$

y-intercept at $\dfrac{1}{2(0+3)^2} - 4 = -\dfrac{71}{18}$

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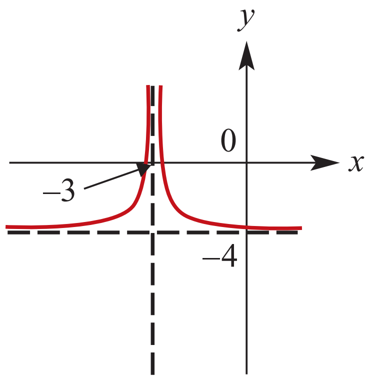


f $y = -\dfrac{2}{(x-2)^2} + 1$

Asymptotes at $x = 2$ and $y = 1$

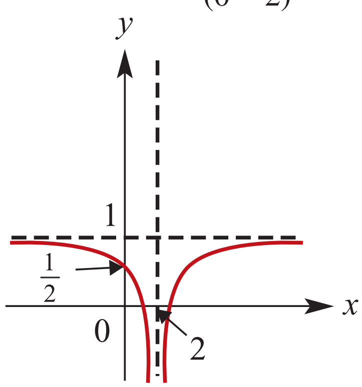
x-intercepts: $y = -\dfrac{2}{(x-2)^2} + 1 = 0$

$\therefore \dfrac{1}{(x-2)^2} = \dfrac{1}{2}$

$\therefore x - 2 = \pm\sqrt{2}$

$\therefore x = 2 \pm \sqrt{2}$

y-intercept at $-\dfrac{2}{(0-2)^2} + 1 = \dfrac{1}{2}$



g $y = \dfrac{3}{(x+3)^2} - 6$

Asymptotes at $x = -3$ and $y = -6$

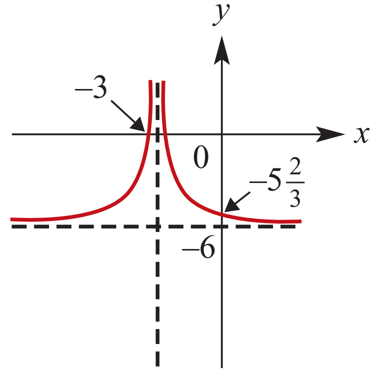
x-intercepts: $y = \dfrac{3}{(x+3)^2} - 6 = 0$

$\therefore \dfrac{1}{(x+3)^2} = 2$

$\therefore x + 3 = \dfrac{\pm\sqrt{2}}{2}$

$\therefore x = -3 \pm \dfrac{\sqrt{2}}{2}$

y-intercept at $\dfrac{3}{(0+3)^2} - 6 = -\dfrac{17}{3}$



h $y = -\dfrac{1}{(x-4)^2} + 2$

Asymptotes at $x = 4$ and $y = 2$

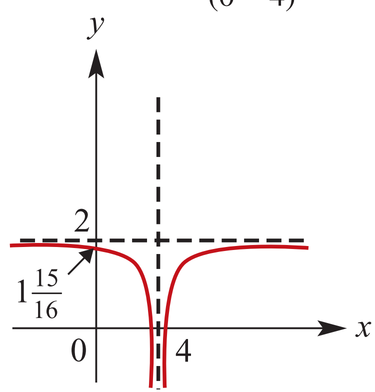
x-intercepts: $y = -\dfrac{1}{(x-4)^2} + 2 = 0$

$\therefore \dfrac{1}{(x-4)^2} = 2$

$\therefore x - 4 = \pm\dfrac{\sqrt{2}}{2}$

$\therefore x = 4 \pm \dfrac{\sqrt{2}}{2}$

y-intercept at $-\dfrac{1}{(0-4)^2} + 2 = \dfrac{31}{16}$



2 Answers given in Question 1.

3 a When $x = 0, y = \dfrac{1}{36}$

Asymptotes at $x = -2$ and $y = 0$

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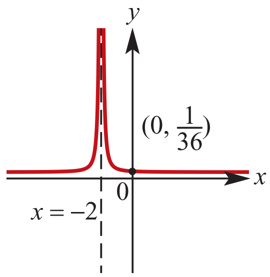
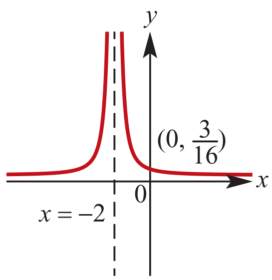


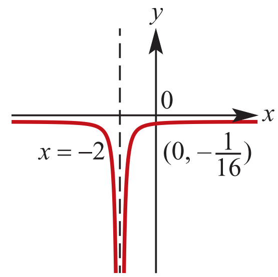
b When $x = 0, y = \frac{3}{16}$

Asymptotes at $x = -2$ and $y = 0$



c When $x = 0, y = -\frac{1}{16}$

Asymptotes at $x = -2$ and $y = 0$



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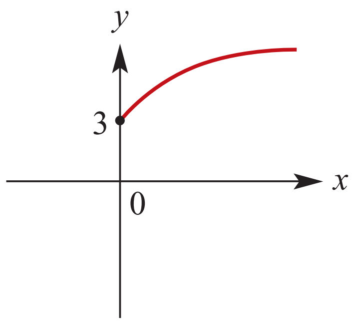
Solutions to Exercise 4C

1 a $y = 2\sqrt{x} + 3$; $\{x: x \geq 0\}$

No x-intercept; y-intercept at (0,3)

y is defined for $\{y: y \geq 3\}$

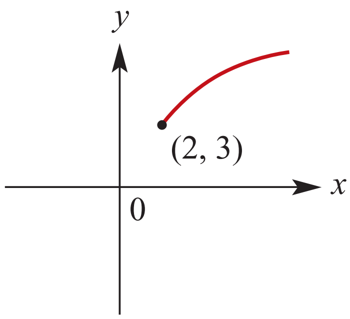


b $y = \sqrt{x-2} + 3$; $\{x: x \geq 2\}$

No axis intercepts

y is defined for $\{y: y \geq 3\}$

Starting point at (2,3)



c $y = \sqrt{x-2} - 3$; $\{x: x \geq 2\}$

x intercept where $\sqrt{x-2} - 3 = 0$

$\therefore \sqrt{x-2} = 3$

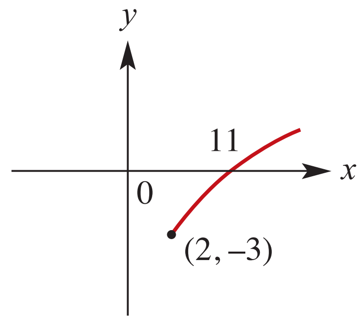
$\therefore x - 2 = 9$

$\therefore x = 11$

x-intercept at (11,0); no y-intercept.

y is defined for $\{y: y \geq -3\}$

Starting point at $(2, -3)$

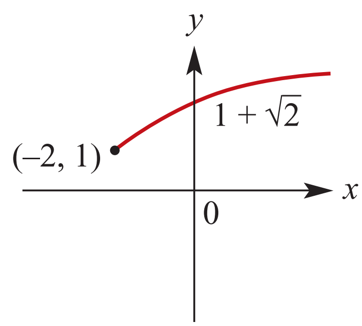


d $y = \sqrt{x+2} + 1$; $\{x: x \geq -2\}$

x-intercept at $(1 + \sqrt{2}, 0)$; no y-intercept.

y is defined for $\{y: y \geq 1\}$

Starting point at $(-2, 1)$



e $y = -\sqrt{x+2} + 3$; $\{x: x \geq -2\}$

y-intercept at $(3 - \sqrt{2}, 0)$

x-intercept where $-\sqrt{x+2} + 3 = 0$

$\therefore \sqrt{x+2} = 3$

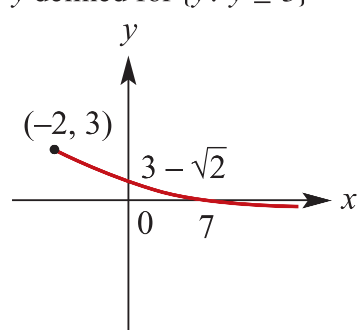
$\therefore x + 2 = 9$

$\therefore x = 7$

x-intercept at $(7, 0)$

Starting point at $(-2, 3)$

y defined for $\{y: y \leq 3\}$



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f $y = 2\sqrt{x+2} - 3;\ \{x\colon x \geq -2\}$

y-intercept at $(3 - 2\sqrt{2}, 0)$

x-intercept where $2\sqrt{x+2} - 3 = 0$

$\therefore \sqrt{x+2} = \dfrac{3}{2}$

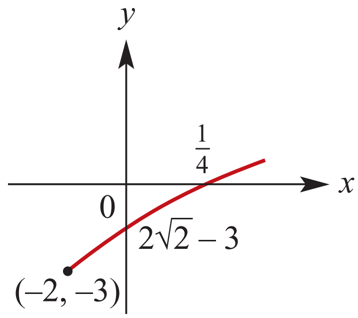
$\therefore x + 2 = \dfrac{9}{4}$

$\therefore x = \dfrac{1}{4}$

x-intercept at $\left(\dfrac{1}{4}, 0\right)$

Starting point at $(-2, -3)$

y defined for $\{y\colon y \geq -3\}$



2 a $y = -\sqrt{x-2} + 3;\ \{x\colon x \geq 2\}$

No y-intercept;

x-intercept where $-\sqrt{x-2} + 3 = 0$

$\therefore \sqrt{x-2} = 3$

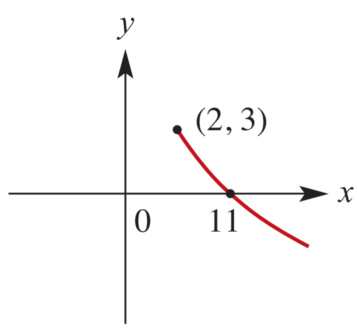
$\therefore x - 2 = 9$

$\therefore x = 11$

x intercept at (11,0)

Starting point at (2,3)

y defined for $\{y\colon y \leq 3\}$



b $y = \sqrt{-(x-4)} - 2;\ \{x\colon x \leq 4\}$

y-intercept at $(0, 0)$

x-intercept where $\sqrt{-(x-4)} - 2 = 0$

$\therefore \sqrt{-(x-4)} = 2$

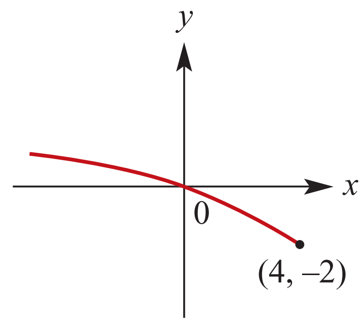
$\therefore -(x-4) = 4$

$\therefore x = 0$

x-intercept at $(0, 0)$

Starting point at $(4, -2)$

y defined for $\{y\colon y \geq -2\}$

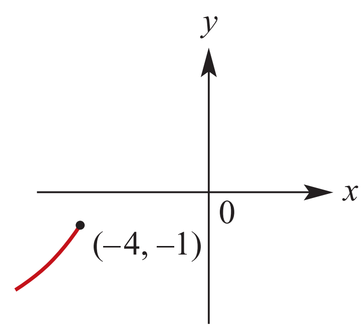


c $y = -2\sqrt{-(x+4)} - 1;\ \{x\colon x \leq -4\}$

No axis intercepts.

Starting point at $(-4, -1)$

y defined for $\{y\colon y \leq -1\}$

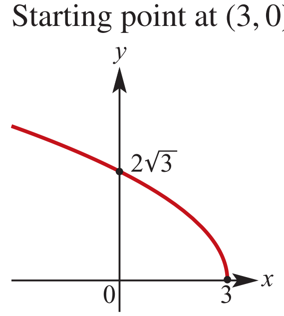


d $y = 2\sqrt{3 - x}$

When $x = 0, y = 2\sqrt{3}$

When $y = 0, x = 3$

Starting point at $(3, 0)$



e $y = -2\sqrt{3 - x}$

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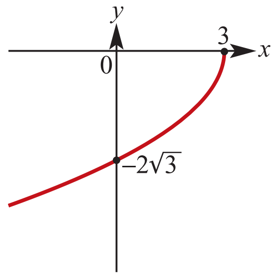
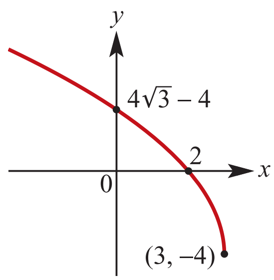
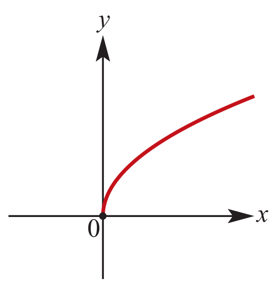
When $x = 0, y = -2\sqrt{3}$
When $y = 0, x = 3$
Starting point at $(3, 0)$



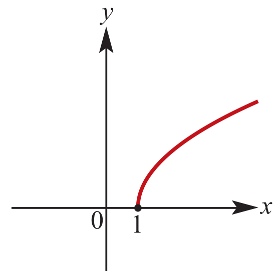
f $y = 4\sqrt{3 - x} - 4$
When $x = 0, y = 4\sqrt{3} - 4$
When $y = 0, x = 2$
Starting point at $(3, -4)$



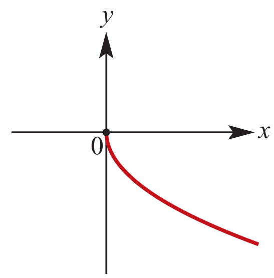
3 a $y = \sqrt{3x}$
y-values $y \geq 0$



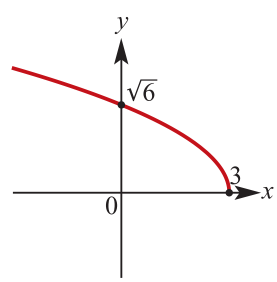
b $y = \sqrt{3(x - 1)}$
Graph of $y = \sqrt{3x}$ translated 1 unit in the positive direction of the x- axis.
x-axis intercept is $(1, 0)$ y-values
$y \geq 0$



c $y = -\sqrt{2x}$
y-values $y \leq 0$



d $y = -\sqrt{2(3 - x)}$
The graph of $y = -\sqrt{-2x}$ translated 3 units in the positive direction of the x- axis
x-axis intercept $(3, 0)$
y -axis intercept $(0, \sqrt{6})$
y-values $y \geq 0$



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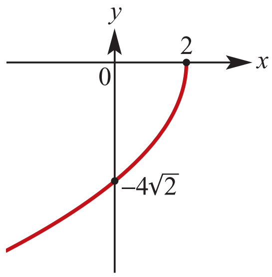
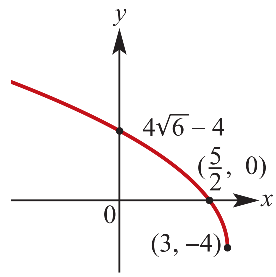
e $y = -2\sqrt{4(2 - x)}$



f $y = 4\sqrt{2(3 - x)} - 4$



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Solutions to Exercise 4D

1 a $C(0, 0),\ r = 3 \therefore\ x^2 + y^2 = 9$

b $C(0, 0),\ r = 4 \therefore\ x^2 + y^2 = 16$

c $C(1, 3),\ r = 5$
$\therefore (x - 1)^2 + (y - 3)^2 = 25$

d $C(2, -4),\ r = 3$
$\therefore (x - 2)^2 + (y + 4)^2 = 9$

e $C(-3, 4),\ r = \dfrac{5}{2}$
$\therefore (x + 3)^2 + (y - 4)^2 = \dfrac{25}{4}$

f $C(-5, -6),\ r = 4.6$
$\therefore (x + 5)^2 + (y + 6)^2 = 4.6^2$

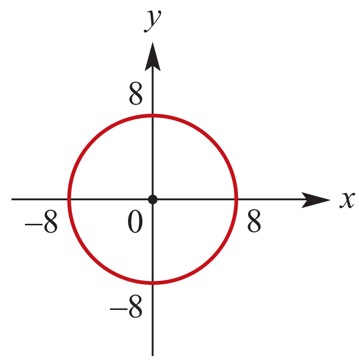
2 a $(x - 1)^2 + (y - 3)^2 = 4$
$C(1, 3),\ r = \sqrt{4} = 2$

b $(x - 2)^2 + (y + 4)^2 = 5$
$C(2, -4),\ r = \sqrt{5}$

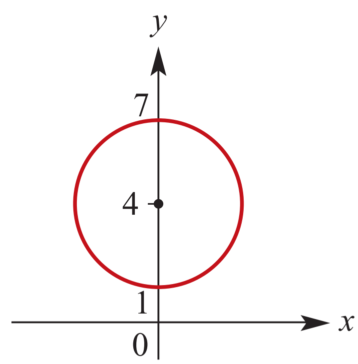
c $(x + 3)^2 + (y - 2)^2 = 9$
$C(-3, 2),\ r = \sqrt{9} = 3$

d $(x + 5)^2 + (y - 4)^2 = 8$
$C(-5, 4),\ r = \sqrt{8} = 2\sqrt{2}$

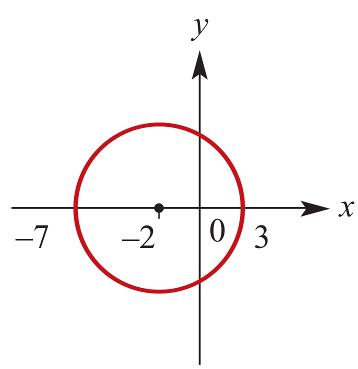
3 a $x^2 + y^2 = 64$
x-intercepts at $(\pm 8, 0)$
y-intercepts at $(0,\ \pm 8)$



b $x^2 + (y - 4)^2 = 9$
No x-intercepts,
y-intercepts at (0,1) and (0,7)



c $(x + 2)^2 + y^2 = 25$
x-intercepts at (3,0) and $(-7, 0)$



d $(x + 1)^2 + (y - 4)^2 - 169 = 0$
Centre at $(-1, 4)$, radius 13

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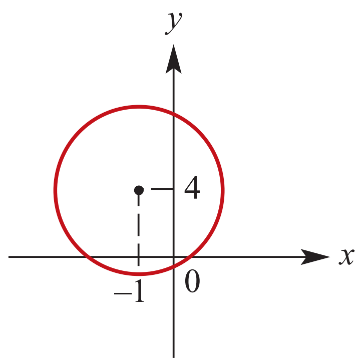
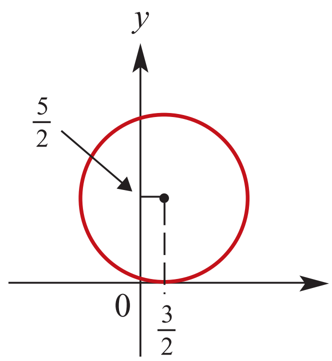


e $(2x-3)^2 + (2y-5)^2 = 36$

Centre at $\left(\frac{3}{2}, \frac{5}{2}\right)$, radius 6

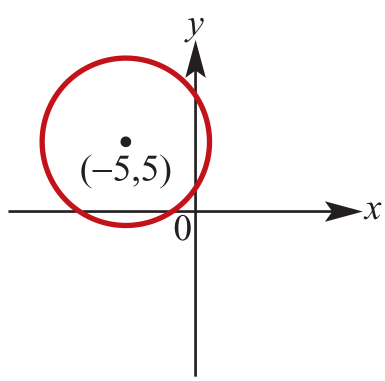


f $(x+5)^2 + (y-5)^2 = 36$

Centre at $(-5, 5)$, radius 6

x-intercepts at $(-5+\sqrt{11}, 0)$ and $(-5-\sqrt{11}, 0)$

y-intercepts at $(5+\sqrt{11}, 0)$ and $(5-\sqrt{11}, 0)$



4 **a** $x^2 + y^2 - 6y - 16 = 0$

$\therefore\ x^2 + y^2 - 6y + 9 - 9 - 16 = 0$

$\therefore\ x^2 + (y-3)^2 - 25 = 0$

$\therefore\ x^2 + (y-3)^2 = 25$

$C(0, 3),\ r = \sqrt{25} = 5$

b $x^2 + y^2 - 8x + 12y + 10 = 0$

$\therefore\ x^2 - 8x + 16 + y^2 + 12y + 36 = 42$

$\therefore\ (x-4)^2 + (y+6)^2 = 25$

$C(4, -6),\ r = \sqrt{42}$

c $x^2 + y^2 - 6x + 4y + 9 = 0$

$\therefore\ x^2 - 6x + y^2 + 4y + 9 = 0$

$\therefore\ x^2 - 6x + 9 + y^2 + 4y + 4 - 4 = 0$

$\therefore\ (x-3)^2 + (y+2)^2 = 0$

$C(3, -2),\ r = \sqrt{4} = 2$

d $x^2 + y^2 + 4x - 6y - 12 = 0$

$\therefore\ x^2 + 4x + 4 + y^2 - 6y + 9 - 12 - 4 - 9 = 0$

$\therefore\ (x+2)^2 + (y-3)^2 - 25 = 0$

$\therefore\ (x+2)^2 + (y-3)^2 = 25$

$C(-2, 3),\ r = \sqrt{25} = 5$

e $x^2 + y^2 - 8x + 4y + 1 = 0$

$\therefore\ x^2 - 8x + 16 + y^2 + 4y + 4 + 1 - 20 = 0$

$\therefore\ (x-4)^2 + (y+2)^2 = 19$

$C(4, -2),\ r = \sqrt{19}$

f $x^2 + y^2 - x + 4y + 2 = 0$

$\therefore\ x^2 - x + \frac{1}{4} + y^2 + 4y + 4 = 2 + \frac{1}{4}$

$\therefore\ (x - \frac{1}{2})^2 + (y+2)^2 = \frac{9}{4}$

$C(\frac{1}{2}, -2),\ r = \frac{3}{2}$

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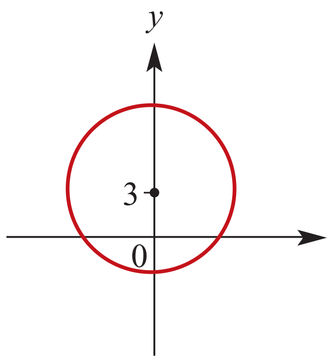
5 **a** $x^2 + y^2 - 6y - 16 = 0$

$\therefore x^2 + (y - 3)^2 = 25$

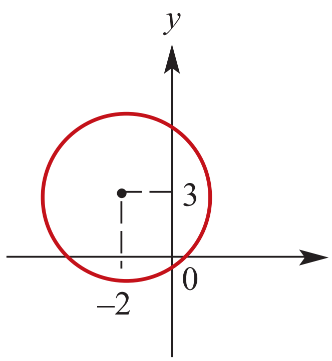
Centre at (0,3), radius 5



b $x^2 + y^2 + 4x - 6y - 3 = 0$

$\therefore (x + 2)^2 + (y - 3)^2 = 16$

Centre at $(-2, 3)$, radius 4

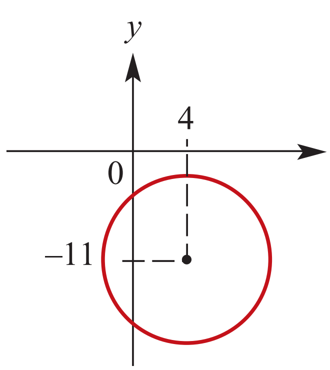


c $x^2 + y^2 - 8x + 22y + 27 = 0$

$\therefore (x - 4)^2 + (y + 11)^2 = 110$

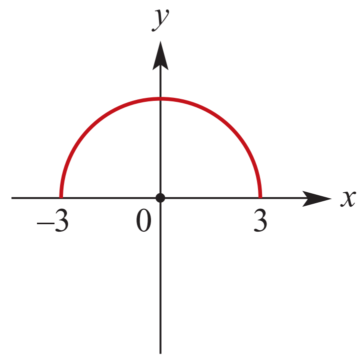
No axis intercepts

Centre (4,-11), radius $\sqrt{110}$

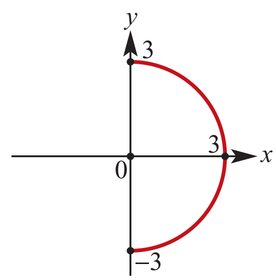


6 **a** $y = +\sqrt{9 - x^2}$

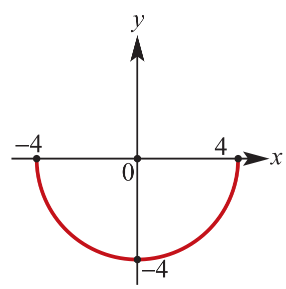
Starting points at $(\pm 3, 0)$



b

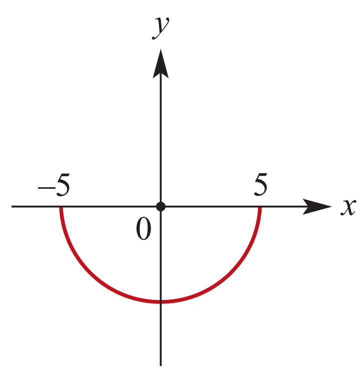


c



d $y = -\sqrt{25 - x^2}$

Starting points at $(\pm 5, 0)$



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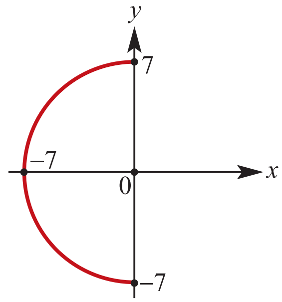
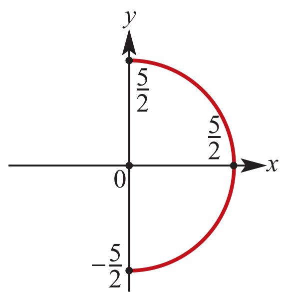
e



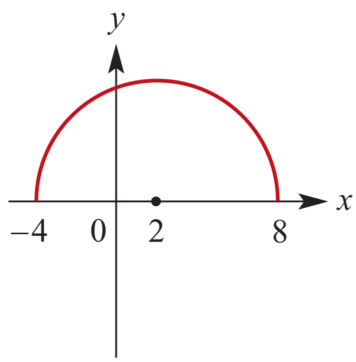
f



7 a $y = \sqrt{36 - (x - 2)^2}$

Starting points at (–4, 0) and (8,0)

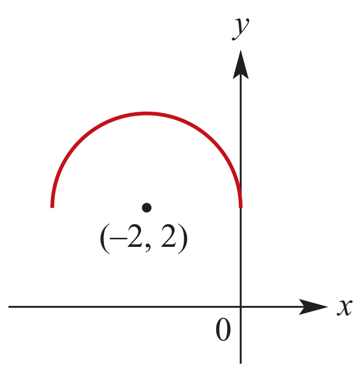
Centre at (2,0)



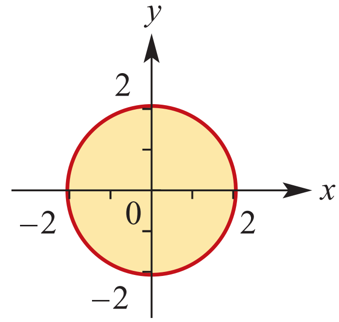
b $y - 2 = \sqrt{4 - (x + 2)^2}$

Starting points at (–4, 2) and (0,2)

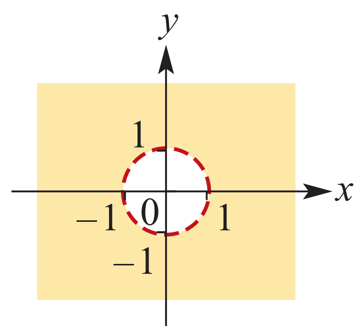
Centre at (–2, 2)



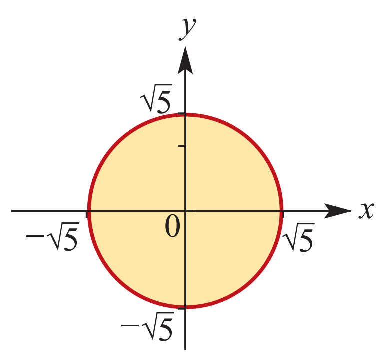
8 a $x^2 + y^2 \leq 4$



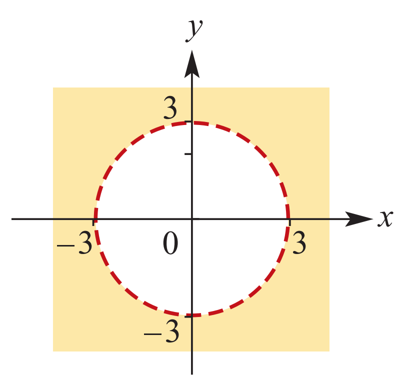
b $x^2 + y^2 > 1$



c $x^2 + y^2 \leq 5$



d $x^2 + y^2 > 9$



e $x^2 + y^2 \geq 6$

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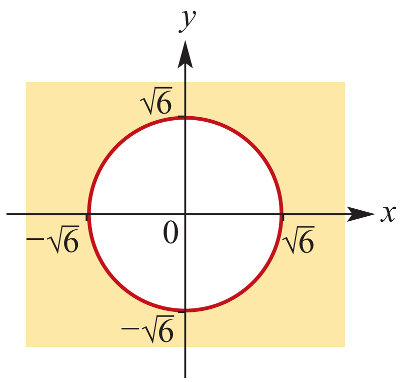
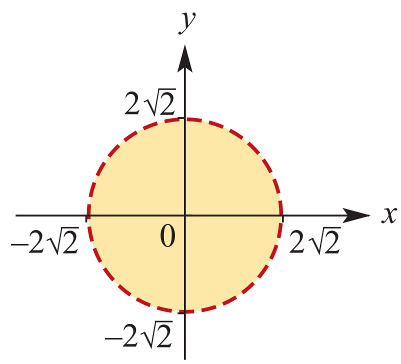


f $x^2 + y^2 < 8$



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Solutions to Exercise 4E

1 $y = \frac{a}{x} + 3$
Passes through $(1, 8)$
$\therefore 8 = a + 3$
$\therefore a = 5$
$\therefore y = \frac{5}{x} + 3$

2 $h = 3, k = 4 \therefore y = \frac{a}{x-3} + 4$
Passes through $(0, 6)$
Therefore, $6 = \frac{a}{-3} + 4$
$\therefore a = -6$
$\therefore y = -\frac{6}{x-3} + 4$

3 $y = \frac{a}{x} + k$
When $x = 1, y = 8$
$\therefore 8 = a + k \ldots (1)$
When $x = -1, y = 7$
$\therefore 7 = -a + k \ldots (2)$
Add (1) and (2)
$15 = 2k$
$k = \frac{15}{2}$ and $a = \frac{1}{2}$

4 $h = 2, k = -4 \therefore y = \frac{a}{x-2} - 4$
Passes through $(0, 4)$
Therefore, $4 = \frac{a}{-2} - 4$
$\therefore a = -16$
$\therefore y = -\frac{16}{x-2} - 4$

5 $y = a\sqrt{x}$
When $x = 2, y = 8$
$\therefore 8 = a\sqrt{2}$
$\therefore a = \frac{8}{\sqrt{2}} = 4\sqrt{2}$

6 $y = a\sqrt{x-h}$
When $x = 1, y = 2$
$\therefore 2 = a\sqrt{1-h} \ldots (1)$
When $x = 10, y = 4$
$\therefore 4 = a\sqrt{10-h} \ldots \ (2)$
Divide (2) by (1)
$2 = \frac{\sqrt{10-h}}{\sqrt{1-h}}$
$2\sqrt{1-h} = \sqrt{10-h}$
Square both sides.
$4(1-h) = 10-h$
$4 - 4h = 10 - h$
$3h = -6$
$h = -2$
Substitute in (1)
$2 = a\sqrt{3}$
$a = 2\frac{\sqrt{3}}{3}$

7 A circle with centre (2,1) has equation:
$(x-2)^2 + (y-1)^2 = a^2$
If it passes through $(4, -3)$, then:
$(4 \ -2)^2 + (-3-1)^2 = a^2$
$\therefore 4 + 16 = a^2, \therefore a = \pm 2\sqrt{5}$
$(x-2)^2 + (y-1)^2 = 20$

8 Circle centre $(-2, 3)$ has equation of the form
$(x+2)^2 + (y-3)^2 = r^2$
Circle passes through $(-3, 3)$
Therefore
$(-3+2)^2 + (3-3)^2 = r^2$
$\therefore r = 1 \therefore (x+2)^2 + (y-3)^2 = 1$
Note: Centre $(-2, 3)$ and passing through $(-3, 3)$ immediately gives you $r = 1$. Think of the horizontal diameter.

9 Again using the simple approach.

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The diameter through the circle with entre $(-2, 3)$ and passing through $(2, 3)$ tells us that the radius is 4. Hence the equation is $(x+2)^2 + (y-3)^2 = 16$

10 A circle with centre $(2, -3)$ has equation:
$(x-2)^2 + (y+3)^2 = a^2$
If it touches the x-axis, then it must be at (2,0):
$\therefore (0+3)^2 = a^2, \therefore a = \pm 3$
$(x-2)^2 + (y+3)^2 = 9$

11 A circle with centre on the line $y = 4$ has equation: $(x-b)^2 + (y-4)^2 = a^2$
If it passes through (2,0) and (6,0) then
$(2 - b)^2 + (0-4)^2 = a^2 \ldots \mathbf{(1)}$
$(6 - b)^2 + (0-4)^2 = a^2 \ldots \mathbf{(2)}$
(2) – **(1)** gives $(6-b)^2 - (2-b)^2 = 0$

$\therefore (36 - 12b + b^2) = (4 - 4b + b^2)$

$\therefore \quad 32 - 12b = -4b$

$\therefore \quad 8b = 32, \therefore b = 4$

Substitute into **(1)**:

$(2-4)^2 + (0-4)^2 = a^2$

$\therefore \quad 4 + 16 = a = 20$

$(x-4)^2 + (y-4)^2 = 20$

12 It touches the x- axis and has radius 5. Let$(a, 5)$ be the centre. It is easy to show it cannot be $(a, -5)$ if it goes through $(0, 8)$.
We also know that $a^2 + (5-8)^2 = 25$.
$\therefore a = 4$ or $a = -4$

The circle has equation
$(x-4)^2 + (y-5)^2 = 25$ or
$(x+4)^2 + (y-5)^2 = 25$

13 The circle has equation of the form
$x^2 + y^2 + bx + cy + d = 0$
When $x = 2, y = 0$

$$4 + 2b + d = 0 \ldots (1)$$

When $x = -4, y = 0$

$$16 - 4b + d = 0 \ldots (2)$$

When $x = 0, y = 2$

$$4 + 2c + d = 0 \ldots (3)$$

From (1) and (3)

we see that $c = b$.

Subtract (1) from (2)

$$12 - 6b = 0$$

$$\therefore b = 2 \text{ and } c = 2$$

$$\therefore d = -8$$

The equation is $x^2 + y^2 + 2x + 2y = 8$
or $(x+1)^2 + (y+1)^2 = 10$

14 **a** $(x-2)^2 + (y+2)^2 = 49$

b $y = 3\sqrt{x-1} - 2$

c $y = \dfrac{1}{x-2} + 2$

d $y = -\dfrac{2}{x-1} - 2$

e $y = \sqrt{2-x} + 1$

f $y = \dfrac{1}{(x-2)^2} - 3$

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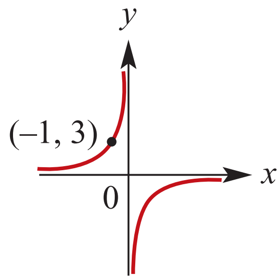
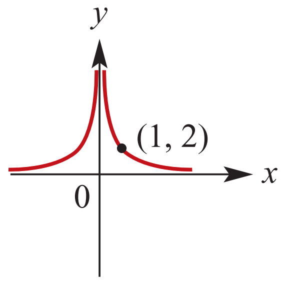
Solutions to Technology-free questions

1 **a** $y = -\frac{3}{x}$; asymptotes at $x = 0,\ y = 0$

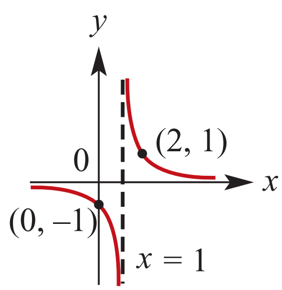


b $y = \frac{2}{x^2}$; asymptotes at $x = 0,\ y = 0$



c $y = \frac{1}{x-1}$; asymptotes at $x = 1,\ y = 0$

y-intercept at $(0, -1)$

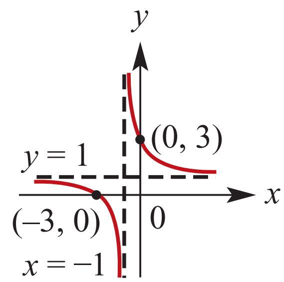


d $y = \frac{2}{x+1} + 1$;

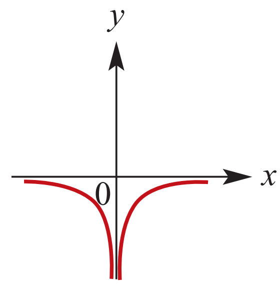
asymptotes at $x = -1,\ y = 1$

x-intercept at $(-3, 0)$

y-intercept at $(0,3)$



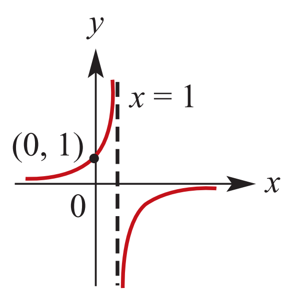
e $y = -\frac{2}{x^2}$; asymptotes at $x = 0,\ y = 0$



f $y = -\frac{1}{x-1}$; asymptotes at

$x = 1,\ y = 0$

y-intercept at $(0, 1)$



g $y = \frac{4}{2-x} + 3$; asymptotes at

$x = 2,\ y = 3$

x-intercept: $\frac{4}{2-x} + 3 = 0$

$\therefore \frac{4}{2-x} = -3$

$\therefore 4 = -3(2-x)$

$\therefore 4 = 3x - 6 \therefore x = \frac{10}{3}$

x-intercept at $\left(\frac{10}{3}, 0\right)$

y-intercept at $(0,5)$

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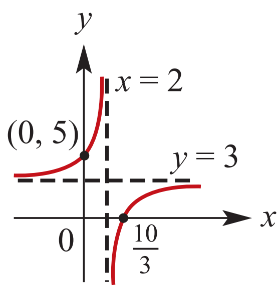
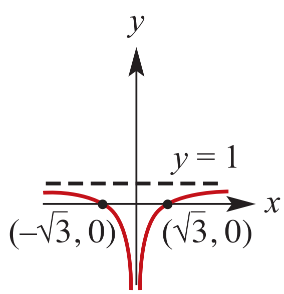


h $y = -\dfrac{3}{x^2} + 1$; asymptotes at $x = 0,\ y = 1$

x-intercepts: $y = -\dfrac{3}{x^2} + 1 = 0$

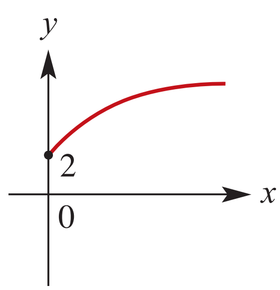
$\therefore \dfrac{3}{x^2} = 1 \quad \therefore x = \pm\sqrt{3}$

x-intercepts at $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$



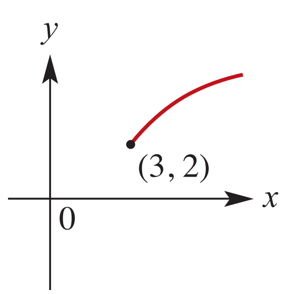
i $y = 2\sqrt{x} + 2$

y-intercept at (0,2)



j $y = 2\sqrt{x-3} + 2$

Starting point at (3,2)



k $y = -2\sqrt{x+2} + 2$

Starting point at $(-2, 2)$

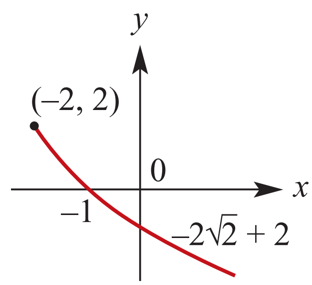
x-intercept: $-2\sqrt{x+2} + 2 = 0$

$\therefore \sqrt{x+2} = 1$

$\therefore x + 2 = 1, \therefore x = -1$

x-intercept at $(-1, 0)$

y-intercept at $(0, 2 - 2\sqrt{2})$



2 **a** $x^2 + y^2 - 6x + 4y - 12 = 0$

$\therefore x^2 - 6x + 9 + y^2 + 4y + 4 - 12 = 13$

$\therefore (x-3)^2 + (y+2)^2 = 5^2$

b $x^2 + y^2 - 3x + 5y - 4 = 0$

$\therefore x^2 - 3x + \dfrac{9}{4} + y^2 + 5y + \dfrac{25}{4} = 4 + \dfrac{34}{4}$

$\therefore \left(x - \dfrac{3}{2}\right)^2 + \left(y + \dfrac{5}{2}\right)^2 = \dfrac{50}{4}$

$\therefore \left(x - \dfrac{3}{2}\right)^2 + \left(y + \dfrac{5}{2}\right)^2 = \left(\dfrac{5\sqrt{2}}{2}\right)^2$

c $2x^2 + 2y^2 - x + y - 4 = 0$

$\therefore x^2 + y^2 - \dfrac{x}{2} + \dfrac{y}{2} = 2$

$\therefore x^2 - \dfrac{x}{2} + \dfrac{1}{16} + y^2 + \dfrac{y}{2} + \dfrac{1}{16} = 2 + \dfrac{1}{8}$

$\therefore \left(x - \dfrac{1}{4}\right)^2 + \left(y + \dfrac{1}{4}\right)^2 = \dfrac{17}{8}$

$\therefore \left(x - \dfrac{1}{4}\right)^2 + \left(y + \dfrac{1}{4}\right)^2 = \left(\dfrac{\sqrt{34}}{4}\right)^2$

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d $x^2 + y^2 + 4x - 6y = 0$
$\therefore x^2 + 4x + 4 + y^2 - 6y + 9 = 13$
$\therefore (x + 2)^2 + (y - 3)^2 = (\sqrt{13})^2$

e $x^2 + y^2 = 6(x + y)$
$\therefore x^2 - 6x + 9 + y^2 - 6y + 9 = 18$
$\therefore (x - 3)^2 + (y - 3)^2 =$
$(\sqrt{18})^2 = (3\sqrt{2})^2$

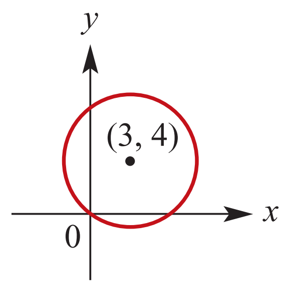
f $x^2 + y^2 = 4x - 6y$
$\therefore x^2 - 4x + 4 + y^2 + 6y + 9 = 13$
$\therefore (x - 2)^2 + (y + 3)^2 = (\sqrt{13})^2$

3 $x^2 + y^2 - 4x + 6y = 14$
$\therefore x^2 - 4x + 4 + y^2 + 6y + 9 = 14 + 13$
$\therefore (x - 2)^2 + (y + 3)^2 = 27$
A diameter must pass through the centre of the circle at $(2, -3)$.
A line connecting $(0,0)$ with $(2, -3)$ has gradient $= -\frac{3}{2}$, hence the equation
$y = -\frac{3x}{2}$ or $3x + 2y = 0$

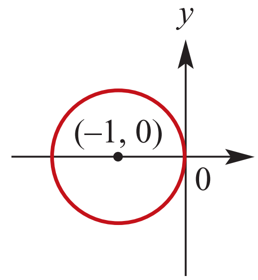
4 $x^2 + y^2 - 3x + 2y = 26$
$\therefore x^2 - 3x + \frac{9}{4} + y^2 + 2y + 1 = 26 + \frac{13}{4}$
$\therefore \left(x - \frac{3}{2}\right)^2 + (y + 1)^2 = \frac{117}{4}$
Centre of circle is at $\left(\frac{3}{2}, -1\right)$.
Diameter of the circle which cuts the x-axis at 45° has gradient = 1.
$\therefore y - (-1) = x - \frac{3}{2}$
$y = x - \frac{5}{2}$
Or $2x + 2y = 1$ if the diameter goes below the axis at an angle of $-45°$. In this case the gradient $= -1$.

6

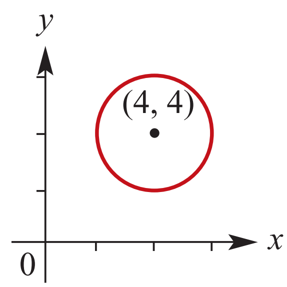
5 **a** $C(3, 4),\ r = 5$
$\therefore (x - 3)^2 + (y - 4)^2 = 25$



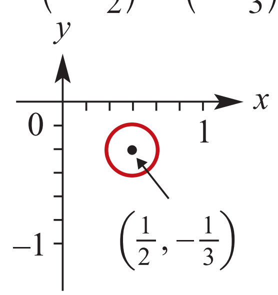
b $C(-1, 0),\ r = 1$
$\therefore (x + 1)^2 + y^2 = 1$



c $C(4, 4),\ r = 2$
$\therefore (x - 4)^2 + (y - 4)^2 = 4$



d $C\left(\frac{1}{2}, -\frac{1}{3}\right),\ r = \frac{1}{6}$
$\therefore \left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{3}\right)^2 = \frac{1}{36}$



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$$x^2 + y^2 + 4x - 6y = 23$$
$$x^2 + 4x + 4 + y^2 - 6y + 9 = 23 + 4 + 9$$
$$(x + 2)^2 + (y - 3)^2 = 36$$

Centre: $(-2, 3)$ Radius: 6

7 $x^2 + y^2 - 2x - 4y = 20$

$\therefore x^2 - 2x + 1 + y^2 - 4y + 4 = 25$

$\therefore (x - 1)^2 + (y - 2)^2 = 5^2$

Length cut off on the x-axis and y-axis = distance between x- and y-intercepts:

$y = 0$: $\therefore (x - 1)^2 + (0 - 2)^2 = 5^2$

$\therefore (x - 1)^2 = 21, \therefore x = 1 \pm \sqrt{21}$

x-axis length $= 2\sqrt{21}$

$x = 0$: $\therefore (0 - 1)^2 + (y - 2)^2 = 5^2$

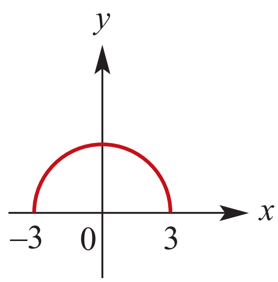
$\therefore (y - 2)^2 = 24, \therefore y = 2 \pm 2\sqrt{6}$

y-axis length $= 4\sqrt{6}$

8 **a** $y = \sqrt{9 - x^2}$

x-intercepts at $(-3, 0)$ and $(3,0)$

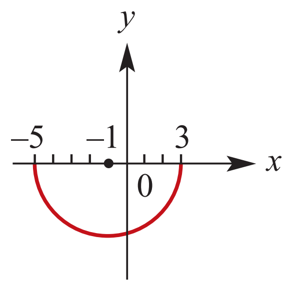
y-intercept at $(0,3)$



b $y = -\sqrt{16 - (x + 1)^2}$

Starting points at $(-5, 0)$ and $(3,0)$

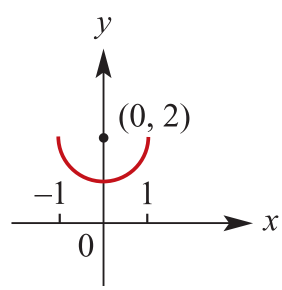
y-intercept at $(0, -\sqrt{15})$, centre at $(-1, 0)$



c $y - 2 = -\sqrt{1 - x^2}$

No x-intercepts, y-intercept at $(0, 1)$

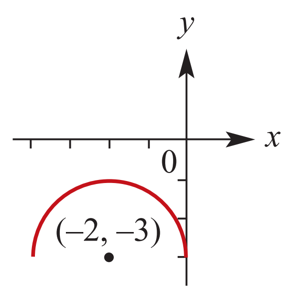
Centre at $(0,2)$



d $y + 3 = \sqrt{4 - (x + 2)^2}$

No x-intercepts, y-intercept at $(0, -3)$

Centre at $(-2, 3)$



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Solutions to multiple-choice questions

1 E $(x-a)^2+(y-b)^2=36$
Centre on the x-axis so $b=0$
Using (6,6): $(6-a)^2+6^2=36$
$\therefore a=6$

2 B $y=5-\dfrac{1}{3x-5}$ has asymptotes at
$y=5$ and $3x-5=0$
$\therefore x=\dfrac{5}{3}$

3 E $y=\dfrac{5}{x^2}+3$
If $x=\dfrac{a}{2}$,
$y=\dfrac{5}{(a/2)^2}+3=\dfrac{20}{a^2}+3$

4 A $(x-a)^2+(y-b)^2=c^2$
y-axis is an axis of symmetry so
$a=0$ Using (0,0):
$(-b)^2=c^2, \therefore b=c; c>0$
Using (0,4):
$(4-b)^2=b^2, \therefore b=2$
$\therefore x^2+(y-2)^2=4$

5 A $y=5+\dfrac{1}{(x-2)^2}$ has asymptotes
at $y=5$ and $x=2$

6 D $(x-5)^2+(y+2)^2=9$
$C(5,-2),\ r=\sqrt{9}=3$

7 D $y=-2\sqrt{x}+3; x\geq 0$
$-2\sqrt{x}\leq 0 \therefore y\leq 3$

8 C Circle end points at $(-2,8)$ and
(6,8);
centre is at $x=\dfrac{-2+6}{2}=2$ and
$y=8$.
Radius $=6-2=4$
$(x-2)^2+(y-8)^2=4^2$

9 E Only $y^2=16-x^2$ has the general
form with $(x-a)^2+(y-b)^2=c^2$

10 B **A** is a full circle, **B** is correct, **C**
isn't circular, and **D** and **E** are
negatives.

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Solutions to extended-response questions

1 **a** The circle has centre (10, 0) and radius 5 and therefore has the equation $(x-10)^2 + y^2 = 25$.

b The line with equation $y = mx$ meets the circle with equation $(x-10)^2 + y^2 = 25$.

Therefore x satisfies the equation $\quad (x-10)^2 + (mx)^2 = 25$

Expanding and rearranging gives $\quad x^2 - 20x + 100 + m^2x^2 = 25$

and therefore $\quad (1+m^2)x^2 - 20x + 75 = 0$

c The discriminant is
$$\begin{aligned}\Delta &= 400 - 4 \times 75 \times (1+m^2)\\ &= 400 - 300(1+m^2)\\ &= 100 - 300\,m^2\end{aligned}$$
As the line is a tangent to the circle, there is only one point of contact and hence only one solution to the equation obtained in part **b**. Therefore the discriminant = 0, which implies
$$m = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$$

d When $m^2 = \frac{1}{3}$, the equation $(1+m^2)x^2 - 20x + 75 = 0$ becomes
$$\frac{4}{3}x^2 - 20x + 75 = 0$$
Multiplying both sides of the equation by 3 gives
$$4x^2 - 60x + 225 = 0$$
The left-hand side is a perfect square and hence
$$(2x-15)^2 = 0$$
The solution is $\quad x = \frac{15}{2}$

The y-coordinate is given by substituting into $y = mx = \pm\frac{\sqrt{3}}{3}x$.
$$\begin{aligned}y &= \pm\frac{\sqrt{3}}{3} \times \frac{15}{2}\\ &= \pm\frac{5\sqrt{3}}{2}\end{aligned}$$
The coordinates of P are $\left(\frac{15}{2}, \pm\frac{5\sqrt{3}}{2}\right)$.

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e Distance of P from the origin $= \sqrt{\left(\frac{15}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2}$

$$= \frac{1}{2}\sqrt{225 + 75}$$

$$= 5\sqrt{3}$$

2 **a** The circle has centre the origin and radius 4.
Hence the equation is $x^2 + y^2 = 16$.

b **i** The general form for a straight line is $y = mx + c$.
When $x = 8$, $y = 0$, hence $0 = 8m + c$ and $c = -8m$.
So the tangents have equations of the form $y = mx - 8m$

ii As in Question 1, consider when the line with equation $y = mx - 8m$ meets the circle $x^2 + y^2 = 16$.
Substitute for y: $(mx - 8m)^2 + x^2 = 16$
Expand and collect like terms to obtain

$$(m^2 + 1)x^2 - 16m^2x + 64m^2 - 16 = 0$$

There will be a tangent when the discriminant is equal to 0, i.e. when there is only one solution.

$$\begin{aligned}\Delta &= 256m^4 - 4(m^2 + 1)(64m^2 - 16)\\ &= 256m^4 - 4(64m^4 + 48m - 16)\\ &= -4(48m^2 - 16)\\ &= 64(-3m^2 + 1)\end{aligned}$$

Thus there is a tangent if $3m^2 = 1$

i.e. $m = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$

Since $y = mx - 8m$, is the equations of the tangents

$$y = \frac{\sqrt{3}}{3}x - \frac{8\sqrt{3}}{3}$$

and $y = -\frac{\sqrt{3}}{3}x + \frac{8\sqrt{3}}{3}$

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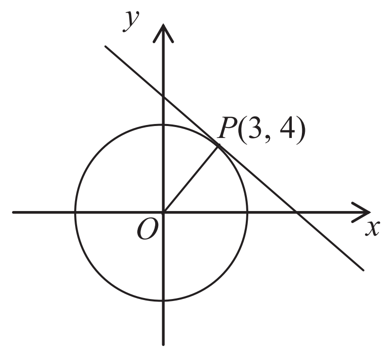
3 a The gradient of $OP = \frac{4}{3}$.



b The tangent is perpendicular to the radius and therefore has gradient of $\frac{-3}{4}$.

c The equation of the tangent is given by

$y - 4 = -\frac{3}{4}(x - 3)$

Therefore $\quad 4y - 16 = -3x + 9$

and $\quad 4y + 3x = 25$

d The coordinates of A are $\left(\frac{25}{3}, 0\right)$ and the coordinates of B are $\left(0, \frac{25}{4}\right)$.

$$AB^2 = \left(\frac{25}{3}\right)^2 + \left(\frac{25}{4}\right)^2$$

$$= 625\left(\frac{1}{9} + \frac{1}{16}\right) = 625 \times \frac{25}{9 \times 16}$$

Therefore $\quad AB = \frac{25 \times 5}{12} = \frac{125}{12}$

4 a i The radius has gradient $\frac{y_1}{x_1}$.

ii The tangent is perpendicular to radius, so using $m_l,\ m_2 = -1$, it has the gradient $-\frac{x_1}{y_1}$.

b The equation of the tangent is given by $\quad y - y_1 = -\frac{x_1}{y_1}(x - x_1)$

Multiply both sides by y_1 $\quad yy_1 - y_1^2 = -x_1x + x_1^2$

Therefore $\quad yy_1 + x_1x = x_1^2 + y_1^2$

But $(x_1,\ y_1)$ is a point on the circle and hence $\quad x_1^2 + y_1^2 = a^2$

This gives $\quad x_1x + y_1y = a^2$

c If $x_1 = y_1$ and $a = 4$,then $x_1^2 + y_1^2 = a^2$ becomes $2x_1^2 = 16$.
Thus $x_1 = \pm 2\sqrt{2}$
The equations of the tangents are $\sqrt{2}x + \sqrt{2}y = 8$ and
$-\sqrt{2}x - \sqrt{2}y = 8$ or $\sqrt{2}x + \sqrt{2}y = -8$.

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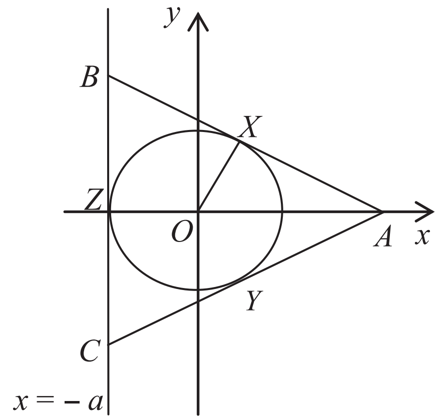
5 a Note that the triangle is equilateral, and that $AX = AY = XB = YC = CZ = ZB$, as these line segments are equal tangents from a point.

Let these equal lengths be b.



Then $\quad AZ^2 = AB^2 - BZ^2$

$$= 4b^2 - b^2 = 3b^2$$

(Pythagoras' theorem in triangle AZB)

Therefore $\quad AZ = \sqrt{3}b$

The gradient of line $BA = -\dfrac{1}{\sqrt{3}}$

and the gradient of $CA = \dfrac{1}{\sqrt{3}}$.

Note also that triangle BZA is similar to triangle AXO.

Therefore $\quad \dfrac{OX}{AX} = \dfrac{BZ}{AZ}$

Therefore $\quad \dfrac{a}{b} = \dfrac{b}{\sqrt{3}b} = \dfrac{1}{\sqrt{3}} \quad$ and $\quad b = \sqrt{3}a$

For line BA: when $x = -a$, $y = \sqrt{3}a$ and, using the form $y = mx + c$,

$$\sqrt{3}a = -\frac{1}{\sqrt{3}} \times -a + c$$

$$\therefore \quad c = \sqrt{3}a - \frac{a}{\sqrt{3}}$$

$$= \frac{2\sqrt{3}a}{3}$$

Therefore the equation of line BA is $\quad y = -\dfrac{\sqrt{3}}{3}x + \dfrac{2\sqrt{3}a}{3}$

For line CA: $\quad y = \dfrac{\sqrt{3}}{3}x - \dfrac{2\sqrt{3}a}{3}$

b The circumcentre has centre O and radius OA. But, from the equation of BA, A has coordinates $(2a, 0)$. Hence the equation of the circle is $x^2 + y^2 = 4a^2$.

6 a If $(a,\ a)$ lies on the line $y = x$ and on the curve with equation $y = \sqrt{x - b} + c$,

then $\quad a = \sqrt{a - b} + c$

Subtract c from both sides and square $\quad (a - c)^2 = a - b$

Expand and rearrange $\quad a^2 - 2ac + c^2 = a - b$

$$a^2 - (2c + 1)a + c^2 + b = 0$$

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b **i** The line meets the curve at one point if the discriminant of the quadratic in a is zero.

$$\begin{aligned}\Delta &= (2c+1)^2 - 4(c^2+b)\\ &= 4c^2 + 4c + 1 - 4c^2 - 4b\\ &= 4c - 4b + 1\end{aligned}$$

If the discriminant is zero, $c = \dfrac{4b-1}{4}$

ii Solving the equation $x = \sqrt{x} - \dfrac{1}{4}$ will give the required coordinates.

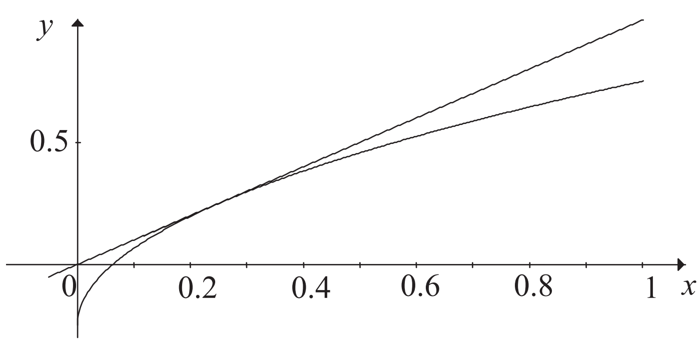
Squaring both sides of $x + \dfrac{1}{4} = \sqrt{x}$ gives $x^2 + \dfrac{1}{2}x + \dfrac{1}{16} = x$

and rearranging gives $x^2 - \dfrac{1}{2}x + \dfrac{1}{16} = 0$

$$\left(x - \frac{1}{4}\right)^2 = 0$$

Therefore $x = \dfrac{1}{4}$

and $y = \dfrac{1}{4}$



c **i** From the above, we know that the line with equation $y = x$ is tangent to the curve with equation $y = \sqrt{x} - \dfrac{1}{4}$.

Hence if $-\dfrac{1}{4} < k < 0$, the line will cross the curve twice.

ii If $k = 0$ or $k < -\dfrac{1}{4}$, the line will cross the curve once.

iii It will not meet the curve if $k > 0$.

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7 **a** From the graphs of $y = kx$ and $y = \sqrt{x} - 1$, it is clear that if $k \le 0$, the line $y = kx$ can only cut the curve once.

For two solutions, consider the equation: $kx = \sqrt{x} - 1$

$$\begin{aligned}\Delta &= (2k-1)^2 - 4k^2 \\ &= 4k^2 - 4k + 1 - 4k^2 \\ &= -4k + 1\end{aligned}$$

Thus for two solutions, $-4k + 1 > 0$, i.e. $k < \dfrac{1}{4}$, and $k > 0$.

Hence $0 < k < \dfrac{1}{4}$.

b There is one solution when $k = \dfrac{1}{4}$ or $k \le 0$.

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Chapter 5 – Functions and relations

Solutions to Exercise 5A

1 $A = \{1, 2, 3, 5, 7, 11, 15\}$
$B = \{7, 11, 25, 30, 32\}$
$C = \{1, 7, 11, 25, 30\}$
$A \cap B$ means must be in both A and B
$A \cup B$ means must be in either or any
$A \backslash B$ means in A but not B

a $A \cap B = \{7, 11\}$

b $A \cap B \cap C = \{7, 11\}$

c $A \cup C = \{1, 2, 3, 5, 7, 11, 15, 25, 30\}$

d $A \cup B =$
$\{1, 2, 3, 5, 7, 11, 15, 25, 30, 32\}$

e $A \cup B \cup C =$
$\{1, 2, 3, 5, 7, 11, 15, 25, 30, 32\}$

f $(A \cap B) \cup C = \{1, 7, 11, 25, 30\}$

2 **a** $A \backslash B = \{1, 2, 3, 5, 15\}$

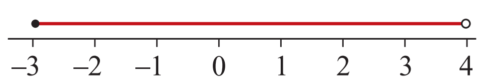
b $B \backslash A = \{25, 30, 32\}$

c $A \backslash C = \{2, 3, 5, 15\}$

d $C \backslash A = \{25, 30\}$

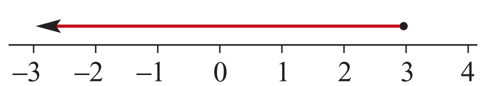
3 **a** $[-3, 4)$

a



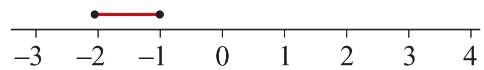
b $(-\infty, 3]$

b



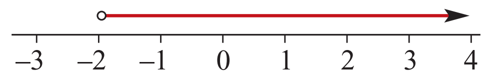
c $[-2, -1]$

c



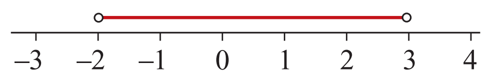
d $(-2, \infty)$

d



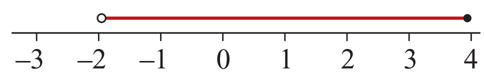
e $(-2, 3)$

e



f $(-2, 4]$

f



4 **a** $(-2, 1]$

b $[-3, 3]$

c $[-3, 2)$

d $(-1, 2)$

5 **a** $\{x\colon -1 \le x \le 2\} = [-1, 2]$

b $\{x\colon -4 < x \le 2\} = (-4, 2]$

c $\{y\colon 0 < y < \sqrt{2}\} = (0, \sqrt{2})$

d $\{y\colon -\frac{\sqrt{3}}{2} < y \le \frac{1}{\sqrt{2}}\} = \left(-\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\right]$

e $\{x\colon x > -1\} = (-1, \infty)$

f $\{x\colon x \le -2\} = (-\infty, -2]$

g $\mathbb{R} = (-\infty, \infty)$

h $\mathbb{R}^+ \cup \{0\} = [0, \infty)$

i $\mathbb{R}^- \cup \{0\} = (-\infty, 0]$

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6 $B = \{7, 11, 25, 30, 32\}$

a $(-2, 10] \cap B = \{7\}$

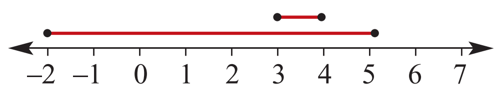
b $(3, \infty) \cap B = \{7, 11, 25, 30, 32\} = B$

c $(2, \infty) \cup B = (2, \infty)$

d $(25, \infty) \cap B = \{30, 32\}$

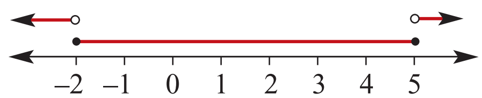
7 **a** $[-2, 5], [3, 4], [-2, 5] \cap [3, 4]$

a



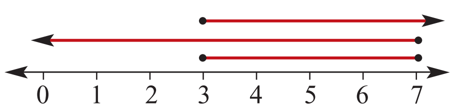
b $[-2, 5], \mathbb{R}\backslash[-2, 5]$

b



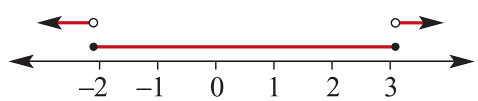
c $[3, \infty), (-\infty, 7], [3, \infty) \cap (-\infty, 7]$

c



d $[-2, 3], \mathbb{R}\backslash[-2, 3]$

d

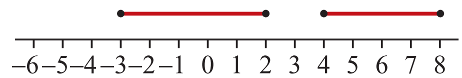


8 **a** $(-\infty, -2) \cup (-2, \infty)$

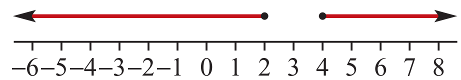
b $(-\infty, 3) \cup (3, \infty)$

c $(-\infty, 4) \cup (4, \infty)$

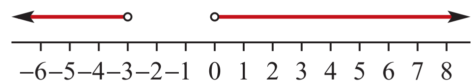
9 **a**



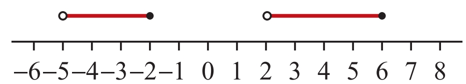
b



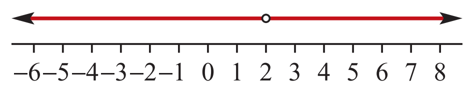
c



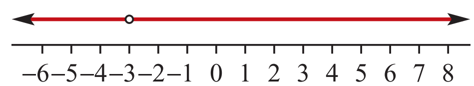
d



e



f



10 **a** $(-6, -3)$

b $\varnothing$

c $[-6, 0]$

d $[-1, 2]$

e $\{1\}$

f $(-10, -1]$

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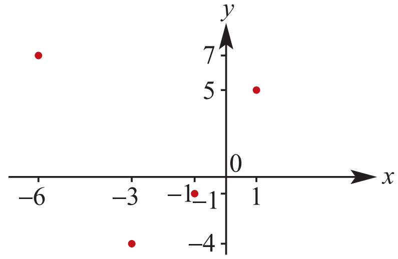
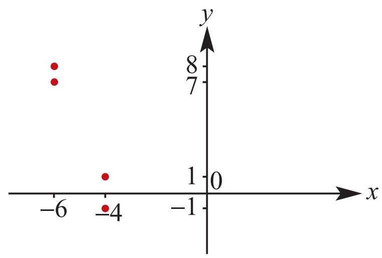
Solutions to Exercise 5B

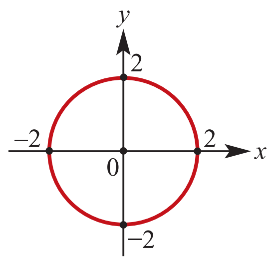
1 a Domain = $\{-3, -1, -6, 1\}$;
Range = $\{-4, -1, 7, 5\}$



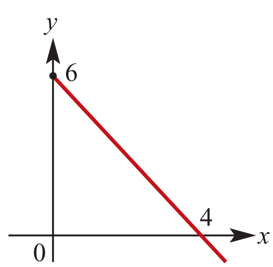
b Domain = $\{-4, -6\}$; Range = $\{-1, 1, 7, 8\}$



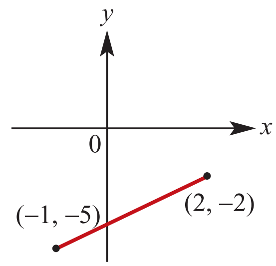
c Domain = $[-2, 2]$
Range = $[-2, 2]$



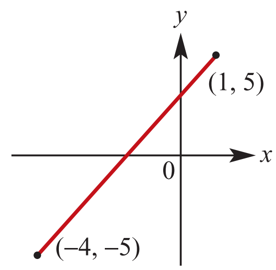
d Domain = $[0, \infty)$
Range = $(-\infty, 6]$



e Domain = $[-1, 2]$
Range = $[-5, -2]$



f Domain = $[-4, 1]$
Range = $[-5, 5]$



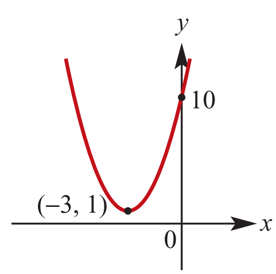
2 a Domain = $[-2, 2]$; Range = $[-1, 2]$

b Domain = $[-2, 2]$; Range = $[-2, 2]$

c Domain = $\mathbb{R}$; Range = $[-1, \infty)$

d Domain = $\mathbb{R}$; Range = $(-\infty, 4]$

3 a $x^2 + 6x + 10 = x^2 + 6x + 9 - 9 + 10$
$= (x - 3)^2 + 1$
$y = (x + 3)^2 + 1$
Range = $[1, \infty)$



b

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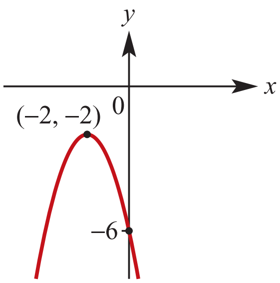
$$-x^2 - 4x - 6 = -(x^2 + 4x + 6)$$
$$= -[x^2 + 4x + 4 - 4 + 6]$$
$$= -(x + 2)^2 - 2$$

$y = -(x + 2)^2 - 2$

Range = $(-\infty, -2]$

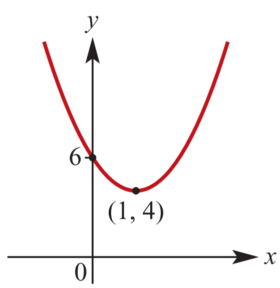


c

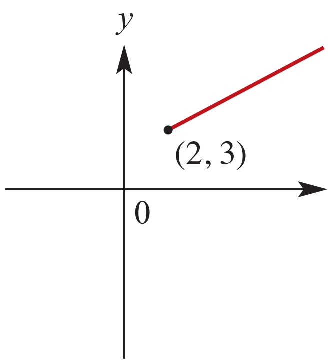
$$2x^2 - 4x + 6 = 2(x^2 - 2x + 3)$$
$$= 2[x^2 - 2x + 1 - 1 + 3]$$
$$= 2(x - 1)^2 + 4$$

$y = 2(x - 1)^2 + 4$

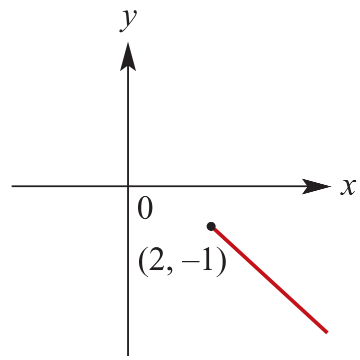
Range = $[4, \infty)$



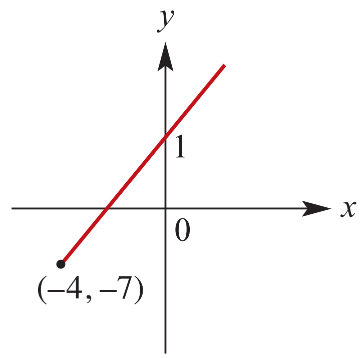
4 **a** $y = x + 1; x \in [2, \infty)$; Range = $[3, \infty)$



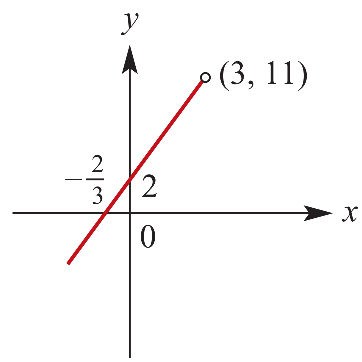
b $y = -x + 1; x \in [2, \infty)$;
Range = $(-\infty, -1]$



c $y = 2x + 1; x \in [-4, \infty)$;
Range = $[-7, \infty)$



d $y = 3x + 2; x \in (-\infty, 3)$;
Range = $(-\infty, 11)$



e $y = x + 1; x \in (-\infty, 3]$;
Range = $(-\infty, 4]$

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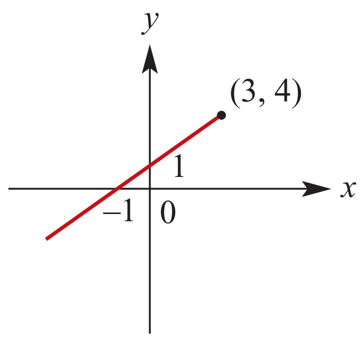
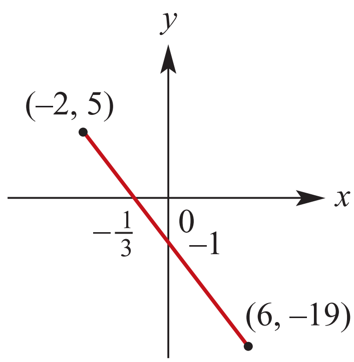
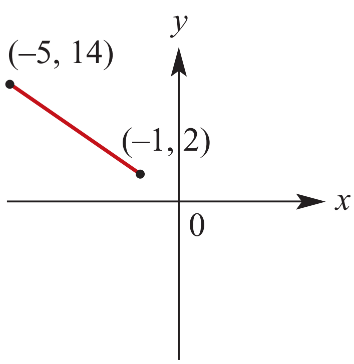


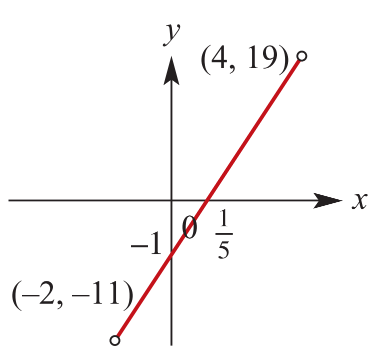
f $y = -3x - 1; x \in [-2, 6]$;
Range $= [-19, 5]$



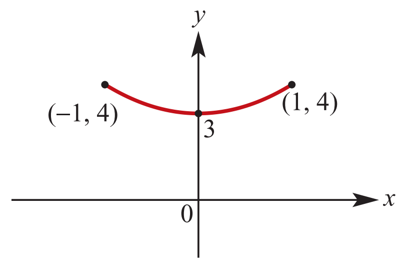
g $y = -3x - 1; x \in [-5, -1]$;
Range $= [2, 14]$



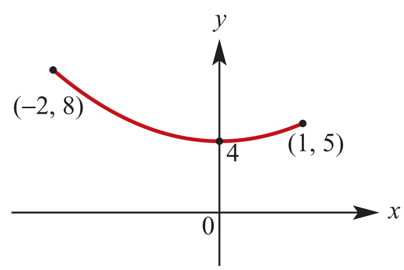
h $y = 5x - 1; x \in (-2, 4)$;
Range $= (-11, 19)$



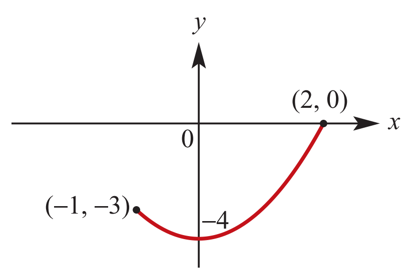
5 a $y = x^2 + 3,\ x \in [-1, 1]$
Range $= [3, 4]$



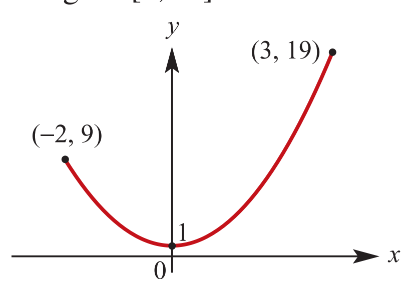
b $y = x^2 + 4,\ x \in [-2, 1]$
Range $= [4, 8]$



c $y = x^2 - 4,\ x \in [-1, 2]$
Range $= [-4, 0]$



d $y = 2x^2 + 1,\ x \in [-2, 3]$
Range $= [1, 19]$



6 a $\{(x, y): y = x^2 + 1\}$;
Range $= [1, \infty)$

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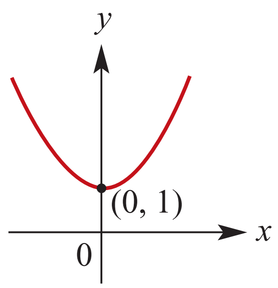
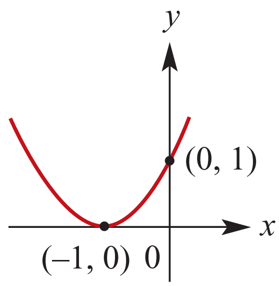
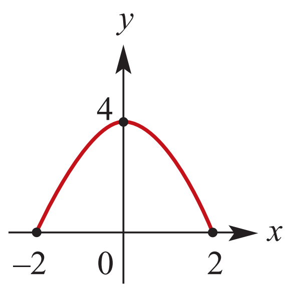


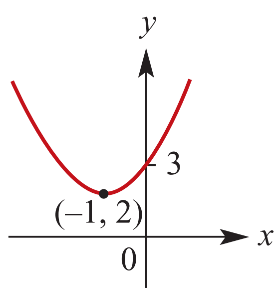
b $\{(x, y): y = x^2 + 2x + 1\}$;
Range = $[0, \infty)$



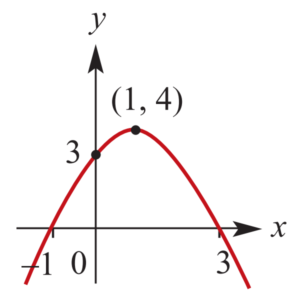
c $\{(x, y): y = 4 - x^2; x \in [-2, 2]\}$;
Range = $[0, 4]$



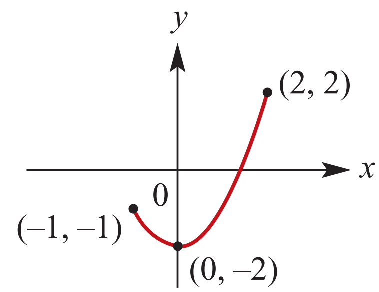
d $\{(x, y): y = x^2 + 2x + 3\}$;
Range = $[2, \infty)$



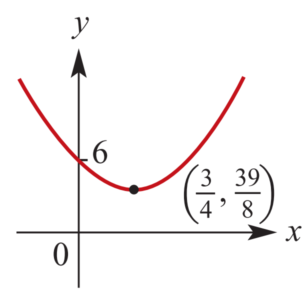
e $\{(x, y): y = -x^2 + 2x + 3\}$;
Range = $(-\infty, 4]$



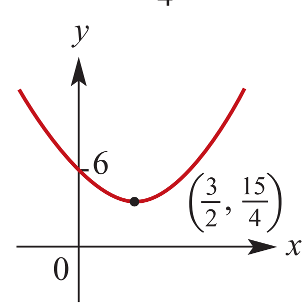
f $\{(x, y): y = x^2 - 2; x \in [-1, 2]\}$;
Range = $[-2, 2]$



g $\{(x, y): y = 2x^2 - 3x + 6\}$;
Range = $[\frac{39}{8}, \infty)$



h $\{(x, y): y = 6 - 3x + x^2\}$;
Range = $[\frac{15}{4}, \infty)$



7 a $\{(x, y): x^2 + y^2 = 9\}$
Max. Domain = $[-3, 3]$

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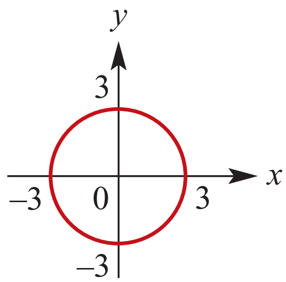
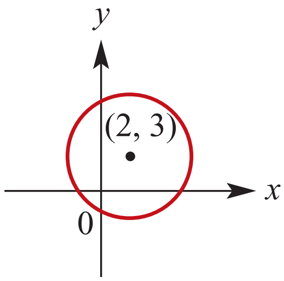
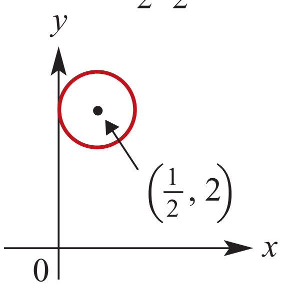
Range = $[-3, 3]$



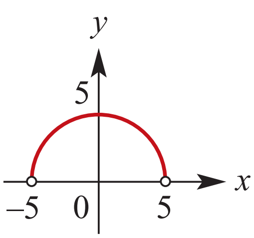
b $\{(x, y): (x-2)^2 + (y-3)^2 = 16\}$
Max. Domain = $[-2, 6]$
Range = $[-1, 7]$



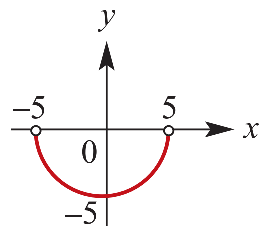
c $\{(x, y): (2x-1)^2 + (2y-4)^2 = 1\}$
Max. Domain = $[0, 1]$
Range = $[\frac{3}{2}, \frac{5}{2}]$



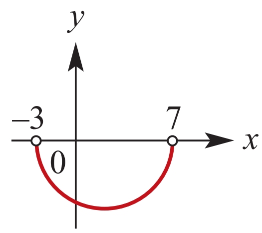
d $\{(x, y): y = \sqrt{25 - x^2}\}$
Max. Domain = $[-5, 5]$,
Range = $[0, 5]$



e $\{(x, y): y = -\sqrt{25 - x^2}\}$
Max. Domain = $[-5, 5]$,
Range = $[-5, 0]$



f $(x, y): y = -\sqrt{25 - (x-2)^2}\}$
Max. Domain = $[-3, 7]$
Range = $[-5, 0]$

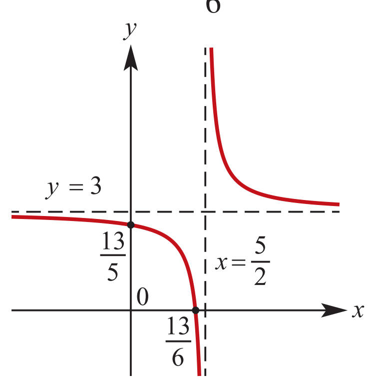


8 a Domain = $\mathbb{R} \setminus \left\{\frac{5}{2}\right\}$; Range = $\mathbb{R} \setminus \{3\}$

When $x = 0, y = -\frac{2}{5} + 3 = \frac{13}{5}$

When $y = 0$

$$\frac{2}{2x-5} + 3 = 0$$
$$\frac{2}{2x-5} = -3$$
$$2 = -3(2x-5)$$
$$2 = -6x + 15$$
$$x = \frac{13}{6}$$



b Domain = $\left[\frac{5}{2}, \infty\right)$

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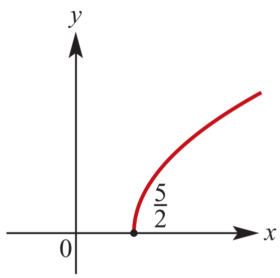
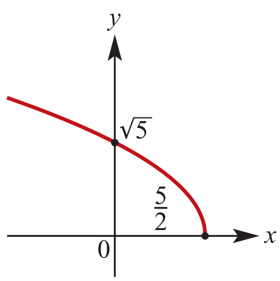
Range $= \mathbb{R}^+ \cup \{0\}$



c $y = \sqrt{5 - 2x} = \sqrt{-(2x - 5)}$

Domain $= \left(-\infty, \dfrac{5}{2}\right]$

Range $= \mathbb{R}^+ \cup \{0\}$



d $y = \sqrt{4 - (x - 5)^2}$

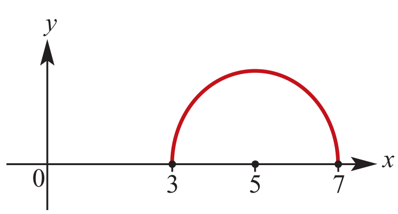
Squaring gives:

$y^2 = 4 - (x - 5)^2$

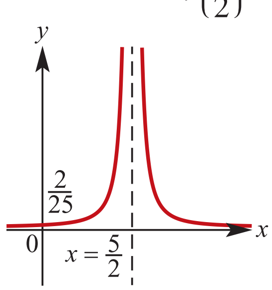
$(x - 5)^2 + y^2 = 4$ This last equation is that of a circle of radius 2 and centre $(5, 0)$.

It is the 'top half' of the circle.

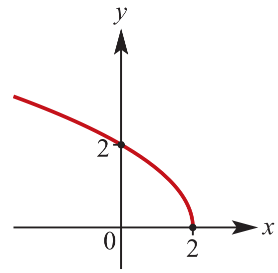
Domain $= [3, 7]$; Range $= [0, 2]$



e Domain $= \mathbb{R} \setminus \left\{\dfrac{5}{2}\right\}$; Range$= (0, \infty)$

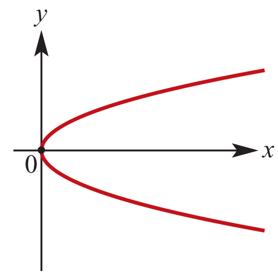


f Domain $= (-\infty, 2]$; Range$= [0, \infty)$



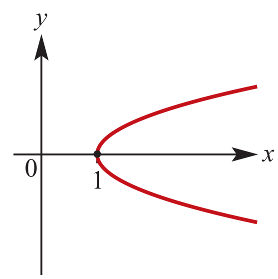
9 **a** Domain $= [0, \infty)$

Range $= \mathbb{R}$



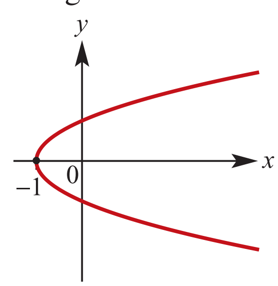
b Domain $= [1, \infty)$

Range $= \mathbb{R}$



c Domain $= [-1, \infty)$

Range $= \mathbb{R}$



d Domain $= [1, \infty)$

Range $= \mathbb{R}$

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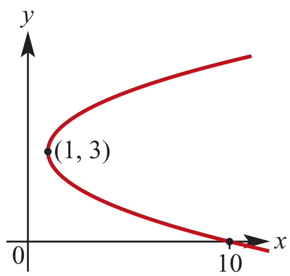

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Solutions to Exercise 5C

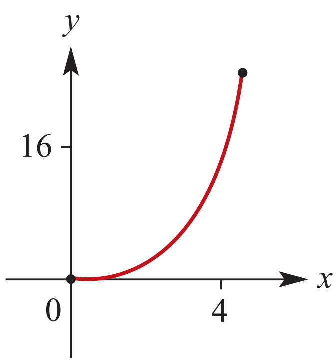
1 a $\{(0,1),(0,2),(1,2),(2,3),(3,4)\}$ is not a function because it is 1– many; (0, 1) and (0,2) Domain = $\{0,1,2,3\}$; Range = $\{1,2,3,4\}$

b $\{(-2,-1),(-1,-2),(0,2),(1,4),(2,-5)\}$ is a function because it is 1 – 1; Domain = $\{-2,-1,0,1,2\}$; Range = $\{-5,-2,-1,2,4\}$

c Not a function; Domain = $\{-1,0,3,5\}$; Range = $\{1,2,4,6\}$

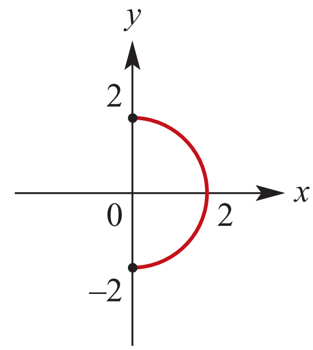
d $\{(1,3),(2,3),(4,3),(5,3),(6,3)\}$ is a function because it is many –1; Domain = $\{1,2,4,5,6\}$; Range = $\{3\}$

2 a $y = x^2; x \in [0,4]$; Range = $[0,16]$; function because 1 – 1 relation



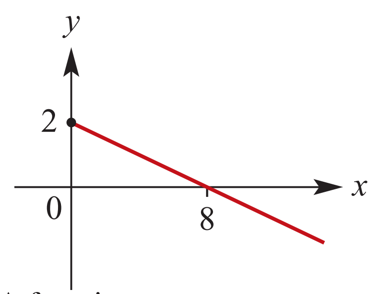
a function
Domain = [0, 4]
Range = [0, 16]

b $\{(x,y): x^2 + y^2 = 4\}; x \in [0,2]$; Range = $[-2,2]$; not a function because 1– many relation



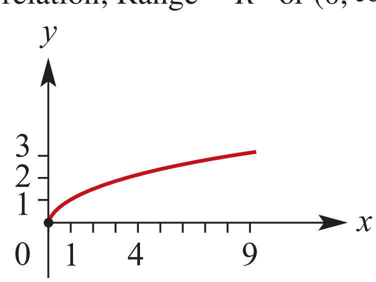
not a function
Domain = [0, 2]
Range = [–2, 2]

c $\{(x,y): 2x + 8y = 16; x \in [0,\infty)\}$; Range = $(-\infty,2]$; function because $1 \to 1$ relation



A function
Domain = $[0,\infty)$
Range = $(-\infty,2]$

d $y = \sqrt{x}; x \in \mathbb{R}^+$, function because $1 \to 1$ relation; Range = R^+ or $(0,\infty)$



a function
Domain = $\{x : x \geq 0\}$
Range = $\{y : y \geq 0\}$

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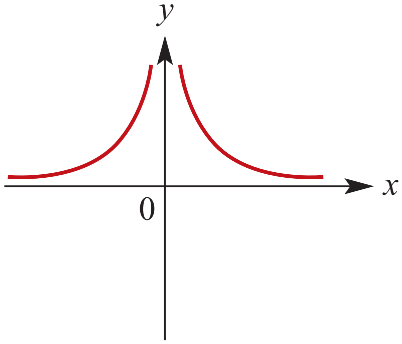
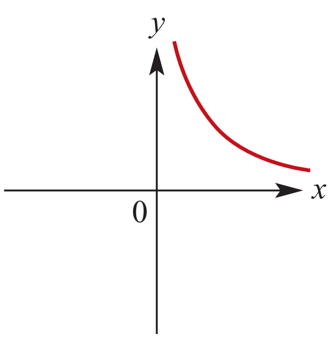
e $\{(x, y) : y = \dfrac{1}{x^2}; x \in R\backslash\{0\}\}$;
function because many $\rightarrow$ 1 relation;
Range $= \mathbb{R}^+$ or $(0, \infty)$

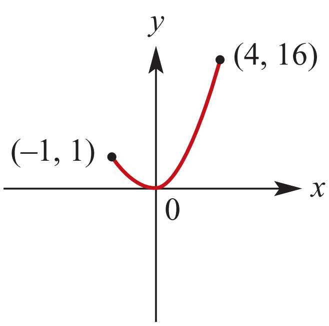


a function
Domain = R\{0}
Range = R+

f $\{(x, y) : y = \dfrac{1}{x}; x \in \mathbb{R}^+\}$; function because $1 \rightarrow 1$ relation; Range $= \mathbb{R}^+$ or $(0, \infty)$
Domain $= \mathbb{R}^+$

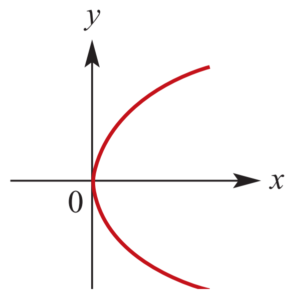


g $y = x^2; x \in [-1, 4]$; Range $= [0, 16]$;
function because many $\rightarrow$ 1 relation



h $\{(x, y) : x = y^2; x \in \mathbb{R}^+\}$;

Range $= \mathbb{R}\backslash\{0\}$; not a function because $1 \rightarrow$ many relation



3 a $\{(x, y) : y = 3x + 2\}$ can be expressed as $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 3x + 2$

b $\{(x, y) : 2y + 3x = 12\}$

$$2y + 3x = 12$$

$$\therefore \quad 2y = 12 - 3x$$

$$\therefore \quad y = 6 - \frac{3x}{2}$$

$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 6 - \dfrac{3x}{2}$

c $\{(x, y) : y = 2x + 3, x \geq 0\}$
can be expressed as
$f : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}, f(x) = 2x + 3$

d $y = 5x + 6, -1 \leq x \leq 2$ can be expressed as
$f : [-1, 2] \rightarrow \mathbb{R}, f(x) = 5x + 6$

e $y + x^2 = 25, -5 \leq x \leq 5$
$\therefore\ y = 25 - x^2$ can be expressed as
$f : [-5, 5] \rightarrow \mathbb{R}, f(x) = 25 - x^2$

f $y = 5x - 7, 0 \leq x \leq 1$ can be expressed as
$f : [0, 1] \rightarrow \mathbb{R}, f(x) = 5x - 7$

4 a $\{(x, -2) : x \in \mathbb{R}\}$ is a function because it is many -1; Domain $= R$,
Range $= \{-2\}$

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b $\{(3, y) : y \in \mathbb{Z}\}$ is not a function because it is $1 \to$ many; Domain $= \{3\}$, Range $= Z$

c $y = -x + 3$ is a function because it is $1 \to 1$; Domain $= \mathbb{R}$, Range $= \mathbb{R}$

d $y = x^2 + 5$ is a function because it is many $\to 1$; Domain $= \mathbb{R}$, Range $= [5, \infty)$

e $\{(x, y) : x^2 + y^2 = 9\}$ is not a function because it is many $\to$ many; Domain $= [-3, 3]$, Range $= [-3, 3]$

5 a $f(x) = 2x - 3$

i $f(0) = 2(0) - 3 = -3$

ii $f(4) = 2(4) - 3 = 5$

iii $f(-1) = 2(-1) - 3 = -5$

iv $f(6) = 2(6) - 3 = 9$

v $f(x - 1) = 2(x - 1) - 3 = 2x - 5$

vi $f\left(\dfrac{1}{a}\right) = \dfrac{2}{a} - 3$

b $g(x) = \dfrac{4}{x}$

i $g(1) = \dfrac{4}{1} = 4$

ii $g(-1) = \dfrac{4}{-1} = -4$

iii $g(3) = \dfrac{4}{3}$

iv $g(2) = \dfrac{4}{2} = 2$

c $g(x) = (x - 2)^2$

i $g(4) = (4 - 2)^2 = 4$

ii $g(-4) = (-4 - 2)^2 = 36$

iii $g(8) = (8 - 2)^2 = 36$

iv $g(a) = (a - 2)^2$

d $f(x) = 1 - \dfrac{1}{x}$

i $f(1) = 1 - \dfrac{1}{1} = 0$

ii $f(1 + a) = 1 - \dfrac{1}{1 + a}$
$= \dfrac{1 + a - 1}{1 + a} = \dfrac{a}{a + 1}$

iii $f(1 - a) = 1 - \dfrac{1}{1 - a}$
$= \dfrac{1 - a - 1}{1 - a}$
$= -\dfrac{a}{1 - a} = \dfrac{a}{a - 1}$

iv $f(\dfrac{1}{a}) = 1 - \dfrac{1}{1/a} = 1 - a$

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6 $f(x) = 2x + 1$

a $f(2) = 2 \times 2 + 1 = 5$ and $f(t) = 2t + 1$

b
$$\begin{aligned} f(x) &= 6 \\ 2x + 1 &= 6 \\ 2x &= 5 \\ x &= \frac{5}{2} \end{aligned}$$

c
$$\begin{aligned} f(x) &= 0 \\ 2x + 1 &= 0 \\ 2x &= -1 \\ x &= -\frac{1}{2} \end{aligned}$$

d
$$\begin{aligned} f(t) &= t \\ 2t + 1 &= t \\ t &= -1 \end{aligned}$$

e
$$\begin{aligned} f(x) &\geq x \\ 2x + 1 &\geq x \\ x &\geq -1 \end{aligned}$$

f
$$\begin{aligned} f(x) &\leq 3x \\ 2x + 1 &\leq 3x \\ -x &\leq -1 \\ x &\geq 1 \end{aligned}$$

7 a $f(x) = 5x - 2 = 3$
$\therefore\ 5x = 5,\ \therefore\ x = 1$

b $f(x) = \dfrac{1}{x} = 6$
$\therefore 1 = 6x, x = \dfrac{1}{6}$

c $f(x) = x^2 = 9$
$\therefore x = \pm\sqrt{9} = \pm 3$

d $f(x) = (x + 1)(x - 4) = 0$
$\therefore x = -1, 4$

e $f(x) = x^2 - 2x = 3$
$\therefore x^2 - 2x - 3 = 0$
$\therefore (x - 3)(x + 1) = 0$
$\therefore x = -1, 3$

f $f(x) = x^2 - x - 6 = 0$
$\therefore (x - 3)(x + 2) = 0$
$\therefore x = -2, 3$

8 $g(x) = x^2 + 2x$ and
$h(x) = 2x^3 - x^2 + 6$

a $g(-1) = (-1)^2 + 2(-1) = -1$
$g(2) = (2)^2 + 2(2) = 8$
$g(-2) = (-2)^2 + 2(-2) = 0$

b $h(-1) = 2(-1)^3 - (-1)^2 + 6 = 3$
$h(2) = 2(2)^3 - (2)^2 + 6 = 18$
$h(-2) = 2(-2)^3 - (-2)^2 + 6 = -14$

c **i** $g(-3x) = (-3x)^2 + 2(-3x) =$
$9x^2 - 6x$

ii $g(x - 5) = (x - 5)^2 + 2(x - 5)$
$= x^2 - 8x + 15$

iii $h(-2x) = 2(-2x)^3 - (-2x)^2 + 6$
$= -16x^3 - 4x^2 + 6$

iv $g(x + 2) = (x + 2)^2 + 2(x + 2)$
$= x^2 + 6x + 8$

v $h(x^2) = 2(x^2)^3 - (x^2)^2 + 6$
$= 2x^6 - x^4 + 6$

9 $f(x) = 2x^2 - 3$

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a $f(2) = 2(2)^2 - 3 = 5$
$f(-4) = 2(-4)^2 - 3 = 29$

b The Range of f is $[-3, \infty)$

10 $f(x) = 3x + 1$

a The image of $2 = 3(2) + 1 = 7$

b The pre-image of 7: $3x + 1 = 7$
so $3x = 6$ and $x = 2$

c $\{x\colon f(x) = 2x\}$:
$3x + 1 = 2x, \therefore\ x = -1$

11 $f(x) = 3x^2 + 2$

a The image of $0 = 3(0)^2 + 2 = 2$

b The pre-image(s) of 5:
$3x^2 + 2 = 5$
$\therefore\ 3x^2 = 3, \therefore\ x = \pm 1$

c $\{x\colon f(x) = 11\}$
$\therefore\ 3x^2 + 2 = 11$
$\therefore\ 3x^2 = 9$
$\therefore\ x^2 = 3, \therefore\ x = \pm\sqrt{3}$

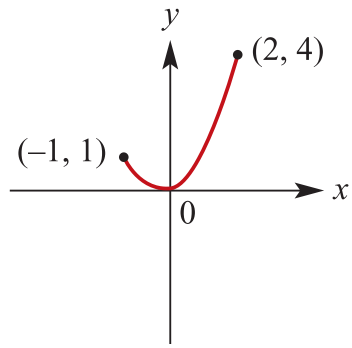
12 $f(x) = 7x + 6$ and $g(x) = 2x + 1$

a $\{x\colon f(x) = g(x)\}$
$\therefore\ 7x + 6 = 2x + 1$
$\therefore\ 5x = -5, \therefore\ x = -1$

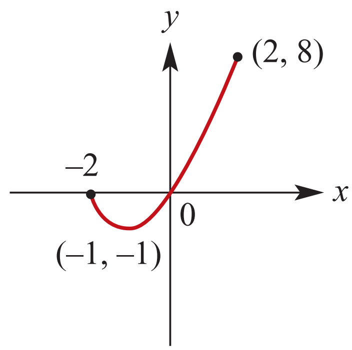
b $\{x\colon f(x) > g(x)\}$
$\therefore\ 7x + 6 > 2x + 1$
$\therefore\ 5x > -5, \therefore\ x > -1$

c $\{x\colon f(x) = 0\}$
$\therefore\ 7x + 6 = 0$
$\therefore\ 7x = -6, \therefore\ x = -\dfrac{6}{7}$

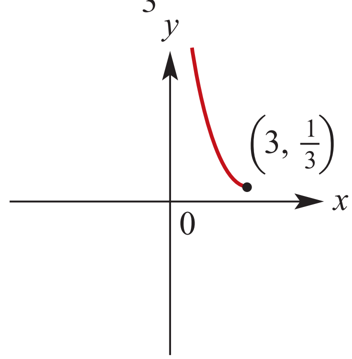
13 **a** $f\colon [-1, 2] \to \mathbb{R}, f(x) = x^2$
Range $= [0, 4]$



b $f\colon [-2, 2] \to \mathbb{R}, f(x) = x^2 + 2x$
Range $= [-1, 8]$



c $f\colon (0, 3] \to \mathbb{R}, f(x) = \dfrac{1}{x}$
Range $= [\frac{1}{3}, \infty)$



d $f\colon \mathbb{R} \to \mathbb{R}, f(x) = x^2 - 2x + 3$
Range $= [2, \infty)$

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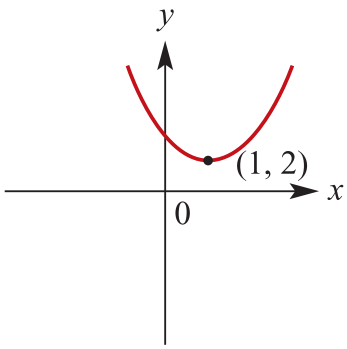
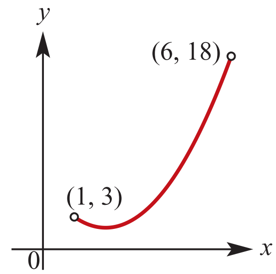
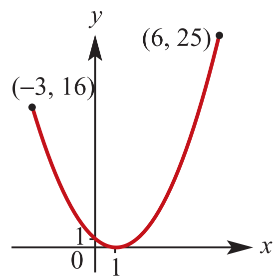


e $f: (1, 6) \to \mathbb{R}, f(x) = x^2 - 4x + 6$
Range $= [2, 18)$



f $f: [-3, 6] \to \mathbb{R}, f(x) = x^2 - 2x + 1$
Range $= [0, 25])$



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Solutions to Exercise 5D

1 **a** Not $1-1$: $(2,4)$, $(4,4)$ both have image $=4$

b $\{(1,3),(2,4),(3,6),(7,9)\}$ is $1-1$

c Not $1-1$: $(-x)^2 = x^2$

d $\{(x,y): y = 3x+1\}$ is $1-1$

e $f(x) = x^3 + 1$ is $1-1$

f Not $1-1$: $1-x^2 = 1-(-x)^2$

g $y = x^2, x \geq 0$ is $1-1$ because $x \geq 0$

2 **a** Is a function but isn't $1-1$(many $\to$ 1)

b Isn't a function: 1– many

c Is a $1-1$ function

d Is a function but isn't $1-1$(many $-$ 1)

e Isn't a function: 1– many

f Is a function but isn't $1-1$(many $\to$ 1)

g Is a $1-1$ function

h Isn't a function: many – many

3 **a** $y = 7 - x$,
Max. Domain $\mathbb{R}$, Range $\mathbb{R}$

b $y = 2\sqrt{x}$
Max. Domain $[0,\infty)$, Range $[0,\infty)$

c $y = x^2 + 1$,
Max. Domain $\mathbb{R}$, Range $[1,\infty)$

d $y = -\sqrt{9-x^2}$,
Max. Domain $[-3,3]$ because
$9 - x^2 \geq 0$,
Range $[-3,0]$

e $y = \dfrac{1}{\sqrt{x}}$,
Max. Domain $\mathbb{R}^+$, Range $\mathbb{R}^+$
(Different from **b** because you can't have $\frac{1}{0}$.)

f $y = 3 - 2x^2$,
Max. Domain $\mathbb{R}$, Range $(-\infty, 3]$

g $y = \sqrt{x-2}$,
Max. Domain $[2,\infty)$ because
$x - 2 \geq 0$,
Range $[0,\infty)$

h $y = \sqrt{2x-1}$,
Max. Domain $[\frac{1}{2},\infty)$ because
$2x - 1 \geq 0$,
Range $[0,\infty)$

i $y = \sqrt{3-2x}$,
Max. Domain $(-\infty, \frac{3}{2}]$ because
$3 - 2x \geq 0$,
Range $[0,\infty)$

j $y = \dfrac{1}{2x-1}$,
Max. Domain $\mathbb{R}\backslash\{\frac{1}{2}\}$ because
$2x - 1 \neq 0$,
Range $\mathbb{R}\backslash\{0\}$ because $\dfrac{1}{2x-1} \neq 0$

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k $y = \dfrac{1}{(2x-1)^2} - 3$,
Max. Domain $\mathbb{R}\backslash\{\frac{1}{2}\}$ because
$2x - 1 \neq 0$,
Range $(-3, \infty)$ because $\dfrac{1}{(2x-1)^2} > 0$

l $y = \dfrac{1}{2x-1} + 2$,
Max. Domain $\mathbb{R}\backslash\{\frac{1}{2}\}$ because
$2x - 1 \neq 0$,
Range $1/\{2\}$ because $\dfrac{1}{2x-1} \neq 0$

4 **a** Domain = $[4, \infty]$; Range = $[0, \infty)$

b Domain = $(-\infty, 4]$; Range = $[0, \infty)$

c Domain = $[2, \infty)$; Range = $[3, \infty)$

d Domain = $\mathbb{R} \setminus \{4\}$; Range = $\mathbb{R} \setminus \{0\}$

e Domain = $\mathbb{R} \setminus \{4\}$; Range = $\mathbb{R} \setminus \{3\}$

f Domain = $\mathbb{R} \setminus \{-2\}$; Range = $\mathbb{R} \setminus \{-3\}$

5 **a** $f(x) = 3x + 4$;
Max. Domain $\mathbb{R}$, Range $\mathbb{R}$

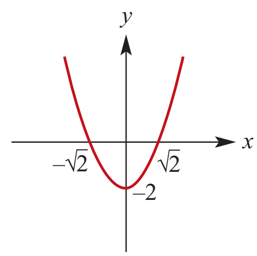
b $g(x) = x^2 + 2$,
Max. Domain $\mathbb{R}$, Range $[2, \infty)$

c $y = -\sqrt{16 - x^2}$,
Max. Domain $[-4, 4]$ because
$16 - x^2 \geq 0$,
Range $[-4, 0]$

d $y = \dfrac{1}{x+2}$,
Max. Domain $\mathbb{R}\backslash\{-2\}$ because
$x + 2 \neq 0$,
Range $\mathbb{R}\backslash\{0\}$ because $\dfrac{1}{x+2} \neq 0$

6 $\{(x, y)\colon y^2 = -x + 2, x \leq 2\}$ is a one $\to$ many relation. Split in two:
$\{f\colon (-\infty, 2], f(x) = \sqrt{2 - x}\}$, Range $[0, \infty)$
$\{f\colon (-\infty, 2], f(x) = -\sqrt{2 - x}\}$, Range $(-\infty, 0]$

7 **a** $\{f\colon \mathbb{R} \to \mathbb{R}; f(x) = x^2 - 2\}$

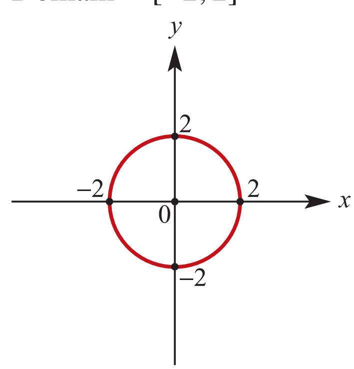


b $\{f\colon [0, \infty) \to \mathbb{R}; f(x) = x^2 - 2\}$ and
$\{f\colon (-\infty, 0] \to \mathbb{R}; f(x) = x^2 - 2\}$

8 $f_1\colon [1, \infty) \to \mathbb{R}$, $f_1(x) = x^2 - 2x + 4$
$f_2\colon (-\infty, 1] \to \mathbb{R}$, $f_2(x) = x^2 - 2x + 4$

9 $f_1\colon (2, \infty) \to \mathbb{R}$, $f_1(x) = \dfrac{1}{(x-2)^2}$
$f_2\colon (-\infty, 2) \to \mathbb{R}$, $f_2(x) = \dfrac{1}{(x-2)^2}$

10 **a** Domain = $[-2, 2]$



b $f_1\colon [0, 2] \to \mathbb{R}$, $f_1(x) = \sqrt{4 - x^2}$
$f_2\colon [0, 2] \to \mathbb{R}$, $f_2(x) = -\sqrt{4 - x^2}$

c $f_1\colon [-2, 0] \to \mathbb{R}$, $f_1(x) = \sqrt{4 - x^2}$
$f_2\colon [-2, 0] \to \mathbb{R}$, $f_2(x) = -\sqrt{4 - x^2}$

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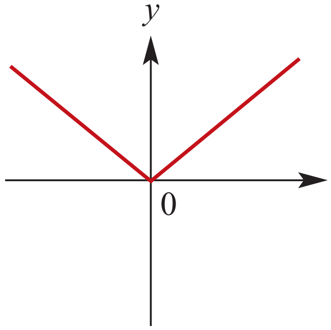
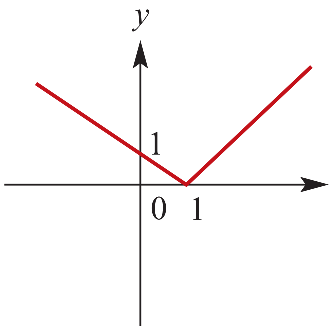
Solutions to Exercise 5E

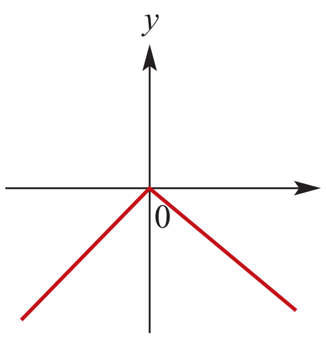
1 a $h(x) = x, x \geq 0$ and $h(x) = -x, x < 0$;
Range $= [0, \infty)$



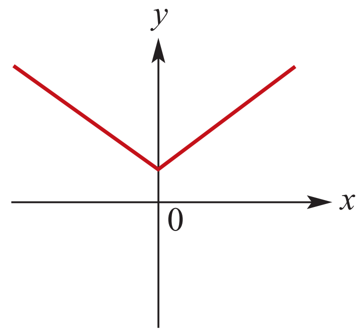
b $h(x) = x - 1, x \geq 1$ and $h(x) = 1 - x$, $x < 1$; Range $= [0, \infty)$



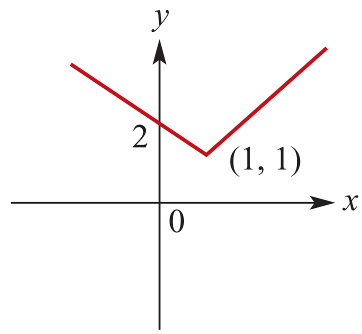
c $h(x) = -x, x \geq 0$ and $h(x) = x, x < 0$;
Range $= (-\infty, 0]$



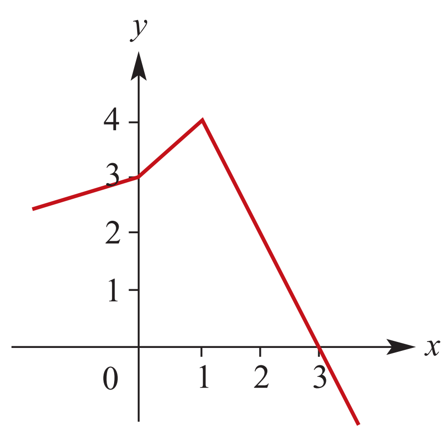
d $h(x) = 1 + x, x \geq 0$ and $h(x) = 1 - x$, $x < 0$; Range $= [1, \infty)$



e $h(x) = x, x \geq 1$ and
$h(x) = 2 - x, x < 1$;
Range $= [1, \infty)$



2 a $f(x) = \dfrac{2}{3}x + 3, x < 0$
$f(x) = x + 3, 0 \leq x \leq 1$
$f(x) = -2x + 6, x > 1$
Axis intercepts at $(-\dfrac{9}{2}, 0)$, (0,3) and (3,0)



b Range $= (-\infty, 4]$

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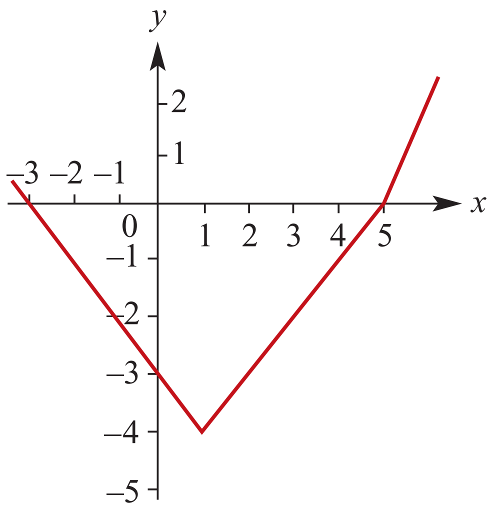
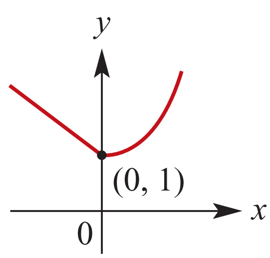
3 $g(x) = -x - 3, x < 1$
$g(x) = x - 5, 1 \le x \le 5$
$g(x) = 3x - 15, x > 5$
Axis intercepts at $(-3, 0)$, $(0, -3)$ and $(5, 0)$

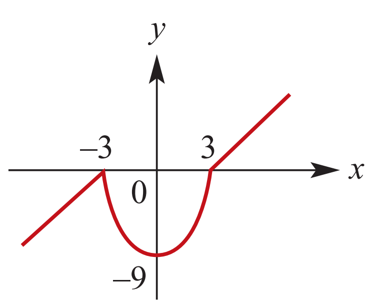


4 a $h(x) = x^2 + 1, x \ge 0$
$h(x) = 1 - x, x < 0$



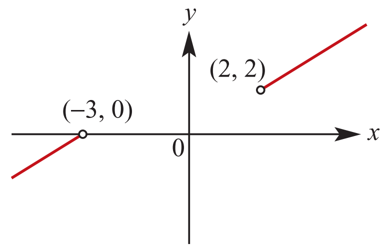
b Range $= [1, \infty)$

5 a $f(x) = -x + 3, x < -3$
$f(x) = x^2 - 9, -3 \le x \le 3$
$f(x) = x - 3, x > 3$

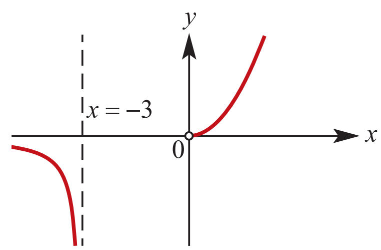


b Range $= \mathbb{R}$

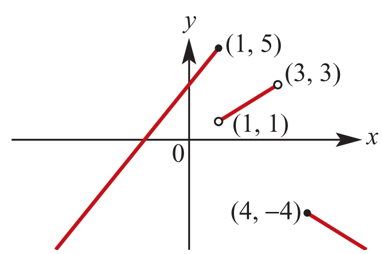
6 a Range $= (-\infty, 0) \cup (2, \infty)$



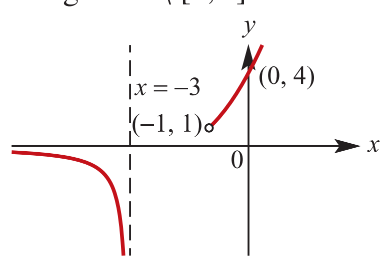
b Range $= \mathbb{R} \setminus \{0\}$



c Range $= (-\infty, 5]$



d Range $= \mathbb{R} \setminus [0, 1]$



7 a $f(x) = \dfrac{1}{x}, x > 1$
$f(x) = x, x \le 1$

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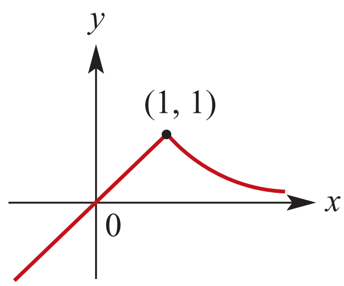

b Range $= (-\infty, 1]$

8 Line connecting $(-3, 0)$ and $(-1, 2)$ has gradient $= \dfrac{2-0}{-1-(-3)} = 1$

Using $(-3, 0)$: $y - 0 = 1(x - -3)$

$\therefore\ y = x + 3$ for $[-3, -1]$

Line connecting $(-1, 2)$ and $(2, -1)$ has gradient $= \dfrac{-1-2}{2-(-1)} = -1$

Using $(-1, 2)$: $y - 2 = -1(x - -1)$

$\therefore\ y = 1 - x$ for $[-1, 2]$

Line connecting $(2, -1)$ and $(4, -2)$ has gradient $= \dfrac{-2 - -1}{4-2} = -\dfrac{1}{2}$

Using $(2, -1)$: $y + 1 = -\dfrac{1}{2}(x - 2)$

$\therefore\ y = -\dfrac{x}{2}$ for $[2, 4]$

$$f(x) = \begin{cases} x + 3; -3 \le x < -1 \\ 1 - x; -1 \le x < 2 \\ -\dfrac{x}{2}; 2 \le x \le 4 \end{cases}$$

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Solutions to Exercise 5F

1 **a** $L(C) = 0.002C + 25; -273 \le C \le 1000$
Most metals will melt at over 1000 degrees and $C = -273$ is absolute zero.

b **i** $L(30) = (0.002)30 + 25 = 25.06$ cm

ii $L(16) = (0.002)16 + 25 = 25.032$ cm

iii $L(100) = (0.002)100 + 25 = 25.20$ cm

iv $L(500) = (0.002)500 + 25 = 26.00$ cm

2 **a** $f(x) = a + bx$

$$\begin{array}{ll} f(4) = -1 & \therefore\ a + 4b = -1 \\ f(8) = 1 & \therefore\ a + 8b = 1 \\ \hline & \therefore\ \quad 4b = 2 \end{array}$$

$\therefore\ b = \dfrac{1}{2}; a = -3$

b $f(x) = 0, \therefore\ \dfrac{x}{2} - 3 = 0$

$\therefore\quad x = 6$

3 If (fx) is parallel to $g(x) = 2 - 5x$ then the gradient of $f(x) = -5$ and
$f(x) = -5x + c$
$f(0) = 7, \therefore c = 7$
$f(x) = -5x + 7$

4 $f(x) = ax + b$

$$\begin{array}{ll} f(-5) = -12 & \therefore\ -5a + b = -12 \\ f(7) = 6 & \therefore\ \quad 7a + b = \ \ 6 \\ \hline & \therefore\ 12a \quad = 18 \end{array}$$

a **i** $f(0) = b = -\dfrac{9}{2}$

ii $f(1) = \dfrac{3}{2} - \dfrac{9}{2} = -3$

b $f(x) = \dfrac{1}{2}(3x - 9) = 0$
$\therefore 3x - 9 = 0, \therefore x = 3$

5 $f(x) = a(x - b)(x - c)$
$f(2) = f(4) = 0$ so $b = 2, c = 4$
If 7 is maximum then $a < 0$; Max. occurs halfway between 2 and 4, i.e. at $x = 3$:
$f(x) = a(3 - 2)(3 - 4) = 7$-
$\therefore a = -7$
$\therefore f(x) = -7(x - 2)(x - 4)$
OR $f(x) = -7x^2 + 42x - 56$

6
$$\begin{aligned} f(x) &= x^2 - 6x + 16 \\ &= x^2 - 6x + 9 + 7 \\ &= (x - 3)^2 + 7 \end{aligned}$$
Range of $f = [7, \infty)$

7 $f : \mathbb{R} \to \mathbb{R}, f(x) = m(x - p)(x - q)$
$f(4) = f(5) = 0$, so $p = 4, q = 5$
$f(0) = 2$, so $mpq = 2$ and $m = 0.1$
$$\begin{aligned} f(x) &= 0.1(x - 4)(x - 5) \\ &= 0.1(x^2 - 9x + 20) \\ &= 0.1x^2 - 0.9x + 2 \end{aligned}$$
$a = 0.1, b = -0.9, c = 2$
OR Use $f(0) = 2$ so $c = 2$:
$f(4) = 0$, so $16a + 4b + 2 = 0$
$f(5) = 0$, so $25a + 9b + 2 = 0$
and use simultaneous equations or matrices.

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8 $f(x) = ax^2 + bx + c$

$f(0) = 10$ so $c = 10$

Max. value $= 18$ at $x = -\dfrac{b}{2a}$:

$$f\left(-\frac{b}{2a}\right) = 18$$

$$= a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + 10$$

$$= \frac{b^2}{4a} - \frac{b^2}{2a} + 10$$

$$= -\frac{b^2}{4a} + 10$$

$\therefore\ -\dfrac{b^2}{4a} = 8$

$\therefore\ b^2 = -32a \ \ldots$ **(1)**

$f(1) = 0$

$\therefore\ a + b + 10 = 0$

$\therefore\ b = -10 - a$

$\therefore\ b^2 = (10 + a)^2 \ldots$ **(2)**

Equate **(1)** and **(2)**:

$\therefore\ (10 + a)^2 = -32a$

$\therefore\ a^2 + 20a + 100 = -32a$

$\therefore\ a^2 + 52a + 100 = 0$

$\therefore\ (a + 50)(a + 2) = 0$

$\therefore\ a = -2, -50$

If $a = -2, b = -8$; if $a = -50, b = 40$

$\therefore\ f(x) = -2x^2 - 8x + 10$

$g(x) = -50x^2 + 40x + 10$

OR $f(x) = -2(x - 1)(x + 5)$

$g(x) = -10(5x + 1)(x - 1)$

9 **a** $f(x) = 3x^2 - 5x - k$

$f(x) > 1$ for all real x

So $f(x) - 1 >$ for all real x

$13x^2 - 5x - (k + 1) > 0$ for all real x.

Then there are two real solutions to the equation $3x^2 - 5x - (k + 1) = 0$,

so $\Delta < 0$.

$\therefore\ 12k < -37$

$\therefore\ k < -\dfrac{37}{12}$

< 0 if $k < -\dfrac{37}{12}$

b $a > 0$ so the curve is an upright parabola, so the vertex is the minimum value which occurs at $x = -\dfrac{b}{2a}$

For $a = 3$ and $b = -5$, $x = \dfrac{5}{6}$

$$f\left(\frac{5}{6}\right) = 3\left(\frac{5}{6}\right)^2 - 5\left(\frac{5}{6}\right) - k = 0$$

$\therefore\ \dfrac{25}{12} - \dfrac{25}{6} - k = 0$

$\therefore\ k = -\dfrac{25}{12}$

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Solutions to Exercise 5G

1 **a** $\{(3, 1), (6, -2), (5, 4), (1, 7)\}$
Domain $= \{3, 6, 5, 1\}$; Range $= \{1, -2, 4, 7\}$

b $\{(3, 2), (6, -1), (-5, 4), (7, 1), (-4, 6)\}$
Domain $= \{3, 6, -5, 7, -4\}$
Range $= \{-1, 1, 2, 4, 6\}$

c $\{(3, 3), (-4, -2), (-1, -1), (1, -8)\}$
Domain $= \{3, 1, -1, -4\}$
Range $= \{3, -2, -1, -8\}$

d $\{(3, 1), (-7, -10), (-6, -7), (8, 2), (4, 11)\}$
Domain $= \{3, -7, -6, 8, 4\}$
Range $= \{1, -10, -7, 2, 11\}$

2 **a** $f : \mathbb{R} \to \mathbb{R}, f(x) = 6 - 2x$ has inverse:
$x = 6 - 2y, \therefore\ y = 3 - \dfrac{x}{2}$
$\therefore\ f^{-1}(x) = 3 - \dfrac{x}{2}$
Domain $= \mathbb{R}$, Range $= \mathbb{R}$

b $f : [1, 5] \to \mathbb{R}, f(x) = 3 - x$ has inverse:
$x = 3 - y, \therefore\ y = 3 - x$
$\therefore\ f^{-1}(x) = 3 - x$
Domain $= [-2, 2]$ (Range of f),
Range $= [1, 5]$ (Domain of f)

c $f : \mathbb{R}^+ \to \mathbb{R}, f(x) = x + 4$ has inverse:
$x = y + 4, \therefore\ y = x - 4$
$\therefore\ f^{-1}(x) = x - 4$
Domain $= (4, \infty)$ (Range of f),
Range $= \mathbb{R}^+$ (Domain of f)

d $f : (-\infty, 4] \to \mathbb{R}, f(x) = x + 4$ has inverse:
$f^{-1}(x) = x - 4$
Domain $= (-\infty, 8]$ (Range of f),
Range $= (-\infty, 4)$ (Domain of f)

e $f : [-1, 7] \to \mathbb{R}, f(x) = 16 - 2x$
$x = 16 - 2y$
$\therefore\ 2y = 16 - x$
$\therefore\ y = 8 - \dfrac{x}{2}$
$\therefore\ f^{-1}(x) = 8 - \dfrac{x}{2}$
Domain $= [2, 18]$ (Range of f),
Range $= [-1, 7]$ (Domain of f)

3 **a** $f : [0, \infty) \to \mathbb{R}, f(x) = x^2$
$x = y^2, \therefore\ y = \sqrt{x}$
$\therefore\ f^{-1}(x) = \sqrt{x}$
Domain $= [0, \infty)$ (Range of f),
Range $= [0, \infty)$ (Domain of f)

b $f : [2, \infty) \to \mathbb{R}, f(x) = (x - 2)^2 + 3$
$x = (y - 2)^2 + 3$
$\therefore\ (y - 2)^2 = x - 3$
$\therefore\ y - 2 = \sqrt{x - 3}$
$\therefore\ y = \sqrt{x - 3} + 2$
$\therefore\ f^{-1}(x) = \sqrt{x - 3} + 2$
Domain $= [3, \infty)$ (Range of f),
Range $= [2, \infty)$ (Domain of f)

c $f(-\infty, 4] \to \mathbb{R}, f(x) = (x - 4)^2 + 6$
$x = (y - 4)^2 + 6$
$\therefore\ x - 6 = (y - 4)^2$
$\therefore\ y - 4 = -\sqrt{x - 6}$
This time we need the negative square root because of the Domain of f, which is restricted to the left-hand side of the graph.
$\therefore\ y = 4 - \sqrt{x - 6}$
$\therefore\ f^{-1}(x) = 4 - \sqrt{x - 6}$

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Domain = $[6, \infty)$ (Range of f),
Range = $(-\infty, 4]$ (Domain of f)

d $f: [0, 1] \to \mathbb{R}, f(x) = \sqrt{1-x}$

$x = \sqrt{1-y}$

$\therefore\ x^2 = 1 - y$

$\therefore\ y = 1 - x^2$

$\therefore f^{-1}(x) = 1 - x^2$

Domain = $[0, 1]$ (Range of f),
Range = $[0, 1]$ (Domain of f)

e $f: [0, 4] \to \mathbb{R}, f(x) = \sqrt{16 - x^2}$

$x = \sqrt{16 - y^2}$

$\therefore\ x^2 = 16 - y^2$

$\therefore\ y^2 = 16 - x^2$

$\therefore\ y = \sqrt{16 - x^2}$

$\therefore\ f^{-1}(x) = \sqrt{16 - x^2}$

Domain = $[0, 4]$ (Range of f),
Range = $[0, 4]$ (Domain of f)

f $f: [0, \infty) \to \mathbb{R}, f(x) = (x+4)^2 + 6$

$x = (y+4)^2 + 6$

$\therefore\ x - 6 = (y+4)^2$

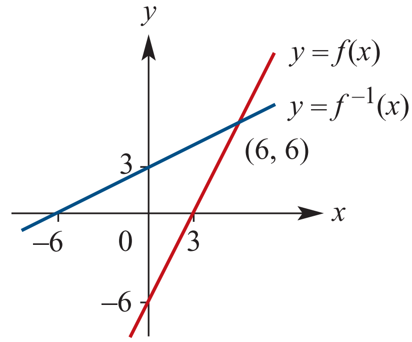
$\therefore\ y + 4 = \sqrt{x-6}$

$\therefore\ y = \sqrt{x-6} - 4$

$\therefore\ f^{-1}(x) = \sqrt{x-6} - 4$

Domain = $[22, \infty)$ (Range of f),
Range = $[0, \infty)$ (Domain of f)

4 a



b $f(x) = f^{-1}(x)$ when $2x - 6 = \dfrac{x}{2} + 3$

$\therefore\ \dfrac{3x}{2} = \therefore\ 9, x = 6$

When $x = 6, y = 6$ so $(6, 6)$

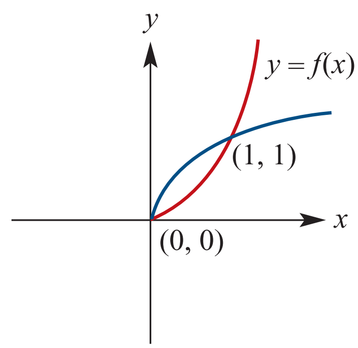
5 a $f: [0, \infty) \to \mathbb{R}, f(x) = x^2$

$\therefore\ f^{-1}(x) = \sqrt{x}$

Positive roots because Domain of f is positive.

$y = f(x)$ (red curve);

$y = f^{-1}(x)$ (blue curve)



b $f(x) = f^{-1}(x)$ where $x^2 = \sqrt{x}$

$\therefore\ x^4 = x,\ \therefore\ x^4 - x = 0$

$\therefore\ x(x^3 - 1) = 0$

$\therefore\ x = 0, 1$

i.e. at $(0, 0)$ and $(1, 1)$

6 $f: \mathbb{R} \to \mathbb{R}, f(x) = ax + b, a, b \neq 0$

$f(1) = 2,\ \therefore\ a + b = 2$

$f^{-1}(x) = \dfrac{x - b}{a}$

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$f^{-1}(1) = \dfrac{1-b}{a} = 3$

$\therefore 1 - b = 3a$

$\therefore 3a + b = 1$

$\quad a + b = 2$

$\therefore 2a = -1$

$\therefore a = -\dfrac{1}{2}; b = \dfrac{5}{2}$

7 $f: (-\infty, a] \to \mathbb{R}, f(x) = \sqrt{a-x}$

a $x = \sqrt{a-y}$

$\therefore x^2 = a - y, \therefore y = a - x^2$

$f^{-1}(x) = a - x^2, x \geq 0$ (to match Range of f)

b At $x = 1$: $\sqrt{a-x} = a - x^2$

$\therefore \sqrt{a-1} = a - 1$

$\therefore a - 1 = (a-1)^2$

$\therefore a^2 - 2a + 1 - a + 1 = 0$

$\therefore a^2 - 3a + 2 = 0$

$\therefore (a-2)(a-1) = 0$

$\therefore a = 1, 2$

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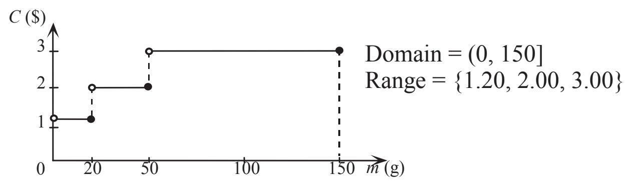
Solutions to Exercise 5H

1 $C = 0.15n + 45$ where n is the number of calls

2 a $C = \begin{cases} 1.2 & 0 < m \le 20 \\ 2 & 20 < m \le 50 \\ 3 & 50 < m \le 150 \end{cases}$

b



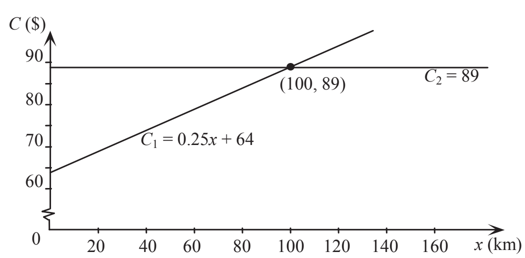
3 a $C_1 = 0.25x + 64$

$C_2 = 89$

b $0.25x + 64 = 89$

implies $0.25x = 25$

$\therefore \quad x = 100$



c Method 2 is cheaper than Method 1 if more than 100 km per day is travelled.

4 a Length = $(50 - x)$ cm

b $A(x) = x(50 - x)$

c $0 \le x \le 50$

d Maximum area = 625 cm^2 when $x = 25$

5 a i $A = (8 + x)y - x^2$

ii $P = y + (8 + x) + (y - x) + x + x + 8 = 2x + 2y + 16$

b i

If $P = 64$, $\quad 64 = 2x + 2y + 16$

$\therefore \quad 48 = 2(x + y)$

$\therefore \quad 24 = x + y$

$\therefore \quad y = 24 - x$

When $y = 24 - x$,

$A = (8 + x)(24 - x) - x^2$

$= 192 + 16x - 2x^2$

ii We know $y = 24 - x$

$, \therefore x < 24$

Also $y - x > 0$, i.e. $24 - 2x > 0$

$\therefore \ x < 12$

The allowable values for x are $\{x \colon 0 < x < 12\}$.

iii Turning point is at $\quad x = \dfrac{-b}{2a}$

and $a = -2, b = 16 \therefore \quad x = \dfrac{-16}{-4}$

$= 4$

When $x = 4$, $\quad A = 192 + 16(4) - 2(4)^2$

$= 192 + 64 - 32 = 224$

When $x = 0$, $\quad A = 192$

When $x = 12$, $\quad A = 192 + 16(12) - 2(12)^2$

$= 192 + 192 - 288 = 96$

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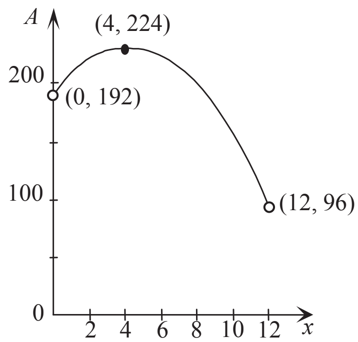

iv The maximum area occurs at the Turning point and is 224 cm^2.

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Solutions to Technology-free questions

1 a $[-2, 4)$

b $[-2, 4]$

c $[1, 8]$

d $(-1, 6]$

e $(-4, -2] \cup (1, 5]$

f $(-4, -2] \cup (2, \infty)$

g $(-\infty, -3] \cup (1, \infty)$

2 a $f(3) = 2 - 6(3) = -16$

b $f(-4) = 2 - 6(-4) = 26$

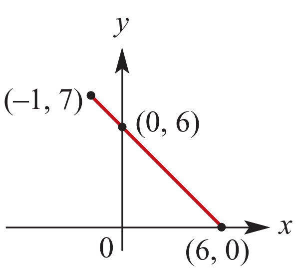
c $f(x) = 2 - 6x = 6$

$\therefore \quad -6x = 4$

$\therefore \quad x = -\dfrac{2}{3}$

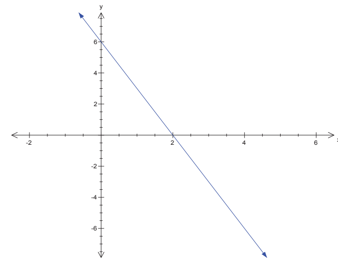
3 $f : [-1, 6] \to \mathbb{R}, f(x) = 6 - x$

a

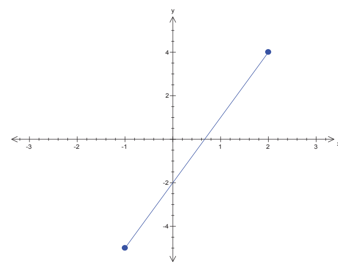


b Range of $f = [0, 7]$

4 a $\{(x, y) : 3x + y = 6\}$; Range $= \mathbb{R}$



b $\{(x, y) : y = 3x - 2; x \in [-1, 2]\}$;
Range $= [-5, 4]$



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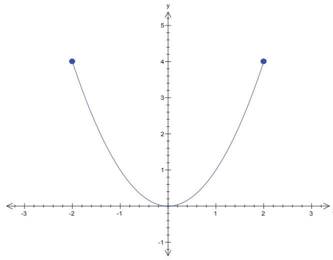
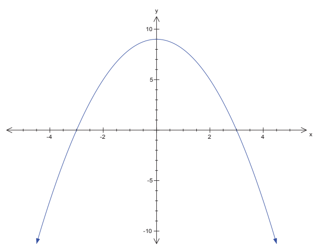
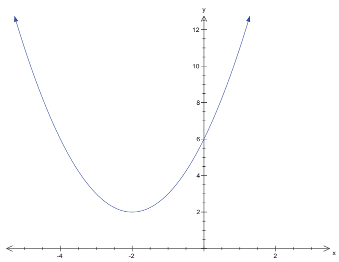
c $\{(x, y) : y = x^2; x \in [-2, 2]\}$;
Range $= [0, 4]$



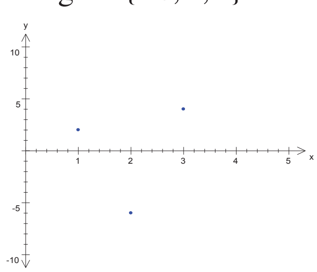
d $\{(x, y) : y = 9 - x^2\}$;
Range $= (-\infty, 9]$



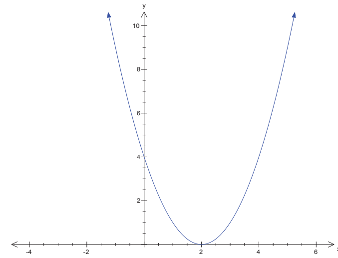
e $\{(x, y) : y = x^2 + 4x + 6\}$;
Range $= [2, \infty)$



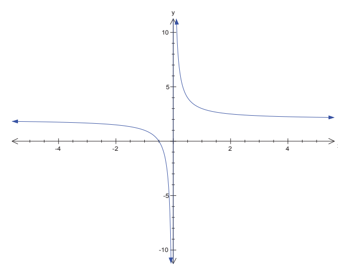
f $\{(1, 2)(3, 4)(2, -6)\}$;
Range $= \{-6, 2, 4\}$



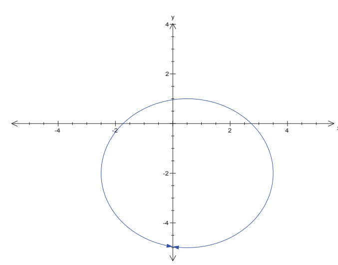
g $f : \mathbb{R} \to \mathbb{R}, f(x) = (x - 2)^2$
Range $= [0, \infty)$



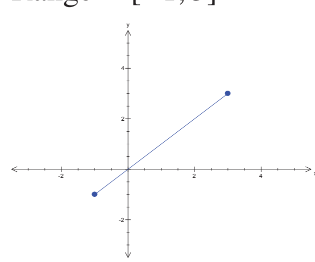
h $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}, f(x) = \dfrac{1}{x} + 2$
Range $= \mathbb{R}\backslash\{2\}$



i $\left(x - \dfrac{1}{2}\right)^2 + (y + 2)^2 = 9$
Range $= [-5, 1]$



j $f : [-1, 3] \to \mathbb{R}, f(x) = x$
Range $= [-1, 3]$



5 a $f(x) = \dfrac{a}{x} + b$

$f(1) = \dfrac{3}{2}, \therefore \ f(1) = a + b = \dfrac{3}{2}$

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$\therefore\ b = \frac{3}{2} - a$

$f(2) = 9, \therefore\ f(2) = \frac{a}{2} + b = 9$

$\therefore\ \frac{a}{2} + \left(\frac{3}{2} - a\right) = 9$

$\therefore\ \frac{3}{2} - \frac{a}{2} = 9$

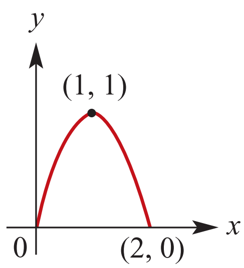
$\therefore\ 3 - a = 18$

$\therefore\ a = -15; b = \frac{33}{2}$

b Implied Domain of f is $\mathbb{R}\backslash\{0\}$.

6 $f\colon [0, 2] \to \mathbb{R}, f(x) = 2x - x^2$

a



b Range = $[0, 1]$

7 $f(x) = ax + b$

$$\begin{array}{ll} f(5) = 10, & \therefore 5a + b = 10 \\ f(1) = -2 & \therefore\ \ a + b = -2 \\ \hline & \therefore 4a \qquad = 12 \end{array}$$

$a = 3, b = -5$

8 $f(x) = ax^2 + bx + c$

$f(0) = 0, \therefore\ c = 0$

$$\begin{array}{ll} f(4) = 0, & \therefore 16a + 4b = 0 \\ & \therefore\ \ 4a + \ \ b = 0 \\ f(-2) = -6 & \therefore\ \ 4a - 2b = -6 \\ \hline & \therefore \qquad\ \ 3b = 6 \end{array}$$

$\therefore\ b = 2; 4a = -2$

$a = -\frac{1}{2}, b = 2, c = 0$

9 **a** $y = \frac{1}{x - 2}$;

implied Domain = $\mathbb{R}\backslash\{2\}$

b $f(x) = \sqrt{x - 2}$;
implied Domain = $[2, \infty)$

c $y = \sqrt{25 - x^2}$;
implied Domain = $[-5, 5]$ since $25 - x^2 \geq 0$

d $f(x) = \frac{1}{2x - 1}$;
implied Domain = $\mathbb{R}\backslash\{\frac{1}{2}\}$

e $g(x) = \sqrt{100 - x^2}$;
implied Domain = $[-10, 10]$

f $h(x) = \sqrt{4 - x}$;
implied Domain = $(-\infty, 4]$

10 **a** $y = x^2 + 2x + 3$ is many $\to$ 1 (full parabola)

b $f\colon [2, \infty) \to \mathbb{R}, f(x) = (x - 2)^2$ is $1 \to 1$ since we only have the right side of the parabola.

c $f(x) = 3x + 2$ is $1 \to 1$ (oblique line)

d $f(x) = \sqrt{x - 2}$ is $1 \to 1$ (half-parabola only)

e $f(x) = \frac{1}{x - 2}$ is $1 \to 1$ (rectangular hyperbola)

f $f\colon [-1, \infty) \to \mathbb{R}, f(x) = (x + 2)^2$ is $1 \to 1$ since we only have part of the right side of the parabola (vertex at $x = -2$).

g $f\colon [-3, 5] \to R, f(x) = 3x - 2$ is $1 \to 1$ (oblique line)

h $f(x) = 7 - x^2$ is many $\to$ 1 (full

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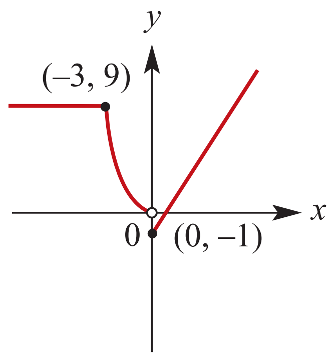
parabola)

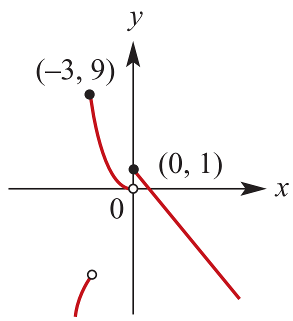
i $f(x) = \dfrac{1}{(x-2)^2}$ is many $\to$ 1 (full truncus)

j $h(x) = \dfrac{1}{x-2} + 4$ is $1 \to 1$ (rectangular hyperbola)

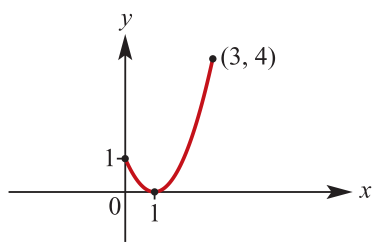
11 a $f(x) = \begin{cases} 3x - 1; x \in [0, \infty) \\ x^2; x \in [-3, 0) \\ 9; x \in (-\infty, -3) \end{cases}$



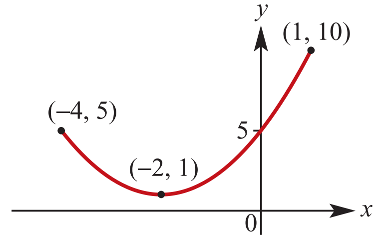
b $h(x) = \begin{cases} 1 - 2x; x \in [0, \infty) \\ x^2; x \in [-3, 0) \\ -x^2; x \in (-\infty, -3) \end{cases}$



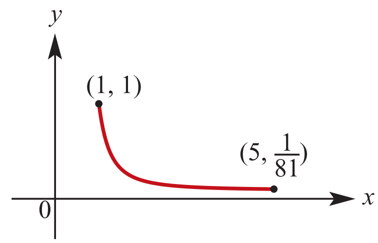
12 a Range $= [0, 4]$



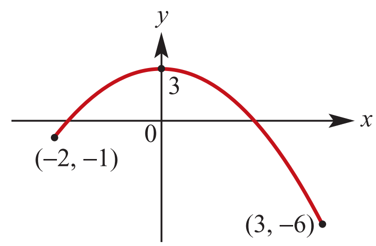
b Range $= [1, 10]$



c Range $= [\frac{1}{81}, 1]$



d Range $= [-6, 3]$



13 a Domain $= [1, \infty)$; Range $= [0, \infty)$

b Domain $= (-\infty, 1]$; Range $= [0, \infty)$

c Domain $= [0, \infty)$; Range $= (-\infty, 1]$

14 a Domain $= \mathbb{R} \setminus \{1\}$; Range $= \mathbb{R} \setminus \{0\}$

b Domain $= \mathbb{R} \setminus \{-1\}$; Range $= \mathbb{R} \setminus \{0\}$

c Domain $= \mathbb{R} \setminus \{1\}$; Range $= \mathbb{R} \setminus \{3\}$

15 a Domain $= [-1, 1]$; Range $= [0, 1]$

b Domain $= [-3, 3]$; Range $= [0, 3]$

c Domain $= [-1, 1]$; Range $= [3, 4]$

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16 **a** $f: [-1, 5] \to \mathbb{R}, f(x) = 3x - 2$

Range of $f = [-5, 13]$

$= $ Domain of inverse

$x = 3y - 2$

$\therefore\ 3y = x + 2$

$\therefore\ y = \dfrac{x+2}{3}$

$\therefore\ f^{-1}(x) = \dfrac{x+2}{3}$

Domain $= [-5, 13]$

b $f: [-2, \infty) \to R, f(x) = \sqrt{x+2} + 2$

Range of $f = [2, \infty)$

$= $ Domain of inverse

$x = \sqrt{y+2} + 2$

$\therefore\ x - 2 = \sqrt{y+2}$

$\therefore\ y + 2 = (x-2)^2$

$\therefore\ f^{-1}(x) = (x-2)^2 - 2$

or (or $x^2 - 4x + 2$)

Domain $= [2, \infty)$

c $f: [-1, \infty) \to R, f(x) = 3(x+1)^2$

Range of $f = [0, \infty)$

$= $ Domain of inverse

$x = 3(y+1)^2$

$\therefore\ (y+1)^2 = \dfrac{x}{3}$

$\therefore\ y + 1 = \sqrt{\dfrac{x}{3}}$

$\therefore\ f^{-1}(x) = \sqrt{\dfrac{x}{3}} - 1$

Domain $= [0, \infty)$

d $f: (-\infty, 1) \to R, f(x) = (x-1)^2$

Range of $f = (0, \infty)$

$= $ Domain of inverse

$x = (y-1)^2$

$\therefore\ -\sqrt{x} = y - 1$

$\therefore\ f^{-1}(x) = 1 - \sqrt{x}$

Domain $= (0, \infty)$

We need the negative root here because $f(x)$ is the left side of the parabola.

17 $f(x) = 2x + 5$

a $f(p) = 2p + 5$

b $f(p+h) = 2p + 2h + 5$

c $f(p+h) - f(p)$

$= (2p + 2h + 5) - (2p + 5) = 2h$

d $f(p+1) - f(p)$

$= (2p + 2 + 5) - (2p + 5) = 2$

18 $f(x) = 3 - 2x$

$f(p+1) - f(p)$

$= (3 - 2(p+1)) - (3 - 2p)$

$= 3 - 2p - 2 - 3 + 2p$

$= -2$

19 **a** $f(x) = -2x^2 + x - 2$

$= -2\left(x^2 - \dfrac{x}{2} + 1\right)$

$= -2\left(x^2 - \dfrac{x}{2} + \dfrac{1}{16} + \dfrac{15}{16}\right)$

$= -2\left(x - \dfrac{1}{4}\right)^2 - \dfrac{15}{8}$

Range of $f = (-\infty, -\dfrac{15}{8}]$

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b $f(x) = 2x^2 - x + 4$

$$= 2\left(x^2 - \frac{x}{2} + 2\right)$$

$$= 2\left(x^2 - \frac{x}{2} + \frac{1}{16} + \frac{31}{16}\right)$$

$$= 2\left(x - \frac{1}{4}\right)^2 + \frac{31}{8}$$

Range of $f = [\frac{31}{8}, \infty)$

c $f(x) = -x^2 + 6x + 11$

$$= -(x^2 - 6x - 11)$$

$$= -(x^2 - 6x + 9 - 20)$$

$$= -(x - 3)^2 + 20$$

Range of $f = (-\infty, 20]$

d $g(x) = -2x^2 + 8x - 5$

$$= -2\left(x^2 - 4x + \frac{5}{2}\right)$$

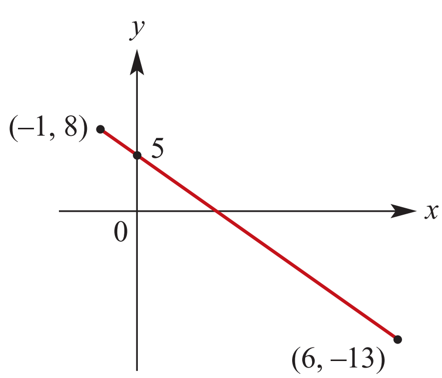
$$= -2\left(x^2 - 4x + 4 - \frac{3}{2}\right)$$

$$= -2(x - 2)^2 + 3$$

Range of $g = (-\infty, 3]$

20 $f: [-1, 6] \to \mathbb{R}, f(x) = 5 - 3x$

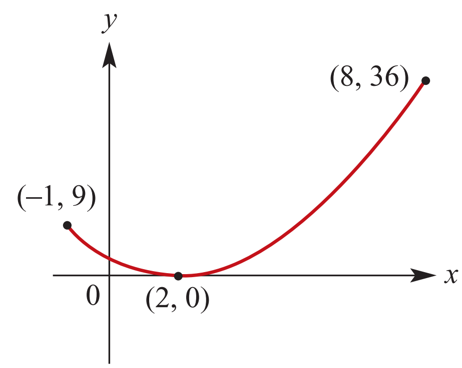
a



b Range of $f = [-13, 8]$

21 a $f: [-1, 8] \to \mathbb{R}, f(x) = (x - 2)^2$

b



c Range of $= [0, 36]$

22 a $x^2 + y^2 = 9$
Circle: radius 3, centre $(0, 0)$
Implied Domain $= [-3, 3]$,
Range $= [-3, 3]$

b $(x - 2)^2 + y^2 = 1$
Circle: radius 1, centre $(2,0)$
Implied Domain $= [1, 3]$,
Range $= [-1, 1]$

c $(2x - 1)^2 + (2y - 1)^2 = 1$
Circle: radius $\frac{1}{2}$, centre $(\frac{1}{2}, \frac{1}{2})$
Implied Domain $= [0, 1]$,
Range $= [0, 1]$

d $(x - 4)^2 + y^2 = 25$
Circle: radius 5, centre $(4, 0)$
Implied Domain $= [-1, 9]$,
Range $= [-5, 5]$

e $(y - 2)^2 + x^2 = 16$
Circle: radius 4, centre $(0,2)$
Implied Domain $= [-4, 4]$,
Range $= [-2, 6]$

23 Domain of the function f is $\{1, 2, 3, 4\}$

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a $f(x) = 2x$ so Range $= \{2, 4, 6, 8\}$

b $f(x) = 5 - x$ so Range $= \{1, 2, 3, 4\}$

c $f(x) = x^2 - 4$ so Range $=$ $\{-3, 0, 5, 12\}$

d $f(x) = \sqrt{x}$ so Range $= \{1, \sqrt{2}, \sqrt{3}, 2\}$

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Solutions to multiple-choice questions

1 **B** $f(x) = 10x^2 + 2$
$\therefore f(2a) = 10(2a)^2 + 2$
$= 40a^2 + 2$

2 **E** Maximal Domain of $f(x) = \sqrt{3x+5}$ is $[-\frac{5}{3}, \infty)$

3 **D** Maximal Domain of $f(x) = \sqrt{6-2x}$ is $[-\infty, 3]$

4 **B** Range of $x^2 + y^2 > 9$ is *all* numbers outside the circle $x^2 + y^2 = 9$
Hence Range is $\mathbb{R}$.

5 **E** Range is [1,5]

6 **C** For $f(x) = 7x - 6$, $f^{-1}(x)$:
$x = 7y - 6$
$x + 6 = 7y$
$\therefore\ y = \dfrac{x+6}{7}$

7 **E** For $f: (a, b] \to \mathbb{R}, f(x) = 3 - x$
Max. value of Range $> 3 - a$
Min. value of Range $= 3 - b$

8 **B** **B** is correct.

A $f(x) = 9 - x^2$ is $1 \to 1$ over $x \geq 0$

B $f(x) = \sqrt{9 - x^2}$ is many $\to 1$ for implied Domain $[-3, 3]$.

C $f(x) = 1 - 9x$ is a line and $1 \to 1$

D $f(x) = \sqrt{x}$ is $1 \to 1$

E $f(x) = \dfrac{9}{x}$ is $1 \to 1$ if Domain is $\mathbb{R}/\{0\}$

9 **D** $y = \dfrac{2}{x} + 3$ is reflected in the x-axis:
$y = -\dfrac{2}{x} - 3$
and then in the y-axis:
$y = -\dfrac{2}{-x} - 3 = \dfrac{2}{x} - 3$

10 **C** For $f: [-1, 5) \to \mathbb{R}, f(x) = x^2$
Min. value at $(0, 0)$;
$f(-1) = 1; f(5) = 25$:
the Range is $[0, 25)$.

11 **D** **D** is correct.

A $y = x^2 - x$ is a many $\to 1$ function

B $y = \sqrt{4 - x^2}$ is a many $\to 1$ function

C $y = 3, x > 0$ is a many $\to 1$ function

D $x = 3$ is a $1 \to$ many relation

E $y = 3x$ is a $1 \to 1$ function

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Solutions to extended-response questions

1 a For the first coach, $d = 80t$ for $0 \le t \le 4$

$$d = 320 \text{ for } 4 < t \le 4\frac{3}{4}$$

$$d = 320 + 80\left(t - 4\frac{3}{4}\right) \text{ for } 4\frac{3}{4} < t \le 7\frac{1}{4}$$

$$= 320 + 80t - 380 = 80t - 60$$

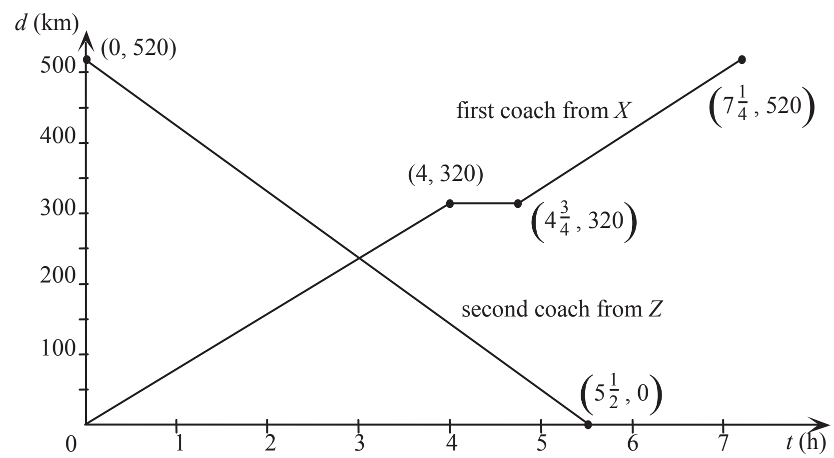
i.e. for the first coach, $d = \begin{cases} 80t & 0 \le t \le 4 \\ 320 & 4 < t \le 4\frac{3}{4} \\ 80t - 60 & 4\frac{3}{4} < t \le 7\frac{1}{4} \end{cases}$ Range: $[0, 520]$

For the second coach, $v = \dfrac{d}{t}$

$$= \frac{520}{5\frac{1}{2}} = \frac{1040}{11}$$

$\therefore\ d = 520 - \dfrac{1040}{11}t, 0 \le t \le 5\dfrac{1}{2}$, Range: $[0, 520]$

b The point of intersection of the two graphs yields the time at which the two coaches pass and where this happens.



$$520 - \frac{1040}{11}t = 80t$$

$$\therefore \quad 520 = \frac{1920t}{11}$$

$$\therefore \qquad t = \frac{520 \times 11}{1920} = \frac{143}{48}$$

$$\text{When } t = \frac{143}{48}, \qquad d = \frac{143}{48}$$

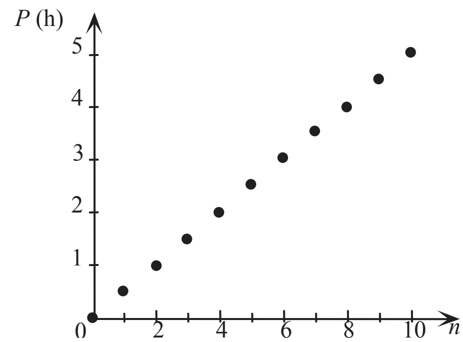
$$= \frac{80 \times 11440}{48} = 238\frac{1}{3}$$

i.e. the two coaches pass each other 23 $8\frac{1}{3}$ km from X.

2 a $P = \frac{1}{2}n, n \le 200, n \in Z$

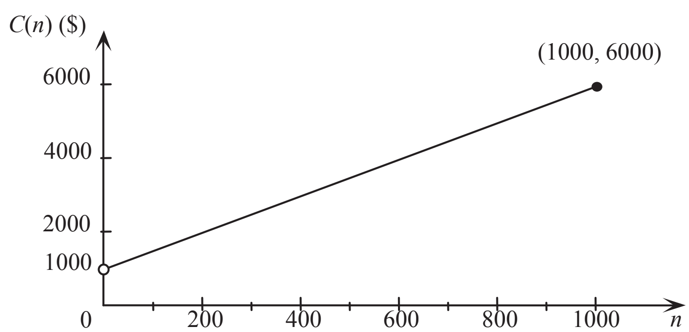
b Domain $= \{n : n \in Z, 0 \le n \le 200\}$

Range $= \{\frac{n}{2} : n \in Z, 0 \le n \le 200\}$



3 a i $C(n) = 5n + 1000, n > 0$

ii

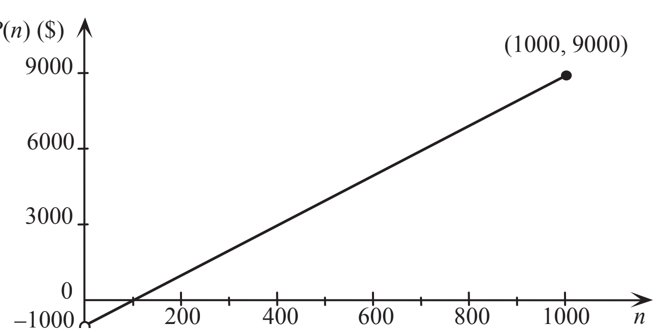


b i $P(n) = 15n - C(n)$

$= 15n - (5n + 1000)$

$= 10n - 1000, n > 0$

ii



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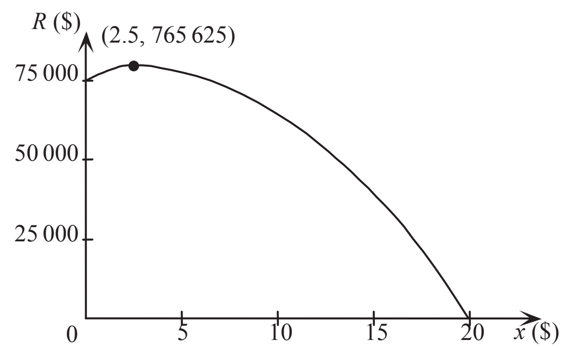
4 $V = 8000 - 0.05 \times 8000 \times n = 8000 - 400n, n \geq 0$

5 **a** R = price of ticket $\times$ number of tickets sold

$= (15 + x)(50\,000 - 2500x)$

$= 2500(x + 15)(20 - x), 0 \leq x \leq 20$

b



x-intercepts occur when $R = 0$ $\quad \therefore x = -15$ or 20(but $x \geq 0$, so $x = 20$)

R-intercept occurs when $x = 0$ $\quad \therefore R = 750000$

Turning point occurs when $\quad x = \dfrac{-15 + 20}{2} = 2.5$

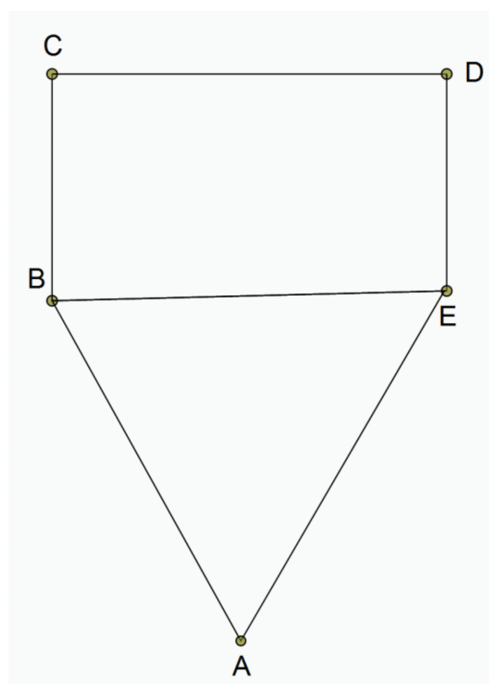
When $x = 2.5$, $\quad R = 2500(2.5 + 15)(20 - 2.5) = 765625$

c The price which will maximise the revenue is \$ 17.50 (i.e. when $x = 2.5$).
This assumes that price can be increased by other than dollar amounts.

6 **a** $BE = CD = x$, as $BCDE$ is a rectangle.
$AB = AE = BE = x$,
as ABE is an equilateral triangle.
$DE = BC$, as $BCDE$ is a rectangle.
Perimeter of pentagon

$= 3x + 2BC = a$

$\therefore \quad BC = \frac{1}{2}(a - 3x)$



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$$A(x) = \text{area of rectangle} + \text{area of triangle}$$
$$= \text{length} \times \text{width} + \frac{1}{2}\text{base} \times \text{height}$$
$$= x \times \frac{1}{2}(a - 3x) + \frac{1}{2}x \times \sqrt{x^2 - \left(\frac{1}{2}x\right)^2}$$
$$= \frac{ax}{2} - \frac{3x^2}{2} + \frac{x}{2}\sqrt{x^2\left(1 - \frac{1}{4}\right)} = \frac{ax}{2} - \frac{3x^2}{2} + \frac{x^2\sqrt{3}}{4}$$
$$= \frac{2ax - 6x^2 + x^2\sqrt{3}}{4}$$
$$= \frac{x}{4}(2a - (6 - \sqrt{3})x)$$

b From geometry: $x > 0$
Also BC > 0
So $a - 3x > 0$
Giving $x < \frac{a}{3}$
Therefore allowable values for x are $\{x : 0 < x < \frac{a}{3}\}$.

c Maximum area occurs when $x = \dfrac{0 + \frac{2a}{6-\sqrt{3}}}{2} = \dfrac{a}{6 - \sqrt{3}}$

When $x = \dfrac{a}{6 - \sqrt{3}}$, $\quad A(x) = \dfrac{a}{4(6 - \sqrt{3})}\left(2a - \dfrac{(6 - \sqrt{3})a}{6 - \sqrt{3}}\right)$
$$= \frac{a^2}{4(6 - \sqrt{3})}$$
i.e. the maximum area is $\dfrac{a^2}{4(6 - \sqrt{3})}$ cm^2.

7 a i $d(x) = AP + PD$
$$= \sqrt{AB^2 + BP^2} + \sqrt{PC^2 + CD^2}$$
$$= \sqrt{5^2 + x^2} + \sqrt{(16 - x)^2 + 3^2}$$
$$= \sqrt{x^2 + 25} + \sqrt{x^2 - 32x + 265}$$

ii $0 \le BO \le BC \qquad \therefore\ 0 \le x \le 16$

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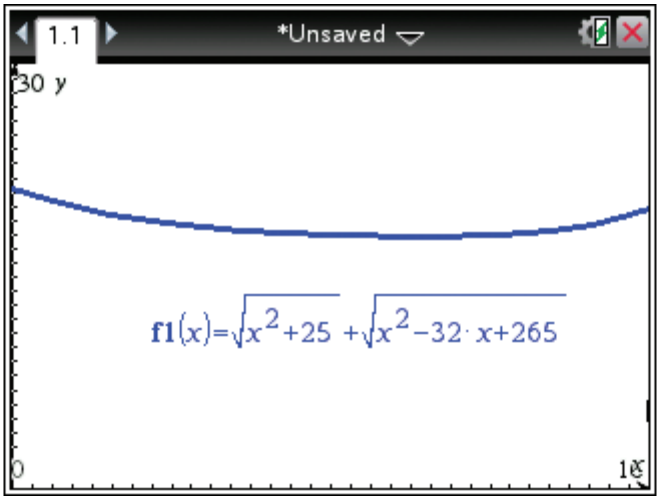
b **i**



ii On a CAS calculator, sketch the graphs of
$\boldsymbol{f1 = \sqrt{}(x\text{^}2 + 25) + \sqrt{}(x\text{^}2 - 32x + 265)}$ and $\boldsymbol{f2 = 20}$. The point of intersection is $(1.5395797, 20)$. Therefore $d(x) = 20$ when $x \approx 1.54$.
Alternatively, enter **solve** $\boldsymbol{(\sqrt{}(x\text{^}2 + 25) + \sqrt{}(x\text{^}2 - 32x + 265) = 20, x)}$ to find

$$x = \frac{80 \pm 25\sqrt{7}}{9} \approx 1.54, 16.24$$

However, $0 \le x \le 16$, just one answer of 1.54.

iii Repeat **b ii** using $\boldsymbol{f2 = 19}$. The points of intersection are $(3.3968503, 19)$ and $(15.041245, 19)$. Therefore $d(x) = 19$ when $x \approx 3.40$ or $x \approx 15.04$.
Alternatively, enter **solve** $\boldsymbol{(\sqrt{}(x\text{^}2 + 25) + \sqrt{}(x\text{^}2 - 32x + 265) = 19, x)}$ to find

$$x = \frac{1936 \pm 19\sqrt{4141}}{210} \approx 15.04, 3.40$$

c **i** with $\boldsymbol{f1 = \sqrt{}(x\text{^}2 + 25) + \sqrt{}(x\text{^}2 - 32x + 265)}$
TI: Press **Menu** → **6:Analyze Graph** → **2:Minimum**
CP: Tap **Analysis** → **G-Solve** → **Min**
to yield $(9.9999998, 17.88544)$. Therefore the minimum value of $d(x)$ is 17.89 when $x \approx 10.00$.

ii Range = $[17.89, 21.28]$. Exact answer is $[8\sqrt{5}, 5 + \sqrt{265}]$.

8 **a** On a CAS calculator, sketch the graphs of $\boldsymbol{f1 = (x + 1)(6 - x)}$ and $\boldsymbol{f2 = 2x}$.
Points of intersection are $(-1.372\,281, -2.744\,563)$ and $(4.372\,281\,3, 8.744\,562\,6)$.
The coordinates of A and B are $(4.37, 8.74)$ and $(-1.37, -2.74)$ respectively, correct to 2 decimal places.

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Or consider $(x+1)(6-x) = 2x$

$\therefore \quad -x^2 + 5x + 6 = 2x$

$\therefore \quad x^2 - 3x - 6 = 0$

$\therefore \quad x^2 - 3x + \frac{9}{4} - \frac{33}{4} = 0$

$\therefore \quad \left(x - \frac{3}{2}\right)^2 - \left(\frac{\sqrt{33}}{2}\right)^2 = 0$

$\therefore \quad \left(x - \frac{3}{2} + \frac{\sqrt{33}}{2}\right)\left(x - \frac{3}{2} - \frac{\sqrt{33}}{2}\right) = 0$

$\therefore \quad x = \frac{3 \pm \sqrt{33}}{2}$ and $y = 3 \pm \sqrt{33}$

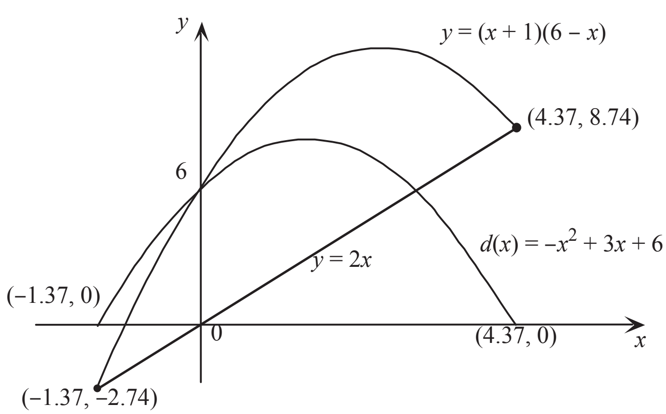
yielding $A\left(\frac{3+\sqrt{33}}{2}, 3+\sqrt{33}\right), B\left(\frac{3-\sqrt{33}}{2}, 3-\sqrt{33}\right)$

b **i** $d(x) = (x+1)(6-x) - 2x$

$= 6x + 6 - x^2 - x - 2x$

$= -x^2 + 3x + 6$

ii



c **i** with $\boldsymbol{f1 = -x\hat{}2 + 3x + 6}$,

TI: Press **Menu → 6:Analyze Graph → 3:Maximum**

CP: Tap **Analysis → G-Solve → Max**

to yield (1.5000015, 8.25).

Therefore the maximum value of $d(x)$ is 8.25.

ii Range = $[0, 8.25]$

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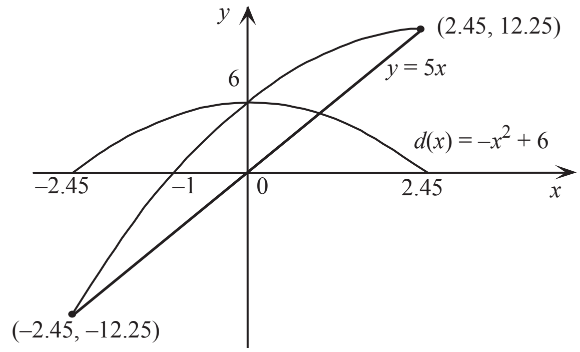
d $d(x) = (x + 1)(6 - x) - 5x$

$= 6x + 6 - x^2 - x - 5x$

$= -x^2 + 6$

The maximum value of $d(x)$ is 6 and the Range is $[0, 6]$.



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Chapter 6 – Polynomials

Solutions to Exercise 6A

1 $P(x) = x^3 - 3x^2 - 2x + 1$

a $P(1) = 1 - 3 - 2 + 1 = -3$

b $P(-1) = (-1)^3 - 3(-1)^2 - 2(-1) + 1$
$= -1 - 3 + 2 + 1$
$= -1$

c $P(2) = (2)^3 - 3(2)^2 - 2(2) + 1$
$= 8 - 12 - 4 + 1$
$= -7$

d $P(-2) = (-2)^3 - 3(-2)^2 - 2(-2) + 1$
$= -8 - 12 + 4 + 1$
$= -15$

2 $P(x) = 8x^3 - 4x^2 - 2x + 1$

a $P(\frac{1}{2}) = 8(\frac{1}{2})^3 - 4(\frac{1}{2})^2 - 2(\frac{1}{2}) + 1$
$= 8 \times \frac{1}{8} - 4 \times \frac{1}{4} - 2 \times \frac{1}{2} + 1$
$= 0$

b
$P(-\frac{1}{2}) = 8(-\frac{1}{2})^3 - 4(-\frac{1}{2})^2 - 2(-\frac{1}{2}) + 1$
$= 8 \times (-\frac{1}{8}) - 4 \times \frac{1}{4} + 2 \times \frac{1}{2} + 1$
$= 0$

3 $P(x) = x^3 + 4x^2 - 2x + 6$

a $P(0) = (0)^3 + 4(0)^2 - 2(0) + 6$
$= 6$

b $P(1) = (1)^3 + 4(1)^2 - 2(1) + 6$
$= 9$

c $P(2) = (2)^3 + 4(2)^2 - 2(2) + 6$
$= 26$

d $P(-1) = (-1)^3 + 4(-1)^2 - 2(-1) + 6$
$= -1 + 4 + 2 + 6$
$= 11$

e $P(a) = (a)^3 + 4(a)^2 - 2(a) + 6$
$= a^3 + 4a^2 - 2a + 6$

f $P(2a) = (2a)^3 + 4(2a)^2 - 2(2a) + 6$
$= 8a^3 + 16a^2 - 4a + 6$

4 a $P(x) = x^3 + 5x^2 - ax - 20$ and $P(2) = 0$
$\therefore 2^3 + 5 \times (2)^2 - 2a - 20 = 0$
$\therefore 8 - 2a = 0$
$\therefore a = 4$

b $P(x) = 2x^3 + ax^2 - 5x - 7$ and $P(3) = 68$
$\therefore 2 \times 3^3 + a \times (3)^2 - 5 \times 3 - 7 = 68$
$\therefore 9a = 36$
$\therefore a = 4$

c $P(x) = x^4 + x^3 - 2x + c$ and $P(1) = 6$
$\therefore 1 + 1 - 2 + c = 6$
$\therefore c = 6$

d $P(x) = 3x^6 - 5x^3 + ax^2 + bx + 10$ and $P(-1) = P(2) = 0$

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$P(-1) = 0$ imples $a - b = -18 \ldots (1)$
$P(2) = 0$ implies $4a + 2b = -162$ and thus $2a + b = -81 \ldots (2)$
Add equations (1) and (2)
$3a = -99$
Hence $a = -33$
Substitute in (1) to find $b = -15$

e Let $P(x) = x^5 - 3x^4 + ax^3 + bx^2 + 24x - 36$
$P(3) = P(1) = 0$
$P(3)$
$= 3^5 - 3 \times 3^4 + 3^3a + 3^2b + 24 \times 3 - 36$
$= 243 - 243 + 27a + 9b + 72 - 36$
$= 9(3a + b + 4)$
$P(1)$
$= 1^5 - 3 \times 1^4 + 1^3a + 1^2b + 24 \times 1 - 36$
$= 1 - 3 + a + b + 24 - 36$
$= a + b - 14$
We have the simultaneous equations
$3a + b = -4 \ldots (1)$
$a + b = 14 \ldots (2)$
Subtract equation (1) from equation (2)
$2a = -18$
$\therefore a = -9$ and $b = 23$.

5 $f(x) = x^3 - 2x^2 + x, g(x) = 2 - 3x$ and $h(x) = x^2 + x$

a $f(x) + g(x) = x^3 - 2x^2 + x + 2 - 3x$
$= x^3 - 2x^2 - 2x + 2$

b $f(x) + h(x) = x^3 - 2x^2 + x + x^2 + x$
$= x^3 - x^2 + 2x$

c $f(x) - g(x) = x^3 - 2x^2 + x - (2 - 3x)$
$= x^3 - 2x^2 + 4x - 2$

d $3f(x) = 3(x^3 - 2x^2 + x)$
$= 3x^3 - 6x^2 + 3x$

e
$f(x)g(x) = (x^3 - 2x^2 + x)(2 - 3x$
$= 2(x^3 - 2x^2 + x) - 3x(x^3 - 2x^2 + x)$
$= -3x^4 + 8x^3 - 7x^2 + 2x$

f $g(x)h(x) = (2 - 3x)(x^2 + x)$
$= 2(x^2 + x) - 3x(x^2 + x)$
$= -3x^3 - x^2 + 2x$

g
$f(x) + g(x) + h(x) = x^3 - 2x^2 + x + 2 - 3x + x^2 + x$
$= x^3 - x^2 - x + 2$

h
$f(x)h(x) = (x^3 - 2x^2 + x)(x^2 + x)$
$= x^3(x^2 + x) - 2x^2(x^2 + x) + x(x^2 + x)$
$= x^5 - x^4 - x^3 + x^2$

6 **a** $(x - 2)(x^2 - 2x + 3)$
$= x(x^2 - 2x + 3) - 2(x^2 - 2x + 3)$
$= x^3 - 2x^2 + 3x - 2x^2 + 4x - 6$
$= x^3 - 4x^2 + 7x - 6$

b $(x - 4)(x^2 - 2x + 3)$
$= x(x^2 - 2x + 3) - 4(x^2 - 2x + 3)$
$= x^3 - 2x^2 + 3x - 4x^2 + 8x - 12$
$= x^3 - 6x^2 + 11x - 12$

c $(x - 1)(2x^2 - 3x - 4)$
$= x(2x^2 - 3x - 4) - 1(2x^2 - 3x - 4)$
$= 2x^3 - 3x^2 - 4x - 2x^2 + 3x + 4$
$= x^3 - 5x^2 - x + 4$

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d $(x-2)(x^2+bx+c)$

$= x(x^2+bx+c) - 2(x^2+bx+c)$

$= x^3 + bx^2 + cx - 2x^2 - 2bx - 2c$

$= x^3 + (b-2)x^2 + (c-2b)x - 2c$

e $(2x+1)(x^2-4x-3)$

$= 2x(x^2-4x-3) + (x^2-4x-3)$

$= 2x^3 - 8x^2 - 6x + x^2 - 4x - 3$

$= 2x^3 - 7x^2 - 10x - 3$

7 a $(x+1)(x^2+bx+c)$

$= x(x^2+bx+c) + (x^2+bx+c)$

$= x^3 + bx^2 + cx + x^2 + bx + c$

$= x^3 + (b+1)x^2 + (c+b)x + c$

b Equating coefficients
$b + 1 = -7$ (coefficients of x^2)
$\therefore b = -8$
Note that $c = 12$. Also as a check
note that: $c + b = 4$ (coefficients of x)
$\therefore c = 12$

c $x^3 - 7x^2 + 4x + 12$

$= (x+1)(x^2-8x+12)$

$= (x+1)(x-6)(x-2)$

8 $x^2 + 6x - 2 = (x-b)^2 + c$

$= x^2 - 2bx + b^2 + c$
Equating coefficients
$-2b = 6$ and $b^2 + c = -2$.
$\therefore b = -3$ and $c = -11$.

9 a We know that
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
$(a+b)^5 = (a+b)(a+b)^4$

$= a(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)$

$+ b(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)$

$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

b

$(a+b)^6 = (a+b)(a+b)^5$

$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

10 We know that
$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

a Let $a = x$ and $b = -y$
$(x-y)^4 = (x + (-y))^4$

$= x^4 + 4x^3(-y) + 6x^2(-y)^2 + 4x(-y)^3 + (-y)^4$

$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

b Let $a = 2x$ and $b = y$
$(2x+y)^4$

$= (2x)^4 + 4(2x)^3y + 6(2x)^2y^2 + 4(2x)y^3 + y^4$

$= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$

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Solutions to Exercise 6B

1 a

$$\begin{array}{r|l} & x^2 + 2x + \dfrac{3}{x-1} \\ \hline x-1 & x^3 + x^2 - 2x + 3 \\ & \underline{x^3 - x^2} \\ & \quad 2x^2 - 2x \\ & \quad \underline{2x^2 - 2x} \\ & \qquad\qquad 0 \end{array}$$

b

$$\begin{array}{r|l} & 2x^2 - x - 3 + \dfrac{6}{x+1} \\ \hline x+1 & 2x^3 + x^2 - 4x + 3 \\ & \underline{2x^3 + 2x^2} \\ & \quad -x^2 - 4x \\ & \quad \underline{-x^2 - x} \\ & \qquad -3x + 3 \\ & \qquad \underline{-3x - 3} \\ & \qquad\qquad 6 \end{array}$$

c

$$\begin{array}{r|l} & 3x^2 - 10x + 22 - \dfrac{43}{x+2} \\ \hline x+2 & 3x^3 - 4x^2 + 2x + 1 \\ & \underline{3x^3 + 6x^2} \\ & \quad -10x^2 + 2x \\ & \quad \underline{-10x^2 - 20x} \\ & \qquad 22x + 1 \\ & \qquad \underline{22x + 44} \\ & \qquad\quad -43 \end{array}$$

d

$$\begin{array}{r|l} & 2x^2 + 3x + 10 + \dfrac{28}{x-3} \\ \hline x-3 & 2x^3 - 3x^2 + x - 2 \\ & \underline{2x^3 - 6x^2} \\ & \quad 3x^2 + x \\ & \quad \underline{3x^2 - 9x} \\ & \qquad 10x - 2 \\ & \qquad \underline{10x - 30} \\ & \qquad\quad 28 \end{array}$$

2 a

$$\begin{array}{r|l} & x^2 - x + 4 - \dfrac{8}{x+1} \\ \hline x+1 & x^3 + 0x^2 + 3x - 4 \\ & \underline{x^3 + x^2} \\ & \quad -x^2 + 3x \\ & \quad \underline{-x^2 - x} \\ & \qquad 4x - 4 \\ & \qquad \underline{4x + 4} \\ & \qquad\quad -8 \end{array}$$

b

$$\begin{array}{r|l} & 2x^2 - 8x + 49 - \dfrac{181}{x+4} \\ \hline x+4 & 2x^3 + 0x^2 + 17x + 15 \\ & \underline{2x^3 + 8x^2} \\ & \quad -8x^2 + 17x \\ & \quad \underline{-8x^2 - 32x} \\ & \qquad 49x + 15 \\ & \qquad \underline{49x + 196} \\ & \qquad\quad -181 \end{array}$$

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c

$$
\begin{array}{r}
x^2+x-3+\dfrac{11}{x+4} \\
x+3\,\big)\overline{x^3+4x^2+0x+2} \\
\underline{x^3+3x^2} \\
x^2+0x \\
\underline{x^2+3x} \\
-3x+2 \\
\underline{-3x-9} \\
11
\end{array}
$$

d

$$
\begin{array}{r}
x^2-x+4+\dfrac{8}{x-2} \\
x-2\,\big)\overline{x^3-3x^2+6x+0} \\
\underline{x^3-2x^2} \\
-x^2+6x \\
\underline{-x^2+2x} \\
4x+0 \\
\underline{4x-8} \\
8
\end{array}
$$

3 a

$$
\begin{array}{r}
x^2-2x+5 \\
x+1\,\big)\overline{x^3-x^2+3x+5} \\
\underline{x^3+x^2} \\
-2x^2+3x \\
\underline{-2x^2-2x} \\
5x+5 \\
\underline{5x+5} \\
0
\end{array}
$$

b

$$
\begin{array}{r}
2x^2-2x-6 \\
x+4\,\big)\overline{2x^3+6x^2-14x-24} \\
\underline{2x^3+8x^2} \\
-2x^2-14x \\
\underline{-2x^2-8x} \\
-6x-24 \\
\underline{-6x-24} \\
0
\end{array}
$$

c

$$
\begin{array}{r}
x^2-2x-6 \\
x-3\,\big)\overline{x^3-5x^2+0x+18} \\
\underline{x^3-3x^2} \\
-2x^2+0x \\
\underline{-2x^2+6x} \\
-6x+18 \\
\underline{-6x+18} \\
0
\end{array}
$$

d

$$
\begin{array}{r}
3x^2-x-6 \\
x-2\,\big)\overline{3x^3-7x^2-4x+12} \\
\underline{-x^3-6x^2} \\
-x^2-4x \\
\underline{-x^2+2x} \\
-6x+12 \\
\underline{-6x+12} \\
0
\end{array}
$$

4 a

$$
\begin{array}{r}
x^2+0x-3 \\
x+2\,\big)\overline{x^3+2x^2-3x+1} \\
\underline{x^3+2x^2} \\
0x^2-3x \\
\underline{0x^2+0x} \\
-3x+1 \\
\underline{-3x-6} \\
7
\end{array}
$$

Quotient $= x^2-3$, Remainder $= 7$

b

$$
\begin{array}{r}
x^2+2x+15 \\
x-5\,\big)\overline{x^3-3x^2+5x-4} \\
\underline{x^3-5x^2} \\
2x^2+5x \\
\underline{2x^2-10x} \\
15x-4 \\
\underline{15x-75} \\
71
\end{array}
$$

Quotient $= x^2+2x+15$,

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Remainder = 71

c
$$\begin{array}{r} 2x^2-3x+0 \\ x+1\,\overline{)\,2x^3-x^2-3x-7} \\ \underline{2x^3+2x^2} \\ -3x^2-3x \\ \underline{-3x^2-3x} \\ 0x-7 \\ \underline{0x+0} \\ -7 \end{array}$$

Quotient = $2x^2 - 3x$,
Remainder = -7

d
$$\begin{array}{r} 5x^2+20x+77 \\ x-4\,\overline{)\,5x^3+0x^2-3x+7} \\ \underline{5x^3-20x^2} \\ 20x^2-3x \\ \underline{20x^2-80x} \\ 77x+7 \\ \underline{77x-308} \\ 315 \end{array}$$

Quotient = $5x^2 + 20x + 77$,
Remainder = 315

5 a
$$\begin{array}{r} \frac{1}{2}x^2+\frac{7}{4}x-\frac{3}{8}-\frac{103}{8(2x+5)} \\ 2x+5\,\overline{)\,x^3+6x^2+8x+11} \\ \underline{2x^3+\frac{5}{2}x^2} \\ \frac{7}{2}x^2+8x \\ \underline{\frac{7}{2}x^2+\frac{35}{4}x} \\ -\frac{3}{4}x+11 \\ \underline{-\frac{3}{8}x-\frac{15}{8}} \\ \frac{103}{8} \end{array}$$

b
$$\begin{array}{r} x^2+2x-3-\frac{2}{2x+1} \\ 2x+1\,\overline{)\,2x^3+5x^2-4x-5} \\ \underline{2x^3+x^2} \\ 4x^2-4x \\ \underline{4x^2+2x} \\ -6x-5 \\ \underline{-6x-3} \\ -2 \end{array}$$

c
$$\begin{array}{r} x^2+2x-15 \\ 2x-1\,\overline{)\,2x^3+3x^2-32x+15} \\ \underline{2x^3-x^2} \\ 4x^2-32x \\ \underline{4x^2-2x} \\ -30x+15 \\ \underline{-30x+15} \\ 0 \end{array}$$

d
$$\begin{array}{r} \frac{1}{3}x^2-\frac{8}{9}x-\frac{8}{27}+\frac{19}{27(3x-1)} \\ 3x-1\,\overline{)\,x^3-3x^2+0x+1} \\ \underline{x^3-\frac{1}{3}x^2} \\ -\frac{8}{3}x^2+0x \\ \underline{-\frac{8}{3}x^2+\frac{8}{9}x} \\ -\frac{8}{9}x+1 \\ \underline{-\frac{8}{9}x+\frac{8}{27}} \\ \frac{19}{27} \end{array}$$

6 a Using equating coefficients.

$$x^3 + 2x^2 + 5x + 1 = (x - 1)(x^2 + 3x + 8) + 9.$$

$$\therefore \frac{x^3 + 2x^2 + 5x + 1}{x - 1} = x^2 + 3x + 8 + \frac{9}{x - 1}$$

$\therefore a = 9$.

b Using equating coefficients.

$$2x^3 - 2x^2 + 5x + 3 = (2x - 1)(x^2 - \frac{x}{2} + \frac{9}{4}) + \frac{21}{4}.$$

$$\therefore \frac{2x^3 - 2x^2 + 5x + 3}{2x - 1} = x^2 - \frac{x}{2} + \frac{9}{4} + \frac{21}{4(2x - 1)}$$

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$$\therefore a = \frac{21}{4}.$$

7 **a**

$$\begin{array}{r l}
 & 2x - 6 \\
x^2 + 0x - 2 & \overline{)\,2x^3 - 6x^2 - 4x + 12} \\
 & \underline{2x^3 + 0x^2 - 4x} \\
 & -6x^2 + 0x + 12 \\
 & \underline{-6x^2 + 0x + 12} \\
 & 0
\end{array}$$

b

$$\begin{array}{r l}
 & x - 6 \\
x^2 + 0x + 1 & \overline{)\,x^3 - 6x^2 + x - 8} \\
 & \underline{x^3 + 0x^2 + x} \\
 & -6x^2 + 0x - 8 \\
 & \underline{-6x^2 + 0x - 6} \\
 & -2
\end{array}$$

c

$$\begin{array}{r l}
 & 2x - 6 \\
x^2 + 0x - 2 & \overline{)\,2x^3 - 6x^2 - 4x + 54} \\
 & \underline{2x^3 + 0x^2 - 4x} \\
 & -6x^2 + 0x + 54 \\
 & \underline{-6x^2 + 0x + 12} \\
 & 42
\end{array}$$

d

$$\begin{array}{r l}
 & x^2 - 4x + 2 \\
x^2 + 2x - 1 & \overline{)\,x^4 - 2x^3 - 7x^2 + 7x + 5} \\
 & \underline{x^4 + 2x^3 - x^2} \\
 & -4x^3 - 6x^2 + 7x \\
 & \underline{-4x^3 - 8x^2 + 4x} \\
 & 2x^2 + 3x + 5 \\
 & \underline{2x^2 + 4x - 2} \\
 & -x + 7
\end{array}$$

e

$$\begin{array}{r l}
 & x^2 - 3x + 7 \\
x^2 + 2x - 1 & \overline{)\,x^4 - x^3 + 0x^2 + 7x + 2} \\
 & \underline{x^4 + 2x^3 - x^2} \\
 & -3x^3 + x^2 + 7x \\
 & \underline{-3x^3 - 6x^2 + 3x} \\
 & 7x^2 + 4x + 2 \\
 & \underline{7x^2 + 14x - 7} \\
 & -10x + 9
\end{array}$$

f

$$\begin{array}{r l}
 & x^2 + x - \tfrac{3}{2} \\
2x^2 - x + 4 & \overline{)\,2x^4 + x^3 + 0x^2 + 13x + 10} \\
 & \underline{2x^4 - x^3 + 4x^2} \\
 & 2x^3 - 4x^2 + 13x \\
 & \underline{2x^3 - x^2 + 4x} \\
 & -3x^2 + 9x + 10 \\
 & \underline{-3x^2 + \tfrac{3}{2}x - 6} \\
 & \frac{15}{2}x + 16
\end{array}$$

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Solutions to Exercise 6C

Use the Remainder Theorem.

1 a $P(x) = x^3 - x^2 - 3x + 1$

Divide by $x - 1$: remainder $= P(1)$

$= 1^3 - 1^2 - 3(1) + 1 = -2$

b $P(x) = x^3 - 3x^2 + 4x - 1$

Divide by $x + 2$: remainder $= P(-2)$

$= (-2)^3 - 3(-2)^2 + 4(-2) - 1 = -29$

c $P(x) = 2x^3 - 2x^2 + 3x + 1$

Divide by $x - 2$: remainder $= P(2)$

$= 2(2)^3 - 2(2)^2 + 3(2) + 1 = 15$

d $P(x) = x^3 - 2x + 3$

Divide by $x + 1$: remainder $= P(-1)$

$= (-1)^3 - 2(-1) + 3 = 4$

e $P(x) = x^3 + 2x - 5$

Divide by $x - 2$: remainder $= P(2)$

$= (2)^3 + 2(2) - 5 = 7$

f $P(x) = 2x^3 + 3x^2 + 3x - 2$

Divide by $x + 2$: remainder $= P(-2)$

$= 2(-2)^3 + 3(-2)^2 + 3(-2) - 2 = -12$

g $P(x) = 6 - 5x + 9x^2 + 10x^3$

Divide by $2x + 3$: remainder $= P(-\frac{3}{2})$

$$= 6 - 5\left(-\frac{3}{2}\right) + 9\left(-\frac{3}{2}\right)^2$$

$$+ 10\left(-\frac{3}{2}\right)^3 = 0$$

h $P(x) = 10x^3 - 3x^2 + 4x - 1$

Divide by $2x + 1$: remainder

$$= P\left(-\frac{1}{2}\right)$$

$$= 10\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2$$

$$+4\left(-\frac{1}{2}\right) - 1 = -5$$

i $P(x) = 108\, x^3 - 27x^2 - 1$

Divide by $3x + 1$: remainder $= P(-\dfrac{1}{3})$

$$= 108\left(-\frac{1}{3}\right)^3 - 27\left(-\frac{1}{3}\right)^2 - 1 = -8$$

2 a $P(x) = x^3 + ax^2 + 3x - 5$

Remainder -3 when divided by $x - 2$

$\therefore\ P(2) = 8 + 4a + 6 - 5 = -3$

$\therefore\ 4a = -12$

$\therefore\ a = -3$

b $P(x) = x^3 + x^2 - 2ax + a^2$

Remainder 8 when divided by $x - 2$

$\therefore\ P(2) = 8 + 4 - 4a + a = 8$

$\therefore\ a - 4a = -4$

$\therefore\ (a - 2)^2 = 0$

$\therefore\ a = 2$

c $P(x) = x^3 - 3x^2 + ax + 5$

Remainder 17 when divided by $x - 3$

$\therefore\ P(3) = 27 - 27 + 3a + 5 = 17$

$\therefore\ 3a = 12$

$\therefore\ a = 4$

d $P(x) = x^3 + x^2 + ax + 8$

Remainder 0 when divided by $x - 1$

$\therefore\ P(1) = 1 + 1 + a + 8 = 0$

$\therefore\ a = -10$

Use the Factor Theorem.

3 a $P(x) = x^3 - x^2 + x - 1$

$\therefore\ P(1) = 1 - 1 + 1 - 1 = 0$

Therefore $P(x)$ is exactly divisible by $x - 1$

b $P(x) = x^3 + 3x^2 - x - 3$

$\therefore\ P(1) = 1 + 3 - 1 - 3 = 0$

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Therefore $P(x)$ is exactly divisible by $x - 1$

c $P(x) = 2x^3 - 3x^2 - 11x + 6$
$\therefore\ P(-2) = -16 - 12 + 22 + 6 = 0$
Therefore $P(x)$ is exactly divisible by $x + 2$

d $P(x) = 2x^3 - 13x^2 + 27x - 18$
$\therefore\ P\left(\frac{3}{2}\right) = \frac{27}{4} - \frac{117}{4} + \frac{81}{2} - 18 = 0$
Therefore $P(x)$ is exactly divisible by $2x - 3$

4 a $P(x) = x^3 - 4x^2 + x + m$
$P(3) = 27 - 36 + 3 + m = 0$
$\therefore\ m = 6$

b $P(x) = 2x^3 - 3x^2 - (m + 1)x - 30$
$P(5) = 250 - 75 - 5(m + 1) - 30 = 0$
$\therefore\ 5(m + 1) = 145$
$\therefore\ m + 1 = 29, \therefore\ m = 28$

c $P(x) = x^3 - (m + 1)x^2 - x + 30$
$P(-3) = -27 - 9(m + 1) + 3 + 30 = 0$
$\therefore 9(m + 1) = 6$
$\therefore\ m + 1 = \frac{2}{3}, \therefore\ m = -\frac{1}{3}$

5 a $2x^3 + x^2 - 2x - 1$
$= x^2(2x + 1) - (2x + 1)$
$= (2x + 1)(x^2 - 1)$
$= (2x + 1)(x + 1)(x - 1)$

b $x^3 + 3x^2 + 3x + 1$
$= (x + 1)^3$

c $P(x) = 6x^3 - 13x^2 + 13x - 6$
$P(1) = 6 - 13 + 13 - 6 = 0$
$(x - 1)$ is a factor.
Long division or calculator:
$P(x) = (x - 1)(6x^2 - 7x + 6)$
No more factors since $\Delta < 0$ for the quadratic term.

d $P(x) = x^3 - 21x + 20$
$P(1) = 1 - 21 + 20 = 0$
$(x - 1)$ is a factor.
Long division or calculator:
$P(x) = (x - 1)(x^2 + x - 20)$
$\therefore\ P(x) = (x - 1)(x - 4)(x + 5)$

e $P(x) = 2x^3 + 3x^2 - 1$
$P(-1) = -2 + 3 - 1 = 0$
$(x + 1)$ is a factor.
Long division or calculator:
$P(x) = (x + 1)(2x^2 + x - 1)$
$\therefore\ P(x) = (x + 1)(x + 1)(2x - 1)$
$= (2x - 1)(x + 1)^2$

f $P(x) = x^3 - x^2 - x + 1$
$= x^2(x - 1) - (x - 1)$
$= (x - 1)(x^2 - 1)$
$= (x + 1)(x - 1)^2$

g $P(x) = 4x^3 + 3x - 38$
$P(2) = 32 + 6 - 38 = 0$
$(x - 2)$ is a factor.
Long division or calculator:
$P(x) = (x - 2)(4x^2 + 8x + 19)$
No more factors since $\Delta < 0$ for the quadratic term.

h $P(x) = 4x^3 + 4x^2 - 11x - 6$
$P(-2) = -32 + 16 + 22 - 6 = 0$
$(x + 2)$ is a factor.
Long division or calculator:
$P(x) = (x + 2)(4x^2 - 4x - 3)$
$= (x + 2)(2x + 1)(2x - 3)$

6 Let $P(x) = (1 + x)^4$. Then

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$P(-2) = (-2)^4 = 1$
The remainder is 1.

7 **a** $P(x) = 2x^3 - 7x^2 + 16x - 15$
Note that $P(x) = 0$ has no integer solutions. Check this by using the factor theorems.
The factor of 2 to be considered is2
The factors of 15 to be considered are $\pm3, \pm5, \pm15, \pm1$.
The values to check using the factor theorem are $\pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}, \pm\frac{1}{2}$
Then using the factor theorem.
$P(\frac{3}{2}) = 0$. No need to try another. We can factorise since we know $2x - 3$ is a factor.
Using the equating coefficients method we find:
$P(x) = (2x - 3)(x^2 - 2x + 5)$

b $P(x) = 2x^3 - 7x^2 + 8x + 5$
Note that $P(x) = 0$ has no integer solutions. Check this by using the factor theorems.
The factor of 2 to be considered is2
The factors of 5 to be considered are $\pm5, \pm1$.
The values to check using the factor theorem are $\pm\frac{5}{2}, \pm\frac{1}{2}$
Then using the factor theorem.
$P(\frac{5}{2}) \neq 0$ but $P\left(-\frac{1}{2}\right) = 0$. No need to try another. We can factorise since we know $2x + 1$ is a factor.
Using the equating coefficients method we find:
$P(x) = (2x + 1)(x^2 - 2x + 5)$

c $P(x) = 2x^3 - 3x^2 - 12x - 5$
Note that $P(x) = 0$ has no integer solutions. Check this by using the factor theorems.
The factor of 2 to be considered is2
The factors of -5 to be considered are $\pm5, \pm1$.
The values to check using the factor theorem are $\pm\frac{5}{2}, \pm\frac{1}{2}$
Then using the factor theorem.
$P(\frac{5}{2}) \neq 0$ but $P\left(-\frac{1}{2}\right) = 0$. No need to try another. We can factorise since we know $2x + 1$ is a factor.
Using the equating coefficients method we find:
$P(x) = (2x + 1)(x^2 - 2x - 5)$

d $P(x) = 2x^3 - x^2 - 8x - 3$
Note that $P(x) = 0$ has no integer solutions. Check this by using the factor theorems.
The factor of 2 to be considered is2
The factors of -3 to be considered are $\pm3, \pm1$.
The values to check using the factor theorem are $\pm\frac{3}{2}, \pm\frac{1}{2}$
Then using the factor theorem.
$P(\frac{1}{2}) \neq 0$ but $P\left(-\frac{3}{2}\right) = 0$. No need to try another. We can factorise since we know $2x + 3$ is a factor.
Using the equating coefficients method we find:
$P(x) = (2x + 3)(x^2 - 2x - 1)$

8 Sum/difference of two cubes:
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

a $x^3 - 1 = (x - 1)(x^2 + x + 1)$

b $x^3 + 64 = (x + 4)(x^2 - 4x + 16)$

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c $27x^3 - 1 = (3x - 1)(9x^2 + 3x + 1)$

d $64x^3 - 125 = (4x - 5)(16x^2 + 20x + 25)$

e $1 - 125x^3 = (1 - 5x)(1 + 5x + 25x^2)$

f $8 + 27x^3 = (2 + 3x)(4 - 6x + 9x^2)$

g $64m^3 - 27n^3 = (4m - 3n)(16m^2 + 12mn + 9n^2)$

h $27b^3 + 8a^3 = (3b + 2a)(9b^2 - 6ab + 4a^2)$

9 **a** $P(x) = x^3 + x^2 - x + 2$
$P(-2) = -8 + 4 + 2 + 2 = 0$
$\therefore\ P(x) = (x + 2)(x^2 - x + 1)$
No more factors since $\Delta < 0$, for the quadratic term.

b $P(x) = 3x^3 - 7x^2 + 4$
$P(1) = 3 - 7 + 4 = 0$
$\therefore\ P(x) = (x - 1)(3x^2 - 4x - 4)$
$= (x - 1)(3x + 2)(x - 2)$

c $P(x) = x^3 - 4x^2 + x + 6$
$P(-1) = -1 - 4 - 1 + 6 = 0$
$\therefore P(x) = (x + 1)(x - 5x + 6)$
$= (x + 1)(x - 2)(x - 3)$

d $P(x) = 6x^3 + 17x^2 - 4x - 3$
$P(-3) = -162 + 153 + 12 - 3 = 0$
$\therefore\ P(x) = (x + 3)(6x^2 - x - 1)$
$= (x + 3)(3x + 1)(2x - 1)$

10 $P(x) = x^3 + ax^2 - x + b$
$P(x)$ is divisible by $x - 1$ and $x + 3$:
$P(1) = 1 + a - 1 + b = 0$
$\therefore a + b = 0$
$P(-3) = -27 + 9a + 3 + b = 0$
$\therefore\ 9a + b = 24$
$a = 3, b = -3$
So $P(x) = x^3 + 3x^2 - x - 3$
$= x^2(x + 3) - (x + 3)$
$= (x + 3)(x^2 - 1)$
$= (x + 3)(x - 1)(x + 1)$

11 **a** $P(x) = x^n - a^n$
$P(a) = a^n - a^n = 0$
By the Factor Theorem, $(x - a)$ is a factor of $P(x)$

b $Q(x) = x^n + a^n$

i If $(x + a)$ is a factor of $Q(x)$, then $Q(-a) = (-a)^n + a^n$,
which is zero if n is an odd number.

ii If $(x + a)$ is a factor of $P(x)$, then $P(-a) = (-a)^n - a^n$,
which is zero if n is an even number.

12 **a** $P(x) = (x - 1)(x - 2)Q(x) + ax + b$
$P(1) = a + b = 2$
$P(2) = 2a + b = 3$
$a = b = 1$

b **i** If $P(x)$ is a cubic with x^3 coefficient = 1:
$P(x) = (x - 1)(x - 2)(x + c) + x + 1$
Since -1 is a solution to $P(1) = 0$:

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$$P(-1) = (-2)(-3)(-1 + c) - 1 + 1 = 0$$

$$\therefore\ c = 1$$

$$\begin{aligned} P(x) &= (x - 1)(x - 2)(x + 1) + x + 1 \\ &= (x + 1)((x - 1)(x - 2) + 1) \\ &= (x + 1)(x^2 - 3x + 2 + 1) \\ &= (x + 1)(x^2 - 3x + 3) \\ &= x^3 - 2x^2 + 3 \end{aligned}$$

ii $P(x) = (x + 1)(x^2 - 3x + 3)$ includes a quadratic where $\Delta < 0$, so $x = -1$ is the only real solution to $P(x) = 0$

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Solutions to Exercise 6D

1 a $(x-1)(x+2)(x-4)=0$
$x=1,-2,4$

b $(x-4)(x-4)(x-6)=0$
$x=4,6$

c $(2x-1)(x-3)(3x+2)=0$
$x=\frac{1}{2},3,-\frac{2}{3}$

d $x(x+3)(2x-5)=0$
$x=0$ or $x=-3$ or $x=\frac{5}{2}$

2 a $x^3-2x^2-8x=0$
$\therefore\ x(x^2-2x-8)=0$
$\therefore\ x(x+2)(x-4)=0$
$x=-2,0,4$

b $x^3+2x^2-11x=0$
$\therefore x(x^2+2x-11)=0$
$\therefore x(x+1-2\sqrt{3})(x+1+2\sqrt{3})=0$
$x=0,-1\pm 2\sqrt{3}$

c $x^3-3x^2-40x=0$
$\therefore x(x^2-3x-40)=0$
$\therefore x(x-8)(x+5)=0$
$x=-5,0,8$

d $x^3+2x^2-16x=0$
$\therefore x(x^2+2x-16)=0$
$\therefore x(x+1-\sqrt{17})(x+1+\sqrt{17})=0$
$x=0,-1\pm\sqrt{17}$

3 a $x^3-x^2+x-1=0$
$\therefore\ x^2(x-1)+(x-1)=0$
$\therefore\ (x-1)(x^2+1)=0$
$x=1$; no other real solutions since $\Delta<0$ for the quadratic term.

b $x^3+x^2+x+1=0$
$\therefore\ x^2(x+1)+(x+1)=0$
$\therefore\ (x+1)(x^2+1)=0$
$x=-1$; no other real solutions since $\Delta<0$ for the quadratic term.

c $x^3-5x^2-10x+50=0$
$\therefore\ x^2(x-5)-10(x-5)=0$
$\therefore\ (x-5)(x^2-10)=0$
$\therefore\ (x-5)(x-\sqrt{10})(x+\sqrt{10})=0$
$x=5,\pm\sqrt{10}$

d $x^3-ax^2-16x+16a=0$
$\therefore\ x^2(x-a)-16(x-a)=0$
$\therefore\ x^2(x-a)(x-16)=0$
$\therefore\ (x-a)(x-4)(x+4)=0$
$x=a,\pm4$

4 a $x^3-19x+30=0$
$P(2)=8-38+30=0$
$\therefore\ P(x)=(x-2)(x^2+2x-15)=0$
$=(x-2)(x-3)(x+5)=0$
$x=-5,2,3$

b $P(x)=3x^3-4x^2-13x-6=0$
$P(-1)=-3-4+13-6=0$
$\therefore\ P(x)=(x+1)(3x^2-7x-6)=0$
$=(x+1)(3x+2)(x-3)=0$
$x=-1,\ -\frac{2}{3},3$

c $x^3-x^2-2x+2=0$
$\therefore\ x^2(x-1)-2(x-1)=0$
$\therefore\ (x-1)(x^2-2)=0$
$\therefore\ (x-1)(x-\sqrt{2})(x+\sqrt{2})=0$
$x=1,\pm\sqrt{2}$

d $P(x)=5x^3+12x^2-36x-16=0$

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$P(2) = 40 + 48 - 72 - 16 = 0$

$\therefore\ P(x) = (x - 2)(5x\ + 22x + 8) = 0$

$= (x - 2)(5x + 2)(x + 4) = 0$

$x = -4, -\frac{2}{5}, 2$

e $P(x) = 6x^3 - 5x^2 - 2x + 1 = 0$

$P(1) = 6 - 5 - 2 + 1 = 0$

$\therefore\ P(x) = (x - 1)(6x^2 + x - 1) = 0$

$= (x - 1)(3x - 1)(2x + 1) = 0$

$x = -\frac{1}{2}, \frac{1}{3}, 1$

f $P(x) = 2x^3 - 3x^2 - 29x - 30 = 0$

$P(-2) = -16 - 12 + 58 - 30 = 0$

$\therefore\ P(x) = (x + 2)(2x^2 - 7x - 15) = 0$

$= (x + 2)(2x + 3)(x - 5) = 0$

$x = -2, -\frac{3}{2}, 5$

5 a $P(x) = x^3 + x^2 - 24x + 36 = 0$

$P(2) = 8 + 4 - 48 + 36 = 0$

$\therefore\ P(x) = (x - 2)(x + 3x - 18) = 0$

$= (x - 2)(x - 3)(x + 6) = 0$

$x = -6, 2, 3$

b $P(x) = 6x^3 + 13x^2 - 4 = 0$

$P(-2) = -48 + 52 - 4 = 0$

$\therefore\ P(x) = (x + 2)(6x + x - 2) = 0$

$= (x + 2)(2x - 1)(3x + 2) = 0$

$x = -2, -\frac{2}{3}, \frac{1}{2}$

c $P(x) = x^3 - x^2 - 2x - 12 = 0$

$P(3) = 27 - 9 - 6 - 12 = 0$

$\therefore\ P(x) = (x - 3)(x^2 + 2x + 4) = 0$

$x = 3$; no other real solutions since $\Delta < 0$ for the quadratic term.

d $P(x) = 2x^3 + 3x^2 + 7x + 6 = 0$

$P(-1) = -2 + 3 - 7 + 6 = 0$

$\therefore\ P(x) = (x + 1)(2x^2 + x + 6) = 0$

$x = -1$; no other real solutions since $\Delta < 0$ for the quadratic term.

e $P(x) = x^3 - x^2 - 5x - 3 = 0$

$P(3) = 27 - 9 - 15 - 3 = 0$

$\therefore\ P(x) = (x - 3)(x^2 + 2x + 1) = 0$

$= (x - 3)(x + 1)^2 = 0$

$x = -1, 3$

f $P(x) = x^3 + x^2 - 11x - 3 = 0$

$P(3) = 27 + 9 - 33 - 3 = 0$

$\therefore\ P(x) = (x - 3)(x^2 + 4x + 1) = 0$

$= (x - 3)(x + 2 - \sqrt{3})(x + 2 + \sqrt{3})$

$= 0$

$x = 3, -2 \pm \sqrt{3}$

6 a $2x^3 = 16x$

$\therefore\ 2x^3 - 16x = 0$

$\therefore\ 2x(x^2 - 8) = 0$

$\therefore\ 2x(x - 2\sqrt{2})(x + 2\sqrt{2}) = 0$

$x = 0, \pm 2\sqrt{2}$

b $2(x - 1)^3 = 32$

$\therefore\ (x - 1)^3 = 16$

$x - 1 = 2\sqrt[3]{2}$

$x = 1 + 2\sqrt[3]{2}$

c $x^3 + 8 = 0$

$\therefore\ (x + 2)(x^2 - 2x + 4)$

$x = -2$; no other real solutions since $\Delta < 0$ for the quadratic term

d $2x^3 + 250 = 0$

$\therefore\ 2(x^3 + 125) = 0$

$\therefore\ 2(x + 5)(x^2 - 5x + 25) = 0$

$x = -5$; no other real solutions since $\Delta < 0$ for the quadratic term.

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e $1000 = \dfrac{1}{x^3}$

$\therefore \quad 1000x^3 = 1$

$\therefore \quad (10x)^3 = 1$

$\therefore \quad 10x = 1, \therefore \ x = \dfrac{1}{10}$

7 a $2x^3 - 22x^2 - 250x + 2574$
$= 2(x-9)(x^2 - 2x - 143)$
$= 2(x-9)(x-13)(x+11)$

b $2x^3 + 27x^2 + 52x - 33$
$= (x+3)(2x^2 + 15x - 11)$
$= (x+3)(2x^2 + 21x - 11)$
$= (x+3)(2x-1)(x+11)$

c $2x^3 - 9x^2 - 242x + 1089$
$= (x-11)(2x^2 + 13x - 99)$
$= (x-11)(2x-9)(x+11)$

d $2x^3 + 51x^2 + 304x - 165$
$= (x+11)(2x^2 + 29x - 15)$
$= (x+11)(2x-1)(x+15)$

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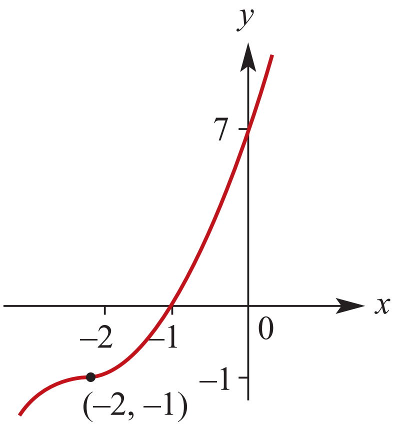
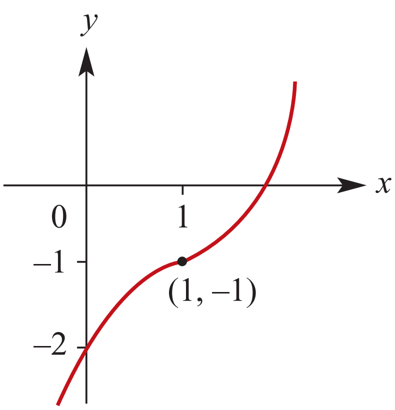
Solutions to Exercise 6E

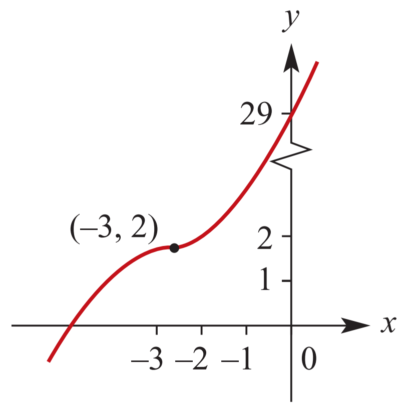
1 a $y = (x + 2)^3 - 1$
Stationary point of inflection at $(-2, -1)$



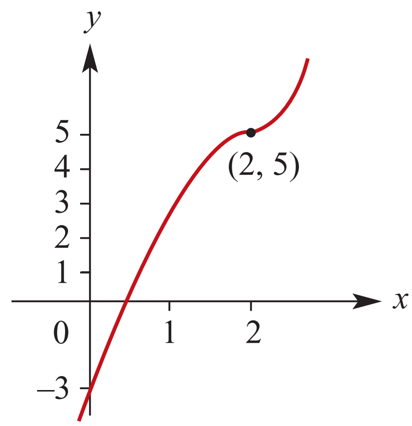
b $y = (x - 1)^3 - 1$
Stationary point of inflection at $(1, -1)$



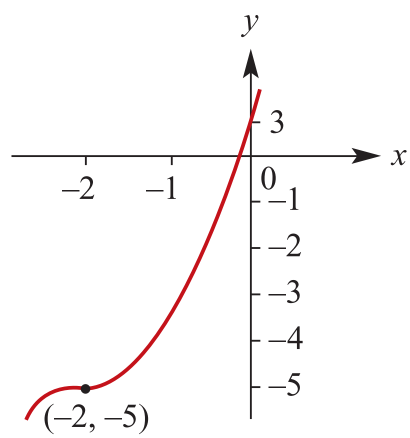
c $y = (x + 3)^3 + 2$
Stationary point of inflection at $(-3, 2)$



d $y = (x - 2)^3 + 5$
Stationary point of inflection at $(2, 5)$



e $y = (x + 2)^3 - 5$
Stationary point of inflection at $(-2, -5)$



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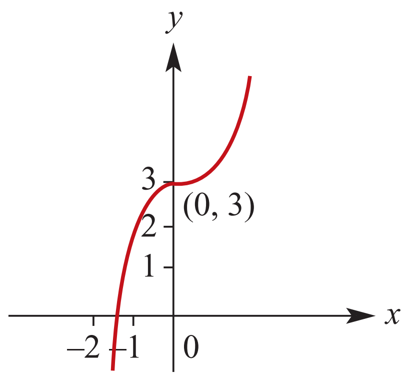
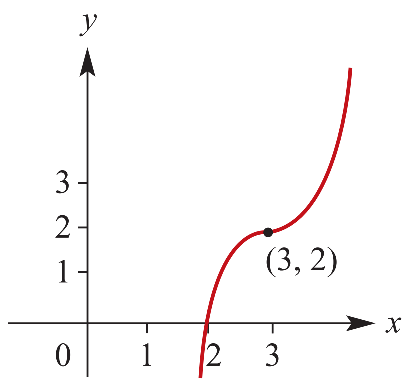
2 **a** $y = 2x^3 + 3$

Stationary point of inflection at $(0, 3)$



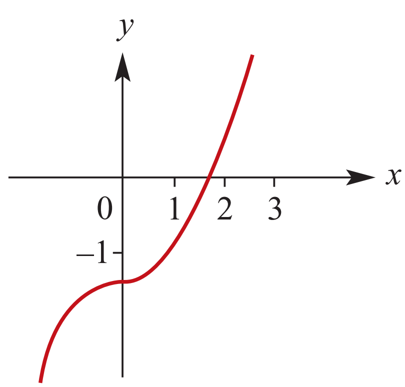
b $y = 2(x - 3)^3 + 2$

Stationary point of inflection at $(3, 2)$



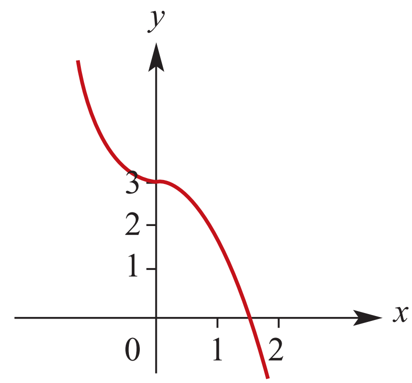
c $3y = x^3 - 5$

Stationary point of inflection at $(0, -\frac{5}{3})$



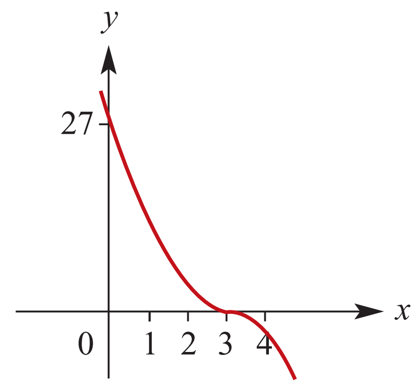
d $y = 3 - x^3$

Stationary point of inflection at $(0, 3)$



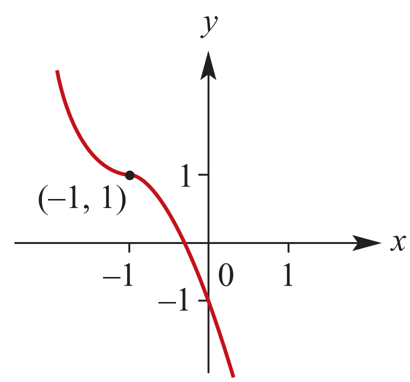
e $y = (3 - x)^3$

Stationary point of inflection at $(3, 0)$



f $y = -2(x + 1)^3 + 1$

Stationary point of inflection at $(-1, 1)$



g $y = \frac{1}{2}(x - 3)^3 + 2$

Stationary point of inflection at (3,2)

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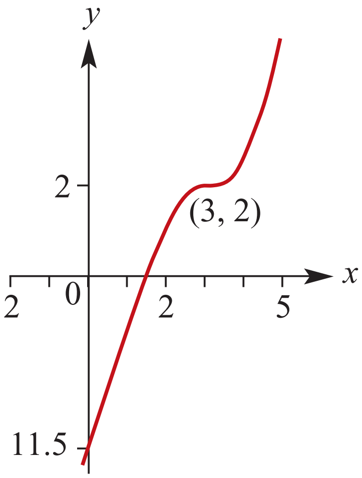
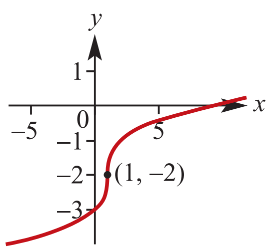


3 a



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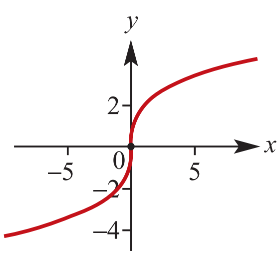

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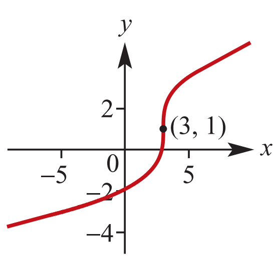
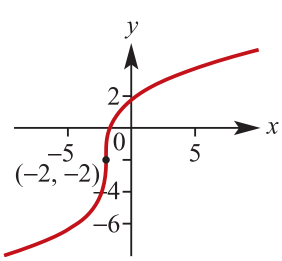


d



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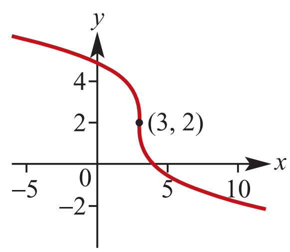

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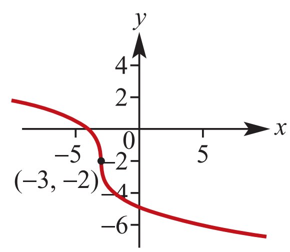

4 a Let $y = f^{-1}(x)$

Then

$$x = 2y^3 + 3$$

$$y^3 = \frac{x-3}{2}$$

$$y = \sqrt[3]{\frac{x-3}{2}}$$

$$\therefore f^{-1}(x) = \sqrt[3]{\frac{x-3}{2}}$$

Maximal domain $= \mathbb{R}$

b Let $y = f^{-1}(x)$

Then

$$x = 3y^{\frac{1}{3}}$$

$$y^{\frac{1}{3}} = \frac{x}{3}$$

$$y = \frac{x^3}{27}$$

$$\therefore f^{-1}(x) = \frac{x^3}{27}$$

Maximal domain $= \mathbb{R}$

c Let $y = f^{-1}(x)$

Then

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$$x = 2(y+1)^3 + 1$$
$$(y+1)^3 = \frac{x-1}{2}$$
$$y + 1 = \sqrt[3]{\frac{x-1}{2}}$$
$$y = \sqrt[3]{\frac{x-1}{2}} - 1$$
$$\therefore f^{-1}(x) = \sqrt[3]{\frac{x-1}{2}} - 1$$

Maximal domain $= \mathbb{R}$

d Let $y = f^{-1}(x)$

Then

$$x = 2(y+3)^{\frac{1}{3}} - 2$$
$$(y+3)^{\frac{1}{3}} = \frac{x+2}{2}$$
$$y + 3 = \frac{(x+2)^3}{8}$$
$$y = \frac{(x+2)^3}{8} - 3$$
$$\therefore f^{-1}(x) = \frac{(x+2)^3}{8} - 3$$

Maximal domain $= \mathbb{R}$

e Let $y = f^{-1}(x)$

Then

$$x = -2(y-1)^{\frac{1}{3}} + 4$$
$$(y-1)^{\frac{1}{3}} = \frac{x-4}{-2}$$
$$y - 1 = -\frac{(x-4)^3}{8}$$
$$y = 1 - \frac{(x-4)^3}{8}$$
$$\therefore f^{-1}(x) = 1 - \frac{(x-4)^3}{8}$$

Maximal domain $= \mathbb{R}$

f Let $y = f^{-1}(x)$

Then

$$x = -2(y+2)^{\frac{1}{3}} - 1$$
$$(y+2)^{\frac{1}{3}} = \frac{x+1}{-2}$$
$$y + 2 = -\frac{(x+1)^3}{8}$$
$$y = -2 - \frac{(x+1)^3}{8}$$
$$\therefore f^{-1}(x) = -2 - \frac{(x+1)^3}{8}$$

Maximal domain $= \mathbb{R}$

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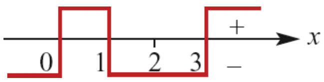
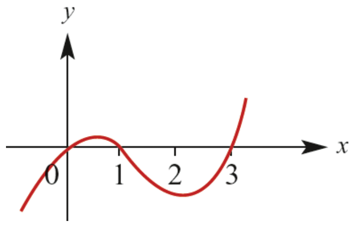
Solutions to Exercise 6F

1 a $y = x(x-1)(x-3)$

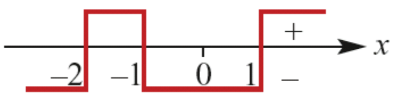
Axis intercepts: $(0,0)$, $(1,0)$ and $(3,0)$

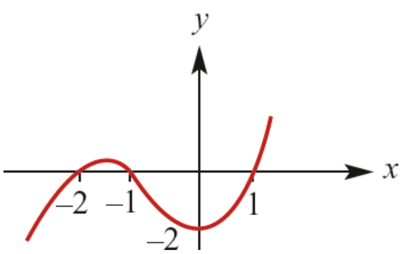




b $y = (x-1)(x+1)(x+2)$

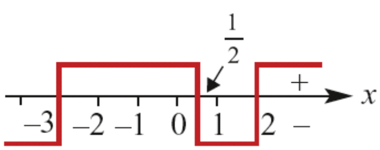
Axis intercepts: $(-2,0)$, $(-1,0)$, $(1,0)$ and $(0,6)$

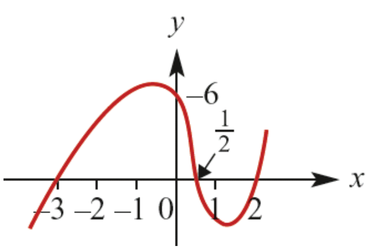




c $y = (2x-1)(x-2)(x+3)$

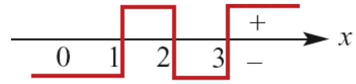
Axis intercepts: $(-3,0)$, $(\frac{1}{2},0)$, $(2,0)$ and $(0,6)$

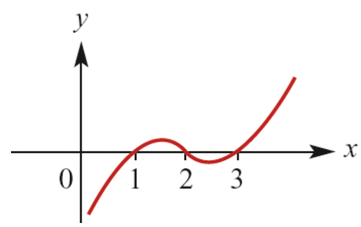




d $y = (x-1)(x-2)(x-3)$

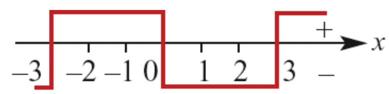
Axis intercepts: $(1,0)$, $(2,0)$, $(3,0)$ and $(0,6)$

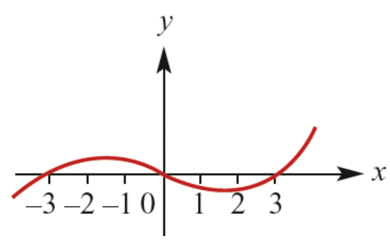




2 a $y = x^3 - 9x = x(x-3)(x+3)$

Axis intercepts: $(0,0)$, $(-3,0)$ and $(3,0)$

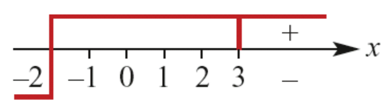


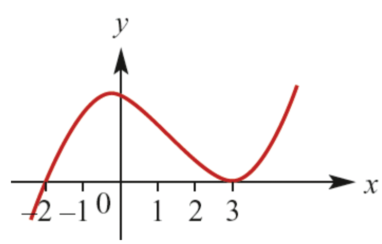


b $y = x^3 - 4x^2 - 3x + 18$

$\therefore y = (x-3)^2(x+2)$

Axis intercepts: $(-2,0)$, $(3,0)$ and $(0,18)$





c $y = -x^3 + x^2 + 3x - 3$

$\therefore\ y = (1-x)(x^2-3)$

$= (1-x)(x-\sqrt{3})(x+\sqrt{3})$

Axis intercepts:

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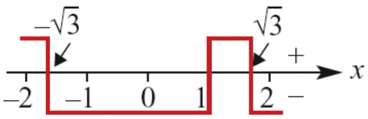
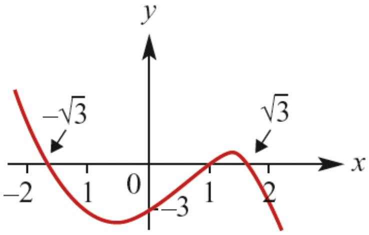
$(1, 0), (-\sqrt{3}, 0), (\sqrt{3}, 0)$ and $(0, -3)$

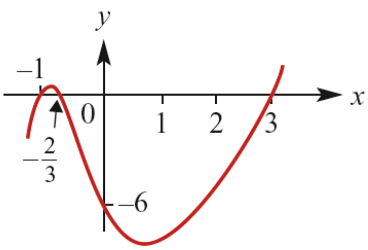




d $y = 3x^3 - 4x^2 - 13x - 6$

$\therefore y = (3x + 2)(x + 1)(x - 3)$

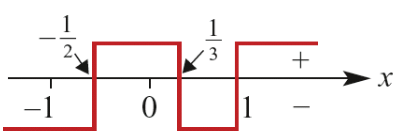
Axis intercepts: $(-1, 0), (-\frac{2}{3}, 0), (3, 0)$ and (0,1)

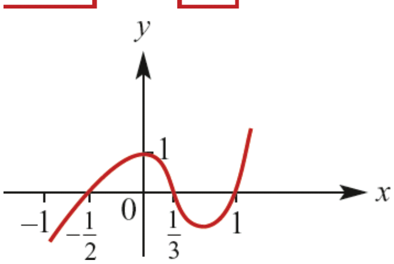


e $y = 6x^3 - 5x^2 - 2x + 1$

$y = (x - 1)(3x - 1)(2x + 1)$

Axis intercepts: $(-\frac{1}{2}, 0), (\frac{1}{3}, 0), (1, 0)$ and $(0, 1)$





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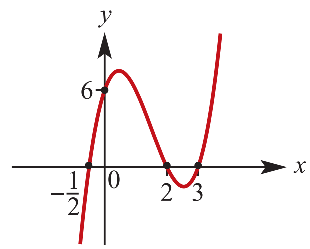
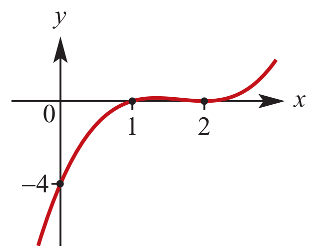
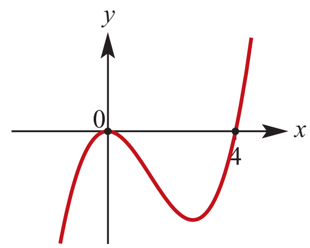


3 a



b



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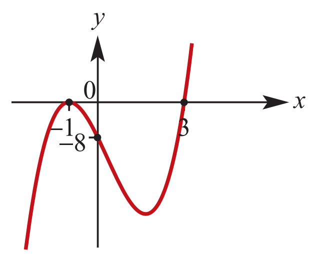
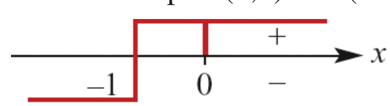
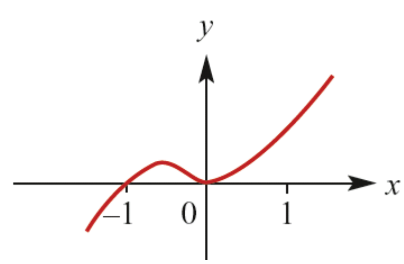


d $y = x^3 + x^2 = x^2(x + 1)$

Axis intercepts: (0,0) and $(-1, 0)$





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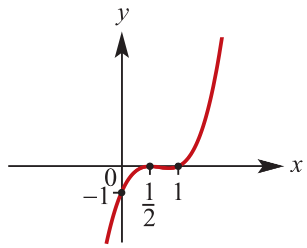
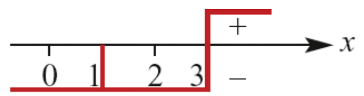
e

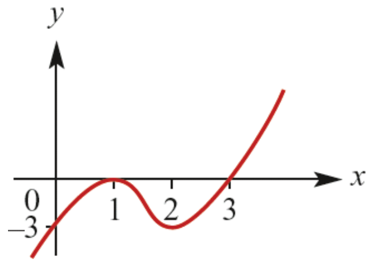


f $y = x^3 - 5x^2 + 7x - 3$

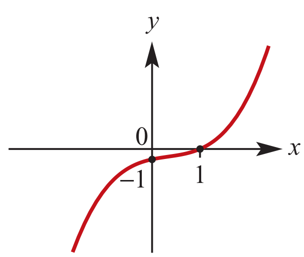
$\therefore\ y = (x-1)^2(x-3)$

Axis intercepts: $(1, 0), (3, 0)$ and $(0, -3)$

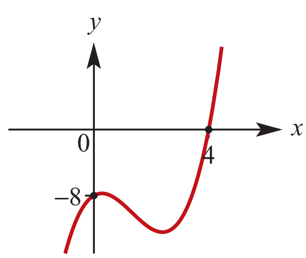




4 a



b

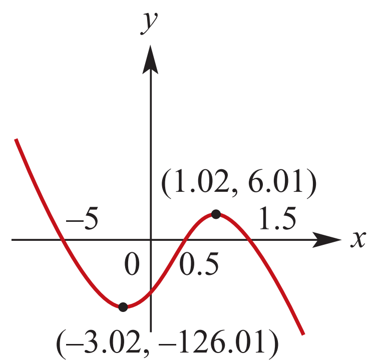


5 a $y = -4x^3 - 12x^2 + 37x - 15$

Intercepts: $(-5, 0)$, $(\frac{1}{2}, 0), (\frac{2}{3}, 0)$ and $(0, -15)$

Max. : $(1.02, 6.01)$

Min. : $(-3.02, -126.01)$

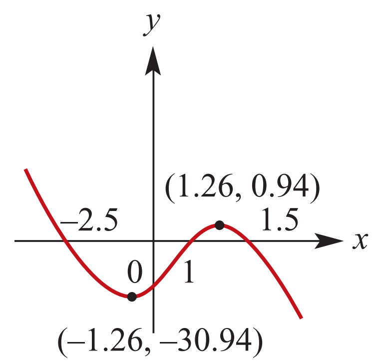


b $y = -4x^3 + 19x - 15$

Intercepts: $(-\frac{5}{2}, 0), (1, 0), (\frac{3}{2}, 0)$ and $(0, -15)$

Max. : $(1.26, 0.94)$

Min. : $(-1.26, -30.94)$

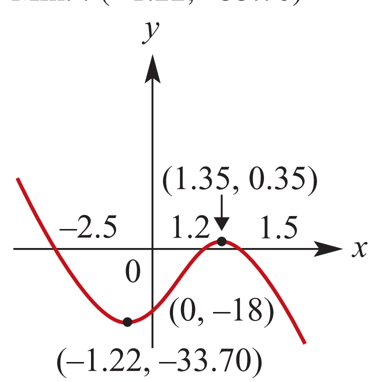


c $y = -4x^3 + 0.8x^2 + 19.8x - 18$

Intercepts: $(-2.5, 0)$, $(1.2, 0)$, $(1.5, 0)$ and $(0, -18)$

Max. : $(1.35, 0.35)$

Min. : $(-1.22, -33.70)$



d $y = 2x^3 + 11x^2 + 15x$

Intercepts: $(-3, 0), (-\frac{5}{2}, 0)$, and $(0, 0)$

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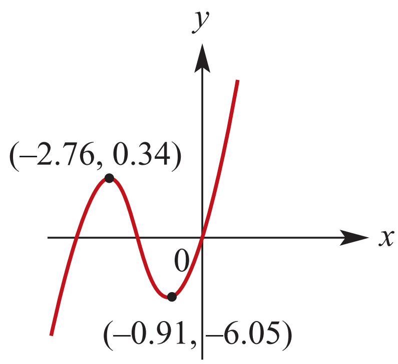
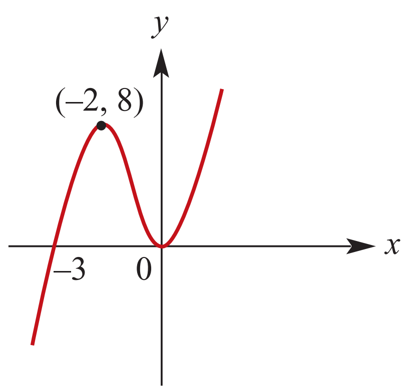
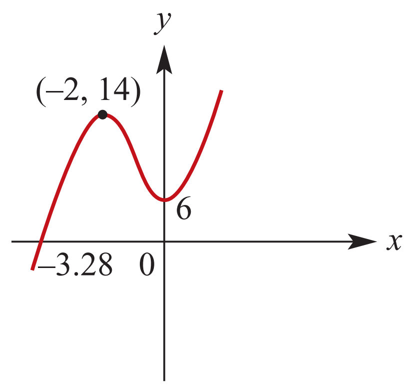
Max : $(-2.76, 0.34)$
Min : $(-0.91, -6.05)$



e $y = 2x^3 + 6x^2$
Intercepts: $(-3, 0)$ and $(0, 0)$
Max : $(-2, 8)$
Min : $(0, 0)$



f $y = 2x^3 + 6x^2 + 6$
Intercepts: $(-3.28, 0)$ and $(0, 6)$
Max : $(-2, 14)$
Min : $(0, 6)$

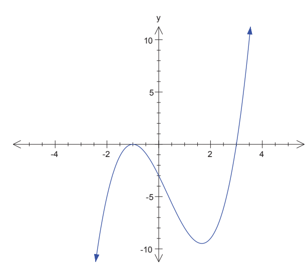


6 $f(x) = x^3 - x^2 - 5x - 3$

$$= (x - 3)(x^2 + 2x + 1)$$

$$= (x - 3)(x + 1)^2$$

$f(x)$ cuts the axis at $x = 3$ and touches the axis at the repeated root $x = -1$.



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Solutions to Exercise 6G

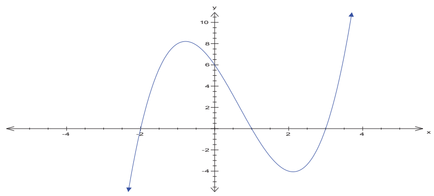
1 **a** $(x-1)(x+2)(x-3) \le 0$

Arrange in order from left to right:

$(x+2)(x-1)(x-3) \le 0$

Upright cubic, so $f(x) \le 0$ for:

$\{x\colon x \le -2\} \cup \{x\colon 1 \le x \le 3\}$



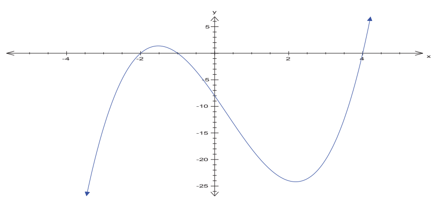
b $(x+1)(x+2)(x-4) \ge 0$

Arrange in order from left to right:

$(x+2)(x+1)(x-4) \ge 0$

Upright cubic so $f(x) \le 0$ for:

$\{x : x \ge 4\} \cup \{x : -2 \le x \le -1\}$

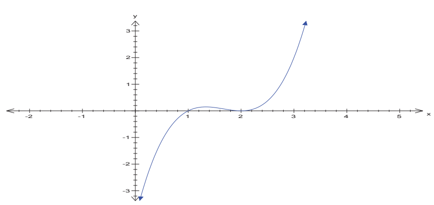


c $(x-1)(x-2)^2 < 0$

Upright cubic, so $f(x) < 0$ for

$\{x : x < 1\}$

Repeated root at $x = 2$ means that the graph is positive or zero for all other x.



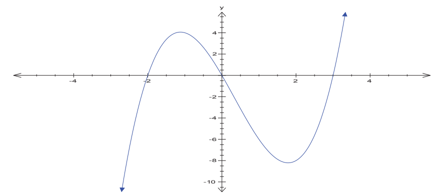
d $x(x+2)(x-3) > 0$

Arrange in order from left to right:

$(x+2)x(x-3) > 0$

Upright cubic, so $f(x) > 0$ for:

$\{x : x > 3\} \cup \{x : -2 < x < 0\}$

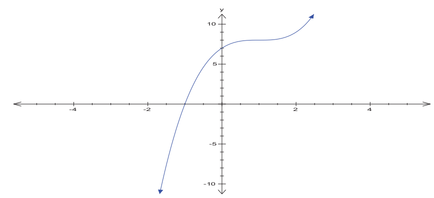


e $(x-1)^3 + 8 \le 0$

Translation of $y = x^3$ which is an increasing function for all x.

$\therefore\ (x-1)^3 \le -8$

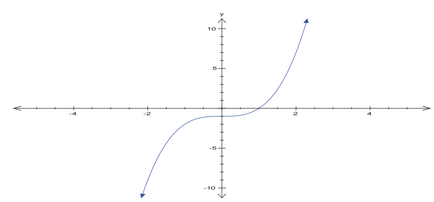
$\therefore\ x - 1 \le -2, \therefore\ x \le -1$



f $x^3 - 1 \ge 0$

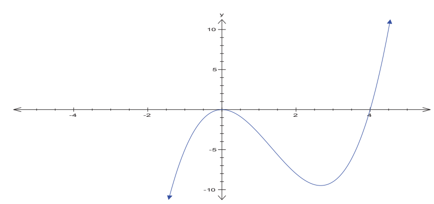
Translation of $y = x^3$ which is an increasing function for all x.

$\therefore x^3 \ge 1, \therefore x \ge 1$



g $x^2(x-4) > 0$

Repeated root at (0,0) so $f(x) > 0$ only for $\{x : x > 4\}$



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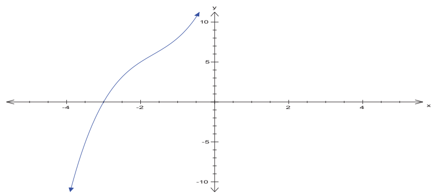
h $(x + 3)(x^2 + 2x + 5) \le 0$

The quadratic expression has no real roots so there is only one x-axis intercept at $x = -3$.

The cubic is upright so $f(x) \le 0$ only for $\{x : x \le -3\}$



2 **a** $x^3 > 4x$

$\Leftrightarrow x^3 - 4x > 0$

$\Leftrightarrow x(x^2 - 4) > 0$

$\Leftrightarrow x(x - 2)(x + 2) > 0$

$\Leftrightarrow x \in (2, \infty) \cup (-2, 0)$

Since coefficient of x^3 is positive

b

$x^3 < 5x^2$

$\Leftrightarrow x^3 - 5x^2 < 0$

$\Leftrightarrow x^2(x - 5) < 0$

$\Leftrightarrow x \in (-\infty, 0) \cup (0, 5)$

Since coefficient of x^3

is positive and 'double root' when $x = 0$

c

$x^3 + 4x \le 4x^2$

$\Leftrightarrow x^3 - 4x^2 + 4x \le 0$

$\Leftrightarrow x(x^2 - 4x + 4) \le 0$

$\Leftrightarrow x(x - 2)^2 \le 0$

$\Leftrightarrow x \in (-\infty, 0] \cup \{2\}$

Since coefficient of x^3

is positive and 'double root' when $x = 2$

d $x^3 > 9x$

$\Leftrightarrow x^3 - 9x > 0$

$\Leftrightarrow x(x^2 - 9) > 0$

$\Leftrightarrow x(x - 3)(x + 3) > 0$

$\Leftrightarrow x \in (3, \infty) \cup (-3, 0)$

Since coefficient of x^3 is positive

e $x^3 - 6x^2 + x \ge 6$

$\Leftrightarrow x^3 - 6x^2 + x - 6 \ge 0$

$\Leftrightarrow x^2(x - 6) + x - 6 \ge 0$

$\Leftrightarrow (x - 6)(x^2 + 1) \ge 0$

$\Leftrightarrow x - 6 > 0$

$\Leftrightarrow x \in (6, \infty)$

f $2x^3 - 6x^2 - 4x < -12$

$\Leftrightarrow 2x^3 - 6x^2 - 4x + 12 < 0$

$\Leftrightarrow 2x^2(x - 3) - 4(x - 3) < 0$

$\Leftrightarrow (x - 3)(2x^2 - 4) < 0$

$\Leftrightarrow 2(x - 3)(x^2 - 2) < 0$

$\Leftrightarrow (x - 3)(x - \sqrt{2})(x + \sqrt{2}) < 0$

$\Leftrightarrow x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, 3)$

Since coefficient of x^3 is positive

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Solutions to Exercise 6H

1 a $y = a(x-3)^3 + 1$

When $x = 4, y = 12$ $12 = a(4-3)^2 + 1$

$\therefore a = 11$

b $y = a(x-2)(x+3)(x-1)$

When $x = 3, y = 24$

$24 = a(3-2)(3+3)(3-1)$

$\therefore a = 2$

c $y = ax^3 + bx$

When $x = 1, y = 16$

When $x = 2, y = 40$

$$16 = a + b \dots (1)$$

$$40 = 8a + 2b \dots (2)$$

Multiply (1) by 2 and subtract from (2)

$$8 = 6a$$

$$\therefore a = \frac{4}{3}$$

$$\therefore b = \frac{44}{3}$$

2 a Equation is of the form

$y = -a(x+2)^3$.

$x = 0, y = -1$:

$-1 = -8a$, so $a = \dfrac{1}{8}$

So $y = -\dfrac{1}{8}(x+2)^3$

b Equation is of the form

$y = -a(x-3)^3 + 2$

$x = 5, y = 0 : 0 = -8a + 2$, so $a = \dfrac{1}{4}$

$\therefore\ y = 2 - \dfrac{1}{4}(x-3)^3$

3 The graph has a repeated root at $(2, 0)$ and cuts $(0, 0)$, $\therefore\ y = ax(x-2)^2$

Using $(3, 6)$: $\quad 3a(3-2) = 6$

$\therefore \quad a = 2$

$\therefore\ y = 2x(x-2)^2$

4 Repeated root at $x = -4$, cuts at $(0, 0)$

$\therefore\ y = ax(x+4)^2$

Using $(-3, 6)$: $\quad -3a(-3+4)^2 = 6$

$\therefore \quad -3a = 6$

$\therefore \quad a = -2$

$\therefore\ y = -2x(x+4)^2$

5 $y = a(x-1)(x-3)(x+1)$

When $x = 0, y = -6$

$\therefore -6 = a(-1)(-3)(1)$

$\therefore a = -2$

$y = -2(x-1)(x-3)(x+1)$

6 $f(x) = (x^2 + a)(x-3)$

$f(6) = 216$

$\therefore 216 = (36 + a)(3)$

$\therefore 72 = 36 + a$

$\therefore a = 36$

7 a $y = a(x-h)^3 + k$

Stationary point of inflection at $(3, 2)$, so $h = 3$.

Using $(3, 2)$: $k = 2$

Using $(0, -25)$:

$$a(-3)^3 + 2 = -25$$

$$\therefore \quad 27a = -27$$

$$\therefore \quad a = 1$$

$\therefore\ y = (x-3)^3 + 2$

b $y = ax^3 + bx^2$

$\therefore\ y = x^2(ax + b)$

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Using $(1, 5)$: $\quad a + b = 5$

Using $(-3, -1)$: $\quad 9(-3a + b) = -1$

$$\therefore \quad 3a - b = \frac{1}{9}$$

$$\begin{array}{rl} a + b &= 5 \\ \hline 4a &= \dfrac{46}{9} \end{array}$$

$\therefore \ a = \dfrac{23}{18}; b = \dfrac{67}{18}$

$y = \dfrac{1}{18}(23x^3 + 67x^2)$

c $y = ax^3$

Using$(1, 5)$: $a(1)^3 = 5, \therefore a = 5$

$y = 5x^3$

8 a Graph has axis intercepts at (0,0) and $(\pm 2, 0)$:

$y = ax(x - 2)(x + 2)$

Using $(1, 1)$:

$$a(1 - 2)(1 + 2) = 1$$

$$\therefore \quad -3a = 1$$

$$\therefore \quad a = -\frac{1}{3}$$

$y = -\dfrac{1}{3}x(x - 2)(x + 2)$

OR $y = -\dfrac{1}{3}x^3 + \dfrac{4}{3}x$

b $y = ax^3 + bx^2 + cx$

Using $(2, 3)$: $\quad 8a + 4b + 2c = 3$

Using $(-2, -3)$: $\quad -8a + 4b - 2c = -3$

$$\begin{array}{rl} \hline 8b &= 0 \end{array}$$

$\therefore \ b = 0$

$\therefore \ y = ax^3 + cx$

and $8a + 2c = 3 \therefore 4a + c = 1.5$

Using $(1, 0.75)$: $\quad a + c = 0.75$

$$\begin{array}{rl} \hline 3a &= 0.75 \end{array}$$

$\therefore \ a = 0.25$

$\therefore \ c = 0.5$

$\therefore \ y = \dfrac{1}{4}x^3 + \dfrac{1}{2}x = \dfrac{1}{4}x(x^2 + 2)$

9 $y = ax^3 + bx^2 + cx + d$

Use CAS calculator **Solve** function.

a $(0, 270)(1, 312)(2, 230)(3, 0)$

$y = -4x^3 - 50x^2 + 96x + 270$

b $(-2, -406)(0, 26)(1, 50)(2, -22)$

$y = 4x^3 - 60x^2 + 80x + 26$

c $(-2, -32)(2, 8)(3, 23)(8, 428)$

$y = x^3 - 2x^2 + 6x - 4$

d $(1, -1)(2, 10)(3, 45)(4, 116)$

$y = 2x^3 - 3x$

e $(-3, -74)(-2, -23)(-1, -2)(1, -2)$

$y = 2x^3 - 3x^2 - 2x + 1$

f $(-3, -47)(-2, -15)(1, -3)(2, -7)$

$y = x^3 - 3x^2 - 2x + 1$

g $(-4, 25)(-3, 7)(-2, 1)(1, -5)$

$y = -x^3 - 3x^2 - 2x + 1$

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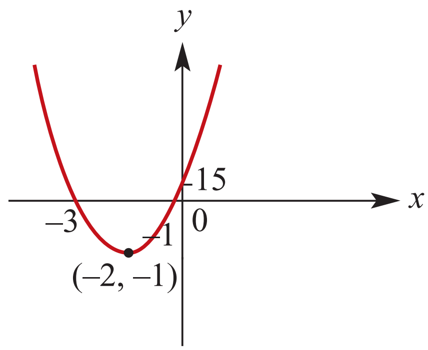
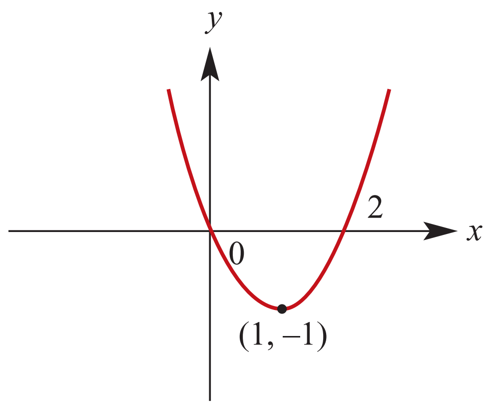
Solutions to Exercise 6I

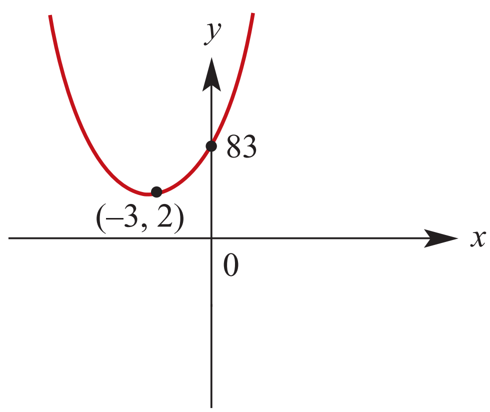
1 **a** $y = (x + 2)^4 - 1$; vertex at $(-2, -1)$



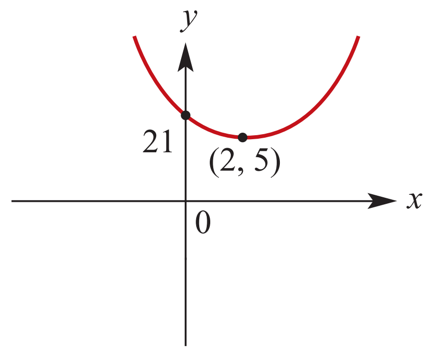
b $y = (x - 1)^4 - 1$; vertex at $(1, -1)$



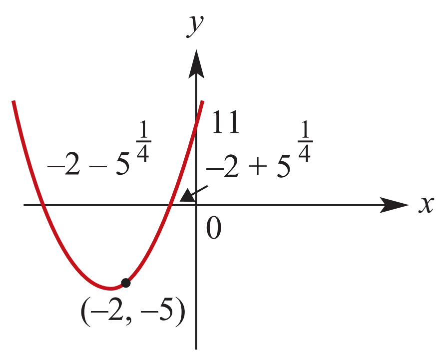
c $y = (x + 3)^4 + 2$; vertex at $(-3, 2)$



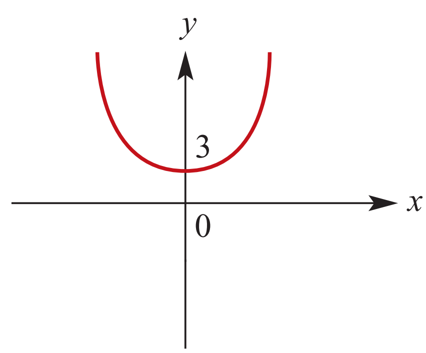
d $y = (x - 2)^4 + 5$; vertex at $(2, 5)$



e $y = (x + 2)^4 - 5$; vertex at $(-2, -5)$



2 **a** $y = 2x^4 + 3$; vertex at $(0, 3)$



b $y = 2(x - 3)^4 + 2$; vertex at $(3, 2)$

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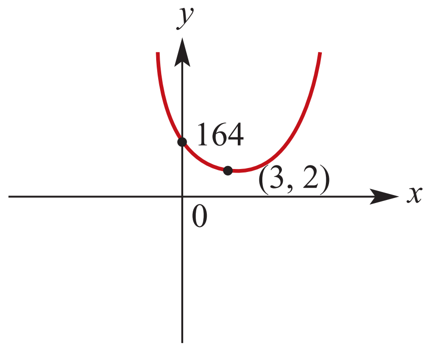
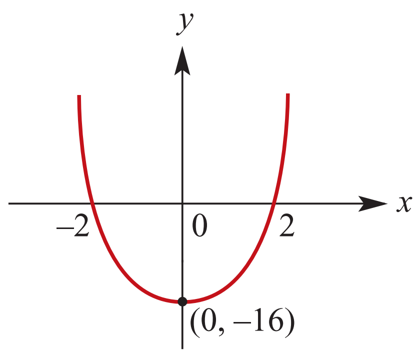
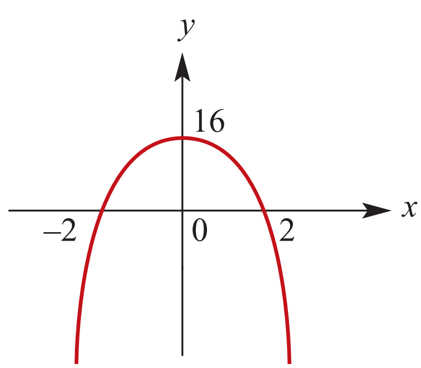


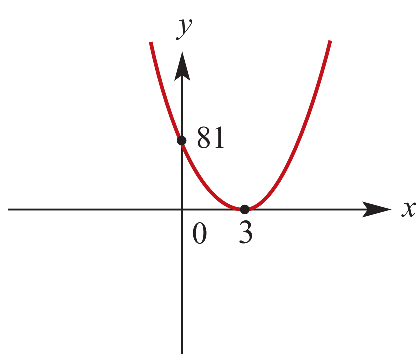
c $y = x^4 - 16$; vertex at $(0, -16)$



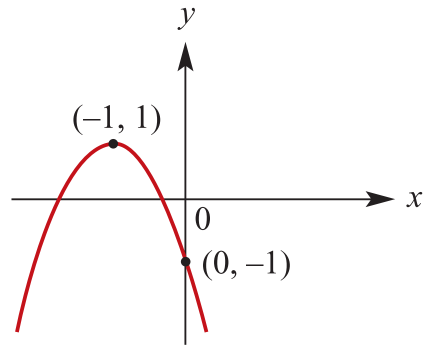
d $y = 16 - x^4$; vertex at $(0, 16)$



e $y = (3 - x)^4$; vertex at $(3,0)$



f $y = -2(x + 1)^4 + 1$; vertex at $(-1, 1)$



3 a $x^4 - 27x = 0$

$\therefore \quad x(x^3 - 27) = 0$

$\therefore \quad x(x - 3)(x^2 + 3x + 9) = 0$

$x = 0, 3$; quadratic has no real solutions

b $(x^2 - x - 2)(x^2 - 2x - 15) = 0$

$\therefore \quad (x - 2)(x + 1)(x - 5)(x + 3) = 0$

$x = -3, -1, 2, 5$

c $x^4 + 8x = 0$

$\therefore \quad (x^3 + 8) = 0$

$\therefore \quad x(x + 2)(x^2 - 2x + 4) = 0$

$x = 0, -2$; quadratic has no real solutions

d $x^4 - 6x^3 = 0$

$\therefore x^3(x - 6) = 0$

$x = 0, 6$

e $x^4 - 9x^2 = 0$

$\therefore \quad x^2(x^2 - 9) = 0$

$\therefore \quad x^2(x - 3)(x + 3) = 0$

$x = 0, \pm 3$

f $81 - x^4 = 0$

$\therefore \quad x^4 - 81 = 0$

$\therefore \quad (x^2 - 9)(x^2 + 9) = 0$

$\therefore \quad (x - 3)(x + 3)(x^2 + 9) = 0$

$x = \pm 3$; quadratic has no real solutions

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g $x^4 - 16x^2 = 0$

$\therefore \quad x^2(x^2 - 16) = 0$

$\therefore \quad x^2(x - 4)(x + 4) = 0$

$x = 0, \pm 4$

h $x^4 - 7x^3 + 12x^2 = 0$

$\therefore \quad x^2(x^2 - 7x + 12) = 0$

$\therefore \quad x^2(x - 3)(x - 4) = 0$

$x = 0, 3, 4$

i $x^4 - 9x^3 + 20x^2 = 0$

$\therefore \quad x^2(x^2 - 9x + 20) = 0$

$\therefore \quad x^2(x - 4)(x - 5) = 0$

$x = 0, 4, 5$

j $(x^2 - 4)(x^2 - 9) = 0$

$\therefore \ (x - 2)(x + 2)(x - 3)(x + 3) = 0$

$x = \pm 2, \pm 3$

k $(x - 4)(x^2 + 2x + 8) = 0$

$x = 4$; quadratic has no real solutions

l $(x + 4)(x^2 + 2x - 8) = 0$

$\therefore \quad (x + 4)(x - 2)(x + 4) = 0$

$\therefore \quad (x + 4)^2(x - 2) = 0$

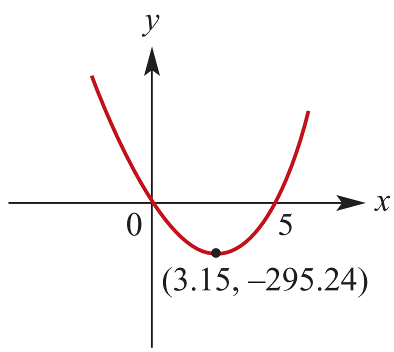
$x = -4, 2$

4 a $y = x^4 - 125x$

$\therefore \ y = x(x^3 - 125)$

x-intercepts: $(0, 0)$ and $(5, 0)$

TP: $(3.15, -295.24)$



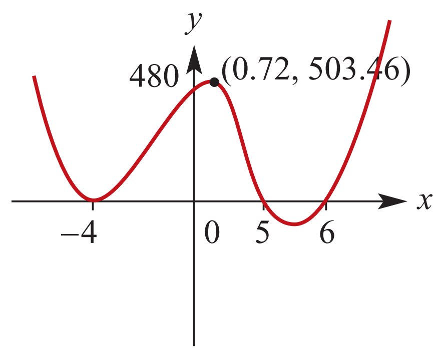
b $y = (x^2 - x - 20)(x^2 - 2x - 24)$

$= (x - 5)(x + 4)(x + 4)(x - 6)$

$= (x - 5)(x + 4)^2(x - 6)$

x-intercepts: $(-4, 0)$, $(5, 0)$ and $(6, 0)$

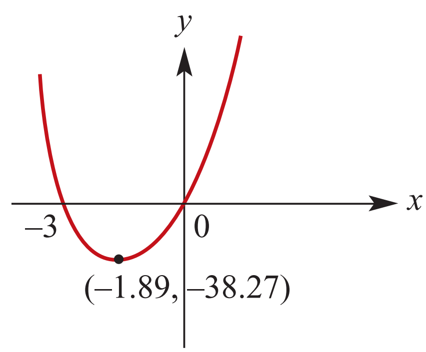
TPs: $(-4, 0)$, $(0.72, 503.5)$ and $(5.53, -22.62)$



c $y = x^4 + 27x$

x intercepts: $(0,0)$ and $(-3, 0)$

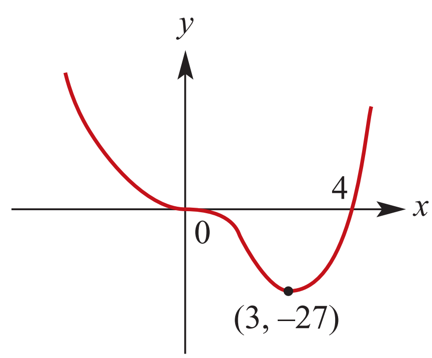
TP: $(-1.89, \ -38.27)$



d $y = x^4 - 4x^3$

x-intercepts: $(0, 0)$ and $(4, 0)$

TP: $(3, -27)$



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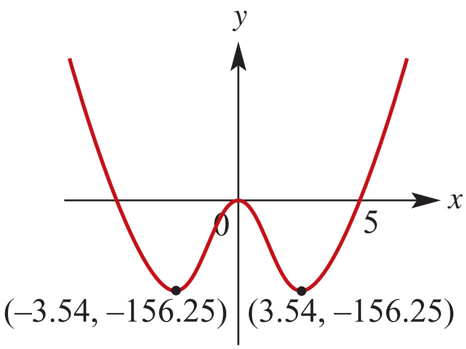
e $y = x^4 - 25x^2$

$= x^2(x^2 - 25)$

$= x^2(x - 5)(x + 5)$

x-intercepts: $(0, 0)(-5, 0)$ and $(5, 0)$

TPs: $(0, 0)$, $(-3.54, -156.25)$ and $(3.54, -156.25)$



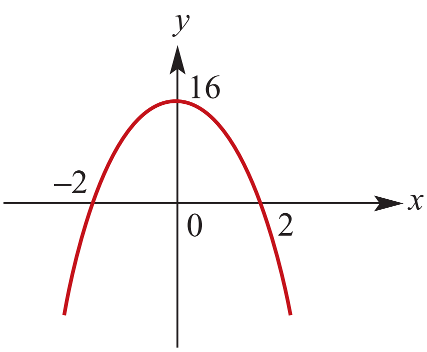
f $y = 16 - x^4$

$= (4 - x^2)(4 + x^2)$

$= (2 - x)(2 + x)(4 + x^2)$

x-intercepts: $(-2, 0)$ and $(2, 0)$

TP: $(0, 16)$



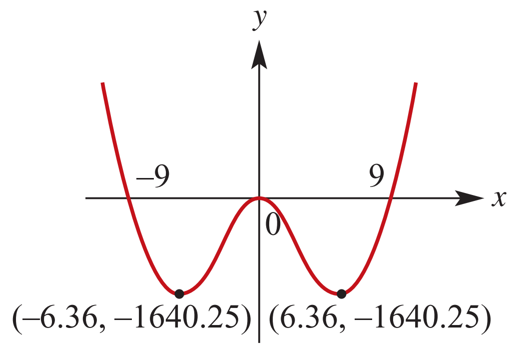
g $y = x^4 - 81x^2$

$= x^2(x^2 - 81)$

$= x^2(x - 9)(x + 9)$

x-intercepts: $(0, 0)$, $(-9, 0)$ and $(9, 0)$

TPs: $(0,0)$, $(-6.36, -1640.25)$ and $(6.36, -1640.25)$



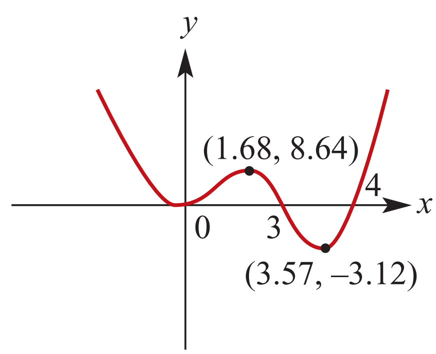
h $y = x^4 - 7x^3 + 12x^2$

$= x^2(x^2 - 7x + 12)$

$= x^2(x - 3)(x - 4)$

x-intercepts: $(0, 0)$, $(3, 0)$ and $(4, 0)$

TPs: $(0, 0)$, $(1.68, 8.64)$ and $(3.57, -3.12)$



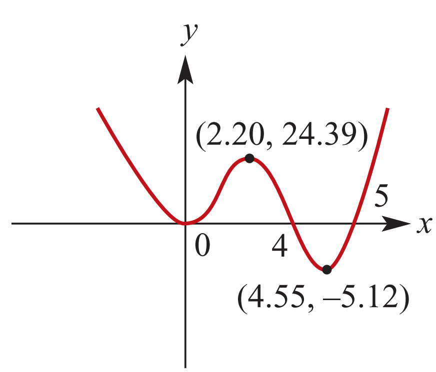
i $y = x^4 - 9x^3 + 20x^2$

$= x^2(x^2 - 9x + 20)$

$= x^2(x - 4)(x - 5)$

x-intercepts: $(0, 0)$, $(4, 0)$ and $(5, 0)$

TPs: $(0, 0)$, $(2.20, 24.39)$ and $(4.55, -5.12)$



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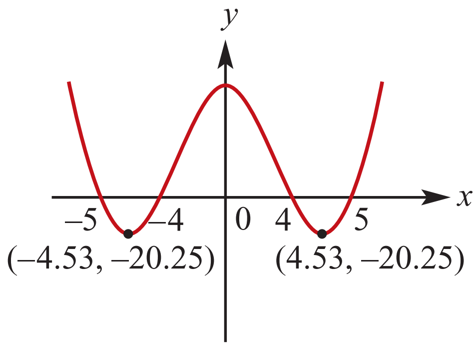
j $y = (x^2 - 16)(x^2 - 25)$

$= (x - 4)(x + 4)(x - 5)(x + 5)$

x-intercepts: $(-5, 0)$, $(-4, 0)$, $(4, 0)$ and $(5, 0)$

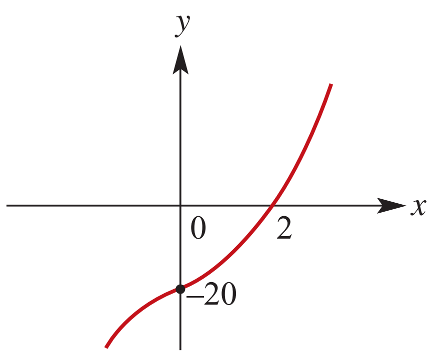
TPs: $(0, 400)$, $(-4.53, -20.25)$ and $(4.53, -20.25)$



k $y = (x - 2)(x^2 + 2x + 10)$ x-intercept: $(2, 0)$

Quadratic has no real solutions.

No Turning points, as shown by reference to a CAS graph.

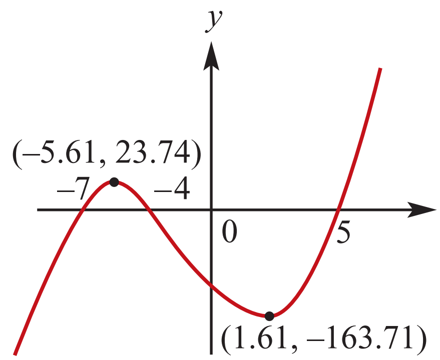


l $y = (x + 4)(x^2 + 2x - 35)$

$= (x + 4)(x + 7)(x - 5)$

x-intercepts: $(-7, 0)$, $(-4, 0)$ and $(5, 0)$

TPs: $(-5.61, 23.74)$ and $(1.61, -163.71)$



5 a
$$f(x) = 5x^2 - 3x^2$$
$$\therefore f(-x) = 5(-x)^2 - 3(-x)^2$$
$$= 5x^2 - 3x^2$$
$$= f(x)$$
$\therefore f(x)$ is even.

b
$$f(x) = 7x^{11} - x^3 + 2x$$
$$\therefore f(-x) = 7(-x)^{11} - (-x)^3 + 2(-x)$$
$$= -7x^{11} + x^3 - 2x$$
$$= -f(x)$$
$\therefore f(x)$ is odd.

c
$$f(x) = x^4 - 3x^2 + 2$$
$$\therefore f(-x) = (-x)^4 - 3(-x)^2 + 2$$
$$= x^4 - 3x^2 + 2$$
$$= f(x)$$
$\therefore f(x)$ is even.

d
$$f(x) = x^5 - 4x^3$$
$$\therefore f(-x) = (-x)^5 - 4(-x)^3$$
$$= -x^5 + 4x^3$$
$$= -f(x)$$
$\therefore f(x)$ is odd.

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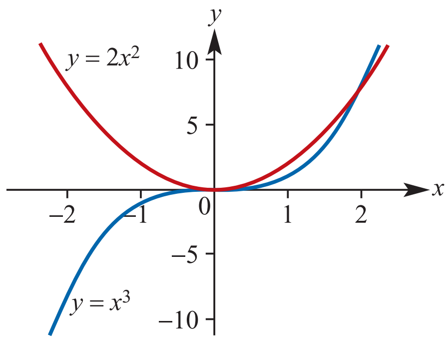
6 a



b
$$\begin{aligned} f(x) &= g(x) \\ x^3 &= 2x^2 \\ x^3 - 2x^2 &= 0 \\ x^2(x-2) &= 0 \\ x = 0 \text{ or } x &= 2 \end{aligned}$$

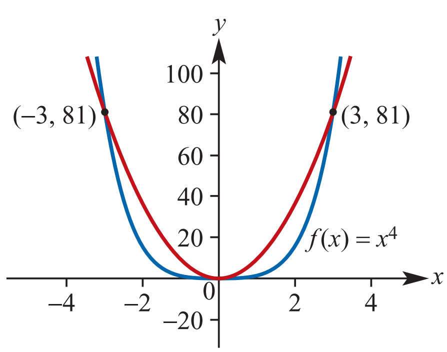
c $f(x) \le g(x)$

$\Leftrightarrow x^3 \le 2x^2$

$\Leftrightarrow x^2(x-2) \le 0$

$\Leftrightarrow x \in (-\infty, 2]$

7 a



b
$$\begin{aligned} f(x) &= g(x) \\ x^4 &= 9x^2 \\ x^4 - 9x^2 &= 0 \\ x^2(x^2 - 9) &= 0 \\ x^2(x-3)(x+3) &= 0 \\ x = -3 \text{ or } x = 3 &\text{ or } x = 0 \end{aligned}$$

c $f(x) \le g(x)$

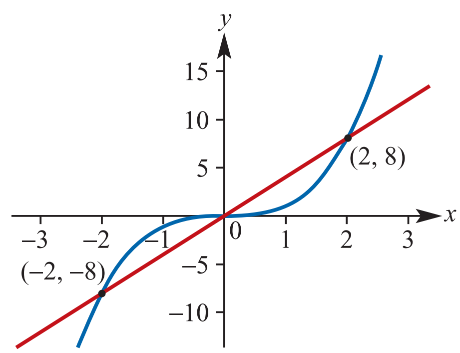
$\Leftrightarrow x^4 \le 9x^2$

$\Leftrightarrow x^2(x^2 - 9) \le 0$

$\Leftrightarrow x^2(x-3)(x+3) \le 0$

$\Leftrightarrow x \in [-3, 3]$

8 a



b
$$\begin{aligned} f(x) &= g(x) \\ x^3 &= 4x \\ x^3 - 4x &= 0 \\ x(x^2 - 4) &= 0 \\ x(x-2)(x+2) &= 0 \\ x = -2 \text{ or } x = 2 &\text{ or } x = 0 \end{aligned}$$

c $f(x) \le g(x)$

$\Leftrightarrow x^3 \le 4x$

$\Leftrightarrow x(x^2 - 4) \le 0$

$\Leftrightarrow x \in (-\infty, -2] \cup [0, 2]$

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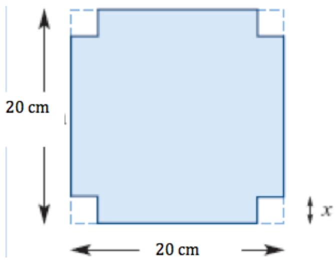
Solutions to Exercise 6J

1 

a $20 - 2x$

b $V = x(20 - 2x)^2$

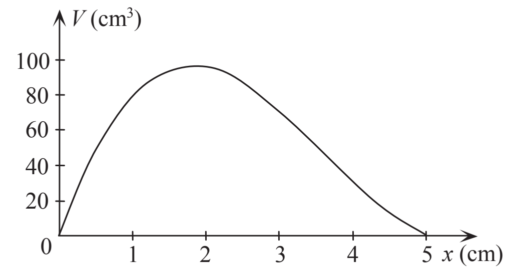
c When $x = 5$,
$V = 5(20 - 2 \times 5)^2 = 500$

d
$$\begin{aligned} x(20-2x)^2 &= 500 \\ 4x(100 - 20x + x^2) &= 500 \\ 100x - 20x^2 + x^3 &= 125 \end{aligned}$$
$x^3 - 20x^2 + 100x - 125 = 0$
We know that $x - 5$ is a factor
Hence
$(x-5)(x^2 - 15x + 25) = 0$
$$(x-5)\left(x^2 - 15x + \left(\frac{15}{2}\right)^2 - \left(\frac{15}{2}\right)^2 + 25\right) = 0$$
$$(x-5)\left(\left(x - \frac{15}{2}\right)^2 - \frac{125}{4}\right) = 0$$
$$(x-5)\left(x - \frac{15}{2} - \frac{5\sqrt{5}}{2}\right)\left(x - \frac{15}{2} + \frac{5\sqrt{5}}{2}\right) = 0$$
The required other value is
$$x = \frac{15 - 5\sqrt{5}}{2}$$

2 a $l = 12 - 2x \qquad w = 10 - 2x$

b
$$\begin{aligned} V &= \text{length} \times \text{width} \times \text{height} \\ &= (12 - 2x)(10 - 2x)x \\ &= 4x(6 - x)(5 - x) \end{aligned}$$

c



d When $x = 1$,
$$\begin{aligned} V &= (12 - 2(1))(10 - 2(1))(1) \\ &= 10 \times 8 \\ &= 80 \end{aligned}$$
Or select **1:Value** from the **F5 Math** menu of the CAS calculator.

e On the CAS calculator, sketch the graphs of $\mathbf{Y_1 = 4X(6 - X)(5 - X)}$ and $\mathbf{Y_2 = 50}$.
The points of intersection are (0.50634849, 50) and (3.5608171, 50).
Therefore $V = 50$ when $x = 0.51$ or $x = 3.56$, correct to 2 decimal places.

f With $\boldsymbol{f1 = 4x \times (6 - x)(5 - x)}$
TI: Press **Menu → 6: Analyze Graph → 3: Maximum**
CP: Tap **Analysis → G-Solve → Max**
to yield (1.810745, 96.770576).
Therefore the maximum volume is 96.77 cm^3 and occurs when $x = 1.81$, correct to 2 decimal places.
Alternatively, type
TI:
fMax(4x × (6 − x)(5 − x), x, 0, 5)
CP:
fMax(4x × (6 − x)(5 − x), x, 0, 5)
to give the maximum when

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$x = \dfrac{11 - \sqrt{31}}{3} \approx 1.81$; then maximum volume is 96.77 cm^3.

3 **a** Surface area $x^2 + 4xh$

b $x^2 + 4xh = 75$

$\therefore h = \dfrac{75 - x^2}{4x}$

c $V = x^2h = \dfrac{x(75 - x^2)}{4}$

d **i** When $x = 2$, $V = \dfrac{71}{2}$

ii When $x = 5$, $V = \dfrac{125}{2}$

iii When $x = 8$, $V = 22$

e It is given that $x = 4$ is a solution of the equation:

$\dfrac{x(75 - x^2)}{4} = 59$

Rearranging we have:

$$\begin{aligned} x(75 - x^2) &= 236 \\ x^3 - 75x + 236 &= 0 \\ (x - 4)(x^2 + 4x - 59) &= 0 \\ (x - 4)(x^2 + 4x + 4 - 63) &= 0 \\ (x - 4)((x + 2)^2 - 63) &= 0 \\ (x - 4)(x + 2 - 3\sqrt{7})(x + 2 + 3\sqrt{7}) &= 0 \end{aligned}$$

The required solution is $x = 3\sqrt{7} - 2$

4 The base is a right-angled triangle $(5x, 12x, 13x)$

a The sum of all the lengths of the prisms edges is180 cm

$\therefore 2(5x + 12x + 13x) + 3h = 180$

$\therefore 60x + 3h = 180$

$\therefore h = \dfrac{180 - 60x}{3} = 60 - 20x$

b The area of the base is $30x^2$.

$\therefore V = 30x^2(60 - 20x) = 600x^2(3 - x)$

c When $x = 3$, $V = 0$

d

$$\begin{aligned} 600x^2(3 - x) &= 1200 \\ x^2(3 - x) &= 2 \\ 3x^2 - x^3 &= 2 \\ x^3 - 3x^2 + 2 &= 0 \\ (x - 1)(x^2 - 2x - 2) &= 0 \\ (x - 1)(x^2 - 2x + 1 - 3) &= 0 \\ (x - 1)((x - 1)^2 - 3) &= 0 \\ (x - 1)((x - 1 - \sqrt{3})(x - 1 + \sqrt{3}) &= 0 \end{aligned}$$

Required solutions $x = 1 + \sqrt{3}$ and $x = 1$

5 **a** Using Pythagoras' theorem,

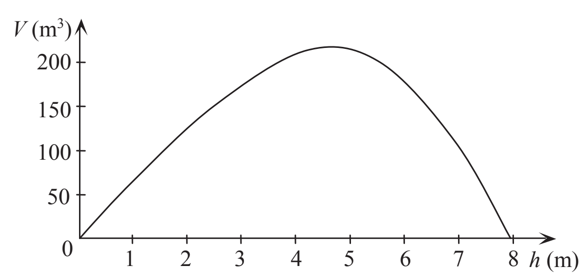
$x^2 + h^2 = 8^2$

$x = \sqrt{64 - h^2}$

b $V = \dfrac{1}{3}\pi x^2 h$

$= \dfrac{1}{3}\pi(64 - h^2)h$

c



d Domain = $\{h : 0 < h < 8\}$

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e When $h = 4$, $V = \frac{1}{3}\pi(64 - 4^2)(4)$

$= 64\pi$

Alternatively, choose **1:Value** from the **F5 Math** menu of the CAS calculator to yield

$V = 201.06193$

$\approx 64\pi$

f On the CAS calculator, sketch the graphs of $\boldsymbol{f1 = 1/3\pi(64 - x^2) \times x}$ and $\boldsymbol{f2 = 150}$. The points of intersection are (2.4750081, 150) and (6.4700086, 0.150).
Therefore $V = 150$ when $h = 2.48$ or $h = 6.47$, correct to 2 decimal places.

g With $\boldsymbol{f1 = 1/3\pi(64 - x\wedge 2) \times x}$,
TI: Press **Menu →**
6:Analyze Graph → 3:Maximum
CP: Tap **Analysis → G – Solve →**
Max
to yield (4.6187997, 206.37006).
Therefore the maximum volume is 206.37 m^3 and occurs when $h = 4.62$, correct to 2 decimal places.
Alternatively, use
fMax(1/3π(64 − x^2) × x, x, 0, 8)
to give the maximum when
$h = \frac{8\sqrt{3}}{3} \approx 4.62$; then maximum volume is 206.37m.

6 a $x + x + h = 160$

$2x + h = 160$

$h = 160 - 2x$

b

$V = x \times x \times h$

$= x^2(160 - 2x)$

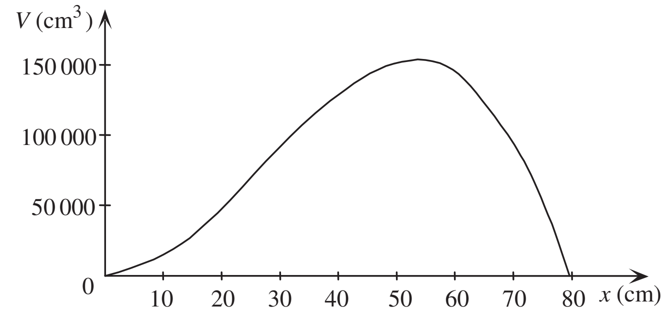
When $V = 0$, $x^2 = 0$ or $160 - 2x = 0$

$\therefore$ $x = 0$ or $160 = 2x$

$\therefore$ $x = 80$

c $\therefore$ Domain $V = \{x : 0 < x < 80\}$

d



e On the CAS calculator, sketch the graphs of $\boldsymbol{f1 = x\wedge 2(160 - 2x)}$ and $\boldsymbol{f2 = 50\,000}$. The points of intersection are (20.497586, 50000) and (75.629199, 50 000).
Therefore $V = 50\,000$ when $x = 20.50$ or $x = 75.63$, correct to 2 decimal places.

f With $\boldsymbol{f1 = x\wedge 2(160 - 2x)}$,
TI: Press **Menu →**
6:Analyze Graph → 3:Maximum
CP: Tap **Analysis → G – Solve →**
Max to yield (53.333336, 151703.7).
Therefore the maximum volume is $151\,704 \text{ cm}^3$ (to the nearest cm^3).
Alternatively, use
fMax(x^2(160 − 2x), x, 0, 80)
to give the maximum when
$h = \frac{160}{3} \approx 53\frac{1}{3}$; then maximum volume is 151703.7 cm^3.

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Solutions to Exercise 6K

See The TI Calculator Appendix for how to complete with a spreadsheet application on the calculator.

1 a The formula for the spreadsheet for solving $x^3 - x - 1 = 0$ in the interval $[1, 2]$ is shown.

	A	B	C	D	E
7	Step 1	1	2	=0.5*(B7+C7)	=0.5*(C7+D7)
8	Step 2	=IF((B7^3-B7-1)*(D7^3-D7-1)<=0,B7,D7)	=IF(B8=B7,D7,C7)	=0.5*(B8+C8)	=0.5*(C8+D8)
9	Step 3	=IF((B8^3-B8-1)*(D8^3-D8-1)<=0,B8,D8)	=IF(B9=B8,D8,C8)	=0.5*(B9+C9)	=0.5*(C9+D9)
10	Step 4	=IF((B9^3-B9-1)*(D9^3-D9-1)<=0,B9,D9)	=IF(B10=B9,D9,C9)	=0.5*(B10+C10)	=0.5*(C10+D10)
11	Step 5	=IF((B10^3-B10-1)*(D10^3-D10-1)<=0,B10,D10)	=IF(B11=B10,D10,C10)	=0.5*(B11+C11)	=0.5*(C11+D11)
12	Step 6	=IF((B11^3-B11-1)*(D11^3-D11-1)<=0,B11,D11)	=IF(B12=B11,D11,C11)	=0.5*(B12+C12)	=0.5*(C12+D12)
13	Step 7	=IF((B12^3-B12-1)*(D12^3-D12-1)<=0,B12,D12)	=IF(B13=B12,D12,C12)	=0.5*(B13+C13)	=0.5*(C13+D13)
14	Step 8	=IF((B13^3-B13-1)*(D13^3-D13-1)<=0,B13,D13)	=IF(B14=B13,D13,C13)	=0.5*(B14+C14)	=0.5*(C14+D14)
15	Step 9	=IF((B14^3-B14-1)*(D14^3-D14-1)<=0,B14,D14)	=IF(B15=B14,D14,C14)	=0.5*(B15+C15)	=0.5*(C15+D15)
16	Step 10	=IF((B15^3-B15-1)*(D15^3-D15-1)<=0,B15,D15)	=IF(B16=B15,D15,C15)	=0.5*(B16+C16)	=0.5*(C16+D16)

The first 10 steps are shown here.

	A	B	C	D	E
7	Step 1	1	2	1.5	1.75
8	Step 2	1	1.5	1.25	1.375
9	Step 3	1.25	1.5	1.375	1.4375
10	Step 4	1.25	1.375	1.3125	1.3438
11	Step 5	1.3125	1.375	1.3438	1.3594
12	Step 6	1.3125	1.34375	1.3281	1.3359
13	Step 7	1.3125	1.328125	1.3203	1.3242
14	Step 8	1.3203125	1.328125	1.3242	1.3262
15	Step 9	1.32421875	1.328125	1.3262	1.3271
16	Step 10	1.32421875	1.3261719	1.3252	1.3257

Answer: 1.32

We go through the first few steps for this question We now return to the function $f(x) = x^3 - x - 1$ and finding the solution of the equation $x^3 - x - 1 = 0$.

Step 1 We start with the interval $[1, 2]$, since we know the solution lies in this interval.
$f(1) = -1 < 0$ and $f(2) = 5 > 0$.
$\dfrac{1+2}{2} = 1.5$.
Since $f(1.5) = 0.875 > 0$, we now know the solution is between 1 and 1.5.

Step 2 Choose 1.5 as the new left endpoint. Therefore the second interval is $[1, 1.5]$.
$\dfrac{1+1.5}{2} = 1.25$ and $f(1.25) = -0.296875 > 0$.

Step 3 Choose 1.25 as the new left endpoint. Thus the third interval is $[1.25, 1.5]$.
Now $\dfrac{1.25 + 1.5)}{2} = 1.375$ and $f(1.375) = 0.224069 < 0$.

Step 4 Choose 1.375 as the new left endpoint. Thus the fourth interval is $[1.25, 1.375]$.

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At this point we know that the solution is in the interval [1.25, 1.375].

b

	A	B	C	D	E
7	Step 1	1	3	2	2.5
8	Step 2	1	2	1.5	1.75
9	Step 3	1	1.5	1.25	1.375
10	Step 4	1	1.25	1.125	1.1875
11	Step 5	1.125	1.25	1.1875	1.2188
12	Step 6	1.125	1.1875	1.1563	1.1719
13	Step 7	1.15625	1.1875	1.1719	1.1797
14	Step 8	1.15625	1.171875	1.1641	1.168
15	Step 9	1.15625	1.1640625	1.1602	1.1621
16	Step 10	1.16015625	1.1640625	1.1621	1.1631
17	Step 11	1.162109375	1.1640625	1.1631	1.1636
18	Step 12	1.163085938	1.1640625	1.1636	1.1638
19	Step 13	1.163574219	1.1640625	1.1638	1.1639
20	Step 14	1.163818359	1.1640625	1.1639	1.164
21	Step 15	1.16394043	1.1640625	1.164	1.164
22	Step 16	1.164001465	1.1640625	1.164	1.164
23	Step 17	1.164031982	1.1640625	1.164	1.1641

Answer: 1.164

c There are two solutions in the interval [1, 2]. Care must be taken. First apply the bisection method in [1, 1.3] and then in [1.3, 2]

	A	B	C	D	E
7	Step 1	1	1.3	1.15	1.225
8	Step 2	1	1.15	1.075	1.1125
9	Step 3	1.075	1.15	1.1125	1.1313
10	Step 4	1.1125	1.15	1.1313	1.1406
11	Step 5	1.1125	1.13125	1.1219	1.1266
12	Step 6	1.121875	1.13125	1.1266	1.1289
13	Step 7	1.121875	1.1265625	1.1242	1.1254
14	Step 8	1.121875	1.1242188	1.123	1.1236
15	Step 9	1.123046875	1.1242188	1.1236	1.1239
16	Step 10	1.123632813	1.1242188	1.1239	1.1241
17	Step 11	1.123925781	1.1242188	1.1241	1.1241
18	Step 12	1.123925781	1.1240723	1.124	1.124
19	Step 13	1.123999023	1.1240723	1.124	1.1241
20	Step 14	1.123999023	1.1240356	1.124	1.124
21	Step 15	1.124017334	1.1240356	1.124	1.124
22	Step 16	1.124026489	1.1240356	1.124	1.124
23	Step 17	1.124026489	1.1240311	1.124	1.124

	A	B	C	D	E
7	Step 1	1.3	2	1.65	1.825
8	Step 2	1.3	1.65	1.475	1.5625
9	Step 3	1.3	1.475	1.3875	1.4313
10	Step 4	1.3875	1.475	1.4313	1.4531
11	Step 5	1.43125	1.475	1.4531	1.4641
12	Step 6	1.43125	1.453125	1.4422	1.4477
13	Step 7	1.4421875	1.453125	1.4477	1.4504
14	Step 8	1.44765625	1.453125	1.4504	1.4518
15	Step 9	1.450390625	1.453125	1.4518	1.4524
16	Step 10	1.450390625	1.4517578	1.4511	1.4514
17	Step 11	1.450390625	1.4510742	1.4507	1.4509
18	Step 12	1.450732422	1.4510742	1.4509	1.451
19	Step 13	1.45090332	1.4510742	1.451	1.451
20	Step 14	1.45098877	1.4510742	1.451	1.4511
21	Step 15	1.451031494	1.4510742	1.4511	1.4511
22	Step 16	1.451031494	1.4510529	1.451	1.451
23	Step 17	1.451042175	1.4510529	1.451	1.4511

Answers: 1.124 and 1.451

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d

	A	B	C	D	E
7	Step 1	2	3	2.5	2.75
8	Step 2	2	2.5	2.25	2.375
9	Step 3	2	2.25	2.125	2.1875
10	Step 4	2.125	2.25	2.1875	2.2188
11	Step 5	2.125	2.1875	2.1563	2.1719
12	Step 6	2.125	2.15625	2.1406	2.1484
13	Step 7	2.140625	2.15625	2.1484	2.1523
14	Step 8	2.1484375	2.15625	2.1523	2.1543
15	Step 9	2.1484375	2.1523438	2.1504	2.1514
16	Step 10	2.150390625	2.1523438	2.1514	2.1519
17	Step 11	2.150390625	2.1513672	2.1509	2.1511
18	Step 12	2.150878906	2.1513672	2.1511	2.1512
19	Step 13	2.150878906	2.151123	2.151	2.1511
20	Step 14	2.150878906	2.151001	2.1509	2.151
21	Step 15	2.150878906	2.1509399	2.1509	2.1509
22	Step 16	2.150909424	2.1509399	2.1509	2.1509

Answer: 2.151

e

	A	B	C	D	E
7	Step 1	-2	-1	-1.5	-1.25
8	Step 2	-2	-1.5	-1.75	-1.625
9	Step 3	-1.75	-1.5	-1.625	-1.5625
10	Step 4	-1.75	-1.625	-1.6875	-1.6563
11	Step 5	-1.75	-1.6875	-1.7188	-1.7031
12	Step 6	-1.75	-1.71875	-1.7344	-1.7266
13	Step 7	-1.75	-1.734375	-1.7422	-1.7383
14	Step 8	-1.75	-1.7421875	-1.7461	-1.7441
15	Step 9	-1.75	-1.7460938	-1.748	-1.7471
16	Step 10	-1.748046875	-1.7460938	-1.7471	-1.7466
17	Step 11	-1.748046875	-1.7470703	-1.7476	-1.7473
18	Step 12	-1.748046875	-1.7475586	-1.7478	-1.7477
19	Step 13	-1.747802734	-1.7475586	-1.7477	-1.7476
20	Step 14	-1.747680664	-1.7475586	-1.7476	-1.7476
21	Step 15	-1.747680664	-1.7476196	-1.7477	-1.7476
22	Step 16	-1.747680664	-1.7476501	-1.7477	-1.7477

Answer -1.75

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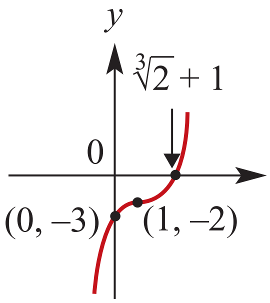
Solutions to Technology-free questions

1 **a** $y = (x-1)^3 - 2$

Stationary point of inflection at $(1, -2)$

x-intercept at $(1 + \sqrt[3]{2}, 0)$

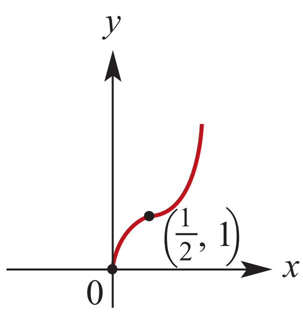
y-intercept at $(0, -3)$



b $y = (2x-1)^3 + 1$

Stationary point of inflection at $(\frac{1}{2}, 1)$

Axis intercept at $(0, 0)$

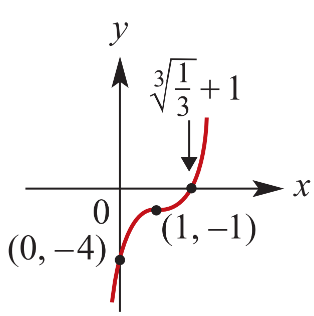


c $y = 3(x-1)^3 - 1$

Stationary point of inflection at $(1, -1)$

x-intercept at $(1 + \sqrt[3]{\frac{1}{3}}, 0)$

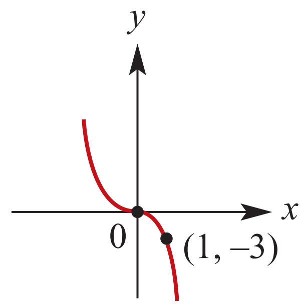
y-intercept at $(0, -4)$



d $y = -3x^3$

Stationary point of inflection at $(0, 0)$

Axis intercept at $(0, 0)$

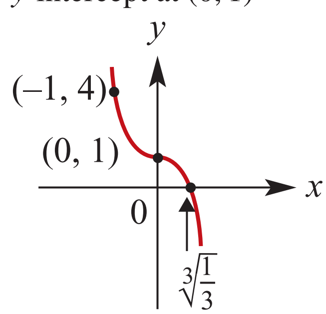


e $y = -3x^3 + 1$

Stationary point of inflection at $(0, 1)$

x-intercept at $(\sqrt[3]{\frac{1}{3}}, 0)$

y-intercept at $(0, 1)$

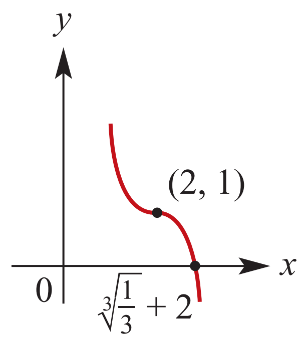


f $y = -3(x-2)^3 + 1$

Stationary point of inflection at $(2,1)$

x-intercept at $(2 + \sqrt[3]{\frac{1}{3}}, 0)$

y-intercept at $(0, 25)$



g $y = 4(x+2)^3 - 3$

Stationary point of inflection at $(-2, -3)$

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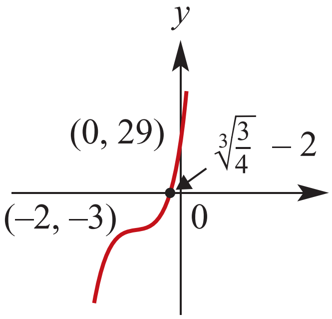
x-intercept at $(-2 + \sqrt[3]{\frac{3}{4}}, 0)$

y-intercept at $(0, 29)$

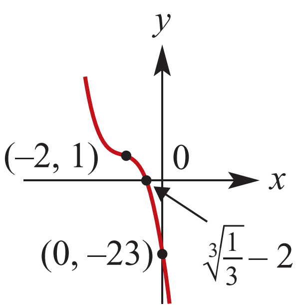


h $y = 1 - 3(x + 2)^3$

Stationary point of inflection at $(-2, 1)$

x-intercept at $(-2 + \sqrt[3]{\frac{1}{3}}, 0)$

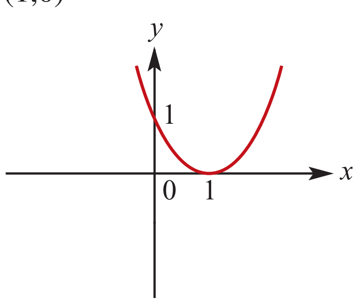
y-intercept at $(0, -23)$



2 a $y = (x - 1)^4$

Turning point at $(1, 0)$

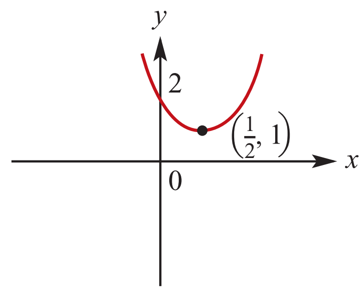
y-intercept at $(0,1)$, x-intercept at $(1,0)$



b $y = (2x - 1)^4 + 1$

Turning point at $(\frac{1}{2}, 1)$

y-intercept at $(0,2)$, no x-intercept

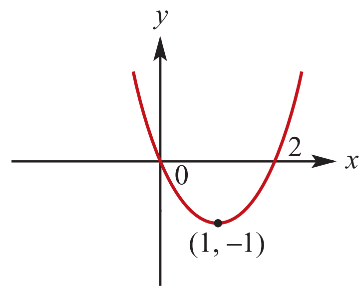


c $y = (x - 1)^4 - 1$

Turning point at $(1, -1)$

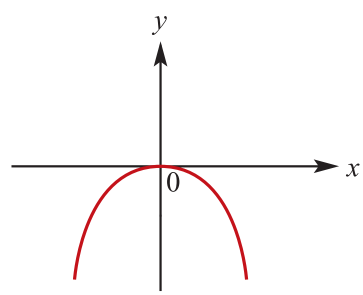
y-intercept at $(0, 0)$,

x-intercept at $(0, 0)$ and $(2, 0)$



d $y = -2x^4$

Turning point and axis intercept at $(0, 0)$



e $y = -3x^4 + 1$

Turning point at $(0, 1)$

x-intercepts at $(\pm\sqrt[4]{\frac{1}{3}}, 0)$

y-intercept at $(0, 1)$

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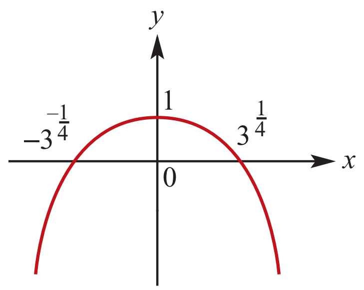
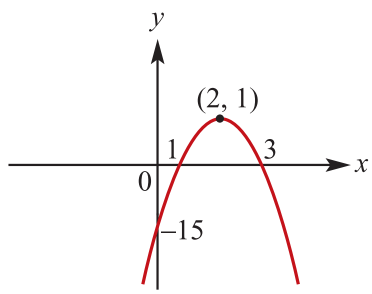
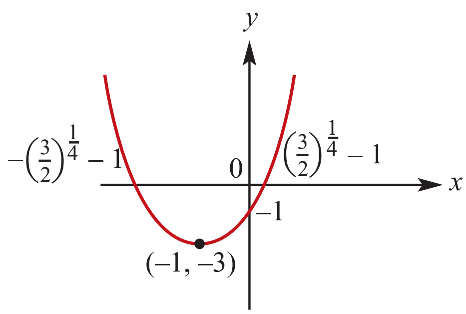


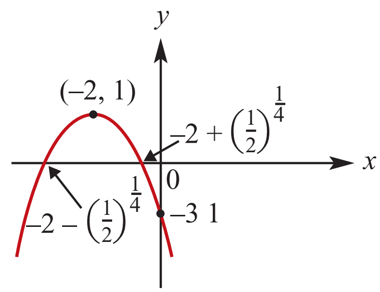
f $y = -(x-2)^4 + 1$
Turning point at $(2, 1)$
y-intercept at $(0, -15)$
x-intercept $(1, 0)$ and $(3, 0)$



g $y = 2(x+1)^4 - 3$
Turning point at $(-1, -3)$
x-intercepts at $\left(-1 \pm \sqrt[4]{\frac{3}{2}}, 0\right)$
y-intercept at $(0, 1)$



h $y = 1 - 2(x+2)^4$
Turning point at $(-2, 1)$
x-intercepts at $(-2 \pm \sqrt[4]{\frac{1}{2}}, 0)$
y-intercept at $(0, -31)$



3 a
$$2x^3 + 3x^2 = 11x + 6$$
$$2x^3 + 3x^2 - 11x - 6 = 0$$
$$(2x+1)(x^2 + x - 6) = 0$$
$$(2x+1)(x+3)(x-2) = 0$$
$$x = -\frac{1}{2} \text{ or } x = -3 \text{ or } x = 2$$

b
$$x^2(5 - 2x) = 4$$
$$5x^2 - 2x^3 - 4 = 0$$
$$2x^3 - 5x^2 + 4 = 0$$
$$(x-2)(2x^2 - x - 2) = 0$$
$$x = 2 \text{ or } 2x^2 - x - 2 = 0$$
$$x = 2 \text{ or } x = \frac{1 \pm \sqrt{17}}{4}$$

c
$$x^3 - 7x^2 + 4x + 12 = 0$$
$$(x-6)(x^2 - x - 2) = 0$$
$$(x-6)(x-2)(x+1) = 0$$
$$x = 6 \text{ or } x = 2 \text{ or } x = -1$$

4 a $P(x) = 6x^3 + 5x^2 - 17x - 6$
$P(-2) = 6(-8) + 5(4) - 17(-2) - 6 = 0$
So $x + 2$ is a factor of $P(x)$.
$P\left(\frac{3}{2}\right) = 6\left(\frac{27}{8}\right) + 5\left(\frac{9}{4}\right) - 17\left(\frac{3}{2}\right) - 6 = 0$
So $2x - 3$ is a factor of $P(x)$.
$\therefore\ P(x) = (x+2)(2x-3)(ax+b)$
$= (ax+b)(2x^2 + x - 6)$
Matching coefficients with $P(x)$:

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$2a = 6, \therefore\ a = 3$
$-6b = -6, \therefore\ b = 1$
So the other factor is $3x + 1$.

b $P(x) = 2x^3 - 3x^2 - 11x + 6 = 0$
$P(-2) = 0$, so $(x + 2)$ is a factor.
$P(3) = 0$, so $(x - 3)$ is a factor.
$$\begin{aligned} P(x) &= (ax + b)(x + 2)(x - 3) \\ &= (ax + b)(x^2 - x - 6) \end{aligned}$$
Matching coefficients with $P(x)$:
$a = 2$
$-6b = 6, \ \therefore\ b = -1$
$\therefore\ P(x) = (2x - 1)(x + 2)(x - 3)$
$x = -2, \dfrac{1}{2}, 3$

c $x^3 + x^2 - 11x - 3 = 8$
$\therefore\ P(x) = x^3 + x^2 - 11x - 11 = 0$
$P(-1) = 0$, so $(x + 1)$ is a factor.
$$\begin{aligned} \therefore\ P(x) &= x^2(x + 1) - 11(x + 1) = 0 \\ &= (x + 1)(x^2 - 11) = 0 \end{aligned}$$
$x = -1, \pm\sqrt{11}$

d **i** $P(x) = 3x^3 + 2x^2 - 19x + 6$
$P\left(\dfrac{1}{3}\right) = \dfrac{3}{27} + \dfrac{2}{9} - \dfrac{19}{3} + 6 = 0$
so $(3x - 1)$ is a factor.

ii $P(2) = 24 + 8 - 38 + 6 = 0$
so $(x - 2)$ is a factor.
$$\begin{aligned} P(x) &= (ax + b)(x - 2)(3x - 1) \\ &= (ax + b)(3x^2 - 7x + 2) \end{aligned}$$
Matching coefficients:
$a = 1, \ b = 3$
$\therefore P(x) = (x + 3)(x - 2)(3x - 1)$

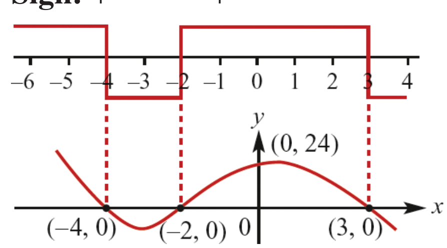
5 **a** $f(x) = x^3 - kx^2 + 2kx - k - 1$
$\therefore\ f(1) = 1 - k + 2k - k - 1 = 0$
By the Factor Theorem, $f(x)$ is divisible by $x - 1$.

b
$$\begin{array}{r@{}l} & x^2 + (1 - k)x + (k + 1) \\ \cline{2-2} x - 1 \big) & x^3 - kx^2 + 2kx - k - 1 \\ & x^3 - x^2 \\ \cline{2-2} & (1 - k)x^2 + 2kx \\ & (1 - k)x^2 - (1 - k)x \\ \cline{2-2} & (k + 1)x - (k + 1) \\ & (k + 1)x - (k + 1) \\ \cline{2-2} & 0 \end{array}$$
$f(x) = (x - 1)(x^2 + (1 - k)x + k + 1)$

6 $P(x) = x^3 + ax^2 - 10x + b$
$P(x)$ is divisible by $Q(x) = x^2 + x - 12$
$Q(x) = (x - 3)(x + 4)$, so
$P(3) = P(-4) = 0$
$P(3) = 27 + 9a - 30 + b = 0$
$\therefore\ 9a + b = 3$
$P(-4) = -64 + 16a + 40 + b = 0$
$\therefore\ 16a + b = 24$
$\therefore\ 7a = 21$
$\therefore\ a = 3; b = -24$

7 Arrange in left-to-right order first.

a $y = (x + 4)(x + 2)(3 - x)$
Inverted cubic.
Axis intercepts: $(-4, 0), (-2, 0), (3, 0)$ and $(0,24)$
Sign: $+ \quad - \quad + \quad -$



b $y = (x + 3)(x - 2)(x - 4)$
Upright cubic.
Axis intercepts: $(-3, 0), (2, 0), (4, 0)$ and $(0, 24)$
Sign: $+ \quad - \quad + \quad -$

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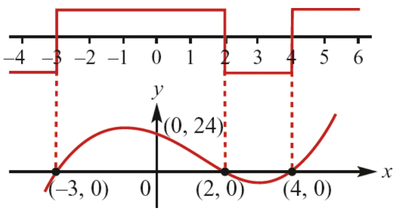


c $y = 6x^3 + 13x^2 - 4$

Upright cubic.

$y(-2) = -48 + 52 - 4 = 0$

So $(x + 2)$ is a factor.

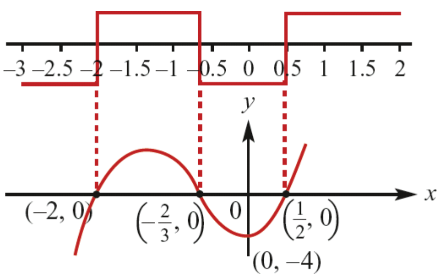
$\therefore\ y = (x + 2)(6x^2 + x - 2)$

$\quad = (x + 2)(3x + 2)(2x - 1)$

Axis intercepts:

$(-2, 0), (-\frac{2}{3}, 0), (\frac{1}{2}, 0)$ and $(0, -4)$

Sign: + – + –



d $y = x^3 + x^2 - 24x + 36$

Upright cubic.

$y(2) = 8 + 4 - 48 + 36 = 0$

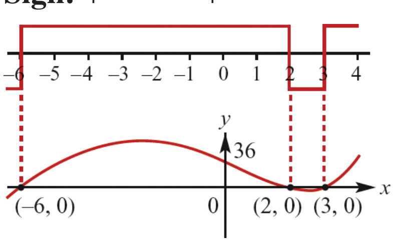
So $(x - 2)$ is a factor.

$\therefore\ y = (x - 2)(x^2 + 3x - 18)$

$\quad = (x + 6)(x - 2)(x - 3)$

Axis intercepts: $(-6, 0), (2, 0), (3, 0)$ and $(0, 36)$

Sign: + – + –



8 **a** $P(x) = x^3 + 4x^2 - 5x + 1.$

Remainder after division by

$(x + 6) = P(-6) = -41$

b $P(x) = 2x^3 - 3x^2 + 2x + 4.$

Remainder after division by

$(x - 2) = P(2) = 12$

c $P(x) = 3x^3 + 2x + 4.$

Remainder after division by

$(3x - 1) = P(\frac{1}{3}) = \frac{43}{9}$

9 $y = a(x + 2)(x - 1)(x - 5)$ accounts for the x intercepts.

At $x = 0, y = a(2)(-1)(-5) = -4$

$\therefore\ a = -\frac{2}{5}$

$y = -\frac{2}{5}(x + 2)(x - 1)(x - 5)$

10 Cubic passes through the origin and touches the x-axis at $(-4, 0)$

$\therefore\ y = ax(x + 4)^2$

Using $(5, 10)$:

$5a(5 + 4)^2 = 10, \therefore\ a = \frac{2}{81}$

$\therefore\ y = \frac{2}{81}x(x + 4)^2$

11 **a** $f(x) = 2x^3 + ax^2 - bx + 3$

$f(1) = 2 + a - b + 3 = 0$

$\therefore\ b - a = 5 \ldots \mathbf{(1)}$

$f(2) = 16 + 4a - 2b + 3 = 15$

$\therefore\ 4a - 2b = -4$

$\therefore\ b - 2a = 2 \ldots \mathbf{(2)}$

(1) – (2) gives $a = 3,\ b = 8$

b $f(x) = 2x^3 + 3x^2 - 8x + 3$

$\quad = (x - 1)(2x^2 + 5x - 3)$

$\quad = (x - 1)(2x - 1)(x + 3)$

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12 **a** $(x-3)^2(x+4) \le 0$

$\Leftrightarrow x+4 \le 0 \text{ or } x = 3$

$\Leftrightarrow x \le -4 \text{ or } x = 3$

b $-(x+3)(x+4)(x-2) \ge 0$

$\Leftrightarrow (x+3)(x+4)(x-2) \le 0$

$\Leftrightarrow x \in (-\infty, -4]) \cup [-3, 2]$

c $x^3 - 4x^2 + x + 6 < 0$

$\Leftrightarrow (x+1)(x^2 - 5x + 6) < 0$

$\Leftrightarrow (x+1)(x-3)(x-2) < 0$

$\Leftrightarrow x \in (-\infty, -1) \cup (2, 3)$

13 $f(x) = x^3$

a Dilation by a factor of 2 from x-axis:

$y = 2x^3$

Translation 1 unit in positive x and 3 units in positive y:

$y = 2(x-1)^3 + 3$

b Reflection in x-axis:

$y = -x^3$

Translation 1 unit in negative x and 2 units in positive y:

$y = -(x+1)^3 + 2$

c Dilation by a factor of $\frac{1}{2}$ from y-axis:

$y = (2x)^3$

Translation $\frac{1}{2}$ unit in negative x and 2 units in negative y:

$y = (2(x + \frac{1}{2}))^3 - 2$

$= (2x+1)^3 - 2$

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Solutions to multiple-choice questions

1 B $P(x) = x^3 + 3x^2 + x - 3$

$\therefore\ P(-2) = (-2)^3 + 3(-2)^2 + (-2) - 3$

$= -1$

2 D $P(x) = (x - a)^2(x - b)(x - c)$,

$a > b > c$

Graph of $y = P(x)$ is an upright quartic with a repeated root at $x = a$, so $P(x) < 0$ only between c and b.

3 A $y = x^3$

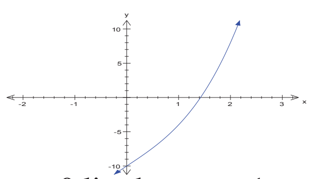
Dilation $\times 2$ from y-axis: $y = \left(\frac{x}{2}\right)^3$

reflection in the y-axis: $y = \left(-\frac{x}{2}\right)^3$

translation of 4 units in negative direction of y-axis:

$y = \left(-\frac{x}{2}\right)^3 - 4 = -\frac{x^3}{8} - 4$

4 D $y = x^3 + 5x - 10$



$y = 0$ lies between 1 and 2

5 A $P(x) = x^4 + ax^2 - 4$

$P(x) = 0$ if $x = -a \pm \sqrt{\frac{a^2}{4} + 4}$

If $P(x) = 0$ when $x = \pm\sqrt{2}$, then $a = 0$

6 C $P(x) = x^3 + ax^2 + bx - 9$

$P(x) = 0$ has zeros at $x = 1$ and $x = -3$.

$\therefore\ P(1) = 1 + a + b - 9 = 0$

$\therefore\ a + b = 8 \ldots \mathbf{(1)}$

$P(-3) = -27 + 9a - 3b - 9 = 0$

$\therefore\ 9a - 3b = 36$

$\therefore\ 3a - b = 12 \ldots \mathbf{(2)}$

(1) + **(2)** gives:

$4a = 20$

$\therefore\ a = 5; b = 3$

7 B $P(x) = ax^3 + 2x^2 + 5$ is divisible by $x + 1$

$\therefore\ P(-1) = -a + 2 + 5 = 0$

$\therefore a = 7$

8 B $P(x) = x^3 + 2x^2 - 5x + d$

$\frac{P(x)}{x-2}$ has a remainder of 10

$\therefore\ P(2) = 10$

$P(2) = 8 + 8 - 10 + d = 10$

$\therefore\ d = 4$

9 D The diagram shows an inverted cubic with a repeated root at $x = b$ and a single root at $x = a$.

$\therefore\ y = -(x - a)(x - b)^2$

10 B The graph of $y = -f(x)$ is a reflection in the x-axis. The graph of $y = 1 - f(x)$ is then a translation up by 1 unit. Only the graph in **B** satisfies these two features.

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Solutions to extended-response questions

1 a $V = \pi r^2 h$

$r + h = 6$

$\therefore V = \pi r^2(6 - r)$

b $0 \le r \le 6$

c $V(3) = 27\pi$

d
$$\begin{aligned} \pi r^2(6 - r) &= 27\pi \\ 6r^2 - r^3 - 27 &= 0 \\ r^3 - 6r^2 + 27 &= 0 \\ (r - 3)(r^2 - 3r - 9) &= 0 \\ \Leftrightarrow r = 3 \text{ or } r &= \frac{3 \pm 3\sqrt{5}}{2} \end{aligned}$$

In the context of the question

$r = 3$ or $r = \dfrac{3 + 3\sqrt{5}}{2}$

2 a At $t = 900$, all the energy is used up.

The point with coordinates $(900, 0)$ is the vertex of the parabola.

Equation of the parabola is $v = a(t - 900)^2 + 0$

$$= a(t - 900)^2$$

When $t = 0$, $v = 25$

$$\begin{aligned} \therefore \quad 25 &= a(0 - 900)^2 \\ \therefore \quad a &= \frac{25}{810\,000} \\ &= \frac{1}{32\,400} \\ \therefore \quad v &= \frac{1}{32\,400}(t - 900)^2 \end{aligned}$$

b $s = vt$ and $\quad v = \dfrac{1}{32\,400}(t - 900)^2$

$$\therefore \quad s = \frac{t}{32\,400}(t - 900)^2$$

(Remember, t is the time in which all energy is used up; v is constant for a given t.)

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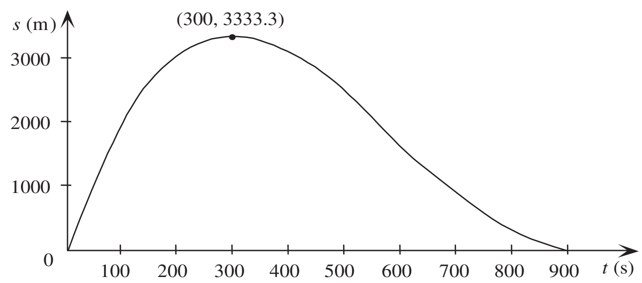
c 

d The maximum distance the t axis can travel is $3\frac{1}{3}$ km, so a proposal to place power sources at 3.5 km intervals is not feasible.

e If the power sources are at 2 km intervals, v_{max} and v min are given for values of t at which $s = 2000$. From the graph, when $s = 2000$, $t1 \approx 105$ and $t2 \approx 560$.

When $t1 = 105$, $v_{\text{max}} \approx \dfrac{2000}{105}$

≈ 19

When $t2 = 560$, $v_{\text{min}} = \dfrac{2000}{560}$

≈ 3.6

Hence, the maximum and minimum speeds recommended for drivers are approximately 19 m/s and 3.6 m/s respectively.

3 a The 'flat spot' is the point of inflexion $\therefore\ (h, k) = (5, 10)$

Hence $R - 10 = a(x - 5)^3$

b At $(0, 0)$, $\quad 0 - 10 = a(0 - 5)^3$

$$\therefore \quad -10 = -125a$$

$$\therefore \quad a = \frac{10}{125} = \frac{2}{25}$$

$$\therefore \quad R - 10 = \frac{2}{25}(x - 5)^3$$

c If $(h, k) = (7, 12)$, then $\quad R - 12 = a(x - 7)^3$

At $(0, 0)$, $\quad 0 - 12 = a(0 - 7)^3$

$$\therefore \quad -12 = -343a \quad \therefore a = \frac{12}{343}$$

$$\therefore \quad R - 12 = \frac{12}{343}(x - 7)^3$$

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4 a Area of net $= \text{length} \times \text{width}$

$$= (l + w + l + w) \times \left(\frac{w}{2} + h + \frac{w}{2}\right)$$
$$= 2(l + w)(w + h)$$
$$= 2(35 + 20)(20 + 23)$$
$$= 2 \times 55 \times 43$$
$$= 4730$$

The area of the net is 4730 cm^2.

b Let V = volume of the box $\therefore V = h \times l \times w$ (1)

Now $2(l + w)(w + h) = 4730$ and $h = l$ (2)

$\therefore \quad 2(l + w)(l + w) = 4730$

$\therefore \quad (l + w)^2 = 2365$

$\therefore \quad l + w = \sqrt{2365}$ (as $l > 0, w > 0$)

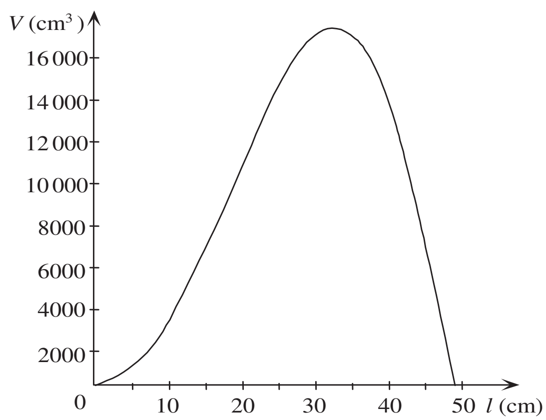
$\therefore \quad w = \sqrt{2365} - l$ (3)

Substitute (2) and (3) in (1)

$$V = l \times l \times (\sqrt{2365} - l)$$

$\therefore \quad V = l^2(\sqrt{2365} - l)$

c



d i On the CAS calculator, sketch **$f1 = x$^2($\sqrt{}$(2365) − x)** and **$f2$ = 14 000**. Points of intersection are (23.694127, 14 000) and (39.787591, 14 000). Therefore the volume is 14 000 cm^3 when $l = 23.69$ or $l = 39.79$.

ii Repeat **d i** using **$f2$ = 10 000**. The points of intersection are (18.096981, 10 000) and (43.296841, 10 000). Therefore the volume is 1 litre when $l = 43.3$ or $l = 18.1$, correct to 1 decimal place..

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e With $\boldsymbol{f1 = x\text{^}2(\sqrt{}(2365) - x)}$,
TI: Press **Menu → 6:Analyze Graph → 3:Maximum**
CP: Tap **Analysis → G-Solve → Max**
to yield (32.420846, 17038.955). The maximum volume is 17039 cm^3 (to the nearest cm^3) and occurs when $l \approx 32.42$.

5 a **TI:** Press **Menu → 1: Actions → 1: Define** then type $\boldsymbol{f(x) = a \times x^3 + b \times x^2 + c \times x + d}$ followed by ENTER.
Now type the following then press ENTER
solve ($\boldsymbol{f(0) = 15.8}$ **and** $\boldsymbol{f(10) = 14.5}$ **and** $\boldsymbol{f(15) = 15.6}$ **and** $\boldsymbol{f(20) = 15, \{a, b, c, d\}}$)
CP: Tap **Action → Command → Define** then type $\boldsymbol{f(x) = a \times x^3 + b \times x^2 + c \times x + d}$ followed by EXE.
Now type the following then press EXE
solve($\boldsymbol{\{f(0) = 15.8, f(10) = 14.5, f(15) = 15.6, f(20) = 15\}, \{a, b, c, d\}}$)

The screen gives $a = -0.00287$, $b = 0.095$, $c = -0.793$ and $d = 15.80$.

b i With $\boldsymbol{f1 - 0.00287x\text{^}3 + 0.095x\text{^}2 - 0.793x + 15.8}$
TI: Press **Menu → 6:Analyze Graph → 2:Minimum**
CP: Tap **Analysis → G-Solve → Min**
to get (5.59, 13.83) as the coordinates of the point closest to the ground.

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ii **TI:** In a Calculator page type $\boldsymbol{f1(0)}$ followed by ENTER

CP: Tap **Analysis → G-Solve → y-Cal** and input 0 for the x-value to get $(0, 15.8)$ as the point furthest from the ground.

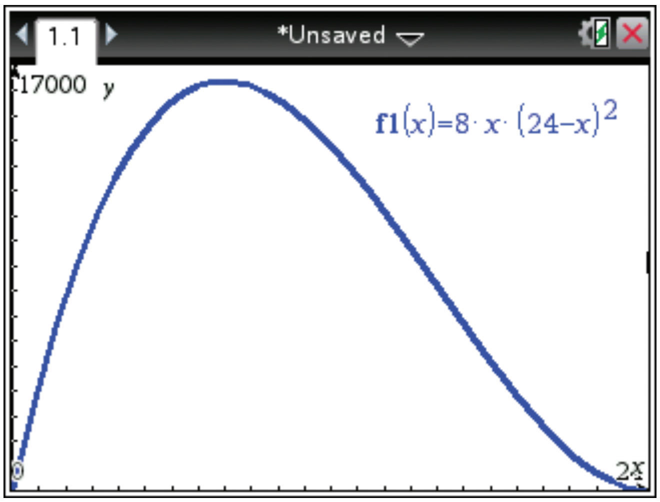
6 **a** The length of the box (in cm) $= 96 - 4x = 4(24 - x)$.

The width of the box (in cm) $= 48 - 2x = 2(24 - x)$.

The height of the box (in cm) $= x$.

Therefore $V = 4(24 - x) \times 2(24 - x) \times x = 8x(24 - x)^2$

b



i The domain of V is $\{x : 0 < x < 24\}$.

ii With $\boldsymbol{f1 = 8x \times (24 - x)\text{^}2}$,

TI: Press **Menu → 6:Analyze Graph → 3:Maximum**

CP: Tap **Analysis → G-Solve → Max** to yield $(8.000002, 16\,384)$.

The maximum volume is 16384 cm^3 (to the nearest cm^3) and occurs when $x \approx 8.00$.

c The volume of the box, when $x = 10$, is $V = 8 \times 10(24 - 10)^2 = 15\,680 \text{ cm}^3$

d The volume is a maximum when $x = 5$. When $x = 5$, $V = 14\,440$.

e The volume is a minimum when $x = 15$. When $x = 15$, $V = 9720$.

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Chapter 7 – Transformations

Solutions to Exercise 7A

1 **a** $(-3, 4) \to (-3+2, 4-3) = (-1, 1)$

b $(-3, 4) \to (-3-2, 4+4) = (-5, 8)$

c $(-3, 4) \to (-3-3, 4-2) = (-6, 2)$

d $(-3, 4) \to (-3-4, 4+5) = (-7, 9)$

e $(-3, 4) \to (-3-2, 4-1) = (-5, 3)$

2 **a** $g(x) = \dfrac{1}{x-2} - 1$

b $g(x) = \dfrac{1}{(x-4)^2} + 3$

c $g(x) = (x+2)^2 - 3$

d $g(x) = (x-4)^2 - 2$

e $g(x) = \sqrt{x-2} - 1$

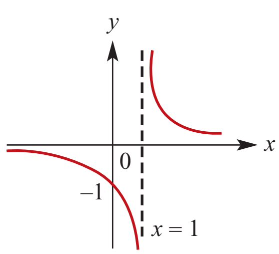
3 $y = f(x) = \dfrac{1}{x}$

a $y = f(x-1) = \dfrac{1}{x-1}$

Asymptotes at $x = 1$ and $y = 0$

y-intercept: $y = \dfrac{1}{0-1} = -1$

No x-intercept because $y = 0$ is an asymptote.



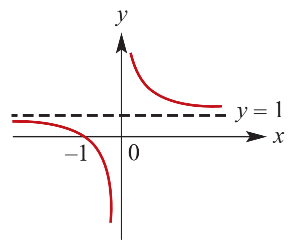
b $y = f(x) + 1 = \dfrac{1}{x} + 1$

Asymptotes at $x = 0$ and $y = 1$

x intercept: $y = \dfrac{1}{x} + 1 = 0$

$\therefore \dfrac{1}{x} = -1, \therefore x = -1$

No y intercept because $y = 0$ is an asymptote

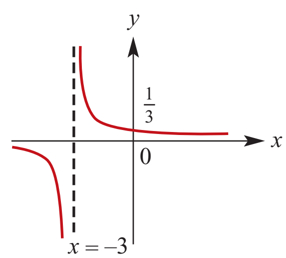


c $y = f(x+3) = \dfrac{1}{x+3}$

Asymptotes at $x = -3$ and $y = 0$

y-intercept: $y = \dfrac{1}{0+3} = \dfrac{1}{3}$

No x-intercept because $y = 0$ is an asymptote.



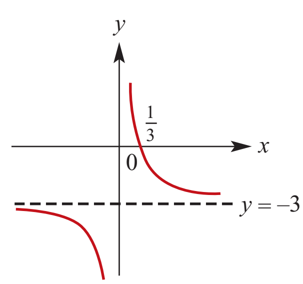
d $y = f(x) - 3 = \dfrac{1}{x} - 3$

Asymptotes at $x = 0$ and $y = -3$

x-intercept: $y = \dfrac{1}{x} - 3 = 0$

$\therefore \dfrac{1}{x} = 3, \therefore x = \dfrac{1}{3}$

No y-intercept because $x = 0$ is an asymptote.



e $y = f(x+1) = \dfrac{1}{x+1}$

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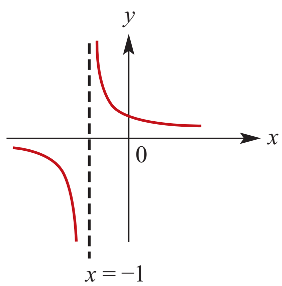
Asymptotes at $x = -1$ and $y = 0$

y-intercept: $y = \dfrac{1}{0+1} = 1$

No x-intercept because $y = 0$ is an asymptote.



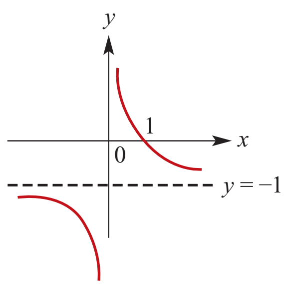
f $y = f(x) - 1 = \dfrac{1}{x} - 1$

Asymptotes at $x = 0$ and $y = -1$

x-intercept: $y = \dfrac{1}{x} - 1 = 0$

$\therefore \dfrac{1}{x} = 1, \therefore x = 1$

No y-intercept because $x = 0$ is an asymptote.



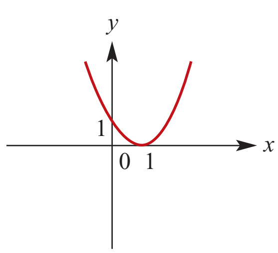
4 $y = f(x) = x^2$

a $y = f(x-1) = (x-1)^2$

x-intercept: $(x-1)^2 = 0, \therefore x = 1$

y-intercept: $f(0-1) = 1$

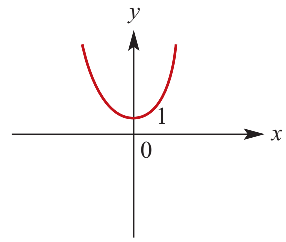
$y = (x-1)^2 = 0, x = 1$



b $y = f(x) + 1 = x^2 + 1$

No x-intercept because $f(x+1) > 0$ for all real x.

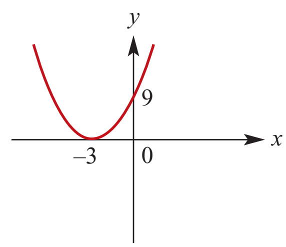
y-intercept: $f(0) + 1 = 1$



c $y = f(x+3) = (x+3)^2$

x-intercept: $(x+3)^2 = 0, \therefore x = -3$

y-intercept: $f(0+3) = 3^2 = 9$



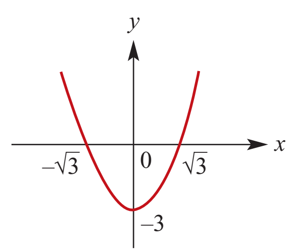
d $y = f(x) - 3 = x^2 - 3$

x-intercepts:

$y = f(x) - 3 = 0, \therefore x^2 - 3 = 0$

$\therefore x^2 = 3, \therefore x = \pm\sqrt{3}$

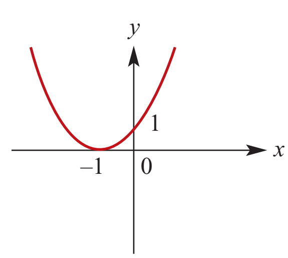
y-intercept: $f(0) - 3 = -3$



e $y = f(x+1) = (x+1)^2$

x-intercept: $(x+1)^2 = 0, \therefore\ x = -1$

y-intercept: $f(0+1) = 1^2 = 1$



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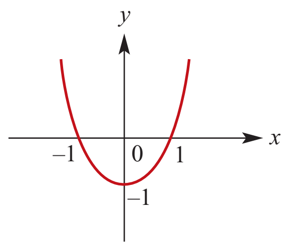
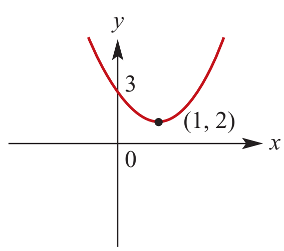
f $y = f(x) - 1 = x^2 - 1$
x-intercepts:
$y = f(x) - 3 = 0, \therefore x^2 - 3 = 0$
$\therefore x^2 = 1, \therefore x = \pm 1$
y-intercept: $f(0) - 1 = -1$

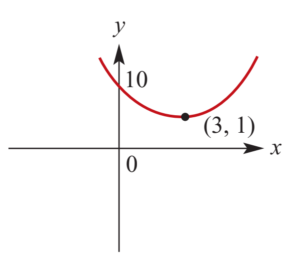


5 $y = f(x) = x^2$

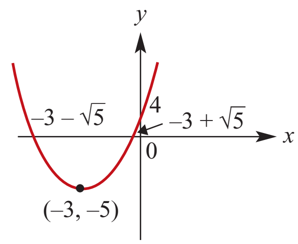
a $y = f(x - 1) + 2 = (x - 1)^2 + 2$
No x-intercepts because
$f(x - 1) + 2 > 0$ for all real x.
y-intercept:
$f(0 - 1) + 1 = (-1)^2 + 2 = 3$



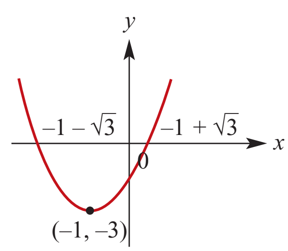
b $y = f(x - 3) + 1 = (x - 3)^2 + 1$
No x-intercepts because
$f(x - 3) + 1 > 0$ for all real x.
y-intercept:
$f(0 - 3) + 1 = (-3)^2 + 1 = 10$



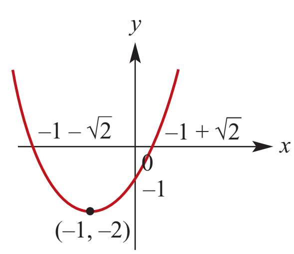
c $y = f(x + 3) - 5 = (x + 3)^2 - 5$
x-intercepts: $y = f(x + 3) - 5 = 0$
$\therefore \ (x + 3)^2 - 5 = 0$
$\therefore \ (x + 3)^2 = 5$
$\therefore \ x + 3 = \pm\sqrt{5}, \ \therefore \ x = -3 \pm \sqrt{5}$
y-intercept: $f(0 + 3) - 5 = 9 - 5 = 4$



d $y = f(x + 1) - 3 = (x + 1)^2 - 3$
x-intercepts: $y = f(x + 1) - 3 = 0$
$\therefore \ (x + 1)^2 - 3 = 0$
$\therefore \ (x + 1)^2 = 3$
$\therefore \ x + 1 = \pm\sqrt{3}, \ \therefore \ x = -1 \pm \sqrt{3}$
y-intercept: $f(0 + 1) - 3 = 1 - 3 = -2$



e $y + 2 = f(x + 1), \therefore \ y = f(x + 1) - 2$
$y = f(x + 1) - 2 = (x + 1)^2 - 2$
x-intercepts: $y = f(x + 1) - 2 = 0$
$\therefore \ (x + 1)^2 - 2 = 0$
$\therefore \ (x + 1)^2 = 2$
$\therefore \ x + 1 = \pm\sqrt{2}, x = -1 \pm \sqrt{2}$
y-intercept: $f(0 + 1) - 2 = 1 - 2 = -1$



f $y = f(x - 5) - 1 = (x - 5)^2 - 1$
x-intercepts: $y = f(x - 5) - 1 = 0$

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$\therefore\ (x-5)^2 - 1 = 0$

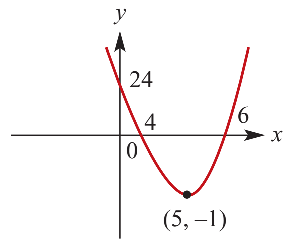
$\therefore\ (x-5)^2 = 1$

$\therefore\ x - 5 = \pm 1$

$\therefore\ x = 5 \pm 1 = 4; 6$

y-intercept:

$f(0-5) - 1 = (-5)^2 - 1 = 24$



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Solutions to Exercise 7B

1 a $(-2,-3) \to (-2,3)$

b $(-2,-3) \to (2,-3)$

c $(-2,-3) \to (-2,-12)$

d $(-2,-3) \to (-8,-3)$

2 a $y = x^2$

i A dilation of factor $\frac{1}{2}$ from the y-axis

$\therefore y = \left(\frac{x}{0.5}\right)^2 = 4x^2$

ii A dilation of factor 5 from the y-axis

$\therefore y = \left(\frac{x}{5}\right)^2 = \frac{x^2}{25}$

iii A dilation of factor $\frac{2}{3}$ from the x-axis

$\therefore y = \frac{2}{3}(x)^2 = \frac{2x^2}{3}$

iv A dilation of factor 4 from the x-axis

$\therefore y = 4(x)^2 = 4x^2$

v A reflection in the x-axis

$\therefore y = -(x)^2 = -x^2$

vi A reflection in the y-axis

$\therefore y = (-x)^2 = x^2$

b $y = \frac{1}{x^2}$

i A dilation of factor $\frac{1}{2}$ from the y-axis

$\therefore y = \left(\frac{0.5}{x}\right)^2 = \frac{1}{4x^2}$

ii A dilation of factor 5 from the y-axis

$\therefore y = \left(\frac{5}{x}\right)^2 = \frac{25}{x^2}$

iii A dilation of factor $\frac{2}{3}$ from the x-axis

$\therefore y = \frac{2}{3}\left(\frac{1}{x^2}\right) = \frac{2}{3x^2}$

iv A dilation of factor 4 from the x-axis

$\therefore y = \frac{4}{x^2}$

v A reflection in the x-axis

$\therefore y = -\frac{1}{x^2}$

vi A reflection in the y-axis

$\therefore y = \left(-\frac{1}{x}\right)^2 = \frac{1}{x^2}$

c $y = \frac{1}{x}$

i A dilation of factor $\frac{1}{2}$ from the y-axis

$\therefore y = \frac{0.5}{x} = \frac{1}{2x}$

ii A dilation of factor 5 from the y-axis

$\therefore y = \frac{5}{x}$

iii A dilation of factor $\frac{2}{3}$ from the x-axis

$\therefore y = \frac{2}{3}\left(\frac{1}{x}\right) = \frac{2}{3x}$

iv A dilation of factor 4 from the

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x-axis

$\therefore y = \frac{4}{x}$

v A reflection in the x-axis

$\therefore y = -\frac{1}{x}$

vi A reflection in the y-axis

$\therefore y = \frac{1}{-x} = -\frac{1}{x}$

d $y = \sqrt{x}$

i A dilation of factor $\frac{1}{2}$ from the y-axis

$\therefore y = \sqrt{\frac{x}{0.5}} = \sqrt{2x}$

ii A dilation of factor 5 from the y-axis

$\therefore y = \sqrt{\frac{x}{5}}$

iii A dilation of factor $\frac{2}{3}$ from the x-axis

$\therefore y = \frac{2}{3}\sqrt{x}$

iv A dilation of factor 4 from the x-axis

$\therefore y = 4\sqrt{x}$

v A reflection in the x-axis

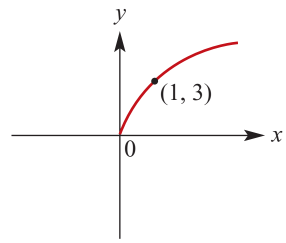
$\therefore y = -\sqrt{x}$

vi A reflection in the y-axis

$\therefore y = \sqrt{-x}; x \le 0$

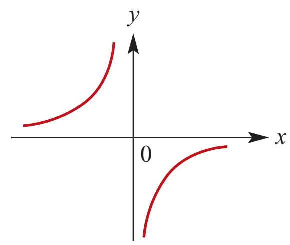
3 a $y = 3\sqrt{x}$

Starting point at (0,0)



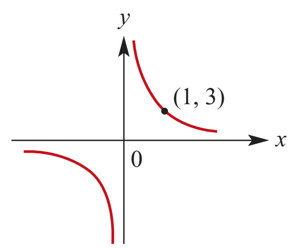
b $y = -\frac{1}{x}$

Asymptotes at $x = 0$ and $y = 0$



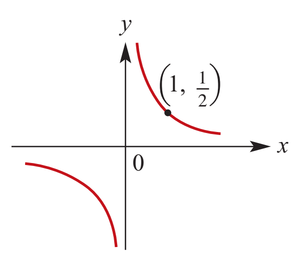
c $y = \frac{3}{x}$

Asymptotes at $x = 0$ and $y = 0$



d $y = \frac{1}{2x}$

Asymptotes at $x = 0$ and $y = 0$



e $y = \sqrt{3x}$

Starting point at $(0, 0)$

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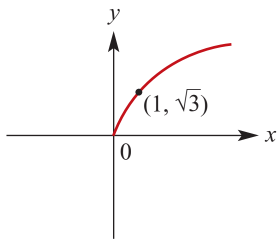
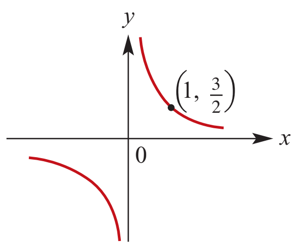


f $y = \dfrac{3}{2x}$
Asymptotes at $x = 0$ and $y = 0$



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Solutions to Exercise 7C

1 **a** Translation of 2 units in the positive direction of the x-axis:
$y = \sqrt{x}$ becomes $y = \sqrt{x-2}$
followed by a dilation of factor 3 from the x-axis: $y = 3\sqrt{x-2}$

b Translation of 3 units in the negative direction of the x-axis:
$y = \sqrt{x}$ becomes $y = \sqrt{x+3}$
followed by a reflection in the x-axis:
$y = -\sqrt{x+3}$

c Reflection in the x-axis:
$y = \sqrt{x}$ becomes $y = -\sqrt{x}$
followed by a dilation of factor 3 from the x-axis: $y = -3\sqrt{x}$

d Reflection in the x-axis: $y = -\sqrt{x}$
followed by a dilation of factor 2 from the y-axis:
$$y = -\sqrt{\frac{x}{2}}$$

e Dilation of factor 2 from the x-axis:
$y = 2\sqrt{x}$
followed by a translation of 2 units in the positive direction of the x-axis:
$y = 2\sqrt{x-2}$
and 3 units in the negative direction of the y-axis: $y = 2\sqrt{x-2} - 3$

f Dilation of factor 2 from the y-axis:
$$y = \sqrt{\frac{x}{2}}$$
followed by a translation of 2 units in the negative direction of the x-axis:
$$y = \sqrt{\frac{x+2}{2}}$$
and 3 units in the negative direction of the y-axis:
$$y = \sqrt{\frac{x+2}{2}} - 3$$

2 $y = \dfrac{1}{x}$

a Translation of 2 units in the positive direction of the x-axis:
$y = \dfrac{1}{x}$ becomes $y = \dfrac{1}{x-2}$
followed by a dilation of factor 3 from the x-axis: $y = \dfrac{3}{x-2}$

b Translation of 3 units in the negative direction of the x-axis: $y = \dfrac{1}{x+3}$
followed by a reflection in the x-axis:
$y = -\dfrac{1}{x+3}$

c Reflection in the x-axis: $y = -\dfrac{1}{x}$
followed by a dilation of factor 3 from the x-axis: $y = -\dfrac{3}{x}$

d Reflection in the x-axis: $y = -\dfrac{1}{x}$
followed by a dilation of factor 2 from the y-axis: $y = -\dfrac{2}{x}$

e Dilation of factor 2 from the x-axis:
$y = \dfrac{2}{x}$
followed by a translation of 2 units in the positive direction of the x-axis:
$y = \dfrac{2}{x-2}$
and 3 units in the negative direction of the y-axis: $y = \dfrac{2}{x-2} - 3$

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f Dilation of factor 2 from the y-axis:
$y = \frac{2}{x}$
followed by a translation of 2 units in the negative direction of the x-axis:
$y = \frac{2}{x+2}$
and 3 units in the negative direction of the y-axis: $y = \frac{2}{x+2} - 3$

3 a $(x, y) \to (x + 2, 3y)$
Let $(x, y) \to (x', y')$.
Then $x' = x + 2$ and $y' = 3y$.
Hence $x = x' - 2$ and $y = \frac{y'}{3}$.
The curve $y = x^{\frac{1}{3}}$ maps to the curve
$\frac{y'}{3} = (x' - 2)^{\frac{1}{3}}$
That is, $y = 3(x - 2)^{\frac{1}{3}}$

b $(x, y) \to (x - 3, -y)$
Let $(x, y) \to (x', y')$.
Then $x' = x - 3$ and $y' = -y$.
Hence $x = x' + 3$ and $y = -y'$.
The curve $y = x^{\frac{1}{3}}$ maps to the curve
$-y' = (x' + 3)^{\frac{1}{3}}$
That is, $y = -(x + 3)^{\frac{1}{3}}$

c $(x, y) \to (-x, 3y)$
Let $(x, y) \to (x', y')$.
Then $x' = -x$ and $y' = 3y$.
Hence $x = -x'$ and $y = \frac{y'}{3}$.
The curve $y = x^{\frac{1}{3}}$ maps to the curve
$\frac{y'}{3} = (-x')^{\frac{1}{3}}$
That is, $y = -3x^{\frac{1}{3}}$

d $(x, y) \to (2x, -y)$
Let $(x, y) \to (x', y')$.
Then $x' = 2x$ and $y' = -y$.
Hence $x = \frac{x'}{2}$ and $y = -y'$.
The curve $y = x^{\frac{1}{3}}$ maps to the curve
$-y = (\frac{x'}{2})^{\frac{1}{3}}$
That is, $y = -\left(\frac{x}{2}\right)^{\frac{1}{3}}$

e $(x, y) \to (x + 2, 2y - 3)$
Let $(x, y) \to (x', y')$.
Then $x' = x + 2$ and $y' = 2y - 3$.
Hence $x = x' - 2$ and $y = \frac{y' + 3}{2}$.
The curve $y = x^{\frac{1}{3}}$ maps to the curve
$\frac{y' + 3}{2} = (x' - 2)^{\frac{1}{3}}$
That is, $y = 2(x - 2)^{\frac{1}{3}} - 3$

f $(x, y) \to (2x - 2, y - 3)$
Let $(x, y) \to (x', y')$.
Then $x' = 2x - 2$ and $y' = y - 3$.
Hence $x = \frac{x' + 2}{2}$ and $y = y' + 3$.
The curve $y = x^{\frac{1}{3}}$ maps to the curve
$y' + 3 = \left(\frac{x' + 2}{2}\right)^{\frac{1}{3}}$
That is, $y = \left(\frac{x + 2}{2}\right)^{\frac{1}{3}} - 3$

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Solutions to Exercise 7D

1 a i Write $y' = 2(x' - 1)^2 + 3$

$\therefore \dfrac{y' - 3}{2} = (x' - 1)^2$

Choose $x = x' - 1$ and $y = \dfrac{y' - 3}{2}$

$\therefore x' = x + 1$ and $y' = 2y + 3$

In summary,

A dilation of factor 2 from the x-axis, then a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

ii Write $y' = -(x' + 1)^2 + 2$

$\therefore -y' + 2 = (x' + 1)^2$

Choose $x = x' + 1$ and $y = -y' + 2$

$\therefore x' = x - 1$ and $y' = -y + 2$

In summary,

A reflection in the x-axis, then a translation of 1 unit in the negative direction of the x-axis and 2 units in the positive direction of the y-axis

iii Write $y' = (2x' + 1)^2 - 2$

$\therefore y' + 2 = (2x' + 1)^2$

Choose $x = 2x' + 1$ and $y = y' + 2$

$\therefore x' = \dfrac{x - 1}{2} = \dfrac{x}{2} - \dfrac{1}{2}$ and

$y' = y - 2$

In summary,

A dilation of factor $\frac{1}{2}$ from the y-axis, then a translation of $\frac{1}{2}$ unit in the negative direction of the x-axis and 2 units in the negative direction of the y-axis

b i Write $y' = \dfrac{2}{x' + 3}$

$\therefore \dfrac{y'}{2} = \dfrac{1}{x' + 3}$

Choose $x = x' + 3$ and $y = \dfrac{y'}{2}$

$\therefore x' = x - 3$ and $y' = 2y$

In summary,

A dilation of factor 2 from the x-axis, then a translation of 3 units in the negative direction of the x-axis

ii Write $y' = \dfrac{1}{x' + 3} + 2$

$\therefore y' - 2 = \dfrac{1}{x' + 3}$

Choose $x = x' + 3$ and $y = y' - 2$

$\therefore x' = x - 3$ and $y' = y + 2$

In summary,

A translation of 3 units in the negative direction of the x-axis and 2 units in the positive direction of the y-axis

iii Write $y' = \dfrac{1}{x' - 3} - 2$

$\therefore y' + 2 = \dfrac{1}{x' - 3}$

Choose $x = x' - 3$ and $y = y' + 2$

$\therefore x' = x + 3$ and $y' = y - 2$

In summary,

A translation of 3 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis

c i Write $y' = \sqrt{x' + 3} + 2$

$\therefore y' - 2 = \sqrt{x' + 3}$

Choose $x = x' + 3$ and $y = y' - 2$

$\therefore x' = x + 3$ and $y' = y - 2$

In summary,

A translation of 3 units in the negative direction of the x-axis and 2 units in the negative direction of the y-axis

d Write $y' = 2\sqrt{3x'}$

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$\therefore \dfrac{y'}{2} = \sqrt{3x'}$

Choose $x = 3x'$ and $y = \dfrac{y'}{2}$

$\therefore x' = \frac{1}{3}x$ and $y' = 2y$

In summary,

A dilation of factor $\frac{1}{3}$ from the y-axis, then a dilation of factor 2 from the x-axis

e Write $y' = -\sqrt{x'} + 2$

$\therefore -y' + 2 = \sqrt{x'}$

Choose $x = x'$ and $y = -y' + 2$

$\therefore x' = x$ and $y' = -y + 2y$

In summary,

A reflection in the x-axis, then a translation of 2 units in the positive direction of the y-axis

2 a Write $y' = \dfrac{1}{x'^2}$ and $\dfrac{y+7}{5} = \dfrac{1}{(x-3)^2}$

Choose $y' = \dfrac{y+7}{5}$ and $x' = x - 3$

$\therefore (x, y) \to \left(x - 3, \dfrac{y+7}{5}\right)$

b Write $y' = (x')^2$ and $y - 5 = (3x + 2)^2$

Choose $y' = y - 5$ and $x' = 3x + 2$

$\therefore (x, y) \to (3x + 2, y - 5)$

c Write $y' = (x')^2$ and

$-y + 7 = 3(3x + 1)^2$

Choose $y' = \dfrac{-y+7}{3}$ and $x' = 3x + 1$

$\therefore (x, y) \to \left(3x + 1, -\dfrac{y-7}{3}\right)$

d Write $y' = \sqrt{x'}$ and $\dfrac{y}{2} = \sqrt{-(x-4)}$

Choose $y' = \dfrac{y}{2}$ and $x' = \sqrt{-(x-4)}$

$\therefore (x, y) \to \left(-(x - 4), \dfrac{y}{2}\right)$

e Write $y' = -\sqrt{x'} + 6$ and

$\dfrac{y-3}{2} = \sqrt{-(x-4)}$

Choose $-y' + 6 = \dfrac{y-3}{2}$ and

$x' = \sqrt{-(x-4)}$

$\therefore (x, y) \to \left(-(x - 4), \dfrac{15-y}{2}\right)$

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Solutions to Exercise 7E

1 $\mathbf{X}+\mathbf{Y}=\begin{bmatrix}1\\-2\end{bmatrix}+\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}4\\-2\end{bmatrix}$

$2\mathbf{X}=\begin{bmatrix}2\\-4\end{bmatrix}$

$4\mathbf{Y}+\mathbf{X}=\begin{bmatrix}12\\0\end{bmatrix}+\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}13\\-2\end{bmatrix}$

$\mathbf{X}-\mathbf{Y}=\begin{bmatrix}1\\-2\end{bmatrix}-\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}-2\\-2\end{bmatrix}$

$-3\mathbf{A}=-3\begin{bmatrix}1&-1\\2&3\end{bmatrix}=\begin{bmatrix}-3&3\\-6&-9\end{bmatrix}$

$-3\mathbf{A}+\mathbf{B}=\begin{bmatrix}-3&3\\-6&-9\end{bmatrix}+\begin{bmatrix}4&0\\-1&2\end{bmatrix}$

$=\begin{bmatrix}1&3\\-7&-7\end{bmatrix}$

2 $2\mathbf{A}=2\begin{bmatrix}1&-1\\0&2\end{bmatrix}=\begin{bmatrix}2&-2\\0&4\end{bmatrix}$

$-3\mathbf{A}=-3\begin{bmatrix}1&-1\\0&2\end{bmatrix}=\begin{bmatrix}-3&3\\0&-6\end{bmatrix}$

$-6\mathbf{A}=-6\begin{bmatrix}1&-1\\0&2\end{bmatrix}=\begin{bmatrix}-6&6\\0&-12\end{bmatrix}$

3 a $2\mathbf{A}=2\begin{bmatrix}3&2\\-2&-2\end{bmatrix}=\begin{bmatrix}6&4\\-4&-4\end{bmatrix}$

b $3\mathbf{B}=3\begin{bmatrix}0&-3\\4&1\end{bmatrix}=\begin{bmatrix}0&-9\\12&3\end{bmatrix}$

c $2\mathbf{A}+3\mathbf{B}=\begin{bmatrix}6&-5\\8&-1\end{bmatrix}$

d $3\mathbf{B}-2\mathbf{A}=\begin{bmatrix}0&-9\\12&3\end{bmatrix}-\begin{bmatrix}6&4\\-4&-4\end{bmatrix}$

$=\begin{bmatrix}-6&-13\\16&7\end{bmatrix}$

4 a $\mathbf{P}+\mathbf{Q}=\begin{bmatrix}1&0\\0&3\end{bmatrix}+\begin{bmatrix}-1&1\\2&0\end{bmatrix}=\begin{bmatrix}0&1\\2&3\end{bmatrix}$

b $\mathbf{P}+3\mathbf{Q}=\begin{bmatrix}1&0\\0&3\end{bmatrix}+\begin{bmatrix}-3&3\\6&0\end{bmatrix}=\begin{bmatrix}-2&3\\6&3\end{bmatrix}$

c $2\mathbf{P}-\mathbf{Q}+\mathbf{R}=\begin{bmatrix}2&0\\0&6\end{bmatrix}-\begin{bmatrix}-1&1\\2&0\end{bmatrix}$

$+\begin{bmatrix}0&4\\1&1\end{bmatrix}$

$=\begin{bmatrix}3&3\\-1&7\end{bmatrix}$

5 $2\mathbf{A}-3\mathbf{X}=\mathbf{B}$

$\therefore 2\mathbf{A}-\mathbf{B}=3\mathbf{X}$

$\therefore \mathbf{X}=\frac{1}{3}(2\mathbf{A}-\mathbf{B})$

$=\frac{1}{3}\left(2\begin{bmatrix}3&1\\-1&4\end{bmatrix}-\begin{bmatrix}0&-10\\-2&17\end{bmatrix}\right)$

$=\frac{1}{3}\begin{bmatrix}6&12\\0&-9\end{bmatrix}$

$=\begin{bmatrix}2&4\\0&-3\end{bmatrix}$

$3\mathbf{A}+2\mathbf{Y}=2\mathbf{B}$

$\therefore 2\mathbf{Y}=2\mathbf{B}-3\mathbf{A}$

$\therefore \mathbf{Y}=\frac{1}{2}(2\mathbf{B}-3\mathbf{A})$

$=\frac{1}{2}\left(2\begin{bmatrix}0&-10\\-2&17\end{bmatrix}-3\begin{bmatrix}3&1\\-1&4\end{bmatrix}\right)$

$=\frac{1}{2}\begin{bmatrix}-9&-23\\-1&22\end{bmatrix}$

$=\begin{bmatrix}-\frac{9}{2}&-\frac{23}{2}\\-\frac{1}{2}&11\end{bmatrix}$

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6 $\mathbf{AX} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

$\mathbf{BX} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

$\mathbf{IX} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\mathbf{AI} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$\mathbf{IB} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$\mathbf{AB} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -3 & 3 \end{bmatrix}$

$\mathbf{BA} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ -1 & 3 \end{bmatrix}$

$\mathbf{A}^2 = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$

$\mathbf{B}^2 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$

7 $\mathbf{AX} = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \end{bmatrix}$

$\mathbf{BX} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$

$\mathbf{CX} = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \end{bmatrix}$

$\mathbf{AC} = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -9 & 1 \end{bmatrix}$

$\mathbf{CB} = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ 6 & 2 \end{bmatrix}$

$\mathbf{AB} = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -6 & -2 \\ 3 & 8 \end{bmatrix}$

$\mathbf{BA} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ -2 & 8 \end{bmatrix}$

$\mathbf{A}^2 = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ -6 & 17 \end{bmatrix}$

$\mathbf{B}^2 = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$

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Solutions to Exercise 7F

1 a $\det \mathbf{A} = 2(2) - 3(1) = 1$

b $\mathbf{A}^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

c $\det \mathbf{B} = (-2)2 - (-2)3 = 2$

d $\mathbf{B}^{-1} = \dfrac{1}{2}\begin{bmatrix} 2 & 2 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{1}{2} \\ -\dfrac{3}{2} & -1 \end{bmatrix}$

2 a $\det\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} = 1$; Inv $= \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$

b $\det\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix} = 14$; Inv $= \begin{bmatrix} \dfrac{2}{7} & -\dfrac{1}{14} \\ \dfrac{1}{7} & \dfrac{3}{14} \end{bmatrix}$

c $\det\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = k$; Inv $= \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{k} \end{bmatrix}$

3 a $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$

$\det\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} = -2$

$\mathbf{A}^{-1} = -\dfrac{1}{2}\begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$

$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$\det\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = 1$

$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

b $\mathbf{AB} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -3 & -1 \end{bmatrix}$

$\det\begin{bmatrix} 5 & 1 \\ -3 & -1 \end{bmatrix} = -2$

Therefore

$(\mathbf{AB})^{-1} = -\dfrac{1}{2}\begin{bmatrix} -1 & -1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$

c $\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$

$= (\mathbf{AB})^{-1}$

4 a $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ so $\det \mathbf{A} = 4(1)$
$-3(2) = -2$

$\mathbf{A}^{-1} = -\dfrac{1}{2}\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{2} & \dfrac{3}{2} \\ 1 & -2 \end{bmatrix}$

b $\mathbf{AX} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} w & x \\ y & Z \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$

$\mathbf{AX} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$

$\mathrm{A}^{-1}\mathrm{AX} = \mathrm{A}^{-1}\begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$

$\mathrm{IX} = \begin{bmatrix} -\dfrac{1}{2} & \dfrac{3}{2} \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$

$\mathrm{X} = \begin{bmatrix} 0 & 7 \\ 1 & -8 \end{bmatrix}$

c $\mathbf{YA} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$

$\mathbf{YAA}^{-1} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}\mathbf{A}^{-1}$

$\mathbf{YI} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} -\dfrac{1}{2} & \dfrac{3}{2} \\ 1 & -2 \end{bmatrix}$

$\mathbf{Y} = \begin{bmatrix} \dfrac{5}{2} & -\dfrac{7}{2} \\ \dfrac{11}{2} & -\dfrac{21}{2} \end{bmatrix}$

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5 **a** $\mathbf{AX} + \mathbf{B} = \mathbf{C}$

$\mathbf{AX} = \mathbf{C} - \mathbf{B}$

$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}(\mathbf{C} - \mathbf{B})$

$\mathbf{IX} = \mathbf{A}^{-1}(\mathbf{C} - \mathbf{B})$

$$\mathbf{A}^{-1} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

$$\mathbf{C} - \mathbf{B} = \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

$$\mathbf{X} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} -6 & 22 \\ 1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{8} & \frac{11}{8} \\ \frac{1}{16} & \frac{7}{16} \end{bmatrix}$$

b $\mathbf{YA} + \mathbf{B} = \mathbf{C}$

$\mathbf{YA} = \mathbf{C} - \mathbf{B}$

$\mathbf{YAA}^{-1} = (\mathbf{C} - \mathbf{B})\mathbf{A}^{-1}$

$\mathbf{YI} = (\mathbf{C} - \mathbf{B})\mathbf{A}^{-1}$

$$\mathbf{A}^{-1} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

$$\mathbf{C} - \mathbf{B} = \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

$$\mathbf{Y} = \frac{1}{16}\begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix} =$$

$$\begin{bmatrix} -\frac{11}{16} & \frac{17}{16} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

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Solutions to Exercise 7G

1 a $\begin{bmatrix} 2 & -3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \end{bmatrix}$

b $\begin{bmatrix} -3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3a - b \\ 3b - a \end{bmatrix}$

2 $\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

$\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$

3 a $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

b $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

c $\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

4 a Dilation of factor 3 from the x-axis

$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 6 \\ 21 \end{bmatrix}$

b Dilation of factor 2 from the y-axis

$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$

c Reflection in the x-axis

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}$

d Reflection in the y-axis

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} -6 \\ 7 \end{bmatrix}$

e Reflection in the line $y = x$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$

5 a $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 18 \\ 5 \end{bmatrix}$

$$\begin{array}{rlrl} \therefore & a - 2b = -4; & & c - 2d = 5 \\ & 3a + 4b = 18; & & 3c + 4d = 5 \\ \hline & 5a \quad = 10; & & 5c \quad = 15 \end{array}$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}$

c $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\therefore a = 1; c = 1; b = 2; d = 2$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

6 Reflection in x-axis = $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Dilation × 2 from x-axis = $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$\mathbf{BA} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

7 a Reflection in the line $x = 0$ $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

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b Reflection in the line $y = x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

c Reflection in the line $y = -x \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

d Dilation of 2 from the x-axis $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

e Dilation of $\frac{1}{2}$ from the origin $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

f Dilation of 3 from the y-axis $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

8 **a** Dilation of factor2 from the x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Reflection in the line $y = -x$: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$$T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$$

b $T(3, 2) = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$

c $\begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$\therefore -2b = 6, \therefore b = -3$

$-a = 2, \therefore a = -2$

9 $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$

$\mathbf{C} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\mathbf{AX} + \mathbf{C} = \mathbf{X}'$, so $\mathbf{X} = \mathbf{A}^{-1}(\mathbf{X}' - \mathbf{C})$

$$\mathbf{X} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x' - 3 \\ y' - 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x' - 3 \\ y' - 4 \end{bmatrix}$$

$\therefore x = \frac{1}{2}(x' - 3); y = y' - 4$

10 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$ and

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -11 \end{bmatrix}$

$a + 3b = -2; c + 3d = -3$

$2a + 4b = -3; 2c + 4d = -11$

These give $a = -\frac{1}{2}, b = -\frac{1}{2}$,

$c = -\frac{21}{2}, d = \frac{5}{2}$

$$T = \frac{1}{2}\begin{bmatrix} -1 & -1 \\ -21 & 5 \end{bmatrix}$$

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Solutions to Exercise 7H

1 $\mathbf{T} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \therefore \mathbf{T}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$

$\mathbf{TX} = \mathbf{X}', \therefore \mathbf{x} = \mathbf{T}^{-1}\mathbf{x}'$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}x' \\ -\frac{1}{2}y' \end{bmatrix}$$

$y = x^2 + x + 2$ maps onto:

$$-\frac{1}{2}y = \left(\frac{x}{3}\right)^2 + \frac{x}{3} + 2$$

$$\therefore \quad y = -\frac{2x^2}{9} - \frac{2x}{3} - 4$$

2 $\mathbf{T} = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \therefore \mathbf{T}^{-1} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$

$\mathbf{TX} = \mathbf{X}', \therefore \mathbf{x} = \mathbf{T}^{-1}\mathbf{x}'$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{4}x' \\ -\frac{1}{2}y' \end{bmatrix}$$

$y = x^3 + 2x$ maps onto:

$$-\frac{1}{2}y = \left(\frac{x}{4}\right)^3 + \frac{2x}{4}$$

$$\therefore \quad y = -\frac{x^3}{32} - x$$

3 $\mathbf{T} = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix} \therefore \mathbf{T}^{-1} = \begin{bmatrix} 0 & -\frac{1}{3} \\ \frac{1}{2} & 0 \end{bmatrix}$

$\mathbf{TX} = \mathbf{X}', \therefore \mathbf{x} = \mathbf{T}^{-1}\mathbf{x}'$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3}y' \\ \frac{1}{3}x' \end{bmatrix}$$

$y = 2x + 3$ maps onto:

$$\frac{1}{2}x = 2\left(-\frac{1}{3}y\right) + 3$$

$$\therefore 3x = -4y + 18$$

$$\therefore \quad y = \frac{-3x + 18}{4}$$

4 $\mathbf{T} = \begin{bmatrix} 0 & 4 \\ -2 & 0 \end{bmatrix} \therefore \mathbf{T}^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{4} & 0 \end{bmatrix}$

$\mathbf{TX} = \mathbf{X}', \therefore \mathbf{x} = \mathbf{T}^{-1}\mathbf{x}'$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2}y' \\ \frac{1}{4}x' \end{bmatrix}$$

$y = -2x + 4$ maps onto:

$$\frac{1}{4}x = y + 4$$

$$\therefore y = \frac{x}{4} - 4$$

5 $\mathbf{T}(\mathbf{X} + \mathbf{B}) = \mathbf{X}'$, where $\mathbf{T} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$

$\therefore \mathbf{T}^{-1} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix} \mathbf{B} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\therefore \mathbf{T}^{-1}\mathbf{T}(\mathbf{X} + \mathbf{B}) = \mathbf{T}^{-1}\mathbf{X}'$,

So $\mathbf{X} = \mathbf{T}^{-1}\mathbf{X}' - \mathbf{B}$

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$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} y' \\ -\frac{1}{2}x' \end{bmatrix}$$

$y = -2x + 6$ maps onto:

$$-\frac{x}{2} - 2 = -2(y + 1) + 6$$

$$\therefore x + 4 = 4y + 4 - 12$$

$$\therefore y = \frac{x}{4} + 3$$

6 $\mathbf{TX} + \mathbf{B} = \mathbf{X'}$, where $\mathbf{T} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$

$$\therefore \mathbf{T}^{-1} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix} \mathbf{B} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$\mathbf{TX} + \mathbf{B} = \mathbf{X'}$, so $\mathbf{X} = \mathbf{T}^{-1}(\mathbf{X'} - \mathbf{B})$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x' + 1 \\ y' - 2 \end{bmatrix}$$

$$= \begin{bmatrix} y' - 2 \\ -\frac{1}{2}x' - \frac{1}{2} \end{bmatrix}$$

$y = -2x + 6$ maps onto:

$$-\frac{x}{2} - \frac{1}{2} = -2(y - 2) + 6$$

$$\therefore x + 1 = 4y - 8 - 12$$

$$\therefore y = \frac{x + 21}{4}$$

7 $\mathbf{TX} + \mathbf{B} = \mathbf{X'}$, where $\mathbf{T} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$$\therefore \mathbf{T}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \boldsymbol{B} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$\mathbf{TX} + \mathbf{B} = \mathbf{X'}$

$\therefore \mathbf{T}^{-1}(\mathbf{TX}) = \mathbf{T}^{-1}(\mathbf{X'} - \mathbf{B})$

$\therefore \mathbf{X} = \mathbf{T}^{-1}(\mathbf{X'} - \mathbf{B})$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x' & + & 2 \\ y' & - & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}x' + 1 \\ \frac{1}{3}(y' - 2) \end{bmatrix}$$

$y = -2x^3 + 6x$ maps onto:

$$\frac{1}{3}(y - 2) = -2\left(\frac{x}{2} + 1\right)^3 + 6\left(\frac{x}{2} + 1\right)$$

$$\therefore y - 2 = -6\left(\frac{x}{2} + 1\right)^3 + 9x + 18$$

$$\therefore y = -\frac{3x^3}{4} - \frac{9x^2}{2} - 9x - 6 + 9x + 20$$

$$= -\frac{3x^3}{4} - \frac{9x^2}{2} + 14$$

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Solutions to Technology-free questions

1 a Dilation of factor 4 from the x-axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 12 \end{bmatrix}$$

b Dilation of factor 3 from the y-axis:

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

c Reflection in the x-axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

d Reflection in the y-axis:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

e Reflection in the line $y = x$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

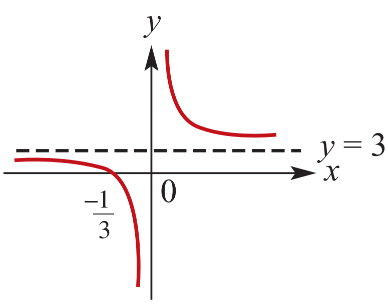
2 a $y = \dfrac{1}{x} + 3$

Asymptotes at $x = 0$ and $y = 3$

x-intercept: $y = \dfrac{1}{x} + 3 = 0$

$\therefore\ \dfrac{1}{x} = -3, \therefore\ x = -\dfrac{1}{3}$

No y-intercept because $x = 0$ is an asymptote.



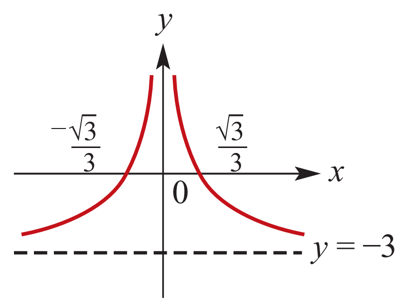
b $y = \dfrac{1}{x^2} - 3$

Asymptotes at $x = 0$ and $y = -3$

x-intercept: $y = \dfrac{1}{x^2} - 3 = 0$

$\therefore\ \dfrac{1}{x^2} = 3, \therefore\ x = \pm\dfrac{1}{\sqrt{3}}$

No y-intercept because $x = 0$ is an asymptote.

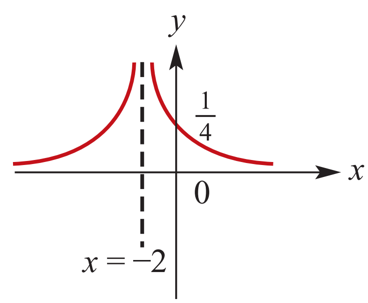


c $y = \dfrac{1}{(x+2)^2}$

Asymptotes at $x = -2$ and $y = 0$

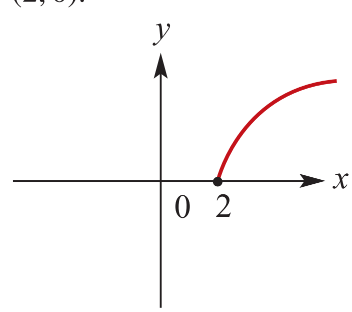
y-intercept: $y = \dfrac{1}{2^2} - 3 = -\dfrac{11}{4}$

No x-intercept because $y = 0$ is an asymptote.



d $y = \sqrt{x - 2}$

No asymptotes, starting point at $(2, 0)$.



e $y = \dfrac{1}{x - 1}$

Asymptotes at $x = 1$ and $y = 0$

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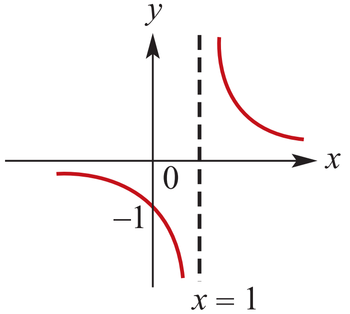
y-intercept: $y = \dfrac{1}{0-1} = -1$

No x-intercept because $y = 0$ is an asymptote.



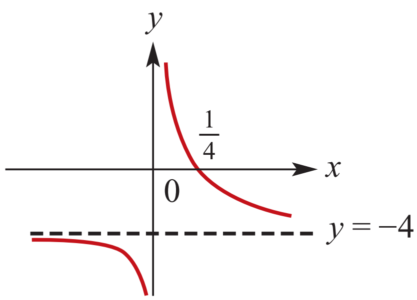
f $y = \dfrac{1}{x} - 4$

Asymptotes at $x = 0$ and $y = -4$

x-intercept: $y = \dfrac{1}{x} - 4 = 0$

$\therefore\ \dfrac{1}{x} = 4, \therefore\ x = \dfrac{1}{4}$

No y-intercept because $x = 0$ is an asymptote.

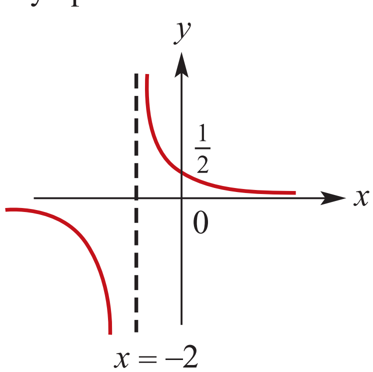


g $y = \dfrac{1}{x+2}$

Asymptotes at $x = -2$ and $y = 0$

y-intercept: $y = \dfrac{1}{0+2} = \dfrac{1}{2}$

No x-intercept because $y = 0$ is an asymptote.

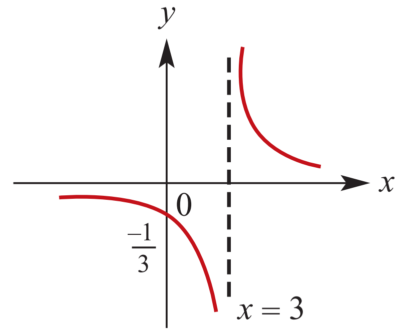


h $y = \dfrac{1}{x-3}$

Asymptotes at $x = 3$ and $y = 0$

y-intercept: $y = \dfrac{1}{0-3} = -\dfrac{1}{3}$

No x-intercept because $y = 0$ is an asymptote.

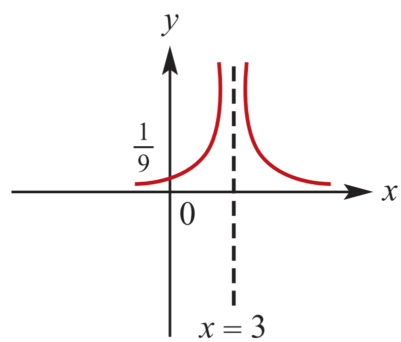


i $f(x) = \dfrac{1}{(x-3)^2}$

Asymptotes at $x = 3$ and $y = f(x) = 0$

y-intercept: $f(0) = \dfrac{1}{(0-3)^2} = \dfrac{1}{9}$

No x-intercept because $y = 0$ is an asymptote.



j $f(x) = \dfrac{1}{(x+4)^2}$

Asymptotes at $x = -4$ and $y = f(x) = 0$

y-intercept: $f(0) = \dfrac{1}{(0+4)^2} = \dfrac{1}{16}$

No x-intercept because $y = 0$ is an asymptote.

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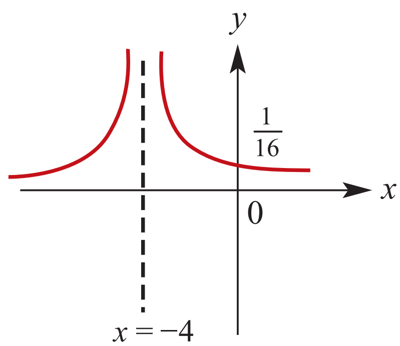


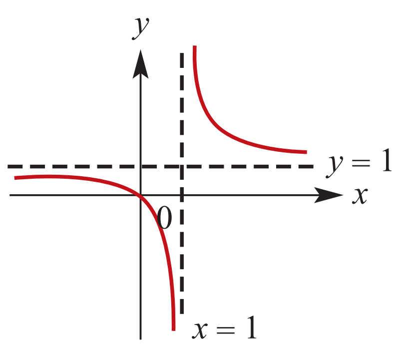
k $f(x) = \dfrac{1}{x-1} + 1$

Asymptotes at $x = 1$ and $y = f(x) = 1$

x-intercept: $y = \dfrac{1}{x-1} + 1 = 0$

$\therefore\ \dfrac{1}{x-1} = -1, \therefore\ x = 0$

y-intercept is also at (0,0).



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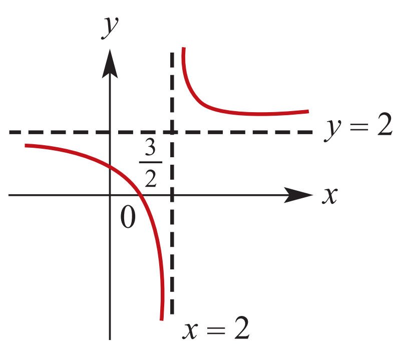


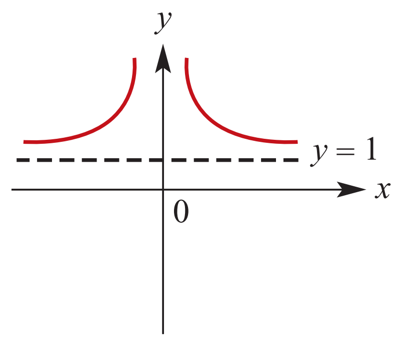
3 **a** $y = \dfrac{1}{x^2} + 1$

Asymptotes at $x = 0$ and $y = 1$

No x-intercept since $y > 1$ for all real x.

No y-intercept since $x = 0$ is an asymptote.

Range is $(1, \infty)$.



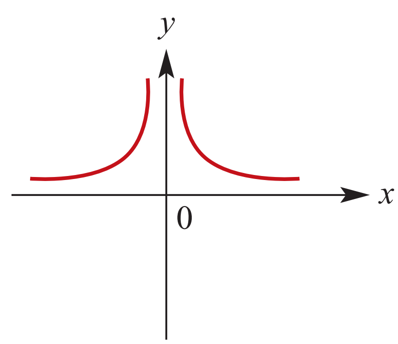
b $y = \dfrac{3}{x^2}$

Asymptotes at $x = 0$ and $y = 0$

No x-intercept since $y > 0$ for all real x.

No y-intercept since $x = 0$ is an asymptote.

Range is $(0, \infty)$.



c $y = \dfrac{1}{(x-1)^2}$

Asymptotes at $x = 1$ and $y = 0$

No x-intercept since $y > 0$ for all real x.

y-intercept at $y = \dfrac{1}{(0-1)^2} = 1$

Range is $(0, \infty)$.

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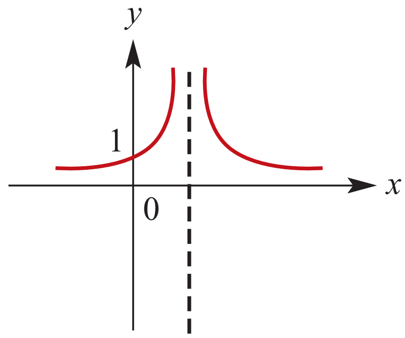


d $y = \dfrac{1}{x^2} - 4$

Asymptotes at $x = 0$ and $y = -4$

x-intercept where $y = \dfrac{1}{x^2} - 4 = 0$

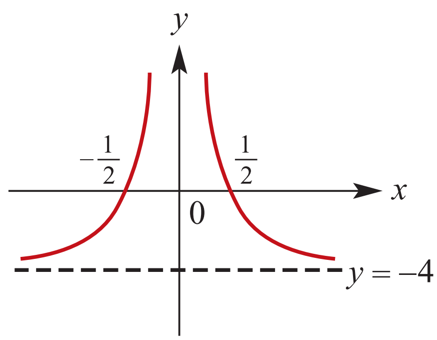
$\therefore \dfrac{1}{x^2} = 4$

$\therefore x^2 = \dfrac{1}{4}$

$\therefore x = \pm\dfrac{1}{2}$

No y-intercept since $x = 0$ is an asymptote.

Range is $(-4, \infty)$.



4 $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$, $x = x' - 2$,

$y = \dfrac{y' - 3}{2}$

5 a **i** $(x, y) \to (x - 1, 3y + 2)$

ii $(x, y) \to (x - 2, -2y + 3)$

iii $(x, y) \to \left(\dfrac{x-1}{3}, y - 1\right)$

b **i** $(x, y) \to (x - 2, 4y)$

ii $(x, y) \to (x - 6, y - 12)$

iii $(x, y) \to (x + 3, 4y - 5)$

c **i** $(x, y) \to (x + 4, y + 2)$

ii $(x, y) \to \left(\dfrac{x}{2}, 2y\right)$

iii $(x, y) \to (x, -2y + 3)$

6 $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$, $x = \dfrac{x' + 2}{3}$,

$y = y' - 3$

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Solutions to multiple-choice questions

1 C $(1, 7) \to (1, 10) \to (1, -10)$

2 D $(4, -3) \to (4, 1) \to (-4, 1)$

3 A $3a - 1 = 8$ and $b + 2 = 8$
$\therefore a = 3$ and $b = 6$

4 B $3a - 1 = a$ and $2b + 2 = b$
$\therefore a = \frac{1}{2}$ and $b = -2$
(Needs to be changed)

5 E 1st matrix is reflection in y-axis:
(1,0) becomes $(-1, 0)$, $(0, 1)$ stays put and $(1, 1)$ becomes $(-1, 1)$.
2nd matrix: $\begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$ maps $(-1, 0)$ onto $(0, 2)$, $(0, 1)$ onto $(-1, 1)$ and $(-1, 1)$ onto $(-1, 3)$.

6 D $(x, y) \to (3x, 2y)$. $\therefore x' = 3x$ and $y' = 2y$

7 B $(x, y) \to (x, 2y) \to (2y, x)$.
$\therefore x' = 2y$ and $y' = x$

8 C $(x, y) \to (x, 2y) \to (x, -2y)$.
$\therefore x' = x$ and $y' = -2y$

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Solutions to extended-response questions

1 **a** $f(x) + k = x^2 + k$

Consider the equations:

$y = x$

$y = x^2 + k$

Solving simultaneously

$$x = x^2 + k$$

$$x^2 - x + k = 0 \ldots (1)$$

For $y = x$ to be a tangent to $y = x^2 + k$ the discriminant of the quadratic in equation (1) is zero. That is $\Delta = 0$

$\therefore 1 - 4k = 0$

$\therefore k = \dfrac{1}{4}$

b $f(x - h) = (x - h)^2$

Consider the equations:

$y = x$

$y = (x - h)^2$

Solving simultaneously

$$x = (x - h)^2$$

$$x = x^2 - 2xh + h^2$$

$$x^2 - 2xh - x - h^2 = 0$$

$$x^2 - (1 + 2h)x + h^2 = 0 \ldots (1)$$

For $y = x$ to be a tangent to $y = (x - h)^2$ the discriminant of the quadratic in equation (1) is zero. That is $\Delta = 0$

$$(1 + 2h)^2 - 4h^2 = 0$$

$$1 + 4h + 4h^2 - 4h^2 = 0$$

$$1 + 4h = 0$$

$$h = -\frac{1}{4}$$

2 **a** $7(1 + h)^2 = 8$

$\therefore \quad 1 + h = \pm\sqrt{8}$

$\therefore \quad h = -1 \pm 2\sqrt{2}$

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b Let $g(x) = f(ax)$

$$= (ax)^2$$

$$= a^2x^2$$

Now $g(1) = 8, \therefore \quad a^2 = 8$

$\therefore \quad a = \pm 2\sqrt{2}$

c $y = ax^2 + bx$

$$= a\left(x^2 + \frac{b}{a}x\right)$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right)$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a}$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a}$$

The vertex has coordinates $(1, 8)$, $\therefore \dfrac{-b}{2a} = 1$ and $\dfrac{-b^2}{4a} = 8$.

Substituting $\dfrac{-b}{2a} = 1$ into $\dfrac{-b^2}{4a} = 8$ gives

$$\frac{b}{2} = 8$$

$$\therefore b = 16$$

Substituting $b = 16$ into $\dfrac{-b}{2a} = 1$ gives

$$\frac{-16}{2a} = 1$$

$$a = -8$$

3 a $g(x) = x^2 + 4x - 6$

$g(x) + k = 0$

$x^2 + 4x - 6 + k = 0$

One solution when $\Delta = 0$ $\quad \Delta = 0$

$$16 - 4(k - 6) = 0$$

$$16 - 4k + 24 = 0$$

$$40 - 4k = 0$$

$$k = 10$$

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b $$x^2 + 4x - 6 = 0$$
$$x^2 + 4x + 4 - 4 - 6 = 0$$
$$(x + 2)^2 - 10 = 0$$
$$(x + 2)^2 = 10$$
$$x = -2 \pm \sqrt{10}$$

i For two positive solutions $h > 2 + \sqrt{10}$

ii for two negative solutions $h < 2 - \sqrt{10}$

iii One positive and one negative $2 - \sqrt{10} < h < 2 + \sqrt{10}$

4 a $f(x - 2) = (x - 5)(x + 2)(x - 7)$
$$f(x - 2) = 0$$
$$(x - 5)(x + 2)(x - 7) = 0$$
$$x = 5 \text{ or } x = -2 \text{ or } x = 7$$

b $f(x + 2) = (x - 1)(x + 6)(x - 3)$
$$f(x + 2) = 0$$
$$(x - 1)(x + 6)(x - 3) = 0$$
$$x = 1 \text{ or } x = -6 \text{ or } x = 3$$

c Since $x = 0$ is a solution of $f(x) + k = 0$
$$f(x) + k = 0$$
$$60 + k = 0$$
$$k = -60$$
$$f(x) - 60 = 0$$
$$(x - 3)(x + 4)(x - 5) - 60 = 0$$
$$x^3 - 4x^2 - 17x = 0$$
$$x(x^2 - 4x - 17) = 0$$
$$x(x^2 - 4x + 4 - 21) = 0$$
$$x((x - 2)^2 - 21) = 0$$
$$x(x - 2 + \sqrt{21})(x - 2 - \sqrt{21}) = 0$$
$$x = 0 \text{ or } x = 2 - \sqrt{21} \text{ or } x = 2 + \sqrt{21}$$

d $f(x - h) = 0$ has a solution when $x = 0$
$\therefore (-h - 3)((-h + 4)(-h - 5) = 0 \therefore h = -3$ or $h = 4$ or $h = -5$

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e The solutions of $f(x-h)=0$ are $h+3, h+5$ and $h-4$

$\therefore -5 < h < -3$

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Chapter 8 – Revision of chapters 1–7

Solutions to Technology-free questions

1 a Vertices $A(-2, 1), B(3, -4), C(5, 7)$

$$\text{Coordinates of M} = \left(\frac{-2+3}{2}, \frac{1+(-4)}{2}\right)$$
$$= \left(\frac{1}{2}, -\frac{3}{2}\right)$$
$$\text{Coordinates of N} = \left(\frac{-2+5}{2}, \frac{1+7}{2}\right)$$
$$= \left(\frac{3}{2}, 4\right)$$

b
$$\text{Gradient of MN} = \frac{4-\left(-\frac{3}{2}\right)}{\frac{3}{2}-\frac{1}{2}}$$
$$= \frac{11}{2}$$
$$\text{Gradient of BC} = \frac{7-(-4)}{5-3}$$
$$= \frac{11}{2}$$

$\therefore BC \parallel MN$

2 $P(x) = 8x^3 + 4x - 3$

a
$$P\left(-\frac{1}{2}\right) = 8\times\left(-\frac{1}{2}\right)^3 + 4\left(-\frac{1}{2}\right) - 3$$
$$= -1 - 2 - 3$$
$$= -6$$

b
$$P(2) = 8\times(2)^3 + 4(2) - 3$$
$$= 64 + 8 - 3$$
$$= 69$$

c
$$Q(x) = P(x+1)$$
$$= 8\times(x+1)^3 + 4(x+1) - 3$$
$$= 64 + 8 - 3$$
$$= 69$$
$$Q(-2) = 8\times(-2+1)^3 + 4(-2+1) - 3$$
$$= -8 - 4 - 3$$
$$= -15$$

3 $g(x) = 3x^2 - 4$

a $g(2a) = 3(2a)^2 - 4 = 12a^2 - 4$

b
$$g(a-1) = 3(a-1)^2 - 4$$
$$= 3(a^2 - 2a + 1) - 4$$
$$= 3a^2 - 6a - 1$$

c
$$g(a+1) - g(a-1)$$
$$= 3(a+1)^2 - 4 - (3(a-1)^2 - 4)$$
$$= 3((a^2 + 2a + 1) - (a^2 - 2a + 1))$$
$$= 12a$$

4 $f(x) = 4 - 5x$ and $g(x) = 7 + 2x$

a $f(2) + f(3) = -6 + (-11) = -17$

$f(2+3) = f(5) = -21$

$\therefore f(2) + f(3) \neq f(2+3)$

b
$$f(x) = g(x)$$
$$4 - 5x = 7 + 2x$$
$$-3 = 7x$$
$$x = -\frac{3}{7}$$

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c $$f(x) \geq g(x)$$
$$4 - 5x \geq= 7 + 2x$$
$$-3 \geq 7x$$
$$x \leq -\frac{3}{7}$$

d $$f(2k) = g(3k)$$
$$4 - 5(2k) = 7 + 2(3k)$$
$$4 - 10k = 7 + 6k$$
$$-3 = 16k$$
$$k = -\frac{3}{16}$$

5 $$x + y = 5 \ldots (1)$$
$$(x+1)^2 + (y+1)^2 = 25 \ldots (2)$$

From equation (1) $y = 5 - x$
Substitute in equation (2)
$$(x+1)^2 + (6-x)^2 = 25$$
$$x^2 + 2x + 1 + 36 - 12x + x^2 = 25$$
$$2x^2 - 10x + 37 = 25$$
$$2x^2 - 10x + 12 = 0$$
$$x^2 - 5x + 6 = 0$$
$$(x-3)(x-2) = 0$$
$$x = 3 \text{ or } x = 2$$

From equation (1)
When $x = 3, y = 2$ and when $x = 2, y = 3$

6 $A(0, -5), B(-1, 2), C(4, 7), D(5, 0)$
$$AB = \sqrt{(7-2))^2 + (4-(-1))^2}$$
$$= \sqrt{25 + 25}$$
$$= 5\sqrt{2}$$
$$BC = \sqrt{(2-(-5))^2 + (-1-0)^2}$$
$$= \sqrt{49 + 1}$$
$$= 5\sqrt{2}$$
$$CD = \sqrt{(0-7)^2 + (5-4)^2}$$
$$= \sqrt{49 + 1}$$
$$= 5\sqrt{2}$$
$$DA = \sqrt{(5-0))^2 + (0-(-5))^2}$$
$$= \sqrt{25 + 25}$$
$$= 5\sqrt{2}$$

This is sufficient to prove $ABCD$ is a rhombus.

7 a $$y = x^2 + 4x - 9$$
$$= x^2 + 4x + 4 - 4 - 9$$
$$= (x+2)^2 - 13$$

b $$y = x^2 - 3x - 11$$
$$= x^2 - 3x + \frac{9}{4} - \frac{9}{4} - 11$$
$$= \left(x - \frac{3}{2}\right)^2 - \frac{53}{4}$$

c $$y = 2x^2 - 3x + 11$$
$$= 2\left[x^2 - \frac{3}{2}x + \frac{11}{2}\right]$$
$$= 2\left[x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{9}{16} + \frac{11}{2}\right]$$
$$= 2\left[\left(x - \frac{3}{4}\right)^2 + \frac{79}{16}\right]$$
$$= 2\left(x - \frac{3}{4}\right)^2 + \frac{79}{8}$$

8 a $$y = 4x + 1 \ldots (1)$$
$$y = x^2 + 3x - 9 \ldots (2)$$

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Substitute in equation (2) from equation 1

$$4x + 1 = x^2 + 3x - 9$$

$$\therefore 0 = x^2 - x - 10$$

$$\therefore x^2 - x - 10 = 0$$

$$\therefore x^2 - x + \frac{1}{4} - \frac{1}{4} - 10 = 0$$

$$\therefore (x - \frac{1}{2})^2 = \frac{41}{4}$$

$$x = \frac{1}{2} \pm \frac{\sqrt{41}}{2}$$

$$x = \frac{1 \pm \sqrt{41}}{2}$$

From equation (1)

When $x = \dfrac{1 + \sqrt{41}}{2}$

$y = 2 + 2\sqrt{41} + 1 = 3 + 2\sqrt{41}$

When $x = \dfrac{1 - \sqrt{41}}{2}$

$y = 2 - 2\sqrt{41} + 1 = 3 - 2\sqrt{41}$

b $y = 2x + 2 \ldots (1)$

$y = x^2 - 2x + 6 \ldots (2)$

Substitute in equation (2) from equation 1

$$2x + 2 = x^2 - 2x + 6 \text{ From}$$

$$\therefore 0 = x^2 - 4x + 4$$

$$\therefore x^2 - 4x + 4 = 0$$

$$\therefore (x - 2)^2 = 0$$

$$x = 2$$

equation (1)

When $x = 2, y = 6$

c $y = -3x + 2 \ldots (1)$

$y = x^2 + 5x + 18 \ldots (2)$

$$-3x + 2 = x^2 + 5x + 18 \text{ From}$$

$$\therefore 0 = x^2 + 8x + 16$$

$$\therefore x^2 + 8x + 16 = 0$$

$$\therefore (x + 4)^2 = 0$$

$$\therefore x = -4$$

equation (1)

When $x = -4, y = 14$

9 a $x^2 + 3x - 5 > 0$

Consider

$$x^2 + 3x - 5 = 0$$

$$x^2 + 3x + \frac{9}{4} - \frac{9}{4} - 5 = 0$$

$$\left(x + \frac{3}{2}\right)^2 = \frac{29}{4}$$

$$x + \frac{3}{2} = \pm\frac{\sqrt{29}}{2}$$

$$x = \frac{-3 \pm \sqrt{29}}{2}$$

The coefficient of x^2 is positive.

Therefore $x^2 + 3x - 5 > 0$ if and only if

$$x \in \left(-\infty, \frac{-3 - \sqrt{29}}{2}\right) \cup \left(\frac{-3 + \sqrt{29}}{2}, \infty\right)$$

b $2x^2 - 5x - 5 \geq 0$

Consider

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$$2\left(x^2 - \frac{5}{2}x - \frac{5}{2}\right) = 0$$

$$x^2 - \frac{5}{2}x - \frac{5}{2} = 0$$

$$x^2 - \frac{5}{2}x + \frac{25}{16} - \frac{25}{16} - \frac{5}{2} = 0$$

$$\left(x - \frac{5}{4}\right)^2 = \frac{65}{16}$$

$$x - \frac{5}{4} = \pm\frac{\sqrt{65}}{4}$$

$$x = \frac{5 \pm \sqrt{65}}{4}$$

The coefficient of x^2 is positive.
Therefore $2x^2 - 5x - 5 \geq 0$ if and only if

$$x \in \left(-\infty, \frac{5 - \sqrt{65}}{4}\right] \cup \left[\frac{5 + \sqrt{65}}{4}, \infty\right)$$

c $(x - 3)^2(x + 4) \geq 0$
$(x - 3)^2 \geq 0$ for all x.
$\therefore (x - 3)^2(x + 4) \geq 0 \Leftrightarrow x + 4 \geq 0$
$\Leftrightarrow x \geq -4$
That is $(x - 3)^2(x + 4) \geq 0$ if and only if $x \in [-4, \infty)$

d $(x - 3)(x + 4)(2x - 1) \leq 0$
The coefficient of x^3 is positive.
Therefore $(x - 3)(x + 4)(2x - 1) \leq 0$ if and only if $x \in [\frac{1}{2}, 3] \cup (-\infty, -4]$

e $(x - 2)^3 - 8 \leq 0$

$$\Leftrightarrow (x - 2)^3 \leq 8$$

$$\Leftrightarrow x - 2 \leq 2$$

$$\Leftrightarrow x \leq 4$$

10 **a** $\mathbb{R} \setminus \{\frac{5}{2}\}$

b $(-\infty, 5]$

c $\mathbb{R}$

d $\mathbb{R} \setminus \{2\}$

e $\mathbb{R}$

f $\mathbb{R} \setminus \{\frac{2}{3}\}$

11 Let $P(x) = 3x^3 + x^2 + px + 24$
$P(-4) = 0$ by the factor theorem.
Hence

$$3(-4)^3 + (-4)^2 + (-4)p + 24 = 0$$

$$-192 + 16 - 4p + 24 = 0$$

$$-4p = 152$$

$$\therefore p = -38$$

$\therefore P(x) = 3x^3 + x^2 - 38x + 24$
$3x^3 + x^2 - 38x + 24 = (x + 4)(3x^2 + bx + 6)$
since $x + 4$ is a factor.
By equating coefficients of x^2
$1 = 12 + b, \therefore b = -11$

$$P(x) = (x + 4)(3x^2 - 11x + 6)$$

$$= (x + 4)(3x - 2)(x - 3)$$

12

$$5x^3 - 3x^2 + ax + 7 = (x + 2)Q_1(x) + R \ldots (1)$$

$$4x^3 + ax^2 + 7x - 4 = (x + 2)Q_2(x) + 2R \ldots (2)$$

Multiply (1) by 2 and subtract (1) from the result.
$6x^3 - (6 + a)x^2 + (2a - 7)x + 18 = (x + 2)(2Q_1 - Q_2)$
When $x = -2$
$6(-2)^3 - (6 + a)(-2)^2 + (2a - 7)(-2) + 18 = 0$
$\therefore -48 - 24 - 4a - 4a + 14 + 18 = 0$

$$\therefore -8a = 40$$

$$\therefore a = -5$$

Substitute in (1)
$5x^3 - 3x^2 - 5x + 7 = (x + 2)Q_1(x) + R$
Substitute $x = -2$
$R = 5(-2)^3 - 3(-2)^2 - 5(-2) + 7 = -35$

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13 a $f : [1, 2] \to \mathbb{R}, f(x) = x^2$
Domain of $f = [1, 2]$
Range of $f = [1, 4]$
Let $y = x^2$
Interchange x and y.
$x = y^2$
Choose $y = \sqrt{x}$, (range of f)
$\therefore f^{-1} : [1, 4] \to \mathbb{R}, f^{-1}(x) = \sqrt{x}$

b $h : [-1, 2] \to \mathbb{R}, f(x) = 2 - x$
Domain of $h = [-1, 2]$
Range of $h = [0, 3]$
Let $y = 2 - x$
Interchange x and y.
$x = 2 - y$
$y = 2 - x$
$\therefore h^{-1} : [0, 3] \to \mathbb{R}, h^{-1}(x) = 2 - x$

c $g : \mathbb{R}^{-1} \to \mathbb{R}, g(x) = x^2 - 4$
Domain of $g = (-\infty, 0)$
Range of $g = (-4, \infty]$
Let $y = x^2 - 4$
Interchange x and y.
$x = y^2 - 4$
$y = -\sqrt{x + 4}$ (range of g)
$\therefore g^{-1} : [0, 3] \to \mathbb{R}, g^{-1}(x) = -\sqrt{x + 4}$

d $f : (-\infty, 2] \to \mathbb{R}, f(x) = \sqrt{2 - x} + 3$
Domain of $f = (-\infty, 2]$
Range of $f = [3, \infty]$
Let $y = \sqrt{2 - x} + 3$
Interchange x and y.
$x = \sqrt{2 - y} + 3$
$y = -(x - 3)^2 + 2$
$\therefore f^{-1} : [3, \infty) \to \mathbb{R}$,
$f^{-1}(x) = -(x - 3)^2 + 2$

e $f : \to \mathbb{R}, f(x) = (x - 2)^3 + 8$
Domain of $f = \mathbb{R}$
Range of $f = \mathbb{R}$
Let $y = (x - 2)^3 + 8$
Interchange x and y.
$x = (y - 2)^3 + 8$
$y = (x - 8)^{\frac{1}{3}} + 2$
$\therefore f^{-1} : \mathbb{R} \to \mathbb{R}$,
$f^{-1}(x) = (x - 8)^{\frac{1}{3}} + 2$

14 Let b be the cost of a Bob's burger.
Let f be the cost of a regular fries.

a $\therefore 3b + 2f = 18.20$

b If $b = 4.2$
$3 \times 4.20 + 2f = 18.20$
$\therefore 2f = 18.20 - 12.60$
$\therefore f = 2.80$
The cost of regular fries is \$2.80

15 $4x + ky = 7$ and $y = 3 - 4x$ The gradient of the line $4x + ky = 7$ is $-\dfrac{4}{k}$
The gradient of the line $y = 3 - 4x$ is -4

a If the lines are parallel, $-\dfrac{4}{k} = -4$
Hence $k = 1$

b If the lines are perpendicular
$-\dfrac{4}{k} \times -4 = -1$
$k = -16$

16 Line ℓ_1 has x-axis intercept $(5, 0)$ and y-axis intercept $(0, -2)$.

a Gradient of $\ell_1 = \dfrac{-2 - 0}{0 - 5} = \dfrac{2}{5}$

b Line ℓ_2 is perpendicular to line line ℓ_1
Hence gradient of ℓ_2 is $-\dfrac{5}{2}$
The line ℓ_2 has equation of the form
$y = -\dfrac{5}{2}x + c$

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When $x = 1, y = 6$ ∴, $6 = -\frac{5}{2} + c$ and hence
$c = \frac{17}{2}$ and $y = -\frac{5}{2}x + \frac{17}{2}$
Rearranging as required
$5x + 2y - 17 = 0$

17 **a** $ax^2 + 2x + a$

$$= a(x^2 + \frac{2}{a}x + 1)$$
$$= a\left(x^2 + \frac{2}{a}x + \frac{1}{a^2} - \frac{1}{a^2} + 1\right)$$
$$= a\left(\left(x + \frac{1}{a}\right)^2 + \frac{a^2 - 1}{a^2}\right)$$
$$= a\left(x + \frac{1}{a}\right)^2 + \frac{a^2 - 1}{a}$$

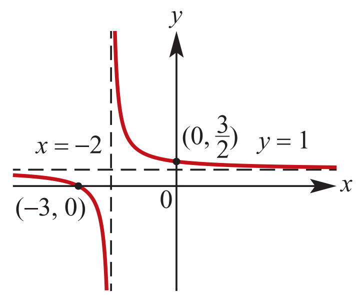
b $\left(-\frac{1}{a}, \frac{a^2 - 1}{a}\right)$

c Perfect squuare when $\frac{a^2 - 1}{a} = 0$.
That is when $a = \pm 1$

d There are two solutions when $\frac{a^2 - 1}{a} < 0$
That is, when $a \in (-1, 1)$.

18 **a** $y = 1 + \frac{1}{2 + x}$
When $x = 0, y = \frac{3}{2}$ When
$y = 0, 1 + \frac{1}{2 + x} = 0$

That is, $\frac{1}{2 + x} = -1$ which implies $x = -3$
The horizontal asymptote has equation $y = 1$
The vertical asymptote has equation $x = -2$



b $A\left(0, \frac{3}{2}\right), B(-3, 0)$

c $y = \frac{1}{2}x + \frac{3}{2}$

d The midpoint
$$\left(\frac{0 + (-3)}{2}, \frac{\frac{3}{2} + 0}{2}\right) = \left(-\frac{3}{2}, \frac{3}{4}\right)$$

e Gradient of line $AB = \frac{\frac{3}{2} - 0}{0 - (-3)} = \frac{1}{2}$
Gradient of a line perpendicular to AB is -2.
Therefore using the general form $y - y_1 = m(x - x_1)$ we have
$y - \frac{3}{4} = -2\left(x + \frac{3}{2}\right)$
That is,
$y = -2x - \frac{9}{4}$

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Solutions to multiple-choice questions

1 B $y = x^2 - ax$

$$= x^2 - ax + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2$$

$$= \left(x - \frac{a}{2}\right)^2 - \frac{a^2}{4}$$

2 D $\Delta = 4a^2 - 4b = 0$

$a^2 = b$

$a = \sqrt{b}$ or $a = -\sqrt{b}$

But a and b are positive constants.

Therefore $a = \sqrt{b}$

3 C Gradients are the same when

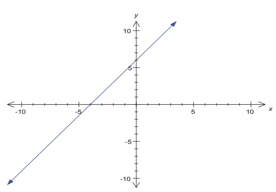
$$\frac{2-m}{3} = \frac{-2}{m+2}$$

$$\frac{m-2}{3} = \frac{2}{m+2}$$

$m^2 - 4 = 6$

$m = \pm\sqrt{10}$

4 A $3x - 2y = -6$



5 D Only (1, 2) is on the line $y = 3x - 1$

6 D $x^3 - 8 = x^3 - 2^3$

$= (x-2)(x^2 + 2x + 4)$

7 C $2x^2 - 5x - 12 = (2x + a)(x - b)$

$a - 2b = -5;\ ab = 12$

$a = 3,\ b = 4: f(x) = (2x + 3)(x - 4)$

8 C $P(x) = 4x^3 - 5x + 5$

$P(-\frac{3}{2}) = -1$

9 C $x^2 + y^2 + 6x - 2y + 6 = 0$

$\therefore\ x^2 + 6x + 9 + y^2 - 2y + 1 = 10 - 6$

$\therefore\ (x+3)^2 + (y-1)^2 = 2^2$

Radius = 2

10 A $2x + 4y - 6 = 0$

$\therefore\ 4y = -2x + 6$

$\therefore\ y = -\frac{1}{2}x + \frac{3}{2}$

Gradient $= -\frac{1}{2}$

11 E $2x + 4y = 3$

$\therefore\ 4y = -2x + 3$

$\therefore\ y = -\frac{1}{2}x + \frac{3}{4}$

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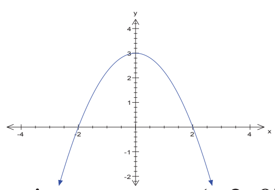
Line has gradient $= -\frac{1}{2}$, so perpendicular has gradient $m = 2$.
Using $(1, 2)$: $y - 2 = 2(x - 1)$
$\therefore\ y = 2x$

12 B $P(x) = x^3 + ax^2 - x - 6$
If $x - 3$ is a factor of $P(x)$ then $P(3) = 0$:
$P(3) = 27 + 9a - 3 - 6 = 0$
$\therefore 9a + 18 = 0,\ \therefore a = -2$

13 A
$$P(x) = x^3 + 8x^2 + 9x - 18$$
$$P(1) = 1 + 8 + 9 - 18 = 0$$
$$\therefore P(x) = (x - 1)(x^2 + 9x + 18)$$
$$= (x - 1)(x + 6)(x + 3)$$

14 E



x-intercepts at $(-2, 0)$ and $(2, 0)$, so
$y = a(x - 2)(x + 2)$
$\therefore\ y = a(x^2 - 4)$
Using y-intercept at $(0, 3)$, $a = -\frac{3}{4}$
$$\therefore\quad y = -\frac{3}{4}(x - 2)(x + 2)$$
OR $4y = -3(x - 2)(x + 2)$

15 B Perpendicular lines have gradients which multiply to -1
$\therefore\ -3m = -1,\ \therefore m = \frac{1}{3}$

16 D
$$f(x) = x^2 - 1$$
$$\therefore\ f(x - 1) = ((x - 1)^2 - 1)$$
$$= x^2 - 2x + 1 - 1$$
$$= x^2 - 2x$$

17 D $y = x^2 + kx + k + 8$ touches the x-axis. Therefore it is a perfect square and $\Delta = 0$:
$$\Delta = k^2 - 4(k + 8)$$
$$= k^2 - 4k - 32$$
$$= (k - 8)(k + 4)$$
$\Delta = 0$ when $k = -4$ or 8

18 E $P(x) = 3x^3 - 4x - k$
If $P(x)$ is divisible by $x - k$, then $P(k) = 0$: $P(k) = 3k^3 - 4k - k = 0$
$$= 3k^3 - 5k = 0$$
Remainder when $P(x)$ is divided by $x + k$:
$$P(-k) = -3k^3 + 4k - k$$
$$= -3k^3 + 3k$$
$$3k^3 - 5k = 0,\ \therefore\ -3k^3 + 5k = 0$$
$$\therefore\ P(-k) = 0 - 2k$$

19 B TP of $y = a(x - b)^2 + c$ is at (b, c)

20 D $y = 3 + 4x - x^2$
meets $y = k$ only once.
$\therefore\ -x^2 + 4x + (3 - k) = 0$ has $\Delta = 0$:
$$\Delta = 16 + 4(3 - k) = 0$$
$\therefore\ 3 - k = -4,\ \therefore\ k = 7$

21 E Midpoint of $(12,7)$ and $(-3, 5)$ is:
$$\left(\frac{12 - 3}{2}, \frac{7 + 5}{2}\right) = \left(\frac{9}{2}, 6\right)$$

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22 D X is at (a, b): $(7, -3) = \left(\frac{5+a}{2}, \frac{4+b}{2}\right)$

$5 + a = 14, \therefore a = 9$

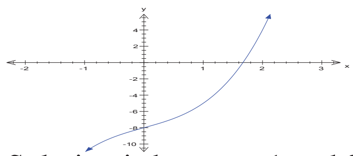
$4 + b = -6, \therefore b = -10$

23 B $y = x^2 + 1$

dom $[-2, 1] \rightarrow$ range $[1, 5]$

24 D $x^3 + 2x - 8 = 0$

Use calculator:



Solution is between 1 and 2.

25 D $f(x) = x(x-2)$

$\therefore f(-3) = (-3)(-5) = 15$

26 A $x^2 + y^2 - 11x - 10y + 24 = 0$

Circle cuts the y-axis at M and N so $x = 0$

$$y^2 - 10y + 24 = 0$$
$$\therefore y^2 - 10y + 25 = 1$$
$$\therefore \quad y - 5 = \pm 1$$
$$\therefore \quad y = 4; 6$$

Distance between M and N is 2

27 B Distance between $(-4, -3)$ and $(-5, -10)$

$$= \sqrt{(-4 - (-5))^2 + (-3 - (-10))^2}$$
$$= \sqrt{1 + 49} = 5\sqrt{2}$$

28 D $y = x^2 + 4x - 3$ cuts the line $y = 4 - 2x$ at

$$x^2 + 4x - 3 = 4 - 2x$$
$$\therefore \quad x^2 + 6x - 7 = 0$$
$$\therefore (x + 7)(x - 1) = 0$$

$x = -7, y = 18$ and $x = 1, y = 2$

Distance between $(-7, 18)$ and $(1,2)$

$$= \sqrt{(-7 - 1)^2 + (18 - 2)^2}$$
$$= \sqrt{8^2 + 16^2} = \sqrt{320}$$

29 D $\{(x, y): y \leq 2x + 3\}$

A $(1, 4)$: $4 < 5$ √

B $(-1, 1)$: $1 = 1$ √

C $\left(\frac{1}{2}, 3\frac{1}{2}\right)$: $3\frac{1}{2} < 4$ √

D $(-\frac{1}{2}, 2\frac{1}{2})$: $2\frac{1}{2} > 2$ **X**

E $(2, 5)$: $5 < 7$ √

30 B $y = k + 2x - x^2$

If the graph touches the x-axis then $\Delta = 0$:

$\Delta = 4 + 4k = 0, \therefore k = -1$

31 C Perpendicular lines have gradients which multiply to -1:

$$kx + y - 4 = 0, \therefore y = 4 - kx$$
$$x - 2y + 3 = 0, \therefore y = \frac{x+3}{2}$$
$$\therefore (-k)\left(\frac{1}{2}\right) = -1, \therefore k = 2$$

32 A $y = x^2 + k$ and $y = x$

$$\therefore \quad x^2 + k = x$$
$$\therefore x^2 - x + k = 0$$

For 1 solution $\Delta = 0$:

$\Delta = 1 - 4k = 0, \therefore k = \frac{1}{4}$

33 C Circle with centre at $(-4, 2)$:

$\therefore (x + 4)^2 + (y - 2)^2 = r^2$

Circle touches the y-axis so $r = 4$:

$$\therefore \quad x^2 + 8x + 16 + y^2 - 4y + 4 = 16$$
$$\therefore \quad x^2 + 8x + y^2 - 4y + 4 = 0$$

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34 **A** $2x - y + 3 = 0$ has gradient = 2.
If $ax + 3y - 1 = 0$ is parallel, its gradient = 2

$\therefore \quad 3y = 1 - ax$

$\therefore \quad y = \dfrac{1 - ax}{3}$

$\therefore \quad -\dfrac{a}{3} = 2, \therefore a = -6$

35 **B** $f(x) = \sqrt{4 - x^2}$ has max. dom. $[-2, 2]$

36 **C** $f(x) = 2x^2 + 3x + 4$

$= 2\left(x^2 + \dfrac{3}{2}x + 2\right)$

$= 2\left(x + \dfrac{3}{2}x + \dfrac{9}{16} + \dfrac{23}{16}\right)$

$= 2\left(x + \dfrac{3}{2}\right)^2 + \dfrac{23}{8}$

Range = $[\dfrac{23}{8}, \infty)$

37 **D** $P(x) = x^3 - kx^2 - 10kx + 25$

$P(2) = 8 - 4k - 20k + 25 = 9$

$\therefore 24k = 24, \therefore k = 1$

38 **E** $f(x) = x^2 - 7x + k$

$f(k) = k^2 - 7k + k = -9$

$\therefore k^2 - 6k + 9 = 0$

$\therefore \quad (k - 3)^2 = 0, \therefore k = 3$

$\therefore f(x) = x^2 - 7x + 3$

$\therefore f(-1) = 1 + 7 + 3 = 11$

39 **E** $2xy - x^2 - y^2$

$= -(x^2 - 2xy + y^2)$

$= -(x - y)^2$

40 **C** $x^2 - x - 12 \leq 0$

$\therefore (x - 4)(x + 3) \leq 0$

Upright parabola so $-3 \leq x \leq 4$

41 **C** $f(x) = \dfrac{1}{2}x(x - 1)$

$\therefore f(x) - f(x + 1)$

$= \dfrac{1}{2}x(x - 1) - \dfrac{1}{2}x(x + 1)$

$= \dfrac{x}{2}((x - 1) - (x + 1))$

$= \dfrac{x}{2}(-2) = -x$

42 **C** $2x^2 - 2 \leq 0$

$\therefore x^2 \leq 1, \therefore -1 \leq x \leq 1$

43 **A** $f(x) = -2\left(\left(x - \dfrac{1}{2}\right)^2 - 3\right)$

$= 6 - 2\left(x - \dfrac{1}{2}\right)^2$

Inverted parabola so max. value = 6

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Solutions to extended-response questions

1 a $$x^2 + y^2 + bx + cy + d = 0$$

At $(-4, 5)$, $16 + 25 - 4b + 5c + d = 0$

$\therefore \quad 4b - 5c - d = 41 \quad (1)$

At $(-2, 7)$, $4 + 49 - 2b + 7c + d = 0$

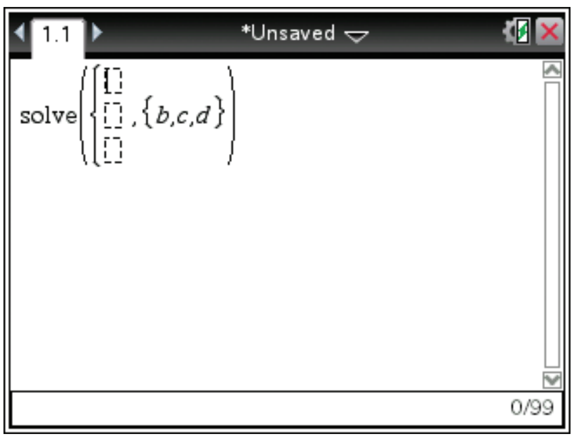
$\therefore \quad 2b - 7c - d = 53 \quad (2)$

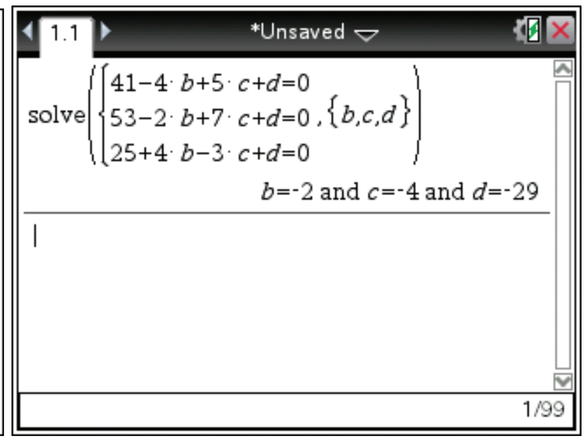
At $(4, -3)$, $16 + 9 + 4b - 3c + d = 0$

$\therefore \quad -4b + 3c - d = 25 \quad (3)$

b **TI**: Press **Menu→3:Algebra→ 7:Solve System of Equations→ 1:Solve System of Equations**

Change the number of equations to 3 and change the variables to b,c, d





Type the equations from part a into the empty boxes then ENTER to yield $b = -2$, $c = -4$ and $d = -29$

CP: Type **solve** ({**41 − 4 × b + 5 × c + d = 0**, **53 − 2 × b + 7 × c + d = 0**, **25 + 4 × b − 3 × c + d = 0**}, {**b,c,d**}) then EXE

Therefore the equation of the circle is $x^2 + y^2 - 2x - 4y - 29 = 0$

2 a $$x^2 + y^2 + bx + cy = 0 \quad (1)$$

At $(4, 4)$, $16 + 16 + 4b + 4c = 0$

$\therefore \quad 32 + 4b + 4c = 0$

$\therefore \quad 4c = -4b - 32$

$\therefore \quad c = -b - 8 \quad (2)$

b To find the x-axis intercept, let $y = 0$ in equation (1),

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$\therefore \quad x^2 + bx = 0$

$\therefore \quad x(x + b) = 0$

$\therefore \quad x = 0 \text{ or } x = -b$

c **i** To find the y-axis intercept, let $x = 0$ in equation (1),

$\therefore \quad y^2 + cy = 0$

$\therefore \quad y(y + c) = 0$

$\therefore \quad y(y - b - 8) = 0 \qquad \text{from (2)}$

$\therefore \quad y = 0 \text{ or } y = b + 8$

ii The circle touches the y-axis when there is one y-axis intercept, i.e. when $b + 8 = 0$, or $b = -8$.

3 **a** For $f(x) = \sqrt{a - x}$, the maximal domain is $x \le a$.

b At the point of intersection, $\sqrt{a - x} = x$

$\therefore \quad a - x = x^2$

$\therefore \quad x^2 + x - a = 0$

Using the general quadratic formula, $x = \dfrac{-1 \pm \sqrt{1 + 4a}}{2}$.

Since the range of $f(x)$ is $[0, \infty)$, the point of intersection of the graphs of $y = f(x)$ and $y = x$ is $\left(\dfrac{-1 + \sqrt{1 + 4a}}{2}, \dfrac{-1 + \sqrt{1 + 4a}}{2}\right)$.

c When $\left(\dfrac{-1 + \sqrt{1 + 4a}}{2}, \dfrac{-1 + \sqrt{1 + 4a}}{2}\right) = (1, 1)$,

$$\frac{-1 + \sqrt{1 + 4a}}{2} = 1$$

$\therefore \quad -1 + \sqrt{1 + 4a} = 2$

$\therefore \quad \sqrt{1 + 4a} = 3$

$\therefore \quad 1 + 4a = 9$

$\therefore \quad 4a = 8$

$\therefore \quad a = 2$

d When $\left(\dfrac{-1 + \sqrt{1 + 4a}}{2}, \dfrac{-1 + \sqrt{1 + 4a}}{2}\right) = (2, 2)$,

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$$\frac{-1+\sqrt{1+4a}}{2} = 2$$

$$\therefore \quad -1+\sqrt{1+4a} = 4$$

$$\therefore \quad \sqrt{1+4a} = 5$$

$$\therefore \quad 1+4a = 25$$

$$\therefore \quad 4a = 24$$

$$\therefore \quad a = 6$$

e When $\left(\frac{-1+\sqrt{1+4a}}{2}, \frac{-1+\sqrt{1+4a}}{2}\right) = (c, c)$,

$$\frac{-1+\sqrt{1+4a}}{2} = c$$

$$\therefore \quad -1+\sqrt{1+4a} = 2c$$

$$\therefore \quad \sqrt{1+4a} = 2c+1$$

$$\therefore \quad 1+4a = (2c+1)^2$$

$$\therefore \quad 1+4a = 4c^2+4c+1$$

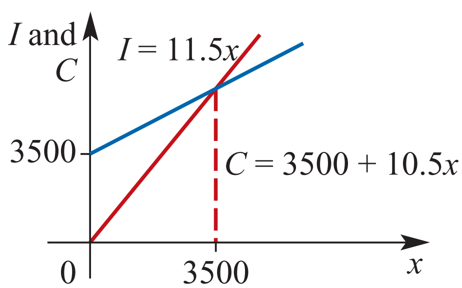
$$\therefore \quad 4a = 4c^2+4c$$

$$\therefore \quad a = c^2+c$$

4 a $C = 3500 + 10.5x$

b $I = 11.5x$

c



d $I = C$

$$\therefore \quad 11.5x = 3500 + 10.5x$$

$$\therefore \quad x = 3500$$

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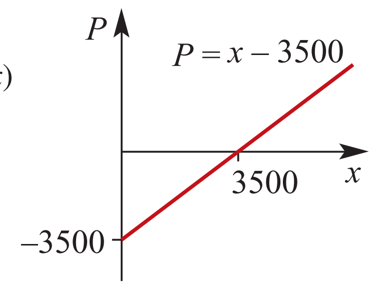
e $P = I - C$

$= 11.5x - (3500 + 10.5x)$

$= x - 3500$

P = profit



f $P = 2000$

$\therefore x - 3500 = 2000 \qquad \therefore x = 5500$

5500 plates must be sold for a profit of \$2000 to be made.

5 a $V = 45\,000 + 40m$

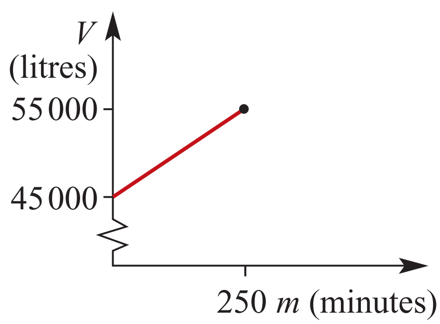
b $45\,000 + 40m = 55\,000$

$\therefore \quad m = \dfrac{10\,000}{40} = 250$

250 min = 4h 10 min

The pool will reach its maximum capacity after 4 hours 10 minutes.

c



6 a When $t = 10$, $V = 20 \times 10 = 200$ litres.

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b For uniform rate, the gradient of the graph is given by the rate.

Hence, $a = 20$

When $t = 10$, $V = 200$ and $b = 15$

Thus $V = bt + c$ gives

$$200 = 15 \times 10 + c, \therefore\ c = 50$$

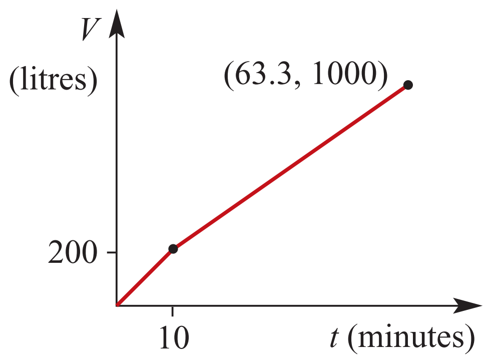
and
$$V = \begin{cases} 20t & 0 \le t \le 10 \\ 15t + 50 & 10 < t \le \dfrac{190}{3} \end{cases}$$

Note: $d = \dfrac{190}{3}$ as $15t + 50 = 1000$

$\Rightarrow$ $15t = 950$

$\Rightarrow$ $t = \dfrac{190}{3}$

c



7 **a** For rectangle, length = $3x$ cm, width = $2x$ cm, area = $6x^2$cm^2

b Side length of square $= \dfrac{1}{4}(42 - 10x)$

$= \dfrac{1}{2}(21 - 5x)$ cm

Area of square $= \left(\dfrac{1}{2}(21 - 5x)\right)^2$

$= (10.5 - 2.5x)^2$ cm^2

c $0 \le 10x \le 42$

$\therefore$ $0 \le x \le 4.2$

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d $A_T = 6x^2 + (10.5 - 2.5x)^2$

$$= 6x^2 + \frac{25}{4}x^2 - \frac{105}{2}x + \frac{441}{4}$$

$$= \frac{49}{4}x^2 - \frac{105}{2}x + \frac{441}{4}$$

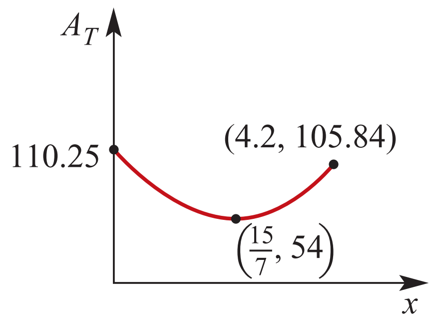
$$= \frac{49}{4}\left(x^2 - \frac{4}{49} \times \frac{105}{2}x + \frac{4}{49} \times \frac{441}{4}\right)$$

$$= \frac{49}{4}\left(x^2 - \frac{30}{7}x + \left(\frac{15}{7}\right)^2 - \frac{225}{49} + \frac{441}{49}\right)$$

$$\therefore A_T = \frac{49}{4}\left(x - \frac{15}{7}\right)^2 + \frac{49}{4} \times \frac{216}{49}$$

$$\therefore A_T = \left(\frac{49}{4}\left(x - \frac{15}{7}\right)^2 + 54\right)\text{cm}^2, \text{ or : } A = (12.25x^2 - 52.5x + 110.25)\text{ cm}^2$$

e



f Maximum total area = 110.25 cm^2 (area of rectangle equals zero)

g
$$\frac{49}{4}x^2 - \frac{105}{2}x + \frac{441}{4} = 63$$

$$\therefore \quad \frac{49}{4}x^2 - \frac{105}{2}x + \frac{441}{4} - \frac{252}{4} = 0$$

$$\therefore \quad \frac{49}{4}x^2 - \frac{105}{2}x + \frac{189}{4} = 0$$

$$\therefore \quad 49x^2 - 210x + 189 = 0$$

$$\therefore \quad 7(7x^2 - 30x + 27) = 0$$

$$\therefore \quad 7(7x - 9)(x - 3) = 0$$

$$\therefore \quad x = \frac{9}{7} \text{ or } x = 3$$

When $x = \frac{9}{7}$, the rectangle has dimensions $3x = \frac{27}{7} \approx 3.9$ and $2x = \frac{18}{7} \approx 2.6$,

i.e. 3.9 cm × 2.6 cm, and the square has dimensions $\frac{1}{2}\left(21 - 5 \times \frac{9}{7}\right) = \frac{51}{7} \approx 7.3$,

i.e. 7.3 cm ×7.3 cm.

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When $x = 3$, the rectangle has dimensions $3x = 9$ and $2x = 6$,
i.e. 9 cm × 6 cm, and the square has dimensions $\frac{1}{2}(21 - 5 \times 3) = 3$,
i.e. 3 cm × 3 cm.

8 $y = -\frac{1}{10}(x + 10)(x - 20), \ x \geq 0$

a When $x = 0$, $y = -\frac{1}{10}(10)(-20)$

$= 20\text{m}$, the height at the point of projection.

b When $y = 0$, $x = 20$ m, the horizontal distance travelled, ($x \neq -10$ as $x \geq 0$).

c
$$\begin{aligned} y &= -\frac{1}{10}(x^2 - 10x - 200) \\ &= -\frac{1}{10}(x^2 - 10x + 25 - 225) \\ &= -\frac{1}{10}(x - 5)^2 + 22.5 \end{aligned}$$

When $x = 5, y = 22.5$ m, the maximum height reached by the stone.

9 **a** If height = x cm, width = $(x + 2)$ cm, length = $2(x + 2)$ cm

$$\begin{aligned} A &= 2x(x + 2) + 2x \times 2(x + 2) + 2(x + 2) \times 2(x + 2) \\ &= 2x^2 + 4x + 4x^2 + 8x + 4x^2 + 16x + 16 \\ &= 10x^2 + 28x + 16 \end{aligned}$$

b **i** When $x = 1$,
$$\begin{aligned} A &= 10(1)^2 + 28(1) + 16 \\ &= 10 + 28 + 16 \\ &= 54\,\text{cm}^2 \end{aligned}$$

ii When $x = 2$,
$$\begin{aligned} A &= 10(2)^2 + 28(2) + 16 \\ &= 40 + 56 + 16 \\ &= 112\,\text{cm}^2 \end{aligned}$$

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c $\quad 10x^2 + 28x + 16 = 190$

$\therefore \quad 10x^2 + 28x - 174 = 0$

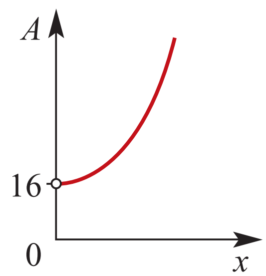
$\therefore \quad 2(5x^2 + 14x - 87) = 0$

$\therefore \quad (5x + 29)(x - 3) = 0$

$\therefore \quad x = \dfrac{-29}{5}$ or 3

But $x > 0$, $\quad \therefore x = 3\text{cm}$

d $A = 10x^2 + 28x + 16$



e $V = 2(x + 2)x(x + 2)$

$= 2x(x + 2)^2$

$= 2x(x^2 + 4x + 4)$

$= 2x^3 + 8x^2 + 8x$

f $\quad 2x^3 + 8x^2 + 8x = 150$

$\therefore 2x^3 + 8x^2 + 8x - 150 = 0$

$P(0) = -150 \qquad \neq 0$

$P(1) = 2(1)^3 + 8(1)^2 + 8(1) - 150$

$= -132 \qquad \neq 0$

$P(2) = 2(2)^3 + 8(2)^2 + 8(2) - 150$

$= 16 + 32 + 16 - 150$

$= -86 \qquad \neq 0$

$P(3) = 2(3)^3 + 8(3)^2 + 8(3) - 150$

$= 54 + 72 + 24 - 150 \qquad = 0$

$\therefore (x - 3)$ is a factor of $2x^3 + 8x^2 + 8x - 150$

When $V = 150$, $x = 3$

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$$
\begin{array}{r|l}
\multicolumn{2}{r}{2x^2 + 14x + 50} \\
\cline{2-2}
x - 3 & 2x^3 + 8x^2 + 8x - 150 \\
 & 2x^3 - 6x^2 \\
\cline{2-2}
 & 14x^2 + 8x - 150 \\
 & 14x^2 - 42x \\
\cline{2-2}
 & 50x - 150 \\
 & 50x - 150 \\
\cline{2-2}
 & 0
\end{array}
$$

$$
\begin{aligned}
\therefore \quad & 2x^3 + 8x^2 + 8x - 150 = (x - 3)(2x^2 + 14x + 50) \\
\text{But} \quad & 2x^2 + 14x + 50 \neq 0 \\
\text{as} \quad & \Delta = 196 - 400 \\
& \quad = -204 < 0 \\
\therefore \quad & x = 3
\end{aligned}
$$

g The answer can be found using a CAS calculator.
Input $\mathbf{Y_1 = 2X\hat{}3 + 8X\hat{}2 + 8X}$ and $\mathbf{Y_2 = 1000}$.
The point of intersection is (6.6627798, 1000). Therefore the volume of the block is 1000 cm^3 when $x = 6.66$, correct to 2 decimal places.

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10 a i $A = 10y + (y - x)x$

$= 10y + yx - x^2$

ii $P = 2y + 20 + 2x$

$= 2(y + 10 + x)$

b i If $P = 100$

$$100 = 2(y + 10 + x)$$

$$\therefore \quad 50 = y + 10 + x$$

$$\therefore \quad y = 40 - x$$

$$\therefore \quad A = (10 + x)(40 - x) - x^2$$

$$= 400 + 30x - x^2 - x^2$$

$$= 400 + 30x - 2x^2$$

ii

$$A = -2(x^2 - 15x - 200)$$

$$= -2\left(x^2 - 15x + \frac{225}{4} - 200 - \frac{225}{4}\right)$$

$$\therefore \quad A = -2\left(\left(x - \frac{15}{2}\right)^2 - \frac{1025}{4}\right)$$

$$= -2(x - \frac{15}{2})^2 + \frac{1025}{2}$$

$$\therefore \text{ maximum possible area } = \frac{1025}{2}\,\text{m}^2$$

$$= 512.5\,\text{m}^2$$

iii $A > 0$ and $y > 0$ and $x \geq 0$ and $y - x \geq 0$

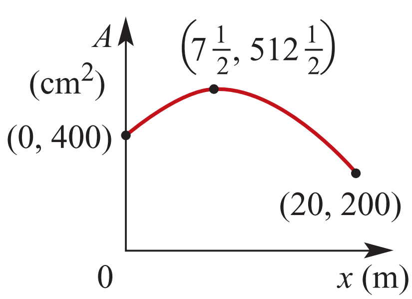
Considering the last inequality, $y \geq x$

$\therefore \quad 40 - x \geq x$

$\therefore \quad x \leq 20$

As $x \geq 0$, the largest possible domain is $0 \leq x \leq 20$.

iv



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11 **a** Let: $A_T(\mathrm{m}^2)$ be the total area of the window.

$$A_T = (2x + y)(3x + 2y)$$
$$= 6x^2 + 7xy + 2y^2$$

b Let $A_W(\mathrm{m}^2)$ be the total area of the dividing wood.

$$A_W = xy + xy + xy + xy + xy + xy + xy + y^2 + y^2$$
$$= 7xy + 2y^2$$

c **i** Area of glass, $A_G = 1.5$

$\therefore \quad 6x^2 = 1.5$

$\therefore \quad x^2 = \frac{3}{2} \times \frac{1}{6} = \frac{1}{4}$

$\therefore \quad x = \frac{1}{2}$ or 0.5 (as $x \geq 0$)

ii Area of wood, $A_w = 1$

$\therefore \quad 7xy + 2y^2 = 1$

As $x = \frac{1}{2}, 7 \times \frac{1}{2} \times y + 2y^2 - 1 = 0$

$\therefore \quad 2y^2 + \frac{7}{2}y - 1 = 0$

$\therefore \quad 4y^2 + 7y - 2 = 0$

$\therefore \quad (4y - 1)(y + 2) = 0$

$\therefore \quad y = \frac{1}{4}$ or $y = -2$

But $y > 0$, $\quad \therefore y = \frac{1}{4} = 0.25$

12 **a** $h(3) = -4.9(3)^2 + 30(3) + 5$

$$= -4.9(9) + 90 + 5$$
$$= -44.1 + 95 = 50.9$$

The drop will be at a height of 50.9 m after 3 seconds.

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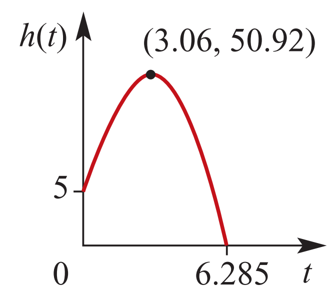
b $$-4.9t^2 + 30t + 5 = 5$$
$$\therefore \quad -4.9t^2 + 30t = 0$$
$$\therefore \quad t(30 - 4.9t) = 0$$
$$\therefore \quad t = 0 \quad \text{or} \quad 30 - 4.9t = 0$$
$$\therefore \quad 4.9t = 30$$
$$\therefore \quad t \approx 6.12$$
The drop will be back at the spout height after approximately 6.12 seconds.

c Turning point at
$$x = \frac{-b}{2a} = \frac{-30}{2(-4.9)} = \frac{300}{98} = \frac{150}{49} \approx 3.06$$



$$h\left(\frac{150}{49}\right) = -4.9\left(\frac{150}{49}\right)^2 + 30\left(\frac{150}{49}\right) + 5 = \frac{2495}{49} \approx 50.92$$

d When $h(t) = 0$,
$$t = \frac{-30 \pm \sqrt{(30)^2 - 4(-4.9)(5)}}{2(-4.9)} = \frac{-30 \pm \sqrt{900 + 98}}{-9.8} \approx \frac{-30 \pm 31.59}{-9.8} \approx \frac{-61.59}{-9.8} \text{ or } \frac{1.5.9}{-9.8} \approx 6.285 \text{ or } -0.162$$

But as $t \geq 0$ $\quad t = 6.285$

It will take a drop of water 6.285 seconds to hit the ground.

13 a height = 7 cm, breadth = x cm, length = $(x + 5)$ cm

b $V = 7x(x + 5)$
$= 7x^2 + 35x$

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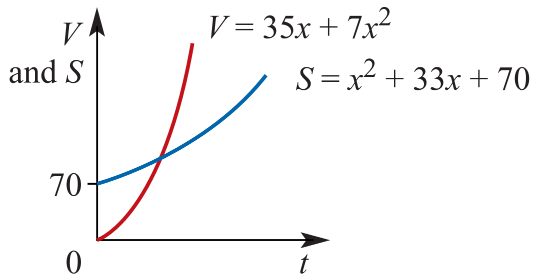
c $S = 7x + 7x + 7(x+5) + 7(x+5) + x(x+5)$

$= 14x + 14x + 70 + x^2 + 5x$

$= x^2 + 33x + 70$

d



e Let $S = V$, $\therefore x^2 + 33x + 70 = 7x^2 + 35x$

$\therefore \quad 6x^2 + 2x - 70 = 0$

$$\therefore \quad x = \frac{-2 \pm \sqrt{4 - 4(6)(-70)}}{12}$$

$$= \frac{-2 \pm \sqrt{1684}}{12}$$

$$= \frac{-2 \pm 41.0366}{12}$$

$$= -3.59, 3.25$$

But $x \geq 0$

$\therefore V = S$ when $x = 3.25$, correct to 2 decimal places.

f Let $S = 500$, $\therefore x^2 + 33x + 70 = 500$

$\therefore \quad x^2 + 33x - 430 = 0$

$\therefore \quad (x - 10)(x + 43) = 0$

$\therefore \quad x = 10$ or $x = -43$

But $x \geq 0$, $\quad \therefore x = 10$

14 a Midpoint of $AC = \left(\dfrac{1+7}{2}, \dfrac{3+7}{2}\right) = (4, 5)$

Gradient of $AC = \dfrac{7-3}{7-1} = \dfrac{4}{6} = \dfrac{2}{3}$

Gradient of a line perpendicular to $AC = \dfrac{-1}{\frac{2}{3}} = \dfrac{-3}{2}$

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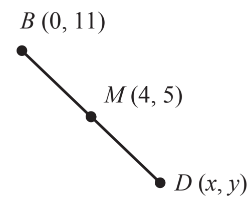
Equation of perpendicular bisector of AC

$$y - 5 = \frac{-3}{2}(x - 4) = \frac{-3x}{2} + 6$$

$$\therefore\ 2y + 3x = 22$$

b **i** When $x = 0, y = 11$

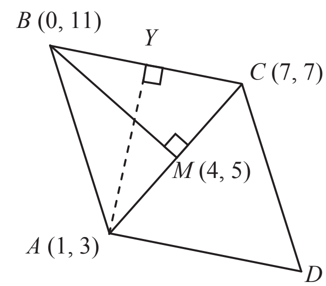
$\therefore B$ has coordinates $(0, 11)$



ii $x = 2 \times 4 - 0 = 8$

$y = 2 \times 5 - 11 = -1$

D has coordinates $(8, -1)$



c Area = length $AC \times$ length BM

$$= \sqrt{6^2 + 4^2} \times \sqrt{4^2 + 6^2}$$

$$= \sqrt{52} \times \sqrt{52}$$

$$= 52 \text{ units}^2$$

d Area $\Delta ABC = \frac{1}{2}$ area of the rhombus

$$= 26 \text{ units}^2$$

Area $\Delta ABC = \frac{1}{2}BC \times AY$

So $\quad AY = \dfrac{2 \times 2b}{BC}$

$$= \frac{52}{\sqrt{7^2 + 4^2}}$$

$$= \frac{52}{\sqrt{65}}$$

$$\approx 6.45$$

The length of AY (the perpendicular distance of A from BC) ≈ 6.45 units.

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15 a Let x hours be the time for the first journey, $\therefore V = \dfrac{300}{x}$.

The time for the second journey = $(x-2)$ hours

$$\begin{aligned}
\therefore\quad V + 5 &= \frac{300}{x-2}\\
\therefore\quad \frac{300}{x} + 5 &= \frac{300}{x-2}\\
\therefore\quad \left(\frac{300}{x} + 5\right)(x-2) &= 300\\
\therefore\quad 300 + 5x - \frac{600}{x} - 10 &= 300\\
\therefore\quad 5x^2 - 600 - 10x &= 0\\
\therefore\quad 5x^2 - 10x - 600 &= 0\\
\therefore\quad 5(x^2 - 2x - 120) &= 0\\
\therefore\quad (x-12)(x+10) &= 0\\
\therefore\quad x = 12 \text{ or } x &= -10\\
\text{But } x \ge 0, \quad \therefore x &= 12\\
\therefore\quad V = \frac{300}{12} &= 25
\end{aligned}$$

The speed of the train travelling at the slower speed is 25 km/h.

b Let t_A minutes be the time it takes tap A to fill the tank, and t_B minutes be the time it takes tap B to fill the tank. $t_B = t_A + 15$
When the taps are running together, it takes $33\frac{1}{3}$ minutes to fill the tank.
Let R_A units/min be the rate of flow of tap A, and R_B units/min be the rate of flow of tap B.

$$\begin{aligned}
\text{Volume to be filled} &= \frac{100}{3}R_A + \frac{100}{3}R_B\\
R_A &= \frac{\text{volume to be filled}}{t_A}\\
R_B &= \frac{\text{volume to be filled}}{t_B}
\end{aligned}$$

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Let V be the volume to be filled.

$$V = \frac{100}{3} \times \frac{V}{t_A} + \frac{100}{3} \times \frac{V}{t_B}$$

$$\therefore \quad \frac{3}{100} = \frac{1}{t_A} + \frac{1}{t_A + 15}$$

$$\therefore \quad 3(t_A + 15)t_A = 100(t_A + 15) + 100t_A$$

$$\therefore \quad 3t_A^2 + 45t_A = 200t_A + 1500$$

i.e. $3t_A^2 - 155t_A - 1500 = 0$

$$t_A = \frac{155 \pm \sqrt{42025}}{6}$$

$$\therefore \quad = \frac{155 \pm 205}{6} = 60 \text{ or } \frac{-25}{3}$$

Tap A takes 60 minutes to fill the tank by itself.
Tap B takes 75 minutes to fill the tank by itself.

c Let x cm be the length of a side of a square tile.
Let A be the floor area of the hall.

Then $\quad A = 200x^2$

and $\quad A = 128(x + 1)^2$

$$\therefore \quad 200x^2 = 128x^2 + 256x + 128$$

$$\therefore \; 72x^2 - 256x - 128 = 0$$

$$\therefore \; 8(9x^2 - 32x - 16) = 0$$

$$\therefore \; 8(9x + 4)(x - 4) = 0$$

$$\therefore \quad x = \frac{-4}{9} \text{ or } x = 4$$

But $x \geq 0$, $\quad \therefore x = 4$

The smaller tiles are $(4 \times 4)\,\text{cm}^2$ and the larger tiles are $(5 \times 5)\,\text{cm}^2$.

16 a $\quad 4(x + 2x + h) = 400$

$$\therefore \quad 3x + h = 100$$

$$\therefore \quad h = 100 - 3x$$

b $V = x \times 2x \times h$

$$= 2x^2(100 - 3x)$$

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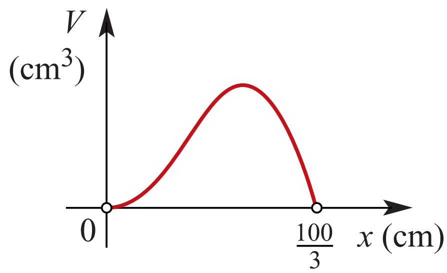
c When $V = 0$, $2x^2(100 - 3x) = 0$

$\therefore$ $x = 0$ or $x = \dfrac{100}{3}$

Now $V > 0$, $\therefore 0 < x < \dfrac{100}{3}$

d



e **i** On a CAS calculator, set ***f*1=2x^2(100-3x)** and ***f*2=30 000**. The points of intersection are (18.142, 30 000) and (25.852, 30 000), correct to 3 decimal places. Thus volume is 30 000 cm^3 when $x = 18.142$ or $x = 25.852$, correct to 3 decimal places.

ii Repeat **e i**, using ***f*2 = 20 000**. Volume is 20 000 cm^3 when $x = 12.715$ or $x = 29.504$, correct to 3 decimal places.

f **TI:** Press **Menu→6:Anaylze Graph→3:Maximum**
CP: Tap **Analysis** → **G-Solve** → Max

g **i**

$$\begin{aligned} S &= 2(x \times 2x + x \times h + 2x \times h) \\ &= 2(2x^2 + x(100 - 3x) + 2x(100 - 3x)) \\ &= 2(2x^2 + 100x - 3x^2 + 200x - 6x^2) \\ &= 2(300x - 7x^2) \\ &= 600x - 14x^2 \end{aligned}$$

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ii On a CAS calculator, sketch $\boldsymbol{f1 = 600x - 14x\text{^}2}$.
TI: Press **Menu** → 6: **Anaylze Graph 3:Maximum**
CP: Tap **Analysis** → **G-Solve→Max**

h Sketch $\boldsymbol{f1{=}600x - 14x\text{^}2}$ and $\boldsymbol{f2{=}2x\text{^}2(100 - 3x)}$ on a CAS calculator. The points of intersection are (3.068, 1708.802) and (32.599, 4681.642). Therefore $S = V$ when $x \approx 3.068$ or $x \approx 32.599$.

17 a **TI:** Press **Menu →1:Actions→1:Define** then type $\boldsymbol{f(x){=}a \times x^3{+}b \times x^2{+}c \times x{+}d}$ followed by ENTER.
Now type the following then press ENTER
solve ($\boldsymbol{f(0){=}0}$ **and** $\boldsymbol{f(100){=}33}$ **and** $\boldsymbol{f(50){=}57}$ **and** $\boldsymbol{f(150){=}-15, \{a, b, c, d\})}$
CP: Tap **Action** → **Command** → **Define** then type $\boldsymbol{f(x){=}a \times x^3{+}b \times x^2{+}c \times x{+}d}$ followed by EXE.
Now type the following then press EXE
solve $\boldsymbol{(\{f(0){=}0, f(100){=}33, f(50){=}57, f(150){=}-15\},\{a, b, c, d\})}$
The screen gives $a = \dfrac{19}{250\,000} = 7.6 \times 10^{-5}$, $b = -\dfrac{69}{2500} = 0.0276$, $c = \dfrac{233}{100} = 2.33$ and $d = 0$.
Therefore the function which passes through the given points is
$y = (7.6 \times 10^{-5})x^3 - 0.0276x^2 + 2.33x$.

b $y = (7.6 \times 10^{-5})x^3 - 0.0276x^2 + 2.33x + 5$

c On a CAS calculator, sketch $\boldsymbol{f1{=}19/250\,000x\text{^}3 - 69/2500x\text{^}2{+}2.33x}$.
TI: Press **Menu→6:Anaylze Graph→3:Maximum**
CP: Tap **Analysis** → **G-Solve** → **Max**
which yields (54.461038, 57.31899). The largest deviation from the x-axis is 57.31 metres, perpendicular to the x-axis and correct to 2 decimal places.

18 a The equation of BC is $y = \dfrac{3}{4}x - 4$

b The equation of AD is
$$y - 6 = -\frac{4}{3}(x - 5) = -\frac{4}{3}x + \frac{20}{3}$$
$$\therefore \quad y = -\frac{4}{3}x + \frac{38}{3} \text{ or } 3y + 4x = 38$$

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c D is on the lines $y = \frac{3}{4}x - 4$ and $3y + 4x = 38$

Substituting $y = \frac{3}{4}x - 4$ into $3y + 4x = 38$ gives

$$3\left(\frac{3}{4}x - 4\right) + 4x = 38$$

$$\therefore \quad \frac{9}{4}x - 12 + 4x = 38$$

$$\therefore \quad \frac{25}{4}x = 50$$

$$\therefore \quad 25x = 200$$

$$\therefore \quad x = 8$$

$$\therefore \quad y = \frac{3}{4}(8) - 4$$

$$= 6 - 4$$

$$= 2$$

The coordinates of D are (8, 2).

d Length of $AD = \sqrt{(8-5)^2 + (6-2)^2}$

$$= \sqrt{25}$$

$$= 5\text{units}$$

e Area of $\Delta ABC = 2 \times$ area of ΔABD

$$= 2 \times \frac{1}{2} \times BD \times AD$$

$$= \sqrt{(8-0)^2 + (2-(-4))^2} \times 5$$

$$= 5\sqrt{64 + 36}$$

$$= 5\sqrt{100}$$

$$= 50 \text{ square units}$$

19 a i $2y + 6x + 4x = 500$

$$\therefore \quad y + 5x = 250$$

$$\therefore \quad y = 5(50 - x)$$

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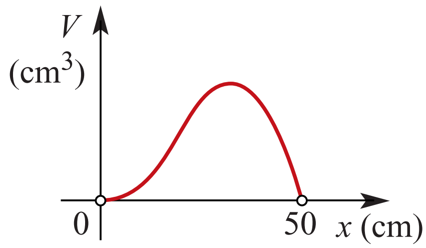
ii $V = x \times x \times y$

$= x^2 \times 5(50 - x)$

$= 5x^2(50 - x)$

b



c Domain $= (0, 50)$

d Sketch ***f*1=5*x*^2(50 − *x*)** and ***f*2=25 000** on a CAS calculator. The points of intersection are (11.378 052, 25 000) and (47.812 838, 25 000).
Therefore $V = 25000$ for $x = 11.38$ and $x = 47.81$, correct to 2 decimal places.

e **TI:** Press **Menu** → **6:Anaylze Graph**→**3:Maximum**
CP: Tap **Analysis** → **G-Solve** → **Max**
to yield the coordinates (33.333 331, 92 592.593). Therefore the maximum volume is 92 592.59 cm^3 when $x = 33.33$, correct to 2 decimal places.
When $x = 33.333\ldots$, $y = 5(50 - 33.333\ldots) \approx 83.33$.

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Chapter 9 – Probability

Solutions to Exercise 9A

1 Toss of a coin: sample space = $\{H, T\}$

2 Die rolled: sample space
$= \{1, 2, 3, 4, 5, 6\}$

3 **a** 52 cards

b 4 suits

c Spades, hearts, diamonds, clubs

d Hearts, diamonds = red;
spades, clubs = black

e 13 cards in each suit.

f 'Picture cards' are Jack, Queen, King and Ace

g 4 aces

h 16 'picture cards'

4 **a** $\{0, 1, 2, 3, 4, 5\}$

b $\{0, 1, 2, 3, 4, 5, 6\}$

c $\{0, 1, 2, 3\}$

5 **a** $\{0, 1, 2, 3, 4, 5 \ldots\}$

b $\{0, 1, 2, 3, 4, 5 \ldots 41\}$

c $\{1, 2, 3, 4, 5 \ldots\}$

6 **a** 'An even number' in die roll
$= \{2, 4, 6\}$

b 'More than two female students'
$= \{FFF\}$

c 'More than four aces' = $\{\}$ or $\varnothing$

7 $\varepsilon = \{1, 2, 3, \ldots, 20\}$, $n(\varepsilon) = 20$

a Let A be the event the number is divisible by 2.
$A = \{2, 4, \ldots, 20\}$, $n(A) = 10$,
$$\Pr(A) = \frac{n(A)}{n(\varepsilon)} = \frac{10}{20} = \frac{1}{2}$$

b Let B be the event the number is divisible by 3.
$B = \{3, 6, \ldots, 18\}$, $n(A) = 6$,
$$\Pr(B) = \frac{n(B)}{n(\varepsilon)} = \frac{6}{20} = \frac{3}{10}$$

c Let C be the event the number is divisible by both 2 and 3.
$C = \{6, 12, 18\}$, $n(C) = 3$,
$$\Pr(C) = \frac{n(C)}{n(\varepsilon)} = \frac{3}{20}$$

8 $\varepsilon = \{1, 2, 3, \ldots, 15\}$, $n(\varepsilon) = 15$

a Let A be the event the number is less than 5.
$A = \{1, 2, 3, 4\}$, $n(A) = 4$,
$$\Pr(A) = \frac{n(A)}{n(\varepsilon)} = \frac{4}{15}$$

b Let B be the event the number is greater than or equal to 6.
$B = \{6, 7, \ldots, 15\}$, $n(B) = 10$,
$$\Pr(B) = \frac{n(B)}{n(\varepsilon)} = \frac{10}{15} = \frac{2}{3}$$

c Let C be the event the number is a

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number from 5 to 8 inclusive.
$C = \{5, 6, 7, 8\}$, $n(C) = 4$,
$\Pr(C) = \dfrac{n(C)}{n(\varepsilon)} = \dfrac{4}{15}$

9 **a** 13 clubs: $\Pr(\clubsuit) = \dfrac{13}{52} = \dfrac{1}{4}$

b 26 red cards: $\Pr(\text{red}) = \dfrac{26}{52} = \dfrac{1}{2}$

c 16 picture cards: $\Pr(\text{picture}) = \dfrac{16}{52} = \dfrac{4}{13}$

d a red picture card Pr(red picture) = $\dfrac{8}{52} = \dfrac{2}{13}$

10 **a** 36 cards < 10: $\Pr(< 10) = \dfrac{36}{52} = \dfrac{9}{13}$

b 40 cards ≤ 10: $\Pr(\leq 10) = \dfrac{40}{52} = \dfrac{10}{13}$

c Even number $= \{2, 4, 6, 8, 10\}$ so 20 evens: $\Pr(\text{even}) = \dfrac{20}{52} = \dfrac{5}{13}$

d 4 aces: $\Pr(\text{ace}) = \dfrac{4}{52} = \dfrac{1}{13}$

11 **a** $\Pr(\text{29 November}) = \dfrac{1}{365}$

b $\Pr(\text{November}) = \dfrac{30}{365} = \dfrac{6}{73}$

c 30 days between 15 January and 15 February, not including either day:
$\therefore \Pr = \dfrac{6}{73}$

d 90 (non-leap) days in the first three months of the year: $\therefore \Pr = \dfrac{90}{365} = \dfrac{18}{73}$

12 $\varepsilon = \{A_1, U, S, T, R, A_2, L, I, A_3\}$, $n(\varepsilon) = 9$

a $\Pr(\{T\}) = \dfrac{1}{9}$

b Pr(an A is drawn) =
$\Pr(\{A_1, A_2, A_3\}) = \dfrac{3}{9} = \dfrac{1}{3}$

c Let V be the event a vowel is drawn
$V = \{A_1, A_2, A_3, U, I\}$, $n(V) = 5$
$\Pr(V) = \dfrac{5}{9}$

d Let C be the event a consonant is drawn
$C = \{S, T, R, L\}$, $n(C) = 4$
$\Pr(C) = \dfrac{4}{9}$

13 $\Pr(1) + \Pr(2) + \Pr(3) + \Pr(5) + \Pr(6) + \Pr(4) = 1$

$$\therefore \frac{1}{12} + \frac{1}{6} + \frac{1}{8} + \frac{1}{6} + \frac{1}{8} + \Pr(4) = 1$$

$$\frac{2 + 4 + 3 + 4 + 3}{24} + \Pr(4) = 1$$

$$\therefore \Pr(4) = 1 - \frac{16}{24} = 1 - \frac{2}{3} = \frac{1}{3}$$

14 $\Pr(1) = 0.2, \Pr(3) = 0.1, \Pr(4) = 0.3$
$\Pr(1) + \Pr(3) + \Pr(4) = 0.6$
$\therefore \Pr(2) = 1 - 0.6 = 0.4$

15 **a** $\Pr(1) = \dfrac{1}{3}$

b $\Pr(1) = \dfrac{1}{8}$

c $\Pr(1) = \dfrac{1}{4}$

16 $\varepsilon = \{M, T, W, Th, F, Sa, Su\}$ $n(\varepsilon) = 7$

a $\Pr(\text{Born on Wednesday}) = \dfrac{1}{7}$

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b Pr(Born on a weekend) =
Pr({Sa,Su}) = $\frac{2}{7}$
Pr(Not born on a weekend)= $1 - \frac{2}{7} = \frac{5}{7}$

17 $n(\varepsilon) = 52$

a $n(\text{Club}) = 13$
$\Pr(\text{Club}) = \frac{13}{52} = \frac{1}{4}$
$\Pr(\text{Not Club}) = 1 - \frac{1}{4} = \frac{3}{4}$

b $n(\text{Red}) = 26$
$\Pr(\text{Red}) = \frac{26}{52} = \frac{1}{2}$
$\Pr(\text{Not Red}) = 1 - \frac{1}{2} = \frac{1}{2}$

c Picture cards are Kings, Queens and Jacks
$n(\text{Picture Card}) = 12$
$\Pr(\text{Picture Card}) = \frac{12}{52} = \frac{3}{13}$
$\Pr(\text{Not Red}) = 1 - \frac{3}{13} = \frac{10}{13}$

d $n(\text{Red Picture}) = 6$
$\Pr(\text{Red Picture}) = \frac{6}{52} = \frac{3}{26}$
$\Pr(\text{Not Red}) = 1 - \frac{3}{26} = \frac{23}{26}$

18 $\varepsilon = \{1, 2, 3, 4\}$
$\Pr(1) = \Pr(2) = \Pr(3) = x$ and
$\Pr(4) = 2x$.
$\therefore x + x + x + 2x = 1$
$\therefore x = \frac{1}{5}$
$\therefore \Pr(1) = \Pr(2) = \Pr(3) = \frac{1}{5}$ and
$\Pr(4) = \frac{2}{5}$

19 $\varepsilon = \{1, 2, 3, 4, 5, 6\}$

a $\Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = x$,
$\Pr(6) = 2x$ and $\Pr(1) = \frac{x}{2}$.
$\therefore x + x + x + x + 2x + \frac{x}{2} = 1$
$\therefore \frac{13x}{2} = 1 \therefore x = \frac{2}{13}$
$\therefore \Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = \frac{2}{13}$
$\Pr(6) = \frac{4}{13}$ and $\Pr(1) = \frac{1}{13}$

b $\frac{9}{13}$

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Solutions to Exercise 9B

1 **a** Pr(head) $= \dfrac{34}{100} = \dfrac{17}{50} = 0.34$

b Pr(ten) $= \dfrac{20}{200} = \dfrac{1}{10} = 0.10$

c Pr(two heads) $= \dfrac{40}{150} = \dfrac{4}{15}$

d Pr(three sixes) $= \dfrac{1}{200}$ or 0.005

2 **a** 20 trials is far too few to obtain reliable data.

b Pr(two heads) $= \dfrac{1}{4}$, Pr(one head) $= \dfrac{1}{2}$, Pr(no heads) $= \dfrac{1}{4}$

c Results may resemble **b**, but could be anything with such a small sample.

d 100 trials is certainly better. For example, with 95% confidence limits, the number of (H, H) results over 20 trials would be between 1 and 9. Over 100 trials we would expect between 16 and 34.

e To find the probabilities exactly would require an infinite number of trials.

3 Die 1 shows Pr(6) $= \dfrac{78}{500} = 0.156$

Die 2 shows Pr(6) $= \dfrac{102}{700} = 0.146$

Die 1 has a higher observed probability of throwing a 6.

4 Total number of balls = 400; 340 red and 60 black.

Proportion of red $= \dfrac{340}{400} = \dfrac{17}{20} = 0.85$

b Proportion of red in sample =

$= \dfrac{48}{60} = \dfrac{4}{5} = 0.8$

c Proportion of red in sample =

$= \dfrac{54}{60} = \dfrac{9}{10} = 0.9$

d Expected number of red balls$= 0.85 \times 60 = 51$

5 Estimate of probability

$= \dfrac{890}{2000} = \dfrac{89}{200} = 0.445$

6 **a** Area of blue section

$= \dfrac{\pi(1)^2}{4} = \dfrac{\pi}{4} \approx 0.7855$

Area of square$= 1 \times 1 = 1$.

Proportion of square that is blue$= \dfrac{\pi}{4} \approx 0.7855$

b Probability of hitting the blue region$= \dfrac{\pi}{4} \approx 0.7855$

7 Area of board$= \pi(14)^2 = 196\pi$

$$\begin{aligned}\text{Area of shaded region } &= \pi(14)^2 - \pi(7)^2\\ &= 196\pi - 49\pi\\ &= 147\pi\end{aligned}$$

Probabilty that the dart will hit the shaded area$= \dfrac{147}{196} = \dfrac{3}{4}$

8 **a** Pr(Red section) $= \dfrac{120}{360} = \dfrac{1}{3}$

b Pr(Yellow section) $= \dfrac{60}{360} = \dfrac{1}{6}$

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c Pr(Not Yellow section) $= 1 - \frac{1}{6} = \frac{5}{6}$

9 Area of square = 1 m^2.
Area of circle $= \pi \times 0.4^2 = 0.16\pi$

a Probability of hitting the shaded part $= 0.16\pi$

b Probability of hitting the unshaded part $= 0.84\pi$

10 **a** **i** Area of square$= x^2$

ii Area of larger circle
$= \pi(\frac{x}{2})^2 = \frac{1}{4}\pi x^2$

iii Area of smaller circle
$= \pi(\frac{x}{4})^2 = \frac{1}{16}\pi x^2$

b **i** Probability of landing inside the smaller circle$= \frac{\frac{1}{16}\pi x^2}{x^2} = \frac{\pi}{16}$

ii Probability of landing inside the smaller circle$= \frac{(\frac{1}{4} - \frac{1}{16})\pi x^2}{x^2} = \frac{3\pi}{16}$

iii Probability of land-
ing in the outer shaded region$= \frac{x^2 - \frac{1}{4}\pi x^2}{x^2} = 1 - \frac{\pi}{4}$

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Solutions to Exercise 9C

1 $\varepsilon = \{HH, HT, TH, TT\}$

a Pr(No heads) = Pr($\{TT\}$) = $\frac{1}{4}$.

b Pr(More than one tail) = Pr($\{TT\}$) = $\frac{1}{4}$.

2 **a** Pr(First toss is a head) = $\frac{1}{2}$

b Pr(Second toss is a head) = $\frac{1}{2}$

c Pr(Both tosses are heads) = $\frac{1}{4}$

3 Sample space = $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
There is only 1 way of getting 2 or 12, 2 ways of getting 3 or 11, 3 ways of getting 4 or 10 etc.

a
$$\begin{aligned}\Pr(\text{even}) &= \Pr(2) + \Pr(4) + \Pr(6) \\ &\quad + \Pr(8) + \Pr(10) + \Pr(12) \\ &= \frac{1+3+5+5+3+1}{36} \\ &= \frac{1}{2}\end{aligned}$$

b $\Pr(3) = \frac{2}{36} = \frac{1}{18}$

c
$$\begin{aligned}\Pr(<6) &= \Pr(2) + \Pr(3) \\ &\quad + \Pr(4) + \Pr(5) \\ &= \frac{1+2+3+4}{36} \\ &= \frac{10}{36} = \frac{5}{18}\end{aligned}$$

4 Sample space = $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

a $\Pr(10) = \frac{3}{36} = \frac{1}{12}$

b
$$\begin{aligned}\Pr(\text{odd}) &= \Pr(3) + \Pr(5) + \Pr(7) \\ &\quad + \Pr(9) + \Pr(11) \\ &= \frac{2+4+6+4+2}{36} \\ &= \frac{1}{2}\end{aligned}$$

c
$$\begin{aligned}\Pr(\leq 7) &= \frac{1+2+3+4+5+6}{36} \\ &= \frac{21}{36} = \frac{7}{12}\end{aligned}$$

5 $\varepsilon = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

a Pr(exactly one tail) = Pr($\{HHT, HTH, THH\}$) = $\frac{3}{8}$

b Pr(exactly two tails) = Pr($\{HTT, TTH, THT\}$) = $\frac{3}{8}$

c Pr(exactly three tails) = Pr($\{TTT\}$) = $\frac{1}{8}$

d Pr(no tails) = Pr($\{HHH\}$) = $\frac{1}{8}$

6 $\varepsilon = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

a Pr(the third toss is a head) = Pr($\{HHH, HTH, THH, TTH\}$) = $\frac{1}{2}$

b
Pr(second and third tosses are heads) =

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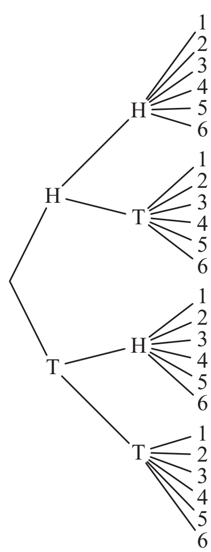
$\Pr(\{HHH, THH\}) = \frac{1}{4}$

c Pr(at least one head and one tail) = $\Pr(\{HHT, HTH, THH, TTH, THT, HTT, \}) = \frac{3}{4}$

7 12 equally likely outcomes:

$$\Pr(\text{even}, H) = \Pr(2, H) + \Pr(4, H) + \Pr(6, H) = \frac{3}{12} = \frac{1}{4}$$

8 a



b **i**

$$\Pr(\text{2 heads and a 6}) = \Pr(\{(H, H, 6)\} = \frac{1}{24}$$

ii

Pr(1 head, 1 tail and an even number)

$= \Pr(\{(H, T, 6), (H, T, 4), (H, T, 2)$

$, (T, H, 6), (T, H, 4), (T, H, 2)\}$

$= \frac{6}{24}$

$= \frac{1}{4}$

iii Pr(2 tails and an odd number)

$= \Pr(\{(T, T, 1), (T, T, 3), (T, T, 5)\}$

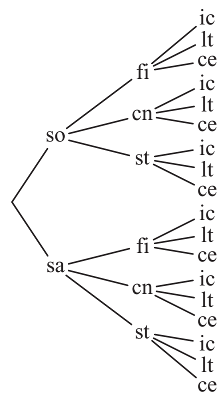
$= \frac{3}{24}$

$= \frac{1}{8}$

iv Pr(an odd number on the die)

$= \frac{1}{2}$

9 a



b **i**

$$\text{Pr(soup, fish and lemon tart)} = \Pr(\{(so, fi, it)\} = \frac{1}{18}$$

ii Pr(fish)

$= \frac{1}{3}$

iii

Pr(salad and chicken)

$= \Pr(\{(sa, c, lt), (sa, c, ic)), (sa, c, ce))\}$

$= \frac{3}{18}$

$= \frac{1}{6}$

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iv Pr(no lemon tart)

$= 1 - \frac{1}{3}$

$= \frac{2}{3}$

c This increases the number of choices for the entree to 3 and the dessert 4. There are $3 \times 3 \times 4 = 36$ choices.

d **i** Pr(soup, fish and lemon tart)

$= \Pr(\{(so, fi, it)\}$

$= \frac{1}{36}$

ii Pr(all courses)

$= \frac{1}{2}$

iii Pr(only two courses)

$= \frac{12}{36}$

$= \frac{1}{3}$

iv Pr(only the main courses)

$= \frac{3}{36}$

$= \frac{1}{12}$

10 **a** $(1, 1)(2, 1)(3, 1)(4, 1)(5, 1)$
$(1, 2)(2, 2)(3, 2)(4, 2)(5, 2)$
$(1, 3)(2, 3)(3, 3)(4, 3)(5, 3)$
$(1, 4)(2, 4)(3, 4)(4, 4)(5, 4)$
$(1, 5)(2, 5)(3, 5)(4, 5)(5, 5)$

b **i** $\Pr(5) = \frac{4}{25}$

ii Pr(different) = 1 − Pr(same) = $1 - \frac{1}{5} = \frac{4}{5}$

iii Pr(second number two more than first number) $= \frac{3}{25}$

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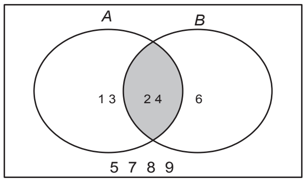
Solutions to Exercise 9D

1 $\in\ = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
$A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6\}$.



a $A \cup B = \{1, 2, 3, 4, 6\}$

b $A \cap B = \{2, 4\}$

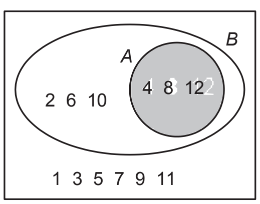
c $A' = \{5, 6, 7, 8, 9, 10\}$

d $A \cap B' = \{1, 3\}$

e $(A \cap B)' = \{1, 3, 5, 6, 7, 8, 9, 10\}$

f $(A \cup B)' = \{5, 7, 8, 9, 10\}$

2 $\in\ = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
$A = \{\text{multiples of four}\}$
$B = \{\text{even numbers}\}$



a $A' = \{1, 2, 3, 5, 6, 7, 9, 10, 11\}$

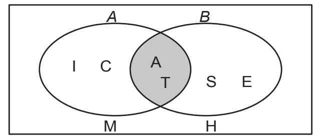
b $B' = \{1, 3, 5, 7, 9, 11\}$

c $A \cup B = \{2, 4, 6, 8, 10, 12\}$

d $(A \cup B)' = B' = \{1, 3, 5, 7, 9, 11\}$

e $A' \cap B' = B' = \{1, 3, 5, 7, 9, 11\}$

3 $\in\ = \{\text{MATHEICS}\}$, $A = \{\text{ATIC}\}$, $B = \{\text{TASE}\}$



a $A' = \{\text{E, H, M, S,}\}$

b $B' = \{\text{C, H, I, M}\}$

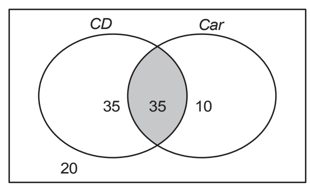
c $A \cup B = \{\text{A, C, E, I, S, T}\}$

d $(A \cup B)' = \{\text{H, M}\}$

e $A' \cup B' = \{\text{C, E, H, I, M, S}\}$

f $A' \cap B' = \{\text{H, M}\}$

4



$\varepsilon = 100$ students

a 20 students own neither a car norsmart phone .

b 45 students own either but not both.

5 $\varepsilon = \{1, 2, 3, 4, 5, 6\}$;
$A = \{2, 4, 6\}$, $B = \{3\}$

a $(A \cup B) = \{2, 3, 4, 6\}$
$\therefore \Pr(A \cup B) = \dfrac{2}{3}$

b $(A \cap B) = \{\}$
$\therefore \Pr(A \cap B) = 0$

c $A' = \{1, 3, 5\}$
$\therefore \Pr(A') = \dfrac{1}{2}$

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d $B' = \{1, 2, 4, 5, 6\} \therefore \Pr(B') = \frac{5}{6}$

6 $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$;
$A = \{2, 4, 6, 8, 10, 12\},\ B = \{3, 6, 9, 12\}$

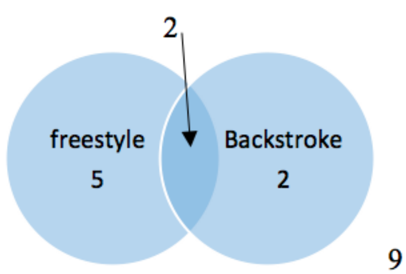
a $\Pr(A) = \frac{6}{12} = \frac{1}{2}$

b $\Pr(B) = \frac{4}{12} = \frac{1}{3}$

c $\{A \cap B\} = \{6, 12\}, \therefore \Pr(A \cap B) = \frac{1}{6}$

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$= \frac{2}{3}$$

7



a $\Pr(\text{Swims freestyle}) = \frac{7}{18}$

b $\Pr(\text{Swims backstroke}) = \frac{4}{18} = \frac{1}{9}$

c $\Pr(\text{Swims freestyle and backstroke}) = \frac{2}{18} = \frac{2}{9}$

d $\Pr(\text{is on the swimming team}) = \frac{9}{18} = \frac{1}{2}$

8 $A = \{1, 2, 3, 4, 6, 12\}$ and $B = \{2, 3, 5, 7\}$

a $\Pr(A) = \frac{6}{20} = \frac{3}{5}$

b $\Pr(B) = \frac{4}{20} = \frac{1}{5}$

c $\Pr(A \cap B) = \frac{2}{20} = \frac{1}{10}$

d $\Pr(A \cup B) = \frac{8}{20} = \frac{2}{5}$

9 $\Pr(A) = 0.5, \Pr(B) = 0.4$, and
$\Pr(A \cap B) = 0.2$.
$\Pr(A \cup B) = 0.5 + 0.4 - 0.2 = 0.7$

10 $\Pr(A) = 0.35, \Pr(B) = 0.24$, and
$\Pr(A \cap B) = 0.12$.
$\Pr(A \cup B) = 0.35 + 0.24 - 0.12 = 0.47$

11 $\Pr(A) = 0.28, \Pr(B) = 0.45$, and $A \subset B$

a $\Pr(A \cap B) = \Pr(B) = 0.28$

b

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$= 0.28 + 0.45 - 0.28$$
$$= 0.45$$

12 $\Pr(A) = 0.58, \Pr(B) = 0.45$, and $B \subset A$

a $\Pr(A \cap B) = \Pr(B) = 0.45$

b

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$= 0.45 + 0.58 - 0.45$$
$$= 0.58$$

13 $\Pr(A) = 0.3, \Pr(B) = 0.4$, and $A \cap B = \varnothing$

a $\Pr(A \cap B) = 0$

b

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$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ &= 0.3 + 0.4 - 0\\ &= 0.7\end{aligned}$$

14 $\Pr(A) = 0.08, \Pr(B) = 0.15$, and $A \cap B = \varnothing$

a $\Pr(A \cap B) = 0$

b

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ &= 0.08 + 0.15 - 0\\ &= 0.23\end{aligned}$$

15 $\Pr(A) = 0.3, \Pr(B) = 0.4$, and $A \cup B = 0.5$

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ 0.5 &= 0.3 + 0.4 - \Pr(A \cap B)\end{aligned}$$

$\therefore \Pr(A \cap B) = 0.2$

16 $\Pr(A) = 0.24, \Pr(B) = 0.44$, and $A \cup B = 0.63$

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ 0.63 &= 0.24 + 0.44 - \Pr(A \cap B)\end{aligned}$$

$\therefore \Pr(A \cap B) = 0.05$

17 $\Pr(A) = 0.3, \Pr(B) = 0.4$, and $A \cap B' = 0.2$

$$\begin{aligned}\Pr(A \cup B') &= \Pr(A) + \Pr(B') - \Pr(A \cap B')\\ &= 0.3 + 0.6 - 0.2\\ &= 0.7\end{aligned}$$

18 Pr(Soccer) = 0.18, Pr(Tennis) = 0.25 and Pr(Soccer and Tennis) = 0.11

Pr(Soccer **or** Tennis) = 0.18 + 0.25 − 0.11
= 0.32

19 Pr(Chinese) = 0.22, Pr(French) = 0.35 and Pr(Chinese and French) = 0.14

a

Pr(Chinese **or** French) = 0.22 + 0.35 − 0.14
= 0.43

b Probability of exactly oneof these languages
$= \Pr(C \cup F) - \Pr(C \cap F) = 0.29$

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Solutions to Exercise 9E

1 $\Pr(A) = 0.6,\ \Pr(A \cap B) = 0.4,$
$\Pr(A' \cap B) = 0.1$

	B	B'	
A	$\Pr(A \cap B) = 0.4$	$\Pr(A \cap B') = 0.2$	$\Pr(A) = 0.6$
A'	$\Pr(A' \cap B) = 0.1$	$\Pr(A' \cap B') = 0.3$	$\Pr(A) = 0.4$
	$\Pr(B) = 0.5$	$\Pr(B') = 0.5$	1

a $\Pr(A \cap B') = 0.2$

b $\Pr(B) = 0.5$

c $\Pr(A' \cap B') = 0.3$

d $\Pr(A \cup B) = 1 - 0.3 = 0.7$

2 $\Pr(A') = 0.25,\ \Pr(A' \cap B) =$
$0.12, \Pr(B) = 0.52:$

	B	B'	
A	$\Pr(A \cap B) = 0.4$	$\Pr(A \cap B') = 0.35$	$\Pr(A) = 0.75$
A'	$\Pr(A' \cap B) = 0.12$	$\Pr(A' \cap B') = 0.13$	$\Pr(A') = 0.25$
	$\Pr(B) = 0.52$	$\Pr(B') = 0.48$	1

a $\Pr(A) = 0.75$

b $\Pr(A \cap B) = 0.4$

c $\Pr(A \cup B) = 1 - 0.13 = 0.87$

d $\Pr(B') = 0.48$

3 $\Pr(C \cup D) = 0.85$
$\therefore \Pr(C' \cap D') = 0.15,\ \Pr(C) = 0.45$
and $\Pr(D') = 0.37$:

	D	D'	
C	$\Pr(C \cap D) = 0.23$	$\Pr(C \cap D') = 0.22$	$\Pr(\mathrm{C}) = 0.45$
C'	$\Pr(C' \cap D) = 0.4$	$\Pr(C' \cap D') = 0.15$	$\Pr(C') = 0.55$
	$\Pr(D) = 0.63$	$\Pr(D') = 0.37$	1

a $\Pr(D) = 0.63$

b $\Pr(C \cap D) = 0.23$

c $\Pr(C \cap D') = 0.22$

d $\Pr(C' \cup D') = 1 - 0.23 = 0.77$

4 $\Pr(E \cup F) = 0.7$
$\therefore \Pr(E' \cap F') = 0.3$
$\Pr(E \cap F) = 0.15, \Pr(E') = 0.55:$

	F	F'	
E	$\Pr(E \cap F) = 0.15$	$\Pr(E \cap F') = 0.3$	$\Pr(E) = 0.45$
E'	$\Pr(E' \cap F) = 0.25$	$\Pr(E' \cap F') = 0.3$	$\Pr(E') = 0.55$
	$\Pr(F) = 0.4$	$\Pr(F') = 0.6$	1

a $\Pr(E) = 0.45$

b $\Pr(F) = 0.4$

c $\Pr(E' \cap F) = 0.25$

d $\Pr(E' \cup F) = 1 - 0.3 = 0.7$

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5 $\Pr(A) = 0.8,\ \Pr(B) = 0.7,$
$\Pr(A' \cap B') = 0.1$:

	B	B'	
A	$\Pr(A \cap B)$ $= \boldsymbol{0.6}$	$\Pr(A \cap B')$ $= \boldsymbol{0.2}$	$\Pr(A)$ $= 0.8$
A'	$\Pr(A' \cap B)$ $= \boldsymbol{0.1}$	$\Pr(A' \cap B')$ $= 0.1$	$\Pr(A')$ $= \boldsymbol{0.2}$
	$\Pr(B) = 0.7$	$\Pr(B') = \boldsymbol{0.3}$	1

a $\Pr(A \cup B) = 1 - 0.1 = 0.9$

b $\Pr(A \cap B) = 0.6$

c $\Pr(A' \cap B) = 0.1$

d $\Pr(A \cup B') = 1 - 0.1 = 0.9$

6 $\Pr(G) = 0.85,\ \Pr(L) = 0.6,$
$\Pr(L \cup G) = 0.5$:

	L	L'	
G	$\Pr(G \cap L)$ $= 0.5$	$\Pr(G \cap L')$ $= \boldsymbol{0.35}$	$\Pr(G)$ $= 0.85$
G'	$\Pr(G' \cap L)$ $= \boldsymbol{0.1}$	$\Pr(G' \cap L')$ $= \boldsymbol{0.05}$	$\Pr(G')$ $= \boldsymbol{0.15}$
	$\Pr(L) = 0.6$	$\Pr(L') = \boldsymbol{0.4}$	1

a $\Pr(G \cup L) = 1 - 0.05 = 0.95$, so 95% favoured at least one proposition.

b $\Pr(G' \cap L') = 0.05$, so 5% favoured neither proposition

7 a

	C	C'	
A	$\Pr(A \cap C)$ $= \frac{4}{52}$	$\Pr(A \cap C')$ $= \frac{12}{52}$	$\Pr(A)$ $= \frac{16}{52}$
A'	$\Pr(A' \cap C)$ $= \frac{9}{52}$	$\Pr(A' \cap C')$ $= \frac{27}{52}$	$\Pr(A')$ $= \frac{36}{52}$
	$\Pr(C) = \frac{13}{52}$	$\Pr(C') = \frac{39}{52}$	1

b **i** $\Pr(A) = \frac{16}{52} = \frac{4}{13}$ (all picture cards)

ii $\Pr(C) = \frac{13}{52} = \frac{1}{4}$ (all hearts)

iii $\Pr(A \cap C) = \frac{4}{52} = \frac{1}{13}$ (picture hearts)

iv $\Pr(A \cup C) = \frac{25}{52}$ (all hearts or pictures)

v $\Pr(A \cup C') = \frac{43}{52}$ (all club, diamond and spades or pictures)

8 $\Pr(M \cap F) = \frac{1}{6}$ or $\frac{10}{60}$

$\Pr(M) = \frac{3}{10} = \frac{18}{60}$

$\Pr(F') = \frac{7}{15} = \frac{28}{60}$

	F	F'	
M	$\Pr(M \cap F)$ $= \frac{10}{60}$	$\Pr(M \cap F')$ $= \frac{8}{60}$	$\Pr(M)$ $= \frac{18}{60}$
M'	$\Pr(M \cap F)$ $= \frac{22}{60}$	$\Pr(M \cap F')$ $= \frac{20}{60}$	$\Pr(M)$ $= \frac{42}{60}$
	$\Pr(F) = \frac{32}{60}$	$\Pr(F')$ $= \frac{28}{60}$	60

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a $\Pr(F) = \dfrac{32}{60} = \dfrac{8}{15}$

b $\Pr(M') = \dfrac{42}{60} = \dfrac{7}{10}$

c $\Pr(M \cap F') = \dfrac{8}{60}$ or $\dfrac{2}{15}$

d $\Pr(M' \cap F') = \dfrac{20}{60}$ or $\dfrac{1}{3}$

9 $\Pr(F) = 0.65$
$\Pr(W) = 0.72$
$\Pr(W' \cap F') = 0.2$

	F	F'	
W	$\Pr(W \cap F) = \boldsymbol{0.57}$	$\Pr(W \cap F') = \boldsymbol{0.15}$	$\Pr(W) = 0.72$
W'	$\Pr(W' \cap F) = \boldsymbol{0.08}$	$\Pr(W' \cap F') = 0.2$	$\Pr(W') = \boldsymbol{0.28}$
	$\Pr(F) = 0.65$	$\Pr(F') = \mathbf{0.35}$	1

a $\Pr(W \cup F) = 1 - 0.2 = 0.8$

b $\Pr(W \cap F) = 0.57$

c $\Pr(W') = 0.28$

d $\Pr(W' \cap F) = 0.08$

10 $\Pr(H \cap N') = 0.05$
$\Pr(H' \cap N) = 0.12$
$\Pr(N') = 0.19$

	N	N'	
H	$\Pr(H \cap N) = \boldsymbol{0.69}$	$\Pr(H \cap N') = 0.05$	$\Pr(H) = \boldsymbol{0.74}$
H'	$\Pr(H' \cap N) = 0.12$	$\Pr(H' \cap N') = \boldsymbol{0.14}$	$\Pr(H') = \boldsymbol{0.26}$
	$\Pr(N) = \mathbf{0.81}$	$\Pr(N') = 0.19$	1

a $\Pr(N) = 0.81$

b $\Pr(H \cap N) = 0.69$

c $\Pr(N) = 0.74$

d $\Pr(H \cup N) = 1 - 0.14 = 0.86$

11 $\Pr(B) = \dfrac{40}{60} = \dfrac{2}{3}$
$\Pr(S) = \dfrac{32}{60} = \dfrac{8}{15}$
$\Pr(B' \cap S') = 0$

	B	B'	
S	$\Pr(S \cap B) = \dfrac{12}{60}$	$\Pr(S \cap B') = \dfrac{20}{60}$	$\Pr(S) = \dfrac{32}{60}$
S'	$\Pr(S' \cap B) = \dfrac{28}{60}$	$\Pr(S' \cap B') = 0$	$\Pr(S') = \dfrac{28}{60}$
	$\Pr(B) = \dfrac{40}{60}$	$\Pr(B') = \dfrac{20}{60}$	60

a $\Pr(B' \cap S') = 0$

b $\Pr(B \cup S) = 1$

c $\Pr(B \cap S) = \dfrac{12}{60} = \dfrac{1}{5} = 0.2$

d $\Pr(B' \cap S) = \dfrac{20}{60} = \dfrac{1}{3}$

12 $\Pr(H) = \dfrac{35}{50} = 0.7$
$\Pr(S) = \dfrac{38}{50} = 0.76$
$\Pr(H' \cap S') = \dfrac{6}{50} = 0.12$

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	H	H'	
S	$\Pr(S \cap H)$ $= \boldsymbol{0.58}$	$\Pr(S \cap H')$ $= \boldsymbol{0.18}$	$\Pr(S)$ $= 0.76$
S'	$\Pr(S' \cap H)$ $= \boldsymbol{0.12}$	$\Pr(S' \cap H')$ $= 0.12$	$\Pr(S')$ $= \boldsymbol{0.24}$
	$\Pr(H)$ $= 0.7$	$\Pr(H')$ $= \boldsymbol{0.3}$	1

a $\Pr(H \cup S) = 1 - 0.12 = 0.88$

b $\Pr(H \cap S) = 0.58$

c $\Pr(H' \cap S) + \Pr(H \cap S')$

$= 0.12 + 0.18$

$= 0.3$

d $\Pr(H \cap S') = 0.12$

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Solutions to Exercise 9F

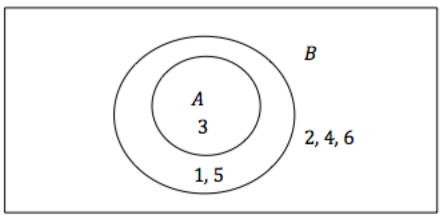
1 $A = \{6\}, \ B = \{3, 4, 5, 6\}$

$\therefore A \cap B = \{6\}$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

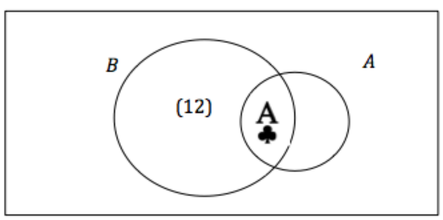
$$= \frac{1}{6} \div \frac{4}{6} = \frac{1}{4}$$

2



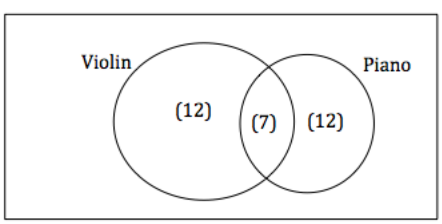
$\Pr(A|B) = \dfrac{1}{3}$

3



$\Pr(A|B) = \dfrac{1}{13}$

4



$\Pr(\text{Violin}|\text{Piano}) = \dfrac{7}{19}$

5 $\Pr(\text{Double six}|\text{ A double}) = \dfrac{1}{6}$

6 **a** $\Pr(\text{iPad}|\text{ iPhone}) = \dfrac{4}{17}$

b $\Pr(\text{iPhone}|\text{ iPad}) = \dfrac{4}{7}$

7 $\Pr(\text{Think yes}|\text{ Male}) = \dfrac{35}{60} = \dfrac{7}{12}$

8 **a** $\Pr(\text{Prefers sport}) = \dfrac{375}{500} = \dfrac{3}{4}$

b $\Pr(\text{Prefers sport}|\text{Male}) = \dfrac{225}{500} = \dfrac{9}{20}$

9

$\cap$	S	A	R	O	T
F	42	61	22	12	137
NF	88	185	98	60	431
Tot	130	246	120	72	568

a $\Pr(S) = \dfrac{130}{568} = \dfrac{65}{284}$

b $\Pr(F) = \dfrac{137}{568}$

c $$\Pr(F|S) = \frac{\Pr(F \cap S)}{\Pr(S)}$$

$$= \frac{42}{568} \div \frac{130}{568}$$

$$= \frac{42}{130} = \frac{21}{65}$$

d $$\Pr(F|A) = \frac{\Pr(F \cap A)}{\Pr(A)})$$

$$= \frac{61}{568} \div \frac{246}{568} = \frac{61}{246}$$

10 $\Pr(A) = 0.6, \Pr(B) = 0.3, \Pr(B|A) = 0.1$

a $\Pr(A \cap B) = \Pr(B|A) \times \Pr(A) = 0.06$

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b $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$

$= \dfrac{0.06}{0.3} = 0.2$

11 a $\Pr(B|A) = \dfrac{\Pr(A \cap B)}{\Pr(A)}$

$= \dfrac{04}{0.7} = \dfrac{4}{7}$

b $\Pr(A \cap B) = \Pr(A|B) \times \Pr(B)$

$= 0.6(0.5) = 0.3$

c $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$

$\therefore \Pr(B) = \dfrac{\Pr(A \cap B)}{\Pr(A|B)}$

$= \dfrac{03}{0.44} = \dfrac{15}{22}$

12 $\Pr(A) = 0.5,\ \Pr(B) = 0.4,\ \Pr(A \cup B) = 0.7$

$\Pr(A \cap B) + \Pr(A \cup B) = \Pr(A) + \Pr(B)$

$\cap$	B	B'		
A	0.2	0.3	0.5	$\Pr(A)$
A'	0.2	0.3	0.5	$\Pr(A')$
	0.4	0.6	1	
	$\Pr(B)$	$\Pr(B')$		

a $\Pr(A \cap B) = 0.5 + 0.4 - 0.7 = 0.2$

b $\Pr(A|B) = \dfrac{0.2}{0.4} = 0.5$

c $\Pr(B|A) = \dfrac{0.2}{0.5} = 0.4$

13 $\Pr(A) = 0.6,\ \Pr(B) = 0.54,$

$\Pr(A \cap B') = 0.4$

$\cap$	B	B'		
A	0.2	0.4	0.6	$= \Pr(A)$
A'	0.34	0.06	0.4	$= \Pr(A')$
	0.54	0.46	1	
	$= \Pr(B)$	$= \Pr(B')$		

a $\Pr(A \cap B) = 0.2$

b $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$

$= \dfrac{0.2}{0.54} = \dfrac{10}{27}$

c $\Pr(B|A) = \dfrac{\Pr(A \cap B)}{\Pr(A)}$

$= \dfrac{0.2}{0.6} = \dfrac{1}{3}$

14 $\Pr(A) = 0.4, \Pr(B) = 0.5, \Pr(A|B) = 0.6$

a $\Pr(A \cap B) = \Pr(A|B) \times \Pr(B) = 0.3$

b $\Pr(B|A) = \dfrac{0.3}{0.4}$

$= \dfrac{3}{4} = 0.75$

15 $\Pr(H) = 0.6,\ \Pr(W|H) = 0.8$

$\therefore\ \Pr(H \cap W) = \Pr(W|H) \times \Pr(H)$

$= 0.8(0.6) = 0.48$

$\Pr(W|H') = 0.4$

$\therefore\ \Pr(H' \cap W) = \Pr(W|H') \times \Pr(H')$

$= 0.4^2 = 0.16$

$\cap$	W	W'		
H	0.48	0.12	0.6	$\Pr(H)$
H'	0.16	0.24	0.4	$\Pr(H')$
	0.64	0.36	1	
	$\Pr(W)$	$\Pr(W')$		

$\Pr(H' \cap W) = 0.16 = 16\%$

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16 $\Pr(C) = 0.15,\ \Pr(F) = 0.08,$
$\Pr(C \cap F) = 0.03$

$$\Pr(F|C) = \frac{\Pr(C \cap F)}{\Pr(C)}$$
$$= \frac{0.03}{0.15} = \frac{1}{5}$$

17 (with replacement)

a $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$

b $\Pr(A,\ A) = \left(\frac{1}{13}\right)^2 = \frac{1}{169}$

c $\Pr(R,\ B) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

d $\Pr(P,\ P) = \left(\frac{4}{13}\right)^2 = \frac{16}{169}$

18 (without replacement)

a $\left(\frac{13}{52}\right)\left(\frac{12}{51}\right) = \frac{1}{17}$

b $\Pr(A,\ A) = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right) = \frac{1}{221}$

c $\Pr(R,\ B) = \left(\frac{26}{52}\right)\left(\frac{26}{51}\right) = \frac{13}{51}$

d $\Pr(P,\ P) = \left(\frac{16}{52}\right)\left(\frac{15}{51}\right) = \frac{20}{221}$

19 $\Pr(W) = 0.652,\ \Pr(A|W) = 0.354$

$$\Pr(A \cap W) = \Pr(A|W) \times \Pr(W)$$
$$= 0.231$$

20 $\varepsilon = 28,\ G = 15,\ B = 14 = (6G + 8G')$
$\therefore B' = (9G + 5G')$

a $\Pr(G) = \frac{15}{28}$

b $\Pr(B) = \frac{14}{28} = \frac{1}{2}$

c $\Pr(B') = 1 - \frac{1}{2} = \frac{1}{2}$

d $$\Pr(B|G) = \frac{\Pr(G \cap B)}{\Pr(G)}$$
$$= \frac{6}{28} \div \frac{15}{28} = \frac{2}{5}$$

e $$\Pr(G|B) = \frac{\Pr(G \cap B)}{\Pr(B)}$$
$$= \frac{6}{28} \div \frac{14}{28} = \frac{3}{7}$$

f $$\Pr(B|G') = \frac{\Pr(G' \cap B)}{\Pr(G')}$$
$$= \frac{8}{28} \div \frac{13}{28} = \frac{8}{13}$$

g $\Pr(B' \cap G') = \frac{5}{28}$

h $\Pr(B \cap G) = \frac{6}{28} = \frac{3}{14}$

21 **a** $\Pr(R) = 0.85$

b $\Pr(L|R) = 0.60$

c $\Pr(L \cap R) = \Pr(L|R) \times \Pr(R) = 0.51$

d $\Pr(L) = 0.51$ since L is a subset of R.

22 U = 'students who prefer not to wear a uniform'
E = 'students in Yr 11'
E' = 'students in Yr 12'
$\Pr(U|E) = 0.25 = \frac{1}{4}$
$\Pr(U|E') = 0.40 = \frac{2}{5}$
$\Pr(E) = 320/600 = \frac{8}{15}$

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$$\Pr(U \cap E) = \Pr(U|E) \times \Pr(E)$$
$$= \left(\frac{8}{15}\right)\frac{1}{4} = \frac{2}{15}$$
$$\Pr(U \cap E') = \Pr(U|E')\Pr(E')$$
$$= \left(\frac{7}{15}\right)\frac{2}{5} = \frac{14}{75}$$
$$\therefore \Pr(U) = \Pr(U \cap E') + \Pr(U \cap E)$$
$$= \frac{2}{15} + \frac{14}{75} = \frac{24}{75} = 32\%$$

However, these are students who prefer *not* to wear uniform.
Students in favour are therefore 68%.

23 $\Pr(B \cap G) = 0.4\left(\frac{4}{9}\right) = 0.178$

$\Pr(B \cap G') = 0.35\left(\frac{5}{9}\right) = 0.194$

$\Pr(B' \cap G) = 0.6\left(\frac{4}{9}\right) = 0.267$

$\Pr(B' \cap G') = 0.65\left(\frac{5}{9}\right) = 0.361$

$\cap$	B	B'	
G	0.178	0.267	0.444
G'	0.194	0.361	0.556
	0.372	0.628	1

a **i** $\Pr(G) = \frac{400}{900} = 0.444$

ii $\Pr(B|G) = 0.40$ (40%)

iii $\Pr(B|G') = 0.35$ (35%)

iv $\Pr(B \cap G) = \Pr(B|G) \times \Pr(G)$
$$= 0.4(0.444) = 0.178$$

v $\Pr(B \cap G') = \Pr(B|G') \times \Pr(G')$
$$= 0.35\left(\frac{500}{900}\right) \cong 0.194$$

b $\Pr(B) = \frac{335}{900} \cong 0.372$

c **i** $\Pr(G|B) = \frac{\Pr(B \cap G)}{\Pr(B)}$
$$= \frac{0.178}{0.372} \cong 0.478$$

ii $\Pr(G|B') = \frac{\Pr(B' \cap G)}{\Pr(B')}$
$$= \frac{0.267}{0.628} = 0.425$$

24 N': 12% of 480 = D;
N: 5% of 620 = D

a **i** $\Pr(N) = \frac{620}{620 + 480}$
$$\approx 0.564$$

ii $\Pr(D|N) = 0.05$ (5%)

iii $\Pr(D|N') = 0.12$ (= 12%)

iv $\Pr(D \cap N) = \Pr(D|N) \times \Pr(N)$
$$= 0.05(0.563)$$
$$= 0.0282$$

v
$$\Pr(D \cap N') = \Pr(D|N') \times \Pr(N')$$
$$= 0.12(0.437) \approx 0.052$$

b $12\%(480) + 5\%(620) = \frac{89}{1100} = 0.081$

c $\Pr(N|D) = \frac{\Pr(D \cap N)}{\Pr(D)}$
$$= \frac{0.028}{0.081} \approx 0.35$$

25 $B1 = 3M,\ 3M'$; $B2 = 3M,\ 2M'$;
$B3 = 2M,\ 1M'$

a $\Pr(M \cap B1) = \frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{6}$

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b $\Pr(M) = \Pr(M \cap B1) + \Pr(M \cap B2)$

$+ \Pr(M \cap B3)$

$= \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3}\left(\frac{3}{5}\right) + \frac{1}{3}\left(\frac{2}{3}\right)$

$= \frac{1}{6} + \frac{1}{5} + \frac{2}{9} = \frac{53}{90}$

c $\Pr(B1|M) = \dfrac{\Pr(M \cap B1)}{\Pr(M)}$

$= \frac{1}{6} \div \frac{53}{90} = \frac{15}{53}$

26 $A,\ B \neq \varnothing$

a $\Pr(A|B) = 1$

$\therefore \Pr(A \cap B) = \Pr(B)$

$\therefore$ B is a subset of A, i.e. $B \subseteq A$

b $\Pr(A|B) = 0$

$\therefore$ A and B are mutually exclusive or $A \cap B = \varnothing$

c $\Pr(A|B) = \dfrac{\Pr(A)}{\Pr(B)}$

$\therefore \Pr(A \cap B) = \Pr(A)$

$\therefore$ A is a subset of B, i.e. $A \subseteq B$

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Solutions to Exercise 9G

1 Do you think private individuals should be allowed to carry guns?

	Male	Female	
Yes	35	30	65
No	25	10	35
Total	60	40	100

Pr(male and support guns) = 0.35;
Pr(male) × Pr(support guns) = 0.65 ≠ 0.35;
therefore not independent

2

	Male	Female	Total
Sport	225	150	375
Music	75	50	125
Total	300	200	500

Pr(male and prefer sport) = 0.45;
Pr(male) × Pr(prefer sport) = 0.45;
therefore independent

3

Type of accident	Speeding Yes	No	Total
Serious	42	61	103
Minor	88	185	273
Total	130	246	376

Pr(speeding and serious) ≈ 0.074;
Pr(speeding) × Pr(serious) = 0.055 ≠ 0.074;
therefore not independent

4 $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
$A = \{1, 2, 3, 4, 5, 6\}$,
$B = \{1, 3, 5, 7, 9, 11\}$,
$C = \{4, 6, 8, 9\}$
$\therefore \Pr(A) = \frac{1}{2}, \ \Pr(B) = \frac{1}{2}, \ \Pr(C) = \frac{1}{3}$

a $A \cap B = \{1, 3, 5\}$
$\therefore \Pr(A \cap B) = \frac{1}{4}$
$\Pr(A)\Pr(B) = \frac{1}{4}$ so A and B are independent.

b $A \cap C = \{4, 6\}$
$\therefore \Pr(A \cap C) = \frac{1}{6}$
$\Pr(A)\Pr(C) = \frac{1}{6}$ so A and C are independent.

c $B \cap C = \{9\}$
$\therefore \Pr(B \cap C) = \frac{1}{12}$
$\Pr(B)\Pr(C) = \frac{1}{6}$ so B and C are not independent.

5 $\Pr(A \cap B)$
= Pr(even number and square number)
$= \Pr(\{4\}) = \frac{1}{6}$
$\Pr(A) = \frac{3}{6} = \frac{1}{2}$
and $\Pr(B) = \Pr(\{1, 4\}) = \frac{2}{6} = \frac{1}{3}$
$\therefore \Pr(A \cap B) = \Pr(A) \times \Pr(B)$

6 $\Pr(A) = 0.3$, $\Pr(B) = 0.1$,
$\Pr(A \cap B) = 0.1$
$\Pr(A)\Pr(B) = 0.03 \neq 0.1$, so A and B are not independent.

7 $\Pr(A) = 0.6$, $\Pr(B) = 0.7$, and A and B are independent

a $\Pr(A|B) = \Pr(A) = 0.6$

b $\Pr(A \cap B) = \Pr(A)\Pr(B)$
$= 0.6(0.7) = 0.42$

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c $\Pr(A \cap B) = \Pr(A) + \Pr(B)$

$- \Pr(A \cap B)$

$\Pr(A \cup B) = 0.6 + 0.7 - 0.42 = 0.88$

8 $\Pr(A \cap B) = \Pr(A)\Pr(B)$

$= 0.5(0.2) = 0.1$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$= 0.5 + 0.2 - 0.1 = 0.6$

9

Blood group	0	A	B	AB
Pr	0.5	0.35	0.1	0.05

a $\Pr(A) = 0.35$

b $\Pr(A,\ B) = 0.35(0.1) = 0.035$

c $\Pr(A,\ A) = 0.35^2 = 0.1225$

d $\Pr(O,\ AB) = 0.05(0.5) = 0.025$

10 $N = 165$:

	H	N	L
M	88	22	10
F	11	22	12

a $\Pr(N) = \dfrac{44}{165} = \dfrac{4}{15}$

b $\Pr(F \cap H) = \dfrac{11}{165} = \dfrac{1}{15}$

c $\Pr(F \cup H) = \Pr(F) + \Pr(H)$

$- \Pr(F \cap H)$

$= \dfrac{45 + 99 - 11}{165} = \dfrac{133}{165}$

d $\Pr(F|L) = \dfrac{\Pr(F \cap L)}{\Pr(L)}$

$= \dfrac{12}{165} \div \dfrac{22}{165} = \dfrac{6}{11}$

e $\Pr(L|F) = \dfrac{\Pr(F \cap L)}{\Pr(F)}$

$= \dfrac{12}{165} \div \dfrac{45}{165} = \dfrac{4}{15}$

F and L are not independent. If they were, then

$\Pr(L|F) = \Pr(L)\Pr(L)$

$= \dfrac{45}{165} = \dfrac{3}{11} \neq \dfrac{4}{15}$

11 $\Pr(A) = \dfrac{20}{36} = \dfrac{5}{9}$

$\Pr(B) = \dfrac{9}{36} = \dfrac{1}{4}$

$\Pr(A \cap B) = \dfrac{5}{36} = \Pr(A)\Pr(B)$

$\therefore$ A and B are independent.

12 $\Pr(W) = 0.4,\ \Pr(M) = 0.5$

$\Pr(W|M) = 0.7 = \dfrac{\Pr(W \cap M)}{\Pr(M)}$

a $\Pr(W \cap M) = \Pr(W|M) \times \Pr(M)$

$= 0.7(0.5) = 0.35$

b $\Pr(M|W) = \dfrac{\Pr(W \cap M)}{\Pr(W)}$

$= \dfrac{035}{0.4} = \dfrac{7}{8}$ or 0.875

13 $N = 65$:

	T	F	S
L	13	4	1
M	8	10	3
H	2	16	8

a $\Pr(L) = \dfrac{18}{65}$

b $\Pr(S) = \dfrac{12}{65}$

c $\Pr(T) = \dfrac{23}{65}$

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d $\Pr(M) = \dfrac{21}{65}$

e $\Pr(L \cap F) = \dfrac{4}{65}$

f $\Pr(T \cap M) = \dfrac{8}{65}$

g $\Pr(L|F) = \dfrac{4}{30} = \dfrac{2}{15}$

h $\Pr(I|M) = \dfrac{8}{21}$

Income is not independent of age, e.g.:

$\Pr(L \cap F) = \dfrac{4}{65} = 0.0615$, but

$\Pr(L)\Pr(F) = \left(\dfrac{18}{65}\right)\left(\dfrac{30}{65}\right) = 0.128$

You would not expect middle managers' income to be independent of age.

14 $N = 150$:

	G	G'
F	48	16
F'	24	62

a **i** $\Pr(G|F) = \dfrac{48}{64} = \dfrac{3}{4} = 0.75$

ii $\Pr(G \cap F) = \dfrac{48}{150} = 0.32$

iii $\Pr(G \cup F) = \dfrac{88}{150} = \dfrac{44}{75} = 0.587$

b $\Pr(G)\Pr(F) = \left(\dfrac{48+24}{150}\right)\left(\dfrac{48+16}{150}\right)$

$= \left(\dfrac{72}{150}\right)\left(\dfrac{64}{150}\right) = 0.2048$

$\Pr(G)\Pr(F) \neq \Pr(G \cap F)$

$\therefore$ G and F are not independent.

c G and F not mutually exclusive:

$\Pr(G \cap F) \neq 0$

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Solutions to Exercise 9H

1 We know the answer is $\frac{1}{8}$. Binomial with $p = \frac{1}{2}$ and $n = 3$.
Simulate with random integers 0 and 1 with your calculator.

2 Binomial, $n = 5$ and $p = \frac{1}{2}$ It can be simulated with using random integers 0 and 1 with your calculator in a .
$\Pr(X \geq 3)$:

X	3	4	5
$\Pr(X = x)$	0.3125	0.15625	0.03125

One in every two simulations would be expected to give this result.

3 Binomial, $n = 10$ and $p = 0.2$
$\Pr(X \geq 5) = 0.032793$:
Simulate with random integers1-5.
Choose one value to be correct for each question.

4 There are many possibilities here, but simplest would be to use a random number table, where each football card is given a number from 0 to 9.
(The average number of purchases needed is exactly given by:
$1 + \frac{10}{9} + \frac{10}{8} + \frac{10}{7} + \ldots + \frac{10}{2} + \frac{10}{1} \approx 29.3$)
This is known as the 'Collector's Problem'

5 **a** Every player plays a match in the first round
Pr(playing 1 match) = 1
Shaun has a 50% chance of winning the first (i.e. playing the second), so
Pr(playing 2 matches) = 0.5
Then he has a probability of 0.5×0.5 of winning two matches (i.e. playing a third), so
Pr(playing 3 matches)
$= 0.5 \times 0.5 = 0.25$
The value of playing a game is 1
$\therefore$

$$\begin{aligned} \text{E(number of matches)} &= 1 \times 1 + 1 \times 0.5 + 1 \times 0.25 \\ &= 1.75 \end{aligned}$$

b Repeating the above for the probability of winning of .7
Pr(playing 1 match) = 1
Pr(playing 2 matches) = 0.7
Pr(playing 3 matches)
$= 0.7 \times 0.7 = 0.49$
$\therefore$

$$\begin{aligned} \text{E(number of matches)} &= 1 \times 1 + 1 \times 0.7 + 1 \times 0.49 \\ &= 2.19 \end{aligned}$$

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Solutions to Technology-free questions

1 a Six ways of getting 7

$\therefore \Pr(7) = \dfrac{6}{36} = \dfrac{1}{6}$

b $\Pr(7') = 1 - \dfrac{1}{6} = \dfrac{5}{6}$

2 $\Pr(O) = 0.993$

$\therefore \Pr(O') = 1 - 0.993 = 0.007$

3 a $\Pr(\text{divisible by } 3) = \dfrac{100}{300} = \dfrac{1}{3}$

b $\Pr(\text{divisible by } 4) = \dfrac{75}{300} = \dfrac{1}{4}$

c $\Pr(\text{divisible by 3 or by 4})$

$= \dfrac{1}{3} + \dfrac{1}{4} - \Pr(\text{divisible by } 12)$

$= \dfrac{7}{12} - \dfrac{25}{300} = \dfrac{1}{2}$

4 $30\,R$, $20\,B$

$\therefore\ \Pr(R) = 0.6$

a $\Pr(R,\ R) = 0.6^2 = 0.36$

b No replacement:

$\Pr(R, R) = \left(\dfrac{3}{5}\right)\left(\dfrac{29}{49}\right) = \dfrac{87}{245}$

5 $A = \{1, 3, 5, 7, 9\},\ B = \{1, 4, 9\}$

If $A + B = C$,

$C = \{2, 5, 10, 4, 7, 12, 6, 9, 14, 8, 11,$

$16, 10, 13, 18\}$

Of these, only $\{6, 9, 12, 18\}$ are divisible by 3.

$\Pr(\text{sum divisible by } 3) = \dfrac{4}{15}$

6 a $\in = \{156, 165, 516, 561, 615, 651\}$

b $\Pr(> 400) = \dfrac{4}{6} = \dfrac{2}{3}$

c $\Pr(\text{even}) = \dfrac{2}{6} = \dfrac{1}{3}$

7 STATISTICIAN has 5 vowels and 7 consonants.

a $\Pr(\text{vowel}) = \dfrac{5}{12}$

b $\Pr(\text{T}) = \dfrac{3}{12} = \dfrac{1}{4}$

8 $\Pr(I) = 0.6,\ \Pr(J) = 0.1,\ \Pr(D) = 0.3$

a $\Pr(I, J, I) = 0.6\,(0.1)\,0.6$

$= 0.036$

b $\Pr(D, D, D) = 0.3^3 = 0.027$

c $\Pr(I,\ D,\ D) + \Pr(J,\ D,\ D) +$

$\Pr(D,\ I,\ D) + \Pr(D,\ J,\ D) +$

$\Pr(D,\ D,\ I) + \Pr(D,\ D,\ J)$

$= 3\,(0.6\ + 0.1)\left(0.3^2\right)$

$= 0.189$

d $\Pr(J') = 0.9$

$\therefore \Pr(J', J', J') = 0.9^3 = 0.729$

9 $\Pr(R) = \dfrac{1}{3},\ \Pr(B) = \dfrac{2}{3}$

a $\Pr(R,\ R,\ R) = \left(\dfrac{1}{3}\right)^3 = \dfrac{1}{27}$

b $\Pr(B,\ R,\ B) = \dfrac{2}{3}\left(\dfrac{1}{3}\right)\left(\dfrac{2}{3}\right) = \dfrac{4}{27}$

c $\Pr(R,\ B,\ B) + \Pr(B,\ R,\ B) +$

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$\Pr(B,\ B,\ R)$

$= 3\left(\dfrac{4}{27}\right) = \dfrac{4}{9}$

d $\Pr(\geq 2B) = \Pr(B,\ B,\ B) + \Pr(2B)$

$$= \left(\frac{2}{3}\right)^3 + \frac{4}{9} = \frac{20}{27}$$

10 $\Pr(A) = 0.6,\ \Pr(B) = 0.5$
If A and B are mutually exclusive,
$\Pr(A \cap B) = 0$
By definition,

$$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$= 1.1 > 0$$

This is impossible, so they cannot be mutually exclusive.

11

$\cap$	B	B'	
A	0.1	0.5	0.6
A'	0.4	0	0.4
	$\Pr(B) = 0.5$	$\Pr(B') = 0.5$	1

a $\Pr(A \cap B') = 0.5$

b $\Pr(A' \cap B') = 0$

c $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B$
$= 0.6 + 0.5 - 0.1$
$= 1$

12 **a** $\dfrac{7}{18}$

b $\dfrac{1}{2}$

13 $\Pr(B) = \dfrac{1}{3}\ \therefore\ \Pr(B') = \dfrac{2}{3}$

a $\Pr(A|B') = \dfrac{\Pr(A \cap B')}{\Pr(B')} = \dfrac{3}{7}$

$\therefore \Pr(A \cap B') = \dfrac{3}{7}\left(\dfrac{2}{3}\right) = \dfrac{2}{7}$

$\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} = \dfrac{2}{3}$

$\therefore \Pr(A \cap B) = \dfrac{2}{3}\left(\dfrac{1}{3}\right) = \dfrac{2}{9}$

b $\Pr(A) = \dfrac{2}{9} + \dfrac{2}{7} = \dfrac{32}{63}$

c $\Pr(B'|A) = \dfrac{\Pr(A \cap B')}{\Pr(A)}$

$= \dfrac{2}{7} \div \dfrac{32}{63} = \dfrac{9}{16}$

14

Pr	O	N	U	Tot
H	0.1	0.08	0.02	0.2
H'	0.15	0.45	0.2	0.8
Tot	0.25	0.53	0.22	1

a $\Pr(H) = 0.2$

b $\Pr(H|O) = \dfrac{\Pr(H \cap O)}{\Pr(O)}$

$= \dfrac{0.1}{0.25} = 0.4$

15 $\Pr(A) = 0.3, \Pr(B) = 0.6, \Pr(A \cap B) = 0.2$

a $\Pr(A \cup B) = \Pr(A) + \Pr(B)$
$- \Pr(A \cap B) = 0.7$

$\cap$	B	B'	
A	0.2	0.1	0.3
A'	0.4	0.3	0.7
	0.6	0.4	1

b $\Pr(A' \cap B') = 0.3$

c $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$

$= \dfrac{0.2}{0.6} = \dfrac{1}{3}$

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d $\Pr(B|A) = \dfrac{\Pr(A \cap B)}{\Pr(A)}$

$= \dfrac{0.2}{0.3} = \dfrac{2}{3}$

16 a $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$

If $\Pr(A|B) = 1$, then $\dfrac{\Pr(A \cap B)}{\Pr(B)} = 1$

$\therefore \Pr(A \cap B) = \Pr(B)$

$\therefore$ B is a subset of A.

b $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$

If $\Pr(A|B) = 0$, then $\dfrac{\Pr(A \cap B)}{\Pr(B)} = 0$

$\therefore \Pr(A \cap B) = 0$

$\therefore$ A and B are mutually exclusive or disjoint.

c $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$

If $\Pr(A|B) = \Pr(A)$, then

$\dfrac{\Pr(A \cap B)}{\Pr(B)} = \Pr(A)$

$\therefore \Pr(A \cap B) = \Pr(A)\Pr(B)$

$\therefore$ A and B are independent.

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Solutions to multiple-choice questions

1 B $\Pr(< 50) = 1 - \Pr(\geq 50)$

$= 1 - 0.7 = 0.3$

2 C $\Pr(G) = 1 - \Pr(G')$

$= 1 - 0.7 = 0.3$

3 A 4 Ts in 10

$\therefore \Pr(T) = \dfrac{2}{5}$

4 C $\Pr(C) = 1 - \Pr(C')$

$= 1 - \dfrac{18}{25} = \dfrac{7}{25}$

5 D $\Pr(J \cup \spadesuit) = \dfrac{16}{52} = \dfrac{4}{13}$

6 A Area outside circle $= 16 - \pi(1.5)^2 \text{ m}^2$

$\therefore \ \Pr = 1 - \dfrac{2.25\pi}{16} \cong 0.442$

7 D $\Pr(\text{Head and a six}) = \dfrac{1}{2} \times \dfrac{1}{6} = \dfrac{1}{12}$

8 E $\Pr(A) = 0.35, \ \Pr(A \cap B) = 0.18,$

$\Pr(B) = 0.38$

$\Pr(A \cup B) = 0.35 + 0.38 - 0.18$

$= 0.55$

9 A $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$= 0.47 + 0.28 - 0.28 = 0.47$

10 B $\Pr(B) = 0.32$

$\Pr(F) = 0.57$

$\Pr(F \cap B) = 0.11$

	B	B'	
F	0.11	0.46	0.57
F'	0.21	0.22	0.43
	0.32	0.68	1

11 B $\Pr(G) = \dfrac{3}{10} = \dfrac{9}{30},$

$\Pr(M) = \dfrac{2}{3} = \dfrac{20}{30},$

$\Pr(G' \cap M) = \dfrac{7}{15} = \dfrac{14}{30},$

	M	M'	
G	$\Pr(G \cap M) = \dfrac{6}{30}$	$\Pr(G \cap M) = \dfrac{3}{30}$	$= \dfrac{9}{30}$
G'	$\Pr(G' \cap M) = \dfrac{14}{30}$	$\Pr(G' \cap M) = \dfrac{7}{30}$	$= \dfrac{21}{30}$
	$\Pr(M) = \dfrac{20}{30}$	$\Pr(M') = \dfrac{10}{30}$	1

$\Pr(G' \cap M') = \dfrac{7}{30}$

12 B

13 A

14 E $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$

$= \dfrac{8}{21} \div \dfrac{4}{7} = \dfrac{2}{3}$

15 C $\Pr(G, \ G) = 0.6(0.7) = 0.42$

16 A $\Pr(G, G) + \Pr(G, G')$

$= 0.42 + (0.4)^2$

$= 0.58$

17 B $\Pr(A \cap B) = \Pr(A)\Pr(B)$

$= 0.35(0.46) = 0.161$

$\Pr(A \cup B) = \Pr(A) + \Pr(B)$

$- \Pr(A \cap B)$

$= 0.35 + 0.46 - 0.161$

$= 0.649$

18 D The reliability

$= 0.85 + 0.95 - 0.85 \times 0.95$

$= 0.9925$

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Solutions to extended-response questions

1 Let A = number of days it takes to build scenery.
Let B = number of days it takes to paint scenery.
Let C = number of days it takes to print programs.

a Pr(building and painting scenery together taking exactly 15 days)

$$\begin{aligned} &= \Pr(A = 7) \times \Pr(B = 8) + \Pr(A = 8) \times \Pr(B = 7) \\ &= \frac{3}{10} \times \frac{1}{10} + \frac{4}{10} \times \frac{3}{10} \\ &= \frac{3 + 12}{100} \\ &= 0.15 \end{aligned}$$

b Pr(all 3 tasks taking exactly 22 days)

$$\begin{aligned} &= \Pr(A = 6) \times \Pr(B = 8) \times \Pr(C = 8) + \Pr(A = 7) \times \Pr(B = 7) \times \Pr(C = 8) + \\ &\quad \Pr(A = 7) \times \Pr(B = 8) \times \Pr(C = 7) + \Pr(A = 8) \times \Pr(B = 6) \times \Pr(C = 8) + \\ &\quad \Pr(A = 8) \times \Pr(B = 7) \times \Pr(C = 7) + \Pr(A = 8) \times \Pr(B = 8) \times \Pr(C = 6) \\ &= \frac{3 \times 1 \times 2 + 3 \times 3 \times 2 + 3 \times 1 \times 4 + 4 \times 6 \times 2 + 4 \times 3 \times 4 + 4 \times 1 \times 4}{1000} \\ &= \frac{6 + 18 + 12 + 48 + 48 + 16}{1000} \\ &= \frac{148}{1000} \\ &= 0.148 \end{aligned}$$

2 **a** For bowl A, $\quad \Pr(\text{2 apples}) = \frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$

For bowl B, $\quad \Pr(\text{2 apples}) = \frac{7}{8} \times \frac{6}{7} = \frac{3}{4}$

b For bowl A, $\quad \Pr(\text{2 apples with replacement}) = \frac{3}{8} \times \frac{3}{8} = \frac{9}{64}$

For bowl B, $\quad \Pr(\text{2 apples with replacement}) = \frac{7}{8} \times \frac{7}{8} = \frac{49}{64}$

c Let A be the event that bowl A is chosen.

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$$\text{Then} \qquad \Pr(A|2\text{ apples}) = \frac{\Pr(A \cap 2\text{ apples without replacement})}{\Pr(2\text{ apples without replacement})}$$

$$= \frac{\frac{1}{2} \times \frac{3}{28}}{\frac{1}{2}\left(\frac{3}{28} + \frac{21}{28}\right)} = \frac{\frac{3}{28}}{\frac{3+21}{28}}$$

$$= \frac{3}{24} = \frac{1}{8} = 0.125$$

d $\Pr(A|2\text{ apples}) = \dfrac{\Pr(A \cap 2\text{ apples with replacement})}{\Pr(2\text{ apples with replacement})}$

$$= \frac{\frac{1}{2} \times \frac{9}{64}}{\frac{1}{2}\left(\frac{9}{64} + \frac{49}{64}\right)}$$

$$= \frac{9}{58}$$

$$\cong 0.125$$

3 **a** $\dfrac{4}{5}$

b $\Pr(\text{running the day after}) = \dfrac{4}{5} \times \dfrac{4}{5} + \dfrac{1}{5} \times \dfrac{1}{4} = 0.69$

c Pr(running exactly twice in the next three days) =

4 **a** The following structure is assumed.

Pr(Player winning 1 match) = 0.5
Pr(Player winning 2 matches) = 0.5×0.5
Pr(Player winning 3 matches) = $0.5 \times 0.5 \times 0.5$
Pr(Player winning 4 matches) = $0.5 \times 0.5 \times 0.5 \times 0.5$

$\therefore$ expected number of matches $= 0.5 \times 1 + 0.5^2 \times 2 + 0.5^3 \times 3 + 0.5^4 \times 4$

$$= \frac{13}{8}$$

b If probability of winning is 0.7, expected number of matches

$$= 0.7 \times 1 + 0.7^2 \times 2 + 0.7^3 \times 3 + 0.7^4 \times 4$$

$$= \frac{18\,347}{5000}$$

$$\approx 3.7$$

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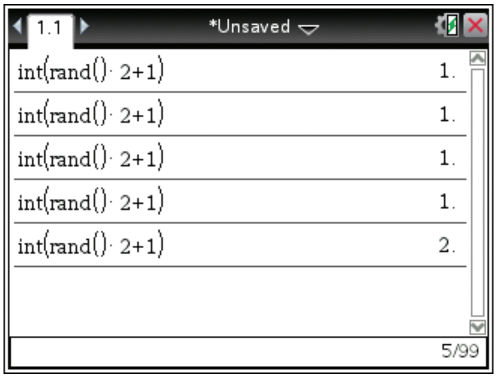
Simulation

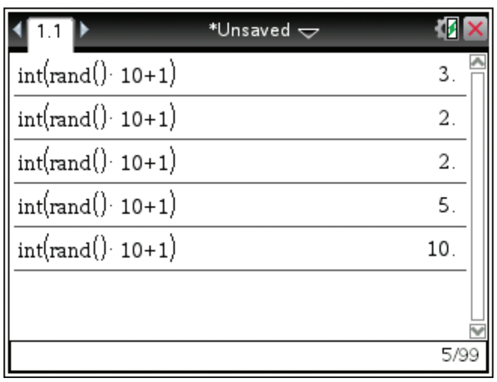
a Use **int(rand()*2 + 1)**

If 1 occurs, a win is recorded. If 2 occurs, a loss is recorded. Sequence stops as soon as 2 is obtained. In the example to the right, the player plays 2 matches.



b Use **int(rand*10 + 1)**

If a digit 1 − 7 inclusive is obtained, a win is recorded. If 8 or 9 or 10 is obtained, a loss is recorded. In the example to the right, the player plays 5 matches.



5 a Theoretical answer

For teams A and B,

$$\begin{aligned}\text{probability of winning} &= \frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\\ &= \frac{1}{4}+\frac{1}{8}\\ &= \frac{3}{8} \text{ or } 0.73\end{aligned}$$

For teams C and D,

$$\text{probability of winning} = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8} \text{ or } 0.125$$

Simulation

TI: The following program simulates the final series and assigns equal probability of winning to each of the two teams in any game. It displays the winner of each game, and lastly the winner of the series.

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```
prog5
Define LibPub prog5()=
Prgm
Local w,x,y,z
Disp "Game 1"
randInt(1,2)→x
If x=1 Then
Disp "A wins"
EndIf
If x=2 Then
Disp "B wins"
EndIf
Disp "Press ENTER to continue"
Disp "Game 2"
randInt(1,2)→y
If y=1 Then
Disp "C wins"
EndIf
If y=2 Then
Disp "D wins"
EndIf
Disp "Press ENTER to continue"
Disp "Game 3"
```

```
prog5
randInt(1,2)→z
If x=1 and z=1 Then
Disp "B wins"
EndIf
If x=2 and z=1 Then
Disp "A wins"
EndIf
If y=1 and z=2 Then
Disp "C wins"
EndIf
If y=2 and z=2 Then
Disp "D wins"
EndIf
Disp "Press ENTER to continue"
Disp "Game 4"
randInt(1,2)→w
If x=1 and w=1 or x=2 and z=1 and w=2 Then
Disp "A wins"
EndIf
If x=2 and w=1 or x=1 and z=1 and w=2 Then
Disp "B wins"
EndIf
```


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```
If y=1 and z=2 and w=2 Then
Disp "C wins"
EndIf
If y=2 and z=2 and w=2 Then
Disp "D wins"
EndIf
EndPrgm
```

b Simulation

TI: The following program uses a simulation of 100 final series to estimate the probability of each team winning a final series. The estimated probabilities are displayed.

```
prog6
Define LibPub prog6()=
Prgm
Local a,b,c,d,n,w,x,y,z
0→a
0→b
0→c
0→d
For n,1,100
randInt(1,2)→x
randInt(1,2)→y
randInt(1,2)→z
randInt(1,2)→w
If x=1 and w=1 or x=2 and z=1 and w=2 Then
a+1→a
ElseIf x=2 and w=1 or x=1 and z=1 and w=2 Then
b+1→b
ElseIf y=1 and z=2 and w=2 Then
c+1→c
ElseIf y=2 and z=2 and w=2 Then
d+1→d
EndIf
EndFor
```

```
Disp "Pr(A wins) =",a/100
Disp "Pr(B wins) =",b/100
Disp "Pr(C wins) =",c/100
Disp "Pr(D wins) =",d/100
EndPrgm
```


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Chapter 10 – Counting Methods

Solutions to Exercise 10A

1 **a** $8 + 3 = 11$

b $3 + 2 + 7 = 12$

c $22 + 14 + 1 = 37$

d $10 + 3 + 12 + 4 = 29$

2 **a** $3 \times 4 \times 5 = 60$ meals

b $10 \times 10 \times 5 = 500$ meals

c $5 \times 7 \times 10 = 350$ meals

d $8^3 = 512$ meals

3 Four choices of entrée, eight of main course and four of dessert.

a $4 \times 8 \times 4 = 128$ meals

b $128 +$ no entrée:$128 + 8 \times 4 =$ 160 meals

4 $3 + 7 + 10 = 20$ choices

5 S_1: $2 \times M, 3 \times L, 4 \times S = 9$ choices
S_2: $2 \times H, 3 \times G, 2 \times A = 7$ choices
Total choices $= 9 \times 7 = 63$ choices

6 M to B: 3 airlines or 3 buses
M to S: 4 airlines $\times$ 5 buses.
Total choices $= 3 + 3 + 4 \times 5 = 26$

7 $5(C) \times 3(T) \times 4(I) \times 2(E) \times 2(\mathrm{A}) = 240$ choices

8 Possible codes $= (26)(10^4) = 260000$

9 No. of plates $= (26^3)(10^3) = 17\,576\,000$

10 2 (dot or dash) + 4 (2 digits) + 8 (3 digits) + 16 (4 digits) = 30

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Solutions to Exercise 10B

1 **a** $3\,! = 3 \times 2 \times 1 = 6$

b $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

c $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

d $2! = 2 \times 1 = 2$

e $0! = 1$

f $1! = 1$

2 **a** $\dfrac{5!}{3!} = \dfrac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = (5)(4) = 20$

b $\dfrac{9!}{7!} = (9)(8) = 72$

c $\dfrac{3!}{0!}! = \dfrac{6}{1} = 6$

d $\dfrac{8!}{6!} = (8)(7) = 56$

e $\dfrac{5!}{0!}! = \dfrac{120}{1} = 120$

f $\dfrac{10!}{7!} = (10)(9)(8) = 720$

3 $5! = 120$ ways

4 $7! = 5040$ ways

5 $4! = 24$ ways

6 $6! = 720$ ways

7 $\dfrac{10!}{7!} = 720$ ways

8 $\dfrac{8!}{5!} = 336$ ways

9 TROUBLE:

a All letters used $= 7! = 5040$ ways

b Three letters only $= \dfrac{7!}{4!} = 210$ ways

10 PANIC:

a All letters used $= 5! = 120$ ways

b Four letters only $= \dfrac{5!}{1!} = 120$ ways

11 COMPLEX:

a No re-use: $\dfrac{7!}{3!} = 840$ ways

b Re-use: $7^4 = 2401$ ways

12 NUMBER:

a No re-use: $\dfrac{6!}{3!}$(3-letter) + $\dfrac{6!}{2!}$(4-letter)
$= 120 + 360 = 480$ codes

b Re-use: $6^3 + 6^4 = 1512$ codes

13 $\epsilon = \{3, 4, 5, 6, 7\}$, no re-use:

a $\dfrac{5!}{2!} = 60$ 3-digit numbers

b Even 3-digit numbers: must end in 4 or 6, so 2 possibilities only for last digit. 4 possibilities then for lst digit and 3 for 2nd digit.
$\therefore\ 4 \times 3 \times 2 = 24$ possible even numbers.

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c Numbers > 700:
3-digit numbers must begin with 7
$\therefore\ 4 \times 3 = 12$
4-digit numbers: $\frac{5!}{1!} = 120$
5-digit numbers: $5! = 120$
Total = 252 numbers

14 $\epsilon = \{3, 4, 5, 6, 7, 8\}$, no re-use:

a 2-digit + 3-digit: $\frac{6!}{4!} + \frac{6!}{3!} = 150$

b 6-digit even: $3 \times 5! = 360$

c >7000: 4-digit numbers must begin with 7 or 8: $2 \times \frac{5!}{2!}! = 120$
5-digit numbers: $\frac{6!}{1!} = 720$
6-digit numbers: $6! = 720$
Total = 1560 numbers

15 4 boys, 2 girls:

a No restrictions: $6! = 720$ ways

b 2 ways for girls at end $\times 4!$ for boys $= 48$ ways

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Solutions to Exercise 10C

1 **a** $(V, C), (V, S), (C, S) = 3$

b $(J, G), (J, W), (G, W) = 3$

c $(T, W), (T, J), (T, P), (W, J), (W, P),$
$(J, P) = 6$

d $(B, G, R), (B, G, W), (B, R, W),$
$(G, R, W) = 4$

2 **a** $^5C_3 = \dfrac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 10$

b $^5C_2 = \dfrac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 3 \times 2 \times 1} = 10$

c $^7C_4 = \dfrac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1} = 35$

d $^7C_3 = \dfrac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1} = 35$
a = **b**, **c** = **d**

3 **a** $^7C_3 = \dfrac{20 \times 19}{2} = 190$

b $^{100}C_{99} = 100$

c $^{100}C_2 = \dfrac{100 \times 99}{2} = 4950$

d $^{250}C_{248} = \dfrac{250 \times 249}{2} = 31\,125$

4 **a** $\dbinom{6}{3} = \dfrac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 3 \times 2 \times 1} = 20$

b $\dbinom{7}{1} = \dfrac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{1 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 7$

c $\dbinom{8}{2} = \dfrac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$
$= 28$

d $\dbinom{50}{48} = \dfrac{50 \times 49}{2} = 1225$

5 $\dbinom{13}{7} = \dfrac{13 \times 12 \times 11 \times 10 \times 9 \times 8}{6 \times 5 \times 4 \times 3 \times 2 \times 1} =$
1716

6 $\dbinom{25}{3} = \dfrac{25 \times 24 \times 23}{3 \times 2 \times 1} = 2300$

7 $\dbinom{52}{7} =$
$\dfrac{52 \times 51 \times 50 \times 49 \times 48 \times 47 \times 46}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$
$= 133\,784\,560$

8 $\dbinom{45}{6} = \dfrac{45 \times 44 \times 43 \times 42 \times 41 \times 40}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$
$= 8\,145\,060$

9 $\dbinom{3}{4}\dbinom{4}{2} = \left(\dfrac{3 \times 2 \times 1}{1 \times 2 \times 1}\right)\left(\dfrac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1}\right)$
$= 18$

10 **a** $\dbinom{30}{8} =$
$\dfrac{30 \times 29 \times 28 \times 27 \times 26 \times 25 \times 24 \times 23}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$
$= 5\,852\,925$

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b Choose 2 men first:
$\binom{10}{2} = \frac{10 \times 9}{2} = 45$

6 women: $\binom{20}{6}$

$= \frac{20 \times 19 \times 18 \times 17 \times 16 \times 15}{720}$

$= 38\,760$

Total $= 45 \times 38\,760 = 1\,744\,200$

11 $2♥$: $\binom{13}{2} = \frac{13 \times 12}{2} = 78$

$5♠$: $\binom{13}{5} = \frac{13 \times 12 \times 11 \times 10 \times 9}{120}$

$= 1287$

7-card hands of $5♠, 2♥ = 1287 \times 78$

$= 100\,386$

12 **a** Without restriction:
$\binom{12}{5} = \frac{12 \times 11 \times 10 \times 9 \times 8}{120} = 792$

b $3W + 2M$: $\binom{8}{3}\binom{4}{2} = (56)(6) = 336$

13 $6F, 5M$, 5 positions:

a $2F + 3M$: $\binom{6}{2}\binom{5}{3} = (15)(10) = 150$

b $4F + 1M$: $\binom{6}{4}\binom{5}{1} = (15)(5) = 75$

c $5F$: $\binom{6}{5} = 6$

d 5 any: $\binom{11}{5} =$
$\frac{11 \times 10 \times 9 \times 8 \times 7 \times 6}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 462$

e $\geq 4\text{F} = n(4\text{F} + 1\text{M}) + n(5\text{F})$

$= 75 + 6 = 81$

14 $15T, 12F$, 10 selections:

a Unrestricted: $\binom{27}{10} = 8\,436\,285$

b $10T$ only: $\binom{15}{10} = 3003$

c $10F$ only: $\binom{12}{10} = 66$

d $5T{+}5F$: $\binom{12}{5}\binom{15}{5} = 2\,378\,376$

15 $6F, 4M$, 5 positions:
$3F + 2M \binom{6}{3}\binom{4}{2} = (20)(6) = 120$

$4F + 1M \binom{6}{4}\binom{4}{1} = (15)(4) = 60$

$5F$ only: $\binom{6}{5} = 6$

Total $= 186$

16 Each of the five times she can choose or refuse
$\therefore$ Total choices $= 2^5 = 32$

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17 Total choices

$$= \binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \cdots + \binom{8}{8}$$

$$= 2^8 = 256$$

18 Total colours (cannot choose no colours)

$$= \binom{5}{1} + \binom{5}{2} + \cdots + \binom{5}{5}$$

$$= 2^5 - 1 = 31$$

19 6 fruits, must choose ≥ 2:

$$\text{choices} = \binom{6}{2} + \binom{6}{3} + \cdots + \binom{6}{6}$$

$$= 2^6 - 7 = 57$$

20 6 people, 2 groups:

a n (two equal groups) $= \binom{6}{3} = 20$

b n (2 unequal groups) $= \binom{6}{1} + \binom{6}{2} = 21$

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Solutions to Exercise 10D

1 $\epsilon = \{1, 2, 3, 4, 5, 6\}$
4 digits, no repetitions; number being even or odd depends only on the last digit.
There are 3 odd and 3 even numbers, so:

a $\Pr(\text{even}) = 0.5$

b $\Pr(\text{odd}) = 0.5$

2 COMPUTER:
$\Pr\text{ (lst letter vowel)} = \frac{3}{8} = 0.375$

3 HEART; 31etters chosen:

a $\Pr(H1\text{st}) = \frac{1}{5} = 0.2$

b $\Pr(H) = 1 - \Pr(H')$
$= \left(1 - \left(\frac{4}{5}\right)\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)\right)$
$= 1 - \frac{2}{5} = 0.6$

c $\Pr\text{ (both vowels)} = 3\left(\frac{2}{5}\right)\left(\frac{1}{4}\right) = 0.3$
(Multiply by 3 because the consonant could be in any of the 3 positions.)

4 There are $6! = 720$ ways of filling the 6 seats, but only $2\binom{3}{2}4! = 144$ have end places with women.
$\therefore \Pr = \frac{144}{720} = 0.2$

5 $7W, 6M$, team of 7:
$\binom{13}{7} = 1716$ possible teams

$3W + 4M : \binom{7}{3}\binom{6}{4} = (35)(15) = 525$

$2W + 5M : \binom{7}{2}\binom{6}{5} = (21)(6) = 126$

$1W + 6M : \binom{7}{1}\binom{6}{6} = 7$

658 arrangements with more men than women
$\therefore \Pr = \frac{658}{1716} = \frac{329}{858}$

6 8 possible combinations, so there are a total of $2^8 - 1 = 255$ possible sandwiches.

a $2^7 = 128$ including H
$\therefore \Pr(H) = \frac{128}{255} = 0.502$

b $\binom{8}{3} = 56$ have 3 ingredients
$\therefore \Pr = \frac{56}{255}$

c $\binom{8}{3} + \binom{8}{4} + \binom{8}{5} + \ldots + \binom{8}{8}$
$= 219$ contain ≥ 3 ingredients
$\therefore \Pr = \frac{219}{255} = \frac{73}{85}$

7 $5W, 6R, 7B$, no replacement:

a $\Pr(R, R, R) = \frac{6}{18} \times \frac{5}{17} \times \frac{4}{16} = \frac{5}{204}$

b There are exactly $\binom{18}{15} = 816$ selections.
$\binom{5}{1}\binom{6}{1}\binom{7}{1} = 210$ have all 3 colours.
$\therefore \Pr\text{ (all different colours)}$
$= \frac{210}{816} = \frac{35}{136}$

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8 $\epsilon = \{0, 1, 2, 3, 4, 5, 6, 7\}$
4 four-digit number (with no repetitions)
$= \frac{8}{4}! = 1680$ possible numbers, but any beginning with zero must be taken out, and there are $\frac{7}{4}! = 210$ of these.
$\therefore$ 1470 numbers

a,b It is easier to find the probability of an odd number first. Begin with the last digit: 4 odd numbers. Then look at the first digit: cannot have zero, so 6 numbers. Then there are 6 choices for the second digit and 5 choices for the first.
Total choices $= 6 \times 6 \times 5 \times 4 = 720$.
So $\Pr(\text{odd}) = \frac{720}{1170} = \frac{24}{49}$
Then $\Pr(\text{even}) = 1 - \frac{24}{49} = \frac{25}{49}$

c $\Pr(< 4000)$: must begin with 1, 2 or 3.
Since there are no other restrictions, $\Pr(< 4000) = \frac{3}{7}$

d $\Pr(< 4000| > 3000) = \dfrac{\Pr(3000 < N < 4000)}{\Pr(N > 3000)}$
For $N > 3000$ it cannot begin with 1 or 2: $\therefore$ 6 possibilities
3 other numbers are unrestricted
$\therefore$ total $(N > 3000) = 6\left(\frac{7!}{4!}\right) = 1260$
For $3000 < N < 4000$ it must begin with 3, so $\frac{7!}{4!} = 210$ satisfy this restriction.
$$\therefore \frac{\Pr(3000 < N < 4000)}{\Pr(N > 3000)} = \frac{210}{1260} = \frac{1}{6}$$

9 $5R, 2B, 3G$, 4 picks,
$\binom{10}{4} = 210$ selections:

a $\Pr(G', G', G', G') = \left(\frac{7}{10}\right)\left(\frac{6}{9}\right)\left(\frac{5}{8}\right)\left(\frac{4}{7}\right) = \frac{1}{6}$

b $\Pr(\geq 1G) = 1 - \Pr(\text{No } G) = \frac{5}{6}$

c $\Pr(\geq 1G \cap \geq 1R)$:
$$N(G + R + B + B) = \binom{3}{1}\binom{5}{1}\binom{2}{2} = 15$$
$$N(G + R + R + B) = \binom{3}{1}\binom{5}{2}\binom{2}{1} = 60$$
$$N(G + G + R + B) = \binom{3}{2}\binom{5}{1}\binom{2}{1} = 30$$
$$N(G + G + R + R) = \binom{3}{2}\binom{5}{2} = 30$$
$$N(G + G + G + R) = \binom{3}{3}\binom{5}{1} = 5$$
$$N(G + R + R + R) = \binom{3}{1}\binom{5}{3} = 30$$
Total = 170
$\therefore \Pr(\geq 1G \cap \geq 1R) = \frac{17}{21}$

d $$\frac{\Pr(\geq 1R| \geq 1G))}{\Pr(\geq 1G)} = \frac{17}{21} \div \frac{5}{6} = \frac{34}{35}$$

10 52 cards, 5 selections, no replacement:

a $\Pr(A', A', A', A', A') = \left(\frac{48}{52}\right)\left(\frac{47}{51}\right)\left(\frac{46}{50}\right)\left(\frac{45}{49}\right)\left(\frac{44}{48}\right) = 0.659$

b $\Pr(\geq 1\text{A}) = 1 - 0.659 = 0.341$

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c $\binom{52}{5}$ hands, $\binom{51}{4}$ contain $A♠$

$\therefore \Pr(A♠)$

$$= \frac{51 \times 50 \times 49 \times 48}{52 \times 51 \times 50 \times 49 \times 48} \div \frac{4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= \frac{5}{52} \cong 0.096$$

d $\Pr(A♠| \geq 1A) = \Pr(A♠) \div \Pr(\geq 1A)$

$$= \frac{5}{52} \div 0.341 \cong 0.282$$

(You cannot simply say that the ace is equally likely to be any of the 4 ($\Pr = \frac{1}{4}$) because there could be more than one ace, hence $\Pr > \frac{1}{4}$.)

11 $5W, 4M$, 3 selections, no replacement:

a $\Pr(W, W, W) = \left(\frac{5}{9}\right)\left(\frac{4}{8}\right)\left(\frac{3}{7}\right) = \frac{5}{42}$

b $\Pr(\geq 1W) = 1 - \Pr(M, M, M)$

$$= 1 - \left(\frac{4}{9}\right)\left(\frac{3}{8}\right)\left(\frac{2}{7}\right)$$

$$= \frac{20}{21}$$

c $\Pr(2M| \geq 1M) = \dfrac{\Pr(2M)}{1 - \Pr(W, W, W)}$

$\Pr(2M)$: $\binom{9}{3} = \dfrac{9 \times 8 \times 7}{6} = 84$ selections;

$\binom{4}{2}\binom{5}{1} = 30$ have $2M$

$\therefore \Pr(2M) = \dfrac{30}{84} = \dfrac{15}{42}$

$\Pr(2M| \geq 1M) = \dfrac{15}{42} \div \dfrac{37}{42} = \dfrac{15}{37}$

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Solutions to Exercise 10E

1 **a** $x^4 + 8x^3 + 24x^2 + 32x + 16$

b $16x^4 + 32x^3 + 24x^2 + 8x + 1$

c $16x^4 - 96x^3 + 216x^2 - 216x + 81$

d $27x^3 - 27x^2 + 9x - 1$

e $16x^4 - 32x^3 + 24x^2 - 8x + 1$

f $-32x^5 + 80x^4 - 80x^3 + 40x^2 - 10x + 1$

g $-243x^5 + 405x^4 - 270x^3 + 90x^2 - 15x + 1$

h $16x^4 - 96x^3 + 216x^2 - 216x + 81$

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Solutions to Technology-free questions

1 **a** ${}^{1000}C_{998} = \dfrac{1000 \times 999}{2} = 499\,500$

b ${}^{1000000}C_{99999} = 1\,000\,000$

c ${}^{100000}C_{1} = 1\,000\,000$

2 Integers 100 to 999 with 3 different digits:
9 ways of choosing the 1st, 9 ways of choosing the 2nd, 8 ways of choosing the 3rd $= 9 \times 9 \times 8 = 648$

3 $1, 2, 3, 4, 5, 6, 3$ digits, no replacement
$= \dfrac{6!}{3!} = 120$

4 n brands, 4 sizes, 2 scents $= 8n$ types

5 9000 integers from 1000 to 9999:
$N(5' + 7') = 7(\text{1st}) \times 8^3(\text{2nd to 4th})$
$= 3584$
$\therefore 9000 - 3584 = 5416$ have at least one 5 or 7

6 $50M, 30W$, choose $2M$, $1\ W$:
$\begin{pmatrix}50\\2\end{pmatrix}\begin{pmatrix}30\\1\end{pmatrix} = 36750$ committees.

7 Choose $2\ V$ from $5, 2C$ from 21, no replacement:
$\begin{pmatrix}5\\2\end{pmatrix}\begin{pmatrix}21\\2\end{pmatrix} = 2100$ possible choices.
Each can be arranged in $4!$ ways
$= 2100 \times 24 = 50400$ possible words

8 **a** 3 toppings from 5, no replacement
$= \begin{pmatrix}5\\3\end{pmatrix} = 10$

b 5 toppings can be present or not
$= 2^5 = 32$

9 7 people to be arranged, always with A and B with exactly one of the others between them:
Arrange (A, X, B) in a block of 3. This can be either (A, X, B) or (B, X, A), and X could be any one of 5 other people.
$\therefore$ 10 possibilities for this block.
There are 4 other people, plus this block, who can be arranged in $5!$ ways.
$\therefore$ Total $N = 10 \times 5!$
$= 1200$ arrangements

10 OLYMPICS: 31etters chosen:

a All letters equally likely
$\therefore \Pr(\text{O, X, X}) = \dfrac{1}{8}$

b $\Pr(\text{Y}') = \dfrac{7}{8}\left(\dfrac{6}{7}\right)\dfrac{5}{6} = \dfrac{5}{8}$
$\therefore \Pr(\text{Y}) = \dfrac{3}{8}$

c $N(\text{O} \cap \text{I})$ has $3!$ arrangements of O, I, X
$\Pr(\text{O, I, X}) = \dfrac{1}{8}\left(\dfrac{1}{7}\right)\dfrac{6}{6} = \dfrac{1}{56}$
$\therefore$ Pr (both chosen) $= \dfrac{6}{56} = \dfrac{3}{28}$

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Solutions to multiple-choice questions

1 E $\binom{8}{1}\binom{3}{1}\binom{4}{1} = 96$

2 D $\binom{3}{1}M \times (\binom{5}{1}L + \binom{3}{1}S) = 24$

3 A 10 people, so possible arrangements $= 10!$

4 D 2 letters, 4 digits, no replacement: $= (\frac{26!}{24!})(\frac{10!}{6!}) = 3276000$

5 C ${}^{21}c_3 = \frac{21!}{3!18!}$

6 B 52 cards, 6 chosen, no replacement: ${}^{52}c_6$

7 C 12 DVDs, 3 chosen, no replays: ${}^{12}c_3 = 220$

8 A $10G, 14B$, 2 of each: ${}^{10}C_2 \times {}^{14}c_2$

9 E METHODS: $\Pr(\text{vowel 1st}) = \frac{2}{7}$

10 E $4M, 4F$, choose 4:

$\binom{8}{4} = 70$ teams

$N(3W, 1M) = \binom{4}{3}\binom{4}{1} = 16$

$\therefore \Pr(3W, 1M) = \frac{16}{70} = \frac{8}{35}$

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Solutions to extended-response questions

1 **a** $1, 2, 3, 4, \ldots, 9$ Even digits: $2, 4, 6, 8$ Odd digits: $1, 3, 5, 7, 9$

The multiplication principle gives
$5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 = 2880$ ways.

b The digits 1 and 2 are considered as a block.
The block can be organised in $2! = 2 \times 1$ ways, i.e. 12 21
There are 8 objects to arrange $\boxed{12}$ 3 4 5 6 7 8 9
$\therefore$ $8!$ arrangements
Total number of arrangements $= 8! \times 2! = 80640$.

2 **a** Three people can be seated in $10 \times 9 \times 8 = 720$ ways in 10 chairs.

b Two end chairs can be occupied in $3 \times 2 = 6$ ways. This leaves 8 chairs for the remaining person to choose from, i.e. $6 \times 8 = 48$ ways of choosing a seat.

c If two end seats are empty it leaves 8 chairs to occupy: $8 \times 7 \times 6 = 336$ ways

3 **a** The odd digits are $1, 3, 5, 7, 9$.
The number three-digit numbers which can be formed $= 5 \times 4 \times 3 = 60$,
i.e. there are 60 different three-digit numbers formed from the odd digits, using each digit only once.

b The numbers greater than 350.
Case 1: Consider numbers greater than or equal to 500.
The first digit can be chosen in 3 ways (it must be 5, 7 or 9). The second digit can be chosen in 4 ways, and the third digit in 3 ways.
There are $3 \times 4 \times 3 = 36$ such numbers.
Case 2: Consider numbers greater than 350 but less than 500.
The first digit can be chosen in one way. It must be 3. The second digit can be chosen in 3 ways (it must be 5, 7 or 9) and the third digit in 3 ways.
There are $1 \times 3 \times 3 = 9$ numbers.
Therefore a total of 45 numbers are greater than 350.

4 **a** There are ${}^{10}C_4 = 210$ ways of choosing a committee of 4 from 10 people.

b there are two men and two women to be selected, there are ${}^5C_2 \times {}^5C_2$ ways of doing this. That is, there are 100 ways of forming the committee.

c When a man is on a committee his wife can't be on it.
Method 1

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- If there are 4 men on the committee, then the condition is satisfied.
  There are 5 ways of choosing such a committee.

- If there are 4 women on the committee, then the condition is satisfied.
  There are 5 ways of choosing such a committee.

- If there are three men on the committee, then the other place can be chosen in 2 ways.
  There are $2 \times {}^5C_3 = 2 \times 10 = 20$ ways having this situation.

- If there are two men on the committee, then the other two places can be chosen in ${}^3C_2$ways.
  There are ${}^5C_2 \times {}^3C_2 = 10 \times 3 = 30$ ways having this situation.

- If there is one man on the committee, then the other three places can be chosen in ${}^4C_3$ways.
  There are ${}^5C_1 \times {}^4C_3 = 5 \times 4 = 20$ ways having this situation.

Therefore the total number of ways = 5 + 5 + 20 + 30 + 20 = 80.

Method 2

Another way of considering this is with order.
The first person can be chosen in 10 ways.
The second in 8 (as the partner is also ruled out).
The third in 6 and the fourth in 4 ways.
This gives $10 \times 8 \times 6 \times 4 = 1920$ ways, but here order has been considered, and so divide by $4! = 24$ to give $\frac{1920}{24} = 80$.

5 **a** There are ${}^{15}C_4 = 1365$ ways of selecting the batteries.

b There are ${}^{10}C_4 = 210$ ways of selecting 10 charged batteries.

c Having at least one flat battery = total number−none flat

$$= 1365 - 210$$

$$= 1155$$

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6 **a** There are ${}^{18}C_4 = 3060$ ways of selecting the lollies.

b There are ${}^{11}C_4 = 330$ ways of choosing the lollies with no mints.

c There are ${}^{11}C_2 \times 7C_2 = 1155$ ways having two mints and two jubes.

7 **Division 1**

The number of ways of choosing 6 winning numbers from 45

$$= {}^{45}C_6$$
$$= 8\,145\,060$$

$$\begin{aligned}\therefore \quad \text{probability of winning Division } 1 &= \frac{1}{8\,145\,060}\\ &= 1.2277\ldots \times 10^{-7}\\ &\approx 1.228 \times 10^{-7}\end{aligned}$$

Division 2

There are 6 winning numbers, 2 supplementary numbers, and 37 other numbers

∴ number of ways of obtaining 5 winning numbers and a supplementary

$$= {}^{6}C_5 \times {}^{2}C_1 \times {}^{37}C_0$$
$$= 6 \times 2$$
$$= 12$$

$$\begin{aligned}\therefore \quad \text{probability of winning Division } 2 &= \frac{12}{8\,145\,060}\\ &= 1.4732\ldots \times 10^{-6}\\ &\approx 1.473 \times 10^{-6}\end{aligned}$$

Division 3

Number of ways of obtaining 5 winning numbers and no supplementary

$$= {}^{6}C_5 \times {}^{2}C_0 \times {}^{37}C_1$$
$$= 6 \times 37$$
$$= 222$$

$$\begin{aligned}\therefore \quad \text{probability of winning Division } 3 &= \frac{222}{8\,145\,060}\\ &= 2.7255\ldots \times 10^{-5}\\ &\approx 2.726 \times 10^{-5}\end{aligned}$$

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Division 4

Number of ways of obtaining 4 winning numbers

$= {}^6C_4 \times {}^{39}C_2$

$= 15 \times 741$

$= 11\,115$

$\therefore \quad$ probability of winning Division $4 = \dfrac{11115}{8\,145\,060}$

$= 0.001\,364\,6\ldots$

$\approx 1.365 \times 10^{-3}$

Division 5

Number of ways of obtaining 3 winning numbers and at least one supplementary

$= {}^6C_3 \times 2C_1 \times 37C_2 + {}^6C_3 \times 2C_2 \times 37C_1$

$= 20 \times 2 \times 666 + 20 \times 37$

$= 27\,380$

$\therefore \quad$ probability of winning Division $5 = \dfrac{27\,380}{8\,145\,060}$

$= 0.003\,3615\ldots$

$\approx 3.362 \times 10^{-3}$

8 a Spot 6

The number of ways of selecting 6 numbers from 80

$= {}^{80}C_6$

$= 300\,500\,200$

20 numbers are winning numbers

The number of ways of selecting 6 numbers from 20

$= {}^{20}C_6$

$= 38760$

$\therefore \quad$ probability of winning with Spot $6 = \dfrac{38\,760}{300\,500\,200}$

$= 1.2898\ldots \times 10^{-4}$

$\approx 1.290 \times 10^{-4}$

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b Spot5

The probability of winning with Spot = $\dfrac{{}^{20}5C_5}{{}^{80}C_5}$

$$= \frac{15504}{24\,040\,016}$$

$$= 6.4492\ldots \times 10^{-4}$$

$$\approx 6.449 \times 10^{-4}$$

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Chapter 11 – Discrete probability distributions

Solutions to Exercise 11A

1 a Not a prob. function, $\sum \Pr \neq 1$

b Not a prob. function, $\sum \Pr \neq 1$

c Prob. function: $\sum \Pr = 1$ and $0 \leq p \leq 1$ for all p

d Not a prob. function, $\sum \Pr \neq 1$

e Not a prob. function because $p(3) < 0$

2 a $\Pr(X = 2)$

b $\Pr(X > 2)$

c $\Pr(X \geq 2)$

d $\Pr(X < 2)$

e $\Pr(X \geq 2)$

f $\Pr(X > 2)$

g $\Pr(X \leq 2)$

h $\Pr(X \geq 2)$

i $\Pr(X \leq 2)$

j $\Pr(X \geq 2)$

k $\Pr(2 < X < 5)$

3 a $\Pr(X = 2)$: $\{2\}$

b $\Pr(X > 2)$: $\{3, 4, 5\}$

c $\Pr(X \geq 2)$: $\{2, 3, 4, 5\}$

d $\Pr(X < 2)$: $\{0, 1\}$

e $\Pr(X \leq 2)$: $\{0, 1, 2\}$

f $\Pr(2 \leq X \leq 5)$: $\{2, 3, 4, 5\}$

g $\Pr(2 < X \leq 5)$: $\{3, 4, 5\}$

h $\Pr(2 \leq X < 5)$: $\{2, 3, 4\}$

i $\Pr(2 < X < 5)$: $\{3, 4\}$

4 a 0.2

b 0.5

c 0.3

d 0.35

e 0.9

5

x	1	2	3	4	5
$\Pr(X = x)$	k	$2k$	$3k$	$4k$	$5k$

a $\sum \Pr = 15k = 1$

$\therefore k = \dfrac{1}{15}$

b $\Pr(2 \leq X \leq 4) = \dfrac{9k}{15k} = \dfrac{3}{5}$

6

r	0	1	2	3	4	5	6	7
p	.09	.22	.26	.21	.13	.06	.02	.01

a $\Pr(X > 4) = 0.06 + 0.02 + 0.01 = 0.09$

b $\Pr(X \geq 2) = 0.26 + 0.13 + 0.21 + 0.09$

$= 0.69$

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7

y	.2	.3	.4	.5	.6	.7	.8	.9
p	.08	.13	.09	.19	.2	.03	.1	.18

a $\Pr(Y \le 0.50)$

$= 0.08 + 0.13 + 0.09 + 0.19$

$= 0.49$

b $\Pr(Y > 0.50) = 1 - 0.49 = 0.51$

c $\Pr(0.30 \le Y \le 0.80)$

$= 1 - (0.08 + 0.18)$

$= 0.74$

8

x	1	2	3	4	5	6
p	0.10	0.13	0.17	0.27	0.20	0.13

a $\Pr(X > 3) = 0.27 + 0.2 + 0.13 = 0.6$

b $\Pr(3 < X < 6) = 0.27 + 0.20 = 0.47$

c

$$\Pr(X \ge 4 | X \ge 2) = \frac{\Pr(X \ge 4)}{\Pr(X \ge 2)} = \frac{0.27 + 0.20 + 0.13}{1 - 0.1} = \frac{0.60}{0.9} = \frac{2}{3}$$

9 **a** $\{(H, H, H), (H, H, T), (H, T, H), (H, T, T), (T, H, H), (T, H, T), (T, T, H), (T, T, T)\}$

b $\Pr(X = 2) = \frac{3}{8}$

c

x	0	1	2	3
$Pr(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

d $\Pr(X \le 2) = 1 - \frac{1}{8} = \frac{7}{8}$

e $\Pr(X \le 1 | X \le 2) = \frac{\Pr(X \le 1)}{\Pr(X \le 2)} = \frac{1}{2} \div \frac{7}{8} = \frac{4}{7}$

10 **a** $Y = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

b $\Pr(Y = 7) = \frac{6}{36} = \frac{1}{6}$

c

y	$\Pr(Y = y)$	y	$\Pr(Y = y)$
2	$\frac{1}{36}$	8	$\frac{5}{36}$
3	$\frac{1}{18}$	9	$\frac{1}{9}$
4	$\frac{1}{12}$	10	$\frac{1}{12}$
5	$\frac{1}{9}$	11	$\frac{1}{18}$
6	$\frac{5}{36}$	12	$\frac{1}{6}$
7	$\frac{1}{6}$		

11 **a** $X = \{1, 2, 3, 4, 5, 6\}$

b $\Pr(X = 4) = \frac{7}{36}$

c (1, 1) gives $X = 1$;
(1, 2), (2, 1), (2, 2) give $X = 2$;
(1, 3), (3, 1)(2, 3), (3, 2), (3, 3) give $X = 3$, etc.

x	1	2	3	4	5	6
p	$\frac{1}{36}$	$\frac{1}{12}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{1}{4}$	$\frac{11}{36}$

12 **a**

y	-3	-2	1	3
$\Pr(Y = y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

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b $\Pr(Y \leq 1) = 1 - \Pr(Y = 3) = \dfrac{7}{8}$

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Solutions to Exercise 11B

1 $\Pr(2M, 3W) = ({}^{12}\mathbf{C}_2) \times ({}^{18}\mathbf{C}_3) \div {}^{30}\mathbf{C}_5$

$$= \frac{66 \times 816}{142\,506} \approx 0.378$$

2 $\Pr(P) = ({}^{4}C_0)({}^{16}C_3) \div {}^{20}C_3$

$$= \frac{84}{71} \approx 0.491$$

3 $7G, 8B, 3$ caught:

$\Pr(G, G, G) = ({}^{8}\mathbf{C}_0)({}^{7}\mathbf{C}_3) \div {}^{15}\mathbf{C}_3 = \dfrac{1}{13}$

$\therefore \Pr(\geq 1B) = 1 - \dfrac{1}{13} = \dfrac{12}{13}$

4 $10T, 15T', 5$ caught:

$\Pr(3T, 2T') = ({}^{10}\mathbf{C}_3)({}^{15}\mathbf{C}_2) \div {}^{25}\mathbf{C}_5$

$$= \frac{420}{1771} \approx 0.237$$

5 $10N, 10N'$, choose 6:

$\Pr(\geq 2N) = 1 - \{\Pr(0N) + \Pr(1N)\}$:

$\Pr(0) = ({}^{10}\mathbf{C}_0)({}^{10}\mathbf{C}_6) \div {}^{20}\mathbf{C}_6$

$$= \frac{7}{1292} \approx 0.0054$$

$\Pr(1) = ({}^{10}\mathbf{C}_1)({}^{10}\mathbf{C}_5) \div {}^{20}\mathbf{C}_6$

$$= \frac{21}{323} \approx 0.0650$$

$\therefore \Pr(\geq 2N) \approx 0.930$

6 $10M, 8F, 6$ chosen:

$\Pr(1F) = ({}^{10}\mathbf{C}_5)({}^{8}\mathbf{C}_1) \div {}^{18}\mathbf{C}_6$

$$= \frac{24}{221} \approx 0.109$$

No reason for suspicion here. You would need a much smaller probability, indicating an unlikely chance, to be concerned.

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Solutions to Exercise 11C

1 Binomial, $n = 6, p = 0.3$:

a $\Pr(X = 3) = {}^6\mathbf{C}_3(0.3)^3(0.7)^3 = 0.185$

b $\Pr(X = 4) = {}^6\mathbf{C}_4(0.3)^4(0.7)^2 = 0.060$

2 Binomial, $n = 10, p = 0.1$:

a $\Pr(X = 2) = {}^{10}\mathbf{C}_2(0.1)^2(0.9)^8 = 0.194$

b $\Pr(X = 2) = 0.194$
$\Pr(X = 1) = {}^{10}\mathbf{C}_1(0.1)(.0.9)^9 = 0.387$
$\Pr(X = 0) = 0.9^{10} = 0.349$
$\therefore \Pr(X \leq 2) = 0.349 + 0.387 + 0.194$
$= 0.930$

3 Binomial, $n = 10, p = \dfrac{1}{6}$, CAS calculator:

a $\Pr(X = 10) = 0.137$

b $\Pr(X < 10) = 0.446$

c $\Pr(X \geq 10) = 1 - 0.446 = 0.554$

4 Binomial, $n = 7, p = 0.35$:

a $\Pr(R, R, R, R', R', R', R')$
$= (0.35)^3(0.65)^4 = 0.00765 \approx 0.008$

b $\Pr(R = 3) = {}^7\mathbf{C}_3(0.35)^3(0.65)^4$
$= 0.268$

c $\Pr(R \geq 3) = \Pr(R = 3) + \Pr(R = 4)$
$+ \Pr(R = 5) + \Pr(R = 6)$
$+ \Pr(R = 7)$
$= 0.468$

5 Binomial, $n = 7, p = \dfrac{1}{6}$

a $\Pr(2 \text{ only on 1 st}) = \left(\dfrac{1}{6}\right)\left(\dfrac{5}{6}\right)^6 = 0.056$

b $\Pr(2) = {}^7\mathbf{C}_1\left(\dfrac{1}{6}\right)\left(\dfrac{5}{6}\right)^6 = 0.391$

6 Binomial, $n = 100, p = 0.5$, CAS calculator:
$\Pr(G > 60) = 0.018$

7 Binomial, $n = 5, p = 0.1$:

a $\Pr(X = 0) = (0.9)^5 = 0.590$
$\Pr(X = 1) = {}^5\mathbf{C}_1(0.1)(0.9)^4 = 0.328$
$\Pr(X = 2) = {}^5\mathbf{C}_2(0.1)^2(0.9)^3 = 0.0729$
$\Pr(X = 3) = {}^5\mathbf{C}_3(0.1)^3(0.9)^2 = 0.0081$
$\Pr(X = 4) = {}^5\mathbf{C}_4(0.1)^4(0.9) = 0.00045$
$\Pr(X = 5) = (0.1)^5 = 0.00001$

b Zero is the most probable number.

8 Binomial, $n = 3, p = 0.48$:
$\Pr(F, F, F) = 0.48^3 = 0.1106$
$\Pr(M, M, M) = 0.52^3 = 0.1406$
$\therefore \Pr(\geq 1 \text{ of each}) = 1 - (0.1406 + 0.1106)$
$= 0.749$

9 Binomial, $n = 100, p = 0.3$, CAS calculator: $\Pr(X \geq 40) = 0.021$

10 Binomial, $n = 100, p = 0.8$, CAS calculator: $\Pr(X \leq 80) = 0.540$

11 Binomial, $n = 4, p = 0.25$:

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$$\Pr(X \ge 1) = 1 - \left(\frac{3}{4}\right)^4 = \frac{175}{256}$$

12 Binomial, $n = 4$, $p = 0.003$:

a $\Pr(X = 0) = (0.997)^4 = 0.9880$

b $\Pr(X = 1) = 4(0.997)^3 \times (0.003)$

$= 0.0119$

$\therefore \Pr(X \le 1) = 0.9999$

c $\Pr(X = 4) = 0.003^4 = 8.1\, x\, 10^{-11}$

13 Binomial, $n = 15$, $p = 0.5$, CAS calculator:

a $\Pr(X \ge 10) = 0.151$

b $\Pr(X \ge 10 \cup X \le 5) = 0.302$

14 Binomial, $n = 10$, $p = 0.04$:

$\Pr(X \ge 2) = 1 - \Pr(X = 0 \text{ or } 1)$

$= 1 - (0.96^{10} + 10(0.96)^9(0.04))$

$= 0.058$

So the percentage is 5.8%.

$\Pr(X \ge 6) < 10^{-5}$, so 6 defectives in a batch of 10 would indicate that the machines aren't working properly.

15 Binomial, $n = 10$, $p = \frac{1}{4}$

a **i** $\Pr(X \ge 3) = 0.474$

ii $\Pr(X \ge 4) = 0.224$

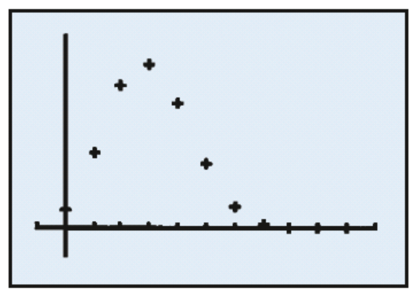
iii $\Pr(X \ge 5) = 0.078$

b 6 or more would be enough. $\Pr(X \ge 6) = 0.020$ so there would be a 98% probability that any such student was not guessing. Even 5 or more seems reasonable with a 92% chance of not guessing.

16 Binomial, $n = 20$, $p = \frac{1}{4}$, CAS calculator: $\Pr(\text{pass}) = \Pr(X \ge 10) = 0.014$

17 Binomial, $n = 20$, $p = 0.3$

x	$\Pr(X = x)$	x	$\Pr(X = x)$
0	0.028	6	0.037
1	0.121	7	0.009
2	0.233	8	0.001
3	0.267	9	< 0.001
4	0.2	10	< 0.001
5	0.103		



18 Binomial, $n = 15$, and $p = 0.6$

x	$\Pr(X = x)$	x	$\Pr(X = x)$
0	< 0.001	8	0.177
1	< 0.001	9	0.207
2	< 0.001	10	0.186
3	0.002	11	0.127
4	0.007	12	0.064
5	0.024	13	0.022
6	0.061	14	0.003
7	0.118	15	< 0.001

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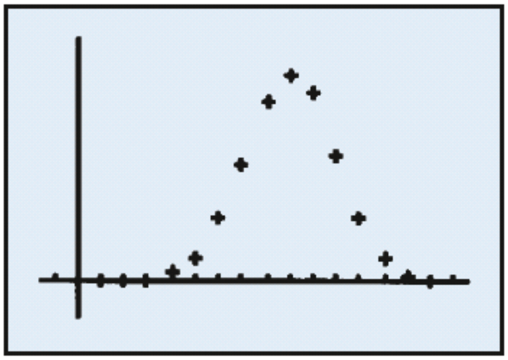

19 Binomial, n = unknown and $p = \dfrac{1}{2}$

a $\Pr(H \geq 1) = 1 - \Pr(H = 0)$
If $\Pr(H = 0) < 0.05$, then $0.5^n < 0.05$
$\therefore 2^n > 20, \therefore n > 4.322$
$\therefore$ 5 tosses needed

b $\Pr(H > 1)$: we know
$\Pr(H = 0) = 0.5^n$
$\Pr(H = 1) = n\,0.5^n$
$\therefore (n + 1)0.5^n < 0.05$

n	5	6	7	8	g
Pr(at most 1)	0.188	0.11	0.06	0.04	0.52
Pr(more than 1)	0.813	0.89	0.94	0.96	0.98

$\therefore$ 8 tosses needed

20 Binomial, n = unknown and $p = \dfrac{1}{6}$

a $\Pr(S \geq 1) = 1 - \Pr(S = 0)$:
If $\Pr(S = 0) < 0.1$, then $\left(\dfrac{5}{6}\right)^n < 0.1$
$\therefore (1.2)^n > 10 \;\therefore\; n > 12.6$
$\therefore$ 13 rolls needed

b $\Pr(S > 1)$: we know
$\Pr(S = 0) = \left(\dfrac{5}{6}\right)^n$

$$\Pr(S = 1) = n\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{n-1}$$
$$= \frac{n}{5}\left(\frac{5}{6}\right)^n$$
$$\therefore \left(\frac{n}{5} + 1\right)\left(\frac{5}{6}\right)^n < 0.1$$

n	19	20	21	22	23
Pr(at most 1)	0.15	0.13	0.11	0.098	0.08
Pr(more than 1)	0.85	0.87	0.89	0.902	0.92

$\therefore$ 22 rolls needed

21 Binomial, n = unknown and $p = 0.1$

a $\Pr(A \geq 1) = 1 - \Pr(A = 0)$:
If $\Pr(A = 0) < 0.2$, then $0.9^n < 0.2$
$\therefore \left(\dfrac{10}{9}\right)^n > 5, \;\therefore\; n > 15.3$
$\therefore$ 16 serves needed

b $\Pr(A > 1)$: we know $\Pr(A = 0) = 0.9^n$

$$\Pr(A = 1) = n(0.1)(0.9)^{n-1}$$
$$= \frac{n}{10}(0.9)^{n-1}$$
$$\therefore 0.9^n + \frac{n}{10}(0.9)^{n-1} < 0.2$$

n	26	27	28	29	30
Pr(at most 1)	0.251	0.23	0.22	0.199	0.18
Pr(more than 1)	0.749	0.77	0.78	0.801	0.82

$\therefore$ 29 serves needed

22 Binomial, n = unknown and $p = 0.05$

a $\Pr(W \geq 1) = 1 - \Pr(W = 0)$
If $Pr(W = 0) < 0.1$, then $0.95^n < 0.1$
$\therefore \left(\dfrac{20}{19}\right)^n > 10, \therefore n > 44.9$
$\therefore$ 45 games needed

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b As for **a:** $0.95^n < 0.05$, $\therefore \left(\frac{20}{19}\right)^n > 20$

$\therefore n > 58.4$

$\therefore$ 59 games needed

23 Binomial, $n = 5$ and $p = 0.7$

a $\Pr(X = 3) = {}^5\mathbf{C}_3(0.7)^3(0.3)^2$

$= 0.3087$

b $\Pr(X = 3|X \geq 1) = \dfrac{\Pr(X = 3)}{\Pr(X \geq 1)}$

$\Pr(X \geq 1) = 1 - \Pr(X = 0) = 0.99757$

$\therefore \Pr(X = 3|X \geq 1) = \dfrac{0.3087}{0.9976}$

$= 0.3095$

24 Binomial, $n = 10$ and $p = 0.2$

X = number of Fridays chosen

a $\Pr(X = 2) = {}^{10}\mathbf{C}_2(0.8)^8(0.2)^2$

$= 0.3020$

b $\Pr(X \geq 2) = 1 - \Pr(X < 2)$

$= 1 - (0.1074 + 0.2684)$

$= 0.6242$

c $\Pr(X = 3|X \geq 2) = \dfrac{\Pr(X = 3)}{\Pr(X \geq 2)}$

$\Pr(X = 3) = {}^{10}\mathbf{C}_3(0.8)^7(0.2)^3$

$= 0.2013$

$\therefore \Pr = 0.3225$

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Solutions to Technology-free questions

1

x	0	1	2	3	4
$\Pr(X = x)$	.12	.25	.43	.12	.08

a $\Pr(X \le 3) = 1 - 0.08 = 0.92$

b $\Pr(X \ge 2) = 0.43 + 0.12 + 0.08$
$= 0.63$

c $\Pr(1 \le X \le 3) = 0.25 + 0.43 + 0.12$
$= 0.80$

2

x	1	2	3	4	Total
$\Pr(X = x)$	0.25	0.28	0.30	0.17	1

3 Four marbles are 1 and two are 2, two chosen, no replacement, $X = \{2, 3, 4\}$

$X = 2$ only if both $= 1$;

$\Pr = \left(\frac{4}{6}\right)\left(\frac{3}{5}\right) = \frac{2}{5}$

$X = 4$ only if both $= 2$;

$\Pr = \left(\frac{2}{6}\right)\left(\frac{1}{5}\right) = \frac{1}{15}$

$X = 3: \Pr = 1 - \left(\frac{1}{15} + \frac{2}{5}\right) = \frac{8}{15}$

x	2	3	4
$\Pr(X = x)$	$\frac{2}{5}$	$\frac{8}{15}$	$\frac{1}{15}$

4 $\{1, 2, 3, 6, 7, 9\}$ 2 chosen, replacement, $X =$ sum.

a

2nd choice \ 1st choice	1	2	3	6	7	9
1	2	3	4	7	8	10
2	3	4	5	8	9	11
3	4	5	6	9	10	12
6	7	8	9	12	13	15
7	8	9	10	13	14	16
9	10	11	12	15	16	18

b $X =$
$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18\}$

c Each pair is equally likely with $\Pr = \frac{1}{36}$

2 can only be (1, 1) $\Pr(2) = \frac{1}{36}$

3 can be (1, 2) or (2, 1) $\Pr(3) = \frac{1}{18}$

4 can be (1, 3), (3, 1) or (2, 2) $\Pr(4) = \frac{1}{12}$

5 can be (2, 3) or (3, 2) $\Pr(5) = \frac{1}{18}$

6 can only be (3, 3) $\Pr(6) = \frac{1}{36}$

7 can be (1, 6) or (6, 1) $\Pr(7) = \frac{1}{18}$

8 can be (2, 6), (6, 2), (1, 7) or (7, 1) $\Pr(8) = \frac{1}{9}$

9 can be (3, 6), (6, 3), (2, 7) or (7, 2) $\Pr(9) = \frac{1}{9}$

10 can be (3, 7), (7, 3), (1, 9) or (9, 1) $\Pr(10) = \frac{1}{9}$

11 can be (2, 9) or (9, 2) $\Pr(11) = \frac{1}{18}$

12 can be (3, 9), (9, 3) or (6, 6) $\Pr(12) = \frac{1}{12}$

13 can be (6, 7) or (7, 6) $\Pr(13) = \frac{1}{18}$

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14 can only be (7, 7) $\Pr(14) = \dfrac{1}{36}$

15 can be (6, 9) or (9, 6) $\Pr(15) = \dfrac{1}{18}$

16 can be (7, 9) or (9, 7) $\Pr(16) = \dfrac{1}{18}$

18 can only be (9, 9) $\Pr(18) = \dfrac{1}{36}$

5 Binomial, $n = 4$ and $p = 0.25$:

a $\Pr(X \geq 3) = 4(0.25)^3(0.75) + (0.25)^4$
$= 0.051$

b $\Pr(X \leq 3) = 1 - \Pr(X = 4)$
$= 1 - 0.25^4 = 0.996$

c $\Pr(X \leq 2) = 1 - 0.051 = 0.949$

6 Binomial, $n = 3$ and $p = \dfrac{1}{4}$:

a $\Pr(X = 2) = 3\left(\dfrac{1}{4}\right)^2\left(\dfrac{3}{4}\right) = \dfrac{9}{64}$

b $\Pr(X \geq 1) = 1 - \left(\dfrac{3}{4}\right)^3 = \dfrac{37}{64}$

7 Binomial, $n = 4$ and $p = \dfrac{1}{3}$:

a $\Pr(X = 0) = \left(\dfrac{2}{3}\right)^4 = \dfrac{16}{81}$

b $\Pr(X = 1) = {}^4\mathbf{C}_1\left(\dfrac{2}{3}\right)^3\left(\dfrac{1}{3}\right) = \dfrac{32}{81}$

c $\Pr(X > 1) = 1 - \dfrac{16 + 32}{81} = \dfrac{11}{27}$

8 Binomial, $n = 7$ and $p = \dfrac{1}{4}$:

a $\Pr(X = 3) = {}^7C_3\,(0.25)^3(0.75)^4$
$= 0.173$

b $\Pr(X < 3) = 0.75^7 + 7(0.75)^6(0.25)$
$+ 21(0.75)^5(0.25)^2$
$= 0.756$

9 Binomial, $n = 15$, p $= 0.01p\%$

a $\Pr(X = 15) = \left(\dfrac{p}{100}\right)^{15}$

b Since Pr(fail) $= 100 - p$ and ${}^{15}\mathbf{C}_1 = 15$

$\Pr(X = 14) = 15\left(1 - \dfrac{p}{100}\right)\left(\dfrac{p}{100}\right)^{14}$

c

$\Pr(X = 13) = {}^{15}\mathbf{C}_2\left(1 - \dfrac{p}{100}\right)^2\left(\dfrac{p}{100}\right)^{13}$

$= 105\left(1 - \dfrac{p}{100}\right)^2\left(\dfrac{p}{100}\right)^{13}$

$\Pr(13 \leq X \leq 15)$

$= \left(\dfrac{p}{100}\right)^{15} + 15\left(1 - \dfrac{p}{100}\right)\left(\dfrac{p}{100}\right)^{14}$
$+ 105\left(1 - \dfrac{p}{100}\right)^2\left(\dfrac{p}{100}\right)^{13}$

10 Binomial, $n = 3$, $p = \dfrac{3}{5}$:

a $\Pr(X \geq 1) = 1 - \Pr(X = 0)$
$= 1 - \left(\dfrac{2}{5}\right)^3 = \dfrac{117}{125}$

b For m games, $\Pr(X = 1) =$
${}^m\mathbf{C}_1\left(\dfrac{3}{5}\right)\left(\dfrac{2}{5}\right)^{m-1}$

and $\Pr(X = 2) = {}^m\mathbf{C}_2\left(\dfrac{3}{5}\right)^2\left(\dfrac{2}{5}\right)^{m-2}$

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Since $\Pr(X = 2) = 3 \times \Pr(X = 1)$,

$${}^m\mathbf{C}_2\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^{m-2} = 3\,{}^m\mathbf{C}_1\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^{m-1}$$

Cancel out all denominators as both $= 5^m$

$$\frac{9}{2}m(m-1)2^{m-2} = 9m\ 2^{m-1}$$

$$\therefore \quad \frac{m-1}{4}2^{m-2} = 2^{m-2}$$

$$\therefore \quad m - 1 = 4$$

$$\therefore \quad m = 5$$

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Solutions to multiple-choice questions

1 B

x	0	1	2	3	4
$\Pr(X = x)$	k	$2k$	$3k$	$2k$	k

$\sum \Pr = 9k = 1$

$\therefore k = \frac{1}{9}$

2 A $\Pr(X \geq 5) = 0.14 + 0.10$
$= 0.24$

3 C Only 2 apples, so $\{0, 1, 2\}$

4 A Only the coin toss = yes/no

5 E Binomial, $n = 6$ and $p = 0.48$:
$\Pr(X = 3) = {}^6\mathbf{C}_3(0.48)^3(0.52)^3$

6 C $\Pr(L \geq 1) = 1 - \Pr(L = 0)$
$= 1 - (0.77)^9$

7 A Binomial, $n = 10$ and $p = 0.2$
Expect a skewed graph with mean and mode $X = 2$.
Only **A** fits; **B** has $p = 0.5$, **C** has $p = 0.8$, **D** is not a probability function and **E** does not tail to zero.

8 D Binomial, $n = 60$ and $p = 0.5$:
$\Pr(X \geq 30) = 0.857$

9 B Binomial, $n = 10$ and $p = 0.1$:
$\Pr(X = 3) + \Pr(X = 4)$
$= {}^{10}\mathbf{C}_3(0.9)^7(0.1)^3 + {}^{10}\mathbf{C}_4(0.9)^6(0.1)^4$
$= 0.0574 + 0.0112$
$= 0.0686$

10 E $\Pr(X \geq 1) = 1 - \Pr(X = 0)$

$\therefore 0.9^n < 0.1$

$\left(\frac{10}{9}\right)^n > 10$

$n > 21.9$

22 games will be needed

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Solutions to extended-response questions

1 $\Pr(A|B) = 0.6, \Pr(A|B') = 0.1, \Pr(B) = 0.4$

$X = 4$ if A and B both occur

$X = 3$ if A occurs but not B

$X = 2$ if B occurs but not A

$X = 1$ if neither A nor B occurs

$$\therefore \quad \Pr(X = 4) = \Pr(A \cap B)$$
$$\Pr(X = 3) = \Pr(A \cap B')$$
$$\Pr(X = 2) = \Pr(B \cap A')$$
$$\Pr(X = 1) = \Pr(A' \cap B')$$

Also $\quad \Pr(A \cap B) = \Pr(A|B) \times \Pr(B)$

$$= 0.6 \times 0.4 = 0.24$$

$$\Pr(A \cap B') = \Pr(A|B') \times \Pr(B')$$
$$= 0.1 \times 0.6 = 0.06$$

	A	A'	
B	$\Pr(A \cap B) = 0.24$	$\Pr(A' \cap B) = 0.16$	$\Pr(B) = 0.4$
B'	$\Pr(A \cap B') = 0.06$	$\Pr(A' \cap B') = 0.54$	$\Pr(B') = 0.6$
	$\Pr(A) = 0.3$	$\Pr(A') = 0.7$	

a Therefore the probability distribution is given by

x	1	2	3	4
$\Pr(X = x)$	0.54	0.16	0.06	0.24

b $\Pr(X \geq 2) = \Pr(X = 2) + \Pr(X = 3) + \Pr(X = 4)$

$$= 1 - \Pr(X = 1)$$
$$= 0.46$$

2 **a** **i** Since X is the random variable of a discrete probability distribution, the sum of the probabilities is 1, therefore $\quad k + 0.9 = 1$

which implies $\quad k = 0.1$

ii $\quad \Pr(X > 3) = \Pr(X = 4) + \Pr(X = 5) + \Pr(X = 6) + \Pr(X = 7)$

Thus $\quad \Pr(X > 3) = 0.2 + 0.3 + 0.1 + 0 = 0.6$

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iii $\Pr(X > 4|X > 3) = \dfrac{\Pr(X > 4)}{\Pr(X > 3)}$

$$= \frac{0.4}{0.6} = \frac{2}{3}$$

b **i** The probability of hitting exactly 4 particular houses

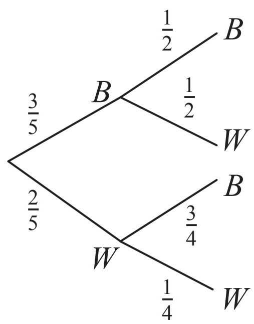
$$= (0.4)^4(0.6)^6 = 0.0012$$

ii Let Y be the number of houses for which the paper boy hits the front step 3 or 4 times. Then Y is a binomial distribution with

$$\Pr(Y = 4) = {}^{10}C_4(0.4)^4(0.6)^6$$

$$= 0.2508 \text{ (correct to 4 decimal places)}$$

3 **a** The bag contains 3 blue cards and 2 white cards.



The result is to be BW or WB.

Probability of different colours $= \dfrac{3}{5} \times \dfrac{1}{2} + \dfrac{2}{5} \times \dfrac{3}{4}$

$$= \frac{3}{5}$$

b If the cards are different colours, then the two coins are tossed once. If the cards are the same colour, the two coins are tossed twice.

Let X be the number of heads achieved. Then X can take values 0, 1, 2, 3 or 4.

i $\Pr(X = 0) = \dfrac{3}{5} \times \left(\dfrac{1}{2}\right)^2 + \dfrac{2}{5} \times \left(\dfrac{1}{2}\right)^4 = \dfrac{3}{20} + \dfrac{1}{40} = \dfrac{7}{40}$

ii $\Pr(X = 2) = \dfrac{3}{5} \times \left(\dfrac{1}{2}\right)^2 + \dfrac{2}{5} \times {}^4C_2\left(\dfrac{1}{2}\right)^4$

$$= \frac{3}{20} + \frac{3}{20}$$

$$= \frac{3}{10}$$

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c A is the event 'two cards of the same colour are drawn', and B is the event '$X \geq 2$'.

i
$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ &= \frac{2}{5} + \Pr(B) - \Pr(A \cap B)\end{aligned}$$

$$\begin{aligned}Pr(B) &= \Pr(X \geq 2) = \Pr(X = 2) + \Pr(X = 3) + \Pr(X = 4)\\ &= \frac{3}{10} + \frac{2}{5} \times {}^4C_3\left(\frac{1}{2}\right)^4 + \frac{2}{5} \times \left(\frac{1}{2}\right)^4\\ &= \frac{3}{10} + \frac{1}{10} + \frac{1}{40} = \frac{17}{40}\end{aligned}$$

$$\begin{aligned}\Pr(A \cap B) &= \Pr(X \geq 2 \textit{ and } \text{two cards the same colour})\\ &= \Pr(\text{cards the same colour}) \times \Pr(X \geq 2|\text{ cards the same colour})\\ &= \Pr(A)\Pr(B|A)\\ &= \frac{2}{5}({}^4C_2(\tfrac{1}{2})^4 + {}^4C_3(\tfrac{1}{2})^4 + (\tfrac{1}{2})^4)\\ &= \frac{3}{20} + \frac{1}{10} + \frac{1}{40} = \frac{11}{40}\end{aligned}$$

Therefore
$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ &= \frac{2}{5} + \Pr(B) - \Pr(\mathrm{A} \cap B)\\ &= \frac{2}{5} + \frac{17}{40} - \frac{11}{40} = \frac{11}{20}\end{aligned}$$

ii
$$\begin{aligned}\Pr(A|B) &= \frac{\Pr(A \cap B)}{\Pr(B)}\\ &= \frac{\frac{11}{40}}{\frac{17}{40}} = \frac{11}{17}\end{aligned}$$

4 Let X be the number of correct answers.
X is the random variable of binomial distribution with $n = 20$ and $p = 0.2$.

a **TI:** Press **Menu → 5:Probability → 5:Distributions → E:Binomial Cdf**
Input $n = 20$, $p = 0.2$, lower bound = 10, upper bound = 20

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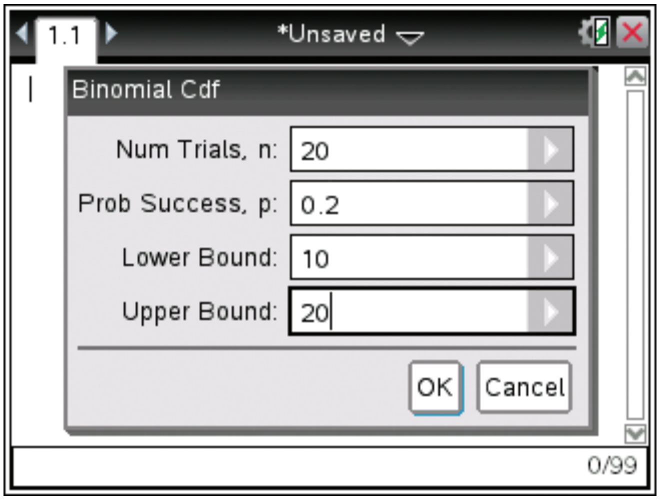

CP: Tap **Interactive → Distribution → binomialCDf**
Input Lower = 10, Upper = 20, Numtrial = 20, pos = 0.2

$$\therefore \Pr(X \geq 10) = 0.002\,594\,827\,379\ldots$$
$$= 0.003 \text{ (to 3 decimal places)}$$

b 80% of 20 = 16, $\Pr(X \geq 16 | X \geq 10) = \dfrac{\Pr(X \geq 16)}{\Pr(X \geq 10)}$

From the CAS calculator, $\Pr(X \geq 16) = 0.000\,000\,013\,803\,464\ldots$

Therefore $\quad \Pr(X \geq 16 | X \geq 10) = \dfrac{0.000\,000\,013\,803\,464\ldots}{0.002\,594\,827\,379\ldots}$

$$\approx 5.320 \times 10^{-6}$$

5 Let p be the probability of winning in any one game.
Then the probability of winning at least once in 5 games

$$= 1 - \text{the probability of losing every one of the 5 games}$$
$$= 1 - (1 - p)^5$$

Therefore $\quad 1 - (1 - p)^5 = 0.999\,68$

Solve for p $\quad 1 - p = (0.000\,32)^{\frac{1}{5}}$

Therefore $\quad 1 - p = 0.2$

which implies $\quad p = 0.8$

6 The probability of a telephone salesperson making a successful call is 0.05.

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a Let X be the number of successful calls out of 10.

$$\begin{aligned}\Pr(X \geq 1) &= 1 - \Pr(X = 0)\\ &= 1 - (0.95)^{10}\\ &= 0.401\end{aligned}$$

b Let Y be the number of successful calls out of n.

$$\begin{aligned}\Pr(Y \geq 1) &= 1 - \Pr(Y = 0)\\ &= 1 - (0.95)^{n}\end{aligned}$$

If $\quad \Pr(Y \geq 1) > 0.9$

then $\quad 1 - (0.95)^n > 0.9$

Therefore $\quad 0.1 > (0.95)^n$

By trial and error,

$$\begin{aligned}(0.95) &> 0.1\\ (0.95)^{10} &> 0.1\\ (0.95)^{20} &> 0.1\\ (0.95)^{40} &> 0.1\\ (0.95)^{44} &= 0.104\,67\ldots > 0.1\\ (0.95)^{45} &= 0.099\ldots\\ n &\geq 45\end{aligned}$$

7 a For a two-engine plane

Let X be the number of engines which will fail.

The plane will successfully complete its journey if 0 or 1 engine fails.

$$\begin{aligned}\Pr(X = 0) + \Pr(X = 1) &= (1 - q)^2 + 2q(1 - q)\\ &= 1 - 2q + q^2 + 2q - 2q^2\\ &= 1 - q^2\end{aligned}$$

b For a four-engine plane

Let Y be the number of engines which will fail.

The plane will successfully complete its journey if 0, 1 or 2 engines fail.

$$\begin{aligned}\Pr(Y = 0) + \Pr(Y = 1) + \Pr(Y = 2) &= (1 - q)^4 + 4q(1 - q)^3 + 6q^2(1 - q)^2\\ &= (1 - q)^2[(1 - q)^2 + 4q(1 - q) + 6q^2]\\ &= (1 - q)^2(1 - 2q + q^2 + 4q - 4q^2 + 6q^2)\\ &= (1 - q)^2(1 + 2q + 3q^2)\\ &= 1 - 4q^3 + 3q^4\end{aligned}$$

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c To find when a two-engine plane is preferable to a four-engine plane consider

$$1 - q^2 > 1 - 4q^3 + 3q^4$$
$$0 > q^2 - 4q^3 + 3q^4$$
$$0 > q^2(3q^2 - 4q + 1)$$
$$0 > (3q - 1)(q - 1)$$
$$\therefore \quad \frac{1}{3} < q < 1$$

A two-engine plane is preferred to a four-engine plane when $\frac{1}{3} < q < 1$.

8 **a** Let the probability of having type O blood be $p = 0.45$

The probability of at least four out of ten blood donors having blood type O can be approximated by simulation in a spreadsheet.

The exact probability can be calculated using the binomial distribution

$$\begin{aligned}
\Pr(X \geq 4) &= \Pr(X = 4) + \Pr(X = 5) + ... + \Pr(X = 10) \\
&= \binom{10}{4} p^4(1-p)^{10-4} + \binom{10}{5} p^5(1-p)^{10-5} + ... + \binom{10}{10} p^{10}(1-p)^{10-10} \\
&= \binom{10}{4} p^4(1-p)^6 + \binom{10}{5} p^5(1-p)^5 + ... + \binom{10}{10} p^{10}(1-p)^0 \\
&= 210 \times (0.45)^4 \times (.55)^6 + ... + 1 \times 0.45^{10} \times 0.55^0 \\
&= 0.734
\end{aligned}$$

b Answers will vary depending on the simulation. To verify the result from the simulation the exact answer can be calculated using the formula

$$\begin{aligned}
E(\text{number of trials for } x \text{ successes}) &= \frac{x}{p} \\
E(\text{number of trials for 4 successes}) &= \frac{4}{0.45} \\
&= 8.89 \\
&\approx 9
\end{aligned}$$

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Chapter 12 – Revision of chapters 9-11

Solutions to Technology-free questions

1 a Sum of numbers showing is 5 means that one of the following four outcomes is observed: $\{(1, 4), (2, 3), (3, 2), (4, 1)\}$.
Since there are 36 possible outcomes $n(\varepsilon) = 36$, and

$$\Pr(\text{sum is } 5) = \frac{4}{36} = \frac{1}{9}$$

b
$$\begin{aligned}\Pr(\text{sum is not } 5) &= 1 - \Pr(\text{sum is } 5)\\ &= 1 - \frac{1}{9}\\ &= \frac{8}{9}\end{aligned}$$

2 a Sample space:
$\{348, 384, 438, 483, 843, 834\}$,
$n(\varepsilon) = 6$

b Number is less than $500 = \{348, 384, 438, 483\}$,
$n(\text{less than } 500) = 4$,
$\Pr(\text{less than } 500) = \frac{4}{6} = \frac{2}{3}$

c Even $= \{348, 384, 438, 834\}$,
$n(\text{Even}) = 4$, $\Pr(\text{Even}) = \frac{2}{3}$

3 a
$$\begin{aligned}\Pr(\text{Not red}) &= \Pr(\text{Black})\\ &= \frac{26}{52}\\ &= \frac{1}{2}\end{aligned}$$

b
$$\begin{aligned}\Pr(\text{Not an ace}) &= 1 - \Pr(\text{Ace})\\ &= 1 - \frac{1}{13}\\ &= \frac{12}{13}\end{aligned}$$

4 a Area circle $= \pi r^2$,

Area $A = \frac{\pi r^2}{4}$,

$\Pr(A) = \frac{\pi r^2}{4} \div (\pi r^2) = \frac{1}{4}$

b Area circle$= \pi r^2$,
Area $A = \frac{135}{360} \times \pi r^2 = \frac{3}{8}\pi r^2$
$\Pr(A) = \frac{3}{8} \times \pi r^2 \div \pi r^2 = \frac{3}{8}$

5 a Let,
$\Pr(1) = \Pr(2) = \Pr(3) = \Pr(5) = x$.
Then $\Pr(4) = 4x$, and $\Pr(6) = \frac{x}{2}$.
Since the sum of probabilities is 1,
$x + x + x + x + 4x + \frac{x}{2} = 1$.

So $x = \frac{2}{17}$.
Thus
$\Pr(1) = \Pr(2) = \Pr(3) = \Pr(5) = \frac{2}{17}$,
$\Pr(4) = \frac{8}{17}$, $\Pr(6) = \frac{1}{17}$

b
$$\begin{aligned}\Pr(\text{ Not a } 4) &= 1 - \frac{8}{17}\\ &= \frac{9}{17}\end{aligned}$$

6 Pr(hitting the blue circle) =
$\pi(10)^2 \div \pi(20)^2 = 100\pi \div 400\pi = \frac{1}{4}$

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7 $\Pr(B) = 0.3, \Pr(H) = 0.4$, and $\Pr(B \cap H) = 0.1$.

a

$$\Pr(B \cup H) = \Pr(B) + \Pr(H) - \Pr(B \cap H)$$
$$= 0.3 + 0.40.1$$
$$= 0.6$$

b $\Pr(H|B) = \dfrac{\Pr(H \cap B)}{\Pr(B)}$

$$= \frac{0.1}{0.3}$$
$$= \frac{1}{3}$$

8

	Music	Not Music	
Painting	$\frac{15}{60}$	$\frac{30}{60}$	$\frac{45}{60}$
Not Painting	$\frac{15}{60}$	0	$\frac{15}{60}$
	$\frac{30}{60}$	$\frac{30}{60}$	1

From the table

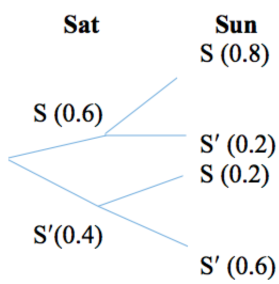
a $\dfrac{30}{60} = \dfrac{1}{2}$

b $\dfrac{45}{60} = \dfrac{3}{4}$

c $\dfrac{30}{60} = \dfrac{1}{2}$

d $\dfrac{15}{60} = \dfrac{1}{4}$

9 Let S be the event that the day is sunny.



a Pr(sunny all weekend) = Pr(SS)

$$= 0.6 \times 0.8$$
$$= 0.48$$

b

$$\Pr(\text{Sunny on Sunday}) = \Pr(SS \text{ or } S'S)$$
$$= 0.6 \times 0.8 + 0.4 \times 0.2$$
$$= 0.48 + 0.08$$
$$= 0.56$$

10 **a** $\Pr(A \cap B) = Pr(B|A)Pr(A)$

$$= 0.1 \times 0.5$$
$$= 0.05$$

b $Pr(A|B) = \dfrac{\Pr(A \cap B)}{Pr(B)}$

$$= \frac{0.05}{0.2}$$
$$= 0.25$$

11 A and B are independent events, and $\Pr(A) = 0.4, \Pr(B) = 0.5$.

a $\Pr(A|B) = \Pr(A) = 0.4$ (since A and B are independent)

b $\Pr(A \cap B) = \Pr(A)\Pr(B)$

(since A and B are independent)

$$= 0.4 \times 0.5$$
$$= 0.2$$

c

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$= 0.4 + 0.5 - 0.2$$
$$= 0.7$$

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12 Since order is important there are $10 \times 9 \times 8 = 720$ ways

13 Since order is not important, there are $\binom{52}{7} = 133784560$ different hands

14 There are $\binom{12}{3} = 220$ different committees (without restrictions)

a If there is one girl then there are two boys. We can choose one girl from 7 girls and two boys from 5 boys in $\binom{7}{1} \times \binom{5}{2} = 7 \times 10 = 70$ ways.
Thus, $\Pr(\text{one girl}) = \dfrac{70}{220} = \dfrac{7}{22}$

b If there are two girls then there is one boy. We can choose two girls from 7 girls and one boy from 5 boys in $\binom{7}{2} \times \binom{5}{1} = 21 \times 5 = 105$ ways.
Thus $\Pr(\text{one girl}) = \dfrac{105}{220} = \dfrac{21}{44}$

15 a

$$\begin{aligned}&\Pr(X \le 2)\\&= \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)\\&= 0.1 + 0.2 + 0.4\\&= 0.7\end{aligned}$$

b

$$\begin{aligned}&\Pr(X \ge 2)\\&= \Pr(X = 2) + \Pr(X = 3) + \Pr(X = 4)\\&= 0.4 + 0.1 + 0.2\\&= 0.7\end{aligned}$$

c

$$\begin{aligned}&\Pr(1 \le X \le 3)\\&= \Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3)\\&= 0.2 + 0.4 + 0.1\\&= 0.7\end{aligned}$$

16 a $\Pr(X = 4) = \binom{4}{4}0.2^4 \times 0.8^0$

$= 0.0016$

b $\Pr(X \ge 3) = \Pr(X = 3) + \Pr(X = 4)$

$$\begin{aligned}\Pr(X = 3) &= \binom{4}{3}(0.2)^3(0.8)^1\\&= 4 \times 0.008 \times 0.9\\&= 0.1536\\\Pr(X \ge 3) &= 0.1536 + 0.0256\\&= 0.0272\end{aligned}$$

17 a $\Pr(X = 3) = \binom{7}{3}(0.2)^3(0.8)^4$

b $\Pr(X < 3) = \binom{7}{3}(0.2)^0(0.8)^7 +$
$\binom{7}{1}(0.2)^1(0.8)^6 + \binom{7}{2}(0.2)^2(0.8)^5$

c $\Pr(X \ge 4) = \binom{7}{4}(0.2)^4(0.8)^3 +$
$\binom{7}{5}(0.2)^5(0.8)^2 + \binom{7}{6}(0.2)^6(0.8)^1 +$
$\binom{7}{7}(0.2)^7(0.8)^0$

18 Records show that x % of people will pass their driver's license on the first attempt.

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a

b $\left(\frac{x}{100}\right)^{10}$

c $10\left(\frac{x}{100}\right)^{9}\left(1-\frac{x}{100}\right)$

d $\left(\frac{x}{100}\right)^{10}+10\left(\frac{x}{100}\right)^{9}\left(1-\frac{x}{100}\right)$
$+45\left(\frac{x}{100}\right)^{8}\left(1-\frac{x}{100}\right)^{2}$

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Solutions to multiple-choice questions

1 E Pr (success) $= \dfrac{1}{12}$ for each

$$\text{Pr (both)} = \left(\frac{1}{12}\right)^2 = \frac{1}{144}$$

2 C $\Pr(WB) + Pr(BW) = \left(\dfrac{2}{5}\right)^2 + \left(\dfrac{3}{5}\right)^2 = \dfrac{13}{25}$

3 E Two dice, $\Pr(X > 12) = 0$,

$$\Pr(X = 12) = \frac{1}{36}$$

4 B $\Pr(G, B) + \Pr(B, G) = \dfrac{3}{7}\left(\dfrac{4}{6}\right) + \dfrac{4}{7}\left(\dfrac{3}{6}\right)$

$$= \frac{4}{7}$$

5 E $\Pr(X \cup Y) = \Pr(X) + \Pr(Y) - \Pr(X \cap Y)$

$$= \Pr(Y') + \Pr(Y) - 0$$

$$= 1$$

6 E Binomial, $n = 500$, $p = \dfrac{1}{2}$:

$$\Pr(X = 250) = {}^{500}\mathrm{C}_{250}\left(\frac{1}{2}\right)^{250}\left(\frac{1}{2}\right)^{250}$$

$$= {}^{500}\mathrm{C}_{250}\left(\frac{1}{2}\right)^{500}$$

7 C Binomial, $n = 6$, $p = \dfrac{1}{6}$:

$$\Pr(X \geq 1) = 1 - \Pr(X = 0)$$

$$= 1 - \left(\frac{5}{6}\right)^6$$

8 C $\Pr(\heartsuit \cup \mathrm{J}) = \Pr(\heartsuit) + \Pr(\mathrm{J}) - \Pr(\mathrm{J}\heartsuit)$

$$= \frac{1}{4} + \frac{1}{13} - \frac{1}{52}$$

$$= \frac{16}{52} = \frac{4}{13}$$

9 B $\Pr(R, R) = \left(\dfrac{k}{k+1}\right)\left(\dfrac{k-1}{k}\right)$

$$= \frac{k-1}{k+1}$$

10 D Replace: $\Pr(A, A) = \left(\dfrac{4}{52}\right)^2 = \dfrac{1}{169}$

No replace: $\Pr(A, A) = \dfrac{4}{52}\left(\dfrac{3}{51}\right)$

$$= \frac{1}{221}$$

Ratio=221:169 = 17:13

11 D Bill: $n = 2$, $p = \dfrac{1}{2}$

Charles: $n = 4$, $p = \dfrac{1}{4}$

$\Pr(\geq 1) = 1 - \Pr(\text{none})$

Bill: $1 - \left(\dfrac{1}{2}\right)^2 = \dfrac{3}{4} = \dfrac{192}{256}$

Charles: $1 - \left(\dfrac{3}{4}\right)^4 = \dfrac{175}{256}$

Bill:Charles = 192:175

12 D N(RAPIDS, vowels together)

= 2!(vowels) × 5!(cons + vowel group)

= 240

13 E n from $(m + n)$: ${}^{m+n}\mathrm{C}_n = \dfrac{(m+n)!}{n!m!}$

14 A Choose 7 from 12 = ${}^{12}\mathrm{C}_7 = 792$

15 E 4 letters, 4 choices, replacement $= 4^4 = 256$

16 E $\Pr(O, O, O) = \dfrac{3}{6}\left(\dfrac{2}{5}\right)\dfrac{1}{4} = \dfrac{1}{20}$

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17 **B** Person 1 has 6×10 possibilities. Person 2 enters by the same gate and can choose 9 exits.

18 **C** $\Pr(A \cap B) = \dfrac{1}{5}, \Pr(B) = \dfrac{1}{2},$

$\Pr(B|A) = \dfrac{1}{3}$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1}{5} \div \frac{1}{2} = \frac{2}{5}$$

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

$$\Pr(A) = \frac{\Pr(A \cap B)}{\Pr(B|A)}$$

$$= \frac{1}{5} \div \frac{1}{3} = \frac{3}{5}$$

19 **C** $\Pr(4, 6) + \Pr(6, 4) + \Pr(5, 5) = \dfrac{3}{36}$

20 **A** $\Pr(A, D, E, H, S) = \dfrac{1}{5!} = \dfrac{1}{120}$

21 **E** $\Pr(G,\ G) = \dfrac{4}{16}\left(\dfrac{3}{15}\right) = \dfrac{1}{20}$

22 **E** Binomial, $n = 6, p = \dfrac{1}{8}$

$$\Pr(X = 4) = {}^6C_4\left(\frac{1}{8}\right)^4\left(\frac{7}{8}\right)^2$$

$$= 15\left(\frac{1}{8}\right)^4\left(\frac{7}{8}\right)^2$$

23 **C** Binomial, $n = 3, p = p$

$$\Pr(X \le 1) = (1 - p)^3 + {}^3C_1 p(1 - p)^2$$

$$= (1 - p)^2(1 - p + 3p)$$

$$= (1 - p)^2(1 + 2p)$$

24 **D** Binomial, $n = 10, p = 0.8$

$$\Pr(X \ge 1) = 1 - \Pr(X = 0)$$

$$= 1 - (0.2)^{10}$$

25 **D** Binomial, $n = n,\ p = 0.15$

$$\Pr(X \ge 1) = 1 - \Pr(X = 0)$$

$$\therefore 0.85^n < 0.1$$

$$\left(\frac{20}{17}\right)^n > 10, \therefore n > 14.2$$

15 shots needed

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Solutions to extended-response questions

1

Interval	No. of plants	Proportion	No. of plants > 30 cm	Proportion
(0, 10]	1	$\frac{1}{56}$		
(10, 20]	2	$\frac{2}{56}$		
(20, 30]	4	$\frac{4}{56}$		
(30, 40]	6	$\frac{6}{56}$	6	$\frac{6}{49}$
(40, 50]	13	$\frac{13}{56}$	13	$\frac{13}{49}$
(50, 60]	22	$\frac{22}{56}$	22	$\frac{22}{49}$
(60, 70]	8	$\frac{8}{56}$	8	$\frac{8}{49}$
Total	56	1	49	1

Let X be the height of the plants (in cm).

a **i** $\Pr(X > 50) = \frac{22}{56} + \frac{8}{56}$

$$= \frac{30}{56} = \frac{15}{28} \approx 0.5357$$

ii $\Pr(X > 50) + \Pr(X \le 30) = \frac{30}{56} + \frac{1}{56} + \frac{2}{56} + \frac{4}{56}$

$$= \frac{37}{56} \approx 0.6607$$

iii $\Pr(X > 40|X > 30) = 1 - \Pr(X \le 40|X > 30)$

$$= 1 - \frac{6}{49} = \frac{43}{49} \approx 0.8776$$

b $\Pr(F) = \frac{6}{7}$ and $\Pr(D) = \frac{1}{4}$

i $\Pr(F \cap D') = \Pr(F) \times \Pr(D')$

$$= \frac{6}{7}(1 - \frac{1}{4}) = \frac{6}{7} \times \frac{3}{4}$$

$$= \frac{9}{14} \approx 0.6429$$

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ii $\Pr(F \cap D' \cap (X > 50)) = \Pr(F) \times \Pr(D') \times \Pr(X > 50)$

$$= \frac{9}{14} \times \frac{15}{28} = \frac{135}{392}$$

$$\approx 0.3444$$

2 a Possible choices

C	B
1	any ball
2	5
4	5
7	no possible choice

$\therefore$ probability that B draws a higher number than C

$$= \frac{1}{3} + \frac{1}{6} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3}$$

$$= \frac{1}{3} + \frac{1}{18} + \frac{1}{9}$$

$$= \frac{6+1+2}{18} = \frac{1}{2}$$

b Possible choices

B	C	A
2	1	3 or 6
2	2	3 or 6
2	4	6
2	7	no possible choice
5	1	6
5	2	6
5	4	6
5	7	no possible choice

$\therefore$ probability that A draws a higher number than B or C

$$= \frac{2}{3} \times \frac{1}{3} \times \frac{5}{6} + \frac{2}{3} \times \frac{1}{6} \times \frac{5}{6} + \frac{2}{3} \times \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{6} \times \frac{1}{6} + \frac{1}{3} \times \square \frac{1}{3} \times \frac{1}{6}$$

$$= \frac{10}{54} + \frac{10}{108} + \frac{2}{54} + \frac{1}{54} + \frac{1}{108} + \frac{1}{54}$$

$$= \frac{20+10+4+2+1+2}{108} = \frac{39}{108} = \frac{13}{36}$$

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3 a

1st card		2nd card	Probability
$\frac{2}{8}$ A	—	$\frac{3}{7}$ L	$\frac{6}{56}$
$\frac{1}{8}$ E	—	$\frac{3}{7}$ L	$\frac{3}{56}$
$\frac{3}{8}$ L	—	$\frac{2}{7}$ L	$\frac{6}{56}$
$\frac{1}{8}$ P	—	$\frac{3}{7}$ L	$\frac{3}{56}$
$\frac{1}{8}$ R	—	$\frac{3}{7}$ L	$\frac{3}{56}$

$$\text{Probability that second card bears L} = \frac{6}{56} + \frac{3}{56} + \frac{6}{56} + \frac{3}{56} + \frac{3}{56}$$
$$= \frac{21}{56} = \frac{3}{8}$$

b $\Pr(\text{A, L, E}) = \frac{2}{8} \times \frac{3}{7} \times \frac{1}{6} = \frac{1}{56}$

c

1st card	2nd card		3rd card	Probability
$\frac{2}{8}$ A	$\frac{3}{7}$ L	—	$\frac{1}{6}$ E	$\frac{6}{336}$
	$\frac{1}{7}$ E	—	$\frac{3}{6}$ L	$\frac{6}{336}$
$\frac{3}{8}$ L	$\frac{2}{7}$ A	—	$\frac{1}{6}$ E	$\frac{6}{336}$
	$\frac{1}{7}$ E	—	$\frac{2}{6}$ A	$\frac{6}{336}$
$\frac{1}{8}$ E	$\frac{2}{7}$ A	—	$\frac{3}{6}$ L	$\frac{6}{336}$
	$\frac{3}{7}$ L	—	$\frac{2}{6}$ A	$\frac{6}{336}$

$$\Pr(\text{A, L, E in any order}) = \frac{36}{336} = \frac{3}{28}\left(\text{or } 3! \times \frac{1}{56} = \frac{6}{56} = \frac{3}{28}\right)$$

d

1st card		2nd card	Probability
$\frac{2}{8}$ A	—	$\frac{6}{7}$ not A	$\frac{12}{56}$
$\frac{1}{8}$ E	—	$\frac{7}{7}$ not E	$\frac{7}{56}$
$\frac{3}{8}$ L	—	$\frac{5}{7}$ not L	$\frac{15}{56}$
$\frac{1}{8}$ P	—	$\frac{7}{7}$ not P	$\frac{7}{56}$
$\frac{1}{8}$ R	—	$\frac{7}{7}$ not R	$\frac{7}{56}$

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Probability that first two cards bear different letters

$$= \frac{12}{56} + \frac{7}{56} + \frac{15}{56} + \frac{7}{56} + \frac{7}{56}$$

$$= \frac{48}{56} = \frac{6}{7}$$

4 Let X be the number of correct predictions, $n = 10$, $p = 0.6$

a Pr(first 8 correct, last 2 wrong) $= (0.6)^8(0.4)^2 \approx 0.0027$

b $\Pr(X = 8) = \binom{10}{8}(0.6)^8(0.4)^2$

$$= \frac{10 \times 9 \times 8!}{8! \times 2 \times 1} \times 0.002\,687\,385$$

$$= 45 \times 0.002\,687\,385$$

$$= 0.120\,932\,352 \approx 0.12$$

c $\Pr(X \geq 8) = \Pr(X = 8) + \Pr(X = 9) + \Pr(X = 10)$

$$= 0.120\,932\,352 + \binom{10}{9}(0.6)^9(0.4)^1 + \binom{10}{10}(0.6)^{10}(0.4)^0$$

$$= 0.120\,932\,352 + 0.040\,310\,784 + 0.006\,046\,617$$

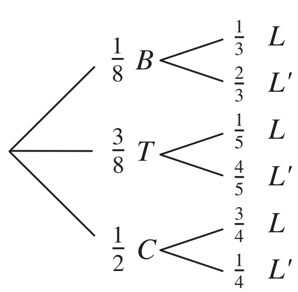
$$= 0.167\,289\,753 \approx 0.17$$

d $\Pr(X = 8|X \geq 8) = \dfrac{\Pr(X = 8)}{\Pr(X \geq 8)}$

$$= \frac{0.120\,932\,352}{0.167\,289\,753}$$

$$\approx 0.722\,891\,568 \approx 0.72$$

5 **a** Let L be the event ‘ an employee is late’, B the event ‘travels by bus’, T the event ‘travels by train’, and C the event ‘travels by car’.



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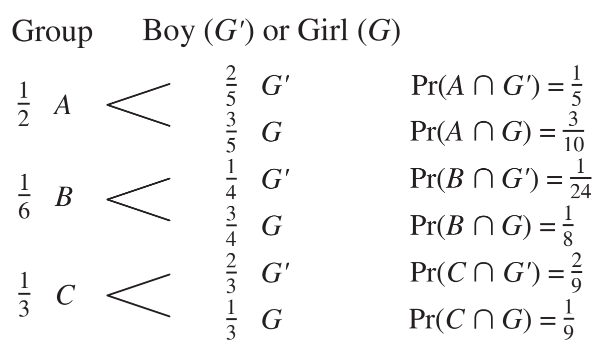
$$\begin{aligned}\Pr(L) &= \Pr(L \cap B) + \Pr(L \cap T) + \Pr(L \cap C)\\ &= \Pr(L|B) \times \Pr(B) + \Pr(L|T) \times \Pr(T) + \Pr(L|C) \times \Pr(C)\\ &= \frac{1}{8} \times \frac{1}{3} + \frac{3}{8} \times \frac{1}{5} + \frac{1}{2} \times \frac{3}{4}\\ &= \frac{1}{24} + \frac{3}{40} + \frac{3}{8}\\ &= \frac{5 + 9 + 45}{120}\\ &= \frac{59}{120} \approx 0.4917\end{aligned}$$

b $$\begin{aligned}\Pr(C|L) &= \frac{\Pr(C \cap L)}{\Pr(L)} = \frac{\Pr(L|C) \times \Pr(C)}{\Pr(L)}\\ &= \frac{\frac{3}{8}}{\frac{59}{120}} = \frac{3 \times 120}{8 \times 59}\\ &= \frac{45}{59} \approx 0.7627\end{aligned}$$

6 Let A be the event 'Group A is chosen', B be the event 'Group B is chosen' and C be the event ' Group C is chosen'



a $$\begin{aligned}\Pr(G') &= \Pr(G' \cap A) + \Pr(G' \cap B) + \Pr(G' \cap C)\\ &= \frac{1}{5} + \frac{1}{24} + \frac{2}{9}\\ &= \frac{216 + 45 + 240}{1080}\\ &= \frac{501}{1080} = \frac{167}{360} \approx 0.639\end{aligned}$$

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b **i** $\Pr(A|G) = \dfrac{\Pr(A \cap G)}{\Pr(G)}$

$$= \frac{\Pr(A \cap G)}{\Pr(A \cap G) + \Pr(B \cap G) + \Pr(C \cap G)}$$

$$= \frac{\frac{3}{10}}{\frac{3}{10} + \frac{1}{8} + \frac{1}{9}}$$

$$= \frac{\frac{3}{10}}{\frac{108 + 45 + 40}{360}}$$

$$= \frac{3}{10} \times \frac{360}{193}$$

$$= \frac{108}{193} \approx 0.596$$

Note: $\Pr(G)$ can also be found by *calculating 1 − Pr*(G') or directly from the tree diagram.

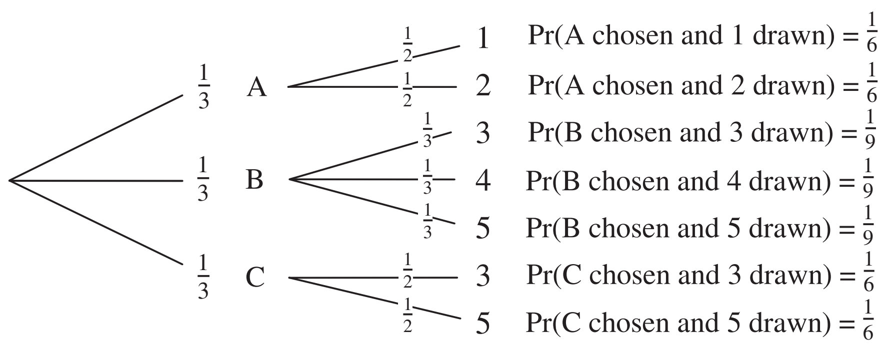
ii $\Pr(B|G) = \dfrac{\Pr(B \cap G)}{\Pr(G)}$

$$= \frac{\frac{1}{8}}{\frac{193}{360}}$$

$$= \frac{1}{8} \times \frac{360}{193}$$

$$= \frac{45}{193} \approx 0.332$$

7 **a**



i Pr(4 drawn) = Pr(B chosen and 4 drawn)

$$= \frac{1}{9}$$

$$\approx 0.1111$$

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ii $\Pr(3 \text{ drawn}) = \Pr(B \text{ chosen and } 3 \text{ drawn}) + \Pr(C \text{ chosen and } 3 \text{ drawn})$

$$= \frac{1}{9} + \frac{1}{6}$$

$$= \frac{5}{18}$$

$$\approx 0.2778$$

b **i** Pr(balls drawn by David and Sally are both 4)

$$= \Pr(B \text{ chosen and } 4 \text{ drawn}) \times \Pr(B \text{ chosen and } 4 \text{ drawn})$$

$$= \frac{1}{9} \times \frac{1}{9} = \frac{1}{81}$$

$$\approx 0.0123$$

ii Pr(David and Sally both draw balls numbered 3 from the same bag)

$$= \Pr(B \text{ chosen and } 3 \text{ drawn}) \times \Pr(B \text{ chosen and } 3 \text{ drawn})$$

$$+ \Pr(C \text{ chosen and } 3 \text{ drawn}) \times \Pr(C \text{ chosen and } 3 \text{ drawn})$$

$$= \frac{1}{9} \times \frac{1}{9} + \frac{1}{6} \times \frac{1}{6}$$

$$= \frac{1}{81} + \frac{1}{36}$$

$$= \frac{36 + 81}{2916}$$

$$= \frac{117}{2916} = \frac{13}{324}$$

$$\approx 0.0401$$

8 **a** **i**

$$m + 10 = 40$$

$$\therefore \quad m = 30$$

$$q + 10 = 45$$

$$\therefore \quad q = 35$$

$$m + q + s + 10 = 100$$

$$\therefore \quad s = 100 - 10 - m - q$$

$$= 100 - 10 - 30 - 35$$

$$\therefore \quad s = 25$$

ii $m + q = 30 + 35$

$$= 65$$

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b Let H be the event 'History is taken'
Let G be the event 'Geography is taken'.

$$\begin{aligned}\Pr(H \cap G') &= \frac{m}{100} \\ &= \frac{30}{100} \\ &= 0.3\end{aligned}$$

c $$\begin{aligned}\Pr(G|H') &= \frac{\Pr(G \cap H')}{\Pr(H')} \\ &= \frac{\frac{q}{100}}{\frac{100 - m - 10}{100}} \\ &= \frac{q}{90 - m} \\ &= \frac{35}{60} = \frac{7}{12} \approx 0.5833\end{aligned}$$

9 **a** Pr(total score = 23) = $\Pr(A = 8) \times \Pr(B = 6) \times \Pr(C = 9)$ as there are no other ways of achieving 23, and spins are independent.

$$\therefore \qquad \begin{aligned}\Pr(\text{total score} = 23) &= \frac{4}{10} \times \frac{7}{10} \times \frac{3}{10} \\ &= \frac{84}{1000} \\ &= 0.084\end{aligned}$$

b Possible combinations for Player B to score more than Player C

Player B	Player C
3	2
6	2
6	5

$$\begin{aligned}\Pr(B > C) &= \Pr(B = 3, C = 2) + \Pr(B = 6, C = 2) + \Pr(B = 6, C = 5) \\ &= \Pr(B = 3) \times \Pr(C = 2) + \Pr(B = 6) \times \Pr(C = 2) + \Pr(B = 6) \\ &\quad \times \Pr(C = 5) \\ &= \frac{3}{10} \times \frac{1}{10} + \frac{7}{10} \times \frac{1}{10} + \frac{7}{10} \times \frac{6}{10}\end{aligned}$$

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$$= \frac{3 + 7 + 42}{100}$$
$$= \frac{52}{100}$$
$$= 0.52$$

c Possible combinations for Player C to score more than Player A

Player C	Player A
2	1
5	1
9	1
5	4
9	4
9	8

$$\begin{aligned}
\Pr(C > A) &= \Pr(C = 2, A = 1) + \Pr(C = 5, A = 1) + \Pr(C = 9, A = 1) \\
&\quad + \Pr(C = 5, A = 4) + \Pr(C = 9, A = 4) + \Pr(C = 9, A = 8) \\
&= \Pr(C = 2) \times \Pr(A = 1) + \Pr(C = 5) \times \Pr(A = 1) + \Pr(C = 9) \\
&\quad \times \Pr(A = 1) + \Pr(C = 5) \times \Pr(A = 4) + \Pr(C = 9) \times \Pr(A = 4) \\
&\quad + \Pr(C = 9) \times \Pr(A = 8) \\
&= \frac{1}{10} \times \frac{2}{10} + \frac{6}{10} \times \frac{2}{10} + \frac{3}{10} \times \frac{2}{10} + \frac{6}{10} \times \frac{4}{10} + \frac{3}{10} \times \frac{4}{10} + \frac{3}{10} \times \frac{4}{10} \\
&= \frac{2 + 12 + 6 + 24 + 12 + 12}{100} \\
&= \frac{68}{100} \\
&= 0.68
\end{aligned}$$

10

a There are $3 \times 4 \times 5 = 60$ different routes from A to D.

b There are $2 \times 2 \times 2 = 8$ routes without roadworks.

c $\Pr(\text{roadworks at each stage}) = \frac{1}{3} \times \frac{2}{4} \times \frac{3}{5}$

$$= \frac{1}{10} = 0.1$$

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11 Let A be the event 'A hits the target', B be the event 'B hits the target', and C be the event 'C hits the target'.

$\therefore \Pr(A) = \frac{1}{5}, \Pr(B) = \frac{1}{4}, \Pr(C) = \frac{1}{3}$

a $\Pr(A \cap B \cap C) = \Pr(A) \times \Pr(B) \times \Pr(C)$ as $A,\ B,\ C$ are independent

$$= \frac{1}{5} \times \frac{1}{4} \times \frac{1}{3}$$

$$= \frac{1}{60} \approx 0.0167$$

b $\Pr(A') = \frac{4}{5}, \Pr(B') = \frac{3}{4}$

$$\Pr(A' \cap B' \cap C) = \Pr(A') \times \Pr(B') \times \Pr(C)$$

$$= \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3}$$

$$= \frac{1}{5} = 0.2$$

c Pr(at least one shot hits the target) = 1 − Pr (no shot hits the target)

$$= 1 - \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3}$$

$$= 1 - \frac{2}{5}$$

$$= \frac{3}{5} = 0.6$$

d $\Pr(C|\text{only one shot hits the target})$

$$= \frac{\Pr(C \cap A' \cap B')}{\Pr(A \cap B' \cap C') + \Pr(A' \cap B \cap C') + \Pr(A' \cap B' \cap C)}$$

$$= \frac{\frac{1}{3} \times \frac{4}{5} \times \frac{3}{4}}{\frac{1}{5} \times \frac{3}{4} \times \frac{2}{3} + \frac{4}{5} \times \frac{1}{4} \times \frac{2}{3} + \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3}}$$

$$= \frac{\frac{12}{60}}{\frac{6}{60} + \frac{8}{60} + \frac{12}{60}}$$

$$= \frac{12}{26}$$

$$= \frac{6}{13} \approx 0.4615$$

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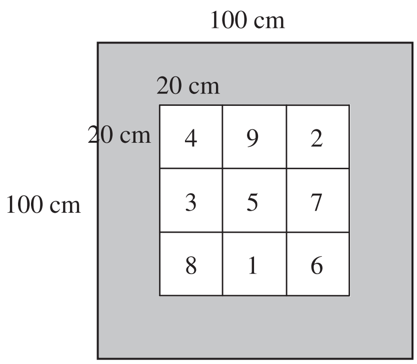
12 a



i Area of large outer square $= 100 \times 100 = 10\,000$ cm^2.

ii Area of one inner square $= 20 \times 20 = 400$ cm^2.

iii Area of shaded region $= 10\,000 - 9 \times 400 = 6400$ cm^2.

b **i** $\Pr(\text{one dart will score } 7) = \dfrac{400}{10\,000}$

$= 0.04$

(i.e. area of small square marked 7 divided by area of large square)

ii $\Pr(\text{at least } 7) = \Pr(7) + \Pr(8) + \Pr(9)$

$= 3 \times 0.4 = 0.12$

iii $\Pr(\text{score will be } 0) = \dfrac{\text{area of shaded region}}{\text{total area of board}}$

$= \dfrac{6400}{10\,000} = 0.64$

c **i** To get 18 from two darts, 9 and 9 need to be thrown.

$\Pr(18) = 0.04 \times 0.04$

$= 0.0016$

ii Throws to score 24 are 6, 9, 9 or 7, 8, 9 or 8, 8, 8 in any order, i.e. possible throws

6 9 9	7 8 9
7 9 8	8 7 9
8 8 8	8 9 7
9 6 9	9 7 8
9 8 7	9 9 6

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There are 10 winning combinations.

$$\Pr(\text{a winning combination}) = (0.04)^3$$

$$\therefore \Pr(\text{scoring } 24) = 10 \times (0.04)^3$$

$$= 10 \times 0.000064$$

$$= 0.00064$$

13 a The possible choices are

c	b
8	11
3	11
3	7

$$\begin{aligned}\Pr(c < b) &= \Pr(c = 8, b = 11) + \Pr(c = 3, b = 11) + \Pr(c = 3, b = 7)\\ &= \frac{1}{3} \times \frac{1}{6} + \frac{2}{3} \times \frac{1}{6} + \frac{2}{3} \times \frac{1}{3}\\ &= \frac{1}{18} + \frac{2}{18} + \frac{2}{9}\\ &= \frac{7}{18} \approx 0.3889\end{aligned}$$

b Possible choices

b	c	a
1	3	6
1	3	10
1	8	10
7	3	10
7	8	10

$$\begin{aligned}\Pr(a > \text{ both } b \text{ and } c) &= \Pr(a = 6, b = 1, c = 3) + \Pr(a = 10, b = 1, c = 3)\\ &\quad + \Pr(a = 10, b = 1, c = 8) + \Pr(a = 10, b = 7, c = 3)\\ &\quad + \Pr(a = 10, b = 7, c = 8)\\ &= \frac{2}{3} \times \frac{1}{2} \times \frac{2}{3} + \frac{1}{6} \times \frac{1}{2} \times \frac{2}{3} + \frac{1}{6} \times \frac{1}{2} \times \frac{1}{3} + \frac{1}{6} \times \frac{1}{3} \times \frac{2}{3}\\ &\quad + \frac{1}{6} \times \frac{1}{3} \times \frac{1}{3}\\ &= \frac{4}{18} + \frac{2}{36} + \frac{1}{36} + \frac{2}{54} + \frac{1}{54}\\ &= \frac{24 + 6 + 3 + 4 + 2}{108}\\ &= \frac{39}{108}\\ &= \frac{13}{36} \approx 0.3611\end{aligned}$$

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c Possible choices

a	b	c
0	1	3
0	1	8
0	7	8
6	1	8

$$\begin{aligned}
\Pr(c > a + b) &= \Pr(c = 3,\ a = 0,\ b = 1) + \Pr(c = 8,\ a = 0,\ b = 1) \\
&\quad + \Pr(c = 8,\ a = 0,\ b = 7) + \Pr(c = 8,\ a = 6,\ b = 1) \\
&= \frac{2}{3} \times \frac{1}{6} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{6} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{6} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} \times \frac{1}{2} \\
&= \frac{2}{36} + \frac{1}{36} + \frac{1}{54} + \frac{2}{18} \\
&= \frac{6 + 3 + 2 + 12}{108} \\
&= \frac{23}{108} \approx 0.2130
\end{aligned}$$

14 a It can be considered as a binomial distribution, with $n = 5$ and $p = 0.2$.
Let X be the number of trout caught in 5 days.

i $\Pr(X = 0) = (0.8)^5 = 0.32768 \approx 0.328$

ii $\Pr(X = 2) = {}^5C_2(0.2)^2(0.8)^3 = 0.2048 \approx 0.205$

iii Probability of at least 1 $= \Pr(X \geq 1)$

$$= 1 - \Pr(X = 0) = 1 - (0.8)^5 = 0.67232 \approx 0.672$$

b i For n days, the probability of catching no trout $= (0.8)^n$.
Therefore the probability of catching at least one $= 1 - (0.8)^n$.

Consider $1 - (0.8)^n > 0.9$

which is equivalent to $(0.8)^n < 0.1$

Using a calculator gives $n = 11$

ii For n days, the probability of catching no trout $= (0.8)^n$.
For n days, the probability of catching one trout $= 0.2n(0.8)^{n-1}$ (binomial distribution).
Probability of catching more than one $= 1 - 0.2n(0.8)^{n-1} - (0.8)^n$.
Use a calculator to find the value of n. It is 18 days.

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15 a Let X be the number of goals scored. X is the random variable of a binomial distribution. The parameter values are $n = 10$ and $p = 0.3$.

i $\Pr(X = 1) = {}^{10}C_1(0.3)(0.7)^9 \approx 0.121$

ii $\Pr(X \geq 2) = 1 - \Pr(X \leq 1)$

$= 1 - 0.149308\ldots \approx= 0.851$

iii $\Pr(X \leq 2) \approx 0.383$

b i The probability of scoring at least one goal

$= 1 -$ probability of scoring no goals

$= 1 - (0.7)^n$

Therefore the least value of n, for which the probability of scoring at least one goal is more than 0.95, is given by the inequality

$$1 - (0.7)^n > 0.95$$

or equivalently $\quad (0.7)^n < 0.05$

The smallest such value is 9.

ii The probability of scoring more than one goal

$= 1 -$ probability of scoring no goals or one goal

$= 1 - (0.7)^n - n \times 0.3 \times (0.7)^{n-1}$

Therefore the least value of n, for which the probability of scoring more than one goal is more than 0.95, is given by the inequality

$$1 - (0.7)^n - n \times 0.3 \times (0.7)^{n-1} > 0.95$$

or equivalently $\quad (0.7)^n + n \times 0.3 \times (0.7)^{n-1} < 0.05$

The smallest such value of n is 14.

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Chapter 13 – Exponential functions and logarithms

Solutions to Exercise 13A

1 a $x^2x^3 = x^{2+3} = x^5$

b $2(x^3x^4)4 = 8x^{3+4} = 8x^7$

c $x^5 \div x^3 = x^{5-3} = x^2$

d $4x^6 \div 2x^3 = 2x^{6-3} = 2x^3$

e $(a^3)^2 = a^{2\times3} = a^6$

f $(2^3)^2 = 2^{3\times2} = 2^6$

g $(xy)^2 = x^2y^2$

h $(x^2y^3)^2 = (x^2)^2(y^3)^2$
$= x^{2\times2}y^{3\times2} = x^4y^6$

i $\left(\frac{x}{y}\right)^3 = \frac{x^3}{y^3}$

j $\left(\frac{x^3}{y^2}\right)^2 = \frac{(x^3)^2}{(y^2)^2}$
$= \frac{x^{3\times2}}{y^{2\times2}} = \frac{x^6}{y^4}$

2 a $3^5 \times 3^{12} = 3^{5+12} = 3^{17}$

b $x^3y^2 \times x^4y^3 = x^{3+4}y^{2+3} = x^7y^5$

c $3^{x+1} \times 3^{3x+2} = 3^{x+1+3x+2} = 3^{4x+3}$

d $5a^3b^2 \times 6a^2b^4 = 30a^{3+2}b^{2+4} = 30a^5b^6$

3 a $\frac{x^5y^2}{x^3y} = x^{5-3}y^{2-1} = x^2y$

b $\frac{b^{5x} \times b^{2x+1}}{b^{3x}} = b^{5x+2x+1-3x} = b^{4x+1}$

c $\frac{8a^2b \times 3a^5b^2}{6a^2b^2} = 4a^{2+5-2}b^{1+2-2} = 4a^5b$

4 a $7^{-2} = \frac{1}{7^2} = \frac{1}{49}$

b $\left(\frac{1}{4}\right)^{-3} = 4^3 = 64$

c $\left(\frac{5}{2}\right)^{-3} = \left(\frac{2}{5}\right)^3 = \frac{8}{125}$

5 a $(b^5)^2 = b^{10}$

b $\left(\left(\frac{1}{3}\right)^{-2}\right)^3 = \left(\frac{1}{3}\right)^{-6} = 3^6 = 729$

c $(b^5)^2 \times (b^2)^{-3} = b^{10} \times b^{-6} = b^4$

6 a $(3a^4b^3)^3 \times (4a^2b^4)^{-2}$
$= 27a^{12}b^9 \times 4^{-2}a^{-4}b^{-8}$
$= \frac{27}{16}a^8b$

b $\left(\frac{5a^3b^3}{ab^2c^2}\right)^3 \div (a^2b^{-1}c)^3$
$= \left(5a^2bc^{-2}\right)^3 \times a^{-6}b^3c^{-3}$
$= 125a^6b^3c^{-6} \times a^{-6}b^3c^{-3}$
$= 125b^6c^{-9}$
$= \frac{125b^6}{c^9}$

7 a $(-2)^6 = 64$

b $(-3a)^3 = -27a^3$

c $(-2a)^5 \times 3a^{-2} = -32a^5 \times 3a^{-2}$
$= -96a^3$

8 **a** $36^n \times 12^{-2n} = 2^{-2n}$

b $\dfrac{2^{-3} \times 8^4 \times 32^{-3}}{4^{-4} \times 2^{-2}} = 2^4$

c $\dfrac{5^{2n} \times 10^n}{8^n \times 5^n} = \dfrac{5^{2n}}{2^{2n}}$

9 **a** $x^3x^4x^2 = x^{3+4+2} = x^9$

b $2^4 4^3 8^2 = 2^4 2^6 2^6$
$= 2^{4+6+6} = 2^{16}$

c $3^4 9^2 27^3 = 3^4 3^4 3^9$
$= 3^{4+4+9} = 3^{17}$

d $(q^2p)^3(qp^3)^2 = q^6p^3q^2p^6$
$= q^{6+2}p^{3+6} = q^8p^9$

e $a^2b^{-3}(a^3b^2)^3 = a^2b^{-3}a^9b^6$
$= a^{2+9}b^{6-3} = a^{11}b^3$

f $(2x^3)^2(4x^4)^3 = 2^2x^{3x2}4^3x^{3x4}$
$= 2^22^6x^6x^{12} = 2^8x^{18}$

g $m^3p^2(m^2n^3)^4(p^{-2})^2 = m^3p^2m^8n^{12}p^{-4}$
$= m^{11}n^{12}p^{-2}$

h $2^3a^3b^2(2a^{-1}b^2)^{-2} = 2^3a^3b^22^{-2}a^2b^{-4}$
$= 2a^5b^{-2}$

10 **a** $\dfrac{x^3y^5}{xy^2} = x^{3-1}y^{5-2} = x^2y^3$

b $\dfrac{16a^5b4a^4b^3}{8ab} = \dfrac{64}{8}a^{5+4-1}b^{1+3-1}$
$= 8a^8b^3$

c $\dfrac{(-2xy)^2 2(x^2y)^3}{8(xy)^3} = \dfrac{4x^2y^2 2x^6y^3}{8x^3y^3}$
$= \dfrac{8}{8}x^{2+6-3}y^{2+3-3}$
$= x^5y^2$

d $\dfrac{(-3x^2y^3)^2}{(2xy)^3}\dfrac{4x^4y^3}{(xy)^3} = \dfrac{9x^4y^6}{8x^3y^3}\dfrac{4x^4y^3}{x^3y^3}$
$= \dfrac{9}{2}x^{4+4-3-3}y^{6+3-3-3}$
$= \dfrac{9x^2y^3}{2}$

11 **a**
$m^3n^2p^{-2}(mn^2p)^{-3} = m^3n^2p^{-2}\ m^{-3}n^{-6}p^{-3}$
$= m^{3-3}n^{2-6}p^{-2-3}$
$= n^{-4}p^{-5} = \dfrac{1}{n^4p^5}$

b
$\dfrac{x^3yz^{-2}2(x^3y^{-2}z)^2}{xyz^{-1}} = \dfrac{2x^3yz^{-2}x^6y^{-4}z^2}{xyz^{-1}}$
$= 2x^{3+6-1}y^{1-4-1}z^{-2+2+1}$
$= 2x^8y^{-4}z = \dfrac{2x^8z}{4}$

c $\dfrac{a^2b(ab^{-2})^{-3}}{(a^{-2}b^{-1})^{-2}} = \dfrac{a^2ba^{-3}b^6}{a^4b^2}$
$= a^{2-3-4}b^{1+6-2}$
$= a^{-5}b^5 = \dfrac{b^5}{a^5}$

d $\dfrac{a^2b^3c^{-4}}{a^{-1}b^2c^{-3}} = a^{2+1}b^{3-2}c^{3-4}$
$= \dfrac{a^3b}{c}$

e $\dfrac{a^{2n-1}b^3c^{1-n}}{a^{n-3}b^{2-n}c^{2-2n}}$
$= a^{2n-1-n+3}b^{3-2+n}c^{1-n-2+2n}$
$= a^{n+2}b^{n+1}c^{n-1}$

12 a $3^{4n}9^{2n}27^{3n} = 3^{4n}3^{4n}3^{9n}$
$= 3^{17n}$

b $\dfrac{2^n 8^{n+1}}{32^n} = \dfrac{2^n 2^{3n+3}}{2^{5n}}$
$= 2^{n+3n+3-5n} = 2^{3-n}$

c $\dfrac{3^{n-1}9^{2n-3}}{6^2 3^{n+2}} = \dfrac{3^{n-1}3^{4n-6}}{6^2 3^{n+2}}$
$= \dfrac{3^{4n-9}}{36} = \dfrac{3^{4n-11}}{2^2}$

d $\dfrac{2^{2n}9^{2n-1}}{6^{n-1}} = \dfrac{2^{2n}3^{4n-2}}{6^{n-1}}$
$= \dfrac{2^{2n}3^{4n-2}}{2^{n-1}3^{n-1}}$
$= 2^{2n-n+1}3^{4n-2-n+1}$
$= 2^{n+1}3^{3n-1}$

e $\dfrac{25^{2n}5^{n-1}}{5^{2n+1}} = \dfrac{5^{4n}5^{n-1}}{5^{2n+1}}$
$= 5^{4n+n-1-2n-1} = 5^{3n-2}$

f $\dfrac{6^{x-3}4^x}{3^{x+1}} = \dfrac{3^{x-3}2^{x-3}2^{2x}}{3^{x+1}}$
$= 3^{x-3-x-1}2^{x-3+2x}$
$= 2^{3x-3}3^{-4}$

g $\dfrac{6^{2n}9^3}{27^n 8^n 16^n} = \dfrac{3^{2n}2^{2n}3^6}{3^{3n}2^{3n}2^{4n}}$
$= 3^{2n+6-3n}2^{2n-3n-4n}$
$= 3^{6-n}2^{-5n}$

h $\dfrac{3^{n-2}9^{n+1}}{27^{n-1}} = \dfrac{3^{n-2}3^{2n+2}}{3^{3n-3}}$
$= 3^{n-2+2n+2-3n+3}$
$= 3^3 = 27$

i $\dfrac{8\,2^5 3^7}{9\,2^7 81} = \dfrac{2^3 2^5 3^7}{3^2 2^7 3^4}$
$= 2^{3+5-7}3^{7-2-4}$
$= (2)(3) = 6$

13 a $\dfrac{(8^3)^4}{(2^{12})^2} = \dfrac{2^{36}}{2^{24}}$
$= 2^{36-24}$
$= 2^{12} = 4096$

b $\dfrac{(125)^3}{(25)^2} = \dfrac{5^9}{5^4}$
$= 5^{9-4}$
$= 5^5 = 3125$

c $\dfrac{(81)^4 \div (27^3)}{9^2} = \dfrac{3^{16} \div 3^9}{3^4}$
$= \dfrac{3^{16} \div 3^9}{3^4}$
$= 3^{16-9-4}$
$= 3^3 = 27$

Solutions to Exercise 13B

1 a $125^{\frac{2}{3}} = 5^2 = 25$

b $243^{\frac{3}{5}} = 3^3 = 27$

c $81^{-\frac{1}{2}} = \frac{1}{\sqrt{81}} = \frac{1}{9}$

d $64^{\frac{2}{3}} = 4^2 = 16$

e $\left(\frac{1}{8}\right)^{\frac{1}{3}} = \frac{1}{2}$

f $32^{-\frac{2}{5}} = \frac{1}{32^{\frac{2}{5}}}$

$= \frac{1}{2^2} = \frac{1}{4}$

g $125^{-\frac{2}{3}} = \frac{1}{125^{\frac{2}{3}}}$

$= \frac{1}{5^2} = \frac{1}{25}$

h $32^{\frac{4}{5}} = 2^4 = 16$

i $1000^{\frac{4}{3}} = \frac{1}{100^{\frac{4}{3}}}$

$= \frac{1}{10^4} = \frac{1}{10\,000}$

j $10\,000^{\frac{3}{4}} = 10^3 = 1000$

k $81^{\frac{3}{4}} = 3^3 = 27$

l $\left(\frac{27}{125}\right)^{\frac{1}{3}} = \left(\frac{3}{5}\right)^{\frac{3}{3}} = \frac{3}{5}$

2 a $(a^2b)^{\frac{1}{3}} \div \sqrt{ab^3} = \frac{a^{\frac{2}{3}}b^{\frac{1}{3}}}{a^{\frac{1}{2}}b^{\frac{3}{2}}}$

$= a^{\frac{2}{3}-\frac{1}{2}}b^{\frac{1}{3}-\frac{3}{2}}$

$= a^{\frac{1}{6}}b^{-\frac{7}{6}}$

b $= a^{-6}b^3b^{\frac{3}{2}}$

$= a^{-6}b^{3+\frac{3}{2}}b^{\frac{3}{2}} = a^{-6}b^{\frac{9}{2}}$

c
$\left(45^{\frac{1}{3}}\right) \div \left(9^{\frac{3}{4}}15^{\frac{3}{2}}\right) = \left(3^{\frac{2}{3}}5^{\frac{1}{3}}\right) \div \left(3^{\frac{3}{2}}3^{\frac{3}{2}}5^{\frac{3}{2}}\right)$

$= 3^{\frac{2}{3}-\frac{3}{2}-\frac{3}{2}}5^{\frac{1}{3}-\frac{3}{2}}$

$= 3^{-\frac{7}{3}}5^{-\frac{7}{6}}$

d $2^{\frac{3}{2}}4^{-\frac{1}{4}}16^{-\frac{3}{4}} = 2^{\frac{3}{2}}2^{-\frac{1}{2}}2^{-3}$

$= 2^{\frac{3}{2}-\frac{1}{2}-3} = 2^{-2} = \frac{1}{4}$

e
$\left(\frac{x^3y^{-2}}{3^{-3}y^{-3}}\right)^{-2} \div \left(\frac{3^{-3}x^{-2}y}{x^4y^{-2}}\right)^2 = \left(\frac{x^{-6}y^4}{3^6y^6}\right)\left(\frac{x^8y^{-4}}{3^{-6}x^{-4}y^2}\right)$

$= 3^{6-6}x^{-6+8+4}y^{4-4-6-2}$

$= x^6y^{-8}$

f $\left((a^2)^{\frac{1}{5}}\right)^{\frac{3}{2}}\left((a^5)^{\frac{1}{3}}\right)^{\frac{1}{5}} = a^{\frac{2}{5}\frac{3}{2}}a^{\frac{5}{3}\frac{1}{5}}$

$= a^{\frac{3}{5}}a^{\frac{1}{3}}$

$= a^{\frac{3}{5}+\frac{1}{3}} = a^{\frac{14}{15}}$

3 a $(2x-1)\sqrt{2x-1} = (2x-1)^{1+\frac{1}{2}}$

$= (2x-1)^{\frac{3}{2}}$

b $(x-1)^2\sqrt{x-1} = (x-1)^{2+\frac{1}{2}}$

$= (x-1)^{\frac{5}{2}}$

c $(x^2+1)\sqrt{x^2+1} = (x^2+1)^{1+\frac{1}{2}}$

$= (x^2+1)^{\frac{3}{2}}$

d $(x-1)^3\sqrt{(x-1)} = (x-1)^{3+\frac{1}{2}}$

$= (x-1)^{\frac{7}{2}}$

e $\dfrac{1}{\sqrt{x-1}} + \sqrt{x-1} = \dfrac{1+x-1}{\sqrt{x-1}}$

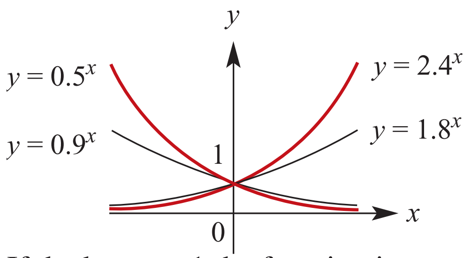
$= x(x-1)^{-\frac{1}{2}}$

f $(5x^2+1)(5x^2+1)^{\frac{1}{3}} = (5x^2+1)^{1+\frac{1}{3}}$

$= (5x^2+1)^{\frac{4}{3}}$

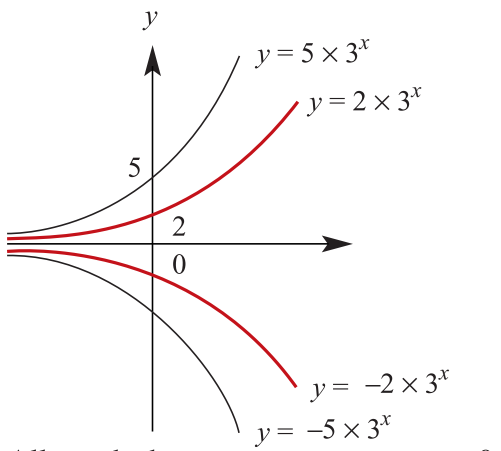
Solutions to Exercise 13C

1



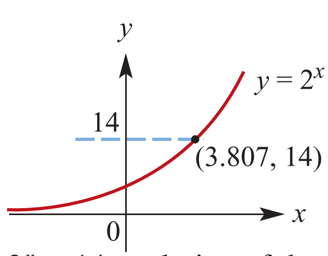
If the bases > 1 the function is increasing; if < 1 they are decreasing.

2



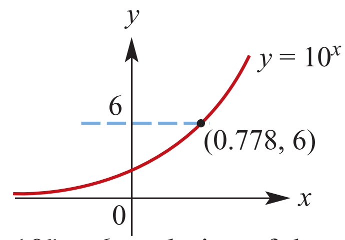
All graphs have an asymptote at $y = 0$. The y-intercepts are wherever the constant is in front of the exponential, however, at $2, -2, 5$ and -5.
The negative values are also below the axis instead of above.

3 $y = 2^x$ for $x \in [-4, 4]$:



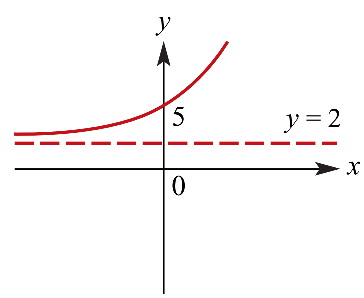
$2^x = 14$: solution of the equation is where the graph cuts the line $y = 14$, i.e. $x = 3.807$

4 $y = 10^x$; $x \in [-0.4, 0.8]$



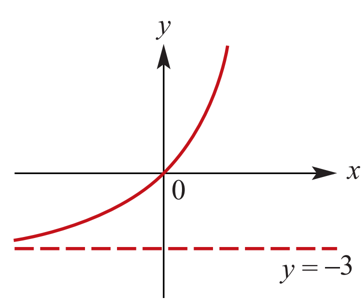
$10^x = 6$: solution of the equation is where the graph cuts the line $y = 6$, i.e. $x = 0.778$

5 **a** $f\colon R \to R;\ f(x) = 3(2^x) + 2$



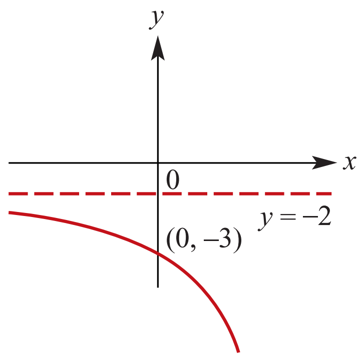
Asymptote at $y = 2$,
y-axis intercept at $(0, 5)$,
range $= (2, \infty)$

b $f\colon R \to R;\ f(x) = 3(2^x) - 3$

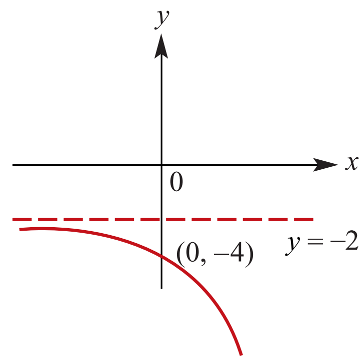


Asymptote at $y = -3$,
y-axis intercept at $(0, 0)$,
range $= (-3, \infty)$

c $f\colon R \to R;\ f(x) = -3^x - 2$

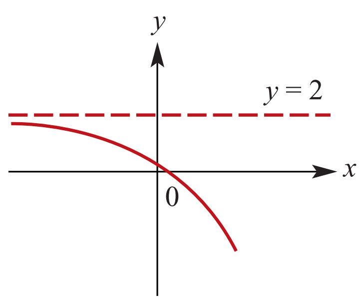


Asymptote at $y = -2$,
y-axis intercept at $(0, -3)$,
range = $(-\infty, -2)$



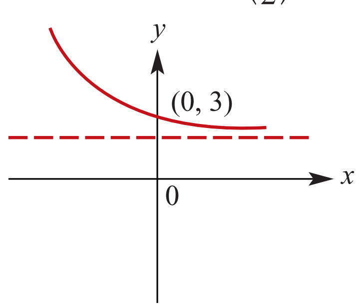
Asymptote at $y = -2$,
y-axis intercept at $(0, -4)$,
range = $(-\infty, -2)$

d $f: R \to R; f(x) = -2(3^x) + 2$



Asymptote at $y = 2$,
y-axis intercept at $(0, 0)$,
range = $(-\infty, 2)$

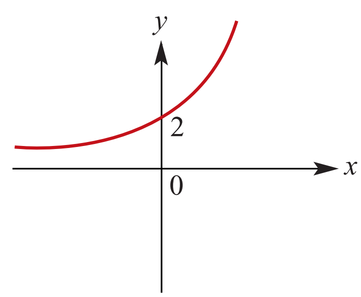
e $f: R \to R; f(x) = \left(\frac{1}{2}\right)^x + 2$



Asymptote at $y = 2$,
y-axis intercept at $(0, 3)$,
range = $(2, \infty)$

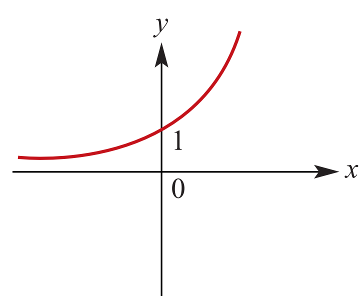
f $f: R \to R; f(x) = -2(3^x) - 2$

6 a $y = 2(5^x)$



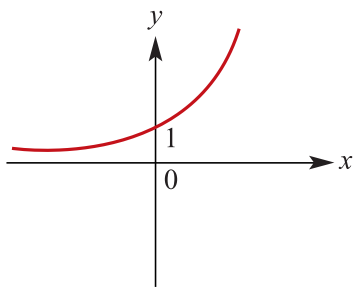
Asymptote at $y = 0$,
y-axis intercept at $(0, 2)$,
range = $(0, \infty)$

b $y = 3^{3x}$



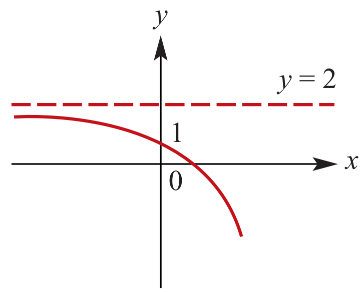
Asymptote at $y = 0$,
y-axis intercept at $(0, 1)$,
range = $(0, \infty)$

c $y = 5^{\frac{x}{2}}$



Asymptote at $y = 0$,
y-axis intercept at $(0, 1)$,
range $= (0, \infty)$

d $y = -3(2^x) + 2$



Asymptote at $y = 2$,
y-axis intercept at $(0, 1)$,
range $= (-\infty, 2)$

Solutions to Exercise 13D

1 a $3^x = 27 = 3^3, \therefore x = 3$

b $4^x = 64 = 4^3, \therefore x = 3$

c $49^x = 7 = 49^{\frac{1}{2}}, \therefore x = \frac{1}{2}$

d $16^x = 8, \therefore 2^{4x} = 2^3$

$\therefore 4x = 3, \therefore x = \frac{3}{4}$

e $125^x = 5, \therefore 5^{3x} = 5$

$\therefore 3x = 1, \therefore x = \frac{1}{3}$

f $5^x = 625 = 5^4, \therefore x = 4$

g $16^x = 256 = 16^2, \therefore x = 2$

h $4^{-x} = \frac{1}{64}, \therefore 4^x = 64$

$\therefore 4^x = 4^3, \therefore x = 3$

i $5^{-x} = \frac{1}{125}, \therefore 5^x = 125$

$\therefore 5^x = 5^3, \therefore x = 3$

2 a $5^n 25^{2n-1} = 125$

$\therefore 5^n 5^{4n-2} = 5^3$

$5^{5n-2} = 5^3$

$5n - 2 = 3, \therefore n = 1$

b $3^{2n-4} = 1$

$\therefore 3^{2n-4} = 3^0$

$2n - 4 = 0, \therefore n = 2$

c $3^{2n-1} = \frac{1}{81}$

$\therefore 3^{2n-1} = 3^{-4}$

$2n - 1 = -4, \therefore n = -\frac{3}{2}$

d $\frac{3^{n-2}}{9^{1-n}} = 1$

$\therefore 3^{n-2} = 9^{1-n}$

$3^{n-2} = 3^{2(1-n)}$

$n - 2 = 2 - 2n$

$3n = 4, n = \frac{4}{3}$

e $3^{3n}9^{-2n+1} = 27$

$\therefore 3^{3n}3^{2-4n} = 3^3$

$3^{3n+2-4n} = 3^3$

$2 - n = 3, \therefore n = -1$

f $2^{-3n}4^{2n-2} = 16$

$\therefore 2^{-3n}2^{4n-4} = 2^4$

$2^{4n-3n-4} = 2^4$

$n - 4 = 4, \therefore n = 8$

g $2^{n-6} = 8^{2-n} = 2^{6-3n}$

$\therefore n - 6 = 6 - 3n$

$4n = 12, \therefore n = 3$

h $9^{3n+3} = 27^{n-2}$

$\therefore 3^{6n+6} = 3^{3n-6}$

$6n + 6 = 3n - 6$

$3n = -12, \therefore n = -4$

i $4^{n+1} = 8^{n-2}$

$\therefore 2^{2n+2} = 2^{3n-6}$

$2n + 2 = 3n - 6, n = 8$

j $32^{2n+1} = 8^{4n-1}$

$\therefore\ 2^{10n+5} = 2^{12n-3}$

$10n + 5 = 12n - 3$

$2n = 8,\ \therefore\ n = 4$

k $25^{n+1} = 5 \times 390\,625$

$\therefore\ 25^{n+1} = (25)^{\frac{1}{2}}(25)^4 = 25^{\frac{9}{2}}$

$n + 1 = \frac{9}{2},\ \therefore\ n = \frac{7}{2} = 3\frac{1}{2}$

l $125^{4-n} = 5^{6-2n}$

$\therefore\ 5^{12-3n} = 5^{6-2n}$

$12 - 3n = 6 - 2n,\ \therefore\ n = 6$

m $4^{2-n} = \frac{1}{2048}$

$\therefore\ 2^{4-2n} = 2^{-11}$

$4 - 2n = -11$

$2n = 15, \therefore\ n = \frac{15}{2}$

3 a $2^{x-1}4^{2x+1} = 32$

$\therefore\ 2^{x-1}2^{4x+2} = 2^5$

$2^{x-1+4x+2} = 2^5$

$5x + 1 = 5, \therefore\ x = \frac{4}{5}$

b $3^{2x-1}9^x = 243$

$\therefore\ 3^{2x-1}3^{2x} = 3^5$

$3^{2x-1+2x} = 3^5$

$4x - 1 = 5$

$4x = 6,\ \therefore\ x = \frac{3}{2}$

c $(27\ 3^x)^2 = 27^x 3^{\frac{1}{2}}$

$\therefore\ (3^3 3^x)^2 = 3^{3x}3^{\frac{1}{2}}$

$3^{6+2x} = 3^{3x+\frac{1}{2}}$

$2x + 6 = 3x + \frac{1}{2}, \therefore\ x = \frac{11}{2} = 5\frac{1}{2}$

4 a $4(2^{2x}) = 8(2^x) - 4,\ A = 2^x$

$\therefore\ 4A^2 = 8A - 4$

$A^2 - 2A + 1 = 0$

$(A - 1)^2 = 0$

$A = 2^x = 1,\ \therefore\ x = 0$

b $8(2^{2x}) - 10(2^x) + 2 = 0,\ A = 2^x$

$\therefore\quad 8A^2 - 10A + 2 = 0$

$4A^2 - 5A + 1 = 0$

$(4A - 1)(A - 1) = 0$

$A = 2^x = \frac{1}{4}, 1$

$\therefore\quad x = -2, 0$

c $3(2^{2x}) - 18(2^x) + 24 = 0,\ A = 2^x$

$\therefore\quad 3A^2 - 18A + 24 = 0$

$A^2 - 6A + 8 = 0$

$(A - 2)(A - 4) = 0$

$A = 2^x = 2, 4$

$\therefore\quad x = 1, 2$

d $9^x - 4(3^x) + 3 = 0,\ A = 3^x$

$\therefore\ (A - 1)(A - 3) = 0$

$A = 3^x = 1, 3$

$\therefore\quad x = 0, 1$

5 **a** $2^x = 5,\ \therefore\ x = 2.32$

b $4^x = 6,\ \therefore\ x = 1.29$

c $10^x = 18,\ \therefore\ x = 1.26$

d $10^x = 56,\ \therefore\ x = 1.75$

6 **a** $7^x > 49,\ \therefore\ 7^x > 7^2$

$$\therefore\ x > 2$$

b $8^x > 2,\ \therefore\ 2^{3x} > 2^1$

$$3x > 1,\ \therefore\ x > \frac{1}{3}$$

c $25^x \le 5,\ \therefore\ 5^{2x} \le 5^1$

$$2x \le 1,\ \therefore\ x \le \frac{1}{2}$$

d $3^{x+1} < 81,\ \therefore\ 3^{x+1} < 3^4$

$$x + 1 < 4, \therefore\ x < 3$$

e $9^{2x+1} < 243, \therefore\ 3^{4x+2} < 3^5$

$$4x + 2 < 5$$

$$4x < 3,\ \therefore\ x < \frac{3}{4}$$

f $4^{2x+1} > 64,\ \therefore\ 4^{2x+1} > 4^3$

$$2x + 1 > 3,\ \therefore\ x > 1$$

g $3^{2x-2} \le 81,\ \therefore\ 3^{2x-2} \le 3^4$

$$2x - 2 \le 4,\ \therefore\ x \le 3$$

Solutions to Exercise 13E

1 **a** $\log_2 128 = 7$

b $\log_3 81 = 4$

c $\log_5 125 = 3$

d $\log_{10} 0.1 = -1$

2 **a** $\log_2 10 + \log_2 a = \log_2 10a$

b $\log_{10} 5 + \log_{10} 2 = \log_{10} 10 = 1$

c $\log_2 9 - \log_2 4 = \log_2\left(\frac{9}{4}\right)$

d $\log_2 10 - \log_2 5 = \log_2\left(\frac{10}{5}\right)$

$= \log_2 2 = 1$

e $\log_2 a^3 = 3\log_2 a$

f $\log_2 8^3 = 3\log_2 8 = 9$

g $\log_5\left(\frac{1}{6}\right) = -\log_5 6$

h $\log_5\left(\frac{1}{25}\right) = -\log_5 25 = -2$

3 **a** $\log_3 27 = \log_3 3^3$

$= 3\log_3 3 = 3$

b $\log_5 625 = \log_5 5^4$

$= 4\log_5 5 = 4$

c $\log_2\left(\frac{1}{128}\right) = \log_2 2^{-7}$

$= -7\log_2 2 = -7$

d $\log_4\left(\frac{1}{64}\right) = \log_4 4^{-3}$

$= -3\log_4 4 = -3$

e $\log_x\ x^4 = 4\log x\ x = 4$

f $\log_2 0.125 = -\log_2 8$

$= -3\log_2 2 = -3$

g $\log_{10} 10000 = \log_{10} 10^4$

$= 4\log_{10} 10 = 4$

h $\log_{10} 0.000001 = \log_{10} 10^{-6}$

$= -6\log_{10} 10 = -6$

i $-3\log_5 125 = -3\log_5 5^3$

$= -9\log_5 5 = -9$

j $-4\log_{16} 2 = -\log_{16} 16 = -1$

k $2\log_3 9 = 4\log_3 3 = 4$

l $-4\log_{16} 4 = -2\log_{16} 16 = -2$

4 **a**

$\frac{1}{2}\log_{10} 16 + 2\log_{10} 5 = \log_{10}(\sqrt{16}\,(5^2))$

$= \log_{10} 100 = 2$

b $\log_2 16 + \log_2 8 = \log_2 2^4 + \log_2 2^3$

$= 4 + 3 = 7$

c $\log_2 128 + \log_3 45 - \log_3 5$

$= \log_2 2^7 + \log_3 5(3^2) - \log_3 5$

$= 7 + 2\log_3 3 + \log_3 5 - \log_3 5$

$= 7 + 2 = 9$

d $\log_4 32 - \log_9 27 = \log_4 2^5 - \log_9 3^3$

$= \log_4 4^{\frac{5}{2}} - \log_9 9^{\frac{3}{2}}$

$= \frac{5}{2} - \frac{3}{2} = 1$

e

$\log_b b^3 - \log_b \sqrt{b} = \log_b b^3 - \log_b\left(b^{\frac{1}{2}}\right)$

$= 3 - \frac{1}{2} = \frac{5}{2}$

f

$2\log_x a + \log_x a^3 = 2\log_x a + 3\log_x a$

$= 5\log_x a$

$= \log_x a^5$

g

$x\log_2 8 + \log_2(8^{1-x}) = \log_2 8^x + \log_2(8^{1-x})$

$= \log_2(8^{x+1-x})$

$= \log_x\ 8 = 3$

h $\frac{3}{2}\log_a a - \log_a \sqrt{a} = \frac{3}{2} - \log_a\left(a^{\frac{1}{2}}\right)$

$= \frac{3}{2} - \frac{1}{2} = 1$

5 **a** $\log_3 9 = x$

$x = \log_3 3^2 = 2$

b $\log_5 x = -3$

$x = 5^{-3},\ \therefore\ x = \frac{1}{125}$

c $\log_3 x = -3$

$x = 3^3,\ \therefore\ x = 27$

d $\log_{10} x = \log_{10} 4 + \log_{10} 2$

$\log_{10} x = \log_{10} 8$

$\therefore\quad x = 8$

e

$\log_{10} 2 + \log_{10} 5 + \log_{10} x - \log_{10} 3 = 2$

$\log_{10}\left(\frac{10x}{3}\right) = 2$

$\frac{10x}{3} = 10^2$

$\therefore\quad x = 30$

f $\log_{10} x = \frac{1}{2}\log_{10} 36 - 2\log_{10} 3$

$\log_{10} x = \log_{10}\ \sqrt{36} - \log_{10} 3^2$

$\log_{10} x = \log_{10} \frac{6}{9}$

$\therefore\quad x = \frac{2}{3}$

g $\log_x 64 = 2$

$64 = x^2$

$x^2 = 64,\ \therefore\ x = 8$

(no negative solutions for log base)

h $\log_5(2x - 3) = 3$

$2x - 3 = 5^3$

$2x - 3 = 125,\ \therefore\ x = 64$

i $\log_3(x + 2) - \log_3 2 = 1$

$\log_3 \frac{x+2}{2} = 1$

$\frac{x+2}{2} = 3^1$

$\frac{x+2}{2} = 3$

$x + 2 = 6,\ \therefore\ x = 4$

j $\log_x 0.01 = -2$

$0.01 = x^{-2}$

$x^{-2} = 0.01$

$x^2 = 100,\ \therefore\ x = 10$

6 a $\log_x\left(\frac{1}{25}\right) = -2$

$\log_x 25 = 2$

$25 = x^2$

$x^2 = 25, \therefore\ x = 5$

(No negative solutions for log base)

b $\log_4(2x - 1) = 3$

$2x - 1 = 4^3$

$2x - 1 = 64, \therefore\ x = \frac{65}{2} = 32.5$

c $\log_4(3x + 2) - \log_4 6 = 1$

$\log_4 \frac{x + 2}{6} = 1$

$\frac{x + 2}{6} = 4^1$

$\frac{x + 2}{6} = 4$

$x + 2 = 24, \therefore\ x = 22$

d $\log_4(3x + 4) + \log_4 16 = 5$

$\log_4(3x + 4) + 2 = 5$

$\log_4(3x + 4) = 3$

$3x + 4 = 4^3$

$3x + 4 = 64, \therefore\ x = 20$

e $\log_3(x^2 - 3x - 1) = 0$

$x^2 - 3x - 1 = 1$

$x^2 - 3x - 2 = 0$

$\therefore \quad x = \frac{3 \pm \sqrt{17}}{2}$

f $\log_3(x^2 - 3x + 1) = 0$

$x^2 - 3x + 1 = 1$

$x^2 - 3x = 0$

$x(x - 3) = 0,\ x = 0, 3$

7 $\log_{10} x = a; \log_{10} y = c$:

$\log_{10}\left(\frac{100x^3y^{-\frac{1}{2}}}{y^2}\right) = \log_{10}\left(100x^3y^{-\frac{5}{2}}\right)$

$= \log_{10}(100x^3) + \log_{10}(y^{-\frac{5}{2}})$

$= \log_{10}(100) + 3\log_{10} x - \frac{5}{2}\log_{10} y$

$= 3a - \frac{5c}{2} + 2$

8 $\log_{10}\frac{ab^2}{c} + \log_{10}\frac{c^2}{ab} - \log_{10}(bc)$

$= \log_{10}\left(\frac{ab^2}{c}\right)\left(\frac{c^2}{ab}\right) - \log_{10}(bc)$

$= \log_{10}(bc) - \log_{10}(bc)$

$= \log_{10}\left(\frac{bc}{bc}\right) = \log_{10} 1 = 0$

9

$\log_a\left(\frac{11}{3}\right) + \log_a\left(\frac{490}{297}\right) - 2\log_a\left(\frac{7}{9}\right) = \log_a(k)$

$\log_a\left(\frac{11}{3}\right)\left(\frac{490}{297}\right) - 2\log_a\left(\frac{7}{9}\right) = \log_a(k)$

$\log_a\left(\frac{490}{81}\right) - \log_a\left(\frac{7}{9}\right)^2 = \log_a(k)$

$\log_a 10 + \log_a 1 = \log_a(k)$

$\log_a 10 = \log_a(k)$

$\therefore \quad k = 10$

10 a $\log_{10}(x^2 - 2x + 8) = 2\log_{10} x$

$\log_{10}(x^2 - 2x + 8) = \log_{10} x^2$

$x^2 - 2x + 8 = x^2$

$-2x + 8 = 0, \therefore\ x = 4$

b

$$\begin{aligned}
\log_{10}(5x) - \log_{10}(3-2x) &= 1 \\
\log_{10}\left(\frac{5x}{3-2x}\right) &= 1 \\
\left(\frac{5x}{3-2x}\right) &= 10^1 \\
5x &= 10(3-2x) \\
x &= 2(3-2x) \\
5x &= 6 \\
\therefore \quad x &= \frac{6}{5}
\end{aligned}$$

c $3\log_{10}(x-1) = \log_{10} 8$

$3\log_{10}(x-1) = 3\log_{10} 2$

$x - 1 = 2, \therefore\ x = 3$

d

$$\begin{aligned}
\log_{10}(20x) - \log_{10}(x-8) &= 2 \\
\log_{10}\left(\frac{20x}{x-8}\right) &= 2 \\
\left(\frac{20x}{x-8}\right) &= 10^2 \\
20x &= 100(x-8) \\
x &= 5x - 40 \\
4x &= 40 \\
\therefore \quad x &= 10
\end{aligned}$$

e

$$\begin{aligned}
\text{LHS} &= 2\log_{10} 5 + \log_{10}(x+1) \\
&= \log_{10} 5^2 + \log_{10}(x+1) \\
&= \log_{10} 25(x+1) \\
\text{RHS} &= 1 + \log_{10}(2x+7) \\
&= \log_{10} 10 + \log_{10}(2x+7) \\
&= \log_{10} 10(2x+7)
\end{aligned}$$

$$\begin{aligned}
\therefore 25(x+1) &= 10(2x+7) \\
5x + 5 &= 4x + 14 \\
x &= 9
\end{aligned}$$

f

$$\begin{aligned}
\text{LHS} &= 1 + 2\log_{10}(x+1) \\
&= \log_{10} 10 + \log_{10}(x+1)^2 \\
&= \log_{10} 10(x+1)^2 \\
\text{RHS} &= \log_{10}(2x+1) + \log_{10}(5x+8) \\
&= \log_{10}(2x+1)(5x+8)
\end{aligned}$$

$$\begin{aligned}
\therefore \quad 10(x+1)^2 &= (2x+1)(5x+8) \\
10x^2 + 20x + 10 &= 10x^2 + 21x + 8 \\
20x + 10 &= 21x + 8 \\
x &= 2
\end{aligned}$$

Solutions to Exercise 13F

1 a $2^x = 7$

$\therefore\ x = \dfrac{\log 7}{\log 2} = 2.81$

b $2^x = 0.4$

$\therefore\ x = \dfrac{\log 0.4}{\log 2} = -1.32$

c $3^x = 14$

$\therefore\ x = \dfrac{\log 14}{\log 3} = 2.40$

d $4^x = 3$

$\therefore\ x = \dfrac{\log 3}{\log 4} = 0.79$

e $2^{-x} = 6$

$\therefore\ x = -\dfrac{\log 6}{\log 2} = -2.58$

f $0.3^x = 2$

$\therefore\ x = \dfrac{\log 2}{\log 0.3} = -0.58$

2 a

$$5^{2x-1} = 90$$
$$\therefore\ (2x-1) = \log_5 90$$
$$2x = \log_5(90) + 1$$
$$x = \frac{1}{2}(\log_5(90) + 1)$$
$$x = 1.90$$

b

$$3^{x-1} = 10$$
$$\therefore\ (x-1)\log 3 = \log 10$$
$$(x-1) = \frac{\log 10}{\log 3}$$
$$x - 1 = 2.10$$
$$x = 3.10$$

or

$$3^{x-1} = 10$$
$$\therefore\ (x-1) = \log_3(10)$$
$$x = \log_3(10) + 1$$
$$x = 3.10$$

c

$$0.2^{x+1} = 0.6$$
$$\therefore\ (x+1)\log 0.2 = \log 0.6$$
$$(x+1) = \frac{\log 0.6}{\log 0.2}$$
$$x + 1 = 0.32$$
$$x = -0.68$$

3 a $2^x > 8,\quad \therefore\ 2^x > 2^3$

$\therefore\quad x > 3$

b $3^x < 5,\quad \therefore\ x\log 3 < \log 5$

$\therefore\quad x < \dfrac{\log 5}{\log 3} < 1.46$

c

$0.3^x > 4,\quad \therefore\ x\log 0.3 < \log 4$

$x < \dfrac{\log 4}{\log 0.3}$

$\therefore\quad x < \dfrac{\log 4}{\log 0.3} < -1.15$

d

$$3^{x-1} \le 7, \quad \therefore \quad (x-1)\log 3 \le \log 7$$

$$(x-1) \le \frac{\log 7}{\log 3}$$

$$(x-1) \le \frac{\log 7}{\log 3} = 1.77$$

$$\therefore \qquad x \le 2.77$$

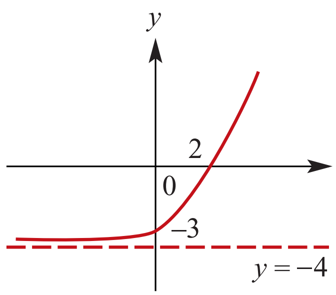
e $0.4^x \le 0.3, \quad \therefore \quad x \le 2.77$

$$\therefore \qquad x \ge \frac{\log 0.3}{\log 0.4} \ge 1.31$$

4 a $f(x) = 2^x - 4$

Asymptote at $y = -4$,

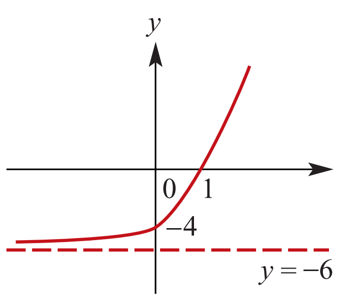
axis intercepts at $(0, -3)$ and $(2, 0)$



b $f(x) = 2(3^x) - 6$

Asymptote at $y = -6$,

axis intercepts at $(0, -4)$ and $(1, 0)$

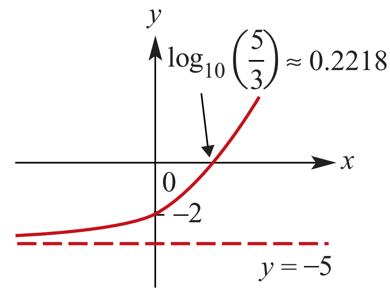


c $f(x) = 3(10^x) - 5$

Asymptote at $y = -5$,

axis intercepts at $(0, -2)$ and

$\left(\log_{10}\left(\frac{5}{3}\right), 0\right)$

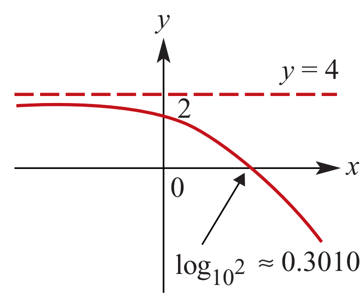


d $f(x) = -2(10^x) + 4$

Asymptote at $y = 4$,

axis intercepts at $(0, 2)$ and

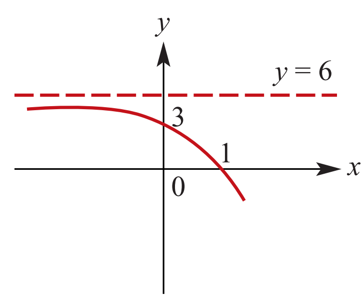
$(\log_{10} 2, 0)$



e $f(x) = -3(2^x) + 6$

Asymptote at $y = 6$,

axis intercepts at $(0, 3)$ and $(1, 0)$

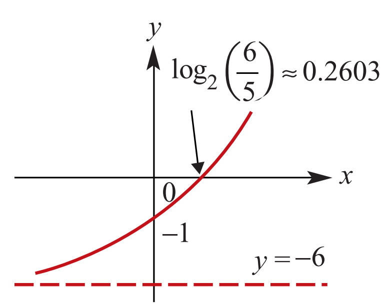


f $f(x) = 5(2^x) - 6$

Asymptote at $y = -6$,

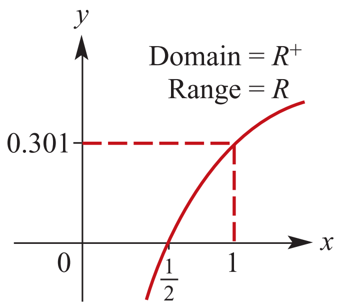
axis intercepts at $(0, -1)$ and

$(\log_2 1.2, 0)$

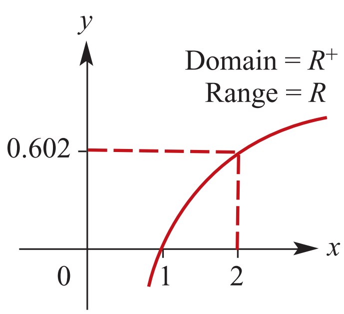


Solutions to Exercise 13G

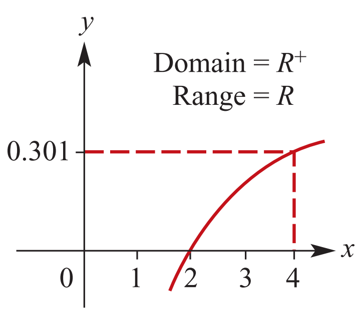
1 **a** $y = \log_{10}(2x)$; domain $(0, \infty)$, range R, x-intercept $\left(\frac{1}{2}, 0\right)$



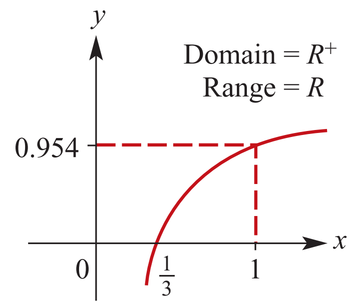
b $y = 2\log_{10} x$; domain $(0, \infty)$, range R, x-intercept $(1, 0)$



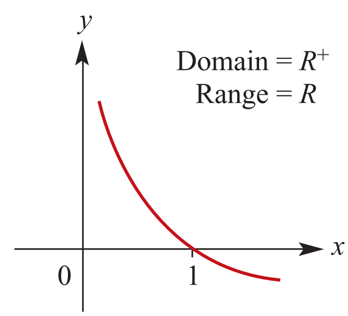
c $y = \log_{10}\left(\frac{x}{2}\right)$; domain $(0, \infty)$ range R, x-intercept $(2, 0)$



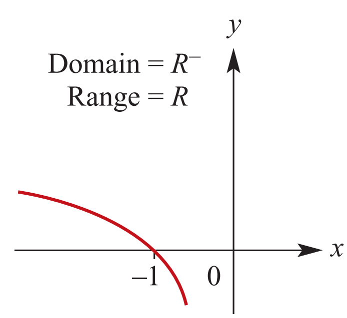
d $y = 2\log_{10}(3x)$; domain $(0, \infty)$, range R, x-intercept $\left(\frac{1}{3}, 0\right)$



e $y = -\log_{10} x$; domain $(0, \infty)$, range R, x-intercept $(1, 0)$



f $y = \log_{10}(-x)$ domain $(-\infty, 0)$, range R, x-intercept $(-1, 0)$



2 **a** $f(x) = 10^{0.5x}$

$f^{-1}(x)\colon\ x = 10^{0.5y}$

$\log_{10} x = 0.5y$

$\therefore\ f^{-1}(x) = 2\log_{10} x$

b $y = 3\log_{10} x$

$f^{-1}(x):\ x = 3\log_{10} y$

$\log_{10} y = \dfrac{x}{3}$

$\therefore\ f^{-1}(x) = 10^{\frac{x}{3}}$

c $f(x) = 10^{3x}$

$f^{-1}(x):\ x = 10^{3y}$

$\log_{10} x = 3y$

$\therefore\ f^{-1}(x) = \dfrac{1}{3}\log_{10} x$

d $y = 2\log_{10} 3x$

$f^{-1}(x):\ x = 2\log_{10} 3y$

$\log_{10} 3y = \dfrac{x}{2}$

$3y = 10^{\frac{x}{2}}$

$\therefore\ f^{-1}(x) = \dfrac{1}{3}10^{\frac{x}{2}}$

3 **a** $f(x) = 3^x + 2$

$f^{-1}(x):\ x = 3^y + 2$

$3^y = x - 2$

$\log_3 3^y = \log_3(x - 2)$

$\therefore\ f^{-1}(x) = \log_3(x - 2)$

b $f(x) = \log_2(x - 3)$

$f^{-1}(x):\ x = \log_2(y - 3)$

$2^x = y - 3$

$\therefore\ f^{-1}(x) = 2^x + 3$

c $f(x) = 4 \times 3^x + 2$

$f^{-1}x:\ (x) = 4 \times 3^y + 2$

$x - 2 = 4 \times 3^y$

$\dfrac{(x-2)}{4} = 3^y$

$\therefore\ f^{-1}(x) = \log_3(\dfrac{x-2}{4})$

d $f(x) = 5^x - 2$

$f^{-1}(x):\ x = 5^y - 2$

$5^y = x + 2$

$\therefore\ f^{-1}(x) = \log_5(x\ + 2)$

e $f(x) = \log_2(3x)$

$f^{-1}(x):\ x = \log_2(3y)$

$2^x = 3y$

$\therefore\ f^{-1}(x) = \dfrac{1}{3}(2^x)$

f $f(x) = \log_2 \dfrac{x}{3}$

$f^{-1}(x):\ x = \log_2 \dfrac{y}{3}$

$\dfrac{y}{3} = 2^x$

$\therefore\ f^{-1}(x) = 3(2^x)$

g $f(x) = \log_2(x + 3)$

$f^{-1}(x):\ x = \log_2(y + 3)$

$2^x = y + 3$

$\therefore\ f^{-1}(x) = 2^x - 3$

h $f(x) = 5(3^x) - 2$

$f^{-1}(x):\ x = 5(3^y) - 2$

$5(3^y) = x + 2$

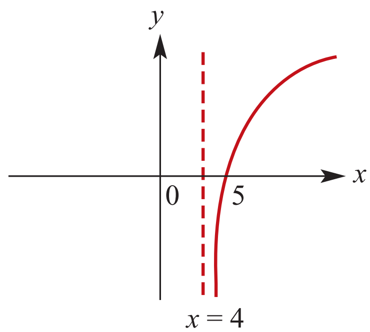
$3^y = \dfrac{x+2}{5}$

$\therefore\ f^{-1}(x) = \log_3 \dfrac{x+2}{5}$

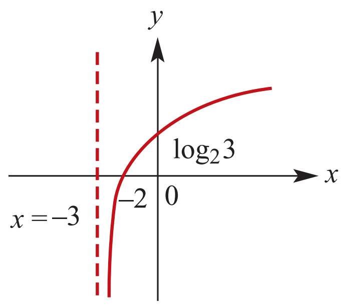
$\therefore\ f^{-1}(x) = \log_3 \dfrac{x+2}{5}$

4 **a** $f(x) = \log_2(x - 4)$

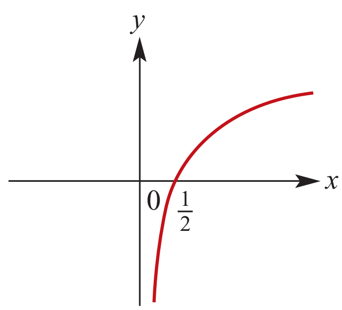
Domain $(4, \infty)$, asymptote $x = 4$, x-intercept at $(5,0)$



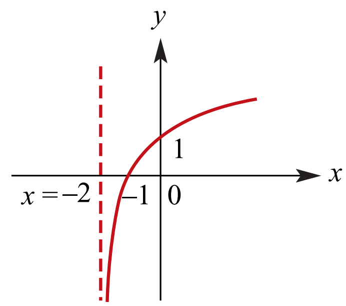
b $f(x) = \log_2(x + 3)$
Domain $(-3, \infty)$, asymptote $x = -3$, x-intercept at $(-2, 0)$, y-intercept at $(0, \log_2 3)$



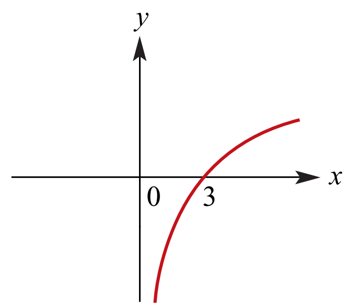
c $f(x) = \log_2(2x)$
Domain $(0, \infty)$, asymptote $x = 0$, x-intercept at $\left(\frac{1}{2}, 0\right)$



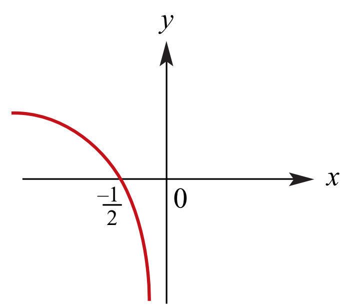
d $f(x) = \log_2(x + 2)$
Domain $(-2, \infty)$, asymptote $x = -2$, x-intercept at $(-1, 0)$, y-intercept at $(0, 1)$



e $f(x) = \log_2\left(\frac{x}{3}\right)$
Domain $(0, \infty)$, asymptote $x = 0$, x-intercept at $(3, 0)$



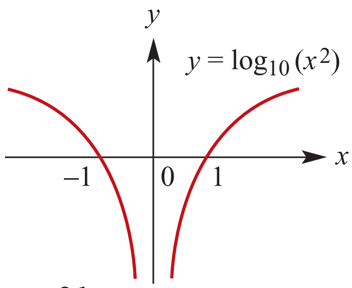
f $f(x) = \log_2(-2x)$
Domain $(-\infty, 0)$, asymptote $x = 0$, x-intercept at $\left(-\frac{1}{2}, 0\right)$



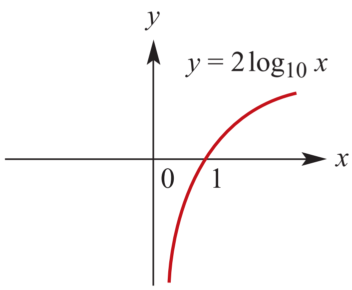
5 **a** $2^{-x} = x, \therefore \; x = 0.64$

b $\log_{10}(x) + x = 0, \therefore \; x = 0.40$

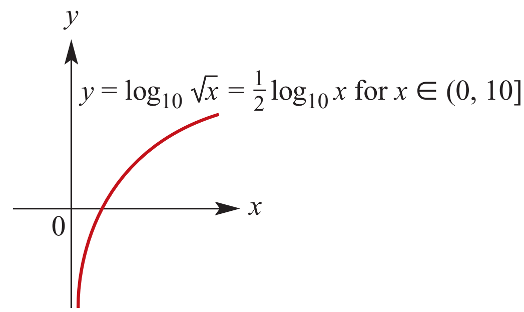
6 $y = \log_{10}(x^2)$;
$x \in [-10, 10], \ x \neq 0$



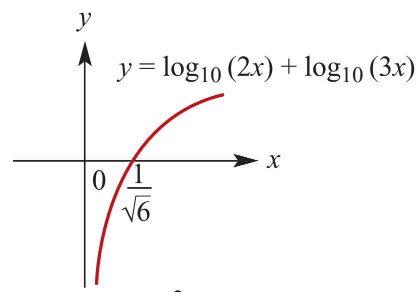
$y = 2\log_{10} x$;
$x \in [-10, 10], \ x \neq 0$



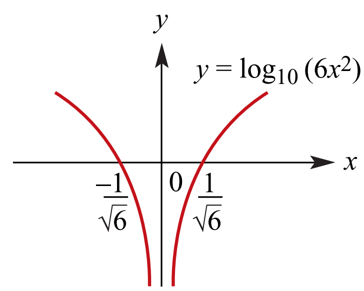
7 $y = \log_{10}\sqrt{x}$;
$x \in (0, 10], \ x \neq 0$
$y = \dfrac{1}{2}\log_{10} x$;
$x \in (0, 10], \ x \neq 0$



8 $y = \log_{10}(2x) + \log_{10}(3x)$



$y = \log_{10}(6x^2)$



Solutions to Exercise 13H

1 Let N be the number of bacteria at time t minutes.

a $N = 1000 \times 2^{\frac{t}{15}}$

b
$$\begin{aligned} 10\,000 &= 1000 \times 2^{\frac{t}{15}} \\ 10 &= 2^{\frac{t}{15}} \\ \frac{t}{15} &= \log_2 10 \\ t &= 49.8289... \\ t &\approx 50. \end{aligned}$$

It will take approximately 50 minutes

2 Choose $A(t) = A_0 \times 10^{-kt}$ as the model where $A_0 = 10$ is the original amount and t is the time in years.

First find k:

$$\begin{aligned} 5 &= 10 \times 10^{-24\,000k} \\ \log_{10} \frac{1}{2} &= -24\,000k \\ k &= -\frac{1}{24\,000} \log_{10} \frac{1}{2}k \quad = 1.254296... \times 10^{-5} \end{aligned}$$

If $A(t) = 1$

$$\begin{aligned} 1 &= 10 \times 10^{-kt} \\ 0.1 &= 10^{-kt} \\ \therefore kt &= 1 \\ \therefore t &= \frac{1}{1.254296 \times 10^{-5}} \\ t &\approx 79\,726. \end{aligned}$$

It will take 79 726 years for there to be 10% of the original.

3 Choose $A(t) = A_0 \times 10^{-kt}$ as the model where A_0 is the original amount and t is the time in years.

First find k:

$$\frac{1}{2}A_0 = A_0 \times 10^{-5730k}$$

$$\log_{10}\frac{1}{2} = -5730k$$

$$k = -\frac{1}{5730}\log_{10}\frac{1}{2}$$

$$k = 5.2535...... \times 10^{-5}$$

When $A(t) = 0.4A_0$

$$0.4A_0 = A_0 \times 10^{-kt}$$

$$0.4 = 10^{-kt}$$

$$\therefore kt = \log_{10} 0.4$$

$$\therefore t = \frac{1}{5.2535... \times 10^{-5}} \times \log_{10} 0.4$$

$$t \approx 7575$$

It is approximately 7575 years old.

4 $P(h) = 1000 \times 10^{-0.0542h}$

a $P(5) = 1000 \times 10^{-0.0542\times 5}$

$$= 535.303\ldots$$

$P(h) \approx 535$millibars

b If $P(h) = 400$

Then $400 = 1000 \times 10^{-0.05428h}$

$$\frac{2}{5} = 10^{-0.05428h}$$

$$\log 10\left(\frac{2}{5}\right) = -0.05428h$$

$h \approx 7331$ metres correct to the nearest metre

5 $N(t) = 500\,000(1.1)^t$ where $N(t)$ is the number of bacteria at time t

$$4\,000\,000 = 500\,000(1.1)^t$$

$$8 = 1.1^t$$

$$t = 21.817\ldots$$

The number will exceed 4 million bacteria after 22 hours.

6 $T = T_0 10^{-kt}$

When $t = 0, T = 100$. Therefore $T_0 = 100$

We have $T = 100 \times 10^{-kt}$

When $t = 5, T = 40$

$\therefore 40 = 100 \times 10^{-5k}$

$$\frac{2}{5} = 10^{-5k}$$

$$k = -\frac{1}{5}\log 10\frac{2}{5}$$

$$k = 0.07958\ldots$$

When $t = 15$

$T = 100 \times 10^{-15k} \quad = 6.4$

The temperature is 6.4°C after 15 minutes.

7 $\quad A(t) = 0.9174^t$

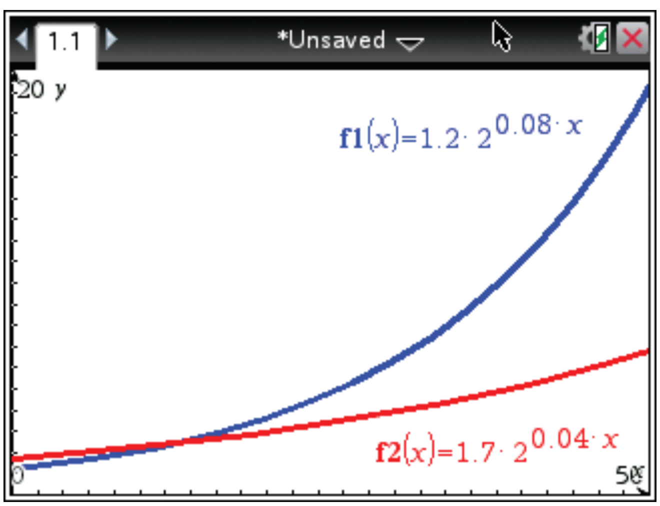
When $A(t) = 0.2$

$$0.2 = 0.9174^t$$

$$t = 18.668\ldots$$

$t > 18.668\ldots$

8 a



b i $\quad p = q$

$$\Leftrightarrow \quad 2^{0.04t} = \frac{17}{12}$$

$\therefore t = 12.56 \qquad$ (mid 1962)

ii Solve the equation $p = 2q$

i.e.
$$1.2 \times 2^{0.08t} = 2(1.7 \times 2^{0.04t})$$
$$\frac{6}{17} \times 2^{0.04t} = 1$$
$$2^{0.04t} = \frac{17}{6}$$
$$t = 37.56 \qquad \text{(mid 1987)}$$

9 **a** We can write

$$a \times b^1 = 15 \qquad (1)$$
$$a \times b^4 = 1875 \qquad (2)$$

Dividing equation (2) by equation (1) gives $b^3 = 125$. Thus $b = 5$, and substituting into equation (1) gives $a = 3$.
$\therefore y = 3 \times 5^x$

b We can write

$$a \times b^2 = 1 \qquad (1)$$
$$a \times b^5 = \frac{1}{8} \qquad (2)$$

Dividing equation (2) by equation (1) gives $b^3 = \frac{1}{8}$. Thus $b = \frac{1}{2}$, and substituting into equation (1) gives $a = 4$.
$\therefore y = 4 \times (\frac{1}{2})^x$

c We can write

$$a \times b^1 = \frac{15}{2} \qquad (1)$$
$$a \times b^{\frac{1}{2}} = \frac{5\sqrt{6}}{2} \qquad (2)$$

Dividing equation (2) by equation (1) gives $b^{-\frac{1}{2}} = \frac{\sqrt{6}}{3}$. Thus $b = \frac{3}{2}$, and substituting into equation (1) gives $a = 5$.
$y = 5 \times (\frac{3}{2})^x$

10 $S = 5 \times 10^{-kt}$

a $S = 3.2$ when $t = 2$

$$3.2 = 5 \times 10^{-2k}$$

$$0.64 = 10^{-2k}$$

$$k = -\frac{1}{2}\log_{10} 0.64$$

$$= 0.0969\ldots$$

b When $S = 1$

$$1 = 5 \times 10^{-0.9969\ldots t}$$

$$10^{(-0.0969\ldots)t} = 0.2$$

$$(-0.0969\ldots)t = \log_{10} 0.2$$

$$t = 7.212...$$

There will be 1 kg of sugar remaining after approximatel 7.21 hours

11 **a** When $t = 0, N = 1000$

$$N = ab^t$$

$$1000 = ab^0$$

$$a = 1000$$

When $t = 5, N = 10\,000$

$$\therefore 10 = b^5$$

$$\therefore b = 10^{\frac{1}{5}}$$

$$\therefore N = 1000 \times 10^{\frac{t}{5}}$$

b When $N = 5000$

$$5 = 10^{\frac{t}{5}}$$

$$\frac{t}{5} = \log_{10} 5$$

$$t = 5\log_{10} 5$$

$$\approx 3.4948 \text{ hours}$$

$$= 210 \text{ minutes}$$

c When $N = 1\,000\,000$

$$\begin{aligned} 1000 &= 10^{\frac{t}{5}} \\ \frac{t}{5} &= \log_{10} 1000 \\ t &= 5 \times 3 \\ &= 15 \text{ hours} \end{aligned}$$

d $N(12) = 1000 \times 10^{\frac{12}{5}} \approx 251188.64$

12 We can write

$a \times 10^{2k} = 6 \qquad (1)$

$a \times 10^{5k} = 20 \qquad (2)$

Dividing equation (2) by equation (1) gives $10^{3k} = \dfrac{10}{3}$. Thus $k = \dfrac{1}{3}\log_{10}\dfrac{10}{3}$, and substituting into equation (1) gives $a = 6 \times \left(\dfrac{10}{3}\right)^{-\frac{2}{3}}$.

13 Use two points, say $(0, 1.5)$ and $(10, 0.006)$ to find $y = ab^x$.

$$\begin{aligned} &\text{at } (0, 1.5) & 1.5 &= a \times b^0 \\ &\therefore & 1.5 &= a \\ &\therefore & y &= 1.5b^x \\ &\text{at } (10, 0.006) & 0.006 &= 1.5b^{10} \\ &\therefore & b^{10} &= \frac{0.006}{1.5} \\ & & &= 0.004 \\ &\therefore & b &= (0.004)^{\frac{1}{10}} \approx 0.5757 \\ &\therefore & y &= 1.5 \times 0.58^x \end{aligned}$$

If CAS is used with exponential regression, $a = 1.5$ and $b = 0.575$, so $y = 1.5(0.575)^x$

14 Use two points, say $(0, 2.5)$ and $(8, 27.56)$ to find $p = ab^t$.

$$\begin{aligned}
\text{at } (0, 2.5) \qquad 2.5 &= a \times b^0 \\
\therefore \qquad 2.5 &= a \\
\therefore \qquad p &= 2.5b^t \\
\text{at } (8, 27.56) \qquad 27.56 &= 2.5b^8 \\
\therefore \qquad b^8 &= \frac{27.56}{2.5} \\
&= 11.024 \\
\therefore \qquad b &= (11.024)^{\frac{1}{8}} \\
&\approx 1.3499 \\
\therefore \qquad p &= 2.5 \times 1.35^t
\end{aligned}$$

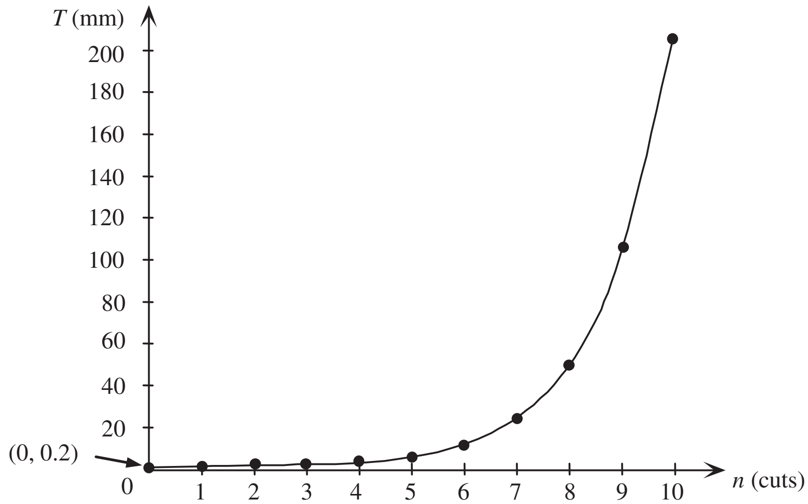
If CAS is used with exponential regression, $a = 1.5$ and $b = 0.575$, so $y = 1.5(0.575)^x$

15 **a**

Cuts, n	Sheets	Total thickness, T (mm)
0	1	0.2
1	2	0.4
2	4	0.8
3	8	1.6
4	16	3.2
5	32	6.4
6	64	12.8
7	128	25.6
8	256	51.2
9	512	102.4
10	1024	204.8

b $T = 0.2 \times 2^n$

c



d When $n = 30$, $T = 0.2 \times 2^{30}$

$= 214\ 748\ 364.8$

Total thickness is 214 748364.8 mm = 214 748.4m

16 $d = d_0(10^{mt})$

$d(1) = 52; d(3) = 80$

$\therefore\ d_0(10^m) = 52; d_0(10^{3m}) = 80$

Take $\log_{10}$ both equations:

(1): $\log_{10} d_0 + m \log_{10} 10 = \log_{10} 52$

$\therefore\quad \log_{10} d_0 + m = \log_{10} 52$

(2): $\log_{10} d_0 + 3m \log_{10} 10 = \log_{10} 80$

$\therefore\quad \log_{10} d_0 + 3m = \log_{10} 80$

(2)–(1) gives

$2m = \log_{10}\left(\frac{80}{52}\right)$

$\therefore\ m = \frac{1}{2}\log_{10}\left(\frac{20}{13}\right) = 0.0935$

Substitute into **(1):**

$\log_{10} d_0 = \log_{10} 52 - 0.0935$

$= \log_{10} 52 - \log_{10}(10^{0.0935})$

$= \log_{10}\left(\frac{52}{1.240}\right) = \log_{10} 41.88$

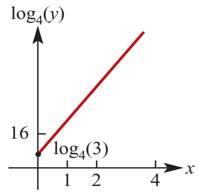
$\therefore\ d_0 = 41.88$ cm

$\therefore\ d_0 = 41.88$ cm

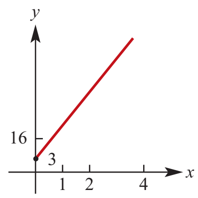
Solutions to Exercise 13I

1 **a** $m = 2$ and $c = \log_4 3$

b

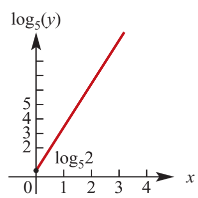


c

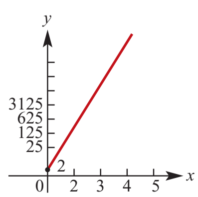


2 **a** $m = 3$ and $c = \log_5 2$

b



c



3 Street 10^{-9} watt/cm^2; Quiet car 10^{-11} watt/cm^2

4 **a** Increases by $10\log_{10} 2 \approx 3$ dB

b Increases by $10\log_{10} 10 = 10$ dB

c $P_{\text{new}} = (P_{\text{old}})^3 \times 10^{32}$

d $P = 10^{-16}$

e $P = 10^{-6}$

5 $P_1 = 10^{\frac{\lambda}{10}} \times P_2$

6 $5 + \log_{10} 5$

7 $7.3 - \log_{10} 4$

8 $[10^{-4}, 10^{-2}]$

Solutions to Technology-free questions

1 a $\dfrac{a^6}{a^2} = a^{6-2} = a^4$

b $\dfrac{b^8}{b^{10}} = b^{8-10}$

$= b^{-2} = \dfrac{1}{b^2}$

c $\dfrac{m^3n^4}{m^5n^6} = m^{3-5}n^{4-6}$

$= m^{-2}n^{-2} = \dfrac{1}{m^2n^2}$

d $\dfrac{a^3b^24}{ab^2} = \dfrac{a^3b^2}{a^4b^8}$

$= a^{3-4}b^{2-8} = \dfrac{1}{ab^6}$

e $\dfrac{6a^8}{4a^2} = \left(\dfrac{6}{4}\right)a^{8-2} = \dfrac{3a^6}{2}$

f $\dfrac{10a^7}{6a^9} = \left(\dfrac{10}{6}\right)a^{7-9} = \dfrac{5}{3a^2}$

g $\dfrac{8(a^3)^2}{2a^3} = \dfrac{8a^6}{8a^3}$

$= a^{6-3} = a^3$

h $\dfrac{m^{-1}n^2}{(mn^{-2})^3} = \dfrac{m^{-1}n^2}{m^3n^{-6}}$

$= m^{-1-3}n^{2+6} = \dfrac{n^8}{m^4}$

i $(p^{-1}q(-2))^2 = p^{-2}q^{-4} = \dfrac{1}{p^2q^4}$

j $\dfrac{(2a^{-4})^3}{5a^{-1}} = \dfrac{8a^{-12}}{5a^{-1}}$

$= \dfrac{8a^{1-12}}{5} = \dfrac{8}{5a^{11}}$

k $\dfrac{6a^{-1}}{3a^{-2}} = \left(\dfrac{6}{3}\right)a^{-1+2} = 2a$

l $\dfrac{a^4 + a^8}{a^2} = \dfrac{a^4}{a^2}(1 + a^4)$

$= a^2(1 + a^4) = a^2 + a^6$

2 a $2^x = 7, \ \therefore \ x = \log_2 7$

b $2^{2x} = 7, 2x = \log_2 7$

$\therefore x = \dfrac{1}{2}\log_2 7$

c $10^x = 2, \ \therefore \ x = \log_{10} 2$

d $10^x = 3.6, \ \therefore \ x = \log_{10} 3.6$

e $10^x = 110, \ \therefore \ x = \log_{10} 110$
(or $1 + \log_{10} 11$)

f $10^x = 1010, \ \therefore \ x = \log_{10} 1010$
(or $1 + \log_{10} 101$)

g $2^{5x} = 100, \ \therefore \ 5x = \log_2 100$
$\therefore \ x = \dfrac{1}{5}\log_2 100$

h $2^x = 0.1, \ \therefore \ x = \log_2 0.1$

$= -\log_2 10$

3 a $\log_2 64 = \log_2 2^6$

$= 6\log_2 2 = 6$

b $\log_{10} 10^7 = 7\log_{10} 10 = 7$

c $\log_a a^2 = 2\log_a a = 2$

d $\log_4 1 = 0$ by definition

e $\log_3 27 = \log_3 3^3$

$= 3\log_3 3 = 3$

f $\log_2 \frac{1}{4} = \log_2 2^{-2}$

$= -2\log_2 2 = -2$

g $\log_{10} 0.001 = \log_{10} 10^{-3}$

$= -3\log_{10} 10 = -3$

h $\log_2 16 = \log_2 2^4$

$= 4\log_2 2 = 4$

4 **a** $\log_{10} 2 + \log_{10} 3 = \log_{10}(2 \times 3) = \log_{10} 6$

b $\log_{10} 4 + 2\log_{10} 3 - \log_{10} 6$

$= \log_{10} 4 + \log_{10}(3^2) - \log_{10} 6$

$= \log_{10} \frac{4(3^2)}{6} = \log_{10} 6$

c

$2\log_{10} a - \log_{10} b = \log_{10} a^2 - \log_{10} b$

$= \log_{10}\left(\frac{a^2}{b}\right)$

d

$2\log_{10} a - 3 - \log_{10} 25$

$= \log_{10} a^2 - \log_{10} 25 - \log_{10} 10^3$

$= \log_{10}\left(\frac{a^2}{25000}\right)$

e $\log_{10} x + \log_{10} y - \log_{10} x = \log_{10} y$

f $2\log_{10} a + 3\log_{10} b - \log_{10} c$

$= \log_{10} a^2 + \log_{10} b^3 - \log_{10} c$

$= \log_{10}\left(\frac{a^2b^3}{c}\right)$

5 **a** $3^x(3^x - 27) = 0$

$3^x = 27, \therefore\ x = 3$

($3^x \neq 0$ for any real x)

b $(2^x - 8)(2^x - 1) = 0$

$2^x = 1, 8, \therefore\ x = 0, 3$

c $2^{2x} - 2^{x+1} = 0$

$(2^x)(2^x - 2) = 0$

$2^x = 2, \therefore\ x = 1$

($2^x \neq 0$ for any real x)

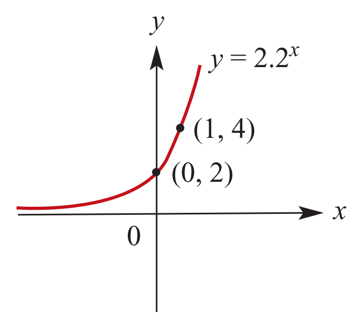
d $2^{2x} - 12(2^x) + 32 = 0$

$(2^x - 8)(2^x - 4) = 0$

$2^x = 4, 8, \therefore\ x = 2, 3$

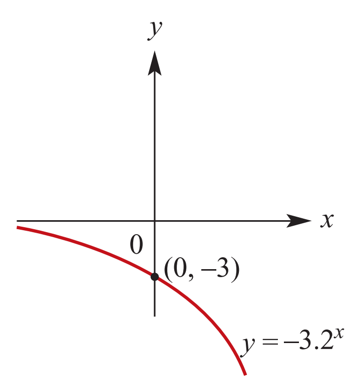
6 **a** $y = 2 \times 2^x$

Asymptote at $y = 0$, y-intercept at (0, 1)



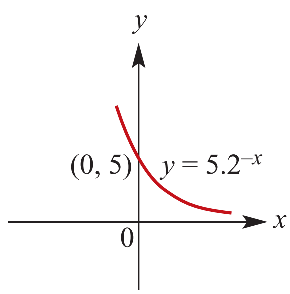
b $y = -3 \times 2^x$

Asymptote at $y = 0$, y-intercept at (0, 1)



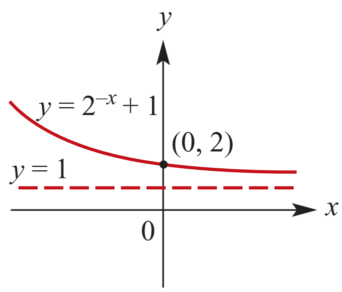
c $y = 5 \times 2^{-x}$

Asymptote at $y = 0$, y-intercept at (0, 1)



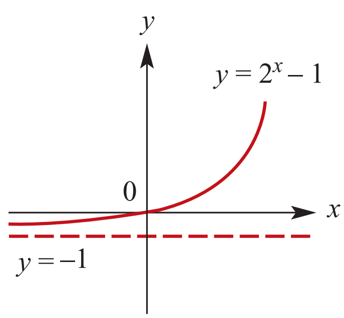
d $y = 2^{-x} + 1$

Asymptote at $y = 1$, y-intercept at $(0, 2)$



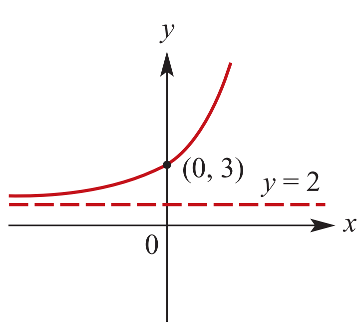
e $y = 2^x - 1$

Asymptote at $y = -1$, y-intercept at $(0, 0)$



f $y = 2^x + 2$

Asymptote at $y = 2$, y-intercept at $(0, 3)$



7

$$\begin{aligned} \log_{10} x + \log_{10} 2x - \log_{10}(x+1) &= 0 \\ \therefore \quad \log_{10} \frac{2x^2}{x+1} &= 0 \\ \frac{2x^2}{x+1} &= 1 \\ 2x^2 &= x + 1 \\ 2x^2 - x - 1 &= 0 \\ (2x+1)(x-1) &= 0 \end{aligned}$$

$\therefore \; x = -\frac{1}{2}, 1$

Since $\log x$ is not defined for $x \le 0$, $x = 1$

8 $3^x = 4^y = 12^z$

$\therefore \; z(\log 12) = x \log 3$

$$x = \frac{z \log 12}{\log 3}$$

$$y = \frac{z \log 12}{\log 4}$$

$$\begin{aligned} \frac{xy}{x+y} &= \frac{(z \log 12)^2}{(\log 3)(\log 4)} \div \left(\frac{z \log 12}{\log 3} + \frac{z \log 12}{\log 4}\right) \\ &= \frac{z \log 12}{(\log 3)(\log 4)} \div \left(\frac{1}{\log 3} + \frac{1}{\log 4}\right) \\ &= \frac{z \log 12}{(\log 3)(\log 4)} \div \frac{\log 4 + \log 3}{(\log 3)(\log 4)} \\ &= (z \log 12) \div (\log 12) = z \end{aligned}$$

$$\therefore \; z = \frac{xy}{x+y}$$

9 $2 \log_2 12 + 3 \log_2 5 - \log_2 15 - \log_2 150$

$$\begin{aligned} &= \log_2 \frac{(12^2)(5^3)}{(15)(150)} \\ &= \log_2 \left(\frac{18000}{2250}\right) \\ &= \log_2 8 = 3 \end{aligned}$$

10 **a** $\log_p 7 + \log_p k = 0, \; \therefore \; \log_p 7k = 0$

$\therefore\ 7k = 1,\ \therefore\ k = \frac{1}{7}$

b $4\log_q 3 + 2\log_q 2 - \log_q 144 = 2$

$$\therefore\ \log_q \frac{(3^4)(2^2)}{144} = 2$$

$$\log_q\left(\frac{9}{4}\right) = 2$$

$$\frac{9}{4} = q^2,\ \therefore\ q = \frac{3}{2}$$

(all log bases > 0)

11 a $2(4^{a+1}) = 16^{2a}$

$$\therefore\ 4^{\frac{1}{2}}(4^{a+1}) = 4^{4a}$$

$$4^{a+\frac{3}{2}} = 4^{4a}$$

$$a + \frac{3}{2} = 4a$$

$$3a = \frac{3}{2},\ \therefore\ a = \frac{1}{2}$$

b $\log_2 y^2 = 4 + \log_2(y + 5)$

$$\therefore\ \log_2 y^2 - \log_2(y + 5) = 4$$

$$\log_2\left(\frac{y^2}{y + 5}\right) = 4$$

$$\frac{y^2}{y + 5} = 2^4$$

$$y^2 = 16y + 80$$

$$y^2 - 16y - 80 = 0$$

$$(y - 20)(y + 4) = 0$$

$\therefore\ y = -4, 20$

(Both solutions must be included here, because the only domain restriction is that $y > -5$)

Solutions to multiple-choice questions

1 **C** $\dfrac{8x^3}{4x^{-3}} = \dfrac{8}{4}x^{3+3} = 2x^6$

2 **A**
$$\begin{aligned}\frac{a^2b}{(2ab^2)^3} \div \frac{ab}{16a^0} &= \frac{a^2b}{8a^3b^6}\,\frac{16}{ab}\\ &= \frac{16}{8}a^{2-3-1}b^{1-6-1}\\ &= 2a^{-2}b^{-6}\\ &= \frac{2}{a^2b^6}\end{aligned}$$

3 **C** The range of $y = 3 \times 2^x$ is $(0, \infty)$ but $f(x) = 3(2^x) - 1$ is translated 1 unit down
$\therefore$ range $= (-1, \infty)$

4 **C** $f(x) = \log_2 3x$
$$\begin{aligned} f^{-1}(x)\colon\ x &= \log_2(3y)\\ 2^x &= 3y\\ \therefore\ f^{-1}(x) &= \frac{1}{3}2^x\end{aligned}$$

5 **A** $\log_{10}(x - 2) - 3\log_{10} 2x = 1 - \log_{10} y$
$$\begin{aligned}\therefore\ \log_{10}\frac{x-2}{(2x)^3} + \log_{10} y &= 1\\ \log_{10}\frac{y(x-2)}{8x^3} &= 1\\ \frac{y(x-2)}{8x^3} &= 10\\ \therefore\ y &= \frac{80x^3}{x-2}\end{aligned}$$

6 **B** $5(2^{5x}) = 10,\ \therefore\ 2^{5x} = 2^1$
$$\therefore\ 5x = 1,\ \therefore\ x = \frac{1}{5}$$

7 **A** The vertical asymptote of $y = \log x$ is at $x = 0$. Here $5x = 0$ so $x = 0$. (y-direction translations don't affect the vertical asymptote.)

8 **A** $f(x) = 2^{ax} + b;\ a, b > 0$
Function must be increasing, with a horizontal asymptote at $y = b$ which the graph approaches at large negative values of x, and there will be no x-intercept because $b > 0$

9 **A** Vertical asymptote, hence log or hyperbola. But B and C both have a vertical asymptote $x = -b$.

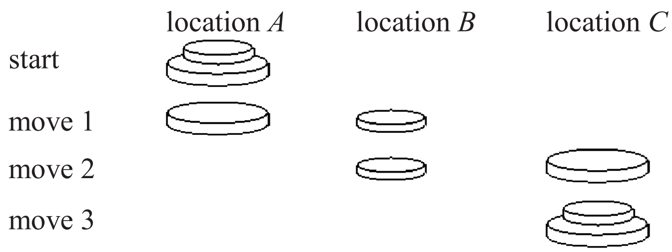
10 **A**
$$\begin{aligned}\frac{2mh}{(3mh^2)^3} \div \frac{mh}{81m^2} &= \frac{2mh}{27m^3h^6}\,\frac{81m^2}{mh}\\ &= 6m^{1+2-3-1}h^{1-6-1}\\ &= 6m^{-1}h^{-6}\\ &= \frac{6}{mh^6}\end{aligned}$$

Solutions to extended-response questions

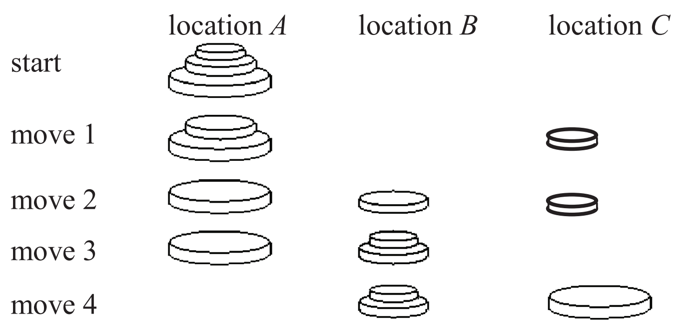
1 a

Number of discs, n	0	1	2	3	4
Minimum no. of moves, M	0	1	3	7	15

For two discs, the following procedure may be used.



For three discs, the procedure is as follows.



Now the problem reduces to taking the two discs from B to C, i.e. three more moves (using the technique for two discs).

$$\therefore \quad \text{total number of moves} = 3 + 4$$
$$= 7$$

This procedure can be generalised for n discs.

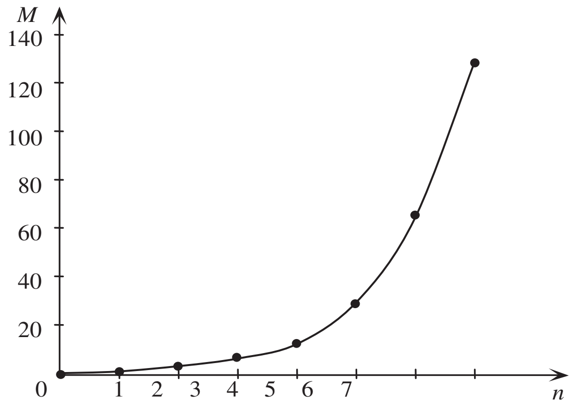
- The top $n - 1$ discs can be moved from A to B in $2^{n-1} - 1$ moves.
- The remaining bottom disc can be moved from A to C.
- The $n - 1$ discs on B can be moved to C in $2^{n-1} - 1$ moves.

$$\therefore \quad \text{total number of moves} = 2^{n-1} - 1 + 1 + 2^{n-1} - 1$$
$$= 2 \times 2^{n-1} - 1$$
$$= 2^n - 1$$

b $M = 2^n - 1$

Number of discs, n	0	5	6	7
Minimum no. of moves, M	0	31	63	127

c



d Let the top disc be called D_1, the next D_2, then D_3 and so on to nth disc, D_n.
For 3 discs, D_1 moves 4 times, D_2 2 times and D_3 once.
For 4 discs, D_1 moves 8 times, D_2 4 times, D_3 2 times and D_4 once.
For n discs, D_1 moves 2^{n-1} times, D_2 2^{n-2} times, ..., D_n 2^0 times.

Three discs	D_1	D_2	D_3
Times moved	4	2	1

Four discs	D_1	D_2	D_3	D_4
Times moved	8	4	2	1

n discs	D_1	D_2	D_3	...	D_{n-1}	D_n
Times moved	2^{n-1}	2^{n-2}	2^{n-3}		2^1	2^0

Note: For n discs, total number of moves $= 1 + 2 + 4 + \dots + 2^{n-1}$

$$= \frac{1(2^n - 1)}{2 - 1} = 2^n - 1$$

2 $$2187 = 9 \times 9 \times 9 \times 3 = 9^3 \times 3^1$$

This gives 3 switches of Type 1 and 1 switch of Type 2.
However, if n of Type 1 and $n + 1$ of Type 2 are used, there needs to be one more 3 than the number of 9s in the factorisation.

$$2187 = 9 \times 9 \times 3 \times 3 \times 3 = 9^2 \times 3^3$$

Two switches of Type 1 and three of Type 2 are needed. Hence, $n = 2$.

3 a $\left(\frac{1}{8}\right)^n = \left(\left(\frac{1}{2}\right)^3\right)^n = \left(\frac{1}{2}\right)^{3n}$

b
$$\begin{aligned}\left(\frac{1}{4}\right)^{n-1}\left(\frac{1}{2}\right)^{3n} &= \left(\left(\frac{1}{2}\right)^2\right)^{n-1}\left(\frac{1}{2}\right)^{3n}\\ &= \left(\frac{1}{2}\right)^{2(n-1)}\left(\frac{1}{2}\right)^{3n}\\ &= \left(\frac{1}{2}\right)^{2n-2}\left(\frac{1}{2}\right)^{3n} = \left(\frac{1}{2}\right)^{5n-2}\end{aligned}$$

c $$\left(\frac{1}{2}\right)^{n-3}\left(\frac{1}{2}\right)^{5n-2} = \left(\frac{1}{2}\right)^{6n-5}$$

Now, $$\left(\frac{1}{2}\right)^{6n-5} = \frac{1}{8192} = \frac{1}{2^{13}} = \left(\frac{1}{2}\right)^{13}$$

$\therefore$ $6n - 5 = 13$

$\therefore$ $6n = 18 \;\therefore\; n = 3$

4 a

Times used	1	2	3	n
Caffeine remaining	$729\left(\frac{1}{4}\right)^1$	$729\left(\frac{1}{4}\right)^2$	$729\left(\frac{1}{4}\right)^3$	$729\left(\frac{1}{4}\right)^n$

b

Times used	1	2	3	n
Tannin remaining	$128\left(\frac{1}{2}\right)^1$	$128\left(\frac{1}{2}\right)^2$	$128\left(\frac{1}{2}\right)^3$	$128\left(\frac{1}{2}\right)^n$

c Can be re-used if amount of tannin $\leq 3\times$ amount of caffeine.

i.e. $$128\left(\frac{1}{2}\right)^n \leq 3 \times 729\left(\frac{1}{4}\right)^n$$

$\Leftrightarrow$ $$128\left(\frac{1}{2}\right)^n \leq 2187\left(\frac{1}{2}\right)^{2n}$$

$\Leftrightarrow$ $$\frac{128}{2187} \leq \frac{\left(\frac{1}{2}\right)^{2n}}{\left(\frac{1}{2}\right)^n}$$

$\Leftrightarrow$ $$\frac{128}{2187} \leq \left(\frac{1}{2}\right)^n$$

$\Leftrightarrow$ $$\log_{10}\left(\frac{128}{2187}\right) \leq \log_{10}\left(\frac{1}{2}\right)^n$$

$\Leftrightarrow$ $$\log_{10}\left(\frac{128}{2187}\right) \leq n\log_{10}\left(\frac{1}{2}\right)$$

$\Leftrightarrow$ $$\frac{\log_{10}\left(\frac{128}{2187}\right)}{\log_{10}\left(\frac{1}{2}\right)} \geq n \text{ as } \log_{10}\left(\frac{1}{2}\right) < 0$$

$\therefore$ $n \leq 4.09$

Hence, the tea leaves can be re-used 4 times.

5 a Brightness Batch 1 after n years $= 15(0.95)^n$
Brightness of Batch 2 after n years $= 20(0.94)^n$

b Let n be the number of years until brightness is the same.

$$
\begin{aligned}
15(0.95)^{n+1} &= 20(0.94)^n \\
\frac{(0.95)^{n+1}}{(0.94)^n} &= \frac{20}{15} \\
\log_{10}\left(\frac{(0.95)^{n+1}}{(0.94)^n}\right) &= \log_{10}\left(\frac{4}{3}\right) \\
\therefore \quad \log_{10}(0.95)^{n+1} - \log_{10}(0.94)^n &= \log_{10}\left(\frac{4}{3}\right) \\
(n+1)\log_{10}(0.95) - n\log_{10}(0.94) &= \log_{10}\left(\frac{4}{3}\right) \\
n\log_{10}(0.95) + \log_{10}(0.95) - n\log_{10}(0.94) &= \log_{10}\left(\frac{4}{3}\right) \\
n(\log_{10}(0.95) - \log_{10}(0.94)) &= \log_{10}\left(\frac{4}{3}\right) - \log_{10}(0.95) \\
n\log_{10}\left(\frac{0.95}{0.94}\right) &= \log_{10}\left(\frac{4}{3\times 0.95}\right) \\
n &= \frac{\log_{10}\left(\frac{400}{285}\right)}{\log_{10}\left(\frac{95}{94}\right)} \\
&= 32.033
\end{aligned}
$$

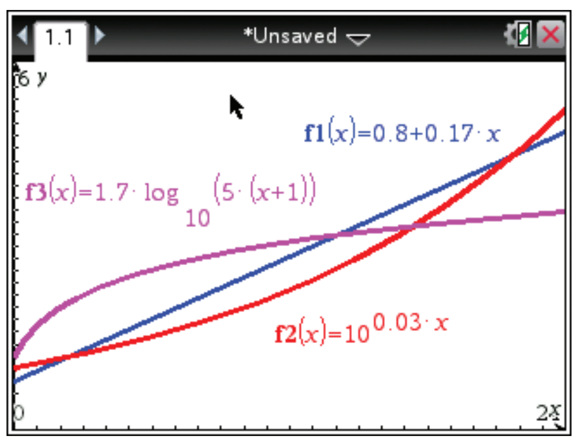
Hence, the brightness is the same early in the 33rd year (i.e. after about 32 years).

6 $x = 0.8 + 0.17t,\ y = 10^{0.03t},\ z = 1.7\log_{10}(5(x+1))$

a For X: $t = 6$ implies
$x = 1.82$
i.e. the value is \$ 1.82

For Y: $t = 6$ implies
$y = 1.51$
i.e. the value is \$ 1.51

For Z: $t = 6$ implies
$z = 2.62$
i.e. the value is \$ 2.62



b For X: $t = 21$ implies $x = 4.37$ i.e. the value is \$ 4.37
For Y: $t = 21$ implies $y = 4.27$ i.e. the value is \$ 4.27
For Z: $t = 21$ implies $z = 3.47$ i.e. the value is \$ 3.47

c The graphs of $x(t)$ and $y(t)$ intersect where $t = 2.09$ and $t = 21.78$.
From the graph, the shares of X have greater value than the shares of Y for $2.09 < t < 21.78$, i.e. from February 2014 to September 2015.

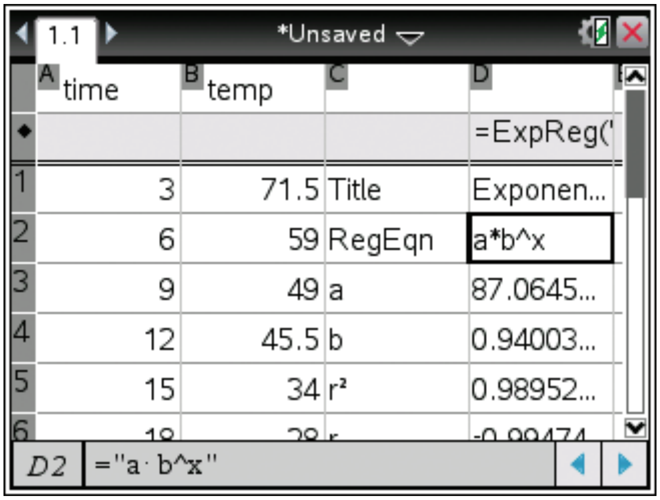
d The graphs of $x(t)$ and $z(t)$ intersect at $t = 14.06$.
From the graphs, it can be seen that shares in X are most valuable for $14.06 < t < 21.78$.
Therefore X were the most valuable shares for 7.72 months, or about 8 months, i.e. from February 1998 to September 2015.

7 Let W be the number of wildebeest and n the number of years.
Then $W = 700(1.03)^n$
Let Z be the number of zebras.
Then
$$Z = (0.96)^n \times 1850$$
$$= 1850(0.96)^n$$

a
$$(0.96)^n \times 1850 = 700(1.03)^n$$
$$\frac{1850}{700} = \left(\frac{1.03}{0.96}\right)^n$$
$$\frac{37}{14} = \left(\frac{103}{96}\right)^n$$
$$\therefore \quad n = 13.81$$
After 13.81 years, the number of wildebeest exceeds the number of zebras.

b Let A be the number of antelopes.
$$A = 1000 + 50n$$
The number of antelopes is greater than the number of zebras when
$$1000 + 50n > 1850(0.96)^n$$
From a CAS calculator,
$$1000 + 50n > 1850(0.96)^n \text{ for } n > 7.38$$
After 7.38 years, the number of antelopes exceeds the number of zebras.

8 **a** **TI:** Type the given data into a new Lists & Spreadsheet application. Call column A *time*, and column B *temp*
Press **Menu → 4:Statistics → 1:Stat Calculations → A:Exponential Regression**
Set **X List** to **time** and **Y List** to **temp** then ENTER



CP: Open the Statistics application. Type the time data into list1 and the temperature data into list2

Tap **Calc → abExponential Reg** and set **XList** to **list1** and **YList** to **list2**

The values of a and b are given as $a = 87.06$ and $b = 0.94$, correct to 2 decimal places,

$$\therefore \qquad T = 87.06 \times 0.94^t$$

b **i** When $t = 0$, $\quad T = 87.06°\text{C}$

ii When $t = 25$, $\quad T = 18.56°\text{C}$

c $(12, 45.5)$ is the reading which appears to be incorrect.

Re-calculating gives $a = 85.724\ldots$ and $b = 0.9400$

$$\therefore \qquad T = 85.72 \times 0.94^t$$

d **i** When $t = 0$, $\quad T = 85.72°\text{C}$

ii When $t = 12$, $\quad T = 40.82°\text{C}$

e When $T = 15$, $\quad t = 28.19$ min

9 **a** At (1, 1) $\quad 1 = a \times b^1$

$\therefore \quad 1 = ab \quad (1)$

At (2, 5) $\quad 5 = a \times b^2 \quad (2)$

Divide (2) by (1) $\quad 5 = b$

Substitute $b = 5$ into (1) $\quad 1 = a \times 5$

$\therefore \quad a = \dfrac{1}{5} = 0.2$

$\therefore \; a = 0.2, \; b = 5$

b Let $\quad b^x = 10^z$

i By the definition of logarithm:

$$z = \log_{10} b^x$$

$$\therefore \quad = x \log_{10} b$$

ii $\quad y = a \times 10^{kx}$

$$= a \times b^x \qquad \text{where } b^x = 10^{kx}$$

From **b i**, $b^x = 10^{kx}$ can be rewritten

$$kx = x \log_{10} b$$

$$\therefore \quad k = \log_{10} b$$

From **a**, $a = 0.2$ and $b = 5$, $\therefore \; k = \log_{10} 5$

10 **a** Use two points, say (0, 2) and (10, 200) to find $y = ab^x$.

At (0, 2) $\quad 2 = a \times b^0$

$\therefore \quad 2 = a$

$\therefore \quad y = 2b^x$

At(10, 200) $\quad 200 = 2b^{10}$

$\therefore \quad b^{10} = \dfrac{200}{2} = 100$

$\therefore \quad b = (100)^{\frac{1}{10}}$

$= 1.5849$ (correct to 4 decimal places)

$\therefore \quad y = 2 \times 1.5849^x$

Using CAS regression $y = 2 \times 1.585^x$

b From Question 9, $k = \log_{10} b$

and from part **a**, $a = 2$ and $b = (100)^{\frac{1}{10}}$

$$\therefore \quad k = \log_{10}(100)^{\frac{1}{10}}$$

$$= \frac{1}{10}\log_{10} 100$$

$$= \frac{1}{10} \times 2 = \frac{1}{5}$$

$$\therefore \quad y = 2 \times 10^{\frac{x}{5}} = 2 \times 10^{0.2x}$$

c $y = 2 \times 10^{\frac{x}{5}}$

can be written $\dfrac{y}{2} = 10^{\frac{x}{5}}$

By definition of logarithms:

$$\frac{x}{5} = \log_{10}\left(\frac{y}{2}\right)$$

$$x = 5\log_{10}\left(\frac{y}{2}\right)$$

Chapter 14 – Circular functions

Solutions to Exercise 14A

1 **a** $60° = \frac{60\pi}{180} = \frac{\pi}{3}$

b $144° = \frac{144\pi}{180} = \frac{4\pi}{5}$

c $240° = \frac{240\pi}{180} = \frac{4\pi}{3}$

d $330° = \frac{330\pi}{180} = \frac{11\pi}{6}$

e $420° = \frac{420\pi}{180} = \frac{7\pi}{3}$

f $480° = \frac{480\pi}{180} = \frac{8\pi}{3}$

2 **a** $\frac{2\pi}{3} = \frac{2\pi}{3}\frac{180°}{\pi} = 120°$

b $\frac{5\pi}{6} = \frac{5\pi}{63}\frac{180°}{\pi} = 150°$

c $\frac{7\pi}{6} = \frac{7\pi}{6}\frac{180°}{\pi} = 210°$

d $0.9\pi = 0.9\pi\frac{180°}{\pi} = 162°$

e $\frac{5\pi}{9} = \frac{5\pi}{9}\frac{180°}{\pi} = 100°$

f $\frac{9\pi}{5} = \frac{9\pi}{5}\frac{180°}{\pi} = 324°$

g $\frac{11\pi}{5} = \frac{11\pi}{5}\frac{180°}{\pi} = 220°$

h $1.8\pi = 1.8\pi\frac{180°}{\pi} = 324°$

3 From calculator:

a $0.6 = 34.38°$

b $1.89 = 108.29°$

c $2.9 = 166.16°$

d $4.31 = 246.94°$

e $3.72 = 213.14°$

f $5.18 = 296.79°$

g $4.73 = 271.01°$

h $6.00 = 343.77°$

4 From calculator:

a $38° = 0.66$

b $73° = 1.27$

c $107° = 1.87$

d $161° = 2.81$

e $84.1° = 1.47$

f $228° = 3.98$

g $136.4° = 2.39$

h $329° = 5.74$

5 a $-\frac{\pi}{3} = -\frac{\pi}{3}\frac{180°}{\pi} = -60°$

b $-4\pi = -4\pi\frac{180°}{\pi} = -720°$

c $-3\pi = -3\pi\frac{180°}{\pi} = -540°$

d $-\pi = -\pi\frac{180°}{\pi} = -180°$

e $\frac{5\pi}{3} = \frac{5\pi}{3}\frac{180°}{\pi} = 300°$

f $-\frac{11\pi}{6} = -\frac{11\pi}{6}\frac{180°}{\pi} = 330°$

g $\frac{23\pi}{6} = \frac{23\pi}{6}\frac{180°}{\pi} = 690°$

h $-\frac{23\pi}{6} = -\frac{23\pi}{6}\frac{180°}{\pi} = -690°$

6 a $-360° = -\frac{360\pi}{180} = -2\pi$

b $-540° = -\frac{540\pi}{180} = -3\pi$

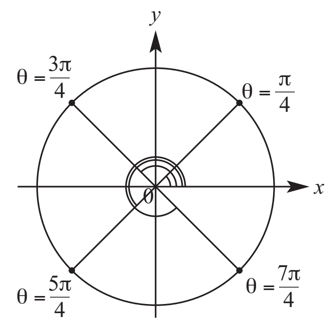
c $-240° = -\frac{240\pi}{180} = -\frac{4\pi}{3}$

d $-720° = -\frac{720\pi}{180} = -4\pi$

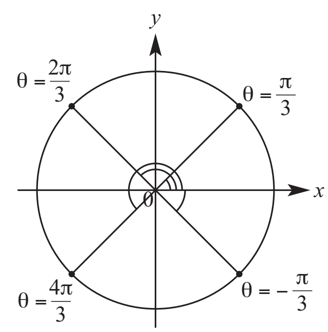
e $-330° = -\frac{330\pi}{180} = -\frac{11\pi}{6}$

f $-210° = -\frac{210\pi}{180} = -\frac{7\pi}{6}$

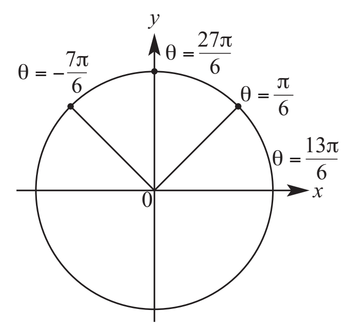
7 a



b



c



Solutions to Exercise 14B

1 **a** $t = 0; \sin t = 0; \cos t = 1$

b $t = \frac{3\pi}{2}; \sin t = -1; \cos t = 0$

c $t = -\frac{3\pi}{2}; \sin t = 1; \cos t = 0$

d $t = \frac{5\pi}{2}; \sin t = 1; \cos t = 0$

e $t = -3\pi; \sin t = 0; \cos t = -1$

f $t = \frac{9\pi}{2}; \sin t = 1; \cos t = 0$

g $t = \frac{7\pi}{2}; \sin t = -1; \cos t = 0$

h $t = 4\pi; \sin t = 0; \cos t = 1$

2 From calculator:

a $\sin 1.9 = 0.95$

b $\sin 2.3 = 0.75$

c $\sin 4.1 = -0.82$

d $\cos 0.3 = 0.96$

e $\cos 2.1 = -0.5$

f $\cos(-1.6) = -0.03$

g $\sin(-2.1) = -0.86$

h $\sin(-3.8) = 0.61$

3 **a** $\theta = 27\pi; \sin\theta = 0; \cos\theta = -1$

b $\theta = -\frac{5\pi}{2}; \sin\theta = -1; \cos\theta = 0$

c $\theta = \frac{27\pi}{2}; \sin\theta = -1; \cos\theta = 0$

d $\theta = -\frac{9\pi}{2}; \sin\theta = -1; \cos\theta = 0$

e $\theta = \frac{11\pi}{2}; \sin\theta = -1; \cos\theta = 0$

f $\theta = 57\pi; \sin\theta = 0; \cos\theta = -1$

g $\theta = 211\pi; \sin\theta = 0; \cos\theta = -1$

h $\theta = -53\pi; \sin\theta = 0; \cos\theta = -1$

Solutions to Exercise 14C

1 a $\tan \pi = \tan 0 = 0$

b $\tan(-\pi) = \tan 0 = 0$

c $\tan\left(\frac{7\pi}{2}\right) = \tan\frac{\pi}{2} = \text{undefined}$

d $\tan(-2\pi) = \tan 0 = 0$

e $\tan\left(\frac{5\pi}{2}\right) = \tan\frac{\pi}{2} = \text{undefined}$

f $\tan -\frac{\pi}{2} = \tan\frac{\pi}{2} = \text{undefined}$

2 From calculator:

a $\tan 1.6 = -34.23$

b $\tan(-1.2) = -2.57$

c $\tan 136° = -0.97$

d $\tan(-54°) = -1.38$

e $\tan 3.9 = 0.95$

f $\tan(-2.5) = 0.75$

g $\tan 239° = 1.66$

3 a $\tan 180° = \tan 0° = 0$

b $\tan 360° = \tan 0° = 0$

c $\tan 0° = 0$

d $\tan(-180°) = \tan 0° = 0$

e $\tan(-540°) = \tan 0° = 0$

f $\tan 720° = \tan 0° = 0$

Solutions to Exercise 14D

1 **a** $\theta = \cos^{-1}\left(\frac{3}{8}\right) = 67.98°$ or $67°59'$

b $x = 5\ \cos 25° = 4.5315$

c $x = 6\ \sin 25° = 2.5357$

d $x = 10\ \cos 50° = 6.4279$

e $\theta = \tan^{-1}\left(\frac{6}{5}\right) = 50.20°$ or $50°12'$

f $x = 10\ \sin 20° = 3.4202$

g $x = \dfrac{5}{\tan 65°} = 2.3315$

h $x = 7\ \sin 70° = 6.5778$

i $x = \dfrac{5}{\cos 40°} = 6.5270$

Solutions to Exercise 14E

1 $\sin\theta = 0.42, \cos x = 0.7, \tan\alpha = 0.38$

a $\sin(\pi + \theta) = -\sin\theta = -0.42$

b $\cos(\pi - x) = -\cos x = -0.7$

c $\sin(2\pi - \theta) = -\sin\theta = -0.42$

d $\tan(\pi - \alpha) = -\tan\alpha = -0.38$

e $\sin(\pi - \theta) = \sin\theta = 0.42$

f $\tan(2\pi - \alpha) = -\tan\alpha = -0.38$

g $\cos(\pi + x) = -\cos x = -0.7$

h $\cos(2\pi - x) = \cos x = 0.7$

2 **a** $\cos x = -\cos\frac{\pi}{6}; \frac{\pi}{2} < x < \pi,$

$\therefore x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

b $\cos x = -\cos\frac{\pi}{6}; \pi < x < \frac{3\pi}{2}$

$\therefore x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

c $\cos x = \cos\frac{\pi}{6}; \frac{3\pi}{2} < x < 2\pi$

$\therefore x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

3 $\sin\theta = \frac{\sqrt{3}}{2}, \cos\theta = \frac{1}{2}$ from diagram

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}$

a $a = \cos(\pi - \theta) = -\cos\theta = -\frac{1}{2}$

b $b = \sin(\pi - \theta) = \sin\theta = \frac{\sqrt{3}}{2}$

c $c = \cos(-\theta) = \cos\theta = \frac{1}{2}$

d $d = \sin(-\theta) = -\sin\theta = -\frac{\sqrt{3}}{2}$

e $\tan(\pi - \theta) = -\tan\theta = -\sqrt{3}$

f $\tan(-\theta) = -\tan\theta = -\sqrt{3}$

4 $\sin\theta = \frac{\sqrt{3}}{2}, \cos\theta = -\frac{1}{2}$ from diagram

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{3}}{2} \div -\frac{1}{2} = -\sqrt{3}$

a $d = \sin(\pi + \theta) = -\sin\theta = -\frac{\sqrt{3}}{2}$

b $c = \cos(\pi + \theta) = -\cos\theta = \frac{1}{2}$

c $\tan(\pi + \theta) = \tan\theta = -\sqrt{3}$

d $\sin(2\pi - \theta) = -\sin\theta = -\frac{\sqrt{3}}{2}$

e $\cos(2\pi - \theta) = \cos\theta = -\frac{1}{2}$

5 **a** $(a, b) = (\cos 40°, \sin 40°)$

$= (0.7660, 0.6428)$

b $(c, d) = (-\cos 40°, \sin 40°)$

$= (-0.7660, 0.6428)$

c **i** $\cos 140° = -0.7660,$

$\sin 140° = 0.6428$

ii $\cos 140° = -\cos 40°$

6 $\sin x° = 0.7, \cos\theta = 0.6°$ and

$\tan \alpha^\circ = 0.4$

a $\sin(180 + x)^\circ = -\sin x^\circ = -0.7$

b $\cos(180 + \theta)^\circ = -\cos \theta^\circ = -0.6$

c $\tan(360 - \alpha)^\circ = -\tan \alpha^\circ = -0.4$

d $\cos(180 - \theta)^\circ = -\cos \theta^\circ = -0.6$

e $\sin(360 - x)^\circ = -\sin x^\circ = -0.7$

f $\sin(-x)^\circ = -\sin x^\circ = -0.7$

g $\tan(360 + \alpha)^\circ = \tan \alpha^\circ = 0.4$

h $\cos(-\theta)^\circ = \cos \theta^\circ = 0.6$

7 **a** $\sin x = \sin 60^\circ$ and $90^\circ < x < 180^\circ$

$\therefore x = 180^\circ - 60^\circ = 120^\circ$

b

$\sin x = -\sin 60^\circ$ and $180^\circ < x < 270^\circ$

$\therefore x = 180^\circ + 60^\circ = 240^\circ$

c $\sin x = -\sin 60^\circ$ and $-90^\circ < x < 0^\circ$

$\therefore x = 0^\circ - 60^\circ = -60^\circ$

d

$\cos x^\circ = -\cos 60^\circ$ and $90^\circ < x < 180^\circ$

$\therefore x = 180^\circ - 60^\circ = 120^\circ$

e

$\cos x^\circ = -\cos 60^\circ$ and $180^\circ < x < 270^\circ$

$\therefore x = 180^\circ + 60^\circ = 240^\circ$

f $\cos x^\circ = \cos 60^\circ$ and $270^\circ < x < 360^\circ$

$\therefore x = 360^\circ - 60^\circ = 300^\circ$

Solutions to Exercise 14F

1

	x	$\sin x$	$\cos x$	$\tan x$
a	$120°$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
b	$135°$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
c	$210°$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
d	$240°$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
e	$315°$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
f	$390°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
g	$420°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
h	$-135°$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
i	-300	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
j	-60	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$

2 a $\sin\frac{2\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

b $\cos\frac{3\pi}{4} = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

c $\tan\frac{5\pi}{6} = -\tan\frac{\pi}{6} = -\frac{\sqrt{3}}{3}$

d $\sin\frac{7\pi}{6} = -\sin\frac{\pi}{6} = -\frac{1}{2}$

e $\cos\frac{5\pi}{4} = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

f $\tan\frac{4\pi}{3} = \tan\frac{\pi}{3} = \sqrt{3}$

g $\sin\frac{5\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

h $\cos\frac{7\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

i $\tan\frac{11\pi}{6} = -\tan\frac{\pi}{6} = -\frac{\sqrt{3}}{3}$

3 a
$$\sin\left(-\frac{2\pi}{3}\right) = -\sin\left(\frac{2\pi}{3}\right)$$
$$= -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

b
$$\cos\frac{11\pi}{4} = \cos\frac{3\pi}{4}$$
$$= -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$

c $\tan\frac{13\pi}{6} = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$

d
$$\tan\frac{15\pi}{6} = \tan\frac{5\pi}{2}$$
$$= \tan\frac{\pi}{2} = \text{undefined}$$

e
$$\cos\frac{14\pi}{4} = \cos\frac{7\pi}{2}$$
$$= \cos\frac{3\pi}{2} = 0$$

f
$$\cos\left(-\frac{3\pi}{4}\right) = \cos\frac{3\pi}{4}$$
$$= -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$

g
$$\sin\frac{11\pi}{4} = \sin\frac{3\pi}{4}$$
$$= \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

h
$$\cos\left(-21\frac{\pi}{3}\right) = \cos(-7\pi)$$
$$= \cos\pi = -1$$

Solutions to Exercise 14G

1 **a** $2\sin\theta$: per $= 2\pi$, ampl $= 2$

b $3\sin 2\theta$: per $= \dfrac{2\pi}{2} = \pi$, ampl $= 3$

c $\dfrac{1}{2}\cos 3\theta$: per $= \dfrac{2\pi}{3}$, ampl $= \dfrac{1}{2}$

d $3\sin\dfrac{\theta}{2}$: per $= \dfrac{2\pi}{\frac{1}{2}} = 4\pi$, ampl $= 3$

e $4\cos 3\theta$: per $= \dfrac{2\pi}{3}$, ampl $= 4$

f $-\dfrac{1}{2}\sin 4\theta$: per $= \dfrac{2\pi}{4} = \dfrac{\pi}{2}$, ampl $= \dfrac{1}{2}$

g $-2\cos\dfrac{\theta}{2}$: per $= \dfrac{2\pi}{\frac{1}{2}} = 4\pi$, ampl $= 2$

h $2\cos\pi t$: per $= \dfrac{2\pi}{\pi} = 2$, ampl $= 2$

i $-3\sin\left(\dfrac{\pi t}{2}\right)$: per $= \dfrac{2\pi}{\frac{\pi}{2}} = 4$, ampl $= 3$

2 **a** $g(x) = 3\sin x$: dilation of 3 from x-axis, amplitude $= 3$, period $= 2\pi$

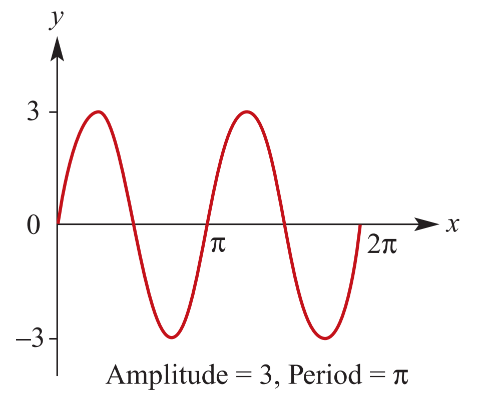
b $g(x) = \sin(5x)$: dilation of $\dfrac{1}{5}$ from y-axis, amplitude $= 1$, period $= \dfrac{2\pi}{5}$

c $g(x) = \sin\left(\dfrac{x}{3}\right)$: dilation of 3 from y-axis, amplitude $= 1$, period $= 6\pi$

d $g(x) = 2\sin 5x$: dilation of 2 from x-axis, dilation of $\dfrac{1}{5}$ from y-axis, amplitude $= 2$, period $= \dfrac{2\pi}{5}$

3 **a** $y = 3\sin 2x$:

per $= \pi$, ampl $= 3$, x-intercepts $0, \dfrac{\pi}{2}, \pi$

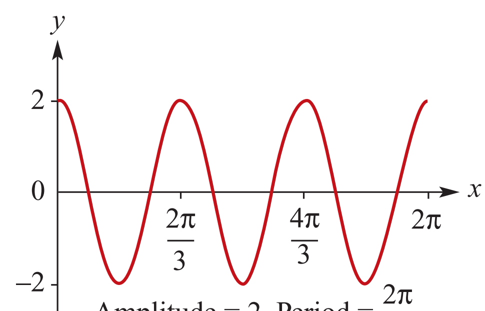


Amplitude = 3, Period = π

b $y = 2\cos 3\theta$:

per $= \dfrac{2\pi}{3}$, ampl $= 2$,

θ intercepts $0, \dfrac{\pi}{6}, \dfrac{\pi}{2}$

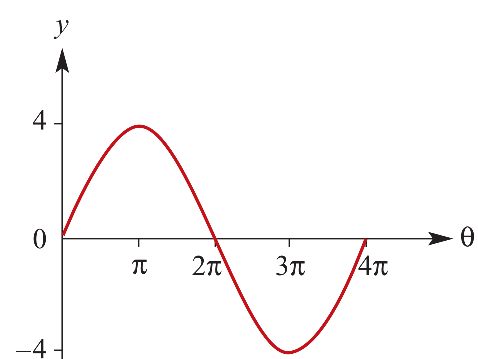


Amplitude = 2, Period = $\dfrac{2\pi}{3}$

c $y = 4\cos\dfrac{\theta}{2}$:

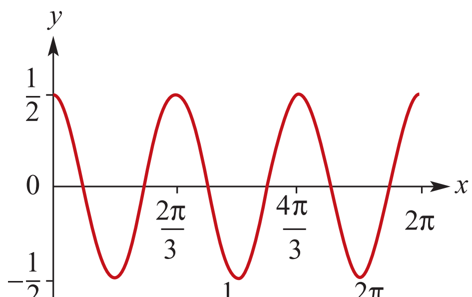
per $= 4\pi$, ampl $= 4$,

θ intercepts $0, \pi, 3\pi$



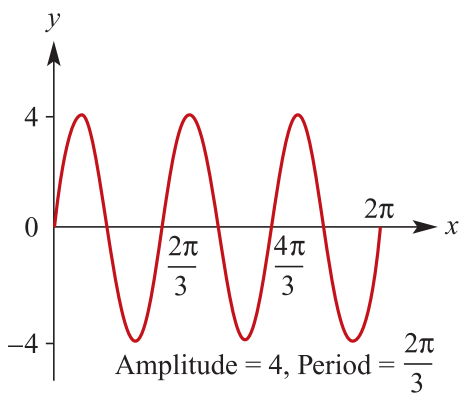
Amplitude = 4, Period = 4π

d $y = \frac{1}{2}\cos 3x$:
per $= \frac{2\pi}{3}$, ampl $= \frac{1}{2}$,
x intercepts $\frac{\pi}{6}, \frac{\pi}{2}$



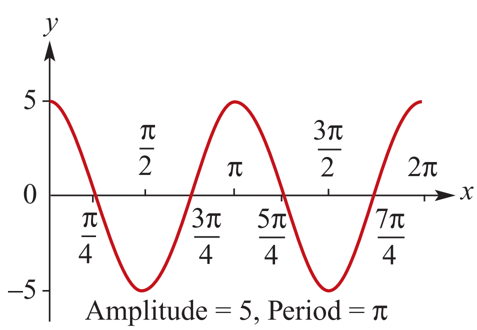
Amplitude $= \frac{1}{2}$, Period $= \frac{2\pi}{3}$

e $y = 4\sin 3x$:
per $= \frac{2\pi}{3}$, ampl $= 4$,
x intercepts $0, \frac{2\pi}{3}, \frac{4\pi}{3}$



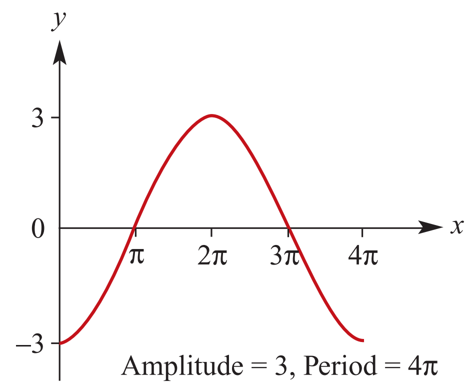
Amplitude $= 4$, Period $= \frac{2\pi}{3}$

f $y = 5\cos 2x$:
per $= \pi$, ampl $= 5$,
x intercepts $0, \frac{\pi}{4}, \frac{3\pi}{4}$



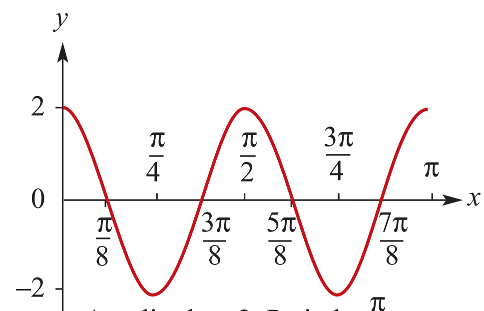
Amplitude $= 5$, Period $= \pi$

g $y = -3\cos\frac{\theta}{2}$:
per $= 4\pi$, ampl $= 3$,
θ intercepts $0, \pi, 3\pi$



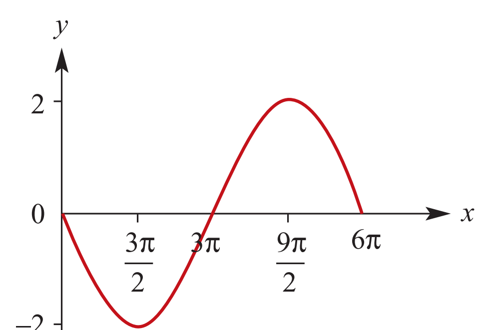
Amplitude $= 3$, Period $= 4\pi$

h $y = 2\cos 4\theta$:
per $= \frac{\pi}{2}$, ampl $= 2$,
θ intercepts $0, \frac{\pi}{8}, \frac{3\pi}{8}$



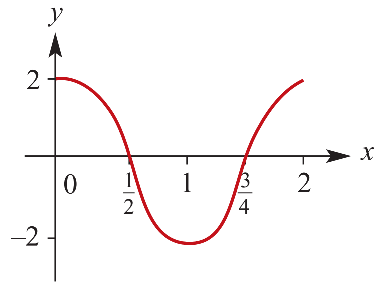
Amplitude $= 2$, Period $= \frac{\pi}{2}$

i $y = -2\sin\frac{\theta}{3}$:
per $= 6\pi$, ampl $= 2$,
θ intercepts $0, 3\pi, 6\pi$

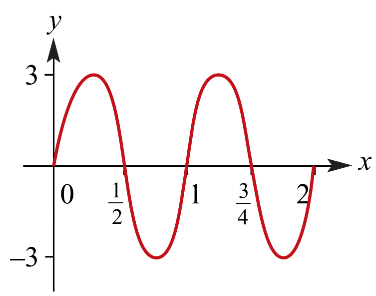


Amplitude $= 2$, Period $= 6\pi$

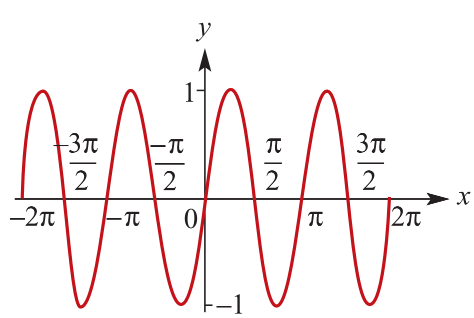
4 a $f\colon [0, 2] \to R, f(t) = 2\cos \pi t$
per $= \dfrac{2\pi}{\pi} = 2$, ampl $= 2$,
range $= [-2, 2]$,
endpoints $(0, 2)$ and $(2, 2)$; x-intercepts $\dfrac{1}{2}, \dfrac{3}{2}$



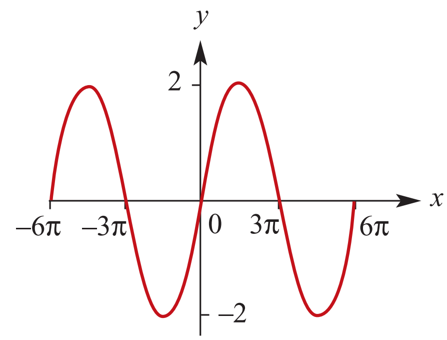
b $f\colon [0, 2] \to R, f(t) = 3\sin(2\pi t)$
per $= \dfrac{2\pi}{2\pi} = 1$, ampl $= 3$,
range $= [-3, 3]$,
endpoints $(0, 0)$ and $(2, 0)$;
x-intercepts $\dfrac{1}{2}, \dfrac{3}{2}, \dfrac{5}{2}$



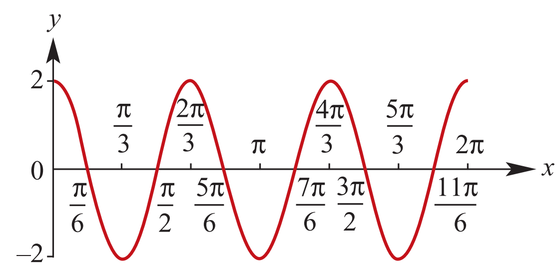
5 a $f(x) = \sin 2x$ for $x \in [-2\pi, 2\pi]$:
endpoints: $(-2\pi, 0), (2\pi, 0)$
x-intercepts: $-\dfrac{3\pi}{2}, -\pi, -\dfrac{\pi}{2}, 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$
ampl $= 1$, range $[-1, 1]$



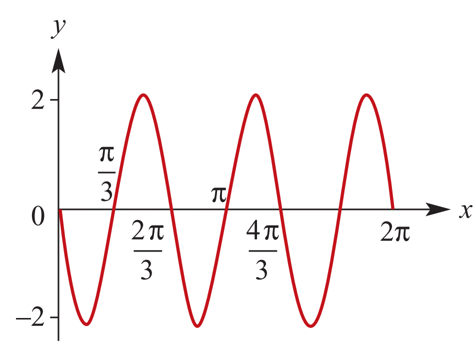
b $f(x) = 2\sin \dfrac{x}{3}$ for $x \in [-6\pi, 6\pi]$:
endpoints: $(-6\pi, 0), (6\pi, 0)$
x-intercepts: $-3\pi, 0, 3\pi$
ampl $= 2$, range $[-2, 2]$



c $f(x) = 2\cos 3x$ for $x \in [0, 2\pi]$:
endpoints: $(0, 1), (2\pi, 1)$
x-intercepts: $\dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{3\pi}{2}, \dfrac{11\pi}{6}$
ampl $= 2$, range $[-2, 2]$



d $f(x) = -2\sin 3x$ for $x \in [0, 2\pi]$:
endpoints: $(0, 0), (2\pi, 0)$
x-intercepts: $\dfrac{\pi}{3}, \dfrac{2\pi}{3}, \pi, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$
ampl $= 2$, range $[-2, 2]$

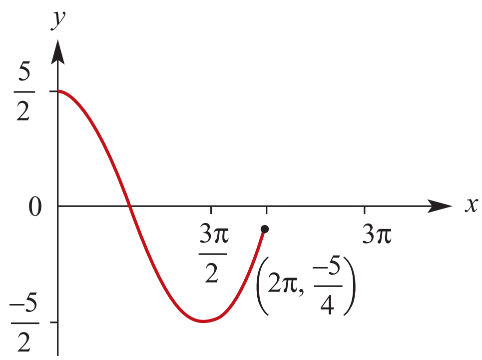


6 $f\colon [0, 2\pi] \to R, f(x) = \dfrac{5}{2}\cos\left(\dfrac{2x}{3}\right)$:

endpoints: $f(0) = \frac{5}{2}$ and $f(2\pi) = -\frac{5}{4}$

per $= \frac{2\pi}{\frac{2}{3}} = 3\pi$ so we only have $\frac{2}{3}$ period

ampl $= \frac{5}{2}$, range $= \left[-\frac{5}{2}, \frac{5}{2}\right]$



7 a $g(x) = -\sin 5x$: dilation of $\frac{1}{5}$ from y-axis, reflection in x-axis, amplitude $= 1$, period $= \frac{2\pi}{5}$

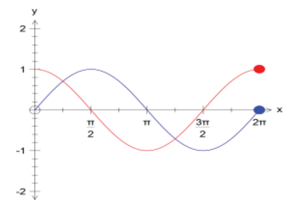
b $g(x) = \sin(-x)$: reflection in y-axis, amplitude $= 1$, period $= 2\pi$

c $g(x) = 2\sin\left(\frac{x}{3}\right)$: dilation of 3 from y-axis, dilation of 2 from x-axis, amplitude $= 2$, period $= 6\pi$

d $g(x) = -4\sin\left(\frac{x}{2}\right)$: dilation of 2 from y-axis, dilation of 4 from x-axis, reflection in x-axis, amplitude $= 4$, period $= 4\pi$

e $g(x) = 2\sin\left(-\frac{x}{3}\right)$: dilation of 3 from y-axis, dilation of 2 from x-axis, reflection in y-axis, amplitude $= 2$, period $= 6\pi$

8 a f: $[0, 2\pi] \to R$, $f(x) = \sin x$;
per $= 2\pi$, ampl $= 1$, range $= [-1, 1]$,
endpoints $(0, 0)$ and $(2\pi, 0)$,
other x-intercept at π



g: $[0, 2\pi] \to R$, $g(x) = \cos x$:
per $= 2\pi$, ampl $= 1$, range $= [-1, 1]$,
endpoints at $(0, 1)$ and $(2\pi, 1)$,
x-intercepts at $\frac{\pi}{2}, \frac{3\pi}{2}$

b $\sin x = \cos x$ when $x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$

Solutions to Exercise 14H

1 a

$$\cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 4\pi - \frac{\pi}{3}, \ldots$$
$$= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$

b

$$\sin x = \frac{1}{\sqrt{2}}$$
$$x = \frac{\pi}{4}, \pi - \frac{\pi}{4}, 2\pi + \frac{\pi}{4}, 3\pi - \frac{\pi}{4}, \ldots$$
$$= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$$

c

$$\sin x = \frac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{3}, \pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 3\pi - \frac{\pi}{3}, \ldots$$
$$= \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$$

2 Solve over $[0, 2\pi]$:

a $\sin x = 0.8, \therefore\ x = 0.93, \pi - 0.93$
$= 0.93, 2.21$

b $\cos x = -0.4, \therefore\ x = \pi \pm 1.16$
$= 1.98, 4.30$

c $\sin x = -0.35,$
$\therefore\ x = \pi + 0.36, 2\pi - 0.36$
$= 3.50, 5.93$

d $\sin x = 0.4, \therefore\ x = 0.41, \pi - 0.41$
$= 0.41, 2.73$

e $\cos x = -0.7, \therefore\ x = \pi \pm 0.80$
$= 2.35, 3.94$

f $\cos x = -0.2, \therefore\ x = \pi \pm 1.39$
$= 1.77, 4.51$

3 Solve over $[0, 360°]$:

a $\cos\theta° = -\frac{\sqrt{3}}{2}, \therefore \theta = 180 \pm 30$
$= 150, 210$

b $\sin\theta° = \frac{1}{2}, \therefore\ \theta = 30, 180 - 30$
$= 30, 150$

c $\cos\theta° = -\frac{1}{2}, \therefore\ \theta = 180 \pm 60$
$= 120, 240$

d $2\cos\theta° + 1 = 0, \therefore \cos\theta° = -\frac{1}{2}$
$\therefore\ \theta = 120, 240$

e $2\sin\theta° = \sqrt{3}, \therefore\ \sin\theta° = \frac{\sqrt{3}}{2}$
$\therefore\ \theta = 60, 180 - 60$
$= 60, 120$

f $\sqrt{2}\sin\theta° - 1 = 0, \therefore\ \sin\theta° = \frac{1}{\sqrt{2}}$
$\theta = 45, 180 - 45$
$= 45, 135$

4 a

$$2\cos x = \sqrt{3}$$
$$\therefore \quad \cos x = -\frac{\sqrt{3}}{2}$$
$$x = \pi - \frac{\pi}{6}, \pi + \frac{\pi}{6}$$
$$= \frac{5\pi}{6}, \frac{7\pi}{6}$$

b $$\sqrt{2}\sin x + 1 = 0$$
$$\therefore \quad \sin x = -\frac{1}{\sqrt{2}}$$
$$x = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$$
$$= \frac{5\pi}{4}.\frac{7\pi}{4}$$

c $$\sqrt{2}\cos x - 1 = 0$$
$$\therefore \quad \cos x = \frac{1}{\sqrt{2}}$$
$$x = \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$$
$$= \frac{\pi}{4}, \frac{7\pi}{4}$$

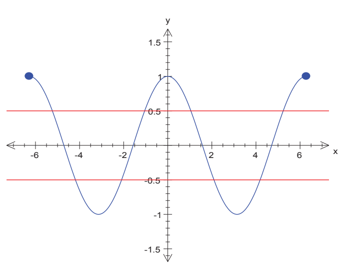
5 Solve over $[-\pi, \pi]$:

a $\cos x = -\frac{1}{\sqrt{2}}, \therefore \ x = \pi - \frac{\pi}{4}, -\pi + \frac{\pi}{4}$
$$= -\frac{3\pi}{4}, \frac{3\pi}{4}$$

b $\sin x = \frac{\sqrt{3}}{2}, \therefore x = \frac{\pi}{3}, \pi - \frac{\pi}{3}$
$$= \frac{\pi}{3}, \frac{2\pi}{3}$$

c $\cos x = -\frac{1}{2}, \therefore x = \pi - \frac{\pi}{3}, -\pi + \frac{\pi}{3}$
$$= -\frac{2\pi}{3}, \frac{2\pi}{3}$$

6 **a** $f : [-2\pi, 2\pi] \to R, f(x) = \cos x$



b Line marked at $y = \frac{1}{2}$, x-values are at:
$$x = \pm\frac{\pi}{3}, \pm\left(2\pi - \frac{\pi}{3}\right) = \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}$$

c Line marked at $y = -\frac{1}{2}$, x-values are at:
$$x = \pm\left(\pi \pm \frac{\pi}{3}\right) = \pm\frac{2\pi}{3}, \pm\frac{4\pi}{3}$$

7 Solve over $[0, 2\pi]$:

a
$$\sin(2\theta) = -\frac{1}{2}$$
$$\therefore \quad 2\theta = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}, 3\pi + \frac{\pi}{6}, 4\pi - \frac{\pi}{6}$$
$$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

b
$$\cos(2\theta) = \frac{\sqrt{3}}{2}$$
$$\therefore \quad 2\theta = \frac{\pi}{6}, 2\pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 4\pi - \frac{\pi}{6}$$
$$\theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$$

c
$$\sin(2\theta) = \frac{1}{2}$$
$$\therefore \quad 2\theta = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}$$
$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

d
$$\sin(3\theta) = -\frac{1}{\sqrt{2}}$$
$$\therefore 3\theta = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 3\pi + \frac{\pi}{4}, 4\pi - \frac{\pi}{4}\ldots$$
$$\theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12}, \frac{21\pi}{12}, \frac{23\pi}{12}$$

e $\cos(2\theta) = -\frac{\sqrt{3}}{2}$

$\therefore \quad 2\theta = \pi \pm \frac{\pi}{6}, 3\pi \pm \frac{\pi}{6}$

$\theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$

f

$\sin(2\theta) = -\frac{1}{\sqrt{2}}$

$\therefore \quad 2\theta = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 3\pi + \frac{\pi}{4}, 4\pi - \frac{\pi}{4}$

$\theta = \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

8 Solve over $[0, 2\pi]$:

a

$\sin(2\theta) = -0.8$

$\therefore \quad 2\theta = \pi + 0.927, 2\pi - 0.927,$

$3\pi + 0.927, 4\pi - 0.927$

$\theta = 2.034, 2.678, 5.176, 5.820$

b

$\sin(2\theta) = -0.6$

$\therefore \quad 2\theta = \pi + 0.644, 2\pi - 0.644,$

$3\pi + 0.644, 4\pi - 0.644$

$\theta = 1.892, 2.820, 5.034, 5.961$

c

$\cos(2\theta) = 0.4$

$\therefore \quad 2\theta = 1.159, 2\pi \pm 1.159, 4\pi - 1.159$

$\theta = 0.580, 2.562, 3.721, 5.704$

d $\cos(3\theta) = 0.6$

$\therefore \quad 3\theta = 0.927, 2\pi \pm 0.927;$

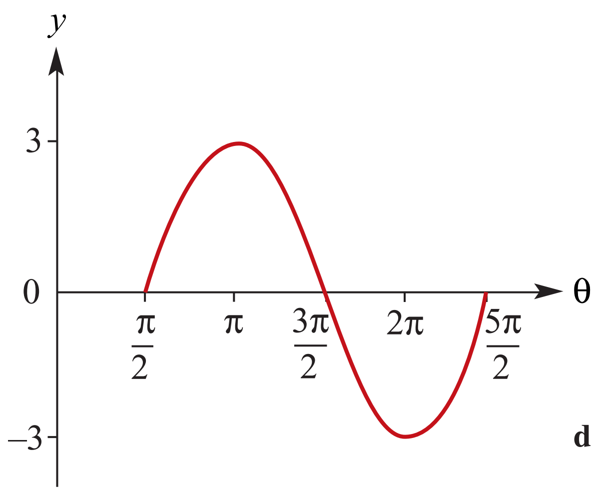
$4\pi \pm 0.927, 6\pi - 0.927$

$\theta = 0.309, 1.785, 2.403,$

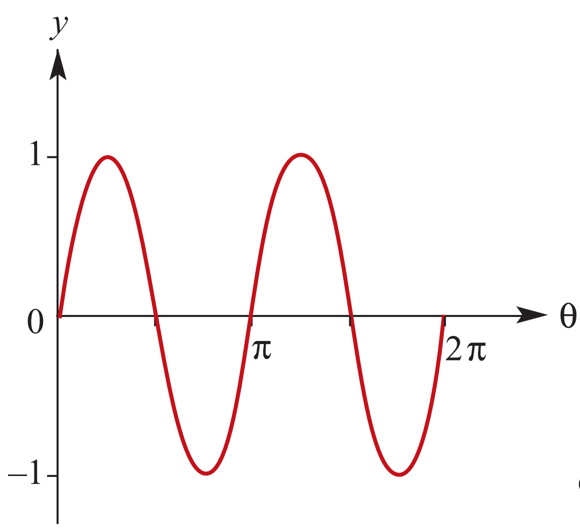
$3.880, 4.498, 5.974$

Solutions to Exercise 14I

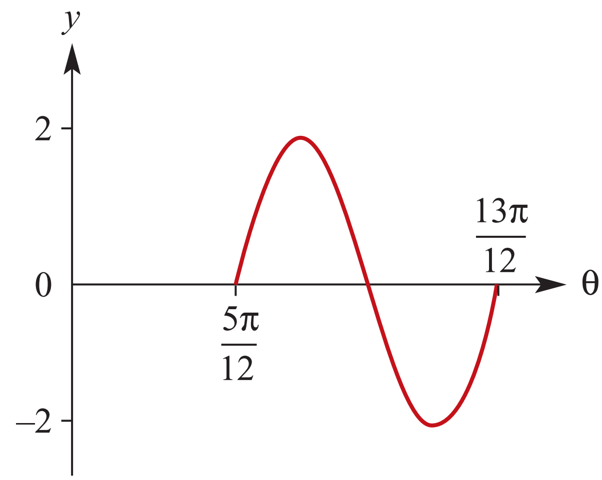
1 a $y = 3\sin\left(\theta - \frac{\pi}{2}\right)$:
per $= 2\pi$, ampl $= 3$, range $= [-3, 3]$



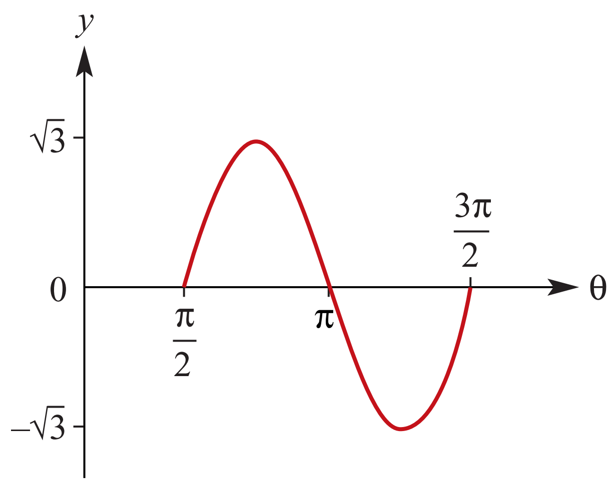
b $y = \sin 2(\theta + \pi)$:
per $= \pi$, ampl $= 1$, range $= [-1, 1]$



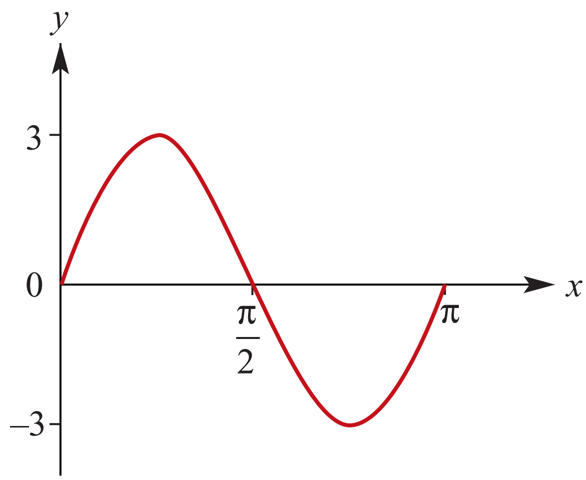
c $y = 2\sin 3\left(\theta + \frac{\pi}{4}\right)$:
per $= \frac{2\pi}{3}$, ampl $= 2$, range $= [-2, 2]$



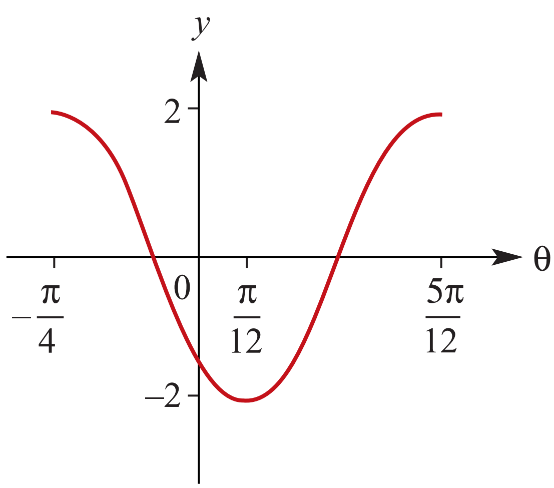
d $y = \sqrt{3}\sin 2\left(\theta - \frac{\pi}{2}\right)$:
per $= \pi$, ampl $= \sqrt{3}$,
range $= [-\sqrt{3}, \sqrt{3}]$



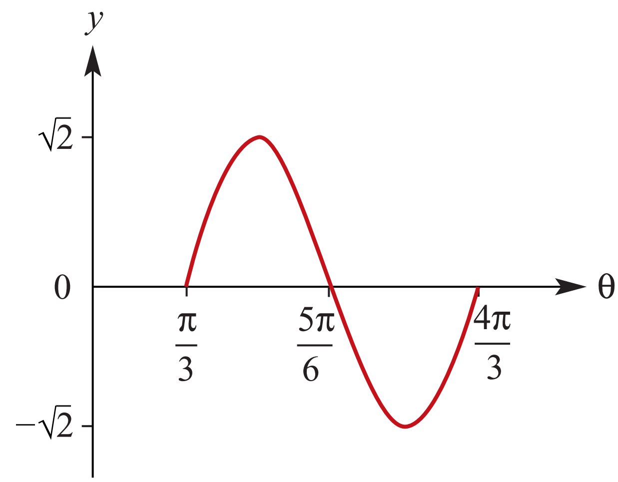
e $y = 3\sin(2x)$:
per $= \pi$, ampl $= 3$, range $= [-3, 3]$



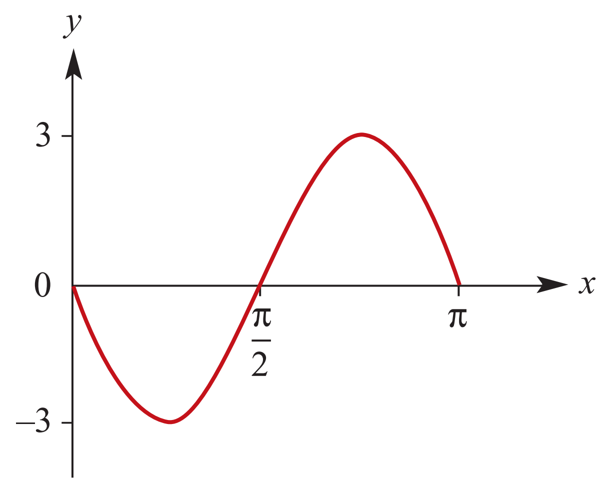
f $y = 2\cos 3\left(\theta + \frac{\pi}{4}\right)$:
per $= \frac{2\pi}{3}$, ampl $= 2$, range $= [-2, 2]$



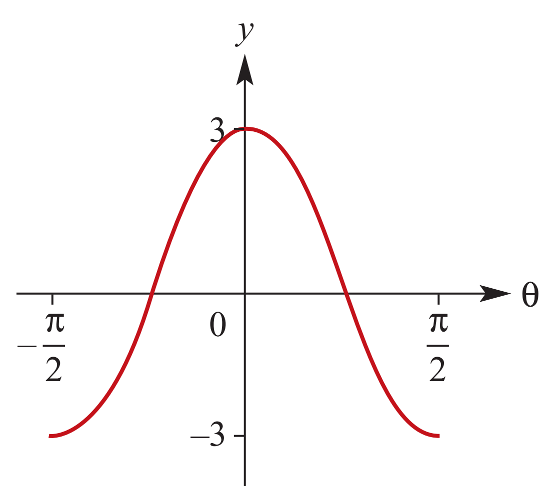
g $y = \sqrt{2}\sin 2\left(\theta - \frac{\pi}{3}\right)$:
per $= \pi$, ampl $= \sqrt{2}$,
range $= [-\sqrt{2}, \sqrt{2}]$



h $y = -3\sin(2x)$:
per $= \pi$, ampl $= 3$, range $= [-3, 3]$



i $y = -3\cos 2\left(\theta + \frac{\pi}{2}\right)$:
per $= \pi$, ampl $= 3$, range $= [-3, 3]$



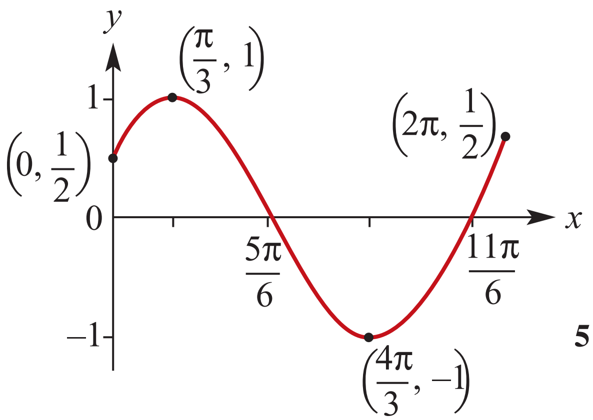
2 $f\colon [0, 2\pi] \to R,\ f(x) = \cos\left(x - \frac{\pi}{3}\right)$

$f(0) = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$

$f(2\pi) = \cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$

per $= 2\pi$, ampl $= 1$, range $[-1, 1]$

x-intercepts at $\frac{5\pi}{6}, \frac{11\pi}{6}$



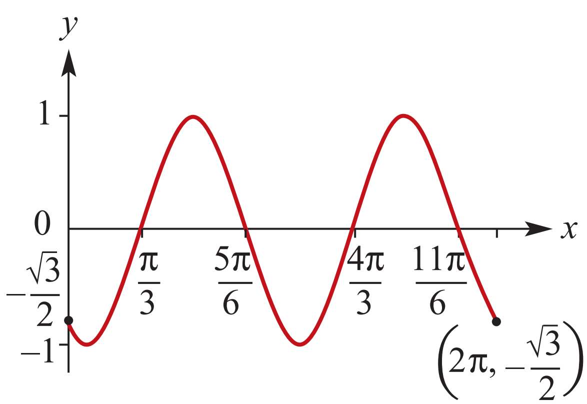
3 $f\colon [0, 2\pi] \to R,\ f(x) = \sin 2\left(x - \frac{\pi}{3}\right)$

$f(0) = \sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$f(2\pi) = \sin\left(\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

per $= \pi$, ampl $= 1$, range $[-1, 1]$

x-intercepts at $\frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$



4 $f\colon [-\pi, \pi] \to R,\ f(x) = \sin\ 3\left(x + \frac{\pi}{4}\right)$:

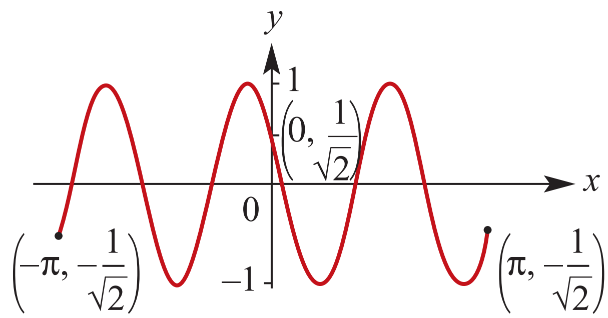
$f(-\pi) = \sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f(\pi) = \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

per $= \frac{2\pi}{3}$, ampl $= 1$, range $[-1, 1]$,

x-intercepts at

$-\frac{11\pi}{12}, -\frac{7\pi}{12}, -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$



5 $y = \sin x$

a Dilation of 2 from y-axis: $y = \sin\left(\frac{x}{2}\right)$;
dilation of 3 from x-axis:
$y = 3\sin\left(\frac{x}{2}\right)$

b Dilation of $\frac{1}{2}$ from y-axis: $y = \sin 2x$;
dilation of 3 from x-axis: $y = 3\sin 2x$

c Dilation of 3 from y-axis: $y = \sin\left(\frac{x}{3}\right)$;
dilation of 2 from x-axis:
$y = 2\sin\left(\frac{x}{3}\right)$

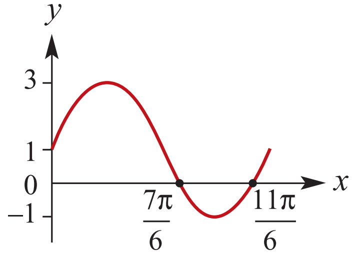
d Dilation of $\frac{1}{2}$ from y-axis: $y = \sin 2x$;
translation of $+\frac{\pi}{3}$ (x-axis):
$y = \sin 2\left(x - \frac{\pi}{3}\right)$

e Dilation of 2 from the y-axis:
$y = \sin\left(\frac{x}{2}\right)$;
translation of $-\frac{\pi}{3}$ (x-axis):
$y = \sin \frac{1}{2}\left(x + \frac{\pi}{3}\right)$

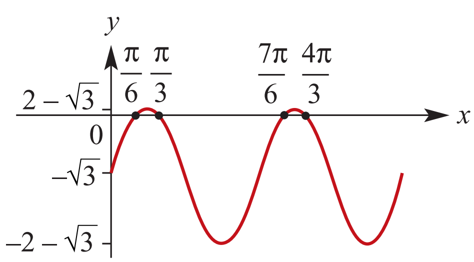
Solutions to Exercise 14J

1 Sketch over $[0, 2\pi]$:

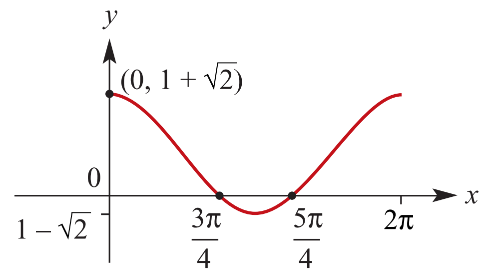
a $y = 2\sin x + 1$;
per $= 2\pi$, ampl $= 2$, range $= [-1, 3]$,
endpoints at $(0, 1)$ and $(2\pi, 0)$
$y = 0$ when $\sin x = -\dfrac{1}{2}$,
i.e. when $x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$



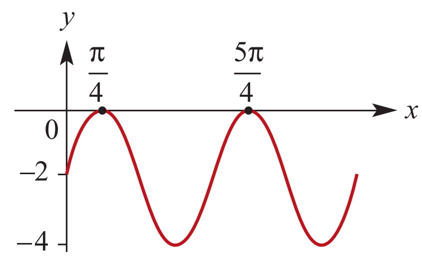
b $y = 2\sin 2x - \sqrt{3}$;
per $= \pi$, ampl $= 2$,
range $= [-2 - \sqrt{3}, 2 - \sqrt{3}]$,
endpoints at $(0, -\sqrt{3})$ and $(2\pi, -\sqrt{3})$
$y = 0$ when $\sin 2x = \dfrac{\sqrt{3}}{2}$,
i.e. when $x = \dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{7\pi}{6}, \dfrac{4\pi}{3}$



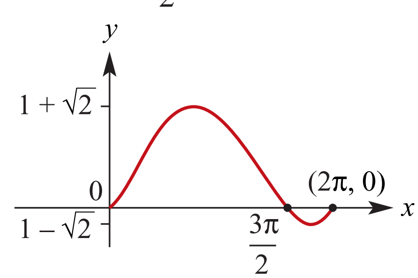
c $y = \sqrt{2}\cos x + 1$;
per $= 2\pi$, ampl $= \sqrt{2}$,
range $= [-\sqrt{2} + 1, \sqrt{2} + 1]$,
endpoints at $(0, \sqrt{2} + 1)$ and
$(2\pi, \sqrt{2} + 1)$
$y = 0$ when $\cos x = -\dfrac{1}{\sqrt{2}}$
i.e. when $x = \dfrac{3\pi}{4}, \dfrac{5\pi}{4}$



d $y = 2\sin 2x - 2$;
per $= \pi$, ampl $= 2$, range $= [-4, 0]$,
endpoints at $(0, -2)$ and $(2\pi, -2)$
$y = 0$ when $\sin 2x = 1$,
i.e. when $x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$



e $y = \sqrt{2}\sin\left(x - \dfrac{\pi}{4}\right) + 1$
per $= 2\pi$, ampl $= \sqrt{2}$,
range $= [1 - \sqrt{2}, 1 + \sqrt{2}]$,
endpoints at $(0, 0)$ and $(2\pi, 0)$
$y = 0$ when $\sin\left(x - \dfrac{\pi}{4}\right) = -\dfrac{1}{\sqrt{2}}$
i.e. when $x - \dfrac{\pi}{4} = -\dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}, 2\pi - \dfrac{\pi}{4}$
i.e. $x = 0, \dfrac{3\pi}{2}, 2\pi$



2 Sketch over $[-2\pi, 2\pi]$:

a $y = 2\sin 3x - 2$;

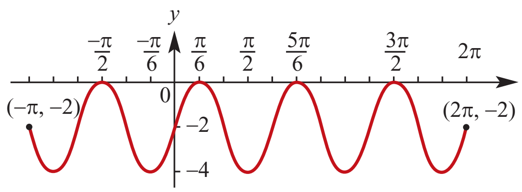
per $= \dfrac{2\pi}{3}$, ampl $= 2$, range $= [-4, 0]$,

endpoints at $(-2\pi, -2)$ and $(2\pi, -2)$

$y = 0$ when $\sin 3x = 1$

i.e. when $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}\dfrac{3\pi}{2},$

$-\dfrac{\pi}{2}, -\dfrac{7\pi}{6}, -\dfrac{11\pi}{6}$



b $y = 2\cos 3\left(x - \dfrac{\pi}{4}\right)$;

per $= \dfrac{2\pi}{3}$, ampl $= 2$, range $= [-2, 2]$,

endpoints at $(-2\pi, -\sqrt{2})$ and

$(2\pi, -\sqrt{2})$

$y = 0$ when $\cos 3\left(x \; - \dfrac{\pi}{4}\right) = 0$

$$\therefore\ 3\left(x - \frac{\pi}{4}\right) = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \cdots \pm \frac{11\pi}{2}$$

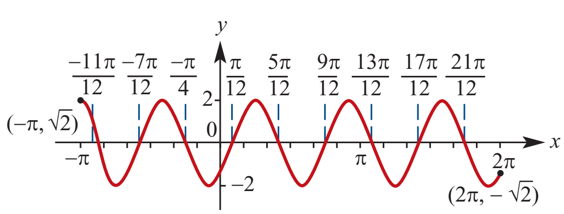
$$x - \frac{\pi}{4} = \pm\frac{\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{5\pi}{6} \cdots \pm \frac{11\pi}{6}$$

$$x = -\frac{23\pi}{12}, -\frac{19\pi}{12}, -\frac{5\pi}{4}, -\frac{11\pi}{12}, -\frac{7\pi}{12},$$

$$-\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$$

The $\dfrac{11\pi}{6}$ solution will drop out, since adding $\dfrac{\pi}{4}$ to it will take the resulting number over 2π.

It must be replaced by the solution $\dfrac{\pi}{4} - \dfrac{13\pi}{6}$.

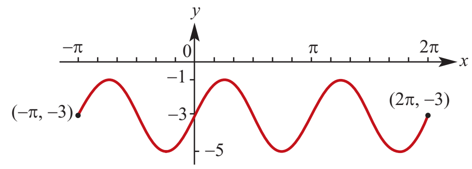


c $y = 2\sin 2x - 3$;

per $= \pi$, ampl $= 2$, range $= [-5, -1]$,

endpoints at $(-2\pi, -3)$ and $(2\pi, -3)$

No x-intercepts since $y < 0$ for all real x



d $y = 2\cos 2x + 1$;

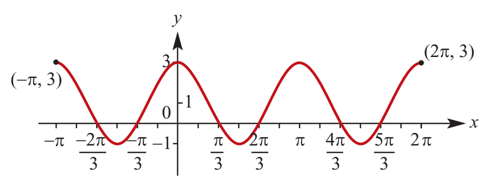
per $= \pi$, ampl $= 2$, range $= [-1, 3]$,

endpoints at $(-2\pi, 3)$ and $(2\pi, 3)$

$y = 0$ when $\cos 2x = -\dfrac{1}{2}$

i.e. when $2x = \pm\left(\pi \pm \dfrac{\pi}{3}, \pm 3\pi \pm \dfrac{\pi}{3}\right)$

$\therefore\ \ x = \pm\dfrac{\pi}{3}, \pm\dfrac{2\pi}{3}, \pm\dfrac{4\pi}{3}, \pm\dfrac{5\pi}{3}$



e $y = 2\cos 2\left(x - \dfrac{\pi}{3}\right) - 1$;

per $= \pi$, ampl $= 2$, range $= [-3, 1]$,

endpoints at $(-2\pi, -2)$ and $(2\pi, -2)$

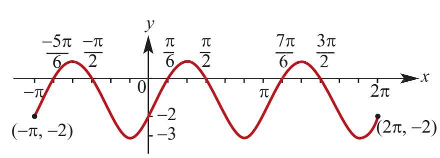
$y = 0$ when $\cos 2(x \; - \dfrac{\pi}{3}) = \dfrac{1}{2}$

$$\therefore\ 2\left(x - \frac{\pi}{3}\right) = \pm\frac{\pi}{3}, \pm\left(2\pi \pm \frac{\pi}{3}\right), \pm\left(4\pi \pm \frac{\pi}{3}\right)$$

$$x - \frac{\pi}{3} = \pm\frac{\pi}{6}, \pm\left(\pi \pm \frac{\pi}{6}\right), \pm\left(2\pi \pm \frac{\pi}{6}\right)$$

$$x = -\frac{11\pi}{6}, -\frac{3\pi}{2}, -\frac{5\pi}{6}, -\frac{\pi}{2},$$

$$\frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$$



f $y = 2\sin 2\left(x + \dfrac{\pi}{6}\right) + 1$;

per $= \pi$, ampl $= 2$, range $= [-1, 3]$,

endpoints at $(-2\pi, 1 + \sqrt{3})$ and

$(2\pi, 1 + \sqrt{3})$

$y = 0$ when $\sin 2(x + \frac{\pi}{6}) = -\frac{1}{2}$

Positive solutions:

$$2\left(x + \frac{\pi}{6}\right) = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}, 3\pi + \frac{\pi}{6}, 4\pi - \frac{\pi}{6}$$

$$x + \frac{\pi}{6} = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

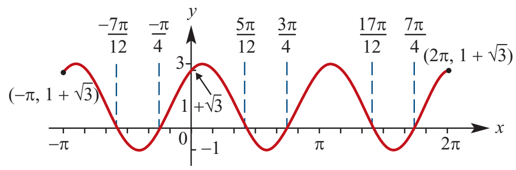
$$x = \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{17\pi}{12}, \frac{7\pi}{4}$$

Negative solutions:

$$2\left(x + \frac{\pi}{6}\right) = -\frac{\pi}{6}, -\pi + \frac{\pi}{6}, -2\pi - \frac{\pi}{6}, -3\pi + \frac{\pi}{6}$$

$$x + \frac{\pi}{6} = -\frac{\pi}{12}, -\frac{5\pi}{12}, -\frac{13\pi}{12}, -\frac{17\pi}{12}$$

$$x = -\frac{\pi}{4}, -\frac{7\pi}{12}, -\frac{5\pi}{4}, -\frac{19\pi}{12}$$



b $y = -2\sin 2\left(x + \frac{\pi}{6}\right) + 1$;
per $= \pi$, ampl $= 2$, range $= [-1, 3]$, endpoints at $(-\pi, 1 - \sqrt{3})$ and $(\pi, 1 - \sqrt{3})$

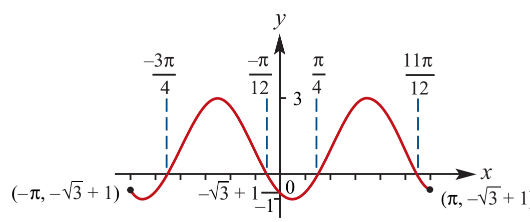
$y = 0$ when $\sin 2(x + \frac{\pi}{3}) = \frac{1}{2}$

$$\therefore 2\left(x + \frac{\pi}{6}\right) = -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$$

$$x + \frac{\pi}{6} = -\frac{7\pi}{12}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}$$

$$x = -\frac{3\pi}{4}, -\frac{\pi}{12}, \frac{\pi}{4}, \frac{11\pi}{12}$$

As with **Q.2b**, $-\frac{11\pi}{6}$ drops out, replaced by $\frac{13\pi}{6}$.



3 Sketch over $[-\pi, \pi]$:

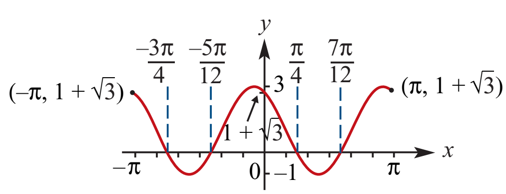
a $y = 2\sin 2\left(x + \frac{\pi}{3}\right) + 1$;
per $= \pi$, ampl $= 2$, range $= [-1, 3]$, endpoints at $(-\pi, 1 + \sqrt{3})$ and $(\pi, 1 + \sqrt{3})$

$y = 0$ when $\sin 2(x + \frac{\pi}{3}) = -\frac{1}{2}$

$$\therefore 2\left(x + \frac{\pi}{3}\right) = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$x + \frac{\pi}{3} = -\frac{5\pi}{12}, -\frac{\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$$

$$x = -\frac{3\pi}{4}, -\frac{5\pi}{12}, \frac{\pi}{4}, \frac{7\pi}{12}$$



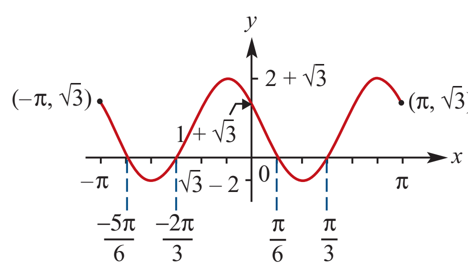
c $y = 2\cos 2\left(x + \frac{\pi}{4}\right) + \sqrt{3}$;
per $= \pi$, ampl $= 2$, range $= [-1, 3]$, endpoints at $(-\pi, \sqrt{3})$ and $(\pi, \sqrt{3})$

$y = 0$ when $\cos 2\left(x + \frac{\pi}{4}\right) = -\frac{\sqrt{3}}{2}$

$$\therefore 2\left(x + \frac{\pi}{4}\right) = -\frac{7\pi}{6}, -\frac{5\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$$

$$x + \frac{\pi}{4} = -\frac{7\pi}{12}, -\frac{5\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}$$

$$x = -\frac{5\pi}{6}, -\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{3}$$



Solutions to Exercise 14K

1 $\sin x = 0.3, \cos\alpha = 0.6$ and $\tan\theta = 0.7$

a $\cos(-\alpha) = \cos\alpha = 0.6$

b $\sin(\frac{\pi}{2} + \alpha) = \cos\alpha = 0.6$

c $\tan(-\theta) = -\tan\theta = -0.7$

d $\cos(\frac{\pi}{2} - x) = \sin x = 0.3$

e $\sin(-x) = -\sin x = -0.3$

f $\tan(\frac{\pi}{2} - \theta) = \dfrac{1}{0.7} = \dfrac{10}{7}$

g $\cos(\frac{\pi}{2} + x) = -\sin x = -0.3$

h $\sin(\frac{\pi}{2} - \alpha) = \cos\alpha = 0.6$

i $\sin(\frac{3\pi}{2} + \alpha) = -\cos\alpha = -0.6$

j $\cos(\dfrac{3\pi}{2} - x) = -\sin x = -0.3$

2 $0 < \theta < \dfrac{\pi}{2}$

a $\cos\theta = \sin\dfrac{\pi}{6}$

$\therefore \theta = (\dfrac{\pi}{2} - \dfrac{\pi}{6}) = \dfrac{\pi}{3}$

b $\sin\theta = \cos\dfrac{\pi}{6}$

$\therefore \theta = (\dfrac{\pi}{2} - \dfrac{\pi}{6}) = \dfrac{\pi}{3}$

c $\cos\theta = \sin\dfrac{\pi}{12}$

$\therefore \theta = (\dfrac{\pi}{2} - \dfrac{\pi}{12}) = \dfrac{5\pi}{12}$

d $\sin\theta = \cos\dfrac{3\pi}{7}$

$\therefore \theta = (\dfrac{\pi}{2} - \dfrac{3\pi}{7}) = \dfrac{\pi}{14}$

3 $\cos x = \dfrac{3}{5}, \dfrac{3\pi}{2} < x < 2\pi$:

$\sin x = \pm\sqrt{1 - \left(\dfrac{3}{5}\right)^2} = \pm\dfrac{4}{5}$

4th quadrant: $\sin x = -\dfrac{4}{5}$

$\tan x = -\dfrac{4}{5} \div \dfrac{3}{5} = -\dfrac{4}{3}$

4 $\sin x = \dfrac{5}{13}, \dfrac{\pi}{2} < x < \pi$:

$\cos x = \pm\sqrt{1 - \left(\dfrac{5}{13}\right)^2} = \pm\dfrac{12}{13}$

2nd quadrant: $\cos x = -\dfrac{12}{13}$

$\tan x = \dfrac{5}{13} \div -\dfrac{12}{13} = -\dfrac{5}{12}$

5 $\cos x = \dfrac{1}{5}, \dfrac{3\pi}{2} < x < 2\pi$:

$\sin x = \pm\sqrt{1 - \left(\dfrac{1}{5}\right)^2} = \pm\dfrac{\sqrt{24}}{5} = \pm\dfrac{2}{5}\sqrt{6}$

4th quadrant: $\sin x = -\dfrac{2}{5}\sqrt{6}$

$\tan x = -\dfrac{2}{5}\sqrt{6} \div \dfrac{1}{5} = -2\sqrt{6}$

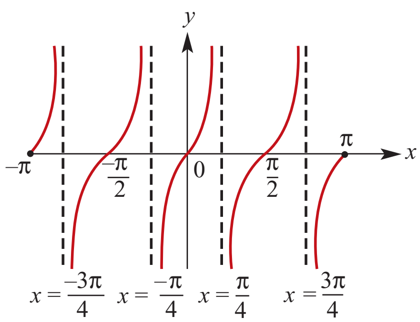
Solutions to Exercise 14L

1 **a** $y = \tan(4x)$, per $= \dfrac{\pi}{4}$

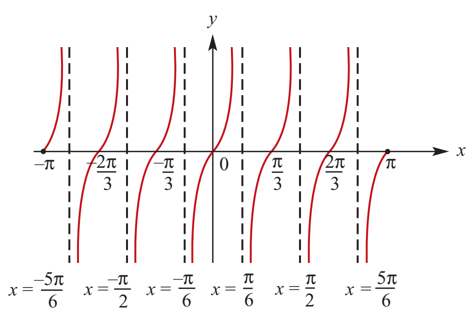
b $y = \tan(\dfrac{2x}{3})$, per $= \dfrac{\pi}{\frac{2}{3}} = \dfrac{3\pi}{2}$

c $y = -3\tan(2x)$, per $= \dfrac{\pi}{2}$

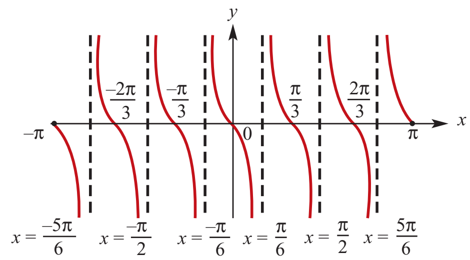
2 **a** $y = \tan(2x)$:
x-intercepts at $0, \pm\dfrac{\pi}{2}, \pm\pi$
Vertical asymptotes at $x = \pm\dfrac{\pi}{4}, \pm\dfrac{3\pi}{4}$



b $y = 2\tan(3x)$:
x-intercepts at $0, \pm\dfrac{\pi}{3}, \pm\dfrac{2\pi}{3}, \pm\pi$
Vertical asymptotes at
$x = \pm\dfrac{\pi}{6}, \pm\dfrac{\pi}{2}, \pm\dfrac{5\pi}{6}$



c $y = -2\tan(3x)$:
x-intercepts at $0, \pm\dfrac{\pi}{3}, \pm\dfrac{2\pi}{3}, \pm\pi$
Vertical asymptotes at
$x = \pm\dfrac{\pi}{6}, \pm\dfrac{\pi}{2}, \pm\dfrac{5\pi}{6}$



3 Solve over $[-\pi, \pi]$:

a $2\tan 2x = 2, \therefore \tan 2x = 1$
$\therefore 2x = -\dfrac{7\pi}{4}, -\dfrac{3\pi}{4}, \dfrac{\pi}{4}, \dfrac{5\pi}{4}$
$x = -\dfrac{7\pi}{8}, -\dfrac{3\pi}{8}, \dfrac{\pi}{8}, \dfrac{5\pi}{8}$

b $3\tan 3x = \sqrt{3}, \therefore \tan 3x = \dfrac{\sqrt{3}}{3}$
$\therefore 3x = -\dfrac{17\pi}{6}, -\dfrac{11\pi}{6}, -\dfrac{5\pi}{6}, \dfrac{\pi}{6}, \dfrac{7\pi}{6}, \dfrac{13\pi}{6}$
$x = -\dfrac{17\pi}{18}, -\dfrac{11\pi}{18}, -\dfrac{5\pi}{18}, \dfrac{\pi}{18}, \dfrac{7\pi}{18}, \dfrac{13\pi}{18}$

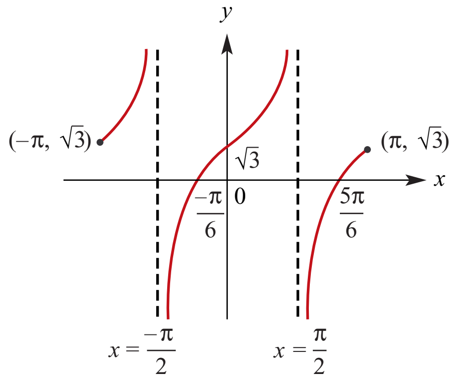
c $2\tan 2x = 2\sqrt{3} \tan 2x = \sqrt{3}$
$\therefore 2x = -\dfrac{5\pi}{3}; -\dfrac{2\pi}{3}; \dfrac{\pi}{3}; \dfrac{4\pi}{3}$
$\therefore x = -\dfrac{5\pi}{6}; -\dfrac{\pi}{3}; \dfrac{\pi}{6}; \dfrac{2\pi}{3}$

d $3\tan 3x = -\sqrt{3}, \therefore \tan 3x = -\dfrac{\sqrt{3}}{3}$
$\therefore 3x = -\dfrac{13\pi}{6}, -\dfrac{7\pi}{6}, -\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{11\pi}{6}, \dfrac{17\pi}{6}$
$x = -\dfrac{13\pi}{18}, -\dfrac{7\pi}{18}, -\dfrac{\pi}{18}, \dfrac{5\pi}{18}, \dfrac{11\pi}{18}, \dfrac{17\pi}{18}$

4 Sketch over $[-\pi, \pi]$:

a $y = 3\tan x + \sqrt{3}$
x-intercepts where $\tan x = -\dfrac{1}{\sqrt{3}}$
$\therefore\ x = -\dfrac{\pi}{6}, \dfrac{5\pi}{6}$

Vertical asymptotes at $x = \pm\dfrac{\pi}{2}$

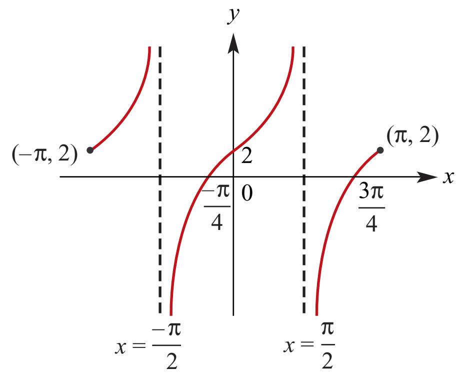


b $y = 2\tan x + 2$

x-intercepts where $\tan x = -1$

$\therefore\ x = -\dfrac{\pi}{4}, \dfrac{3\pi}{4}$

Vertical asymptotes at $x = \pm\dfrac{\pi}{2}$

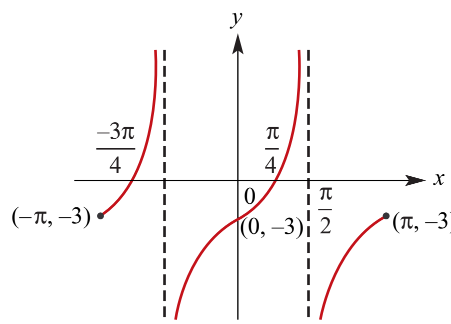


c $y = 3\tan x - 3$

x-intercepts where $\tan x = 1$

$\therefore\ x = \dfrac{\pi}{4}, -\dfrac{3\pi}{4}$

Vertical asymptotes at $x = \pm\dfrac{\pi}{2}$



Solutions to Exercise 14M

1 From calculator:

a $\cos x = x, \therefore x = 0.74$

b $\sin x = 1 - x, \therefore x = 0.51$

c $\cos x = x^2, \therefore x = \pm 0.82$

d $\sin x = x^2, x = 0, 0.88$

2 $y = a\sin(b\theta + c) + d$ From calculator:

a $y = 1.993\sin(2.998\,\theta + 0.003) + 0.993$

b $y = 3.136\sin(3.051\,\theta + 0.044) - 0.140$

c $y = 4.971\sin(3.010\,\theta + 3.136) + 4.971$

Solutions to Exercise 14N

1 a $\sin x = 0.5$, $\therefore\ x = \frac{\pi}{6}$ (primary solution)

2nd solution is $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\therefore\ x = (12n+1)\frac{\pi}{6}, (12n+5)\frac{\pi}{6};\ n \in \mathbb{Z}$

b $2\cos 3x = \sqrt{3}$, $\therefore\ 3x = \pm\frac{\pi}{6}$

$$x = \pm\frac{\pi}{18}$$

$\therefore\ x = (12n \pm 1)\frac{\pi}{18};\ n \in \mathbb{Z}$

c $\sqrt{3}\tan x = -3$, $\therefore$ $\tan x = -\sqrt{3}$

$$x = \frac{2\pi}{3}$$

$\therefore x = (3n+2)\frac{\pi}{3};\ n \in \mathbb{Z}$

2 a $\sin x = 0.5$, $\therefore\ x = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$$= \frac{\pi}{6}, \frac{5\pi}{6}$$

b $2\cos 3x = \sqrt{3}$, $\therefore\ \cos 3x = \frac{\sqrt{3}}{2}$

$$3x = \frac{\pi}{6}, \frac{11\pi}{6}$$

$\therefore\ x = \frac{\pi}{18}, \frac{11\pi}{18}$

c $\sqrt{3}\tan x = -3$, $\therefore\ \tan x = -\sqrt{3}$

$\therefore\ x = \frac{2\pi}{3}, \frac{5\pi}{3}$

3

$$2\cos\left(2x + \frac{\pi}{4}\right) = \sqrt{2}$$

$$\therefore\quad \cos\left(2x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$2x + \frac{\pi}{4} = (8n \pm 1)\frac{\pi}{4}$$

$$2x = \frac{8n\pi}{4}, (8n-2)\frac{\pi}{4}$$

$$2x = 2n\pi, (4n-1)\frac{\pi}{2}$$

$$\therefore\quad x = n\pi, (4n-1)\frac{\pi}{4};\ n \in \mathbb{Z}$$

Over $(-2\pi, 2\pi)$ the solutions are:

$x = \frac{5\pi}{4}, -\pi, -\frac{\pi}{4}, 0, \pi, \frac{3\pi}{4}, \frac{7\pi}{4}$

4

$$\sqrt{3}\tan\left(\frac{\pi}{6} - 3x\right) - 1 = 0$$

$$\therefore\quad \tan\left(\frac{\pi}{6} - 3x\right) = \frac{1}{\sqrt{3}}$$

$$\frac{\pi}{6} - 3x = (6n+1)\frac{\pi}{6}$$

$$-3x = \frac{6n\pi}{6} = n\pi$$

$$\therefore\quad x = \frac{n\pi}{3};\ n \in \mathbb{Z}$$

Over $[-\pi, 0]$ the solutions are:

$x = -\pi, -\frac{2\pi}{3}, -\frac{\pi}{3}, 0$

5

$$2\sin(4\pi x) + \sqrt{3} = 0$$

$$\therefore\quad \sin(4\pi x) = -\frac{\sqrt{3}}{2}$$

$$4\pi x = (6n+4)\frac{\pi}{3}, (6n+5)\frac{\pi}{3}$$

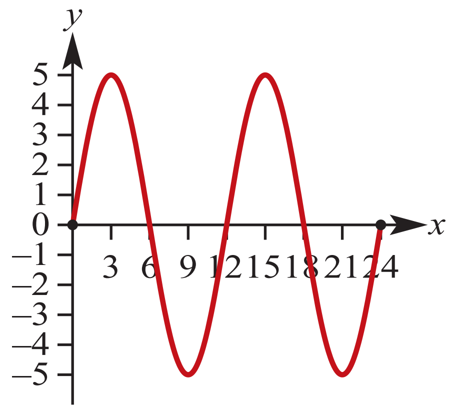
$\therefore\ x = \frac{3n+2}{6}, \frac{6n+5}{12}; n \in Z$

Over $[-1, 1]$ the solutions are:

$x = -\frac{2}{3}, -\frac{7}{12}, -\frac{1}{6}, -\frac{1}{12}, \frac{1}{3}, \frac{5}{12}, \frac{5}{6}, \frac{11}{12}$

Solutions to Exercise 14O

1 a



b Maximum values occur when $\sin\left(\frac{\pi}{6}t\right) = 1$

That is when $t = 3$ and $t = 15$

c $h(3) = h(15) = 5$. The aximum height is5 m above mean sea level

d $h(2) = 5\sin\left(\frac{\pi}{3}\right) = \frac{5\sqrt{3}}{2}$ m above mean sea level

e $h(14) = 5\sin\left(\frac{7\pi}{3}\right) = \frac{5\sqrt{3}}{2}$ m above mean sea level

f $5\sin\left(\frac{\pi}{6}t\right) = 2.5$

$\sin\left(\frac{\pi}{6}t\right) = \frac{1}{2}$

$\frac{\pi}{6}t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

$t = 1, 5, 13, 17$

Tide is higher than 2.5 m for $t \in [1, 5] \cup [13, 17]$

2 a $x = 3 + 2\sin 3t$

When $\sin 3t = 1$, $x = 3 + 2 = 5$, the greatest distance from O.

b When $\sin 3t = -1$, $x = 3 - 2 = 1$, the least distance from O.

c When $x = 5$, $3 + 2\sin 3t = 5$

$\therefore\ \sin 3t = 1$

$\therefore\ 3t = \frac{\pi}{2}$ or $\frac{5\pi}{2}$ or $\frac{9\pi}{2}$ or ...

$\therefore\ t = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $\frac{9\pi}{6}$ or ...

$\therefore\ t = 0.524$ or 2.618 or 4.712 seconds for $t \in [0, 5]$

d When $x = 3, 3 + 2\sin 3t = 3$

$\therefore\ \sin 3t = 0$

$\therefore\ 3t = 0$ or π or 2π or ...

$\therefore\ t = 0$ or $\dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$ or ...

$\therefore\ t = 0$ or 1.047 or 2.094 seconds for $t \in [0, 3]$

e Particle oscillates about $x = 3$, from $x = 1$ to $x = 5$.

3 $x = 5 + 2\sin(2\pi t)$. Note that the particle oscillates between $x = 3$ and $x = 7$

a Greatest distance from O when $\sin(2\pi t) = 1$. Therefore greatest distance from O is 7 m

b Least distance from O when $\sin(2\pi t) = -1$. Therefore least distance from O is 7 m

c
$$\begin{aligned} 5 + 2\sin(2\pi t) &= 7 \\ 2\sin(2\pi t) &= 2 \\ \sin(2\pi t) &= 1 \\ 2\pi t &= \frac{\pi}{2}, \frac{5\pi}{2}, \ldots \\ t &= \frac{1}{4}, \frac{5}{4}, \frac{9}{4}, \frac{13}{4}, \frac{17}{4} \end{aligned}$$

d
$$\begin{aligned} 5 + 2\sin(2\pi t) &= 6 \\ 2\sin(2\pi t) &= 1 \\ \sin(2\pi t) &= \frac{1}{2} \\ 2\pi t &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \ldots \\ t &= \frac{1}{12}, \frac{5}{12}, \frac{13}{12}, \frac{17}{12}, \frac{25}{12}, \frac{29}{12} \end{aligned}$$

e Particle oscillates between $x = 3$ and $x = 7$

4 $h(t) = 10\sin\left(\dfrac{\pi t}{3}\right) + 10$

a **i** $h(0) = 10\sin\left(0\right) + 10 = 10$

ii $h(1) = 10\sin\left(\frac{\pi}{3}\right) + 10 = 10 + 5\sqrt{3}$

iii $h(2) = 10\sin\left(\frac{2\pi}{3}\right) + 10 = 10 + 5\sqrt{3}$

iv $h(4) = 10\sin\left(\frac{4\pi}{3}\right) + 10 = 10 - 5\sqrt{3}$

v $h(5) = 10\sin\left(\frac{5\pi}{3}\right) + 10 = 10 - 5\sqrt{3}$

b Period $= 2\pi \div \frac{\pi}{3} = 6$seconds

c Greatest height = 20 m

d
$$\begin{aligned} 10\sin\left(\frac{\pi t}{3}\right) + 10 &= 15 \\ 10\sin\left(\frac{\pi t}{3}\right) &= 5 \\ \sin\left(\frac{\pi t}{3}\right) &= \frac{1}{2} \\ \frac{\pi t}{3} &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \ldots \\ t &= \frac{1}{2}, \frac{5}{2}, \frac{13}{2}, \frac{17}{4} \end{aligned}$$

e
$$\begin{aligned} 10\sin\left(\frac{\pi t}{3}\right) + 10 &= 5 \\ 10\sin\left(\frac{\pi t}{3}\right) &= -5 \\ \sin\left(\frac{\pi t}{3}\right) &= -\frac{1}{2} \\ \frac{\pi t}{3} &= \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6} \ldots \\ t &= \frac{7}{2}, \frac{11}{2}, \frac{19}{2}, \frac{23}{2} \end{aligned}$$

5 $T = 17 - 8\cos\left(\frac{\pi t}{12}\right)$

a $T(0) = 17 - 8\cos\left(0\right) = 9$
The temperature was 9° C at midnight

b Maximum temperature 25°

Minimum temperature 9°

c $17 - 8\cos\left(\frac{\pi t}{12}\right) = 20$

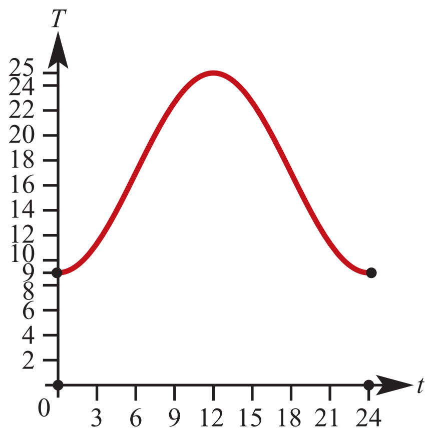
$$-8\cos\left(\frac{\pi t}{12}\right) = 3$$

$$\cos\left(\frac{\pi t}{12}\right) = -\frac{3}{8}$$

$$\frac{\pi t}{12} = \pi - \cos^{-1}\frac{3}{8}, \pi + \cos^{-1}\frac{3}{8}, \ldots$$

$$t = 7.468\ldots, 16.53\ldots$$

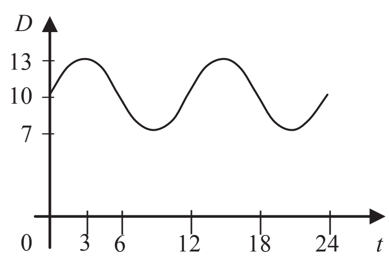
d



6 **a** $D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right), 0 \le t \le 24$

period $= \frac{2\pi}{\frac{\pi}{6}} = 12$; amplitude $= 3$;

translation in the positive direction of the $D(t)$-axis $= 10$



b When $D(t) = 8.5$, $10 + 3\sin\left(\frac{\pi t}{6}\right) = 8.5$

$\therefore\ 3\sin\left(\frac{\pi t}{6}\right) = -1.5$

$\therefore\ \sin\left(\frac{\pi t}{6}\right) = -\frac{1}{2}$

$\therefore\ \frac{\pi t}{6} = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$ or $\frac{19\pi}{6}$ or $\frac{23\pi}{6}$ or $\ldots$

$\therefore t = 7$ or 11 or 19 or 23 or ...

From the graph, $D(t) \geq 8.5$ implies

$0 \leq t \leq 7$, or $11 \leq t \leq 19$, or $23 \leq t \leq 24$, for $0 \leq t \leq 24$

$\therefore \{t\colon D(t) \geq 8.5\} = \{t\colon 0 \leq t \leq 7\} \cup \{t\colon 11 \leq t \leq 19\} \cup \{t : 23 \leq t \leq 24\}$

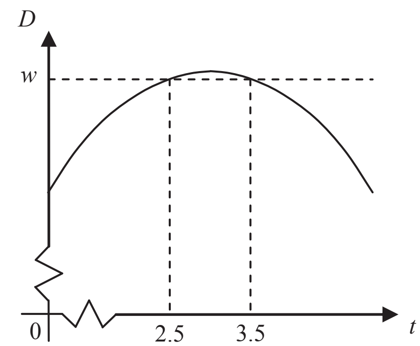
c The maximum depth is 13 m.

From the graph, the required period of time is $[2.5, 3.5]$.

The largest value of w occurs for $t = 2.5$.

$w = 10 + 3\sin\left(\frac{2.5\pi}{6}\right) \approx 12.9$

The largest value of w is 12.9, correct to 1 decimal place.



7 a period $= 2 \times 6$, and also period $= \frac{360}{r}$

$\therefore \frac{360}{r} = 12 \qquad \therefore r = 30$

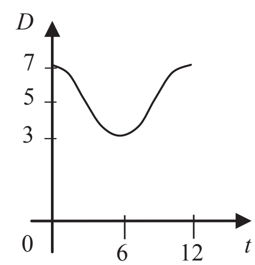
translation parallel to D-axis $= \frac{7+3}{2} = 5$

$\therefore p = 5$

amplitude $= \frac{7-3}{2} = 2 \qquad \therefore q = 2$

b When $t = 0, D = 7$

When $t = 6, D = 3$



c $D = 5 + 2\cos(30t)^\circ$

When $D = 4, 5 + 2\cos(30t)^\circ = 4$

$\therefore 2\cos(30t)^\circ = -1 \qquad \therefore \cos(30t)^\circ = -\frac{1}{2}$

$\therefore (30t)^\circ = 120^\circ$ or 240°

$\therefore t = 4$ or 8 (from graph, only two values required)

Low tide is at $t = 6$, hence it will be 2 hours before the ship can enter the harbour.

Solutions to Technology-free questions

1 **a** $330^\circ = 330\left(\frac{\pi}{180}\right) = \frac{11\pi}{6}$

b $810^\circ = 810\left(\frac{\pi}{180}\right) = \frac{9\pi}{2}$

c $1080^\circ = 1080\left(\frac{\pi}{180}\right) = 6\pi$

d $1035^\circ = 1035\left(\frac{\pi}{180}\right) = \frac{23\pi}{4}$

e $135^\circ = 135\left(\frac{\pi}{180}\right) = \frac{3\pi}{4}$

f $405^\circ = 405\left(\frac{\pi}{180}\right) = \frac{9\pi}{4}$

g $390^\circ = 390\left(\frac{\pi}{180}\right) = \frac{13\pi}{6}$

h $420^\circ = 420\left(\frac{\pi}{180}\right) = \frac{7\pi}{3}$

i $80^\circ = 80\left(\frac{\pi}{180}\right) = \frac{4\pi}{9}$

2 **a** $\frac{5\pi}{6} = \frac{5\pi}{6}\left(\frac{180^\circ}{\pi}\right) = 150^\circ$

b $\frac{7\pi}{4} = \frac{7\pi}{4}\left(\frac{180^\circ}{\pi}\right) = 315^\circ$

c $\frac{11\pi}{4} = \frac{11\pi}{4}\left(\frac{180^\circ}{\pi}\right) = 495^\circ$

d $\frac{3\pi}{12} = \frac{3\pi}{12}\left(\frac{180^\circ}{\pi}\right) = 45^\circ$

e $\frac{15\pi}{2} = \frac{15\pi}{2}\left(\frac{180^\circ}{\pi}\right) = 1350^\circ$

f $-\frac{3\pi}{4} = -\frac{3\pi}{4}\left(\frac{180^\circ}{\pi}\right) = -135^\circ$

g $-\frac{\pi}{4} = -\frac{\pi}{4}\left(\frac{180^\circ}{\pi}\right) = -45^\circ$

h $-\frac{11\pi}{4} = -\frac{11\pi}{4}\left(\frac{180^\circ}{\pi}\right) = -495^\circ$

i $-\frac{23\pi}{4} = -\frac{23\pi}{4}\left(\frac{180^\circ}{\pi}\right) = -1035^\circ$

3 **a** $\sin\frac{11\pi}{4} = \sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$

b $\cos\left(-\frac{7\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

c $\sin\frac{11\pi}{6} = -\sin\frac{\pi}{6} = -\frac{1}{2}$

d $\cos\left(-\frac{7\pi}{6}\right) = \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

e $\cos\left(\frac{13\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

f $\sin\frac{23\pi}{6} = -\sin\frac{\pi}{6} = -\frac{1}{2}$

g $\cos\left(-\frac{23\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

h $\sin\left(-\frac{17\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

4 **a** $2\sin\left(\frac{\theta}{2}\right)$

Ampl $= 2$, per $= \frac{2\pi}{\frac{1}{2}} = 4\pi$

b $-3\sin 4\theta$

Ampl $= 3$, per $= \frac{2\pi}{4} = \frac{\pi}{2}$

c $\frac{1}{2}\sin 3\theta$

Ampl $= \frac{1}{2}$, per $= \frac{2\pi}{3}$

d $-3\cos 2x$

Ampl $= 3$, per $= \dfrac{2\pi}{2} = \pi$

e $-4\sin\left(\dfrac{x}{3}\right)$

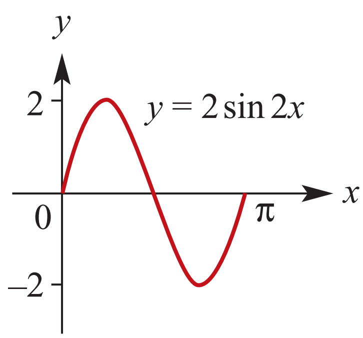
Ampl $= 4$, per $= \dfrac{2\pi}{\frac{1}{3}} = 6\pi$

f $\dfrac{2}{3}\sin\left(\dfrac{2x}{3}\right)$

Ampl $= \dfrac{2}{3}$, per $= \dfrac{2\pi}{\frac{2}{3}} = 3\pi$

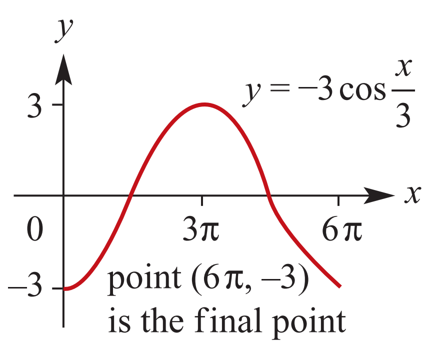
5 a $y = 2\sin 2x$

Per $= \pi$, ampl $= 2$, x-intercepts at $0, \dfrac{\pi}{2}, \pi$



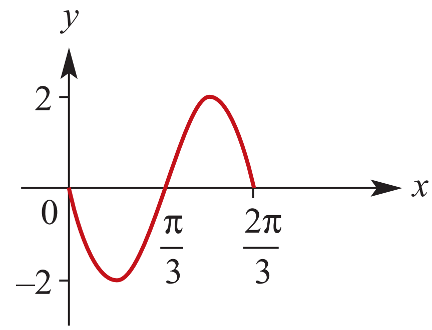
b $y = -3\cos\left(\dfrac{x}{3}\right)$

Per $= 6\pi$, ampl $= 3$, x-intercepts at $\dfrac{3\pi}{2}, \dfrac{9\pi}{2}$



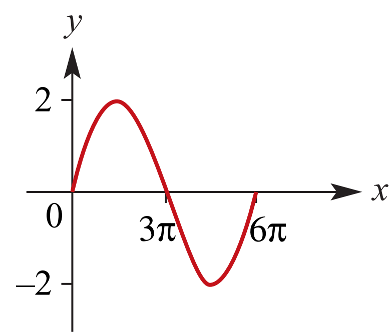
c $y = -2\sin 3x$

Per $= \dfrac{2\pi}{3}$, ampl $= 2$, x-intercepts at $0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$



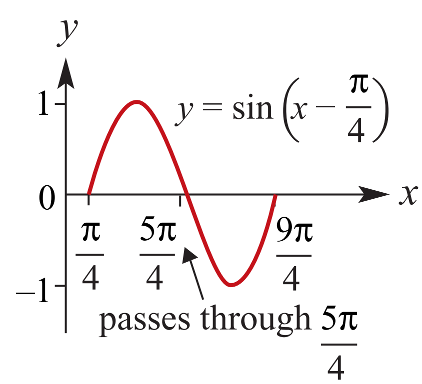
d $y = 2\sin\left(\dfrac{x}{3}\right)$

Per $= 6\pi$, ampl $= 2$, x-intercepts at $0, 3\pi, 6\pi$



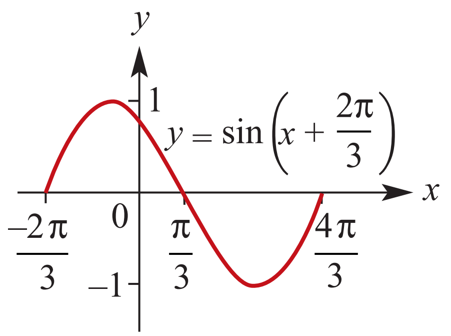
e $y = \sin\left(x - \dfrac{\pi}{4}\right)$

Per $= 2\pi$, ampl $= 1$, x-intercepts at $\dfrac{\pi}{4}, \dfrac{5\pi}{4}$

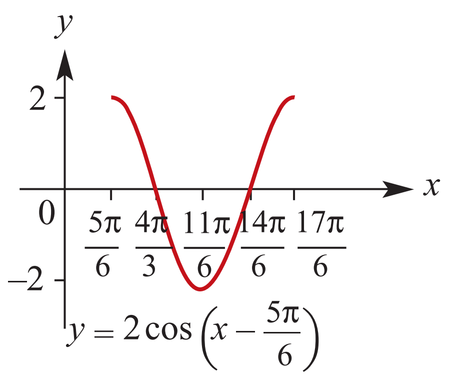


f $y = \sin\left(x + \dfrac{2\pi}{3}\right)$

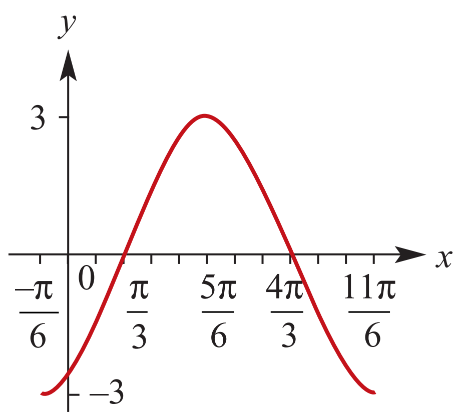
Per $= 2\pi$, ampl $= 1$, x-intercepts at $\dfrac{\pi}{3}, \dfrac{4\pi}{3}$



g $y = 2\cos\left(x - \frac{5\pi}{6}\right)$
Per $= 2\pi$, ampl $= 2$, x-intercepts at $\frac{\pi}{3}, \frac{4\pi}{3}$



h $y = -3\cos\left(x + \frac{\pi}{6}\right)$
Per $= 2\pi$, ampl $= 3$, x-intercepts at $\frac{\pi}{3}, \frac{4\pi}{3}$



6 **a** $\sin\theta = -\frac{\sqrt{3}}{2}, \therefore\ \theta = -\frac{2\pi}{3}, -\frac{\pi}{3}$
(No solutions over $[0, \pi]$)

b
$$\sin(2\theta) = -\frac{\sqrt{3}}{2}$$
$$\therefore\quad 2\theta = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$
$$\theta = -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}$$

c
$$\sin\left(\theta - \frac{\pi}{3}\right) = -\frac{1}{2}$$
$$\therefore\quad \theta - \frac{\pi}{3} = -\frac{\pi}{6}, \frac{7\pi}{6}$$
$$\theta = \frac{\pi}{6}, \frac{3\pi}{2}$$

d
$$\sin\left(\theta + \frac{\pi}{3}\right) = -1$$
$$\therefore\quad \theta + \frac{\pi}{3} = \frac{3\pi}{2}$$
$$\theta = \frac{7\pi}{6}$$
(Only 1 solution for -1 and 1)

e
$$\sin\left(\frac{\pi}{3} - \theta\right) = -\frac{1}{2}$$
$$\therefore\quad \frac{\pi}{3} - \theta = -\frac{\pi}{6}, -\frac{5\pi}{6}$$
$$-\theta = -\frac{\pi}{2}, -\frac{7\pi}{6}$$
$$\theta = \frac{\pi}{2}, \frac{7\pi}{6}$$

Solutions to multiple-choice questions

1 C $\sin^{-1}\left(\dfrac{3}{5}\right) \approx 37°$

2 D $3 - 10\cos 2x$ has range $[3 - 10, 3 + 10]$.
So the minimum value is $3 - 10$

3 E $4\sin\left(2x - \dfrac{\pi}{2}\right)$ has range $[-4, 4]$.

4 C $3\sin\left(\dfrac{x}{2} - \pi\right) + 4$ has per $= \dfrac{2\pi}{\frac{1}{2}} = 4\pi$

5 E $y = \sin x$:
Dilation of $\frac{1}{2}$ from y-axis:
$y = \sin 2x$
Translated $+\frac{\pi}{4}$ units in x-axis:
$y = \sin 2\left(x - \dfrac{\pi}{4}\right)$

6 D $f(x) = a\sin(bx) + c$: per $= \dfrac{2\pi}{b}$

7 E $y = \tan ax$ has vertical asymptotes at $y = \pm\dfrac{\pi}{2a}$
If $\dfrac{\pi}{2a} = \dfrac{\pi}{6}$, then a could be 3

8 E $3\sin x + 1 = b$
If $b > 0$ the only value of b possible is 4, since the only positive value of y for $\sin x$ with one solution over a period is 1.
$3\sin x + 1 = 4, \therefore \sin x = 1$

9 C $b = a\sin x, x \in [-2\pi, 2\pi], a > b > 0$
2 periods, each with 2 solutions = 4

10 B $D(t) = 8 + 2\sin\left(\dfrac{\pi t}{6}\right), 0 \le t \le 24$
Find primary solution for $D = 9$:

$$8 + 2\sin\frac{\pi t}{6} = 9$$
$$\sin\frac{\pi t}{6} = \frac{1}{2}$$
$$\frac{\pi t}{6} = \frac{\pi}{6}$$

$\therefore\ t = 1$

Solutions to extended-response questions

1 a i When $t = 5.7$, $\quad d = 12 + 12\cos\frac{1}{6}\pi\left(5.7 + \frac{1}{3}\right)$

$= 0.001\,83 = 1.83 \times 10^{-3}$ hours

ii When $t = 2.7$, $\quad d = 12 + 12\cos\frac{1}{6}\pi\left(2.7 + \frac{1}{3}\right)$

$= 11.79$ hours

b When $d = 5$, $\quad 12 + 12\cos\frac{1}{6}\pi\left(t + \frac{1}{3}\right) = 5$

$\therefore \quad 12\cos\frac{1}{6}\pi\left(t + \frac{1}{3}\right) = -7$

$\therefore \quad \cos\frac{1}{6}\pi\left(t + \frac{1}{3}\right) = -\frac{7}{12}$

$\therefore \quad \frac{1}{6}\pi\left(t + \frac{1}{3}\right) = 2.193\,622\,912, 4.089\,562\,395$

(first two positive values required)

$\therefore \quad t = \frac{2.193\,622\,912 \times 6}{\pi} - \frac{1}{3}, \frac{4.089\,562\,395 \times 6}{\pi} - \frac{1}{3}$

$\therefore \quad t = 3.856, 7.477$

When $t = 3.856$, the date is 26 April. When $t = 7.477$, the date is 14 August.

2 a When $t = 4$, $\quad A = 21 - 3\cos\left(\frac{4\pi}{12}\right) = 19.5$

The temperature inside the house is 19.5°C at 8 am.

b

$$D = A - B = 21 - 3\cos\left(\frac{\pi t}{12}\right) - \left(22 - 5\cos\left(\frac{\pi t}{12}\right)\right)$$

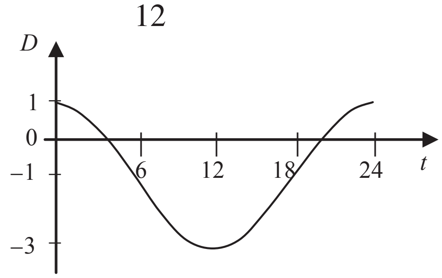
$$= 21 - 3\cos\left(\frac{\pi t}{12}\right) - 22 + 5\cos\left(\frac{\pi t}{12}\right)$$

$$\therefore \quad D = 2\cos\left(\frac{\pi t}{12}\right) - 1, 0 \le t \le 24$$

c amplitude = 2,

translation in positive direction of D-axis $= -1$,

period $= \dfrac{2\pi}{\frac{\pi}{12}} = 24$



d When $A < B$, $\quad D < 0$

When $D = 0$, $\quad 2\cos\left(\frac{\pi t}{12}\right) - 1 = 0$

$$\therefore \quad \cos\left(\frac{\pi t}{12}\right) = \frac{1}{2}$$

$$\therefore \quad \frac{\pi t}{12} = \frac{\pi}{3} \text{ or } \frac{5\pi}{3} \text{ or } \ldots$$

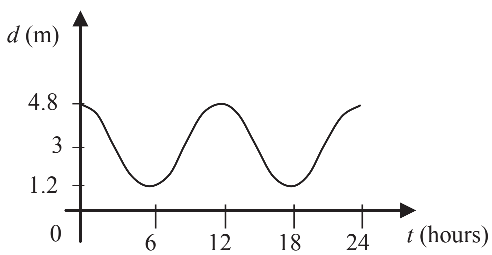
$$\therefore \quad t = 4 \text{ or } 20 \text{ for } t \in [0, 24]$$

When $D < 0, 4 < t < 20$

$\therefore \{t\colon A < B\} = \{t\colon 4 < t < 20\}$

3 **a** $d = 3 + 1.8\cos\left(\frac{\pi t}{6}\right)$

amplitude $= 1.8$; period $= 2\pi \div \frac{\pi}{6} = 12$



b High tides occur when $t = 0, t = 12$ and $t = 24$, i.e. at 3 am, 3 pm and 3 am.

c Low tides occur when $t = 6$ and $t = 18$, i.e. at 9 am and 9 pm.

d The ferry operates from $t = 5$ to $t = 17$.

Consider $\quad 3 + 1.8\cos\left(\frac{\pi t}{6}\right) = 2$

$$\therefore \quad \cos\left(\frac{\pi t}{6}\right) = \frac{-1}{18}$$

$$= \frac{-5}{9}$$

$$\therefore \quad \frac{\pi t}{6} = \pi - \cos^{-1}\left(\frac{5}{9}\right), \pi + \cos^{-1}\left(\frac{5}{9}\right), 3\pi - \cos^{-1}\left(\frac{5}{9}\right), 3\pi + \cos^{-1}\left(\frac{5}{9}\right)$$

$$t = 6 - \frac{6}{\pi}\cos^{-1}\left(\frac{5}{9}\right), 6 + \frac{6}{\pi}\cos^{-1}\left(\frac{5}{9}\right), 18 - \frac{6}{\pi}\cos^{-1}\left(\frac{5}{9}\right) \text{ or } 18 + \frac{6}{\pi}\cos^{-1}\left(\frac{5}{9}\right)$$

$$\approx 4.125 \text{ or } 7.875 \text{ or } 16.125 \text{ or } 19.875$$

$\therefore$ earliest time, $7.875 - \frac{50}{60} = 7.04$

$\therefore$ ferry can leave Main Beach at 10.03 am.

e **i** Ferry must be in and out harbour by $t = 16.125$. It must leave 55 minutes earlier, i.e. at $t = 15.208\ldots$ It can leave Main Beach no later than 6.12 pm.

ii Starts at 10.03 am and last trip leaves at 6.12 pm. Five trips are possible.

4 $D = p - 2\cos(rt)$

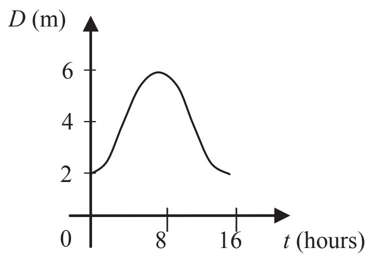
a Low tide depth is 2 m. High tide is 8 hours later, and the depth is 6 m.

period is 16 $\quad \therefore \quad \dfrac{2\pi}{r} = 16$

$\therefore \quad r = \dfrac{\pi}{8}$

The centre is 4, as the upper value is 6 and the lower value is 2, $\therefore\ p = 4$.
The amplitude is 2.

b



c The first low tide is at 4 am. The second low tide will be at 8 pm.

d The depth is equal to 4 metres when $t = 4$ and $t = 12$, i.e. at 8 am and 4 pm.

e **i** $7.5 - 6 = 1.5$ metres

ii At 2 pm, $t = 10$, and the depth is $5.414\ldots$ metres.

$7.5 - 5.414\ldots = 2.085\ldots$

The length of pole exposed = 2.086 m.

f When $d = 3.5$, $t = 3.356\ldots \therefore$ by symmetry,

$$\text{total time} = 6.713\ldots = 6 \text{ hours } 42 \text{ minutes } 47 \text{ seconds}$$

$$\therefore \quad \text{time covered} = 16 - 6.713 = 9.287$$

$$= 9 \text{ hours } 17 \text{ minutes}$$

Chapter 15 – Revision of chapters 13–14

Solutions to Technology-free questions

1 a $(-2a^2)^3 \times 3a^4 = -8a^6 \times 3a^4$
$= -24a^{10}$

b $\dfrac{5a^4 \times 2ab^2}{20a^2b^4} = \dfrac{10a^5b^2}{20a^2b^4}$
$= \dfrac{a^3}{2b^2}$

c $\dfrac{(xy^{-2})^{-1}}{y} \times \dfrac{3x^{-1}y^2}{4(xy)^3} = \dfrac{x^{-1}y^2}{y} \times \dfrac{3x^{-1}y^2}{4x^3y^3}$
$= \dfrac{3y^3}{4x^5y^3}$
$= \dfrac{3}{4x^5}$

d
$\left(\dfrac{4a^2}{ab}\right)^3 \div (2ab^{-1})^3 = \left(\dfrac{64a^6}{a^3b^3}\right) \div (8a^3b^{-3})$
$= \dfrac{64a^6}{a^3b^3} \times \dfrac{1}{8a^3b^{-3}}$
$= \dfrac{64a^6}{8a^6}$
$= 8$

e $\sqrt{x^{-1}y^2} \times \left(\dfrac{y}{x}\right)^{-\frac{1}{3}} = x^{-\frac{1}{2}}y \times y^{-\frac{1}{3}}x^{\frac{1}{3}}$
$= x^{-\frac{1}{6}}y^{\frac{2}{3}}$
$= \dfrac{y^{\frac{2}{3}}}{x^{\frac{1}{6}}}$

f
$\sqrt{2x-1} \times (2x-1)^{-1} = (2x-1)^{\frac{1}{2}}(2x-1)^{-1}$
$= \dfrac{1}{(2x-1)^{\frac{1}{2}}}$

2 a $\left(\dfrac{3}{5}\right)^{-2} = \left(\dfrac{5}{3}\right)^2 = \dfrac{25}{9}$

b $\left(\dfrac{4^2}{2^6}\right)^{-2} = \dfrac{4^{-4}}{2^{-12}} = \dfrac{2^{-8}}{2^{-12}} = 2^4 = 16$

c $\dfrac{27^2 \times 9^3}{81^2} = \dfrac{3^6 \times 3^6}{3^8} = 3^4 = 81$

d $(-27)^{-\frac{1}{3}} = \dfrac{1}{-27^{\frac{1}{3}}} = -\dfrac{1}{3}$

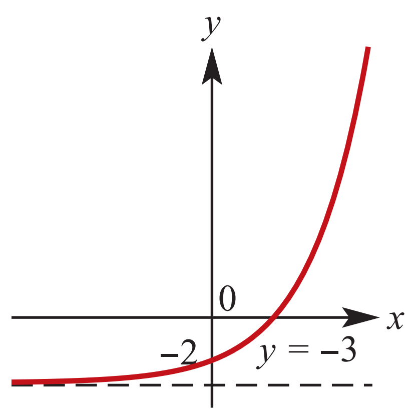
3 a $\dfrac{9^{2n} \times 8^n \times 16^n}{6^n} = \dfrac{3^{4n} \times 2^{3n} \times 2^{4n}}{3^n2^n}$
$= 2^{6n}3^{3n}$

b $3\log_2(16) = 3 \times 4 = 12$

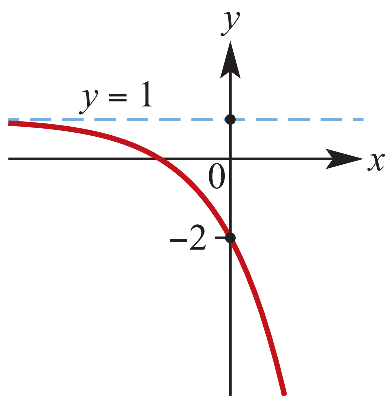
c $2\log_{10} 3 + \log_{10} 4 = \log_{10}(3^2 \times 4)$
$= \log_{10} 36$

d $\log_3\left(\dfrac{1}{27}\right) = \log_3(3^{-3}) = -3$

4 a $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2^x - 3$ Range $(-3, \infty)$



b $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = -3 \times 2^x + 1$
Range $(-\infty, 1)$



5 **a** $4^x = 8^{x-1}$

$2^{2x} = 2^{3x-3}$

$2x = 3x - 3$

$x = 3$

b $4^x = 5 \times 2^x - 4$

$2^{2x} - 5 \times 2^x + 4 = 0$

$(2^x - 4)(2^x - 1) = 0$

$x = 2$ or $x = 0$

c $5^{x-1} > 125$

$\Leftrightarrow 5^{x-1} > 5^3$

$\Leftrightarrow x - 1 > 3$

$\Leftrightarrow x > 4$

d $\log_2(x + 1) = 3$

$x + 1 = 2^3$

$x = 7$

e $\log_4(2x) - \log_4(x + 1) = 0$

$\log_4 \dfrac{2x}{x+1} = 0$

$\dfrac{2x}{x+1} = 4^0$

$2x = x + 1$

$x = 1$

6 **a** $2^x = 5 \Leftrightarrow x = \log_2(5)$

b $5^{3x+1} = 10$

$5^{3x} = 2$

$3x = \log_5(2)$

$x = \dfrac{1}{3}\log_5(2)$

c $0.6^x < 0.2$

$\Leftrightarrow x\log_{10}(0.6) < \log_{10} 0.2$

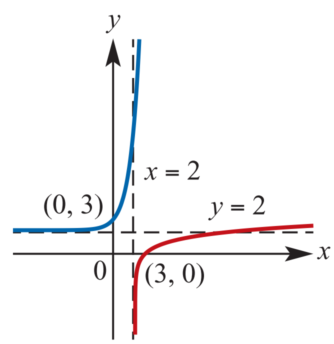
$\Leftrightarrow x > \dfrac{\log_{10} 0.2}{\log_{10}(0.6)}$

7 $f(f^{-1}(x)) = x$

$3^{f^{-1}(x)} + 2 = x$

$3^{f^{-1}(x)} = x - 2$

$f^{-1}(x) = \log_3(x - 2)$



8 **a** $60° = 60 \times \dfrac{\pi}{180}$ radians $= \dfrac{\pi}{3}$ radians

b $270° = \dfrac{3\pi}{2}$ radians

c $140° = 140 \times \dfrac{\pi}{180}$ radians $= \dfrac{7\pi}{9}$ radians

9 **a** $\sin\left(-\dfrac{\pi}{2}\right) = -1$

b $\cos\left(\frac{3\pi}{2}\right) = 0$

c $\tan(3\pi) = 0$

d $\tan\left(-\frac{\pi}{2}\right)$ undefined

10 a $\tan\theta = \frac{2}{5}$
$\theta \approx 21.8°$

b $x = 4\cos 40° \approx 3.06$

c $\frac{6}{x} = \sin 37°$
$x = \frac{6}{\sin 37°} \approx 9.97$

11 a $\sin(2\pi - \theta) = -\sin\theta = -0.3$

b $\cos(-\theta) = \cos\theta = -0.5$

c $\tan(\pi + \theta) = \tan\theta = 1.6$

d $\sin(\pi + \theta) = -\sin\theta = -0.6$

e $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta = 0.1$

f $\cos\theta = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$ (since $0 < \theta < \frac{\pi}{2}$)

12 a $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

b $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

c $\tan\left(\frac{-\pi}{4}\right) = -1$

d $\sin\left(\frac{-7\pi}{6}\right) = \frac{1}{2}$

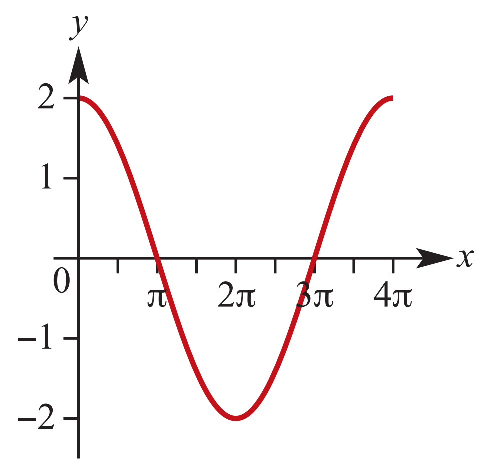
e $\cos\left(\frac{-7\pi}{4}\right) = \frac{1}{\sqrt{2}}$

f $\tan\left(\frac{5\pi}{3}\right) == -\sqrt{3}$

13 $f(x) = 2\cos\left(\frac{x}{2}\right)$.

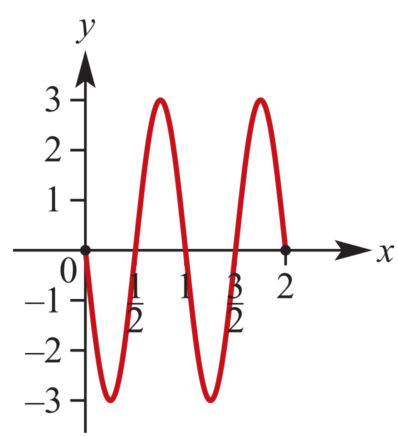
a Period $= 4\pi$; Amplitude $= 2$

b



c Dilation of factor 2 from the x-axis and dilation of factor 2 from the y-axis

14



15 a $\cos\theta = -\frac{\sqrt{3}}{2}$

$\theta = -\frac{5\pi}{6}, -\frac{7\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6},$

b $\sqrt{2}\sin\theta = 1$

$$\sin\theta = \frac{1}{\sqrt{2}}$$

$$\theta = -\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$$

c

$$\sin(2\theta) = -\frac{1}{2}$$

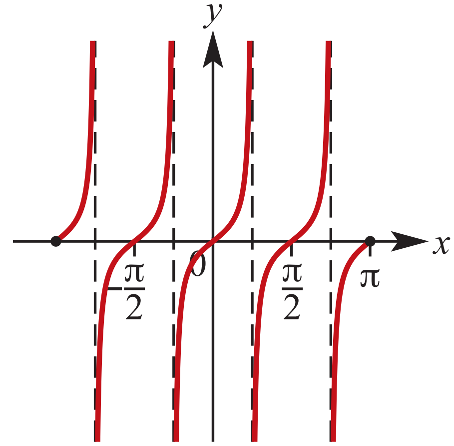
$$2\theta = -\frac{17\pi}{6}, -\frac{13\pi}{6}, -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$$

$$\theta = -\frac{17\pi}{12}, -\frac{13\pi}{12}, -\frac{5\pi}{12}, -\frac{\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

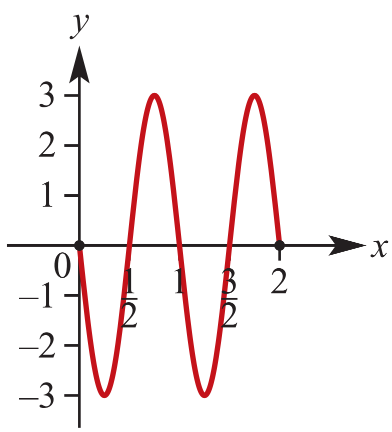
d $\tan\theta = -\sqrt{3}$

$$\theta = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$$

17



16



18 a

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6} + 2k\pi \text{ or } x = \frac{5\pi}{6} + 2k\pi, k \in \mathbb{Z}$$

b

$$2\cos x = \sqrt{3}$$

$$\cos x = \frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{6} + 2k\pi \text{ or } x = -\frac{\pi}{6} + 2k\pi, k \in \mathbb{Z}$$

c $\tan(2x) = -1$

$$2x = -\frac{\pi}{4} + k\pi$$

$$x = -\frac{\pi}{8} + k\frac{\pi}{2}, k \in \mathbb{Z}$$

Solutions to multiple-choice questions

1 B $\log_a 8 = 3, \ \therefore a^3 = 8$

$a = 2$

2 B $5^{n-1}5^{n+1} = 5^{n-1+n+1}$

$= 5^{2n}$

3 B $2^x = \frac{1}{64}, \ \therefore 2^x = 2^{-6}$

$\therefore \ x = -6$

4 E $125^a 5^b = 5^{3a}5^b$

$= 5^{3a+b}$

5 D $4^x = 10 - 4^{x+1}$

$\therefore \ 4^x + 4^{x+1} = 10$

$4^x(1+4) = 10$

$5(4^x) = 10$

$4^x = 2 = 4^{0.5}$

$\therefore x = 0.5$

6 A $\dfrac{7^{n+2} - 35(7^{n+1})}{44(7^{n+2})} = \dfrac{7^{n+2} - 5(7^n)}{44(7^{n+2})}$

$= \dfrac{7^n(49-5)}{44(7^{n+2})}$

$= \dfrac{7^n}{7^{n+2}} = \dfrac{1}{49}$

7 D $f(x) = 2 + 3^x$

$\therefore \ f(2x) - f(x) = (2 + 3^{2x}) - (2 + 3^x)$

$= 3^{2x} - 3^x$

$= 3^x(3^x - 1)$

8 C $(7^{2x})(49^{2x-1}) = 1$

$\therefore \ 7^{2x}7^{4x-2} = 1$

$7^{6x-2} = 1 = 7^{\circ}$

$\therefore \ 6x - 2 = 0, \therefore \ x = \frac{1}{3}$

9 B $y = 2^x$ and; $y = \left(\frac{1}{2}\right)^x$

y-intercept at $(0, 1)$

10 A $f(x) = (2x)^0 + x^{-\frac{2}{3}}$

$= 1 + x^{-\frac{2}{3}}$

$\therefore \ f(8) = 1 + 8^{-\frac{2}{3}}$

$= 1 + \frac{1}{4} = \frac{5}{4}$

11 A $\log a^2 + \log \ b^2 - 2\log \ ab$

$= \log \dfrac{(a^2b^2)}{(ab)^2}$

$= \log 1 = 0$

12 D $2x = 2x\left(\dfrac{180}{\pi}\right)^{\circ}$

$= \left(\dfrac{360x}{\pi}\right)^{\circ}$

$= \dfrac{360x^{\circ}}{\pi}$

13 A $y = \sin 2x + 1$

Q is at the 1st maximum:

$x = \frac{\pi}{4}, \ y = \sin\frac{\pi}{2} + 1 = 2$

14 D $1 - 3\cos\theta$

range $= [1-3, 1+3] = [-2, 4]$,

so min value $= -2$

15 D $y = 16 + 15\sin\dfrac{\pi x}{60}$

$\therefore \ y(10) = 16 + 15\sin\dfrac{10\pi}{60}$

$= 16 + \dfrac{15}{2}$

$= 23.5$ m

16 D $\sin(\pi + \theta) + \cos(\pi + \theta)$

$= -\sin\theta - \cos\theta$

17 **A** $\sin x = 0, \therefore x = 0, \pi$
Over $[0, \pi]$, **B**, **C**, **E** have 1 solution and **D** has none.

18 **E** $y = \sin\dfrac{\theta}{2}$ has per $= 4\pi$

19 **D** $2 - 3\sin\theta$
range $= [2 - 3, 2 + 3] = [-1, 5]$

20 **D** $y = \cos x^\circ$ with translation of 30° in negative x direction
$\therefore y = \cos(x + 30)^\circ$

21 **E** $f(x) = -2\cos 3x$:
per $\dfrac{2\pi}{3}$, ampl 2

22 **A**
$$C^d = 3$$
$$\therefore C^{4d} - 5 = 3^4 - 5$$
$$= 76$$

23 **E** $\log_2 56 - \log_2 7 + \log_2 2$
$$= \log_2\left(\frac{56 \times 2}{7}\right)$$
$$= \log_2 16$$
$$= 4$$

24 **B** $\log_b a = c;\ \log_x b = c$
$$\therefore\ a = b^c,\ b = x^c$$
$$\therefore\ \log_a b = c\log_a x$$
$$\therefore \log_a x = \frac{1}{c}\log_a b = \frac{1}{c^2}\log_a b^c$$
$$\therefore \log_a x = \frac{1}{c^2}\log_a a = \frac{1}{c^2}$$

25 **D** $\cos\theta - \sin\theta = \dfrac{1}{4}$
$$\therefore\ (\cos\theta - \sin\theta)^2 = \frac{1}{16}$$
$$\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta = \frac{1}{16}$$
$$1 - 2\sin\theta\cos\theta = \frac{1}{16}$$
$$2\sin\theta\cos\theta = 1 - \frac{1}{16}$$
$$\therefore\ \sin\theta\cos\theta = \frac{15}{32}$$

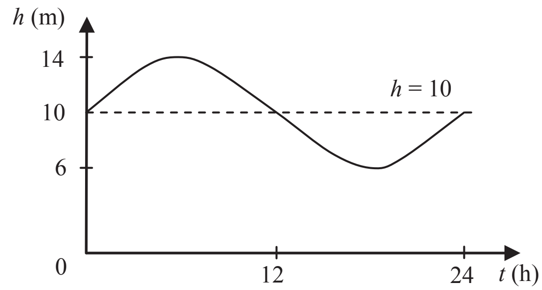
26 **B** $y = \dfrac{1}{2}\sin 2x$ and $y = \dfrac{1}{2}$ meet at
$\sin 2x = 1$
$$\therefore 2x = \frac{\pi}{2}, \frac{5\pi}{2}\dots$$
$$x = \frac{\pi}{4}, \frac{5\pi}{4}\dots$$

Solutions to extended-response questions

1 **a** $h(t) = 10 + 4\sin(15t)°,\ 0 \le t \le 24$

period $= \frac{360}{15} = 24$, amplitude $= 4$

translation of 10 units in the positive direction of the h-axis



b When $h = 13$, $10 + 4\sin(15t) = 13$

$$\therefore\ 4\sin(15t) = 3 \qquad \therefore\ \sin(15t) = \frac{3}{4}$$

$$\therefore\ 15t = \sin^{-1}\left(\frac{3}{4}\right) \quad \text{or} \quad 15t = 180 - \sin^{-1}\left(\frac{3}{4}\right)$$

$$\text{and} \quad t = \frac{1}{15}\sin^{-1}\left(\frac{3}{4}\right) \quad \text{or} \quad t = \frac{1}{15}\left(180 - \sin^{-1}\left(\frac{3}{4}\right)\right)$$

From the graph it can be seen that only two solutions are required.

$$\therefore\ t \approx \frac{1}{15}(48.5904) \quad \text{or} \quad t \approx \frac{1}{15}(180 - 48.5904)$$

$$\approx 3.2394 \qquad\qquad \approx 8.7606$$

Hence, $h = 13$ after approximately 3.2394 hours and 8.7606 hours.

c When $h = 11$, $10 + 4\sin(15t) = 11$

$$\therefore\ 4\sin(15t) = 1 \qquad \therefore\ \sin(15t) = \frac{1}{4}$$

$$\therefore\ 15t = \sin^{-1}(0.25) \quad \text{or} \quad 15t = 180 - \sin^{-1}(0.25)$$

$$\text{and} \quad t = \frac{1}{15}\sin^{-1}(0.25) \quad \text{or} \quad t = \frac{1}{15}(180 - \sin^{-1}(0.25))$$

From the graph only two solutions are required for the domain $0 \le t \le 24$.

$$\therefore\ t \approx \frac{1}{15}(14.4775) \quad \text{or} \quad t \approx \frac{1}{15}(180 - 14.4775)$$

$$\approx 0.9652 \qquad\qquad \approx 11.0348$$

For $h \ge 11, 0.9652 \le t \le 11.0348$ (approximately).

Hence a boat can leave the harbour between 0.9652 hours and 11.0348 hours.

2 a At the start of the experiment, $t = 0$.

$$\therefore N(0) = 40 \times 2^{1.5(0)}$$
$$= 40 \times 2^0$$
$$= 40 \times 1 = 40$$

Hence there are 40 bacteria present at the start of the experiment.

b i When $t = 2$, $N(2) = 40 \times 2^{1.5(2)}$
$$= 40 \times 2^3$$
$$= 40 \times 8$$
$$= 320$$

After 2 hours, there are 320 bacteria present.

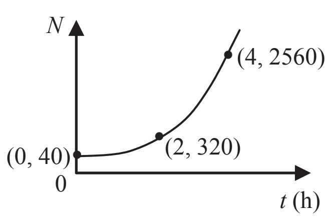
ii When $t = 4$, $N(4) = 40 \times 2^{1.5(4)}$
$$= 40 \times 2^6$$
$$= 40 \times 64$$
$$= 2560$$

After 4 hours, there are 2560 bacteria present.

iii When $t = 12$, $N(12) = 40 \times 2^{1.5(12)}$
$$= 40 \times 2^{18}$$
$$= 40 \times 262\,144$$
$$= 10\,485\,760$$

After 12 hours, there are 10 485 760 bacteria present.

c



d When $N = 80$, $80 = 40 \times 2^{1.5(t)}$

$$\therefore \quad 2^{1.5(t)} = 2^1$$
$$\therefore \quad 1.5t = 1$$
$$\therefore \quad t = \frac{2}{3}$$

The number of bacteria doubles after $\frac{2}{3}$ of an hour (40 minutes).

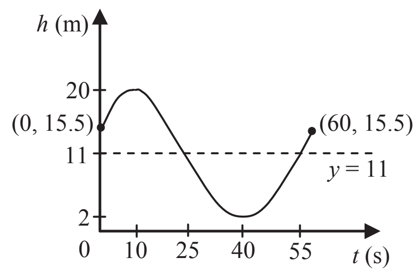
3 a The Ferris wheel makes one revolution after one period.

$$\begin{aligned}\text{Period} &= \frac{2\pi}{n},\ \text{where } n = \frac{\pi}{30}\\ &= 2\pi \div \frac{\pi}{30}\\ &= \frac{2\pi \times 30}{\pi}\\ &= 60\end{aligned}$$

i.e. the Ferris wheel takes 60 seconds for one revolution.

b Period = 60, amplitude = 9

The graph is translated 10 units in the positive direction of the t-axis and 11 units in the positive direction of the h-axis.



c Range = $[2, 20]$

d At $h = 2$,
$$11 + 9\cos\left(\frac{\pi}{30}(t-10)\right) = 2$$
$$\therefore \quad 9\cos\left(\frac{\pi}{30}(t-10)\right) = -9$$
$$\therefore \quad \cos\left(\frac{\pi}{30}(t-10)\right) = -1$$
$$\therefore \quad \frac{\pi}{30}(t-10) = \pi \text{ or } 3\pi \text{ or } 5\pi \text{ or } \ldots$$
$$\therefore \quad t - 10 = 30 \text{ or } 90 \text{ or } 150 \text{ or } \ldots$$
$$\therefore \quad t = 40 \text{ or } 100 \text{ or } 160 \text{ or } \ldots$$

i.e. the height of the person above the ground is 2 m after 40 seconds and then again after each further 60 seconds.

e At $h = 15.5$, $\quad 11 + 9\cos\left(\frac{\pi}{30}(t-10)\right) = 15.5$

$$\therefore \quad 9\cos\left(\frac{\pi}{30}(t-10)\right) = 4.5$$

$$\therefore \quad \cos\left(\frac{\pi}{30}(t-10)\right) = \frac{1}{2}$$

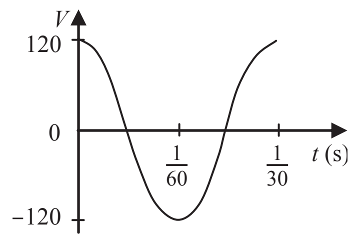
$$\therefore \quad \frac{\pi}{30}(t-10) = \frac{-\pi}{3} \text{ or } \frac{\pi}{3} \text{ or } \frac{5\pi}{3} \text{ or } \frac{7\pi}{3} \text{ or } \ldots$$

$$\therefore \quad t - 10 = -10 \text{ or } 20 \text{ or } 50 \text{ or } 70 \text{ or } \ldots$$

$$\therefore \quad t = 0 \text{ or } 20 \text{ or } 60 \text{ or } 80 \text{ or } \ldots$$

i.e. the height of the person above the ground is 15.5 m at the start and each 60 seconds thereafter, and also at 20 seconds and each 60 seconds after that.

4 a $V = 120\cos(60\pi t)$, period $= \frac{2\pi}{60\pi} = \frac{1}{30}$, amplitude $= 120$



b At $V = 60$, $\quad 120\cos(60\pi t) = 60$

$$\therefore \quad \cos(60\pi t) = \frac{1}{2}$$

$$\therefore \quad 60\pi t = \frac{\pi}{3}$$ (Only smallest positive solution is required.)

$$\therefore \quad t = \frac{\pi}{3 \times 60\pi} = \frac{1}{180}$$

i.e. the first time the voltage is 60 is at $\frac{1}{180}$ second.

c The voltage is maximised when $\quad V = 120$

$$\therefore \quad 120\cos(60\pi t) = 120$$

$$\therefore \quad \cos(60\pi t) = 1$$

$$\therefore \quad 60\pi t = 0 \text{ or } 2\pi \text{ or } 4\pi \text{ or } \ldots$$

$$\therefore \quad t = \frac{0}{60\pi} \text{ or } \frac{2\pi}{60\pi} \text{ or } \frac{4\pi}{60\pi} \text{ or } \ldots$$

$$= 0 \text{ or } \frac{1}{30} \text{ or } \frac{1}{15} \text{ or } \ldots$$

i.e. the voltage is maximised when $t = 0$ seconds, and every $\frac{1}{30}$ second thereafer $(t = \frac{k}{30}, k = 0, 1, 2, \dots)$.

5 $d = a + b \sin c(t - h)$

a **i** period $= \dfrac{60 \text{ seconds}}{4 \text{ revolutions}}$

$= 15$ seconds

ii amplitude = radius of waterwheel

$= 3$ metres

iii period $= \dfrac{2\pi}{c} = 15$

$\therefore \quad c = \dfrac{2\pi}{15}$

b At $(0, 0)$, $\quad 0 = a + b \sin\left(\dfrac{2\pi}{15}(0 - h)\right)$

Now amplitude $= 3$, $\quad \therefore b = 3$
and the translation in the positive direction of the y-axis is 2,

$\therefore \quad a = 2$

$\therefore \quad 0 = 2 + 3 \sin \dfrac{-2\pi h}{15}$

$\therefore \quad 3 \sin \dfrac{-2\pi h}{15} = -2$

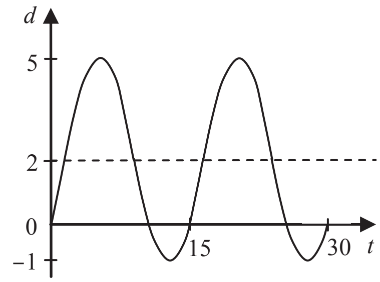
$\therefore \quad \sin \dfrac{-2\pi h}{15} = \dfrac{-2}{3}$

$\therefore \quad \dfrac{-2\pi h}{15} \approx -0.729\,727\,656$

$\therefore \quad h \approx \dfrac{-0.729\,727\,656 \times 15}{-2\pi}$

$\approx 1.742\,10$

c $d = 2 + 3\sin\left(\frac{2\pi}{15}(t - 1.742\,10)\right)$



6 a i When $t = 0$, $h = 30(1.65)^0$

$= 30 \times 1$

$= 30$

ii When $t = 1$, $h = 30(1.65)^1$

$= 30 \times 1.65$

$= 49.5$

iii When $t = 2$, $h = 30(1.65)^2$

$= 30 \times 2.7225$

$= 81.675$

b

$$\begin{aligned} h(N) &= 30(1.65)^N \\ h(N+1) &= 30(1.65)^{N+1} \\ &= 30(1.65)^N \times 1.65 \\ &= 1.65h(N) \end{aligned}$$

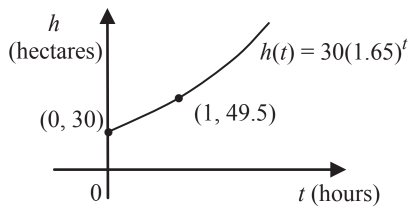
$\therefore \quad h(N+1) = kh(N)$

implies $\quad k = 1.65$

c When $h = 900$, $\quad 30(1.65)^t = 900$

$$\begin{aligned} \therefore \quad 1.65^t &= 30 \\ \therefore \quad \log_{10} 1.65^t &= \log_{10} 30 \\ \therefore \quad t\log_{10} 1.65 &= \log_{10} 30 \\ \therefore \quad t &= \frac{\log_{10} 30}{\log_{10} 1.65} \\ &\approx 6.792 \end{aligned}$$

i.e. it takes approximately 6.792 hours for 900 hectares to be burnt.

d 

7 **a** When $t = 0$, $\theta = 80(2^{-0}) + 20$

$= 80 + 20$

$= 100$

When $t = 1$, $\theta = 80(2^{-1}) + 20$

$= 40 + 20$

$= 60$

When $t = 2$, $\theta = 80(2^{-2}) + 20$

$= 20 + 20$

$= 40$

When $t = 3$, $\theta = 80(2^{-3}) + 20$

$= 10 + 20$

$= 30$

When $t = 4$, $\theta = 80(2^{-4}) + 20$

$= 5 + 20$

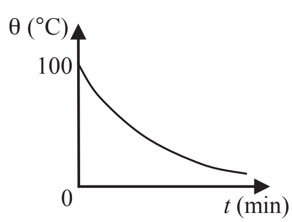
$= 25$

When $t = 5$, $\theta = 80(2^{-5}) + 20$

$= 2.5 + 20$

$= 22.5$

t	0	1	2	3	4	5
θ	100	60	40	30	25	22.5

b 

c When $\theta = 60°$, $t = 1$

i.e. the temperature is 60°C after 1 minute.

d When $t = 3.5$, $\theta = 80(2^{-3.5}) + 20$

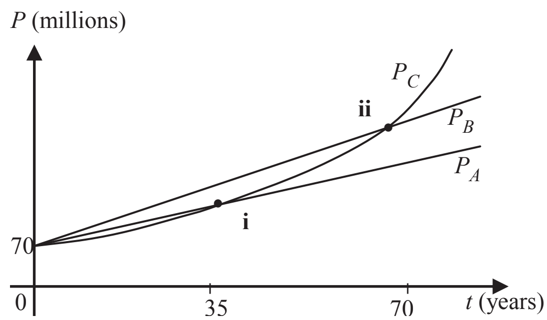
$$\approx \frac{80}{11.313\,708\,5} + 20$$

$$\approx 27.071$$

8 **a** $P_A = 70\,000\,000 + 3\,000\,000t$,

$P_B = 70\,000\,000 + 5\,000\,000t$

$P_C = 70\,000\,000(1.3)^{\frac{t}{10}}$

b 

c From the graph, the population of C overtakes the population of

i A after approximately 35 years

ii B after approximately 67 years.

9 **a** **i** When $t = 1975$, $P = 4(2)^{\frac{1975-1975}{35}}$

$$= 4(2)^0$$

$$= 4 \times 1$$

$$= 4 \text{ billion}$$

ii When $t = 1995$, $P = 4(2)^{\frac{1995-1975}{35}}$

$$= 4(2)^{\frac{20}{35}}$$

$$\approx 4 \times 1.485\,99$$

$$\approx 5.944 \text{ billion}$$

iii When $t = 2005$, $P = 4(2)^{\frac{2005-1975}{35}}$

$$= 4(2)^{\frac{30}{35}}$$

$$\approx 7.246 \text{ billion}$$

b When $t = 1997$, $P = 4(2)^{\frac{1997-1975}{35}}$

$$= 4(2)^{\frac{22}{35}}$$

In 1997, $P = 4(2)^{\frac{1997-1975}{35}}$

$$= 4(2)^{\frac{22}{35}}$$

Double this is $2 \times 4(2)^{\frac{22}{35}}$

$$= 4(2)^{1+\frac{22}{35}}$$

$$= 4(2)^{\frac{57}{35}}$$

Solve for t: $4(2)^{\frac{t-1975}{35}} = 4(2)^{\frac{57}{35}}$

Then $t - 1975 = 57$

$t = 2032$

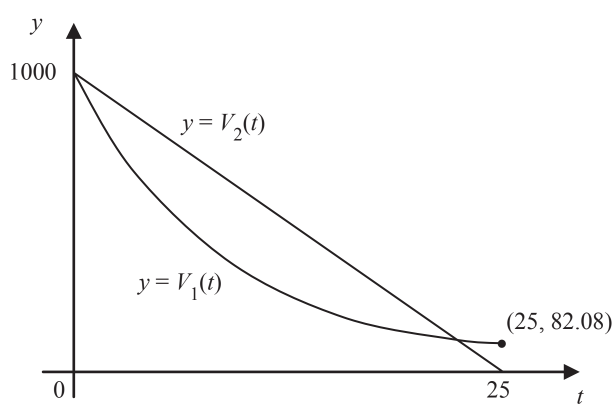
i.e. in 2032 the population of Earth will be twice the population it was in 1997.

10 $V_1(t) = 1000e^{\frac{-t}{10}}$, $t \geq 0$

$V_2(t) = 1000 - 40t$, $0 \leq t \leq 25$

a $V_1(0) = 1000$, $V_2(0) = 1000$

b



c Tank B is empty when $t = 25$, i.e. when $1000 - 40t = 0$.

$V_1(25) = 10003^{\frac{-25}{10}}$

$= 64.15\ldots$

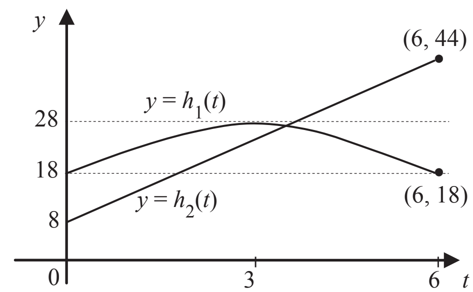
Tank A has 64.15 litres in it when B is first empty.

d On a CAS calculator, with $\boldsymbol{f1 = 10003\text{^}(-x/10)}$ and $\boldsymbol{f2 = 1000 - 40x}$, and using
TI: Press **Menu → 6:Analyze Graph → 4:Intersection**
CP: Tap **Analysis → G-Solve → Intersect**
$t = 0,$ and $V_1(0) = V_2(0) = 1000$

$t = 23,$

11
$$h_1(t) = 18 + 10\sin\left(\frac{\pi t}{6}\right)$$
$$h_2(t) = 8 + 6t$$

a period of $y = h_1(t)$
$$= 2\pi \div \frac{\pi}{6}$$
$$= 12$$



b On a CAS calculator, with $\boldsymbol{f1 = 18 + 10\sin(\pi x/6)}$ and $\boldsymbol{f2 = 8 + 6x}$, and using
TI: Press **Menu → 6:Analyze Graph → 4:Intersection**
CP: Tap **Analysis → G-Solve → Intersect**
the coordinates of the intersection point are (3.311, 27.867). (3.19 am)

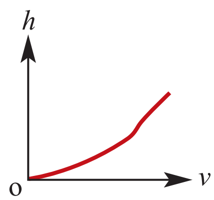
c **i** When $t = 9$ (9.00 am), $h_1(t)$ reaches its minimum value of 8.

ii The original function satisfies this with t redefined, i.e. $h(t) = 8 + 6t$.

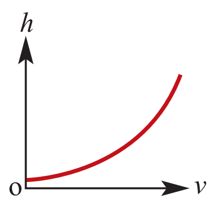
Chapter 16 – Rates of change

Solutions to Exercise 16A

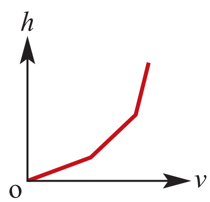
1 **a**



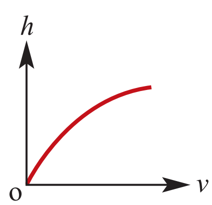
b



c



d

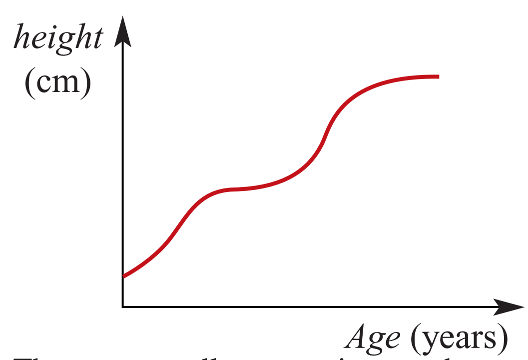


2 For the first 2 minutes, the particle travels a distance of 4 m with its speed increasing. For the next 4 minutes, it travels 4 m at constant speed. Then it turns back and returns to its starting point O, travelling at a constant speed and taking 8 minutes to reach O.

3 **a** **C** and **D** are most likely, since putting the price up is most likely to result in fewer customers.
In **C** the present price of admission is clearly too low, whereas in the case of **D** the present price is about right.
B, **E** and **F** assume that people will keep coming in the same numbers however expensive the tickets are. This seems very unlikely.
A is possible, but the shape seems wrong.

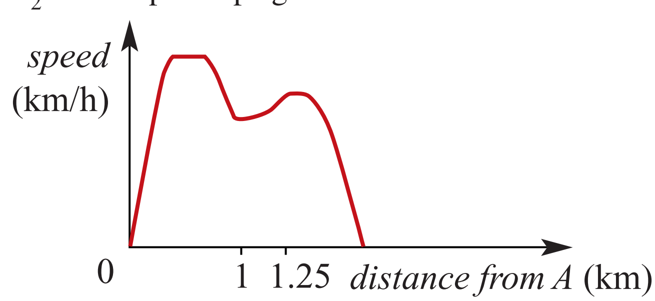
b The axis intersection must refer to current profits and current prices. This can hardly be zero profit and zero prices since it is scarcely possible that net overheads are zero.

4



There are usually two main growth spurts, one before and during puberty. Height decreases during old age as bones diminish in size and strength.

5 **a** The car accelerates up to 100 km/hr and slows considerably just before 1 km and again at 1.8 km. From 1 to $1\frac{1}{2}$ it will speed up again.



6 **a** **A** begins very slowly and smoothly accelerates continually until maximum speed at the end of the race.

B sets off fast, slows very slightly, then maintains a constant speed until the final sprint.

C is like **B** except that **C**'s start and finish are a little slower than the average speed.

D starts quickly, slows down a little and maintains constant speed until the finish.

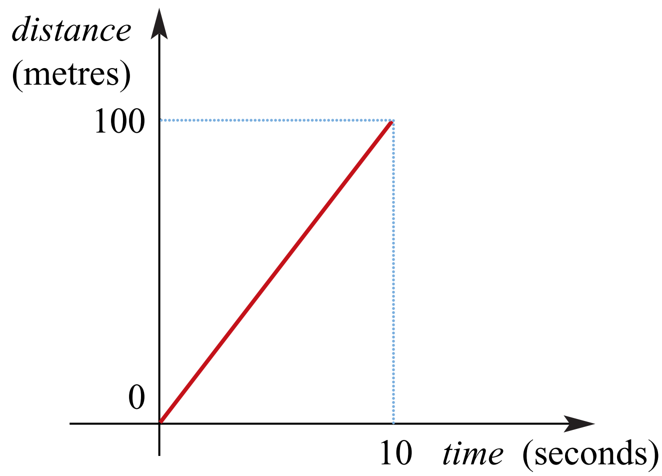
E begins fast and progressively slows almost to a complete stop.

F begins fast, slows down progressively until the middle of the race, and then slows down at an increasing rate until the last lap, when **D** (presumably) walks slower and slower until the end.

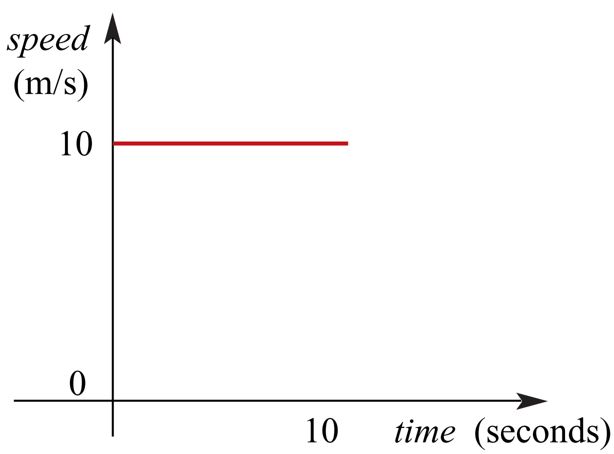
b **B** is most likely to win because the final sprint generally decides the race, and **B** is the only one apart from **A** who is accelerating at this point. **A** will be, in all probability, several laps behind **B** by then.

C is also a possibility to win.

7 **a** distance–time graph:



b speed–time graph:



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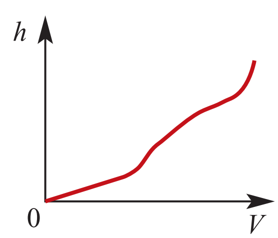
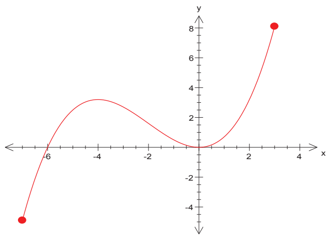


9 Distance-time graph is a straight line, therefore the car travels at constant speed. **D**

10 Only **C** shows the rate of cost of living slowing down. **B** and **D** show the rate actually decreasing and**A** shows an acceleration.

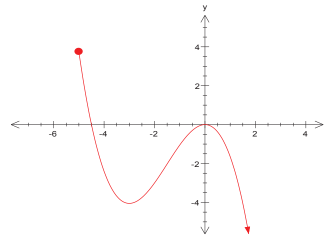
11



a $(-4, 0)$

b y increases with x:
$[-7, -4) \cup (0, 3]$

12



a $(-3, 0)$

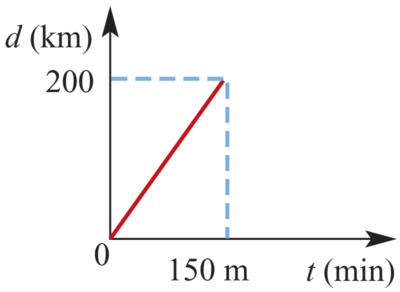
b y decreases as x increases:
$[-5, \ -3) \cup (0, 2]$

Solutions to Exercise 16B

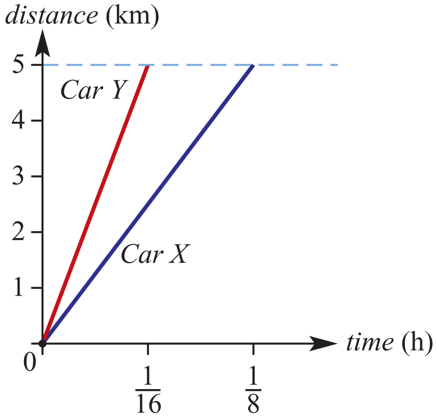
1 Speed $= \dfrac{200}{150} = \dfrac{4}{3}$ km/min

$= \dfrac{4}{3}(60)$

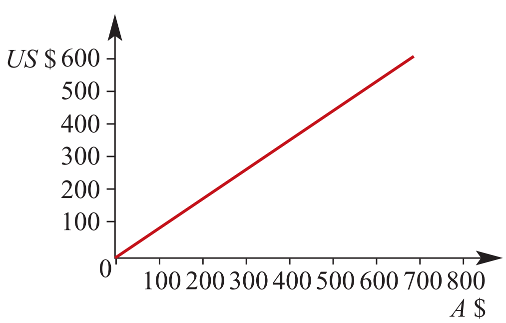
$= 80$ km/h



2



3 A\$1=US\$0.75



4 **a** 120 km in 2 hours $= \dfrac{120}{2}$

$= 60$ km/h

b 60 m in 20 seconds $= \dfrac{60}{20}$

$= 3$ m/s

c 8000 m in 20 minutes

$= \dfrac{8}{1/3} = 24$ km/h

OR $\dfrac{8000}{20 \times 60} = \dfrac{20}{3}$ m/s $= 6\dfrac{2}{3}$ m/s

d 200 km in 5 hours 40 minutes

$= \dfrac{200}{17/3} \approx 35.29$ km/h

e 6542 m in 5 minutes 20 seconds

$= \dfrac{6542}{320} = 20.44$ m/s

5 **a** 40 L in 5 minutes $= \dfrac{40}{5} = 8$ L/min

b 600 L in 12 minutes

$= \dfrac{600}{12} = 50$ L/min

c 200 L in 17 minutes

$= \dfrac{200}{17} \approx 11.8$ L/min

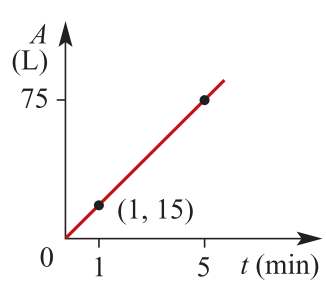
d 180 L in $\dfrac{52}{3}$ minutes

$= \dfrac{135}{13} \approx 1.04$ L/min

6 **a**

t	0	0.5	1	1.5	2	3	4	5
A	0	7.5	15	22.5	30	45	60	75

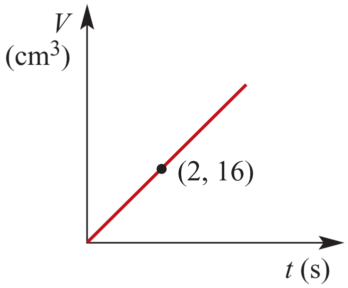
b



7 \$200 for 13 hours $= \dfrac{\$200}{13} = \$15.38$ per hour

8 5000 m in 24 sec $= \dfrac{5000}{24} = 208\frac{1}{3}$ m/s

9



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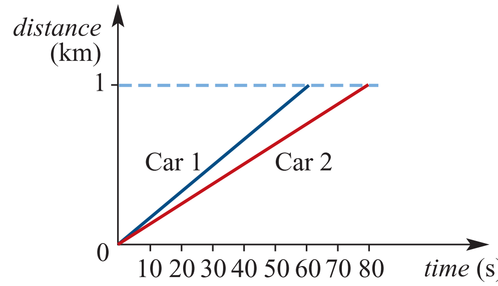


Solutions to Exercise 16C

1 Average speed $= \dfrac{48-32}{8-3} = \dfrac{16}{5}$ m/s^2

2 a $f(x) = 2x + 5$

Av. rate of change:

$\dfrac{f(3)-f(0)}{3-0} = \dfrac{11-5}{3} = 2$

b $f(x) = 3x^2 + 4x - 2$

Av. rate of change:

$\dfrac{f(2)-f(-1)}{2-(-1)} = \dfrac{18-(-3)}{3}$

$= \dfrac{21}{3} = 7$

c $f(x) = \dfrac{2}{x-3} + 4$

Av. rate of change:

$\dfrac{f(7)-f(4)}{7-4} = \dfrac{4.5-6}{3} = -\dfrac{1}{2}$

d $f(x) = \sqrt{5-x}$

Av. rate of change:

$\dfrac{f(4)-f(0)}{4-0} = \dfrac{1-\sqrt{5}}{4}$

3 a Av. rate of change: $\dfrac{5-30}{2-(-5)} = -\dfrac{25}{7}$

b Av. rate: $\dfrac{5-14}{2-(-1.5)} = -\dfrac{9}{3.5} = -\dfrac{18}{7}$

c Av. rate: $\dfrac{15-3}{3-0} = \dfrac{12}{3} = 4$

d Av. rate: $\dfrac{5b-b}{2a-(-a)} = \dfrac{4b}{3a}$

4 $S(t) = t^3 + t^2 - 2t,\ t > 0$

a Av. rate: $\dfrac{S(2)-S(0)}{2-0} = \dfrac{8}{2} = 4$m/s

b Av. rate: $\dfrac{S(4)-S(2)}{4-2} = \dfrac{72-8}{2}$

$= 32$m/s

5 \$2000 dollars, 7% per year over 3 years

$\therefore I = 2000(1.07^t)$

a $I(3) = 2000(1.07^3) = \$2450.09$

b Av. return $= \dfrac{2450.09-2000}{3} =$

\$150.03

6 $d(t) = -\dfrac{300}{t+6} + 50,\ t > 0$

$d(10) = (50 - \dfrac{300}{16}) = 31.25$ cm

$d(0) = \left(50 - \dfrac{300}{6}\right) = 0$ cm

Av. rate: $\dfrac{31.25}{10} = 3.125$ cm/min

7 C $d(3) = 2$m, $d(0) = 0$m

Av. speed $= \dfrac{2}{3}$m/s

Solutions to Exercise 16D

1 $y = x^3 + x^2$; chord from $x = 1.2$ to 1.3:

$$\cong \frac{y(1.3) - y(1.2)}{1.3 - 1.2} = \frac{3.887 - 3.168}{0.1}$$
$$= 7.19$$

2 **a**

From 0 to 1200, av. rate $= \dfrac{19 - 5}{1200}$

≈ 0.012L/kgm

b $C(600) = 15$L/min, $C(0) = 5$L/min.

$W = 450$, est. rate $= \dfrac{15 - 5}{600} = \dfrac{1}{60}$

≈ 0.0167L/kg m

3 $y = 10^x$

a Average rate of change over:

i $[0, 1] : \dfrac{y(1) - y(0)}{1} = \dfrac{10 - 1}{1}$
$= 9$

ii

$[0, 0.5] : \dfrac{y(0.5) - y(0)}{0.5} = \dfrac{\sqrt{10} - 1}{0.5}$
$\cong 4.3246$

iii $[0, 0.1] : \dfrac{y(0.1) - y(0)}{0.1} \cong 2.5893$

b Even smaller intervals suggest the instantaneous rate of change at $x = 0$ is about 2.30

4 **a** $T \approx 25°$ at $t = 16$ hours, i.e. at 16:00.

b $T(14) = 23°$, $T(10) = 9°$ (approx.)

Est. rate $= \dfrac{23 - 10}{14 - 10} = 3°$C/hr

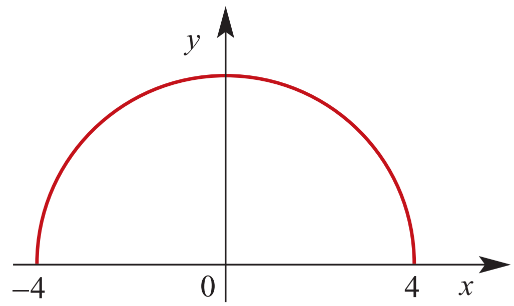
c $T(20) = 15.2°$, $T(16) = 25.2°$

Est. rate $= \dfrac{15.2 - 25.2}{20 - 16} = -2.5°$C/hr

5 Using chord $x = 1.2$ to 1.4, av. rate of change

$$= \left(\frac{1}{1.4} - \frac{1}{1.2}\right) \div (1.4 - 1.2)$$
$$\cong \frac{0.714 - 0.833}{0.2} = -0.5952$$

6 $y = \sqrt{16 - x^2}$, $-4 \le x \le 4$



a Gradient at $x = 0$ must be zero, as a tangent drawn at that point is horizontal.

b $x = 2$; chord connecting $x = 1.9$ and 2.1.

$y(2.1) = \sqrt{16 - 2.1^2} \cong 3.40$

$y(1.9) = \sqrt{16 - 1.9^2} \cong 3.52$

Av. rate $= \dfrac{3.40 - 3.52}{2.1 - 1.9} \cong -0.6$

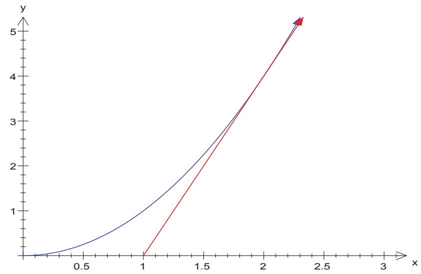
c $x = 3$; chord connecting $x = 2.9$ and 3.1.

$y(3.1) = \sqrt{16 - 3.1^2} \cong 2.53$

$y(2.9) = \sqrt{16 - 2.9^2} \cong 2.76$

Av. rate $= \dfrac{2.53 - 2.76}{3.1 - 2.9} \cong -1.1$

7 $y = x^2$ and $y = 4x - 4$:



Graphs meet at (2, 4), where the line is a tangent.
Gradient = 4 (= gradient of $y = 4x - 4$)

8 $V = 3t^2 + 4t + 2$

a Av. rate of change from $t = 1$ to $t = 3$:

$$\frac{V(3) - V(1)}{3 - 1} = \frac{41 - 9}{2}$$
$$= 16$$
$$= 16 \text{ m}^3/\text{min}$$

b Est. rate of change at $t = 1$, chord 0.9 to 1.1:

$$\frac{V(1.1) - V(0.9)}{1.1 - 0.9} = \frac{10.03 - 8.03}{0.2}$$
$$= 10$$
$$= 10 \text{ m}^3/\text{min}$$

9 $P = 3(2^t)$

a Av. rate of change from $t = 2$ to $t = 4$:

$$\frac{P(4) - P(2)}{4 - 2} = \frac{48 - 12}{2}$$
$$= 18$$
$$= 18 \text{ million/min}$$

b Est. rate of change at $t = 2$, chord 1.9 to 2.1:

$$\frac{P(2.1) - P(1.9)}{2.1 - 1.9} \cong \frac{12.86 - 11.20}{0.2}$$
$$= 8.30$$
$$= 8.30 \text{ million/min}$$

10 $V = 5 \times 10^5 - 10^2(2^t),\ 0 \leq t \leq 12$

a Av. rate of change from $t = 0$ to $t = 5$:

$$\frac{V(5) - V(0)}{5 - 0} = \frac{-3200 + 100}{5}$$
$$= -620 \text{ m}^3/\text{min}$$

i.e. 620 m^3/min flowing out

b Est. rate of change at $t = 6$, chord 5.9 to 6.1:

$$\frac{V(1.1) - V(0.9)}{6.1 - 5.9} = \frac{-686 + 597}{0.2}$$
$$\cong -4440 \text{ m}^3/\text{ min}$$

i.e. 4440 m^3/min flowing out

c Est. rate of change at $t = 12$, chord 11.9 to 12:

$$\frac{V(12) - V(11.9)}{12 - 11.9} = \frac{-409600 + 382200}{0.1}$$
$$\cong -284000 \text{ m}^3/\text{min}$$

i.e. 284 000 m^3/min flowing out

11 **a** $y = x^3 + 2x^2$; chord from $x = 1$ to 1.1:

$$\cong \frac{y(1.1) - y(1)}{1.1 - 1} = \frac{3.751 - 3}{0.1}$$
$$= 7.51$$

b $y = 2x^3 + 3x$;
chord from $x = 1$ to $1.\ 1$:

$$\cong \frac{y(1.1) - y(1)}{1.1 - 1} = \frac{5.962 - 5}{0.1}$$
$$= 9.62$$

c $y = -x^3 + 3x^2 + 2x$;
chord from $x = 2$ to 2.1:

$$\cong \frac{y(2.1) - y(2)}{2.1 - 2} = \frac{8.169 - 8}{0.1}$$
$$= 1.69$$

d $y = 2x^3 - 3x^2 - x + 2$;
chord from $x = 3$ to 3.1:

$$\cong \frac{y(3.1) - y(3)}{3.1 - 3} = \frac{29.7 - 26}{0.1}$$
$$= 37$$

(Using smaller chords give answers which approach $a7, b9, c2, d35$)

12 $V = x^3$

a Av. rate of change from $x = 2$ to $x = 4$:

$$\frac{V(4) - V(2)}{4 - 2} = \frac{64 - 8}{2}$$
$$= 28$$

b Est. rate of change at $t = 2$, chord 1.9 to 2.1:

$$\frac{V(2.1) - V(1.9)}{2.1 - 1.9} \cong \frac{9.261 - 6.859}{0.2}$$
$$= 12.01$$

13 $y = 2x^2 - 1$

a Av. rate of change from $x = 1$ to $x = 4$:

$$\frac{y(4) - y(1)}{4 - 1} = \frac{31 - 1}{3}$$
$$= 10$$

b Est. rate of change at $x = 1$, chord 0.9 to 1.1:

$$\frac{y(1.1) - y(0.9)}{1.1 - 0.9} = \frac{1.42 - 0.62}{0.2}$$
$$= 4.00$$

14 a **i** $\frac{2}{\pi} \approx 0.637$

ii $\frac{2\sqrt{2}}{\pi} \approx 0.9003$

iii 0.959

iv 0.998

b 1

Solutions to Exercise 16E

1 $s(t) = 6t - 2t^3$

a Av. velocity over $[0, 1]$
$$= \frac{s(1) - s(0)}{1} = \frac{4 - 0}{1} = 4 \text{ m/s}$$

b Av. velocity over $[0.8, 1]$
$$\frac{s(1) - s(0.8)}{1 - 0.8} = \frac{4 - 3.776}{0.2} = 1.12 \text{ m/s}$$

2 **a** Train's velocity over:

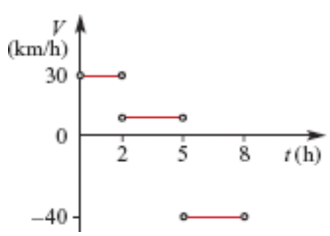
i $[0, 2] = \frac{60}{2} = 30\text{km/h}$

ii $[2, 5] = \frac{20}{3} = 6.67\text{km/h}$

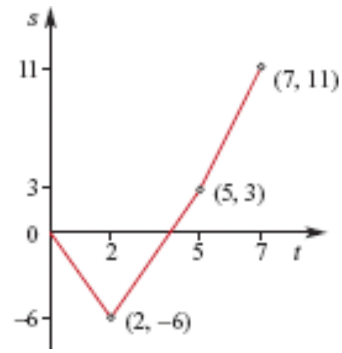
iii $[5, 8] = -\frac{120}{3} = -40\text{km/h}$

b The train journey travelled steadily for 2 hrs at 30 km/hr, and at 6.67 km/h for another 3 hours.
It then turned around and headed back to Jimbara at 40 km/h, reaching the station after 7 hours.
It went back past the station at the same speed for another hour.

c



3 Over $(0, 2)$, $v = -3$; over $(2, 5)$, $v = 3$; over $(5, 7)$, $v = 4$



4 **a** From graph: $v = 0$ at $t = 2.5$ seconds

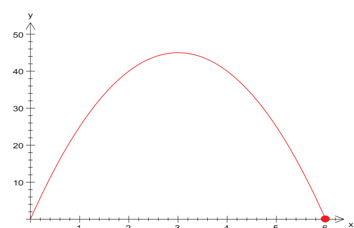
b $v > 0$ for $0 \le t < 2.5$ seconds

c 6 m (maximum value of x)

d 5 seconds (2nd x-intercept)

e $v(1) \cong 3$ m/s

5



a Ball returns to starting point at $t = 6$ sec

b Av. velocity $t = 1$ to $t = 2$:
$$\frac{40 - 25}{2 - 1} = 15 \text{ m/s}$$

c $t = 1$ to $t = 1.5$:
$$\frac{33.75 - 25}{1.5 - 1} = 17.5 \text{ m/s}$$

d $v(1) = 20$ m/s

e $v(4) = -10$ m/s

f $v(5) = -20$ m/s

6 **a** $v(0) \cong 11$ m/s

b h max. = 15 m

c h max. occurs at $t = 1$ sec

d The stone hit the ground at $t = 2.8$ seconds

e The stone hits the ground at 15 m/s

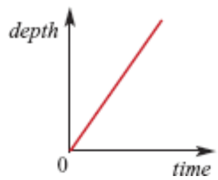
7 **a** Particle is at O at x-intercepts: $t = 2, 3, 8$ seconds

b Particle moves right when gradient is positive: $\{t: 0 < t < 2.5\} \cup \{t: t > 6\}$

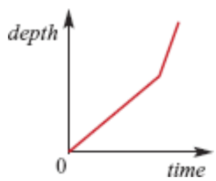
c Particle is stationary at gradient zero: $t = 2.5, 6$

Solutions to Technology-free questions

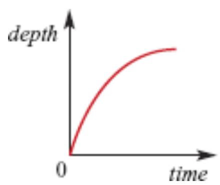
1 a



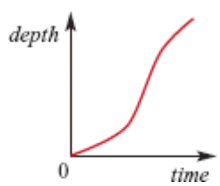
b



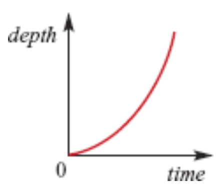
c



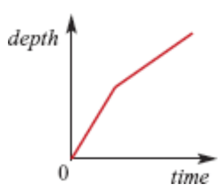
d



e



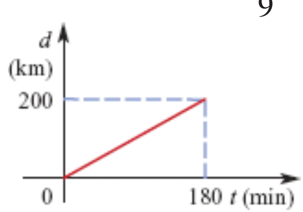
f



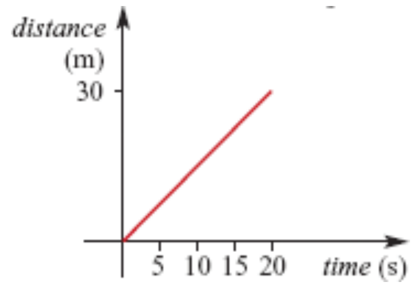
2 a Constant speed $= \dfrac{200}{3}$ km/h

$= \dfrac{200}{180}$ km/min

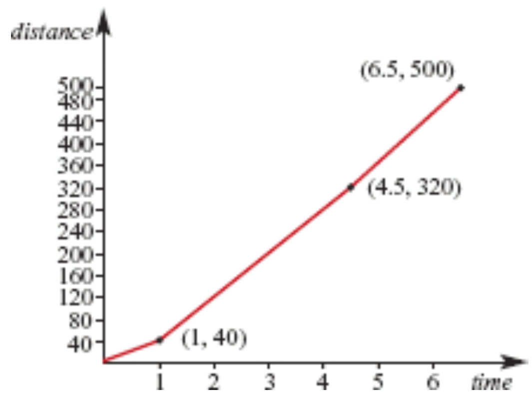
$= \dfrac{10}{9}$ km/min



b



c



3 $s = 6x^2$

Av. rate of change from $x = 2$ to $x = 4$:

$$\frac{s(4) - s(2)}{4 - 2} = \frac{6(16 - 4)}{2}$$
$$= 36 \text{ cm}^2/\text{cm}$$

4 $y = x^3$

a Av. rate over $[0, 1]$: $\dfrac{1^3 - 0^3}{1 - 0} = 1$

b Av. rate over $[1, 3]$: $\dfrac{3^3 - 1^3}{3 - 1} = 13$

5 $s(t) = 4t - 6t^3$

a Av. v over $[0, 1]$: $\dfrac{-2 - 0}{1 - 0} = -2$

b av v over $[0.9, 1]$: $\dfrac{-2 + 0.774}{1 - 0.9} =$ -12.26

c Smaller intervals suggest a good estimate of the instantaneous velocity for $t = 1 = -14$m/s

Solutions to multiple-choice questions

1 C Av. $v = \dfrac{12 + 0 + 8}{2 + 0.75 + 1.25} = \dfrac{20}{4}$

$= 5$ km/h

2 B Av. rate $= \dfrac{12000 + 2500}{8 + 2}$

$= 1450$ letters/hour

3 D OA oblique line ∴ constant speed

4 E AB = horizontal line ∴ stationary

5 D Horizontal lines at AB and DE

6 B Av. rate of change of $y = 3(2^x)$ over $[0, 2]$:

$\dfrac{3(2^2 - 2^0)}{2 - 0} = \dfrac{9}{2} = 4.5$

7 C $P = 10(1.1^t)$

Av. rate of growth in 5th week:

$\dfrac{P(5) - P(4)}{5 - 4} = 10(1.1^5 - 1.1^4)$

$= 1.5$

8 E $f(x) = 2x^3 + 3x$

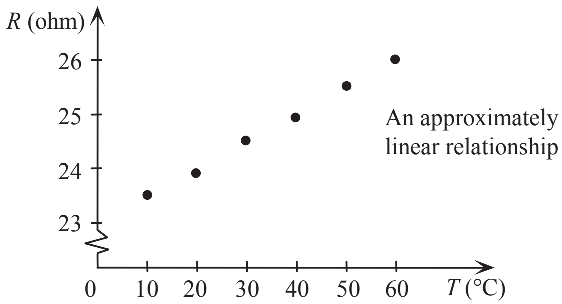
Av. rate over $[-2, 2]$:

$\dfrac{f(2) - f(-2)}{2 - (-2)} = \dfrac{22 - (-22)}{4}$

$= \dfrac{44}{4} = 11$

Solutions to extended-response questions

1 a



b Rate of increase $= \dfrac{\text{change in resistance}}{\text{change in temperature}}$

$$= \frac{26 - 23.4}{60 - 10} = 0.05\text{ohm/°C}$$

2 a i When $t = 0$, $y = 4.9(0)^2 = 0$

When $t = 2$, $y = 4.9(2)^2 = 19.6$

Average speed between $t = 0$ and $t = 2$ is

$$\frac{19.6 - 0}{2 - 0} = 9.8 \text{ m/s}$$

ii When $t = 4$, $y = 4.9(4)^2 = 78.4$

Average speed between $t = 2$ and $t = 4$ is

$$\frac{78.4 - 19.6}{4 - 2} = 29.4 \text{ m/s}$$

b i When $t = 4 - h$,

$$\begin{aligned} y &= 4.9(4 - h)^2 \\ &= 4.9(16 - 8h + h^2) \\ &= 78.4 - 39.2h + 4.9h^2 \end{aligned}$$

Distance rock has fallen between $t = 4 - h$ and $t = 4$

$$\begin{aligned} &= 78.4 - (78.4 - 39.2h + 4.9h^2) \\ &= 39.2h - 4.9h^2 \\ &= 4.9h(8 - h) \end{aligned}$$

ii Average speed $= \dfrac{\text{distane}}{\text{time}}$

$$\begin{aligned} &= \frac{4.9h(8 - h)}{4 - (4 - h)} \\ &= \frac{4.9h(8 - h)}{h} \\ &= 4.9(8 - h) \end{aligned}$$

iii When $h = 0.2$, average speed $= 4.9(8 - 0.2) = 38.22$

When $h = 0.1$, average speed $= 4.9(8 - 0.1) = 38.71$

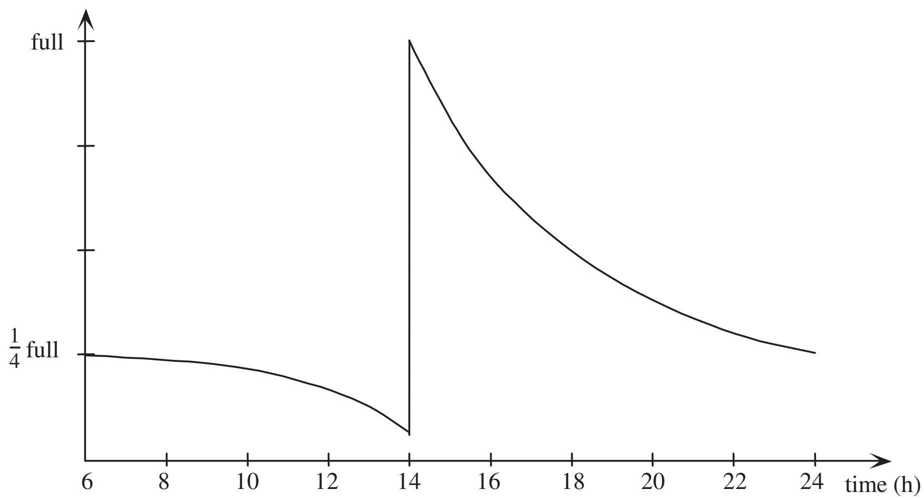
When $h = 0.05$, average speed $= 4.9(8 - 0.05) = 38.955$

When $h = 0.01$, average speed $= 4.9(8 - 0.01) = 39.151$

When $h = 0.001$, average speed $= 4.9(8 - 0.001) = 39.1951$

Hence, the speed of impact will be 39.2 m/s.

3 From 6 am to 2 pm business should gradually improve, hence a negative gradient getting steeper. Fill up at 2 pm. From 2 pm to midnight business should gradually decrease, hence a negative gradient getting less steep. At midnight the machine has slightly more cans than at 6 am next morning.



4 a

$$\text{Gradient of } PQ = \frac{b^2 - a^2}{b - a}$$
$$= \frac{(b - a)(b + a)}{b - a}$$
$$= b + a \quad \text{for } a \neq b$$

b When $a = 1$, $b = 2$,

$$\text{gradient of } PQ = 2 + 1$$
$$= 3$$

c When $a = 2$, $b = 2.01$,

$$\text{gradient of } PQ = 2.01 + 2$$
$$= 4.01$$

5 **a** When $x = 1.5$,

$$\begin{aligned} y &= \frac{4}{1.5} \\ &= \frac{8}{3} \\ &= 2\frac{2}{3} \end{aligned}$$

When $x = 2.5$,

$$\begin{aligned} y &= \frac{4}{2.5} \\ &= \frac{8}{5} \\ &= 1\frac{3}{5} \end{aligned}$$

Coordinates of $A_1 = \left(1.5, \frac{8}{3}\right)$ and coordinates of $A_2 = \left(2.5, \frac{8}{5}\right)$

$$\begin{aligned} \therefore \qquad \text{gradient of } A_1A_2 &= \frac{\frac{8}{5} - \frac{8}{3}}{2.5 - 1.5} \\ &= \frac{8 \times 3 - 8 \times 5}{15} \\ &= -\frac{16}{15} \\ &= -1\frac{1}{15} \\ &\approx -1.07 \end{aligned}$$

b When $x = 1.9$,

$$\begin{aligned} y &= \frac{4}{1.9} \\ &= \frac{40}{19} \approx 2.1053 \end{aligned}$$

When $x = 2.1$,

$$\begin{aligned} y &= \frac{4}{2.1} \\ &= \frac{40}{21} \approx 1.9048 \end{aligned}$$

Coordinates of $B_1 = \left(1.9, \frac{40}{19}\right)$ and coordinates of $B_2 = \left(2.1, \frac{40}{21}\right)$

$$\begin{aligned} \therefore \qquad \text{gradient of } B_1B_2 &= \frac{\frac{40}{21} - \frac{40}{19}}{2.1 - 1.9} \\ &= -\frac{400}{399} \\ &= -1\frac{1}{399} \\ &\approx -1.003 \end{aligned}$$

c When $x = 1.99$, $\quad y = \dfrac{4}{1.99}$

$$= \frac{400}{199}$$

When $x = 2.01$, $\quad y = \dfrac{4}{2.01}$

$$= \frac{400}{201}$$

Coordinates of $C_1 = \left(1.99, \dfrac{400}{199}\right)$ and coordinates of $C_2 = \left(2.01, \dfrac{400}{201}\right)$

$\therefore$ gradient of $C_1C_2 = \dfrac{\frac{400}{201} - \frac{400}{199}}{2.01 - 1.99}$

$$= -\frac{40\,000}{39\,999}$$

$$= -1\frac{1}{39\,999}$$

$$\approx -1.000\,025$$

d When $x = 1.999$, $\quad y = \dfrac{4}{1.999}$

$$= \frac{4000}{1999}$$

When $x = 2.001$, $\quad y = \dfrac{4}{2.001}$

$$= \frac{4000}{2001}$$

Coordinates of $D_1 = \left(1.999, \dfrac{4000}{1999}\right)$ and coordinates of $D_2 = \left(2.001, \dfrac{4000}{2001}\right)$

$\therefore$ gradient of $D_1D_2 = \dfrac{\frac{4000}{2001} - \frac{4000}{1999}}{2.001 - 1.999}$

$$= -\frac{4\,000\,000}{3\,999\,999}$$

$$= -1\frac{1}{3\,999\,999}$$

$$\approx -1.000\,000\,3$$

6 Let t = age (in years), let x = Andrea's weight (kg), let y = Clive's weight (kg).

a When $t = 0$, $x \approx 2$ and when $t = 18$, $x \approx 62$.

Average rate of change for Andrea $\approx \dfrac{62 - 2}{18 - 0}$

$$\approx \frac{60}{18}\text{kg/year} \approx 3\frac{1}{3}\text{kg/year}$$

b When $t = 0$, $y \approx 2$ and when $t = 18$, $y \approx 81$.

$$\text{Average rate of change for Clive} \approx \frac{81 - 2}{18 - 0}$$
$$\approx \frac{79}{18}\text{kg/year} \approx 4.4\text{kg/year}$$

c Andrea weighed more than Clive between the ages of 0 and 5 and between the ages of 10 and 12, i.e. $\{t: 0 < t < 5\} \cup \{t: 10 < t < 12\}$.

d Clive was growing more rapidly than Andrea between the ages of 5 and 7 and between the ages of ll and $17\frac{1}{2}$, i.e. $\left\{t: 5 < t < 7\right\} \cup \left\{t: 11 < t < 17\frac{1}{2}\right\}$.

7 **a** **i** 0.24 billion

ii 0.52 billion

b
$$\text{Average annual population increase} = \frac{0.52 - 0.24}{2000 - 1960}$$
$$= \frac{0.28}{40}$$
$$= \frac{7}{1000}$$
$$= 0.007 \text{ billion/year}$$

c **i** Draw a tangent to the curve at 1960. Select 2 points on the tangent, e.g. (1950, 0.2) and (1960, 0.24).

$$\text{Rate of population increase} = \frac{0.24 - 0.2}{1960 - 1950}$$
$$= \frac{4}{1000}$$
$$= 0.004 \text{ billion/year}$$

ii Draw a tangent to the curve at 2000. Select 2 points on the tangent, e.g. (1990, 0.38) and (2000, 0.52).

$$\text{Rate of population increase} = \frac{0.52 - 0.38}{2000 - 1990}$$
$$= \frac{14}{1000}$$
$$= 0.014 \text{ billion/year}$$

d If the curve is continued with ever-increasing gradient, an estimation of how long it will take to double the 2020 population is 25 years, i.e. in 2045.

8 When $x = 2.5$, $y = 10^{2.5}$

≈ 316.23

When $x = 2.6$, $y = 10^{2.6}$

≈ 398.11

When $x = 2.7$, $y = 10^{2.7}$

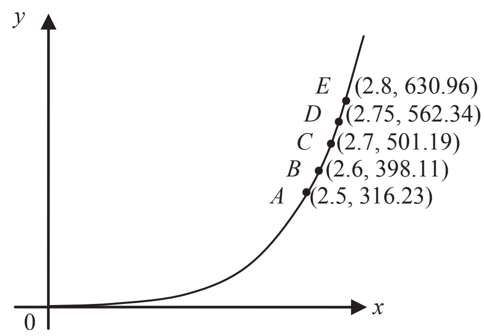
≈ 501.19

When $x = 2.75$, $y = 10^{2.75}$

≈ 562.34

When $x = 2.8$, $y = 10^{2.8}$

≈ 630.9573445



a **i** Gradient of $AE = \dfrac{630.96 - 316.23}{2.8 - 2.5} \approx 1049.1$

ii Gradient of $BE = \dfrac{630.96 - 398.11}{2.8 - 2.6} \approx 1164.3$

iii Gradient of $CE = \dfrac{630.96 - 501.19}{2.8 - 2.7} \approx 1297.7$

iv Gradient of $DE = \dfrac{630.96 - 562.34}{2.8 - 2.75} \approx 1372.4$

b The gradients are approaching the gradient of the curve at $x = 2.8$ as the intervals are made smaller.

When $x = 2.79$, $y = 10^{2.79} \approx 616.595$

$$\text{Gradient} = \frac{630.957 - 616.595}{2.8 - 2.79} \approx 1436.2$$

When $x = 2.799$, $y = 10^{2.799} \approx 629.5062$

$$\text{Gradient} = \frac{630.9573 - 629.5062}{2.8 - 2.799} \approx 1451.2$$

When $x = 2.7999$, $y = 10^{2.7999} \approx 630.812\,08$

$$\text{Gradient} = \frac{630.957\,34 - 630.812\,08}{2.8 - 2.7999} \approx 1452.7$$

When $x = 2.799\,99$, $y = 10^{2.799\,99} \approx 630.942\,816\,3$

$$\text{Gradient} = \frac{630.957\,3.445 - 630.942\,816\,3}{28 - 2.799\,99} \approx 1452.8$$

When $x = 2.799\,999$, $y = 10^{2.799\,999} \approx 630.955\,891\,6$

$$\text{Gradient} = \frac{630.957\,344\,5 - 630.955\,891\,6}{2.8 - 2.799\,999} \approx 1452.8$$

When $x = 2.799\,999\,9$, $y = 10^{2.799\,999\,9} \approx 630.957\,199\,2$

$$\text{Gradient} = \frac{630.957\,344\,5 - 630.957\,199\,2}{2.8 - 2.799\,999\,9} \approx 1452.8$$

When $x = 2.799\,999\,99$, $y = 10^{2.799\,999\,99} \approx 630.957\,33$

$$\text{Gradient} = \frac{630.957\,344\,5 - 630.957\,33}{2.8 - 2.799\,999\,99} \approx 1452.8$$

Hence the gradient of the curve at $x = 2.8$ has been shown to be 1452.8.

9 **a** Gradient of $QP = \dfrac{a^3 - b^3}{a - b}$

$$= \frac{(a-b)(a^2 + ab + b^2)}{a - b}$$

$$= a^2 + ab + b^2 \quad \text{for } a \neq b$$

b When $a = 1,\ b = 2$

$$\begin{aligned} \text{gradient} &= 1^2 + 1 \times 2 + 2^2 \\ &= 1 + 2 + 4 \\ &= 7 \end{aligned}$$

c When $a = 2,\ b = 2.01$

$$\begin{aligned} \text{gradient} &= 2^2 + 2 \times 2.01 + 2.01^2 \\ &= 4 + 4.02 + 4.0401 \\ &= 12.0601 \end{aligned}$$

d $\text{gradient} = a^2 + ab + b^2$

If $a = b$, then

$$\begin{aligned} \text{gradient} &= b^2 + b \times b + b^2 \\ &= 3b^2 \end{aligned}$$

At the point with coordinates $(b,\ b^3)$ the gradient is $3b^2$.

10 **a** B wins the race.

b A is in front at the 50 metre mark.

c From the graph, approximately 25 m separates 1st and 3rd placegetters when 1st finishes the race.

d From the graph, approximately 45 s separates 1st and 3rd finishing times.

e From the graph, average speed of $A \approx \dfrac{100 - 0}{102 - 0} \approx 0.980\text{m/s}$

From the graph, average speed of $B \approx \dfrac{100 - 0}{58 - 0} \approx 1.724\text{m/s}$

From the graph, average speed of $C \approx \dfrac{100-0}{88-0} \approx 1.136$m/s

f A got a fine start, for an early lead, with B second and C trailing third. A started strongly; perhaps too strongly because his pace is slowing, allowing B and C to gain ground. B and C are swimming consistently, maintaining a constant speed, although B is faster and increasing the gap. At the 70 metre mark now, and A is tiring visibly as B powers past him. A has his head down and is swimming much more consistently but his early sprint has cost him dearly. The crowd is cheering wildly as B wins this race very comfortably, with A still 25 m to go and C a further 10 m behind him. The excitement is building further as C closes the gap on A and with 15 m to go surges past him to finish in second place. A finishes third and would be most disappointed with this result. I'd say he has a lot of promise and if he can get his timing right, he'll be a serious contender against B in the next competition.

11 a The graph of $y = f(x) + c$ is obtained from the graph of $y = f(x)$ by a translation of c units in the positive direction of the y-axis. Hence the average rate of change of $y = f(x) + c$ is m for the interval $[a, b]$.
The computation is:

$$\begin{aligned}\text{average rate of change} &= \frac{f(b)+c-(f(a)+c)}{b-a}\\ &= \frac{f(b)-f(a)}{b-a}\\ &= m\end{aligned}$$

b The graph of $y = cf(x)$ is obtained from the graph of $y = f(x)$ by a dilation of c units from the x-axis. Hence the average rate of change of $y = cf(x)$ is cm for the interval $[a,\ b]$.
The computation is:

$$\begin{aligned}\text{average rate of change} &= \frac{cf(b)-cf(a)}{b-a}\\ &= c \times \frac{f(b)-cf(a)}{b-a}\\ &= cm\end{aligned}$$

c The graph of $y = -f(x)$ is obtained from the graph of $y = f(x)$ by a reflection in the x-axis. Hence the average rate of change of $y = -f(x)$ is $-m$ for the interval $[a, b]$.

The computation is:

$$\begin{aligned}
\text{average rate of change} &= \frac{-f(b) - (-f(a))}{b - a} \\
&= -1 \times \frac{f(b) - f(a)}{b - a} \\
&= -m
\end{aligned}$$

Chapter 17 – Differentiation of polynomials

Solutions to Exercise 17A

1 **a**
$$\text{Gradient} = \frac{-(3+h)^2 + 4(3+h) - 3}{3+h-3}$$
$$= \frac{-(9+6h+h^2) + 12 + 4h - 3}{h}$$
$$= \frac{-9 - 6h - h^2 + 12 + 4h - 3}{h}$$
$$= \frac{-2h - h^2}{h}$$
$$= -2 - h$$

b $\lim\limits_{h \to 0}(-2 - h) = -2$

2 **a**
$$\text{Gradient} = \frac{(4+h)^2 - 3(4+h) - 4}{4+h-4}$$
$$= \frac{16 + 8h + h^2 - 12 - 3h - 4}{h}$$
$$= \frac{5h + h^2}{h}$$
$$= 5 + h$$

b $\lim\limits_{h \to 0}(5 + h) = 5$

3 Gradient
$$= \frac{(x+h)^2 - 2(x+h) - (x^2 - 2x)}{x+h-x}$$
$$= \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h}$$
$$= \frac{2xh + h^2 - 2h}{h}$$
$$= 2x + h - 2$$
$\lim\limits_{h \to 0}(2x + h - 2) = 2x - 2$

4
$$\text{Gradient} = \frac{(2+h)^4 - 16}{2+h-2}$$
$$= \frac{16 + 32h + 24h^2 + h^4 - 16}{h}$$
$$= \frac{32h + 24h^2 + h^4}{h}$$
$$= 32 + 24h + h^3$$
$\lim\limits_{h \to 0}(32 + 24h + h^3) = 32$

5 $y = 4t^4$
Chord between $t = 4$ and $t = 5$ has gradient $\dfrac{4(5^4 - 4^4)}{5-4} = 2244$ (around 2000 m/s)

6 $P = 1000 + t^2 + t,\ t > 0$
$$P(3+h) - P(3) = (3+h)^2 - 9 + (3+h) - 3$$
$$= 6h + h^2 + h$$
$$= 7h + h^2$$
$$\text{Chord gradient} = \frac{7h + h^2}{3+h-3}$$
$$= 7 + h$$
Growth rate at $t = 3$ is 7 insects/day

7 **a** $\lim\limits_{h \to 0} \dfrac{2x^2h^3 + xh^2 + h}{h}$
$= \lim\limits_{h \to 0} 2x^2h^2 + xh + 1 = 1$

b $\lim\limits_{h \to 0} \dfrac{3x^2h - 2xh^2 + h}{h}$
$= \lim\limits_{h \to 0} 3x^2 - 2xh + 1 = 3x^2 + 1$

c $\lim\limits_{h \to 0} 20 - 10h = 20$

d $\lim_{h\to 0} \dfrac{30hx^2 + 2h^2 + h}{h}$
$= \lim_{h\to 0} 30x^2 + 2h + 1 = 30x^2 + 1$

e $\lim_{h\to 0} 5 = 5$

f $\lim_{h\to 0} \dfrac{30hx^3 + 2h^2 + 4h}{h} =$
$\lim_{h\to 0} 30x^3 + 2h + 4 = 30x^3 + 4$

8 **a** $\lim_{h\to 0} \dfrac{(x+h)^2 + 2(x+h) - (x^2 + 2x)}{h}$
$= \lim_{h\to 0} \dfrac{2hx + h^2 + 2h}{h} = 2x + 2$

b $\lim_{h\to 0} \dfrac{(5+h)^2 + 3(5+h) - 40}{h}$
$= \lim_{h\to 0} \dfrac{10h + h^2 + 3h}{h}$
$= \lim_{h\to 0} 13 + h = 13$

c $\lim_{h\to 0} \dfrac{(x+h)^3 + 2(x+h)^2 - (x^3 + 2x^2)}{h}$
$= \lim_{h\to 0} \dfrac{3x^2h + 3xh^2 + h^3 + 4xh + 2h^2}{h}$
$= \lim_{h\to 0} 3x^2 + 3xh + h^2 + 4x + 2h$
$= 3x^2 + 4x$

9 $y = 3x^2 - x$

a Gradient of chord PQ:

$$= \frac{3(1+h)^2 - (1+h) - 2}{1 + h - 1}$$
$$= \frac{3(1 + 2h + h^2) - 1 - h - 2}{h}$$
$$= \frac{6h + 3h^2 - h}{h} = 5 + 3h$$

b Gradient of PQ when $h = 0.1$ is 5.3

c Gradient of the curve at $P = 5$

10 $y = \dfrac{2}{x}$

a Gradient of chord AB:

$$= \frac{\frac{2}{2+h} - 1}{2 + h - 2}$$
$$= \frac{2 - (2+h)}{h(2+h)}$$
$$= \frac{-h}{h(2+h)} = \frac{-1}{2+h}$$

b Gradient of AB when $h = 0.1 \cong -0.48$

c Gradient of the curve at $A = -\dfrac{1}{2}$

11 $y = x^2 + 2x - 3$

a Gradient of chord PQ:

$$= \frac{(2+h)^2 + 2(2+h) - 3 - 5}{2 + h - 2}$$
$$= \frac{4 + 4h + h^2 + 4 + 2h - 8}{h}$$
$$= \frac{6h + h^2}{h} = 6 + h$$

b Gradient of PQ when $h = 0.1$ is 6.1

c Gradient of the curve at $P = 6$

12 Derivatives from first principles

a $\lim_{h\to 0} \dfrac{3(x+h)^2 - 3x^2}{h}$
$= \lim_{h\to 0} \dfrac{3x^2 + 6xh + 3h^2 - 3x^2}{h}$
$= \lim_{h\to 0} \dfrac{6xh + 3h^2}{h}$
$= \lim_{h\to 0} 6x + 3h = 6x$

b $\lim_{h\to 0} \dfrac{4(x+h) - 4x}{h}$
$= \lim_{h\to 0} \dfrac{4h}{h} = 4$

c $\lim_{h\to 0} \frac{3-3}{h} = 0$

d

$$\lim_{h\to 0} \frac{3(x+h)^2 + 4(x+h) - 3 - 3x^2 - 4x + 3}{h}$$
$$= \lim_{h\to 0} \frac{6hx + 3h^2 + 4h}{h}$$
$$= \lim_{h\to 0} 6x + 3h + 4 = 6x + 4$$

e $\lim_{h\to 0} \frac{2(x+h)^3 - 4 - 2x^3 + 4}{h}$

$$= \lim_{h\to 0} \frac{6x^2h + 6xh^2 + 2h^3}{h}$$
$$= \lim_{h\to 0} 6x^2 + 6xh + 2h^2 = 6x^2$$

f $\lim_{h\to 0} \frac{4(x+h)^2 - 5(x+h) - 4x^2 + 5x}{h}$

$$= \lim_{h\to 0} \frac{4x^2 + 8hx + 4h^2 - 5x - 5h - 4x^2 + 5x}{h}$$
$$= \lim_{h\to 0} \frac{8hx + 4h^2 - 5h}{h}$$
$$= \lim_{h\to 0} 8x + 4h - 5 = 8x - 5$$

g

$$\lim_{h\to 0} \frac{3 - 2(x+h) + (x+h)^2 - 3 + 2x - x^2}{h}$$
$$= \lim_{h\to 0} \frac{-2x - 2h + x^2 + 2hx + h^2 + 2x - x^2}{h}$$
$$= \lim_{h\to 0} \frac{-2h + 2hx + h^2}{h}$$
$$= \lim_{h\to 0} -2 + 2x + h = 2x - 2$$

13 Gradient

$$= \frac{(x+h)^4 - x^4}{x + h - x}$$
$$= \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$
$$= \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$
$$= 4x^3 + 6x^2h + 4xh^2 + h^3$$
$$\lim_{h\to 0}(4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3$$

Solutions to Exercise 17B

1 Derivatives using $\frac{d}{dx}x^n = nx^{n-1}$

a $\frac{d}{dx}(x^2 + 4x) = 2x + 4$

b $\frac{d}{dx}(2x + 1) = 2$

c $\frac{d}{dx}(x^3 - x) = 3x^2 - 1$

d $\frac{d}{dx}\left(\frac{1}{2}x^2 - 3x + 4\right) = x - 3$

e $\frac{d}{dx}(5x^3 + 3x^2) = 15x^2 + 6x$

f $\frac{d}{dx}(-x^3 + 2x^2) = -3x^2 + 4x$

2 **a** $f(x) = x^{12}, \therefore f'(x) = 12x^{11}$

b $f(x) = 3x^7, \therefore f'(x) = 21x^6$

c $f(x) = 5x, \therefore f'(x) = 5$

d $f(x) = 5x + 3, \therefore f'(x) = 5$

e $f(x) = 3, \therefore f'(x) = 0$

f $f(x) = 5x^2 - 3x, \therefore f'(x) = 10x - 3$

g $f(x) = 10x^5 + 3x^4,$
$\therefore f'(x) = 50x^4 + 12x^3$

h $f(x) = 2x^4 - \frac{1}{3}x^3 - \frac{1}{4}x^2 + 2$
$\therefore f'(x) = 8x^3 - x^2 - \frac{1}{2}x$

3 **a** $f(x) = x^6, \therefore f'(x) = 6x^5, \therefore f'(1) = 6$

b $f(x) = 4x^5, \therefore f'(x) = 20x^4,$
$\therefore f'(1) = 20$

c $f(x) = 5x, \therefore f'(x) = 5, \therefore f'(1) = 5$

d $f(x) = 5x^2 + 3, \therefore f'(x) = 10x,$
$\therefore f'(1) = 10$

e $f(x) = 3, \therefore f'(x) = 0, \therefore f'(1) = 0$

f $f(x) = 5x^2 - 3x, \therefore f'(x) = 10x - 3,$
$\therefore f'(1) = 7$

g $f(x) = 10x^4 - 3x^3,$
$\therefore f'(x) = 40x^3 - 9x^2, \therefore f'(1) = 31$

h $f(x) = 2x^4 - \frac{1}{3}x^3, \therefore f'(x) = 8x^3 - x^2,$
$\therefore f'(1) = 7$

i $f(x) = -10x^3 - 2x^2 + 2,$
$\therefore f'(x) = -30x^2 - 4x, \therefore f'(1) = -34$

4 **a** $f(x) = 5x^3, \therefore f'(x) = 15x^2,$
$\therefore f'(-2) = 60$

b $f(x) = 4x^2, \therefore f'(x) = 8x,$
$\therefore f'(-2) = -16$

c $f(x) = 5x^3 - 3x, \therefore f'(x) = 15x^2 - 3,$
$\therefore f'(-2) = 57$

d $f(x) = -5x^4 - 2x^2,$
$\therefore f'(x) = -20x^3 - 4x,$
$\therefore f'(-2) = 168$

5 **a** $f(x) = x^2 + 3x, \therefore f'(x) = 2x + 3,$
$\therefore f'(2) = 7$

b $f(x) = 3x^2 - 4x, \therefore f'(x) = 6x - 4,$
$\therefore f'(1) = 2$

c $f(x) = -2x^2 - 4x, \therefore f'(x) = -4x - 4,$
$\therefore f'(3) = -16$

d $f(x) = x^3 - x, \therefore f'(x) = 3x^2 - 1,$
$\therefore f'(2) = 11$

6 a $y = -x, \therefore \dfrac{dy}{dx} = -1$

b $y = 10, \therefore \dfrac{dy}{dx} = 0$

c $y = 4x^3 - 3x + 2, \therefore \dfrac{dy}{dx} = 12x^2 - 3$

d $y = \dfrac{1}{3}(x^3 - 3x + 6)$
$= \dfrac{1}{3}x^3 - x + 2$
$\therefore \dfrac{dy}{dx} = x^2 - 1$

e $y = (x + 1)(x + 2)$
$= x^2 + 3x + 2$
$\therefore \dfrac{dy}{dx} = 2x + 3$

f $y = 2x(3x^2 - 4)$
$= 6x^3 - 8x$
$\therefore \dfrac{dy}{dx} = 18x^2 - 8$

g $y = \dfrac{10x^5 + 3x^4}{2x^2}$
$= 5x^3 + \dfrac{3}{2}x^2, \ x \neq 0$
$\therefore \dfrac{dy}{dx} = 15x^2 + 3x$

7 a $y = (x + 4)^2 = x^2 + 8x + 16$
$\dfrac{dy}{dx} = 2x + 8$

b $z = (4t - 1)^2(t + 1)$
$= (16t^2 - 8t + 1)(t + 1)$
$= 16t^3 - 8t^2 + t + 16t^2 - 8t + 1$
$= 16t^3 + 8t^2 - 7t + 1$
$\therefore \dfrac{dz}{dt} = 48t^2 + 16t - 7$

c $\dfrac{x^3 + 3x}{x} = x^2 + 3 \therefore \dfrac{dy}{dx} = 2x$

8 a $y = x^3 + 1, \therefore \dfrac{dy}{dx} = 3x^2$

i Gradient at $(1, 2) = 3$

ii Gradient at $(a, a^3 + 1) = 3a^2$

b Derivative $= 3x^2$

9 a $y = x^3 - 3x^2 + 3x$
$\therefore \dfrac{dy}{dx} = 3x^2 - 6x + 3$
$= 3(x + 1)^2 \geq 0$
The graph of $y = x^3 - 3x^2 + 3x$ will have a positive gradient for all x, except for a saddle point at $x = -1$ where the gradient = 0.

b $y = \dfrac{x^2 + 2x}{x} = x + 2, \ x \neq 0$
$\therefore \dfrac{dy}{dx} = 1, \ x \neq 0$

c $y = (3x + 1)^2 = 9x^2 + 6x + 1$
$\therefore \dfrac{dy}{dx} = 18x + 6 = 6(3x + 1)$

10 a $y = x^2 - 2x + 1, \therefore \dfrac{dy}{dx} = 2x - 2$
$\therefore y(2) = 1, \ y'(2) = 2$

b $y = x^2 + x + 1, \therefore \dfrac{dy}{dx} = 2x + 1$
$\therefore y(0) = 1, \ y'(0) = 1$

c $y = x^2 - 2x, \therefore \dfrac{dy}{dx} = 2x - 2$
$\therefore y(-1) = 3, \ y'(-1) = -4$

d $y = (x + 2)(x - 4) = x^2 - 2x - 8$

$\therefore \dfrac{dy}{dx} = 2x - 2$
$\therefore y(3) = -5,\ y'(3) = 4$

e $y = 3x^2 - 2x^3,\ \therefore \dfrac{dy}{dx} = 6x - 6x^2$
$\therefore y(-2) = 28,\ y'(-2) = -36$

f $y = (4x - 5)^2 = 16x^2 - 40x + 25$
$\therefore \dfrac{dy}{dx} = 32x - 40 = 8(4x - 5)$
$\therefore y\left(\dfrac{1}{2}\right) = 9, y'\left(\dfrac{1}{2}\right) = -24$

11 a i $f(x) = 2x^2 - x,\ \therefore f'(x) = 4x - 1$
$\therefore f'(1) = 3$
Gradient = 1 when $4x - 1 = 1$
$\therefore x = \dfrac{1}{2}$ and $f\left(\dfrac{1}{2}\right) = 0$
Gradient = 1 at $\left(\dfrac{1}{2}, 0\right)$

ii $f(x) = 1 + \dfrac{1}{2}x + \dfrac{1}{3}x^2$
$\therefore f'(x) = \dfrac{2}{3}x + \dfrac{1}{2},\ \therefore f'(1) = \dfrac{7}{6}$
Gradient = 1 when $\dfrac{2}{3}x + \dfrac{1}{2} = 1$
$\therefore x = \dfrac{1}{2}\left(\dfrac{3}{2}\right) = \dfrac{3}{4}$ and $f\left(\dfrac{3}{4}\right) = \dfrac{25}{16}$
Gradient = 1 at $\left(\dfrac{3}{4}, \dfrac{25}{16}\right)$

iii $f(x) = x^3 + x, \therefore f'(x) = 3x^2 + 1$
$\therefore f'(1) = 4$
Gradient = 1 when $3x^2 + 1 = 1$
$\therefore x = 0$ and $f(0) = 0$
Gradient = 1 at $(0, 0)$

iv $f(x) = x^4 - 31x,$
$\therefore f'(x) = 4x^3 - 31$
$\therefore f'(1) = -27$
Gradient = 1 when $4x^3 - 31 = 1$
$\therefore 4x^3 = 32$
$\therefore x = 2$ and $f(2) = -46$
Gradient = 1 at $(2, -46)$

b Points where the gradients equal 1 are where a tangent makes an angle of 45° to the axes.

12 a $\dfrac{d}{dt}(3t^2 - 4t) = 6t - 4$

b $\dfrac{d}{dx}(4 - x^2 + x^3) = -2x + 3x^2$

c $\dfrac{d}{dz}(5 - 2z^2 - z^4) = -4z - 4z^3$
$= -4z(z^2 + 1)$

d $\dfrac{d}{dy}(3y^2 - y^3) = 6y - 3y^2$
$= 3y(2 - y)$

e $\dfrac{d}{dx}(2x^3 - 4x^2) = 6x^2 - 8x$
$= 2x(3x - 4)$

f $\dfrac{d}{dt}(9.8t^2 - 2t) = 19.6t - 2$

13 a $y = x^2,\ \therefore \dfrac{dy}{dx} = 2x$
Gradient = 8 at $(4, 16)$

b $y = x^3,\ \therefore \dfrac{dy}{dx} = 3x^2 = 12,\ \therefore x = \pm 2$
Gradient = 12 at $(-2, -8), (2, 8)$

c $y = x(2 - x) = 2x - x^2,\ \therefore \dfrac{dy}{dx} = 2 - 2x$
Gradient = 2 where $x = 0$, i.e. at $(0, 0)$

d $y = x^2 - 3x + 1,\ \therefore \dfrac{dy}{dx} = 2x - 3$
Gradient = 0 where
$x = \dfrac{3}{2}$, i.e. at $\left(\dfrac{3}{2}, -\dfrac{5}{4}\right)$

e $y = x^3 - 6x^2 + 4,\ \dfrac{dy}{dx} = 3x^2 - 12x$
Gradient = -12 where

$3x^2 - 12x + 12 = 0$

$\therefore x^2 - 4x + 4 = 0$

$(x - 2)^2 = 0, \ \therefore x = 2$

i.e. at $(2, -12)$

f $y = x^2 - x^3 \ \therefore \dfrac{dy}{dx} = 2x - 3x^2$

Gradient $= -1$ where

$-3x^2 + 2x + 1 = 0$

$\therefore 3x^2 - 2x - 1 = 0$

$(3x + 1)(x - 1) = 0$

$x = -\dfrac{1}{3}, 1$

i.e. at $\left(-\dfrac{1}{3}, \dfrac{4}{27}\right)$ and $(1, 0)$

Solutions to Exercise 17C

1 a

$$\frac{f(x+h)-f(x)}{x+h-x} = \frac{\dfrac{1}{x+h-3} - \dfrac{1}{x-3}}{x+h-x}$$

$$= \frac{\dfrac{(x-3-(x+h-3)}{x+h-3)(x-3)}}{h}$$

$$= \frac{-h}{(x+h-3)(x-3)} \times \frac{1}{h}$$

$$= \frac{-1}{(x+h-3)(x-3)}$$

$$\lim_{h\to 0} \frac{-1}{(x+h-3)(x-3)} = -\frac{1}{(x-3)^2}$$

b

$$\frac{f(x+h)-f(x)}{x+h-x} = \frac{\dfrac{1}{x+h+2} - \dfrac{1}{x+2}}{x+h-x}$$

$$= \frac{\dfrac{(x+2-(x+h+2)}{x+h+2)(x+2)}}{h}$$

$$= \frac{-h}{(x+h+2)(x+2)} \times \frac{1}{h}$$

$$= \frac{-1}{(x+h+2)(x+2)}$$

$$\lim_{h\to 0} \frac{-1}{(x+h+2)(x+2)} = -\frac{1}{(x+2)^2}$$

2 a

$$\frac{f(x+h)-f(x)}{x+h-x} = \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{x+h-x}$$

$$= \frac{\dfrac{x^2-(x+h)^2}{x+h)^2x^2}}{h}$$

$$= \frac{x^2-x^2-2xh-h^2}{(x+h)^2x^2} \times \frac{1}{h}$$

$$= \frac{-2xh-h^2}{(x+h)^2x^2} \times \frac{1}{h}$$

$$= \frac{-2x-h}{(x+h)^2x^2}$$

$$\lim_{h\to 0} \frac{-2x-h}{(x+h)^2x^2} = -\frac{2}{x^3}$$

b

$$\frac{f(x+h)-f(x)}{x+h-x} = \frac{\dfrac{1}{(x+h)^4} - \dfrac{1}{x^4}}{x+h-x}$$

$$= \frac{\dfrac{x^2-(x+h)^4}{x+h)^2x^2}}{h}$$

$$= \frac{x^4-(x^4+4x^3h+6x^2h^2+4xh^3+h^4}{(x+h)^2x^2}$$

$$= \frac{-(4x^3h+6x^2h^2+4xh^3+h^4)}{(x+h)^4x^4} \times \frac{1}{h}$$

$$= \frac{-(4x^3+6xh+4xh^2+h^3)}{(x+h)^4x^4}$$

$$\lim_{h\to 0} \frac{-(4x^3+6xh+4xh^2+h^3)}{(x+h)^4x^4} = -\frac{4x^3}{x^8}$$

$$= -\frac{4}{x^5}$$

3 a $\dfrac{d}{dx}(3x^{-2}+5x^{-1}+6) = -6x^{-3}-5x^{-2}$

b $\dfrac{d}{dx}\left(\dfrac{3}{x^2}+5x^2\right) = -\dfrac{6}{x^3}+10x$

c $\dfrac{d}{dx}\left(\dfrac{5}{x^3}+\dfrac{4}{x^2}+1\right) = -\dfrac{15}{x^4}-\dfrac{8}{x^3}$

d $\dfrac{d}{dx}\left(3x^2+\dfrac{5}{3}x^{-4}+2\right) = 6x-\dfrac{20}{3}x^{-5}$

e $\dfrac{d}{dx}(6x^{-2}+3x) = -12x^{-3}+3$

f

$$\frac{d}{dx}\frac{3x^2+2}{x} = \frac{d}{dx}\left(3x+\frac{2}{x}\right)$$

$$= 3-\frac{2}{x^2}$$

4 $z \neq 0$ throughout

a $\dfrac{d}{dz}\dfrac{3z^2+2z+4}{z^2} = \dfrac{d}{dz}\left(3+\dfrac{2}{Z}+\dfrac{4}{z^2}\right)$
$= -\dfrac{2}{z^2}-\dfrac{8}{z^3}$

b $\dfrac{d}{dz}\dfrac{3+z}{z^3} = \dfrac{d}{dz}\left(\dfrac{3}{z^3}+\dfrac{1}{z^2}\right)$
$= -\dfrac{9}{z^4}-\dfrac{2}{z^3}$

c $\dfrac{d}{dz}\dfrac{2z^2+3z}{4z} = \dfrac{d}{dz}\left(\dfrac{Z}{2}+\dfrac{3}{4}\right) = \dfrac{1}{2}$

d $\dfrac{d}{dz}(9z^2+4z+6z^{-3}) = 18z+4-18z^{-4}$

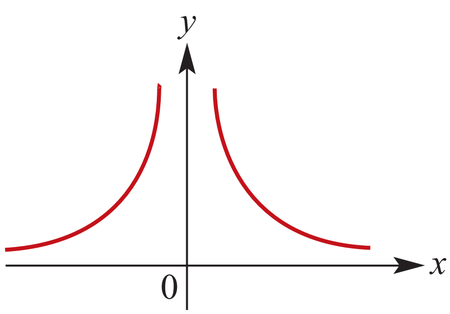
e $\dfrac{d}{dz}(9-z^{-2}) = -2z^{-3}$

f $\dfrac{d}{dz}\dfrac{5z-3z^2}{5z} = \dfrac{d}{dz}\left(5-\dfrac{3z}{5}\right) = -\dfrac{3}{5}$

5 **a** $f'(x) = 12x^3+18x^{-4}-x^{-2}$

b $f'(x) = 20x^3-8x^{-3}-x^{-2}$

6 $f(x) = \dfrac{1}{x^2}; x \neq 0$



a $P = (1, f(1)); Q = (1+h, f(1+h))$

$PQ's \text{ gradient } = \dfrac{\dfrac{1}{(1+h)^2}-1}{h}$
$= \dfrac{1-1-2h-h^2}{h(1+h^2)}$
$= \dfrac{-2-h}{1+h^2}$

b $f(x) = \dfrac{1}{x^2}$ has gradient of -2 at $x = 1$

c Normal at (1, 1) has gradient $= \dfrac{1}{2}$:
$y-1 = \dfrac{1}{2}(x-1), \therefore y = \dfrac{1}{2}(x+1)$

7 **a** $y = x^{-2}+x^3, \therefore\ y' = -2x^{-3}+3x^2$
$\therefore\ y'(2) = -\dfrac{2}{8}+12 = \dfrac{47}{4}$

b $y = \dfrac{x-2}{x} = 1-\dfrac{2}{x}, \therefore\ y' = \dfrac{2}{x^2}$
$\therefore y'(4) = \dfrac{2}{16} = \dfrac{1}{8}$

c $y = x^{-2}-\dfrac{1}{x}, \therefore\ y' = -\dfrac{2}{x^3}+\dfrac{1}{x^2}$
$\therefore\ y'(1) = -2+1 = -1$

d $y = x(x^{-1}+x^2-x^{-3}) = 1+x^3-x^{-2}$
$\therefore\ y' = 3x^2+2x^{-3}$
$\therefore\ y'(1) = 3+2 = 5$

8 $f(x) = x^{-2},\ \therefore\ f'(x) = -2x^{-3}; x > 0$

a $f'(x) = -2x^{-3} = 16,\ \therefore x^3 = -\dfrac{1}{8}$
$\therefore\ x = -\dfrac{1}{2}$

b $f'(x) = -2x^3 = -16, \therefore\ x^3 = \dfrac{1}{8}$
$\therefore x = \dfrac{1}{2}$

9 $f'(x) = -x^{-2} = -\dfrac{1}{x^2} < 0$ for all non-zero x

Solutions to Exercise 17D

1 $\frac{dy}{dx}$ = gradient

a $\frac{dy}{dx} < 0$ for all x

b $\frac{dy}{dx} > 0$ for all x

c $\frac{dy}{dx}$ varies in sign

d $\frac{dy}{dx} > 0$ for all x

e $\frac{dy}{dx} > 0$ for all $x > 0$ and $\frac{dy}{dx} < 0$ for all $x < 0$

Gradient is uniformly positive for **b** and **d** only.

2 $\frac{dy}{dx}$ = gradient:

a $\frac{dy}{dx} < 0$ for all x

b $\frac{dy}{dx} < 0$ for all x

c $\frac{dy}{dx}$ varies in sign

d $\frac{dy}{dx}$ varies in sign

e $\frac{dy}{dx} < 0$ for all x

f $\frac{dy}{dx} = 0$ for all x

Gradient is uniformly negative for **a**, **b** and **e** only.

3 $f(x) = 2(x-1)^2$

a $f(x) = 0, \therefore 2(x-1)^2 = 0$

$x = 1$

b $f'(x) = 4x - 4 = 0, \therefore x = 1$

c $f'(x) = 4x - 4 > 0, \therefore x > 1$

d $f'(x) = 4x - 4 < 0, \therefore x < 1$

e $f'(x) = 4x - 4 = -2$

$4x = 2, \therefore x = \frac{1}{2}$

4 **a** $\{x: h'(x) > 0\}$

$= \{x: x < -3\} \cup \{x: \frac{1}{2} < x < 4\}$

b $\{x: h'(x) < 0\}$

$= \{x: -3 < x < \frac{1}{2}\} \cup \{x: x > 4\}$

c $\{x: h'(x) = 0\} = \{-3, \frac{1}{2}, 4\}$

5 **a** $\frac{dy}{dx} < 0$ for $x < 0$, $\frac{dy}{dx} = 0$ at $x = 0$,

$\frac{dy}{dx} > 0$ for $x > 0$

$\therefore \frac{dy}{dx}$ = line $y = kx,\ k > 0$ **B**

b $\frac{dy}{dx} > 0$ for $x < 0$ and $x > a > 0$,

$\frac{dy}{dx} = 0$ at $x = 0$ and a, $\frac{dy}{dx} < 0$ for $0 < x < a$

$\therefore \frac{dy}{dx}$ = curve like a parabola

$y = kx(x - a)$ **C**

c $\frac{dy}{dx} < 0$ for all x except $\frac{dy}{dx} = 0$ at $x = 0$ and a,

$\therefore \frac{dy}{dx}$ = curve where $y \le 0$ for all x **D**

d $\frac{dy}{dx} > 0$ for $x < a > 0$, $\frac{dy}{dx} = 0$ at $x = a$,

$\frac{dy}{dx} > 0$ for $x > a$,

$\therefore \frac{dy}{dx}$ = line $y = -k(x - a)$, $k > 0$ **A**

e $y = -k$, $k > 0$ for all x so $\frac{dy}{dx} = 0$ **F**

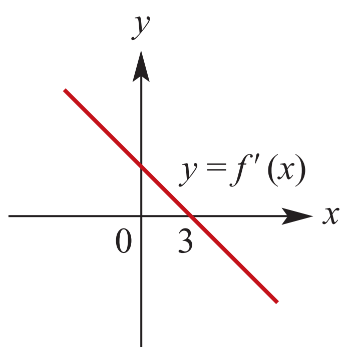
f $y = kx + c$; k, $c > 0$ so $\frac{dy}{dx} = k$ **E**

6 a $\{x\colon f'(x) > 0\} = \{x\colon x < 3\}$

$\{x\colon f'(x) < 0\} = \{x\colon x > 3\}$

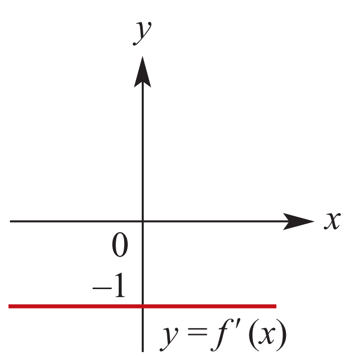
$\{x\colon f'(x) = 0\} = \{3\}$

$\therefore f'(x) = -k(x - 3)$, $k > 0$



b $f(x) = 1 - x$

$\therefore f'(x) = -1$



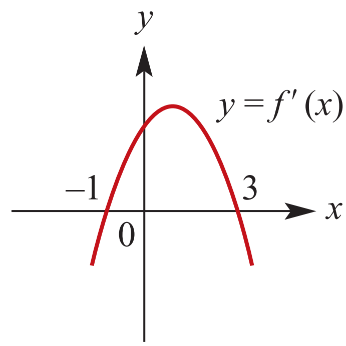
c

$\{x\colon f'(x) > 0\} = \{x\colon -1 < x < 3\}$

$\{x\colon f'(x) < 0\} = \{x\colon x < -1\} \cup \{x\colon x > 3\}$

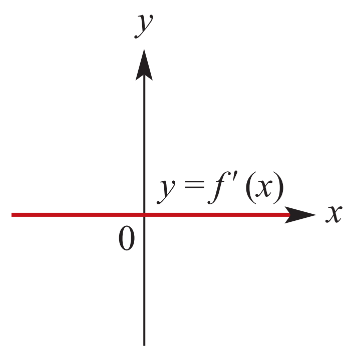
$\{x\colon f'(x) = 0\} = \{-1, 3\}$

$\therefore f'(x) = -k(x - 3)(x + 1)$, $k > 0$



d $f(x) = 3$

$\therefore f'(x) = 0$



7 a $\{x\colon f'(x) > 0\} = \{x\colon -1 < x < 1.5\}$

b $\{x\colon f'(x) < 0\}$
$= \{x\colon x < -1\} \cup \{x\colon x > 1.5\}$

c $\{x\colon f'(x) = 0\} = \{-1, 1.5\}$

8 $y = x^2 - 5x + 6, \therefore \frac{dy}{dx} = 2x - 5$

a Tangent makes an angle of 45° with the positive direction of the x-axis

$\therefore$ gradient = 1

$\therefore \frac{dy}{dx} = 2x - 5 = 1, \therefore x = 3$

$y(3) = 0$ so coordinates are $(3, 0)$.

b Tangent parallel to $y = 3x + 4$

$\therefore$ gradient $= 3$

$\therefore \dfrac{dy}{dx} = 2x - 5 = 3, \therefore x = 4$

$y(4) = 2$ so coordinates are $(4, 2)$.

9 $y = x^2 - x - 6, \ \therefore \dfrac{dy}{dx} = 2x - 1$

a $\dfrac{dy}{dx} = 2x - 1 = 0, \therefore \ x = \dfrac{1}{2}$

$y\left(\dfrac{1}{2}\right) = -\dfrac{25}{4}$ so coordinates are $\left(\dfrac{1}{2}, -\dfrac{25}{4}\right)$.

b Tangent parallel to $x + y = 6$

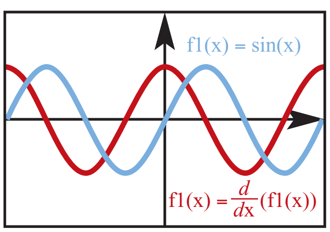
$\therefore$ gradient $= -1$

$\therefore \dfrac{dy}{dx} = 2x - 1 = -1, \ \therefore x = 0$

$y(0) = -6$ so coordinates are $(0, -6)$.

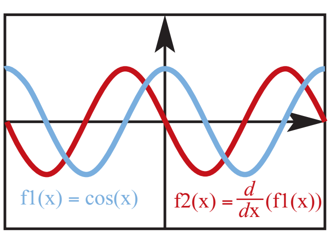
10 **a** $f(x) = \sin x$

$f'(x)$



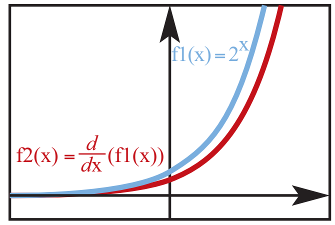
b $f(x) = \cos x$

$f'(x)$



c $f(x) = 2^x$

$f'(x)$



11 **a** **i** $66.80°$

ii $42.51°$

b $(0.5352, 0.2420)$

c No

12 $S(t) = (0.2)t^3$ m

a Speed $= s'(t) = 0.6t^2$ m/s

b $s'(1) = 0.6(1)^2 = 0.6$ m/s

$S'(3) = 0.6(3)^2 = 5.4$ m/s

$S'(5) = 0.6(5)^2 = 15$ m/s

13 $y = ax^2 + bx$

a $y(2) = -2, \ \therefore 4a + 2b = -2$

$y'(x) = 2ax + b$

$\therefore y'(2) = 4a + b = 3$

$\therefore b = -5, \ a = 2$

b $\dfrac{dy}{dx} = 4x - 5 = 0, \therefore x = \dfrac{5}{4}$

$y\left(\dfrac{5}{4}\right) = 4\left(\dfrac{5}{4}\right)^2 - 5\left(\dfrac{5}{4}\right)$

$= \dfrac{25}{4} - \dfrac{25}{8} = -\dfrac{25}{8}$

Coordinates are $\left(\dfrac{5}{4}, -\dfrac{25}{8}\right)$.

14 $h(t) = 20t^2, 0 \leq t \leq 150$

a $h(150) = 20(150)^2 = 450\,000$ m
$h'(t) = 40t$, $\therefore h'(150) = 6000$ m/s

b $h'(t) = 40t = 1000$
$\therefore t = \dfrac{1000}{40} = 25$ sec

Solutions to Exercise 17E

1 **a** $\int \frac{1}{2}x^3 dx = \frac{1}{8}x^4 + c$

b $\int 3x^2 - 2 dx = x^3 - 2x + c$

c $\int 5x^3 - 2x dx = \frac{5}{4}x^4 - x^2 + c$

d $\int \frac{4}{5}x^3 - 2x^2 dx = \frac{1}{5}x^4 - \frac{2}{3}x^3 + c$

e $\int (x-1)^2 dx = \int x^2 - 2x + 1\, dx$

$$= \frac{x^3}{3} - x^2 + x + c$$

f $\int x\left(x + \frac{1}{x}\right) dx = \int x^2 + 1\, dx$

$$= \frac{1}{3}x^3 + x + c$$

g $\int 2z^2(z-1) dz = \int 2z^3 - 2z^2 dz$

$$= \frac{1}{2}z^4 - \frac{2}{3}z^2 + c$$

h $\int (2t-3)^2 dt = \int 4t^2 - 12t + 9\, dt$

$$= \frac{4t^3}{3} - 6t^2 + 9t + c$$

i $\int (t-1)^3 dt = \int t^3 - 3t^2 + 3t - 1\, dt$

$$= \frac{t^4}{4} - t^3 + \frac{3t^2}{2} - t + c$$

2

$$f'(x) = 4x^3 + 6x^2 + 2$$

$$\therefore f(x) = x^4 + 2x^3 + 2x + c$$

We have, $f(0) = 0$

$$\therefore c = 0$$

$$\therefore f(x) = x^4 + 2x^3 + 2x$$

3

$$f'(x) = 6x^2$$

$$\therefore f(x) = 2x^3 + c$$

We have, $f(0) = 12$

$$\therefore c = 12$$

$$\therefore f(x) = 2x^3 + 12$$

4 **a** $\frac{dy}{dx} = 2x - 1, \therefore y = x^2 - x + c$

$y(1) = c = 0, \therefore y = x^2 - x$

b $\frac{dy}{dx} = 3 - x, \therefore y = 3x - \frac{1}{2}x^2 + c$

$y(0) = c = 1, \therefore y = -\frac{1}{2}x^2 + 3x + 1$

c $\frac{dy}{dx} = x^2 + 2x, \therefore y = \frac{1}{3}x^3 + x^2 + c$

$y(0) = c = 2, \therefore y = \frac{1}{3}x^3 + x^2 + 2$

d

$\frac{dy}{dx} = 3 - x^2, \therefore y = 3x - \frac{1}{3}x^3 + c$

$y(3) = c = 2, \therefore y = -\frac{1}{3}x^3 + 3x + 2$

e $\frac{dy}{dx} = 2x^4 + x, \therefore y = \frac{2}{5}x^5 + \frac{1}{2}x^2 + c$

$y(0) = c = 0, \therefore y = \frac{2}{5}x^5 + \frac{1}{2}x^2$

5 $\frac{dV}{dt} = t^2 - t, \; t > 1$

a $V(t) = \frac{1}{3}t^3 - \frac{1}{2}t^2 + c$

$V(3) = 9 - \frac{9}{2} + c = 9$

$$c = \frac{9}{2}$$

b $V(10) = \frac{1000}{3} - \frac{100}{2} + \frac{9}{2}$

$= \frac{1727}{6} \approx 287.33$

6 $f'(x) = 3x^2 - 1, \; \therefore f(x) = x^3 - x + c$

$f(1) = c = 2, \; \therefore f(x) = x^3 - x + 2$

7 **a** Only **B** has the correct gradient (negative) with the correct axis intercept.

b $\frac{dw}{dt} = 2000 - 20t,\ t > 0$

$w = 2000t - 10t^2 + c, t \geq 0$

$w(0) = c = 100\,000$

$\therefore w = -10t^2 + 2000t + 100\,000$

8 $\frac{dy}{dx} = 5 - x,\ \therefore f(x) = 5x - \frac{1}{2}x^2 + c$

$f(0) = c = 4,\ \therefore f(x) = -\frac{1}{2}x^2 + 5x + 4$

9 $f(x) = x^2(x - 3) = x^3 - 3x^2$

$\therefore f(x) = \frac{1}{4}x^4 - x^3 + c$

$f(2) = 4 - 8 + c = -6,\ \therefore c = -2$

$\therefore f(x) = \frac{1}{4}x^4 - x^3 - 2$

10 $f'(x) = 4x + k,\ \therefore f(x) = 2x^2 + kx + c$

a $f'(-2) = -8 + k = 0$

$k = 8$

b $f(-2) = 8 - 16 + c = -1,\ \therefore c = 7$

$\therefore\ f(x) = 2x^2 + 8x + 7$

$\therefore\ f(0) = 7$

Curve meets y-axis at $(0, 7)$

11 $\frac{dy}{dx} = ax^2 + 1,\ \therefore y = \frac{a}{3}x^3 + x + c$

$y'(1) = a + 1 = 3,\ \therefore a = 2$

$y(1) = \frac{2}{3} + 1 + c = 3,\ \therefore c = \frac{4}{3}$

$\therefore y(2) = \frac{2}{3}(2)^3 + 2 + \frac{4}{3}$

$= \frac{26}{3}$

12 $\frac{dy}{dx} = 2x + k,\ \therefore y'(3) = 6 + k$

a Tangent: $y - 6 = (6 + k)(x - 3)$

$y = (6 + k)x - 12 - 3k$

Tangent passes through $(0, 0)$,

$\therefore k = -4$

b $y = \int 2x - 4dx = x^2 - 4x + c$

$y(3) = 9 - 12 + c = 6,\ \therefore c = 9$

$\therefore y = x^2 - 4x + 9$

13 $f'(x) = 16x + k$

a $y'(2) = 32 + k = 0$

$k = -32$

b $f(x) = \int 16x - 32dx$

$= 8x^2 - 32x + c$

$f(2) = 32 - 64 + c = 1,\ \therefore c = 33$

$\therefore f(7) = 8(7)^2 - 32(7) + 33$

$= 201$

14 $f'(x) = x^2,\ \therefore f(x) = \frac{1}{3}x^3 + c$

$f(2) = \frac{8}{3} + c = 1,\ \therefore c = -\frac{5}{3}$

$\therefore\ f(x) = \frac{1}{3}(x^3 - 5)$

Solutions to Exercise 17F

1 a $\lim_{x\to 3} 15 = 15$

b $\lim_{x\to 6} (x-5) = 6-5 = 1$

c $\lim_{x\to \frac{1}{2}} (3x-5) = \frac{3}{2} - 5 = -\frac{7}{2}$

d $\lim_{t\to -3} \frac{t-2}{t+5} = \frac{-3-2}{-3+5} = -\frac{5}{2}$

e $\lim_{t\to -1} \frac{t^2+2t+1}{t+1} = \frac{(t+1)^2}{t+1}$
$= \lim_{t\to -1} t+1 = 0$

f $\lim_{x\to 0} \frac{(x+2)^2-4}{x} = \frac{x^2+4x}{x}$
$= \lim_{x\to 0} x+4 = 4$

g $\lim_{t\to 1} \frac{t^2-1}{t-1} = \frac{(t-1)(t+1)}{t-1}$
$= \lim_{t\to 1} t+1 = 2$

h $\lim_{x\to 9} \sqrt{x+3} = \sqrt{12} = 2\sqrt{3}$

i $\lim_{x\to 0} \frac{x^2-2x}{x} = x-2 = -2$

j $\lim_{x\to 2} \frac{x^3-8}{x-2} = \frac{(x-2)(x^2+2x+4)}{x-2}$
$= \lim_{x\to 2} x^2+2x+4 = 12$

k $\lim_{x\to 2} \frac{3x^2-x-10}{x^2+5x-14} = \frac{(x-2)(3x+5)}{(x-2)(x+7)}$
$= \lim_{x\to 2} \frac{3x+5}{x+7} = \frac{11}{9}$

l $\lim_{x\to 1} \frac{x^2-3x+2}{x^2-6x+5} = \frac{(x-1)(x-2)}{(x-1)(x-5)}$
$= \lim_{x\to 1} \frac{x-2}{x-5} = \frac{1}{4}$

2 a Discontinuities at $x = 3$ and 4, because at $x = 3$, $f(x)$ is not defined, and the right limit of $f(x)$ at $x = 4$ is not equal to $f(4)$. ($x = 1$ is not a discontinuity, although the function is not differentiable there.)

b There is a discontinuity at $x = 7$, because the right limit of $f(x)$ at $x = 7$ is not equal to $f(7)$.

3 a $f(x) = 3x$ if $x \geq 0$, $-2x+2$ if $x < 0$
Discontinuity at $x = 0$: $f(0) = 0$, but
$\lim_{x\to 0^+} f(x) = 0$, $\lim_{x\to 0^-} f(x) = 2$

b $f(x) = x^2+2$ if $x \geq 1$, $-2x+1$ if $x < 1$
Discontinuity at $x = 1$: $f(1) = 3$, but
$\lim_{x\to 1^+} f(x) = 3$, $\lim_{x\to 1^-} f(x) = -1$

c $f(x) = -x$ if $x \leq -1$

$f(x) = x^2$ if $-l < x < 0$

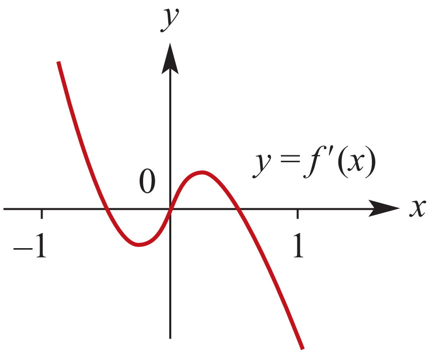
$f(x) = -3x+1$ if $x \geq 0$
Discontinuity at $x = 0$: $f(0) = 1$, but
$\lim_{x\to 0^+} f(x) = 1$, $\lim_{x\to 0^-} f(x) = 0$
$x = -1$ is not a discontinuity, since
$f(-1) = 1$
$\lim_{x\to (-1)^+} f(x) = 1$, $\lim_{x\to (-1)^-} f(x) = 1$

4 $y = \begin{cases} 2; & x < 1 \\ (x-4)^2 - 9; & 1 \leq x < 7 \\ x-7; & x \geq 7 \end{cases}$
Discontinuity at $x = 1$: $y(1) = 0$, but
$\lim_{x\to 1^+} y(x) = 0$, $\lim_{x\to 1^+} y(x) = 2$
$x = 7$ is not a discontinuity, since
$y(7) = 0$
$\lim_{x\to 7^+} y(x) = 0$, $\lim_{x\to 7^+} y(x) = 0$

Solutions to Exercise 17G

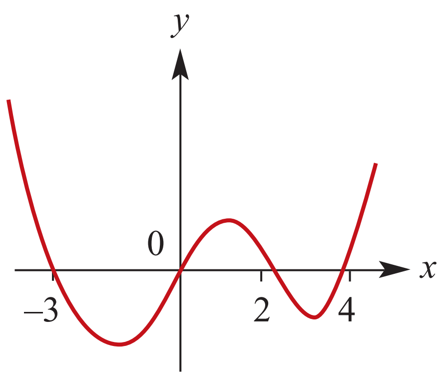
1 a

x	-1	-0.5	0.2	0	0.2	0.5	1
$f'(x)$	$+$	0	$-$	0	$+$	0	$-$

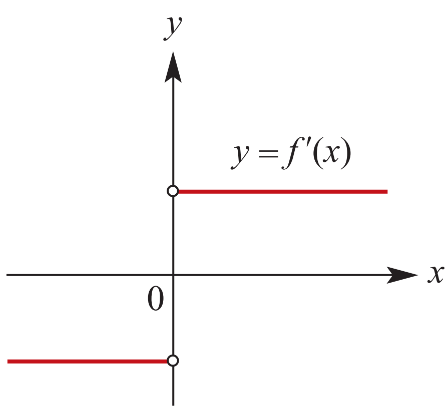


b

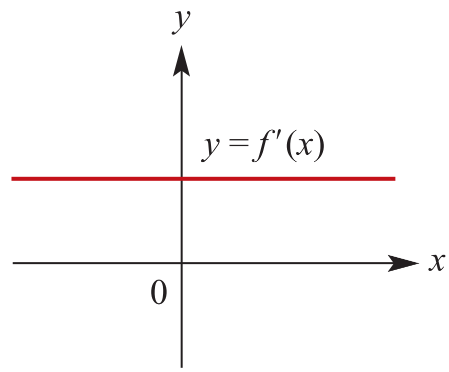
x	-4	-3	-2	0	1	2	3	4	5
$f'(x)$	$+$	0	$-$	0	$+$	0	$-$	0	$+$



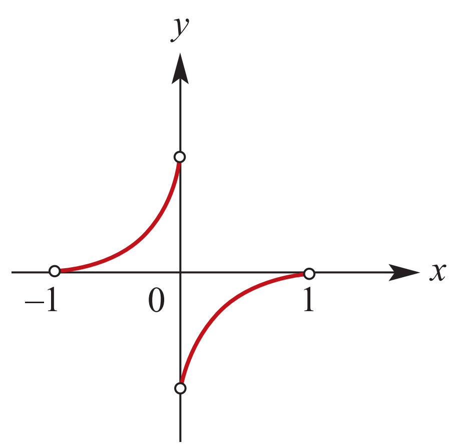
c For $x < 0$, $f'(x) = -k$, $k > 0$
For $x > 0$, $f'(x) = k$, $k > 0$
At $x = 0$, $f'(x)$ is undefined since $f(x)$ is not differentiable at that point.



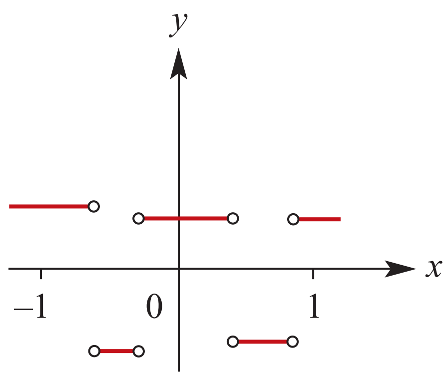
d For all x, $f'(x) = k$, $k > 0$



e $f'(x)$ only exists for $\{x: -1 < x < 1\}/\{0\}$
For $-1 < x < 0$, $f'(x) < 0$
For $0 < x < 1$, $f'(x) > 0$
At $x = 0$, $f'(x)$ is undefined since $f(x)$ is not differentiable at that point.



f $f'(x)$ is undefined at four points over $[-1, 1]$ and is positive at both ends, alternating $+$ to $-$ between the undefined points:



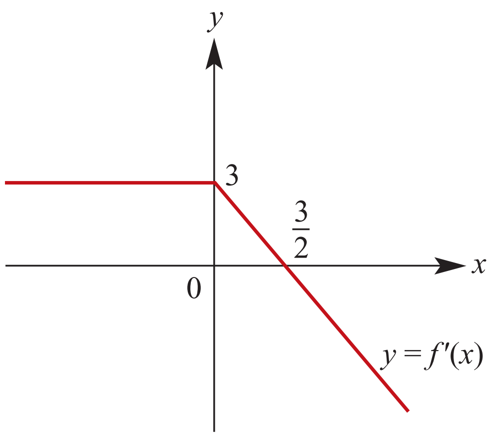
2 $f(x) = -x^2 + 3x + 1$ if $x \geq 0$

$f(x) = 3x + 1$ if $x < 0$

$\therefore\ f'(x) = -2x + 3$ if $x \geq 0$

$f'(x) = 3$ if $x < 0$

$f(x)$ is differentiable at $x = 0$ because both $f(x)$ and $f'(x)$ are continuous at that point.



3 $f(x) = x^2 + 2x + 1$ if $x \geq 1$

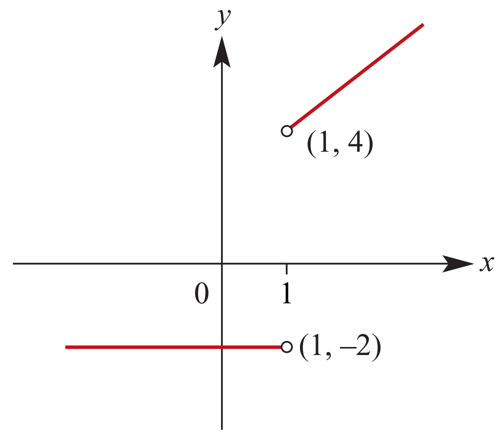
$f(x) = -2x + 3$ if $x < 1$

$\therefore\ f'(x) = 2x + 2$ if $x > 1$

$f'(x) = -2$ if $x < 1$

$f(x)$ is not differentiable at $x = 0$ because both $f(x)$ and $f'(x)$ are discontinuous at that point.

$\therefore\ f'(x)$ is defined over $R/\{1\}$



4 $f(x) = -x^2 - 3x + 1$ if $x \geq -1$

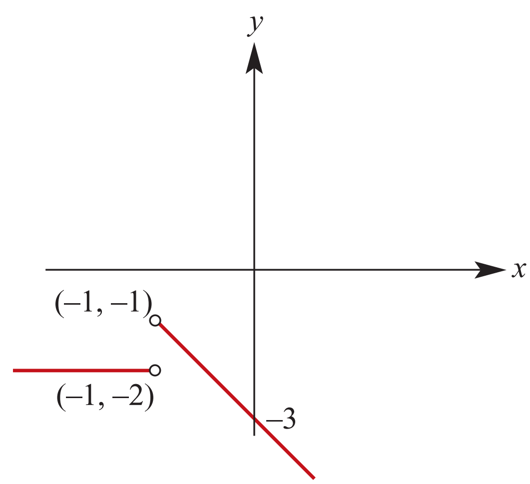
$f(x) = -2x + 3$ if $x < -1$

$\therefore\ f'(x) = -2x - 3$ if $x > -1$

$f(x) = -2$ if $x < -1$

$f(x)$ is not differentiable at $x = -1$ because both $f(x)$ and $f'(x)$ are discontinuous at that point.

$\therefore\ f'(x)$ is defined over $R/\{-1\}$



Solutions to Technology-free questions

1 a

$$\frac{f(x+h)-f(x)}{x+h-x} = \frac{3(x+h)+1-(3x+1)}{x+h-x}$$
$$= \frac{3h}{h}$$
$$= 3$$
$$\lim_{h\to 0} 3 = 3$$

b

$$\frac{f(x+h)-f(x)}{x+h-x} = \frac{4-(x+h)^2-(4-x^2}{x+h-x}$$
$$= \frac{4-x^2-2xh-h^2-4+x^2}{h}$$
$$= \frac{-2xh-h^2}{h}$$
$$= -2x-h$$
$$\lim_{h\to 0} -2x-h = -2x$$

c

$$\frac{f(x+h)-f(x)}{x+h-x} = \frac{(x+h)^2+5(x+h)-(x^2+5x)}{x+h-x}$$
$$= \frac{x^2+2xh+h^2+5x+5h-x^2-5x}{h}$$
$$= \frac{2xh+h^2+5h}{h}$$
$$= 2x+h+5$$
$$\lim_{h\to 0} 2x+h+5 = 2x+5$$

d

$$\frac{f(x+h)-f(x)}{x+h-x} = \frac{(x+h)^3+(x+h)-(x^3+x)}{x+h-x}$$
$$= \frac{x^3+3x^2h+3xh^2+x+h-x^3-x}{h}$$
$$= \frac{3x^2h+3xh^2+h}{h}$$
$$= 3x^2+3xh+1$$
$$\lim_{h\to 0} 3x^2+3xh+1 = 3x^2+1$$

e

$$\frac{f(x+h)-f(x)}{x+h-x} = \frac{(x+h)^2+2(x+h)+1-(x^2+2x+1)}{x+h-x}$$
$$= \frac{x^2+2xh+h^2+2x+2h+1-x^2-2x-1}{h}$$
$$= \frac{2xh+h^2+2h}{h}$$
$$= 2x+h+2$$
$$\lim_{h\to 0} 2x+h+2 = 2x+2$$

f

$$\frac{f(x+h)-f(x)}{x+h-x} = \frac{3(x+h)^2-(x+h)-(3x^2-x)}{x+h-x}$$
$$= \frac{3x^2+6xh+3h^2-x-h-3x^2+x}{h}$$
$$= \frac{6xh+3h^2-h}{h}$$
$$= 6x+3h-1$$
$$\lim_{h\to 0} 6x+3h-1 = 6x-1$$

2 a $y = 3x^2-2x+6$

$$\therefore \frac{dy}{dx} = 6x-2$$

b $y = 5, \ \therefore \dfrac{dy}{dx} = 0$

c $y = 2x(2-x) = 4x-2x^2$

$$\therefore \frac{dy}{dx} = 4-4x$$

d $y = 4(2x-1)(5x+2)$

$$= 40x^2-4x-8$$
$$\therefore \frac{dy}{dx} = 80x-4 = 4(20x-1)$$

e $y = (x+1)(3x-2)$

$$= 3x^2+x-2$$
$$\therefore \frac{dy}{dx} = 6x+1$$

f $y = (x+1)(2-3x)$

$= -3x^2 - x + 2$

$\therefore \dfrac{dy}{dx} = -6x - 1$

3 a $y = -x, \ \therefore \dfrac{dy}{dx} = -1$

b $y = 10, \ \therefore \dfrac{dy}{dx} = 0$

c $y = \dfrac{(x+3)(2x+1)}{4}$

$= \dfrac{1}{2}x^2 + \dfrac{7}{4}x + \dfrac{3}{4}$

$\therefore \dfrac{dy}{dx} = x + \dfrac{7}{4}$

d $y = \dfrac{2x^3 - x^2}{31} = \dfrac{2}{3}x^2 - \dfrac{1}{3}x, \ x \neq 0$

$\therefore \dfrac{dy}{dx} = \dfrac{4}{3}x - \dfrac{1}{3} = \dfrac{1}{3}(4x-1), \ x \neq 0$

e $y = \dfrac{x^4 + 3x^2}{2x^2} = \dfrac{1}{2}x^2 + 3, \quad x \neq 0$

$\therefore \dfrac{dy}{dx} = x, \ x \neq 0$

4 a $y = x^2 - 2x + 1, \ \therefore \dfrac{dy}{dx} = 2x - 2$

At $x = 2, \ y = 1$ and gradient $= 2$

b $y = x^2 - 2x, \ \therefore \dfrac{dy}{dx} = 2x - 2$

At $x = -1, \ y = 3$ and gradient $= -4$

c $y = (x+2)(x-4) = x^2 - 2x - 8$

$\therefore \dfrac{dy}{dx} = 2x - 2$

At $x = 3, \ y = -5$ and gradient $= 4$

d $y = 3x^2 - 2x^3, \ \therefore \dfrac{dy}{dx} = 6x - 6x^2$

At $x = -2, \ y = 28$ and gradient $= -36$

5 a $y = x^2 - 3x + 1, \therefore \dfrac{dy}{dx} = 2x - 3$

$\dfrac{dy}{dx} = 0, \ \therefore 2x - 3 = 0$

$x = \dfrac{3}{2}$

$y\left(\dfrac{3}{2}\right) = \dfrac{9}{4} - \dfrac{9}{2} + 1 = -\dfrac{5}{4}$

Coordinates are $\left(\dfrac{3}{2}, \ -\dfrac{5}{4}\right)$

b $y = x^3 - 6x^2 + 4, \therefore \dfrac{dy}{dx} = 3x^2 - 12x$

$\dfrac{dy}{dx} = -12, \therefore 3x^2 - 12x = -12$

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0, \ \therefore x = 2$

$y(2) = 8 - 24 + 4 = -12$

Coordinates are $(2, -12)$

c $y = x^2 - x^3, \ \therefore \dfrac{dy}{dx} = 2x - 3x^2$

$\dfrac{dy}{dx} = -1, \ \therefore -3x^2 + 2x + 1 = 0$

$3x^2 - 2x - 1 = 0$

$(3x+1)(x-1) = 0$

$\therefore x = -\dfrac{1}{3}, 1$

$y\left(-\dfrac{1}{3}\right) = \dfrac{4}{27}, \ y(1) = 0$

Coordinates are $\left(-\dfrac{1}{3}, \ \dfrac{4}{27}\right)$ and $(1, 0)$

d $y = x^3 - 2x + 7, \therefore \dfrac{dy}{dx} = 3x^2 - 2$

$\dfrac{dy}{dx} = 1, \therefore 3x^2 - 2 = 1$

$3x^2 = 3, \ \therefore x = \pm 1$

$y(-1) = 8; y(1) = 6$

Coordinates are $(-1, 8)$ and $(1, 6)$

e $y = x^4 - 2x^3 + 1, \ \therefore \dfrac{dy}{dx} = 4x^3 - 6x$

$\frac{dy}{dx} = 0, \therefore 4x^3 - 6x^2 = 0$

$2x^2(2x - 3) = 0, \therefore x = 0, \frac{3}{2}$

$y(0) = 1; y\left(\frac{3}{2}\right) = \frac{81}{16} - \frac{27}{4} + 1 = -\frac{11}{16}$

Coordinates are (0, 1) and $\left(\frac{3}{2}, -\frac{11}{16}\right)$

f

$y = x(x - 3)^2 = x^3 - 6x^2 + 9x$

$\therefore \frac{dy}{dx} = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$

$\frac{dy}{dx} = 0, \therefore x^2 - 4x + 3 = 0$

$(x - 1)(x - 3) = 0 \therefore x = 1, 3$

$y(1) = 4; y(3) = 0$

Coordinates are (1, 4) and (3, 0)

6 $f(x) = 3(2x - 1)^2 = 12x^2 - 12x + 3$

$\therefore f'(x) = 24x - 12 = 12(2x - 1)$

a $f(x) = 0, \therefore 2x - 1 = 0$

$x = \frac{1}{2}$

b $f'(x) = 0, \therefore 2x - 1 = 0$

$x = \frac{1}{2}$

c $f'(x) > 0, \therefore 2x - 1 > 0$

$x > \frac{1}{2}$

d $f'(x) < 0, \therefore 2x - 1 < 0$

$x < \frac{1}{2}$

e $f'(x) > 0, \therefore 3(2x - 1)^2 > 0$

$\{x : x \in R \backslash \{\frac{1}{2}\}\}$

f $f'(x) = 3, \therefore 24x - 12 = 3$

$24x = 15$

$x = \frac{5}{8}$

7 a $\frac{d}{dx}x^{-4} = -4x^{-5}$

b $\frac{d}{dx}2x^{-3} = -6x^{-4}$

c $\frac{d}{dx} - \frac{1}{3x^2} = -\frac{1}{3}\frac{d}{dx}x^{-2} = \frac{2}{3x^3}$

d $\frac{d}{dx} - \frac{1}{x^4} = -(-4)x^{-5} = \frac{4}{x^5}$

e $\frac{d}{dx}\frac{3}{x^5} = -15x^{-6} = -\frac{15}{x^6}$

f $\frac{d}{dx}\frac{x^2 + x^3}{x^4} = \frac{d}{dx}x^{-2} + x^{-1} = -\frac{2}{x^3} - \frac{1}{x^2}$

g $\frac{d}{dx}\frac{3x^2 + 2x}{x^2} = \frac{d}{dx}\left(3 + \frac{2}{x}\right) = -\frac{2}{x^2}$

h $\frac{d}{dx}\left(5x^2 - \frac{2}{x}\right) = 10x + \frac{2}{x^2}$

8 $y = ax^2 + bx$

$\therefore \frac{dy}{dx} = 2ax + b$

a Using (1, 1): $a + b = 1$
Gradient = 3: $2a + b = 3$
$\therefore a = 2, b = -1$

b $\frac{dy}{dx} = 0, \therefore 2ax + b = 0$

$\therefore 4x - 1 = 0$

$x = \frac{1}{4}$

$y = 2x^2 - x$

$\therefore y\left(\frac{1}{4}\right) = \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$

Coordinates are $\left(\frac{1}{4}, -\frac{1}{8}\right)$

9 **a** $\int \frac{1}{2} dx = \frac{x}{2} + c$

b $\int \frac{x^2}{2} dx = \frac{x^3}{6} + c$

c $\int x^2 + 3x\, dx = \frac{x^3}{3} + \frac{3x^2}{2} + c$

d $\int (2x+3)^2 dx = \int 4x^2 + 12x + 9\, dx$

$$= \frac{4x^3}{3} + 6x^2 + 9x + c$$

e $\int atdt = \frac{1}{2}at^2 + c$

f $\int \frac{1}{3}t^3 dt = \frac{1}{12}t^4 + c$

g

$$\int (t+1)(t-2)dt = \int t^2 - t - 2\, dt$$

$$= \frac{1}{3}t^3 - \frac{1}{2}t^2 - 2t + c$$

h

$$\int (2-t)(t+1)dt = \int -t^2 - t_2 + 2dt$$

$$= -\frac{1}{3}t^3 - \frac{1}{2}t^2 + 2t + c$$

10 $f'(x) = 2x + 5$

$\therefore\ f(x) = x^2 + 5x + c$

$f(3) = 9 + 15 + c = -1$

$\therefore\quad c = -25$

$f(x) = x^2 + 5x - 25$

11 $f'(x) = 3x^2 - 8x + 3$

$\therefore\ f(x) = x^3 - 4x^2 + 3x + c$

a $f(0) = 0,\ \therefore c = 0$

$\therefore f(x) = x^3 - 4x^2 + 3x$

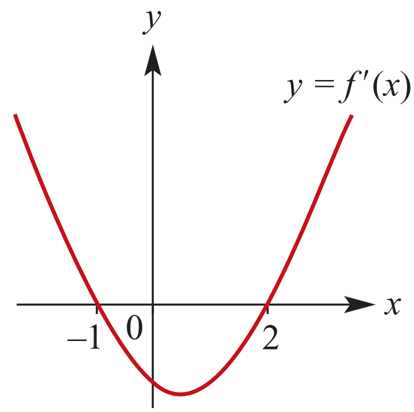
b

$f(x) = 0,\ \therefore x(x-1)(x-3) = x = 0$

$x = 0, 1, 3$

12

x	-2	-1	0	2	3
$f'(x)$	+	0	−	0	+



13 **a** $\{x\colon h'(x) > 0\} = \{x\colon\ -1 < x < 4\}$

b $\{x\colon h'(x) < 0\}$

$= \{x\colon x < -1\} \cup \{x\colon x > 4\}$

c $\{x\colon h'(x) = 0\} = \{-1, 4\}$

Solutions to multiple-choice questions

1 D $y = x^3 + 4x, \therefore \dfrac{dy}{dx} = 3x^2 + 4$

$\therefore y'(2) = 12 + 4 = 16$

2 B $y = 2x^2$

$$\therefore \text{ chord gradient} = \frac{2(1+h)^2 - 2(1)^2}{h} = \frac{4h + 2h^2}{h} = 4 + 2h$$

3 E $y = 2x^4 - 5x^3 + 2$

$\therefore \dfrac{dy}{dx} = 8x^3 - 15x^2$

4 B $f(x) = x^2(x+1) = x^3 + x^2$

$\therefore f'(x) = 3x^2 + 2x$

$\therefore f'(-1) = 3 - 2 = 1$

5 C $f(x) = (x-3)^2 = x^2 - 6x + 9$

$\therefore f'(x) = 2x - 6$

6 C $y = \dfrac{2x^4 + 9x^2}{3x}$

$= \dfrac{2}{3}x^3 + 3x; x \neq 0$

$\therefore \dfrac{dy}{dx} = 2x^2 + 3; x \neq 0$

7 A $y = x^2 - 6x + 9$

$\therefore \dfrac{dy}{dx} = 2x - 6 \geq 0$ if $x \geq 3$

8 E $y = 2x^4 - 36x^2$

$\therefore \dfrac{dy}{dx} = 8x^3 - 72x = 8x(x^2 - 9)$

Tangent to curve parallel to x-axis where

$8x(x^2 - 9) = 0$

$\therefore x = 0, \pm 3$

9 A $y = x^2 + 6x - 5, \therefore \dfrac{dy}{dx} = 2x + 6$

Tangent to curve parallel to $y = 4x$ where

$\dfrac{dy}{dx} = 2x + 6 = 4$

$\therefore 2x = -2 < \therefore x = -1$

$y(-1) = (-1)^2 + 6(-1) - 5 = -10$

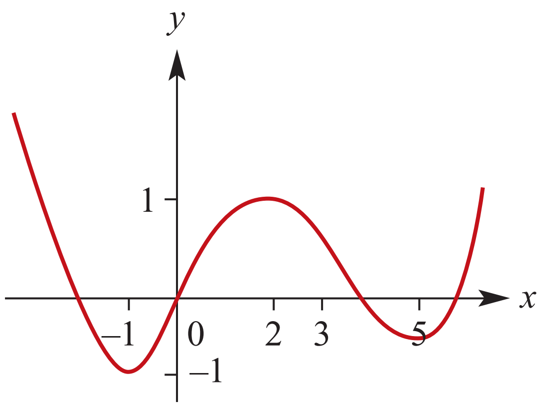
Coordinates are $(-1, -10)$

10 D $y = -2x^3 + 3x^2 - x + 1$

$\therefore \dfrac{dy}{dx} = -6x^2 + 6x - 1$

Solutions to extended-response questions

1 For $x < -1$, the gradient is negative, becoming less steep as x approaches -1.
For $x = -1$, the gradient is zero.
For $-1 < x < 2$, the gradient is positive, getting steeper as x approaches 0.5 (approximately) then becoming less steep as x approaches 2.
For $x = 2$, the gradient is zero.
For $2 < x < 5$, the gradient is negative, getting steeper as x approaches 4 (approximately) then becoming less steep as x approaches 5.
For $x = 5$, the gradient is zero.
For $x > 5$, the gradient is positive and becoming steeper.



2 $P(x) = ax^3 + bx^2 + cx + d$

At $(0, 0)$,	$0 = 0 + 0 + 0 + d$	
$\therefore$	$d = 0$	
At $(-2, 3)$,	$3 = a(-2)^3 + b(-2)^2 + c(-2)$	
$\therefore$	$3 = -8a + 4b - 2c$	(1)
At $(1, -2)$,	$-2 = a(1)^3 + b(1)^2 + c(1)$	
$\therefore$	$-2 = a + b + c$	(2)
	$P'(x) = 3ax^2 + 2bx + c$	
At $x = -2$, $P'(x) = 0$,	$\therefore 0 = 3a(-2)^2 + 2b(-2) + c$	
$\therefore$	$0 = 12a - 4b + c$	(3)

$$
\begin{array}{lll}
(3) - (2) & 12a - 4b + c = 0 & \\
 & \underline{-a + b + c = -2} & \\
(1) + 2 \times (2) & 11a - 5b = 2 & (4) \\
 & -8a + 4b - 2c = 3 & \\
 & \underline{+2a + 2b + 2c = -4} & \\
6 \times (4) + 5 \times (5) & -6a + 6b = -1 & (5) \\
 & 66a - 30b = 12 & \\
 & \underline{+ \quad -30a + 30b = -5} & \\
 & 36a = 7 & \\
\therefore & a = \dfrac{7}{36} & (6) \\
\text{Substitute (6) into (5)} & -6\left(\dfrac{7}{36}\right) + 6b = -1 & \\
\therefore & 6b = -1 + \dfrac{7}{6} & \\
\therefore & b = \dfrac{1}{36} & (7) \\
\text{Substitute (6) and (7) into (2)} & -2 = \dfrac{7}{36} + \dfrac{1}{36} + c & \\
\therefore & c = \dfrac{-72 - 7 - 1}{36} & \\
 & = \dfrac{-80}{36} & \\
 & = \dfrac{-20}{9} & \\
\text{Hence} & a = \dfrac{7}{36}, b = \dfrac{1}{36},\ c = \dfrac{-20}{9},\ d = 0 & \\
\text{so} & P(x) = \dfrac{7}{36}x^3 + \dfrac{1}{36}x^2 - \dfrac{20}{9}x &
\end{array}
$$

3 a

$$y = \frac{1}{5}x^5 + \frac{1}{2}x^4$$

$$\frac{dy}{dx} = x^4 + 2x^3$$

i When $x = 1$, $\quad \dfrac{dy}{dx} = 1^4 + 2(1)^3$

$$= 1 + 2$$

$$= 3$$

$\therefore \quad \tan\theta = 3$ where θ is the angle required

$\therefore \quad \theta \approx 71°34'$

ii When $x = 3$, $\quad \dfrac{dy}{dx} = 3^4 + 2(3)^3$

$$= 81 + 54$$

$$= 135$$

$\therefore \quad \tan\theta = 135$

$\therefore \quad \theta \approx 89°35'$

b Consider $\quad \dfrac{dy}{dx} = 32$

which implies $\quad x^4 + 2x^3 = 32$

i.e. $\quad x^4 + 2x^3 - 32 = 0$

The factor theorem gives that $x - 2$ is a factor

$\therefore \quad (x - 2)(x^3 + 4x^2 + 8x + 16) = 0$

i.e. $\dfrac{dy}{dx} = 32$ when $x = 2$.

So, gradient path is 32 when $x = 2$ km.

4 a

$$y = 2 + 0.12x - 0.01x^3$$

$$\frac{dy}{dx} = 0.12 - 0.03x^2$$

At the beginning of the trail, $x = 0$

$$\therefore \frac{dy}{dx} = 0.12 - 0.03(0)^2 = 0.12$$

Hence, the gradient at the beginning of the trail is 0.12.

At the end of the trail, $x = 3$

$\therefore \quad \dfrac{dy}{dx} = 0.12 - 0.03(3)^2$

$$= 0.12 - 0.27$$

$$= -0.15$$

Hence, the gradient at the end of the trail is -0.15.

b The trail climbs at the beginning and goes downwards at the end, suggesting a peak in between (i.e. for $0 < x < 3$) where the gradient will be zero.

Gradient is zero where $\frac{dy}{dx} = 0$

$\therefore \quad 0.03x^2 = 0.12$

$\therefore \quad x^2 = 4$

$\therefore \quad x = \pm 2$

$\therefore \quad x = 2$ as $0 < x < 3$

At $x = 2$,

$$\begin{aligned} y &= 2 + 0.12(2) - 0.01(2)^3 \\ &= 2 + 0.24 - 0.08 \\ &= 2.16 \end{aligned}$$

From the above, $\frac{dy}{dx} > 0$ for $x < 2$

and $\frac{dy}{dx} < 0$ for $x > 2$

Hence the gradient is zero when $x = 2$, i.e. 2 km from the beginning of the trail, and the height of the pass is 2.16 km.

5 a Let $y = 25 - 0.1t^3$

At the surface of the pond $y = 0$

$\therefore \quad 25 - 0.1t^3 = 0$

$\therefore \quad 0.1t^3 = 25$

$\therefore \quad t^3 = 250$

$\therefore \quad t = \sqrt[3]{250} \approx 6.30$

Hence it takes the tadpole approximately 6.30 seconds to reach the surface.

$$\begin{aligned} \text{Speed} &= \frac{dy}{dt} \\ &= -0.3t^2 \end{aligned}$$

At $t = \sqrt[3]{250}$,

$$\begin{aligned} \frac{dy}{dt} &= -0.3(\sqrt[3]{250})^2 \\ &\approx -11.9 \end{aligned}$$

The tadpole's speed as it reaches the surface is approximately 11.9 cm/s.

b When $t_1 = 0$, $\quad y_1 = 25 - 0.1(0)^3 = 25$

When $t_2 = \sqrt[3]{250}$, $\quad y_2 = 0$

$$\text{Average speed} = \frac{y_2 - y_1}{t_2 - t_1}$$

$$= \frac{0 - 25}{\sqrt[3]{250} - 0}3 \approx -3.97$$

Hence the average speed over this time is 3.97 cm/s.

6 a

$$y = x(x - 2)$$

$$= x^2 - 2x$$

$$\frac{dy}{dx} = 2x - 2$$

At $(0, 0)$ $\quad \dfrac{dy}{dx} = 2(0) - 2$

$$= -2$$

At $(2, 0)$ $\quad \dfrac{dy}{dx} = 2(2) - 2$

$$= 2$$

Geometrically, the angles of inclination between the positive direction of the x-axis and the tangents to the curve at $(0, 0)$ and $(2, 0)$ are supplementary (i.e. add to 180°).

b

$$y = x(x - 2)(x - 5)$$

$$= x(x^2 - 5x - 2x + 10)$$

$$= x(x^2 - 7x + 10)$$

$$= x^3 - 7x^2 + 10x$$

$$\frac{dy}{dx} = 3x^2 - 14x + 10$$

At $(0, 0)$ $\quad \dfrac{dy}{dx} = l$

$\therefore \quad l = 3(0)^2 - 14(0) + 10$

$$= 10$$

At $(2, 0)$ $\quad \dfrac{dy}{dx} = m$

$\therefore \quad m = 3(2)^2 - 14(2) + 10$

$$= 12 - 28 + 10$$

$$= -6$$

$$\text{At } (5, 0) \qquad \frac{dy}{dx} = n$$

$$\therefore \qquad n = 3(5)^2 - 14(5) + 10$$

$$= 75 - 70 + 10$$

$$= 15$$

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = \frac{1}{10} + \frac{1}{-6} + \frac{1}{15}$$

$$= \frac{3 - 5 + 2}{30}$$

$$= 0 \text{ as required.}$$

Chapter 18 – Applications of differentiation of polynomials

Solutions to Exercise 18A

1 a $f(x) = x^2, \therefore f'(x) = 2x$

$$f'(2) = 4$$

Tangent at (2, 4) has equation:

$y - 4 = 4(x - 2)$

$\therefore \quad y = 4x - 4$

Normal at (2, 4) has equation:

$$y - 4 = -\frac{1}{4}(x - 2)$$

$$y = -\frac{1}{4}x + \frac{9}{2}$$

$\therefore \ 4x + y = 18$

b $f(x) = (2x - 1)^2 = 4x^2 - 4x + 1$

$\therefore f'(x) = 8x - 4$

$f'(2) = 12$

Tangent at (2, 9) has equation:

$y - 9 = 12(x - 2)$

$\therefore y = 12x - 15$

Normal at (2, 9) has equation:

$$y - 9 = -\frac{1}{12}(x - 2)$$

$$y = -\frac{1}{12}x + \frac{55}{6}$$

$\therefore \quad 12y + x = 110$

c $f(x) = 3x - x^2, \therefore f'(x) = 3 - 2x$

$$f'(2) = -1$$

Tangent at (2, 2) has equation:

$y - 2 = -(x - 2)$

$\therefore y = -x + 4$

Normal at (2,2) has equation:

$y - 2 = x - 2$

$\therefore \quad y = x$

d $f(x) = 9x - x^3, \therefore f'(x) = 9 - 3x^2$

$$f'(1) = 6$$

Tangent at (1, 8) has equation:

$y - 8 = 6(x - 1)$

$\therefore \quad y = 6x + 2$

Normal at (1, 8) has equation:

$$y - 8 = -\frac{1}{6}(x - 1)$$

$$y = -\frac{1}{6}x + \frac{49}{6}$$

$\therefore \quad 6y + x = 49$

2 $y = 3x^3 - 4x^2 + 2x - 10$

$\therefore \dfrac{dy}{dx} = 9x^2 - 8x + 2$

Intersection with the y-axis is at $(0, -10)$

$\therefore$ gradient = 2

Tangent equation: $y + 10 = 2(x - 0)$

$\therefore \quad y = 2x - 10$

3 $y = x^2, \therefore \dfrac{dx}{dy} = 2x$

Tangent at (1,1) has grad = 2 and equation:

$y - 1 = 2(x - 1)$

$\therefore \quad y = 2x - 1$

$y = \dfrac{x^3}{6}, \therefore \dfrac{dy}{dx} = \dfrac{x^2}{2}$

Tangent at $\left(2, \dfrac{4}{3}\right)$ has grad = 2 and equation:

$$y - \frac{4}{3} = 2(x - 2)$$

$$\therefore y = 2x - \frac{8}{3}$$

Tangents are parallel, since both have gradient = 2.
To find the perpendicular distance between them we need to measure the normal between, which has a gradient of $-\frac{1}{2}$.
From (1,1) the normal is:

$$y - 1 = -\frac{1}{2}(x - 1)$$
$$\therefore y = -\frac{x}{2} + \frac{3}{2}$$

This cuts the 2nd tangent where:

$$-\frac{x}{2} + \frac{3}{2} = 2x - \frac{8}{3}$$
$$\frac{5x}{2} = \frac{8}{3} + \frac{3}{2}$$
$$15x = 16 + 9, \therefore x = \frac{5}{3}$$

$\therefore$ Normal cuts 2nd tangent at $\left(\frac{5}{3}, \frac{2}{3}\right)$
Distance between (1,1) and $\left(\frac{5}{3}, \frac{2}{3}\right)$ is
$$\sqrt{\left(\frac{5}{3} - 1\right)^2 + \left(\frac{2}{3} - 1\right)^2} = \frac{\sqrt{5}}{3}$$

4
$$y = x^3 - 6x^2 + 12x + 2$$
$$\therefore \frac{dy}{dx} = 3x^2 - 12x + 12$$
Tangents parallel to $y = 3x$ have gradient = 3

$$\therefore\ 3x^2 - 12x + 12 = 3$$
$$3x^2 - 12x + 9 = 0$$
$$3(x - 1)(x - 3) = 0, \therefore x = 1, 3$$

$y(1) = 9; y(3) = 11$
Tangents are:

$$y - 9 = 3(x - 1), \therefore y = 3x + 6$$
$$y - 11 = 3(x - 3), \therefore y = 3x + 2$$

5 **a**
$$y = (x - 2)(x - 3)(x - 4)$$
$$= x^3 - 9x^2 + 26x + 24$$
$$\therefore \frac{dy}{dx} = 3x^2 - 18x + 26$$
$\frac{dy}{dx} = 2$ at P, 4 at R and -1 at Q.
Gradients at P and R are equal, so tangents are parallel.

b Normal at $Q(3, 0)$ has gradient = ±1:
$y = x - 3$ which cuts the y-axis at $(0, -3)$.

6 $y = x^2 + 3, \therefore \frac{dy}{dx} = 2x$
Gradient at $x = a$ is $2a; y(a) = a^2 + 3$
Tangent has equation:

$$y - (a^2 + 3) = 2a(x - a)$$
$$\therefore \quad y = 2ax - 2a^2 + a^2 + 3$$
$$= 2ax - a^2 + 3$$

Tangents pass through (2, 6)

$$\therefore \quad 6 = 2a(2) - a^2 + 3$$
$$a^2 - 4a + 3 = 0$$
$$(a - 1)(a - 3) = 0, \therefore a = 1, 3$$

If $a = 1$, the point is (1, 4)
If $a = 3$, the point is (3, 12)

7 **a** $y = x^3 - 2x, \therefore \frac{dy}{dx} = 3x^2 - 2$
At (2, 4), gradient = 10
Equation of tangent:

$$y - 4 = 10(x - 2)$$
$$\therefore y = 10x - 16$$

b The tangent meets the curve again where

$$y = x^3 - 2x = 10x - 16$$

$$\therefore \quad x^3 - 12x + 16 = 0$$

$$(x - 2)(x^2 + 2x - 8) = 0$$

$$(x - 2)^2(x + 4) = 0$$

$$\therefore \quad x = 2, -4$$

Tangent cuts the curve again at $x = -4$

$y(-4) = (-4)^3 - 2(-4) = -56$

Coordinates are $(-4, -56)$.

8 a $$y = x^3 - 9x^2 + 20x - 8$$

$$\therefore \frac{dy}{dx} = 3x^2 - 18x + 20$$

At $(1, 4)$, gradient $= 5$

Equation of tangent:

$$y - 4 = 5(x - 1)$$

$$\therefore \quad y = 5x - 1$$

b $4x + y - 3 = 0$ has gradient $= -4$

$$\therefore \frac{dy}{dx} = 3x^2 - 18x + 20 = -4$$

$$3x^2 - 18x + 24 = 0$$

$$x^2 - 6x + 8 = 0$$

$$(x - 2)(x - 4) = 0$$

$$\therefore x = 2, 4$$

If $x = 2, y = 2^3 - 9(2)^2 + 20(2) - 8$
$= 4$

If $x = 4, y = 4^3 - 9(4)^2 + 20(4) - 8$
$= -8$

Coordinates are $(2, 4)$ and $(4, -8)$.

Solutions to Exercise 18B

1 **a** $y = 35 + 12x^2$

$\therefore y(2) = 83, y(1) = 47$

Av. rate of change

$= \dfrac{y(2) - y(1)}{2 - 1} = \dfrac{83 - 47}{1} = 36$

b $y(2 - h) = 35 + 12(2 - h)^2$

$= 35 + 12(4 - 4h + h^2)$

$= 83 - 48h + 12h^2$

Av. rate of change $= \dfrac{y(2) - y(2 - h)}{2 - (2 - h)}$

$= \dfrac{83 - \left(83 - 48h + 12h^2\right)}{h} = 48 - 12h$

c Rate of change at $x = 2$ is $y'(2)$:

$y'(x) = 24x, \therefore y'(2) = 48$

(Alternatively, let $h \to$ 0 in **part b** answer)

2 **a** $M = 200\,000 + 600\,t^2 - \dfrac{200}{3}t^3$

$\therefore \dfrac{dM}{dt} = 1200t - 200t^2 = 200t(6 - t)$

b At $t = 3$, $\dfrac{dM}{dt} = \$1800$/month

c $\dfrac{dM}{dt} = 0$ at $t = 0$ and $t = 6$

3 **a** $R = 30P - 2P^2, \therefore \dfrac{dR}{dP} = 30 - 4P$

$\dfrac{dR}{dP}$ means the rate of change of profit per dollar increase in list price.

b $\dfrac{dR}{dP}$ is 10 at $P = 5$ and -10 at $P = 10$

c Revenue is rising for

$0 < P < 7.5\left(= \dfrac{30}{4}\right)$

4 $P = 100(5 + t - 0.25t^2)$

$\therefore \dfrac{dP}{dt} = 100(1 - 0.5t)$

i At 1 year $\dfrac{dP}{dt} = 100(1 - 0.5) =$ 50 people/yr

ii At 2 years $\dfrac{dP}{dt} = 100(1 - 1) =$ 0 people/yr

iii At 3 years $\dfrac{dP}{dt} = 100(1 - 1.5) = -50$

i.e. decreasing by 50 people/yr

5 **a** $V(t) = \dfrac{5}{8}\left(10t^2 - \dfrac{t^3}{3}\right), 0 \le t \le 20$

i $V(0) = 0$

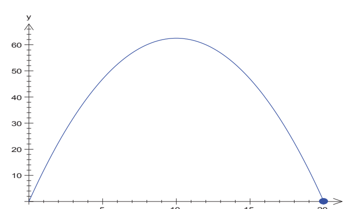
ii $V(t) = \dfrac{5}{8}\left(10(20)^2 - \dfrac{20^3}{3}\right)$

$= \dfrac{5}{8}\left(4000 - \dfrac{8000}{3}\right)$

$= \dfrac{2500}{3} = 833\dfrac{1}{3}$ mL

b $V'(t) = \dfrac{5}{8}(20t - t^2)$

c



6 $A(t) = \dfrac{t}{2} + \dfrac{1}{10}t^2$ km^2

$\therefore A'(t) = \dfrac{1}{2} + \dfrac{t}{5}$km^2/h

a $A(1) = \dfrac{1}{2} + \dfrac{1}{10} = 0.6$ km^2

b $A'(1) = \frac{1}{2} + \frac{1}{5} = 0.7\text{km}^2/\text{hr}$

Solutions to Exercise 18C

1 a $f(x) = x^2 - 6x + 3$

$\therefore f'(x) = 2x - 6$

$2x - 6 = 0, \therefore x = 3$

$f(3) = -6$

Coordinates of stationary pt are $(3, -6)$.

b $y = x^3 - 4x^2 - 3x + 20, x > 0$

$\therefore y'(x) = 3x^2 - 8x - 3$

$= (3x + 1)(x - 3)$

$y' = 0$ for $x = 3$ since $x = -\frac{1}{3} < 0$

$y(3) = 2$

Coordinates of stationary pt are $(3, 2)$.

c $z = x^4 - 32x + 50$

$\therefore z' = 4x^3 - 32$

$4x^3 - 32 = 0, \therefore x = 2$

$z(2) = 2$

Coordinates of stationary pt are $(2, 2)$.

d $q = 8t + 5t^2 - t^3, t > 0$

$\therefore q' = 8 + 10t - 3t^2$

$= (4 - t)(3t + 2)$

$q' = 0$ for $t = 4$ since $x = -\frac{2}{3} < 0$

$q(4) = 48$

Coordinates of stationary pt are $(4, 48)$.

e $y = 2x^2(x - 3)$

$= 2x^3 - 6x^2$

$\therefore y' = 6x^2 - 12x$

$= 6x(x - 2)$

$y' = 0$ for $x = 0, 2$

$y(0) = 0$; $y(2) = -8$

Stationary pts at $(0, 0)$ and $(2, -8)$.

f $y = 3x^4 - 16x^3 + 24x^2 - 10$

$\therefore y = 12x^3 - 48x^2 + 48x$

$= 12x(x - 2)^2$

$y' = 0$ for $x = 0, 2$

$y(0) = -10$; $y(2) = 6$

Stationary pts at $(0, -10)$ and $(2, 6)$.

2 $y = ax^2 + bx + c, \therefore y' = 2ax + b$

Using $(0, -1)$: $c = -1$

Using $(2, -9)$: $4a + 2b = -8$

$y'(2) = 0, \therefore 4a + b = 0$

$\therefore a = 2, b = -8, c = -1$

3 $y = ax^2 + bx + c, \therefore y' = 2ax + b$

When $x = 0$, the slope of the curve is 45°.

$y'(0) = 1, \quad \therefore b = 1$

$y'(1) = 0, \quad \therefore 2a + b = 0$

$a = -\frac{1}{2}$

$y(1) = 2, \quad \therefore -\frac{1}{2} + 1 + c = 2$

$c = \frac{3}{2}$

$\therefore a = -\frac{1}{2}, \quad b = 1, c = \frac{3}{2}$

4 a $y = ax^2 + bx, \therefore y' = 2ax + b$

$y'(2) = 3, \therefore 4a + b = 3$

$y(2) = -2, \therefore 4a + 2b = -2$

$\therefore a = 2, b = -5$

b $y'(x) = 4x - 5 = 0, \therefore x = \dfrac{5}{4}$

$y\left(\dfrac{5}{4}\right) = 2\left(\dfrac{5}{4}\right)^2 - 5\left(\dfrac{5}{4}\right) = -\dfrac{25}{8}$

Coordinates of stationary pt are $\left(\dfrac{5}{4}, -\dfrac{25}{8}\right)$.

5 $y = x^2 + ax + 3, \therefore\ y' = 2x + a$

$y' = 0$ when $x = 4$

$\therefore\ a = -8$

6 $y = x^2 - ax + 4, \therefore\ y' = 2x - a$

$y' = 0$ when $x = 3$

$\therefore a = 6$

7 a $y = x^2 - 5x - 6, \therefore\ y' = 2x - 5$

$y' = 0$ when $x = 2.5: y\left(\dfrac{5}{2}\right) = -12.25$

Stationary pt at $(2.5, -12.25)$.

b
$$\begin{aligned} y &= (3x-2)(8x+3) \\ &= 24x^2 - 7x - 6 \end{aligned}$$

$$y' = 48x - 7 = 0, \therefore\ x = \frac{7}{48}$$

$$\begin{aligned} y\left(\frac{7}{48}\right) &= \left(\frac{7}{16} - 2\right)\left(\frac{7}{6} + 3\right) \\ &= -\frac{625}{96} \end{aligned}$$

Stationary pt at $\left(\dfrac{7}{48}, -\dfrac{625}{96}\right)$.

c
$$y = 2x^3 - 9x^2 + 27$$

$$\begin{aligned} \therefore\ y' &= 6x^2 - 18x \\ &= 6x(x-3) \end{aligned}$$

$y' = 0$ at $x = 0, 3: y(0) = 27, y(3) = 0$

Stationary pts at $(0, 27)$ and $(3, 0)$.

d
$$y = x^3 - 3x^2 - 24x + 20$$

$$\begin{aligned} \therefore\ y' &= 3x^2 - 6x - 24 \\ &= 3(x+2)(x-4) \end{aligned}$$

$y' = 0$ when $x = -2, 4$:

$y(-2) = -48, y(4) = -60$

Stationary pts at $(-2, 48)$ and $(4, -60)$.

e
$$\begin{aligned} y &= (x+1)^2(x+4) \\ &= x^3 + 6x^2 + 9x + 4 \end{aligned}$$

$$\begin{aligned} \therefore\ y' &= 3x^2 + 12x + 9 \\ &= 3(x+1)(x+3) \end{aligned}$$

$y' = 0$ when $x = -3, -1$:

$y(-3) = 4, y(-1) = 0$

Stationary pts at $(-3, 4)$ and $(-1, 0)$.

f
$$\begin{aligned} y &= (x+1)^2 + (x+2)^2 \\ &= 2x^2 + 6x + 5 \end{aligned}$$

$\therefore\ y' = 4x + 6 = 0, x = -1.5$

$y(-1.5) = 0.5$

Stationary pt at $(-1.5, 0.5)$.

8 $y = ax^2 + bx + 12, \therefore\ y' = 2ax + b$

$y' = 0$ at $x = 1: 2a + b = 0$

Using (1, 13): $a + b = 1$

$\therefore a = -1, b = 2$

9
$$y = ax^3 + bx^2 + cx + d$$

$\therefore\ y'(x) = 3ax^2 + 2bx + c$

$y' = -3$ at $x = 0$: $\qquad c = -3$

$y' = 0$ at $x = 3$: $\qquad 27a + 6b - 3 = 0$

$\qquad 9a + 2b = 1 \ldots$ **(1)**

$y(0) = \frac{15}{2}$: $d = \frac{15}{2}$

$y(3) = 6$, $\therefore\ 27a + 9b - 9 + \frac{15}{2} = 6$

$$9a + 3b = \frac{5}{2} \ldots \mathbf{(2)}$$

From **(1)** and **(2)**: $b = \frac{3}{2}, \therefore a = -\frac{2}{9}$

$a = -\frac{2}{9}, b = \frac{3}{2}, c = -3, d = \frac{15}{2}$

Solutions to Exercise 18D

1 a $y = 9x^2 - x^3$

$\therefore\ y' = 18x - 3x^2 = 3x(6 - x)$

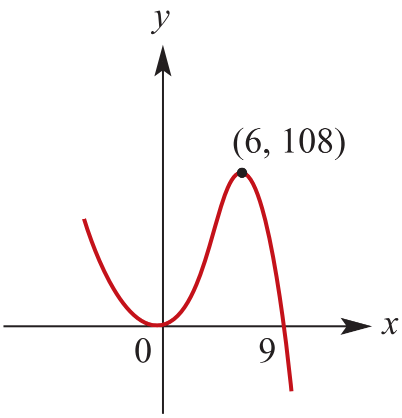
$y' = 0$ at $x = 0, 6$:

$y(0) = 0$; $y(6) = 108$

x	-3	0	3	6	9
y'	$-$	0	$+$	0	$-$

$(0, 0)$ is a local minimum.

$(6, 108)$ is a local maximum.



b $y = x^3 - 3x^2 - 9x$

$\therefore y' = 3x^2 - 6x - 9 = 3(x + 1)(x - 3)$

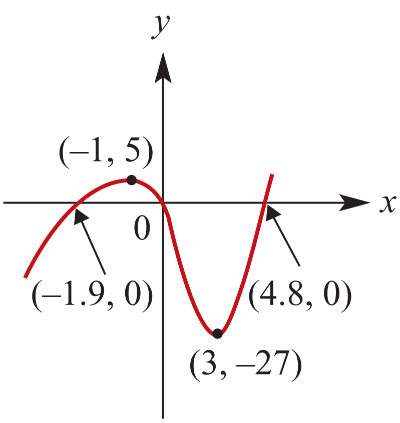
$y' = 0$ at $x = -1, 3$:

$y(-1) = 5$; $y(3) = -27$

x	-2	-1	0	3	4
y'	$+$	0	$-$	0	$+$

$(-1, 5)$ is a local maximum.

$(3, -27)$ is a local minimum.



c $y = x^4 - 4x^3$

$\therefore\ y' = 4x^3 - 12x^2 = 4x^2(x - 3)$

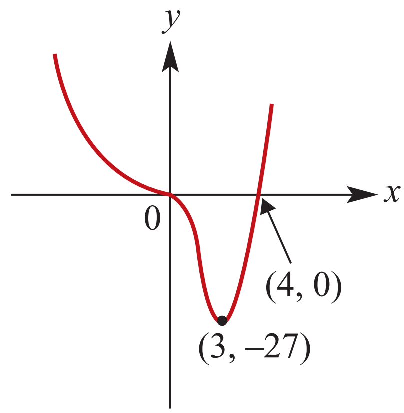
$y' = 0$ at $x = 0, 3$:

$y(0) = 0$; $y(3) = -27$

x	-1	0	1	3	4
y'	$-$	0	$-$	0	$+$

$(0, 0)$ is a stationary pt of inflexion.

$(3, -27)$ is a local minimum.



2 a $y = x^2(x - 4) = x^3 - 4x^2$

$\therefore\ y' = 3x^2 - 8x = x(3x - 8)$

$y' = 0$ at $x = 0, \dfrac{8}{3}$:

$y(0) = 0$; $y\left(\dfrac{8}{3}\right) = -\dfrac{256}{27}$

x	-1	0	1	$\frac{8}{3}$	3
y'	$+$	0	$-$	0	$+$

$(0, 0)$ is a local maximum.

$\left(\dfrac{8}{3}, -\dfrac{256}{27}\right)$ is a local minimum.

b $y = x^2(3 - x) = 3x^2 - x^3$

$\therefore\ y' = 6x - 3x = 3x(2 - x)$

$y' = 0$ at $x = 0, 2$

$y(0) = 0$; $y(2) = 4$

x	-1	0	1	2	3
y'	$-$	0	$+$	0	$-$

$(0, 0)$ is a local minimum

(2, 4) is a local maximum

c $y = x^4$

$\therefore\ y' = 4x^3$

$y' = 0$ at $x = 0$; $y(0) = 0$

$y'(-1) = -4$; $y'(1) = 4$

(0, 0) is a local minimum.

d $y = x^5(x-4) = x^6 - 4x^5$

$\therefore\ y' = 6x^5 - 20x^4 = 2x^4(3x - 10)$

$y' = 0$ at $x = 0, \dfrac{10}{3}$

$$y(0) = 0; y\left(\frac{10}{3}\right) = \left(\frac{10}{3}\right)^5\left(-\frac{2}{3}\right) = -\frac{200\,000}{729}$$

x	-1	0	1	$\frac{10}{3}$	4
y'	$-$	0	$-$	0	$+$

(0, 0) is a stationary pt of inflexion.

$\left(\dfrac{10}{3}, -\dfrac{200\,000}{729}\right)$ is a local minimum.

e $y = x^3 - 5x^2 + 3x + 2$

$\therefore\ y' = 3x^2 - 10x + 3$

$= (3x-1)(x-3) = 0,$

$\therefore x = \dfrac{1}{3}, 3$

$y\left(\dfrac{1}{3}\right) = \dfrac{67}{27}$; $y(3) = -7$

x	0	$\frac{1}{3}$	1	3	4
y'	$+$	0	$-$	0	$+$

$\left(\dfrac{1}{3}, \dfrac{67}{27}\right)$ is a local maximum.

$(3, -7)$ is a local minimum.

f $y = x(x-8)(x-3)$

$= x^3 - 11x^2 + 24x$

$\therefore\ y' = 3x^2 - 22x + 24$

$= (3x-4)(x-6)$

$y' = 0$ at $x = \dfrac{4}{3}, 6$

$$y\left(\frac{4}{3}\right) = \frac{4}{3}\left(-\frac{20}{3}\right)\left(-\frac{5}{3}\right) = \frac{400}{27}$$

$y(6) = -36$

x	0	$\frac{4}{3}$	2	6	9
y'	$+$	0	$-$	0	$+$

$\left(\dfrac{4}{3}, \dfrac{400}{27}\right)$ is a local maximum.

$(6, -36)$ is a local minimum.

3 a $y = 2 + 3x - x^3 = (x+1)^2(2-x)$

Axis intercepts at (0, 2), $(-1, 0)$ and (2, 0)

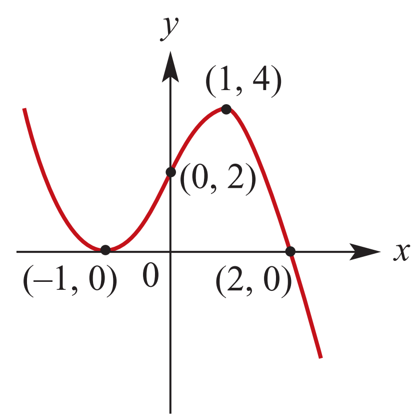
$y' = 3 - 3x^2 = 0, \therefore x = \pm 1$

$y(-1) = 0$; $y(1) = 4$

x	-2	-1	0	1	2
y'	$-$	0	$+$	0	$-$

$(-1, 0)$ is a local minimum.

(1, 4) is a local maximum.



b $y = 2x^2(x-3) = 2x^3 - 6x^2$

Axis intercepts at (0, 0) and (3, 0)

$y' = 6x^2 - 12x = 6x(x-2)$

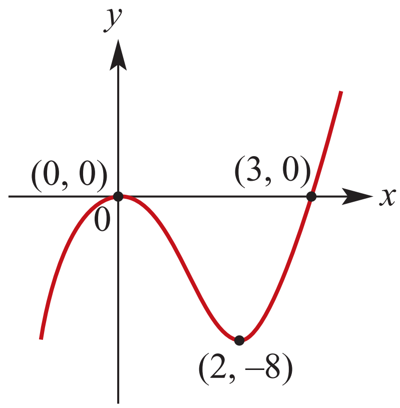
$y' = 0$ when $x = 0, 2$:

$y(0) = 0$; $y(2) = -8$

x	-1	0	1	2	3
y'	$+$	0	$-$	0	$+$

(0, 0) is a local maximum.

$(2, -8)$ is a local minimum.



c

$y = x^3 - 3x^2 - 9x + 11$

$= (x - 1)(x^2 - 2x - 11)$

$= (x - 1)(x - 1 - 2\sqrt{3})(x - 1 + 2\sqrt{3})$

Axis intercepts at $(0, 11)$, $(1, 0)$, $(1 - 2\sqrt{3}, 0)$ and $(1 + 2\sqrt{3}, 0)$.

$y' = 3x - 6x - 9$

$= 3(x + 1)(x - 3)$

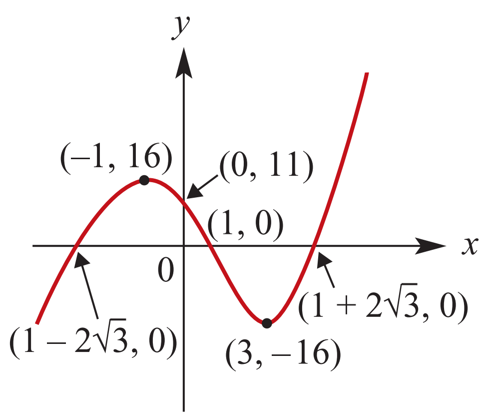
$y' = 0$ when $x = -1, 3$:

$y(-1) = 16$; $y(3) = -16$

x	-2	-1	0	3	4
y'	$+$	0	$-$	0	$+$

$(-1, 16)$ is a local maximum.

$(3, -16)$ is a local minimum.



4 Graphs with a stationary point at $(-2, 10)$

a $y = 2x^3 + 3x^2 - 12x - 10$

$\therefore\ y' = 6x^2 + 6x - 12$

$= 6(x + 2)(x - 1)$

x	-3	-2	0	1	2
y'	$+$	0	$-$	0	$+$

$(-2, 10)$ is a local maximum.

b $y = 3x^4 + 16x^3 + 24x^2 - 6$

$\therefore\ y = 12x^3 + 48x^2 + 48$

$= 12(x + 2)^2$

$y' > 0; x \neq -2$

$(-2, 10)$ is a stationary pt of inflexion.

5 $y = x^3 - 6x^2 + 9x + 10$

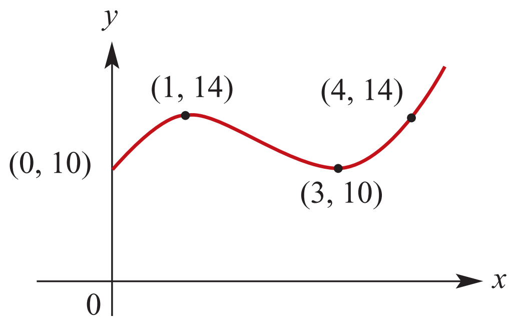
a $y' = 3x^2 - 12x + 9$

$= 3(x - 1)(x - 3)$

x	0	1	2	3	4
y'	$+$	0	$-$	0	$+$

$\{x\colon \frac{dy}{dx} > 0\} = \{x\colon x < 1\} \cup \{x\colon x > 3\}$

b $y(1) = 14, y(3) = 10$
(1, 14) is a local maximum.
(3, 10) is a local minimum.



6 $f(x) = 1 + 12x - x^3$

$f'(x) = 12 - 3x^2$

$= 3(2 - x)(2 + x)$

x	-3	-2	0	2	3
f'	$-$	0	$+$	0	$-$

$\{x: f'(x) > 0\} = \{x: -2 < x < 2\}$

7 $f(x) = 3 + 6x - 2x^3$

a $f'(x) = 6 - 6x^2$

$= 6(1 - x)(1 + x)$

x	-2	-1	0	1	2
f'	$-$	0	$+$	0	$-$

$\{x: f'(x) > 0\} = \{x: -1 < x < 1\}$

b $(-\infty, -1) \cup (1, \infty)$

8 **a**
$$\begin{aligned} f(x) &= x(x+3)(x-5) \\ &= x^3 - 2x^2 - 15x \\ \therefore\ f'(x) &= 3x^2 - 4x - 15 \\ &= (3x+5)(x-3) \end{aligned}$$

$f'(x) = 0$ for $x = -\frac{5}{3}, 3$

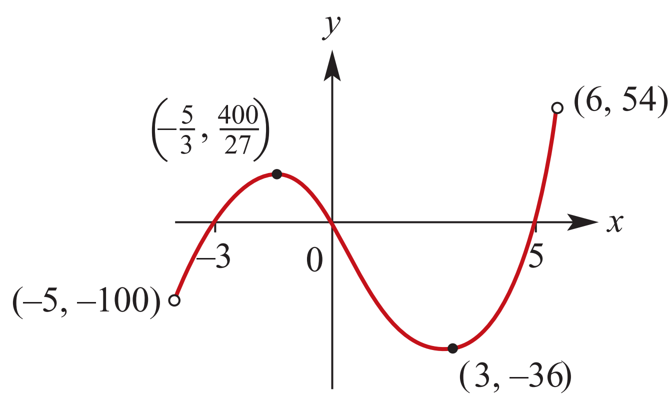
b Axis intercepts at $(0, -15)$, $(-3, 0)$ and $(5, 0)$

$$\begin{aligned} f'(-\tfrac{5}{3}) &= (-\tfrac{5}{3})(\tfrac{4}{3})(-\tfrac{20}{3}) \\ &= \frac{400}{27} \end{aligned}$$

$f(3) = -36$

x	-2	$-\frac{5}{3}$	0	3	4
f'	$+$	0	$-$	0	$+$

$(-\frac{5}{3}, \frac{400}{27})$ is a local maximum.
$(3, -36)$ is a local minimum.



9
$$\begin{aligned} y &= x^3 - 6x^2 + 9x - 4 \\ &= (x-1)^2(x-4) \end{aligned}$$

Axis intercepts at $(0, -4)$, $(1, 0)$ and $(4, 0)$

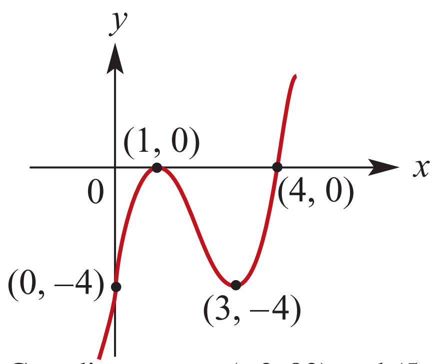
$$\begin{aligned} y' &= 3x^2 - 12x + 9 \\ &= 3(x-1)(x-3) \end{aligned}$$

$y(1) = 0; y(3) = -4$

x	0	1	2	3	4
y'	$+$	0	$-$	0	$+$

(1, 0) is a local maximum.

(3, −4) is a local minimum.



Coordinates are: (−3, 83) and (5, −173).

10 $y = x^3 - 3x^2 - 45x + 2$

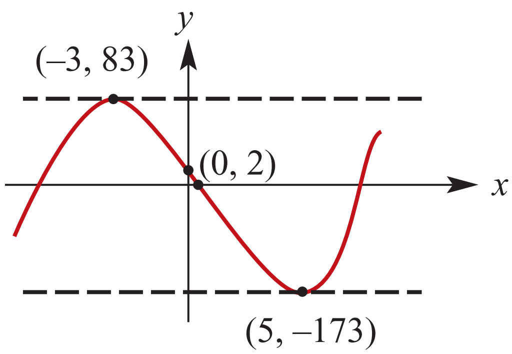
$\therefore\ y' = 3x^2 - 6x - 45$

$= 3(x + 3)(x - 5)$

If tangent is parallel to the x-axis then $y' = 0$

$\therefore x = -3, 5$

$y(-3) = 83; y(5) = -173$



11 $f(x) = x^3 - 3x^2$

$\therefore\ f'(x) = 3x^2 - 6x = 3x(x - 2)$

$f'(x) = 0$ for $x = 0, 2$

$f(0) = 0; f(2) = -4$

x	−1	0	1	2	3
f'	+	0	−	0	+

(0, 0) is a local maximum.

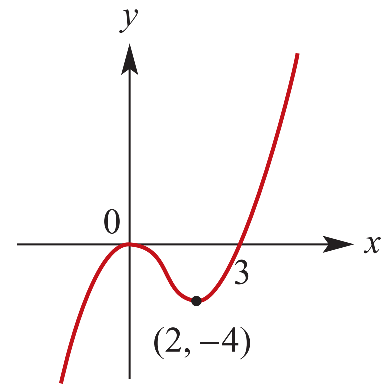
(2, −4) is a local minimum.

a **i** $\{x: f'(x) < 0\} = \{x: 0 < x < 2\}$

ii $\{x: f'(x) > 0\} = \{x: x < 0\} \cup \{x: x > 2\}$

iii $\{x: f'(x) = 0\} = \{0, 2\}$

b



12 $y = x^3 - 9x^2 + 27x - 19$

$= (x - 1)(x^2 - 8x + 19)$

Axis intercepts: (0, −19) and (1, 0)

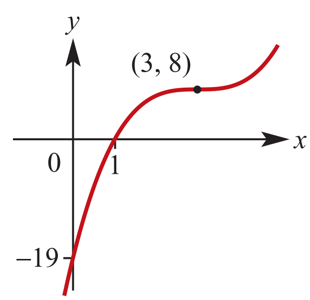
$y' = 3x^2 - 18x + 27$

$= 3(x - 3)^2$

$y' = 0$ when $x = 3$; $y(3) = 8$

$y' > 0$ for all $x \neq 0$

Stationary pt of inflexion at (3, 8)



13 $y = x^4 - 8x^2 + 7$

$= (x^2 - 1)(x^2 - 7)$

$= (x - 1)(x + 1)(x - \sqrt{7})(x + \sqrt{7})$

Axis intercepts: (0, 7), $(-\sqrt{7}, 0)$, (−1, 0), (1, 0) and $(\sqrt{7}, 0)$

$y' = 4x^3 - 16x$

$= 4x(x - 2)(x + 2)$

$y' = 0$ when $x = -2, 0, 2$

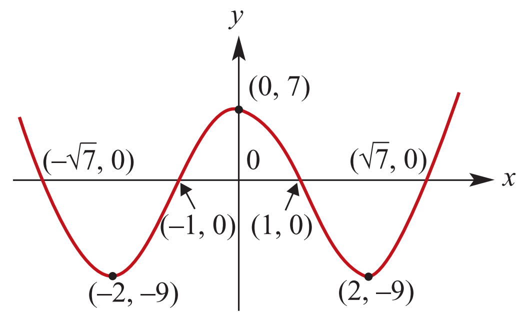
$y(-2) = -9, y(0) = 7, y(2) = -9$

x	-3	-2	-1	0	1	2	3
y'	$-$	0	$+$	0	$-$	0	$+$

$(-2, -9)$ is a local minimum.

$(0, 7)$ is a local maximum.

$(2, -9)$ is a local minimum.



Solutions to Exercise 18E

1 Let x cm be the width and y cm be the the length.
Then $2x + 2y = 200$ which implies that $y = 100 - x$
We note that $0 \le x \le 100$

$$\begin{aligned} \text{Area} &= xy \\ &= x(100 - x) \\ &= 100x - x^2 \end{aligned}$$

Turning point of parabola with negative coefficient of x^2. Therefore a maximum.

$$\frac{dA}{dx} = 100 - 2x$$

$$\frac{dA}{dx} = 0 \text{ implies that}$$

$$100 - 2x = 0$$

$$\therefore x = 50$$

Maximum area of $50 \times 50 = 2500$ cm^2 when $x = 50$

2 Let $P = x(10 - x) = 10x - x^2$

Then $\dfrac{dP}{dx} = 10 - 2x$

$$\frac{dP}{dx} = 0 \text{ implies that}$$

$$10 - 2x = 0$$

$$\therefore x = 5$$

Turning point of parabola with negative coefficient of x^2. Therefore a maximum.
Maximum value of $P = 25$

3 Let $M = x^2 + y^2$and it is given that $x + y = 2$
$\therefore y = 2 - x$ and $M = x^2 + (2 - x)^2 = 2x^2 - 4x + 4$

Then $\dfrac{dM}{dx} = 4x - 4$ Turn-

$$\frac{dM}{dx} = 0 \text{ implies that}$$

$$4 - 4x = 0$$

$$\therefore x = 1$$

ing point of parabola with positive coefficient of x^2. Therefore a minimum.
Therefore minimum value of $M = 1 + 1 = 2$

4 **a** Let x cm be the length of the sides of the squares which are being removed.
The base of the box is a square with side lengths $6 - 2x$ cm and the height of the box is x cm.
Therefore the volume Vcm^3 is given by

$$\begin{aligned} V &= (6 - 2x)^2 x \\ &= (36 - 24x + 4x^2)x \\ &= 36x - 24x^2 + 4x^3 \end{aligned}$$

Note that $0 \le x \le 3$

b

$$\frac{dV}{dx} = 12x^2 - 48x + 36$$

$$\frac{dV}{dx} = 0 \text{ implies that}$$

$$12x^2 - 48x + 36 = 0$$

$$\therefore x^2 - 4x + 3 = 0$$

$$\therefore (x - 1)(x - 3) = 0$$

$$\therefore x = 1 \text{ or } x = 3$$

The maximum value occurs when $x = 1$
We note that $V(3) = 0$
Maximum value $= V(1) = 16$.
The maximum value of the volume of the box is 16 cm^3

5 $y(x) = \frac{x^2}{400}(20 - x), 0 \le x \le 20$

a **i** $y(5) = \frac{5^2}{400}(20 - 5)$

$= \frac{15}{16} = 0.9375$ m

ii $y(10) = \frac{10^2}{400}(20 - 10)$

$= \frac{5}{2} = 2.5$ m

iii $y(15) = \frac{15^2}{400}(20 - 15)$

$= 2.8125$ m

b Use a CAS calculator to find the gradient function:

$$y'(x) = \frac{x(20 - x)}{200} - \frac{x^2}{400}$$
$$= \frac{x(40 - 3x)}{400}$$

$y' = 0$ when $x = 0, \frac{40}{3}$
(0, 0) is the local minimum.

$$y(\frac{40}{3}) = \frac{40^2}{3600}(20 - \frac{40}{3})$$
$$= (\frac{4}{9})(\frac{20}{3}) = \frac{80}{27}$$

Local maximum at $\left(\frac{40}{3}, \frac{80}{27}\right)$.

c **i** $y' = \frac{x(40 - 3x)}{400} = \frac{1}{8}$

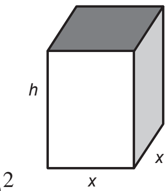
$\therefore 40x - 3x^2 - 50 = 0$
$x = 1.396, 11.397$

ii $y' = \frac{x(40 - 3x)}{400} = -\frac{1}{8}$

$\therefore 40x - 3x^2 + 50 = 0$
$x = 14.484$ (since $x > 0$)

6 TSA = 150 cm^2

a Area of top & base
$= 2x^2$
Area of 4 sides
$= 4xh$
$\therefore\ 2x^2 + 4xh = 150$ cm^2



$$2xh = 150 - 4x^2$$
$$h = \frac{75 - x^2}{2x}$$

b $V(x) = x^2h = x^2\left(\frac{75 - x^2}{2x}\right)$

$= \frac{x}{2}(75 - x^2)$

$= \frac{1}{2}(75x - x^3)$

c $V'(x) = \frac{1}{2}(75 - 3x^2)$
$v'(x) = 0, \therefore x^2 = 25$

$x = 5$ cm
$\therefore\ V(5) = 125$ cm^3
$V'(4) > 0; V'(6) < 0$
$\therefore$ stationary pt must be a maximum.

d Since $5 > 4$ and V is still increasing at $x = 4$,
V max . is $V(4) = \frac{4}{2}(75 - 4^2)$

$= 118$ cm^3

7 $V = \pi r^2 h$ and $r + h = 12$
$h = 12 - r$
$\therefore V = \pi r^2(12 - r) = \pi(12r^2 - r^3)$
Note $0 \le r \le 12$

$$\frac{dV}{dr} = \pi(24r - 3r^2)$$

$\frac{dV}{dr} = 0$ implies that

$24r - 3r^2 = 0$

$\therefore 3r(8 - r) = 0$

$\therefore r = 0$ or $r = 8$

Maximum occurs when $r = 8$

Maximum volume= $8^2(12 - 8)\pi = 256\pi$

8 The lengths of the sides of the base of the tray are $50 - 2x$ cm and $40 - 2x$ cm. The height of the tray is x cm. Therefore the volume $V\text{cm}^3$ of the tray is given by

$V = (50 - 2x)(40 - 2x)x =$

$4(x^3 - 45x^2 + 500x)$

We note: $20 \le x \le 25$

$$\frac{dV}{dx} = 4(3x^2 - 90x + 500)$$

$$\frac{dV}{dx} = 0 \text{ implies that}$$

$$3x^2 - 90x + 500 = 0$$

$$\therefore x = \frac{5(9 - \sqrt{21})}{3} \text{ or } x = \frac{5(9 + \sqrt{21})}{3}$$

Maximum occurs when $x = \dfrac{5(9 - \sqrt{21})}{3}$

9 $f(x) = 2 - 8x^2, -2 \le x \le 2$

$\therefore f'(x) = -16x = 0, \therefore x = 0$

For $x < 0, f'(x) > 0$; for $x > 0, f(x) < 0$

Local and absolute maximum for $f(0) = 2$ Absolute minimum at $f(\pm 2) = -30$.

10 $f(x) = x^3 + 2x + 3, -2 \le x \le 1$

$\therefore f'(x) = 3x^2 + 2 > 0, x \in R$

Function is constantly increasing, so absolute maximum is $f(1) = 6$.

Absolute minimum $\neq f(-2) = -9$.

11 $f(x) = 2x^3 - 6x^2, 0 \le x \le 4$

$\therefore f'(x) = 6x^2 - 12x = 6x(x - 2)$

x	0	1	2	3
f'	0	−	0	+

$x = 0$ is a local maximum $f(0) = 0$, but $f(4) = 32$, so the absolute maximum is 32. $x = 2$ is an absolute minimum of $f(2) = -8$.

12 $f(x) = 2x^4 - 8x^2, -2 \le x \le 5$

$\therefore f'(x) = 8x^3 - 16x$

$= 8x(x - \sqrt{2})(x + \sqrt{2})$

x	−2	$-\sqrt{2}$	−1	0	1	$\sqrt{2}$	2
f'	−	0	+	0	−	0	+

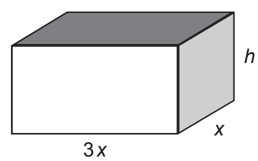
$x = 0$ is a local maximum $f(0) = 0$, but $f(5) = 1050$, so the absolute maximum is 1050.

At the other boundary condition,

$f(-2) = 0 < 1050$.

$f(\pm\sqrt{2}) = -8$ are local and absolute minima.

13



Total edges: $4h + 4x + 12x = 20$ cm.

$\therefore\ h = \dfrac{20 - 16x}{4} = 5 - 4x$

a $v = x(3x)h = 3x^2(5 - 4x)$

$= 15x^2 - 12x^3$

b $\dfrac{dV}{dx} = 30x - 36x^2 = 6x(5 - 6x)$

c Sign diagram for $x \in [0, 1.25]$:

x	0	$\frac{1}{2}$	$\frac{5}{6}$	1
V'	0	+	0	−

Local maximum = $V(\frac{5}{6}) = \dfrac{125}{36}$ cm^3

d If $x \in [0, 0.8]$, then $0.8 < \dfrac{5}{6}$ and V is still increasing

$\therefore V \max = V(0.8) = \dfrac{432}{125}$ cm^3

e If $x \in [0, 1]$, $V \max = V(\dfrac{5}{6}) = \dfrac{125}{36}$ cm^3

14 $x + y = 20, \therefore y = 20 - x$

a If $x \in [2, 5]$, $y \in [15, 18]$

b $z = xy = x(20 - x) = 20x - x^2$

$\dfrac{dz}{dx} = 20 - 2x = 0$

$\therefore\ x = 10$ for a stationary point.

However, with x restricted to [2, 5], $\dfrac{dz}{dx} > 0$ So the minimum value of z is $z(2) = 36$ and maximum value is $z(5) = 75$.

15 $2x + y = 50, \therefore y = 50 - 2x$

$\therefore z = x^2y = x^2(50 - 2x) = 50x^2 - 2x^3$

$\dfrac{dz}{dx} = 100x - 6x^2 = 2x(50 - 3x)$

Inverted cubic, so z has a local minimum at $(0, 0)$ and a local maximum at $\left(\dfrac{50}{3}, \dfrac{125\,000}{27}\right)$

a $x \in [0, 25]$, So max. $z = \dfrac{125\,000}{27}$

b $x \in [0, 10]$, so max. $z = z(10) = 3000$

c $x \in [5, 20]$, so max. $z = \dfrac{125\,000}{27}$

16 **a** 1st piece has length x metres, so 2nd piece has length $(10 - x)$ metres, each folded into 4 to make a square:

$$\begin{aligned}\text{Total area } A &= (\frac{x}{4})^2 + (\frac{10 - x}{4})^2\\ &= \frac{x^2}{16} + \frac{100 - 20x + x^2}{16}\\ &= \frac{1}{8}(x^2 - 10x + 50)\text{ m}^2\end{aligned}$$

b $\dfrac{dA}{dx} = \dfrac{1}{8}(2x - 10)$

$= \dfrac{x - 5}{4}$

c Upright parabola, so turning point is a minimum.

$\dfrac{dA}{dx} = \dfrac{x - 5}{4} = 0$

$\therefore\ x = 5$

d For $x \in [4, 7]$, check end points:

$A(4) = \dfrac{26}{8}, A(7) = \dfrac{29}{8}$ so $A_{\max} = \dfrac{29}{8}$

$= 3.625$ m^2.

Solutions to Exercise 18F

1 **a** When $t = 0, x = t^2 - 12t + 11 = 11$.
The initial position is 11 cm to the right of O

b When $t = 3, x = t^2 - 12t + 11 =$
$3^2 - 12 \times 3 + 11 = -16$. The position when $t = 3$ is 16 cm to the left of O

2 $x = t^2 - 12t + 11, t \geq 0$

a Velocity $= v = \dfrac{dx}{dt} = 2t - 12$
When $t = 0, v = -12$
The particle is moving to the left at 12 cm/s

b $v = 0$ implies $2t - 12 = 0$. That is $t = 6$
When $t = 6, x = 36 - 72 + 11 = -25$.
The particle is 25 cm to the left of O.

c Average velocity for the first three seconds =

$$\frac{x(3) - x(0)}{3 - 0} = -\frac{27}{3} = -9 \text{ cm/s.}$$

d The particle moves to the left for the first three seconds and sdoesn't change direction. The speed is 9 cm/s

3 $x = \dfrac{1}{3}t^3 - 12t + 6, t \geq 0$

a Therefore $\dfrac{dx}{dt} = t^2 - 12$ When
$t = 3, v = \dfrac{dx}{dt} = -3$

b
$$\frac{dx}{dt} = 0$$
$$\Rightarrow t^2 = 12$$
$\Rightarrow t = \pm 2\sqrt{3}$
But $t \geq 0$. Therefore $t = 2\sqrt{3}$
The velocity is zero at time $t = 2\sqrt{3}$ seconds

4 $x = 4t^3 - 6t^2 + 5$
Velocity $: v = \dfrac{dx}{dt} = 12t^2 - 12t$
Acceleration : $a = \dfrac{dv}{dt} = 24t - 12$

a When $t = 0, x = 5, v = 0, a = -12$
The particle is initially at rest at $x = 5$ and starts moving to the left.
It is instantaneously at rest when $12t(t - 1) = 0$. That is, when $t = 0$ and $t = 1$
When $t > 1$ it is moving to the right.

5 $s = t^4 + t^2$
$v = \dfrac{ds}{dt} = 4t^3 + 2t$
$a = \dfrac{dv}{dt} = 12t^2 + 2$

a When $t = 0$, acceleration is 2m/s^2

b When $t = 2$, acceleration is 50m/s^2

6 **a** $s = 10 + 15t - 4.9t^2$

$$\therefore v = \frac{ds}{dt} = 15 - 9.8t \text{ m/s}$$

b $a = \dfrac{dv}{dt} = -9.8 \text{ m/s}^2$

7 $x(t) = t^2 - 7t + 10, t \geq 0$

a $v(t) = 2t - 7 = 0$

$\therefore t = 3.5$ s

b $a(t) = 2$ m/s^2 at all times

c 14.5 m

d $v(t) = 2t - 7 = -2$

$\therefore \quad t = 2.5$ s

$x(2.5) = 2.5^2 - 7(2.5) + 10$ cm

$= 1.25$ m to the left of O.

8 **a** $s = t^3 - 3t^2 + 2t$

$= t(t-1)(t-2)$

$\therefore s = 0$ at $t = 0, 1$ and 2

b $v(t) = 3t^2 - 6t + 2; a(t) = 6t - 6$
$t = 0: v = 2$ m/s and $a = -6$ m/s^2
$t = 1: v = -1$ m/s and $a = 0$ m/s^2
$t = 2: v = 2$ m/s and $a = 6$ m/s^2

c Av. v in 1st second
$= s(1) - s(0) = 0$ m/s

9 **a** $x = t^2 - 7t + 12$
$\therefore x(0) = 12$ cm to the right of O.

b $x(5) = 5^2 - 7(5) + 12$

$= 2$ cm to the right of O.

c $v(t) = 2t - 7$

$\therefore v(0) = -7$

$= 7$ cm/s moving to the left of O.

d $v = 0$ when $t = 3.5$ s

$x(3.5) = 3.5^2 - 7(3.5) + 12$

$= -0.25$

$= 0.25$ cm to the left.

e $\text{Av.}v = \dfrac{x(5) - x(0)}{5}$

$= \dfrac{2 - 12}{5}$

$= -2$ cm/s

f Total distance traveled

$= x(0) - x(3.5) + x(5) - x(3.5)$

$= 12.25 + 2.25 = 14.5$ cm.

Total time = 5 seconds
$\therefore$ av. speed $= \dfrac{14.5}{5} = 2.9$ cm/s

10 $s = t^4 + 3t^2$
$v = \dfrac{ds}{dt} = 4t^3 + 6t$
$a = \dfrac{dv}{dt} = 12t^2 + 6$

a When $t = 1$, acceleration is 18m/s^2
When $t = 2$, acceleration is 54 m/s^2
When $t = 3$, acceleration is 114 m/s^2

b Average acceleration
$= \dfrac{v(3) - v(1)}{3 - 1} = \dfrac{116}{2} = 58$ m/s^2

11 $x(t) = t^3 - 11t^2 + 24t - 3, t \geq 0$

a $v(t) = 3t^2 - 22t + 24, t \geq 0$

$x(0) = -3$ cm; $v(0) = 24$ cm/s
Particle is 3 cm to the left of O
moving to the right at 24 cm/s.

b See **a**: $v(t) = 3t^2 - 22t + 24, t \geq 0$

c $v(t) = 3t^2 - 22t + 24 = 0$

$= (3t - 4)(t - 6) = 0,$

$\therefore \quad t = \frac{4}{3}, 6 \text{ s}$

d $x\left(\frac{4}{3}\right) = \left(\frac{4}{3}\right)^3 - 11\left(\frac{4}{3}\right)^2 + 24\left(\frac{4}{3}\right) - 3$

$= \frac{64}{27} - \frac{176}{9} + 32 - 3$

$= \frac{319}{27}$ cm right of O

$x(6) = (6)^3 - 11(6)^2 + 24(6) - 3$

$= 216 - 396 + 144 - 3$

$= 39$ cm left of O

e Velocity negative for $t \in \left(\frac{4}{3}, 6\right)$,

i.e. for $\frac{14}{3}$ s $= 4\frac{2}{3}$ s.

f $a(t) = 6t - 22 \text{ cm/s}^2$

g $a(t) = 6t - 22 = 0, \therefore \; t = \frac{11}{3}$s

$x\left(\frac{11}{3}\right) = \left(\frac{11}{3}\right)^3 - 11\left(\frac{11}{3}\right)^2 + 24\left(\frac{11}{3}\right) - 3$

$= \frac{1331}{27} - \frac{1331}{9} + \frac{264}{3} - 3$

$= -\frac{313}{27}$

$= \frac{313}{27}$ cm to the left of O

$v\left(\frac{11}{3}\right) = 3\left(\frac{11}{3}\right)^2 - 22\left(\frac{11}{3}\right) + 24$

$= \frac{121}{3} - \frac{242}{3} + 24$

$= -\frac{49}{3}$

$= \frac{49}{3}$ cm/s moving to the left.

12 $x(t) = 2t^3 - 5t^2 + 4t - 5, t \geq 0$

$\therefore \; v(t) = 6t^2 - l0t + 4, t \geq 0$

$\therefore \; a(t) = 12t - 10, t \geq 0$

a $v(t) = 6t^2 - 10t + 4 = 0$

$2(3t - 2)(t - 1) = 0$

$\therefore t = \frac{2}{3}, 1$

$a\left(\frac{2}{3}\right) = 12\left(\frac{2}{3}\right) - 10$

$= -2 \text{ cm/s}^2$

$a(1) = 12 - 10 = 2 \text{ cm/s}^2$

b $a(t) = 12t - 10 = 0, \quad \therefore t = \frac{5}{6}$ s

$v\left(\frac{5}{6}\right) = 6\left(\frac{5}{6}\right)^2 - 10\left(\frac{5}{6}\right) + 4$

$= \frac{25}{6} - \frac{50}{6} + 4 = -\frac{1}{6}$

Particle is moving to the left at $\frac{1}{6}$ cm/s

13 $x(t) = t^3 - 13t^2 + 46t - 48, t \geq 0$

$\therefore \; v(t) = 3t^2 - 26t + 46, t \geq 0$

$\therefore \; a(t) = 6t - 26, t \geq 0$

The particle passes through O where $x = 0$:

$x(t) = (t - 2)(t - 3)(t - 8) = 0$

$\therefore \quad t = 2, 3, 8$ s

At $t = 2$: $v = 6$ cm/s, $a = -14 \text{ cm/s}^2$
At $t = 3$: $v = -5$ cm/s, $a = -8 \text{ cm/s}^2$
At $t = 8$: $v = 30$ cm/s, $a = 22 \text{ cm/s}^2$

14 P1: $x(t) = t + 2, \therefore v(t) = 1$
P2: $x(t) = t^2 - 2t - 2, \therefore v(t) = 2t - 2$

a Particles at same position when

$$t + 2 = t^2 - 2t - 2$$
$$t^2 - 3t - 4 = 0$$
$$(t - 4)(t + 1) = 0$$
$$\therefore \quad t = -1, 4$$

(No restricted domain: both correct)

b Velocities equal when $2t - 2 = 1$ so $t = 1.5$ s

Solutions to Exercise 18G

1 $v = 4t - 6; t \geq 0$

a $x = \int 4t - 6dt = 2t^2 - 6t; v(0) = 0$

b $x(3) = 18 - 18 = 0$ cm
Body is at O.

c $v = 0, \therefore t = 1.5$ s
$x(1.5) = \dfrac{9}{2} - 9 = -4.5$ cm
$x(3) = 0$
Body goes 4.5 cm in each direction = 9 cm total.

d Av. v over $[0, 3] = 0$ since $x(3) = x(0)$

e Av. speed over $[0, 3] = \dfrac{9}{3} = 3$ cm/s

2 $v = 3t^2 - 8t + 5; t \geq 0$
$x(0) = 4$ m + direction to the right of O.

a $x = t^3 - 4t^2 + 5t + 4$

$a = 6t - 8$

b $v(t) = (3t - 5)(t - 1) = 0, \therefore t = 1, \dfrac{5}{3}$
$x(1) = 1 - 4 + 5 + 4 = 6$ m
$$x\left(\frac{5}{3}\right) = \frac{125}{27} - \frac{100}{9} + \frac{25}{3} + 4$$
$$= \frac{158}{27} \text{ m}$$

c $a(1) = -2$ m/s^2; $a\left(\dfrac{5}{3}\right) = 2$ m/s^2

3 $a = 2t - 3; t \geq 0$
$\therefore\ v = t^2 - 3t + 3;\ v(0) = 3$
$\therefore\ x = \dfrac{1}{3}t^3 - \dfrac{3}{2}t^2 + 3t + 2; x(0) = 2$

$$x(10) = \frac{1000}{3} - 150 + 30 + 2$$
$$= \frac{646}{3} \text{ m}$$

4 $a = -10$ m/s^2

a $v(0) = 25, \therefore v(t) = 25 - 10t$ m/s

b $h(0) = 0, \therefore h(t) = 25t - 5t^2$ m

c h is max. when $v = 0$ at $t = 2.5$ s

d $h\left(\dfrac{5}{2}\right) = \dfrac{125}{4} = 31.25$ m

e $h = 0, \therefore 5t(5 - t) = 0$

$t = 0, 5$

Body returns to its start after 5 seconds.

5 $a = \dfrac{t-5}{9}, \therefore v = \dfrac{t^2}{18} - \dfrac{5t}{9} + c$ m/s

$v(0) = -8, \therefore c = -8$

$$\therefore\ v(t) = \frac{t^2}{18} - \frac{5t}{9} - 8 \text{ m/s}$$

$$\therefore\ x(t) = \frac{t^3}{54} - \frac{5t^2}{18} - 8t + 300 \text{ m}$$

since $x(0) = 300$ m

$v = 0$ when $\dfrac{t^2}{18} - \dfrac{5t}{9} = 8$

$\therefore\quad t^2 - 10t - 144 = 0$

$(t - 18)(t + 10) = 0$

$t = 18;\ t > 0$

$x(18) = 108 - 90 - 144 + 300$

$= 174$ m

$\dfrac{174}{6} = 29, \therefore$ lift stops at the 29th floor.

Solutions to Exercise 18H

1 $f(x) = (x-2)^2(x-b)$, $b > 2$

a Use CAS calculator to find gradient function.
$f'(x) = (x-2)(3x-2(b+l))$

b For stationary points $f'(x) = 0$
$\therefore x = 2, c$ where $c = \frac{2}{3}(b+1)$

$$f(2) = 0; f(c) = (c-2)^2(c-b)$$
$$= -\frac{4}{27}(b-2)^3$$

$(2, 0)$ and $(\frac{2}{3}(b+1), -\frac{4}{27}(b-2)^3)$

c $f(x)$ is an upright cubic and the lst stationary pt is always a maximum. Since $\frac{2}{3}(b+1) > 0$ by definition, this is the 2nd stationary pt and is thus a minimum.
A sign diagram confirms this:

x	0	2		c	
f'	+	0	−	0	+

$\therefore (2, 0)$ is always a local maximum.

d $\frac{2}{3}(b+1) = 4$
$\therefore \quad b+1 = 6$
$b = 5$

2 a $y = x^4 - 12x^3$
$$\frac{dy}{dx} = 4x^3 - 36x^2 = 4x^2(x-9)$$
$$\frac{dy}{dx} = 0$$
$\Rightarrow 4x^2(x-9) = 0$
$\Rightarrow x = 0$ or $x = 9$
$\frac{dy}{dx} > 0$ for $x > 9$ and $\frac{dy}{dx} < 0$ for $x < 9$. There is a stationary point of inflextion at $(0, 0)$ and a local minimum at $(9, -2187)$.

b $(a, b)$and $(9 + a, -2187 + b)$.

3 $f(x) = x - ax^2, a > 0$
$\therefore f'(x) = 1 - 2ax$

a **i** f is an increasing function if $1 - 2ax > 0$
$$\therefore 2ax < 1, \therefore x < \frac{1}{2a}$$
(since $a > 0$)

ii f is a decreasing function if $1 - 2ax < 0$
$$\therefore 2ax > 1, \therefore x > \frac{1}{2a}$$
(since $a > 0$)

b Tangent at $(\frac{1}{a}, 0)$ has gradient $= -1$
$\therefore \; y - 0 = -1\left(x - \frac{1}{a}\right)$
$y = \frac{1}{a} - x$

c Normal at $(\frac{1}{a}, 0)$ has gradient = 1:
$\therefore \; y = x - \frac{1}{a}$

d Local maximum occurs at $x = \frac{1}{2a}$
$f\left(\frac{1}{2a}\right) = \frac{1}{2a} - \frac{a}{4a^2} = \frac{1}{4a}$
$\therefore$ Range of $f = (-\infty, \frac{1}{4a}]$

4 a Using a CAS:

$f'(x) = (x - a)(x - a + 2(x - 1))$

$= (x - a)(3x - a - 2)$

$f'(x) = 0, \therefore x = a, \dfrac{a + 2}{3}$

$f(a) = 0;$

$f\left(\dfrac{a+2}{3}\right) = \left(\dfrac{2}{3}\right)^2 (a-1)^2 \left(\dfrac{a-1}{3}\right)$

$= \dfrac{4}{27}(a - 1)^3$

Turning pts at $(a, 0)$ and $\left(\dfrac{a+2}{3}, \dfrac{4}{27}(a-1)^3\right)$

b $(a, 0)$ is a local minimum. $\left(\dfrac{a+2}{3}, \dfrac{4}{27}(a-1)^3\right)$ is a local maximum.

c **i** Tangent at $x = 1$ has gradient $(a - 1)^2$:
$y(1) = 0, \therefore y = (a - 1)^2(x - 1)$

ii Tangent at $x = a$ has gradient 0:
$y(0) = 0, \ \therefore y = 0$

iii Tangent at $x = \dfrac{a+1}{2}$ has gradient:

$= \left(\dfrac{a+1}{2} - a\right)\left(\dfrac{3}{2}(a + 1) - a - 2\right)$

$= \dfrac{1-a}{2}\left(\dfrac{a-1}{2}\right)$

$= -\dfrac{1}{4}(a - 1)^2$

$y(\dfrac{a+1}{2}) = (\dfrac{a+1}{2} - a)^2(\dfrac{a+1}{2} -)$

$= (\dfrac{1-a}{2})^2(\dfrac{a-1}{2})$

$= \dfrac{1}{8}(a - 1)^3$

Tangent equation:

$y - \dfrac{1}{8}(a - 1)^3$

$= -\dfrac{1}{4}(a - 1)^2(x - \dfrac{a+1}{2})$

$\therefore y = -\dfrac{1}{4}(a - 1)^2(x - \dfrac{a+1}{2})$

$+ \dfrac{1}{8}(a - 1)^3$

$= -\dfrac{1}{4}(a - 1)^2(x - \dfrac{a+1}{2} - \dfrac{a-1}{2})$

$= -\dfrac{1}{4}(a - 1)^2(x - a)$

5 $y = (x - 2)^2$
$y = mx + c$ is a tangent to the curve at point P.

a **i** $y'(x) = 2(x - 2)$

$\therefore y'(a) = 2(a - 2)$
where $0 \le a < 2$

ii $m = 2(a - 2)$

b $P = (a, (a - 2)^2)$

c $y - (a - 2)^2 = 2(a - 2)(x - a)$

$\therefore \ y = 2(a - 2)x - 2a(a - 2) + (a - 2)^2$

$= 2(a - 2)x + (a - 2)(a - 2 - 2a)$

$= 2(a - 2)x + (a - 2)(-a - 2)$

$= 2(a - 2)x + 4 - a^2$

d x-axis intercept of the tangent is where $y = 0$

$2(a - 2)x + 4 - a^2 = 0$

$\therefore x = \dfrac{a^2 - 4}{2(a - 2)} = \dfrac{a+2}{2}$

(since $a \neq 2$)

6 a $f(x) = x^3 \rightarrow y = f(x+h)$

$$f(1+h) = 27, \therefore\ (1+h)^3 = 27$$
$$1 + h = 3$$
$$h = 2$$

b $f(x) = x^3 \rightarrow y = f(ax)$

$f(ax)$ passes through (1, 27)

$\therefore ax = 3$

$\therefore\ a = 3$ since $x = 1$

c $y = ax^3 - bx^2 = x^2(ax - b)$

$\therefore\ y' = 3ax^2 - 2bx = x(3ax - 2b)$

Using (1, 8): a − b = 8 . . . **(1)**

$y'(1) = 0, \therefore\ 3a - 2b = 0 \ldots$ **(2)**

From **(1)**: $3a - 3b = 24$

$\therefore a = -16, b = -24$

7 $y = x^4 + 4x^2$

Translation + a in x direction, and + b in y direction:

$y = (x-a)^4 + 4(x-a)^2 + b$

a $y' = 4x^3 + 8x = 4x(x^2 + 2)$

Turning pt at(0, 0) only, since

$x^2 + 2 > 0; x \in R$

b Turning point of image = (a, b)

8 a

$$f(x) = (x-1)^2(x-b)^2, b > 1$$
$$\therefore\ f'(x) = 2(x-1)(x-b)^2$$
$$+ 2(x-b)^2(x-1)$$
$$= 2(x-1)(x-b)(2x-b-1))$$

Use a CAS calculator to determine the gradient function.

b $f'(x) = 0$ when $x = 1, b, \dfrac{b+1}{2}$

$f(1) = f(b) = 0$

$$f'(\frac{b+1}{2}) = (\frac{b+1}{2} - 1)^2(\frac{b+1}{2} - b)^2$$
$$= (\frac{b-1}{2})^2(\frac{1-b}{2})^2$$
$$= \frac{1}{16}(b-1)^4$$

Turning pts: $(1, 0)$, $(b, 0)$ and

$(\dfrac{b+1}{2}, \dfrac{1}{16}(b-1)^4)$

c Turning pt at (2, 1) must mean

$\dfrac{b+1}{2} = 2$

$\therefore\ b + 1 = 4, \therefore\ b = 3$

Solutions to Exercise 18I

1 **a** Let $f(x) = x^3 - x - 1$

$f'(x) = 3x^2 - 1$

Using $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

We start with $x_0 = 1.5$, $f(x_0) = 0.875$

Step 1

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 1.5 - \frac{0.875}{5.75}$$
$$= 1.347\ldots$$

Step 2

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 1.347\cdots - \frac{0.100\ldots}{4.449\ldots}$$
$$= 1.325\ldots$$

A section of spreadsheet is given here:

n	xn	f(xn)	f'(xn)
0	1.5	0.875	5.75
1	1.347826086957	0.100682173	4.449905482
2	1.32520039895091	0.002058362	4.268468292
3	1.32471817399905	9.24378E-07	4.264634722
4	1.32471795724479	1.86517E-13	4.264632999
5	1.32471795724475	0	4.264632999

In the remainder of Question 1 only the spreadsheet is presented

b $f(x) = x^4 + x - 3$ We start with $x_0 = 2$, $f(x_0) = 15$

n	xn	f(xn)	f'(xn)
0	2	15	33
1	1.545454545455	4.250051226	15.76483847
2	1.27586401112589	0.925691183	9.307553621
3	1.17640810463650	0.091687036	7.512294242
4	1.16420317321405	0.001228377	7.311699696
5	1.16403517169033	2.29506E-07	7.308967641
6	1.16403514028977	7.10543E-15	7.308967131
7	1.16403514028977	0	7.308967131

c $f(x) = x^3 - 5x + 4.2$ We start with $x_0 = 1.5$, $f(x_0) = 0.075$

n	xn	f(xn)	f'(xn)
0	1.5	0.075	1.75
1	1.457142857143	0.008186589	1.369795918
2	1.45116635450366	0.000155928	1.317651365
3	1.45104801685409	6.09639E-08	1.316621042
4	1.45104797055081	9.76996E-15	1.316620639
5	1.45104797055080	0	1.316620639

We note that a second solution also exists. We start with $x_0 = 1.1, f(x_0) = 0.031$

n	xn	f(xn)	f'(xn)
0	1.1	0.031	-1.37
1	1.122627737226	0.001701234	-1.219120891
2	1.12402319650002	6.56102E-06	-1.209715561
3	1.12402862010434	9.91909E-11	-1.209678984
4	1.12402862018633	0	-1.209678983

d $f(x) = x^3 - 2x^2 + 2x - 5$ We start with $x_0 = 2.5, f(x_0) = 3.125$

n	xn	f(xn)	f'(xn)
0	2.5	3.125	10.75
1	2.209302325581	0.440212811	7.805840995
2	2.15290701702655	0.014539377	7.293397804
3	2.15091351865629	1.77112E-05	7.27563282
4	2.15091108433957	2.63869E-11	7.275611141
5	2.15091108433594	0	7.275611141

e $f(x) = 2x^4 - 3x^2 + 2x - 6$ We start with $x_0 = -1.5, f(x_0) = -5.625$

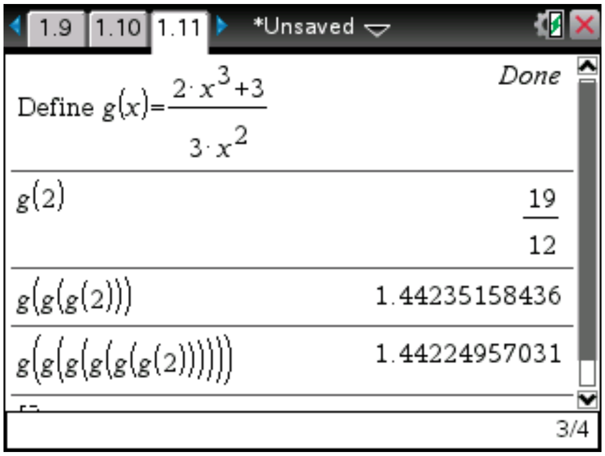
n	xn	f(xn)	f'(xn)
0	-1.5	-5.625	-16
1	-1.851562500000	3.518282063	-37.67207718
2	-1.75817019655487	0.320742748	-30.92929622
3	-1.74780000429266	0.003650831	-30.22670348
4	-1.74767922265566	4.9098E-07	-30.21857364
5	-1.74767920640803	8.88178E-15	-30.21857255
6	-1.74767920640803	0	-30.21857255

2 The equation $x^3 = 3$ can be rearrranged as follows;

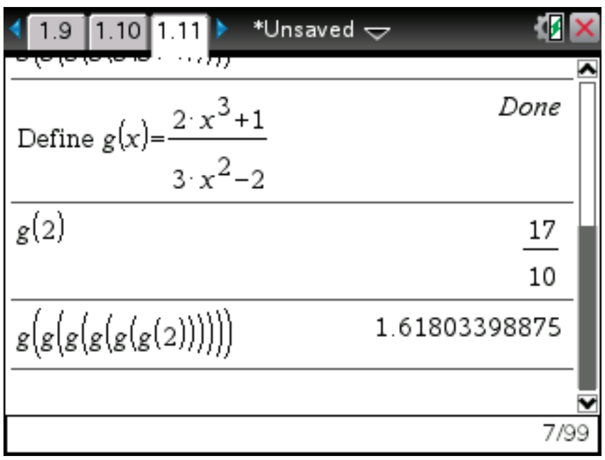
$$x^3 = 3$$

$$3x^3 = 2x^3 + 3$$

$$x = \frac{2x^3 + 3}{3x^2}$$



3 The equation $x^3 - 2x - 1 = 0$ can be rearranged to $x = \dfrac{2x^3 + 1}{3x^2 - 2}$



Solutions to Technology-free questions

1 $y = 4x - x^2$

a $\dfrac{dy}{dx} = 4 - 2x$

b Gradient at $Q(1, 3) = 4 - 2 = 2$

c Tangent at $Q : y - 3 = 2(x - 1)$
$\therefore\ y = 2x + 1$

2 $y = x^3 - 4x^2$

a $\dfrac{dy}{dx} = 3x^2 - 8x$

b Gradient at $(2, -8) = 3(2)^2 - 8(2)$

$= -4$

c Tangent at $(2, -8) = y + 8$

$= -4(x - 2)$

$y = -4x$

d Tangent meets curve when $y = x^3 - 4x^2$

$= -4x$

$\therefore x(x - 2)^2 = 0$

$x = 0, 2$

Tangent cuts curve again at $(0, 0)$.

3 $y = x^3 - 12x + 2$

a $\dfrac{dy}{dx} = 3x^2 - 12$

$= 3(x - 2)(x + 2)$

$\dfrac{dy}{dx} = 0, \therefore\ x = \pm 2$

$y(-2) = 18, y(2) = -14$

b Upright cubic.
$(-2, 18)$ is a local maximum and $(2, -14)$ is a local minimum.

c Upright cubic.
$(-2, 18)$ is a local maximum and $(2, -14)$ is a local minimum.

4 a $\dfrac{dy}{dx} = 3x^2$
Stationary pt of inflexion at $x = 0$:

x	-1	0	1
$\dfrac{dy}{dx}$	+	0	+

b $\dfrac{dy}{dx} = -3x^3$
Local maximum at $x = 0$:

x	-1	0	1
$\dfrac{dy}{dx}$	+	0	−

c $f'(x) = (x - 2)(x - 3)$

x	0	2	2.5	3	4
f'	+	0	−	0	+

Local maximum at $x = 2$, minimum at $x = 3$

d $f'(x) = (x - 2)(x + 2)$

x	-3	-2	0	2	3
f'	+	0	−	0	+

Local maximum at $x = -2$, minimum at $x = 2$

e $f'(x) = (2 - x)(x + 2)$

x	-3	-2	0	2	3
f'	−	0	+	0	−

Local minimum at $x = -2$, maximum at $x = 2$

f $f'(x) = -(x - 1)(x - 3)$

x	0	1	2	3	4
f'	−	0	+	0	−

Local minimum at $x = 1$, maximum at $x = 3$

g $\dfrac{dy}{dx} = -x^2 + x + 12 = (4 - x)(x + 3)$

x	-4	-3	0	4	5
$\dfrac{dy}{dx}$	−	0	+	0	−

Local minimum at $x = -3$, maximum at $x = 4$

h $\dfrac{dy}{dx} = 15 - 2x - x^2 = (3 - x)(x + 5)$

x	-6	-5	0	3	4
$\frac{dy}{dx}$	$-$	0	$+$	0	$-$

Local minimum at $x = -5$, maximum at $x = 3$

5 a $y = 4x - 3x^3, \therefore\ y' = 4 - 9x^2$

$y' = 0, \therefore x = \pm\frac{2}{3}$

$y\left(-\frac{2}{3}\right) = -\frac{16}{9}, y\left(\frac{2}{3}\right) = \frac{16}{9}$

Inverted cubic:

$\left(-\frac{2}{3}, -\frac{16}{9}\right)$ is a local minimum, $\left(\frac{2}{3}, \frac{16}{9}\right)$ is a local maximum.

b $$\begin{aligned} y &= 2x^3 - 3x^2 - 12x - 7 \\ \therefore\ y' &= 6x^2 - 6x - 12 \\ &= 6(x-2)(x+1) \end{aligned}$$

$y' = 0, \therefore x = -1, 2$

$y(-1) = 0, y(2) = -27$

Upright cubic:

$(-1, 0)$ is a local maximum,

$(2, -27)$ is a local minimum.

c $$\begin{aligned} y &= x(2x-3)(x-4) \\ &= 2x^3 - 11x^2 + 12x \\ \therefore y' &= 6x^2 - 22x + 12 \\ &= 2(3x-2)(x-3) \end{aligned}$$

$y' = 0, \therefore x = \frac{2}{3}, 3$

$y(3) = -9,$

$$\begin{aligned} y\left(\frac{2}{3}\right) &= \frac{2}{3}\left(-\frac{5}{3}\right)\left(-\frac{10}{3}\right) \\ &= \frac{100}{27} \end{aligned}$$

Upright cubic:

$\left(\frac{2}{3}, \frac{100}{27}\right)$ is a local maximum,

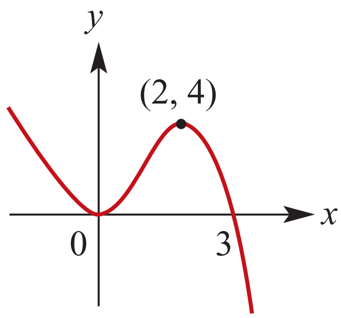
$(3, -9)$ is a local minimum.

6 a $$\begin{aligned} y &= 3x^2 - x^3 \\ &= x^2(3-x) \end{aligned}$$

Axis intercepts at $(0, 0)$ and $(3, 0)$.

$$y' = 6x - 3x^2$$
$$= 3x(2 - x)$$

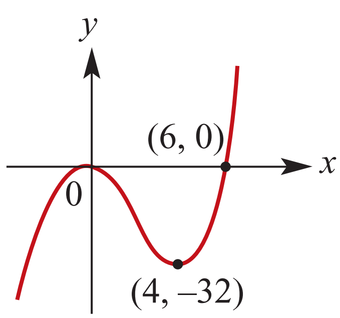
Stationary pts at $(0, 0)$ and $(2, 4)$.
Inverted cubic:
local min. at $(0, 0)$, max. at $(2, 4)$.



b $y = x^3 - 6x^2$
$= x^2(x - 6)$
Axis intercepts at $(0, 0)$ and $(6, 0)$.

$$y' = 3x^2 - 12x$$
$$= 3x(x - 4)$$

Stationary pts at $(0, 0)$ and $(4, -32)$.
Upright cubic:
local max. at $(0, 0)$, min. at $(4, -32)$.

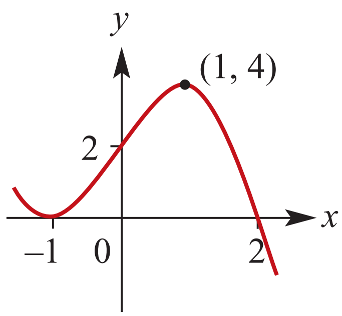


c $y = (x + 1)^2(2 - x)$
$= 2 + 3x - x^3$
Axis intercepts at $(0, 2)$, $(-1, 0)$ and $(2, 0)$.

$$y' = 3 - 3x^2$$
$$= 3(1 - x)(1 + x)$$

Stationary pts at $(-1, 0)$ and $(1, 4)$.

Inverted cubic:
local min. at $(-1, 0)$, max. at $(1, 4)$.



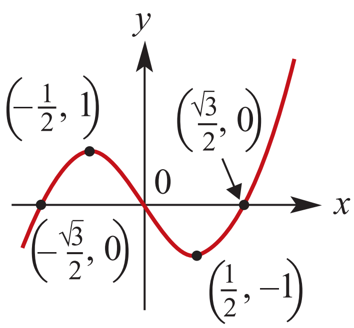
d $y = 4x^3 - 3x$

$= x(2x - \sqrt{3})(2x + \sqrt{3})$
Axis intercepts at $(0, 0)$, $(-\sqrt{3}/2, 0)$ and $(\sqrt{3}/2, 0)$.

$y' = 12x^2 - 3$

$= 3(2x - 1)(2x + 1)$

Stationary pts at $(-\frac{1}{2}, 1)$ and $(\frac{1}{2}, -1)$.
Upright cubic:
local max. $(-\frac{1}{2}, 1)$ min. $(\frac{1}{2}, -1)$.



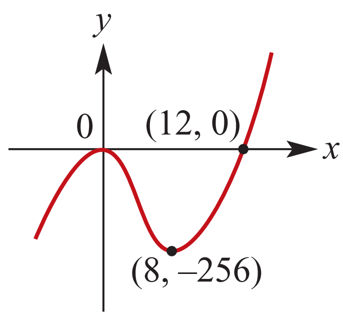
e $y = x^3 - 12x^2$

$= x^2(x - 12)$
Axis intercepts at $(0, 0)$ and $(12, 0)$.

$y' = 3x^2 - 24x$

$= 3x(x - 8)$

Stationary pts at $(0, 0)$ and $(8, -256)$.
Inverted cubic:
local max. $(0, 0)$, min. $(8, -256)$.



7 **a** C

b A

c B

8 $h = 20t - 5t^2$

$v = \dfrac{dh}{dt} = 20 - 10t$

a $\dfrac{dh}{dt} = 0$

$\Rightarrow 20 - 10t = 0$

$\Rightarrow t = 2$

Stone reaches the maximum height when $t = 2$

$h(2) = 4020 = 20$

The maximum height is 20 m.

b $20t - 5t^2 = -60$

$\Leftrightarrow 5t^2 - 20t - 60 = 0$

$\Leftrightarrow t^2 - 4t - 12 = 0$

$\Leftrightarrow (t - 6)(t + 2) = 0$

$\Leftrightarrow t = 6 \text{ or} t = -2$

$t \geq 0, \therefore t = 6$

It takes 6 seconds to hit the beach.

c When$t = 6, v = 20 - 60$

The speed is 40 m/s

9 $x + y = 12 \Rightarrow y = 12 - x$

Let $M = x^2 + y^2$

Then $M = x^2 + 144 - 24x + x^2 = 2x^2 - 24x + 144$

Minimum value when $\dfrac{dM}{dx} = 0$

$\dfrac{dM}{dx} = 4x - 24$

$\therefore$ minimum value when $x = 6$

Therefore minimum value is 72

10 $\dfrac{dv}{dt} = a = 4 - t$

$\therefore v = 4t - \dfrac{1}{2}t^2 + c$

When $t = 0, v = 0$

$\therefore v = 4t - \dfrac{1}{2}t^2$

$v = \dfrac{dx}{dt} = 4t - \dfrac{1}{2}t^2$

$\therefore x = 2t^2 - \dfrac{1}{6}t^3 + c$

When $t = 0, v = 0$

$\therefore x = 2t^2 - \dfrac{1}{6}t^3$

When $t = 3, v = \dfrac{15}{2}$ m/s

b Comes to rest when $v = 0$

$4t - \dfrac{1}{2}t^2 = 0$

$t(4 - \dfrac{1}{2}t) = 0$

$t = 0$ or $t = 8$

When $t = 8$

$$x = 2 \times 8^2 - \frac{1}{6} \times 8^3$$

$$x = \frac{128}{3} \text{ m}$$

c When $t = 12, x = 0$

d Moves to the right for the first 8 seconds of motion. $x(8) = \dfrac{128}{3}$

It then returns to the origin in the next 4 seconds.

Hence the total distance travelled $= \dfrac{256}{3}$ m

The average speed $= \dfrac{256}{36} = \dfrac{64}{9}$ m/s

Solutions to multiple-choice questions

1 D $y = x^3 + 2x, \therefore\ y' = 3x^2 + 2$

Tangent at (1, 3) has gradient

$y'(1) = 5$

$$y - 3 = 5(x - 1)$$
$$\therefore y = 5x - 2$$

2 E Normal at (1, 3) has gradient $= -\dfrac{1}{5}$

$$y - 3 = -\frac{1}{5}(x - 1)$$
$$\therefore\ y = -\frac{1}{5}x + \frac{16}{5}$$

3 E $y = 2x - 3x^3, \therefore\ y' = 2 - 9x^2$

Tangent at (0,0) has gradient

$y'(0) = 2$

$\therefore y = 2x$

4 A $f(x) = 4x - x^2$

Av. rate of change over [0, 1]

$= \dfrac{f(1) - f(0)}{1} = 3$

5 C
$$S(t) = 4t^3 + 3t - 7$$
$$\therefore\ S'(t) = 12t^2 + 3$$
$$\therefore\ S(0) = 3\,\text{m/s}$$

6 D
$$y = x^3 - 12x$$
$$\therefore\ y' = 3x^2 - 12$$
$$= 3(x - 2)(x + 2)$$
$$y' = 0 \text{ for } x = \pm 2$$

7 D
$$y = 2x^3 - 6x$$
$$\therefore y' = 6x^2 - 6 = 6$$

So $6x^2 = 12$

$$\therefore x = \pm\sqrt{2}$$

8 A
$$f(x) = 2x^3 - 5x^2 + x$$
$$\therefore f'(x) = 6x^2 - 10x + 1$$
$$\therefore f'(2) = 5$$

9 A $y = \dfrac{1}{2}x^4 + 2x^2 - 5$

Av. rate of change over [−2, 2]

$= \dfrac{y(-2) - y(-2)}{2 - (-2)} = 0$

10 C
$$y = x^2 - 8x + 1$$
$$\therefore y' = 2x - 8$$

Minimum value is $y(4) = -15$

11 A

12 A

Solutions to extended-response questions

1 **a** $s = 2 + 10t - 4t^2$

$$v = \frac{ds}{dt} = 10 - 8t, \text{ where } v \text{ is velocity}$$

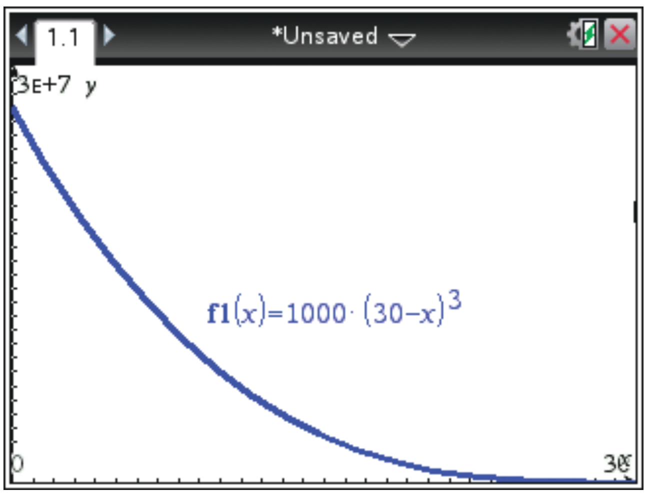
$$\text{When } t = 3, \quad v = 10 - 8(3) = -14$$

After 3 seconds, the velocity of the stone is -14 m/s (i.e. the stone is falling).

b $a = \dfrac{dv}{dt} = -8$

The acceleration due to gravity is -8 m/s^2.

2 **a**



b **i**

$$\begin{aligned} 2\,000\,000 &= 1000(30 - t)^3 \\ 2000 &= (30 - t)^3 \\ (2000)^{\frac{1}{3}} &= 30 - t \\ \therefore \quad t &= 30 - (2000)^{\frac{1}{3}} \\ &\approx 17.4 \text{ min} \end{aligned}$$

ii

$$\begin{aligned} 20\,000\,000 &= 1000(30 - t)^3 \\ 20\,000 &= (30 - t)^3 \\ (20\,000)^{\frac{1}{3}} &= 30 - t \\ \therefore \quad t &= 30 - (20000)^{\frac{1}{3}} \\ &\approx 2.9 \text{ min} \end{aligned}$$

c $V = 1000(30 - t)^3$

$= 1000(30 - t)(900 - 60t + t^2)$

$= 1000(27\,000 - 1800t + 30t^2 - 900t + 60t^2 - t^3)$

$= 1000(27\,000 - 2700t + 90t^2 - t^3)$

$= 27\,000\,000 - 2700\,000t + 90\,000t^2 - 1000t^3$

$\dfrac{dV}{dt} = -2700\,000 + 180\,000t - 3000t^2$

$= -3000(900 - 60t + t^2)$

$= -3000(30 - t)^2, \qquad t \geq 0$

At any time t, the dam is being emptied at the rate of $3000(30 - t)^2$ litres/min.

d When $V = 0$, $1000(30 - t)^3 = 0$

$\therefore\ 30 - t = 0$

$\therefore\qquad t = 30$

It takes 30 minutes to empty the dam.

e When $\dfrac{dV}{dt} = -8000 \qquad -3000(30 - t)^2 = -8000$

$\therefore\qquad (30 - t)^2 = \dfrac{8}{3}$

$\therefore\qquad 30 - t = \pm\dfrac{\sqrt{8}}{\sqrt{3}}$

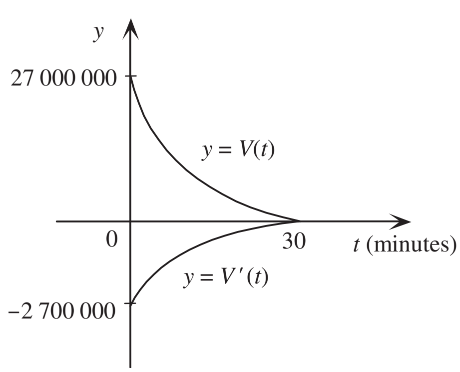
$\therefore\qquad t = 30 \pm \dfrac{2\sqrt{2}}{\sqrt{3}}$

$\therefore\qquad t = 30 - \dfrac{2\sqrt{2}}{\sqrt{3}},$ as $t \leq 30$

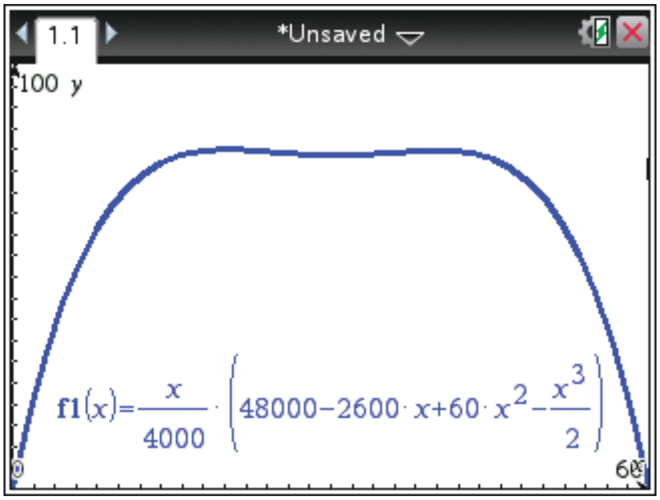
$\therefore\qquad t \approx 28.37$

Water is flowing out of the dam at 8000 litres per minute when t is approximately 28.37 minutes.

f



3 a



b Sketch the graph of $f2 = 50$ and

TI: Press **Menu → 6:Analyze Graph → 4:Intersection**

CP: Tap **Analysis → G-Solve → Intersect**

After 5.71 days; quantity drops below this after 54.29 days.

c

$$W = \frac{x}{4000}\left(48000 - 2600x + 60x^2 - \frac{x^3}{2}\right)$$

$$= 12x - \frac{13}{20}x^2 + \frac{3}{200}x^3 - \frac{1}{8000}x^4$$

$$\frac{dW}{dx} = 12 - \frac{26}{20}x + \frac{9}{200}x^2 - \frac{4}{8000}x^3$$

$$= 12 - \frac{13}{10}x + \frac{9}{200}x^2 - \frac{1}{2000}x^3$$

When $x = 20$, $$\frac{dW}{dx} = 12 - \frac{13}{10}(20) + \frac{9}{200}(20)^2 - \frac{1}{2000}(20)^3$$

$$= 12 - 26 + 18 - 4$$

$$= 0$$

When $x = 40$, $$\frac{dW}{dx} = 12 - \frac{13}{10}(40) + \frac{9}{200}(40)^2 - \frac{1}{2000}(40)^3$$

$$= 12 - 52 + 72 - 32$$

$$= 0$$

When $x = 60$, $$\frac{dW}{dx} = 12 - \frac{13}{10}(60) + \frac{9}{200}(60)^2 - \frac{1}{2000}(60)^3$$

$$= 12 - 78 + 162 - 108$$

$$= -12$$

The rate of increase of W, when $x = 20, 40$ and 60 is $0, 0$ and -12 tonnes per day respectively.

d When $x = 30$, $W = 12(30) - \frac{13}{20}(30)^2 + \frac{3}{200}(30)^3 - \frac{1}{8000}(30)^4$

$= 360 - 585 + 405 - 101.25 = 78.75$

4 a When $t = 0$, $y = 15 + \frac{1}{80}(0)^2(30 - 0) = 15$

When $t = 0$, the temperature is 15°C.

b

$$
\begin{aligned}
y &= 15 + \frac{1}{80}t^2(30 - t) \\
&= 15 + \frac{3}{8}t^2 - \frac{1}{80}t^3 \\
\frac{dy}{dt} &= \frac{3}{4}t - \frac{3}{80}t^2
\end{aligned}
$$

When $t = 0$, $\frac{dy}{dt} = \frac{3}{4}(0) - \frac{3}{80}(0)^2 = 0$

When $t = 5$, $\frac{dy}{dt} = \frac{3}{4}(5) - \frac{3}{80}(5)^2$

$= \frac{15}{4} - \frac{75}{80} = \frac{45}{16}$

When $t = 10$, $\frac{dy}{dt} = \frac{3}{4}(10) - \frac{3}{80}(10)^2$

$= \frac{30}{4} - \frac{300}{80} = \frac{15}{4}$

When $t = 15$, $\frac{dy}{dt} = \frac{3}{4}(15) - \frac{3}{80}(15)^2$

$= \frac{45}{4} - \frac{675}{80} = \frac{45}{16}$

When $t = 20$, $\frac{dy}{dt} = \frac{3}{4}(20) - \frac{3}{80}(20)^2$

$= \frac{60}{4} - \frac{1200}{80} = 0$

The rate of increase of y with respect to t when $t = 0, 5, 10, 15$ and 20 is 0, $\frac{45}{16}, \frac{15}{4}, \frac{45}{16}$ and 0°C per minute respectively.

c

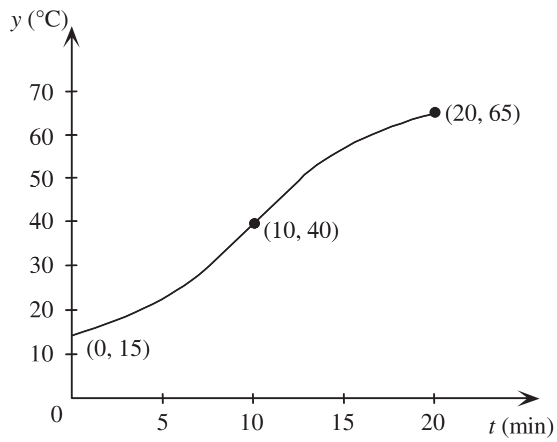
t	0	5	10	15	20
y	15	22.8125	40	57.1875	65

When $t = 5$, $y = 15 + \frac{1}{80}(5)^2(30 - 5)$

$= 22.8125$

When $t = 20$, $y = 15 + \frac{1}{80}(20)^2(30 - 20)$

$= 65$



5 a

$$\begin{aligned} S &= 4000 + (t - 16)^3 \\ &= 4000 + (t - 16)(t^2 - 32t + 256) \\ &= 4000 + t^3 - 32t^2 + 256t - 16t^2 + 512t - 4096 \\ &= t^3 - 48t^2 + 768t - 96 \\ \frac{dS}{dt} &= 3t^2 - 96t + 768 \end{aligned}$$

When $t = 0$,

$$\begin{aligned} \frac{dS}{dt} &= 3(0)^2 - 96(0) + 768 \\ &= 768 \end{aligned}$$

Sweetness was increasing by 768 units/day when $t = 0$.

b When $t = 4$,

$$\begin{aligned} \frac{dS}{dt} &= 3(4)^2 - 96(4) + 768 \\ &= 48 - 384 + 768 \\ &= 432 \end{aligned}$$

When $t = 8$,

$$\begin{aligned} \frac{dS}{dt} &= 3(8)^2 - 96(8) + 768 \\ &= 192 - 768 + 768 \\ &= 192 \end{aligned}$$

When $t = 12$,

$$\begin{aligned} \frac{dS}{dt} &= 3(12)^2 - 96(12) + 768 \\ &= 432 - 1152 + 768 \\ &= 48 \end{aligned}$$

When $t = 16$,

$$\begin{aligned} \frac{dS}{dt} &= 3(16)^2 - 96(16) + 768 \\ &= 768 - 1536 + 768 \\ &= 0 \end{aligned}$$

c The rate of increase of sweetness is zero after 16 days.

d When $t = 0$, $S = 4000 + (0 - 16)^3$

$= -96$

When $t = 4$, $S = 4000 + (4 - 16)^3$

$= 2272$

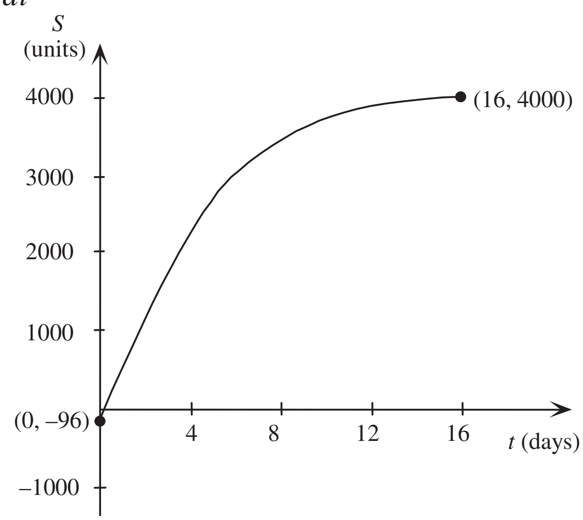
When $t = 8$, $S = 4000 + (8 - 16)^3$

$= 3488$

When $t = 16$, $S = 4000 + (16 - 16)^3$

$= 4000$

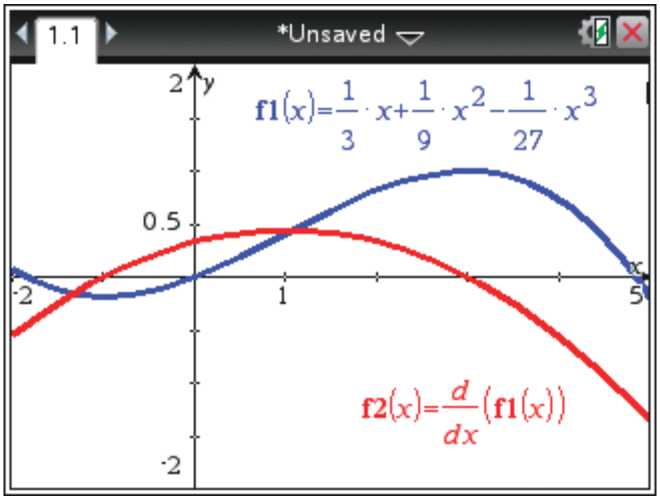
Note that $\dfrac{dS}{dt} = 3t^2 - 96t + 768 = 3(t^2 - 32t + 256) = 3(t - 16)^2$ which implies $\dfrac{dS}{dt} > 0$ for $0 \le t < 16$.



6 a $\dfrac{ds}{dt} = \dfrac{1}{3} + \dfrac{2}{9}t - \dfrac{1}{9}t^2$

$= -\dfrac{1}{9}(t^2 - 2t - 3)$

$= -\dfrac{1}{9}(t - 3)(t + 1)$



b When the train stops at stations,

$$\frac{ds}{dt} = 0$$

$$\therefore \quad -\frac{1}{9}(t-3)(t+1) = 0$$

$$\therefore \quad t = 3 \text{ or } t = -1$$

When $t = -1$, the time is 1 minute before noon, i.e. 11.59 am is the time of departure from the first station.

When $t = 3$, the time is 3 minutes past noon, i.e. 12.03 pm is the time of arrival at the second station.

c When $t = -1$,

$$s = \frac{1}{3}(-1) + \frac{1}{9}(-1)^2 - \frac{1}{27}(-1)^3$$

$$= -\frac{1}{3} + \frac{1}{9} + \frac{1}{27} = -\frac{5}{27}$$

The first station is $\frac{5}{27}$ km before the signal box.

When $t = 3$,

$$s = \frac{1}{3}(3) + \frac{1}{9}(3)^2 - \frac{1}{27}(3)^3$$

$$= 1 + 1 - 1 = 1$$

The second station is 1 km after the signal box.

d $$\text{Average velocity} = \frac{s_2 - s_1}{t_2 - t_1}$$

where $s_2 = 1, s_1 = \frac{5}{27}, t_2 = 3, t_1 = -1$

$$\text{average velocity} = \frac{1 - \frac{-5}{27}}{3 - (-1)}$$

$$= \frac{\frac{32}{27}}{4} = \frac{8}{27}$$

$$\frac{8}{27}\text{ km/min} = \left(\frac{8}{27} \times 60\right)\text{ kW/h} = \frac{160}{9}\text{km/h}$$

$$\therefore \quad \text{average velocity} = 17\frac{7}{9}\text{km/h}$$

The average velocity between the stations is $17\frac{7}{9}$km/h.

e $$v = \frac{ds}{dt} = -\frac{1}{9}(t - 3)(t + 1)$$

When the train passes the signal box, $t = 0$

i.e. $$v = -\frac{1}{9}(0 - 3)(0 + 1) = \frac{1}{3}$$

$$\text{velocity} = \frac{1}{3}\text{km/min} = \left(\frac{1}{3} \times 60\right)\text{km/h} = 20\text{km/h}$$

The train passes the signal box at 20 km/h.

7 a $$V(t) \geq 0$$

$$\therefore \quad 1000 + (2 - t)^3 \geq 0$$

$$\therefore \quad (2 - t)^3 \geq -1000$$

$$\therefore \quad 2 - t \geq -10$$

$$\therefore \quad 2 \geq t - 10$$

$$\therefore \quad t \leq 12$$

Now $t \geq 0$ so the possible values of t are $0 \leq t \leq 12$.

b Rate of change in volume over time $= \frac{dV}{dt}$

Now $\quad V = 1000 + (2 - t)^3$

$= 1000 + (2 - t)(4 - 4t + t^2)$

$= 1000 + 8 - 8t + 2t^2 - 4t + 4t^2 - t^3$

$= 1008 - 12t + 6t^2 - t^3$

$\therefore \quad \frac{dV}{dt} = -12 + 12t - 3t^2$

$= -3(t^2 - 4t + 4)$

$= -3(t - 2)^2$

i When $t = 5$, $\frac{dV}{dt} = -3(5 - 2)^2 = -27$
The rate of draining is 27 L/h when $t = 5$.

ii When $t = 10$, $\frac{dV}{dt} = -3(10 - 2)^2 = -192$
The rate of draining is 192 L/h when $t = 10$.

8 **a** When $x = 0$, $\quad y = \frac{1}{5}(4(0)^3 - 8(0)^2 + 192(0) + 144)$

$= \frac{1}{5} \times 144$

$= \frac{144}{5}$

$= 28.8$

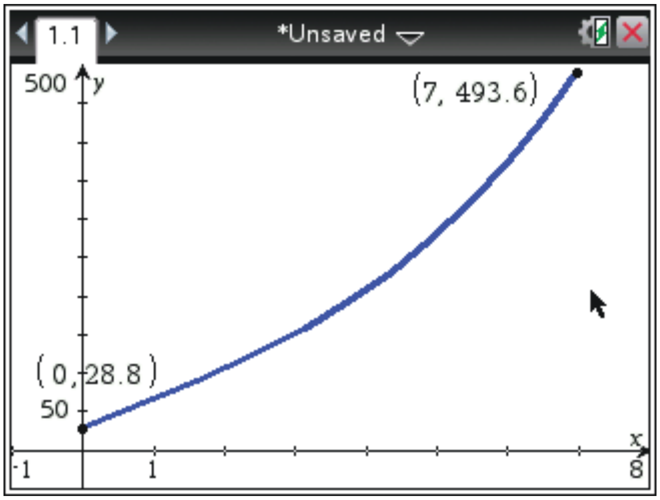
The start of the track is 28.8 m above sea level.

b When $x = 6$, $\quad y = \frac{1}{5}(4(6)^3 - 8(6)^2 + 192(6) + 144)$

$= \frac{1}{5}(864 - 288 + 1152 + 144)$

$= \frac{1870}{5}$

$= 374.4$

c 

d The graph gets very steep for $x > 7$, which would not be practical.

e

$$y = \frac{4}{5}x^3 - \frac{8}{5}x^2 + \frac{192}{5}x + \frac{144}{5}$$

$$\text{Gradient} = \frac{dy}{dx} = \frac{12}{5}x^2\frac{16}{5}x + \frac{192}{5}$$

i When $x = 0$,

$$\begin{aligned}\frac{dy}{dx} &= \frac{12}{5}(0)^2 - \frac{16}{5}(0) + \frac{192}{5}\\ &= 38.4\end{aligned}$$

$$\begin{aligned}38.4 \text{ m/km} &= \left(38.4\frac{1}{1000}\right) \text{ m/m}\\ &= 0.0384 \text{ m/m}\end{aligned}$$

The gradient of the graph is 0.0384 for $x = 0$.

ii When $x = 3$,

$$\begin{aligned}\frac{dy}{dx} &= \frac{12}{5}(3)^2 - \frac{16}{5}(3) + \frac{192}{5}\\ &= \frac{108}{5} - \frac{48}{5} + \frac{192}{5}\\ &= 50.4\end{aligned}$$

$$\begin{aligned}50.4 \text{ m/km} &= \left(50.4 \times \frac{1}{1000}\right) \text{ m/m}\\ &= 0.0504 \text{ m/m}\end{aligned}$$

The gradient of the graph is 0.0504 for $x = 3$.

iii When $x = 7$,

$$\frac{dy}{dx} = \frac{12}{5}(7)^2 - \frac{16}{5}(7) + \frac{192}{5}$$
$$= \frac{588}{5} - \frac{112}{5} + \frac{192}{5}$$
$$= \frac{668}{5}$$
$$= 133.6$$

$$133.6\text{m/km} = \left(133.6 \times \frac{1}{1000}\right)\text{m/m}$$
$$= 0.1336\ \text{m/m}$$

The gradient of the graph is 0.1336 for $x = 7$.

9 a $y = x^3$

Point of inflexion at $(0, 0)$

When $x = -1$, $\quad y = -1 \quad (-1, -1)$

When $x = 1$, $\quad y = 1 \quad (1, 1)$

$$y = 2 + x - x^2$$
$$= -(x^2 - x - 2)$$
$$= -(x - 2)(x + 1)$$

When $x = 0$, $\quad y = -(0 - 2)(0 + 1)$
$$= 2$$

$\therefore$ y-axis intercept is 2.

When $y = 0$, $\quad -(x - 2)(x + 1) = 0$

$\therefore \quad x - 2 = 0 \quad \text{or} \quad x + 1 = 0$

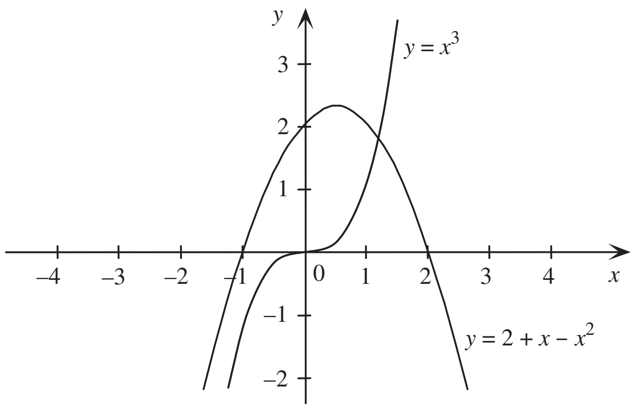
$\therefore \quad x = 2 \quad \text{or} \quad x = -1$

$\therefore$ x-axis intercepts are -1 and 2.

By symmetry, turning point is at $x = \dfrac{2 + -1}{2} = \dfrac{1}{2}$.

When $x = \dfrac{1}{2}$, $\quad y = -\left(\dfrac{1}{2} - 2\right)\left(\dfrac{1}{2} + 1\right)$
$$= -\left(\frac{-3}{2}\right)\left(\frac{3}{2}\right)$$
$$= \frac{9}{4}$$

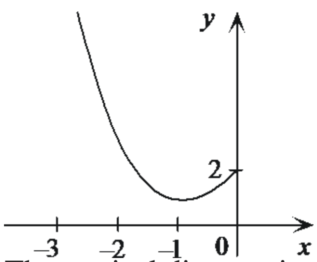
$\therefore$ turning point is $\left(\dfrac{1}{2}, \dfrac{9}{4}\right)$.



b For $x \leq 0, 2 + x - x^2 \geq x^3$.

The vertical distance between the two curves is given by $y = 2 + x - x^2 - x^3, x \leq 0$.

x	−3	−2	−1	0
y	17	4	1	2



The vertical distance is a minimum when y is a minimum.

This occurs where $\dfrac{dy}{dx} = 0$.

Now $\quad \dfrac{dy}{dx} = 1 - 2x - 3x^2$

Consider $\quad 0 = 1 - 2x - 3x^2$

$\therefore \quad 0 = (-3x + 1)(x + 1)$

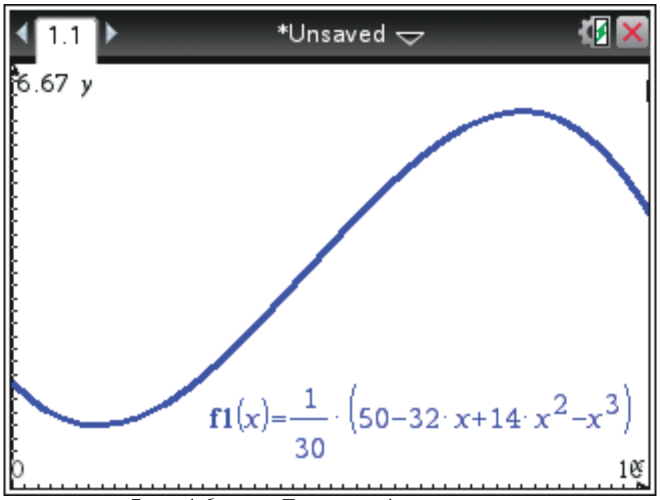
$\therefore \quad x + 1 = 0 \quad$ or $\quad -3x + 1 = 0$

$\therefore \quad x = -1 \quad$ or $\quad x = \dfrac{1}{3}$

But $x \leq 0$, so $x = -1$ and $y = 2 + (-1) - (-1)^2 - (-1)^3 = 1$.

Hence the minimum distance between the two curves is 1 unit when $x = -1$.

10



$$M(x) = \frac{5}{3} - \frac{16}{15}x + \frac{7}{15}x^2 - \frac{1}{30}x^3 \le x \le 10$$

$$M'(x) = -\frac{16}{15} + \frac{14}{15}x - \frac{1}{10}x^2$$

Stationary points occur where $\quad M'(x) = 0$

$$\therefore \quad -\frac{16}{15} + \frac{14}{15}x - \frac{1}{10}x^2 = 0$$

$$\therefore \quad -\frac{1}{10}x^2 + \frac{14}{15}x - \frac{16}{15} = 0$$

$$\therefore \quad -\frac{1}{30}(3x^2 - 28x + 32) = 0$$

$$\therefore \quad -\frac{1}{30}(3x - 4)(x - 8) = 0$$

$$\therefore \quad 3x - 4 = 0 \quad \text{or} \quad x - 8 = 0$$

$$x = \frac{4}{3} \quad \text{or} \quad x = 8$$

x	0	$\frac{4}{3}$	2	8	10
Sign of $M'(x)$	–	0	+	0	–
Shape	\	—	/	—	\

Hence the minimum number of mosquitoes is produced when rainfall is $\frac{4}{3}$ mm and the maximum number is produced when rainfall is 8 mm.

11 **a** $\quad x + y = 5$

$\therefore \quad y = 5 - x$

b $\quad P = xy$

$\therefore \quad P = x(5 - x)$

c $P = 5x - x^2$

$$\frac{dP}{dx} = 5 - 2x$$

Stationary points occur where $\frac{dP}{dx} = 0$

$\therefore \quad 5 - 2x = 0$

$\therefore \quad x = 2.5$

As coefficient of x^2 is negative, there is a local maximum at $x = 2.5$.

When $x = 2.5$, $\quad y = 5 - 2.5$

$\quad = 2.5$

and $\quad P = xy$

$\quad = 2.5 \times 2.5$

$\quad = 6.25$, the maximum value of P.

12 a $2x + y = 10$

$\therefore \quad y = 10 - 2x$

b $A = x^2y$

$\therefore \quad A = x^2(10 - 2x)$

c $A = 10x^2 - 2x^3$

$$\frac{dA}{dx} = 20x - 6x^2$$

Stationary points occur where $\frac{dA}{dx} = 0$

$\therefore \quad 20x - 6x^2 = 0$

$\therefore \quad 2x(10 - 3x) = 0$

$\therefore \quad x = 0 \quad \text{or} \quad 10 - 3x = 0$

$\quad x = \frac{10}{3}$

x	0	1	$\frac{10}{3}$	4
Sign of $\frac{dA}{dx}$	0	+	0	−
Shape	—	/	—	\

The maximum value of A occurs at $x = \frac{10}{3}$.

When $x = \frac{10}{3}$, $\quad y = 10 - 2x = 10 - 2\left(\frac{10}{3}\right) = \frac{10}{3}$

$$A = x^2y$$

$$= \left(\frac{10}{3}\right)^2 \times \frac{10}{3} = \frac{1000}{27}$$

Maximum value of A of $\frac{1000}{27}$ occurs where $x = y = \frac{10}{3}$.

13 $xy = 10 \qquad \therefore \qquad y = \frac{10}{x}$

$$T = 3x^2y - x^3$$

$$= 3x^2 \times \frac{10}{x} - x^3$$

$$= 30x - x^3$$

$$\frac{dT}{dx} = 30 - 3x^2$$

Stationary points occur where $\quad \frac{dT}{dx} = 0$

$$30 - 3x^2 = 0$$

$$30 = 3x^2$$

$$x^2 = 10$$

$$x = \pm\sqrt{10}$$

$\therefore \qquad x = \sqrt{10} \quad$ as $0 < x < \sqrt{30}$

x	0	$\sqrt{10}$	4
Sign of $\frac{dT}{dx}$	+	0	−
Shape	/	—	\

Hence the maximum value of T occurs when $x = \sqrt{10}$.

When $x = \sqrt{10}$, $\quad T = 30\sqrt{10} - (\sqrt{10})^3$

$$= \sqrt{10}(30 - 10)$$

$$= 20\sqrt{10}$$

$$\approx 63.25$$

14 **a** $x + y = 8 \qquad \therefore \qquad y = 8 - x$

b $\qquad s = x^2 + y^2$

$$= x^2 + (8 - x)^2$$

c

$$\begin{aligned} s &= x^2 + (8 - x)^2 \\ &= x^2 + 64 - 16x + x^2 \\ &= 2x^2 - 16x + 64 \\ \frac{ds}{dx} &= 4x - 16 \end{aligned}$$

Stationary points occur where $\frac{ds}{dx} = 0$

$\therefore$

$$\begin{aligned} 4x - 16 &= 0 \\ 4x &= 16 \\ x &= 4 \end{aligned}$$

x	0	4	5
Sign of $\frac{ds}{dx}$	$-$	0	+
Shape	\	—	/

or $x = 4$ is a local minimum because coefficient of x^2 is positive.
Hence the least value of the sum of the squares occurs at $x = 4$.
When $x = 4$,

$$\begin{aligned} s &= x^2 + (8 - x)^2 \\ &= 4^2 + (8 - 4)^2 \\ &= 16 + 4^2 \\ &= 16 + 16 \\ &= 32 \end{aligned}$$

15 Let x and y be the two numbers.

$x + y = 4 \qquad \therefore y = 4 - x$

$s = x^3 + y^2 \qquad \therefore s = x^3 + (4 - x)^2$

$$\begin{aligned} &= x^3 + 16 - 8x + x^2 \\ &= x^3 + x^2 - 8x + 16 \\ \frac{ds}{dx} &= 3x^2 + 2x - 8 \end{aligned}$$

When $\frac{ds}{dx} = 0$, $\quad 3x^2 + 2x - 8 = 0$

$\therefore \quad (3x - 4)(x + 2) = 0$

$\therefore 3x - 4 = 0 \quad$ or $x + 2 = 0$

$x = \frac{4}{3} \quad$ or $x = -2$

x	-3	-2	0	$\frac{4}{3}$	2
Sign of $\frac{ds}{dx}$	+	0	–	0	+
Shape	/	—	\	—	/

Note: Positive numbers are considered, so $x = -2$ need not be considered.

s will be as small as possible when $x = \frac{4}{3}$

and

$$y = 4 - x$$
$$= 4 - \frac{4}{3} = \frac{8}{3}$$

Hence the two numbers are $\frac{4}{3}$ and $\frac{8}{3}$.

16 Let x be the length, y the width and A the area of the rectangle.

$$\therefore \quad 2(x + y) = 100$$
$$\therefore \quad x + y = 50$$
$$\therefore \quad y = 50 - x$$
$$A = xy$$
$$= x(50 - x) = 50x - x^2$$
$$\frac{dA}{dx} = 50 - 2x$$

When $\frac{dA}{dx} = 0$, $\quad 50 - 2x = 0 \quad \therefore x = 25$

A local maximum at $x = 25$, as the coefficient of x^2 is negative.

When $x = 25$, $\quad y = 50 - x = 25$

The area is a maximum (625 m^2) when the rectangle is a square of side length 25 m.

17 Let y be the second number and P the product of the two numbers.

$$x + y = 24$$
$$\therefore \quad y = 24 - x$$
$$P = xy$$
$$= x(24 - x)$$
$$= 24x - x^2$$
$$\frac{dP}{dx} = 24 - 2x$$

When $\frac{dP}{dx} = 0$, $\quad 24 - 2x = 0$

$$\therefore \quad x = 12$$

There is a local maximum at $x = 12$, as the coefficient of x^2 is negative.
Hence, the product of the two numbers is a maximum when $x = 12$.

18 Let C = overhead costs (\$/h)

$$\therefore \quad C = 400 - 16n + \frac{1}{4}n^2$$

$$\frac{dC}{dn} = -16 + \frac{1}{2}n$$

When $\frac{dC}{dn} = 0$, $\quad -16 + \frac{1}{2}n = 0$

$$\therefore \quad n = 32$$

There is a local minimum at $n = 32$, as the coefficient of n^2 is positive.
Hence, 32 items should be produced per hour to keep costs to a minimum.

19

$$x + y = 100$$

$$\therefore \quad y = 100 - x$$

$$\begin{aligned} P &= xy \\ &= x(100 - x) \\ &= 100x - x^2 \end{aligned}$$

$$\frac{dP}{dx} = 100 - 2x$$

When $\frac{dP}{dx} = 0$, $\quad 100 - 2x = 0$

$$\therefore \quad x = 50$$

There is a local maximum at $x = 50$, as the coefficient of x^2 is negative.

When $x = 50$,
$$\begin{aligned} y &= 100 - x \\ &= 100 - 50 \\ &= 50 \end{aligned}$$

Hence $\quad x = y$

When $x = 50$,
$$\begin{aligned} P &= xy \\ &= 50 \times 50 \\ &= 2500, \text{ the maximum value of } P. \end{aligned}$$

20 Let x be the length of river (in km) to be used as a side of the enclosure and let y be the side length of the rectangle (in km) perpendicular to the river.

$$\therefore \quad x + 2y = 4$$

$$\therefore \quad y = \frac{1}{2}(4 - x)$$

Let $A = xy$, the area of the land enclosed.

$$\therefore \quad A = x \times \frac{1}{2}(4 - x)$$
$$= 2x - \frac{1}{2}x^2$$
$$\frac{dA}{dx} = 2 - x$$

When $\frac{dA}{dx} = 0$, $\quad 2 - x = 0$

$$x = 2$$

There is a local maximum at $x = 2$, as the coefficient of x^2 is negative.

When $x = 2$, $\quad y = \frac{1}{2}(4 - x)$

$$= \frac{1}{2}(4 - 2) = 1$$

Hence the maximum area of land of 2 km^2 will be enclosed if the farmer uses a 2 km stretch of river and a width of 1 km for his land.

21 $\quad p^3q = 9$ and $p, q > 0$

$$\therefore \quad q = \frac{9}{p^3}$$
$$= 9p^{-3}$$
$$z = 16p + 3q$$
$$= 16p + 3(9p^{-3})$$
$$= 16p + 27p^{-3}$$

We know that $\frac{d}{dx}(x^n) = yx^{n-1}$ when n = 1, 2, 3

Now suppose this also true for $n = -1, -2, -3, \ldots$

So $\frac{d}{dx}(p^{-3}) = -3x^{-4}$

$$\text{Hence } \frac{dz}{dp} = 16 - 81p^{-4}$$

When $\frac{dz}{dp} = 0$, $\quad 16 - 81p^{-4} = 0$

$$16 = 81p^{-4}$$
$$p^{-4} = \frac{16}{81}$$

$$p^4 = \frac{81}{16}$$

$$p = \frac{\sqrt[4]{81}}{\sqrt[4]{16}} = \frac{3}{2}$$

$$= \pm\frac{3}{2}$$

p	1	$\frac{3}{2}$	2
Sign of $\frac{dz}{dp}$	–	0	+
Shape	\	—	/

Hence z is a minimum when $p = \frac{3}{2}$

and $q = \frac{9}{p^3}$

$$= \frac{9}{(\frac{3}{2})^3}$$

$$= \frac{9\text{x}8}{27}$$

$$= \frac{8}{3}$$

So $p = \frac{3}{2}$ and $q = \frac{8}{3}$.

22 **a** $2(x + y) = 120$

$x + y = 60$

$y = 60 - x$

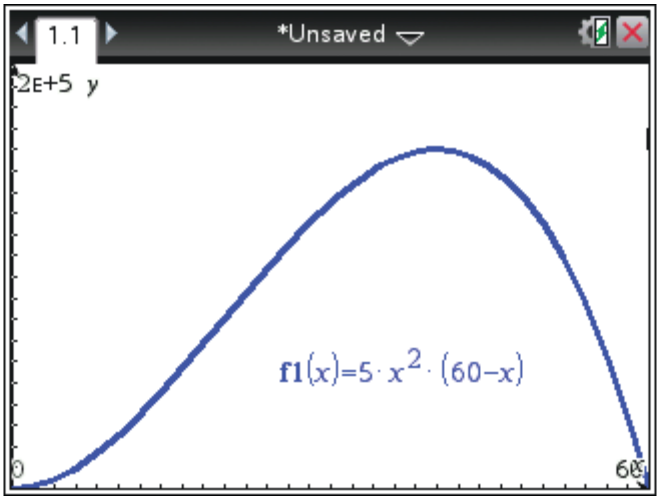
b $S = 5x^2y$

$= 5x^2(60 - x)$

c $S > 0$, $\therefore$ $5x^2(60 - x) > 0$

$\therefore$ $0 < x < 60$

d



e $S = 5x^2(60 - x)$

$= 300x^2 - 5x^3$

$\frac{dS}{dx} = 600x - 15x^2$

$= 0$

if $x = 0$ or $x = 40$

From the graph, the maximum occurs at $x = 40$

$\therefore \quad y = 60 - x$

$= 60 - 40$

$= 20$

Hence $x = 40$ and $y = 20$ give the strongest beam.

f For $x \le 19$, the maximum strength of the beam occurs when $x = 19$.

$\therefore \quad S = 5 \times 19^2(60 - 19)$

$= 74\,005$

The maximum strength of the beam, if the cross-sectional depth of the beam must be less than 19 cm, is 74 005.

23

$$s'(x) = -3x^2 + 6x + 360$$
$$= -3(x^2 - 2x - 120)$$
$$= -3(x + 10)(x - 12)$$

When $s'(x) = 0, -3(x + 10)(x - 12) = 0$

$$\therefore x + 10 = 0 \text{ or } x - 12 = 0$$
$$\therefore x = -10 \text{ or } x = 12$$

But $6 \le x \le 20$, so $x = 12$.

x	10	12	14
Sign of $s'(x)$	+	0	–
Shape	/	—	\

Hence the maximum number of salmon swimming upstream occurs when the water temperature is 12°C.

24 **a** Let x (cm) be the breadth of the box, $2x$ (cm) be the length of the box, and h (cm) be the height of the box.

$$4(x + 2x) + 4h = 360$$

$$\therefore \quad 4h = 360 - 4(3x)$$

$$\therefore \quad h = 90 - 3x$$

$$\begin{aligned} V &= x \times 2x \times h \\ &= 2x^2(90 - 3x) \\ &= 180x^2 - 6x^3 \text{ as required} \end{aligned}$$

b

$$V = 6x^2(30 - x)$$

$$\therefore \quad \text{Domain } V = \{x\text{: } 0 < x < 30\}$$

c

$$\begin{aligned} V &= 180x^2 - 6x^3 \\ \frac{dV}{dx} &= 360x - 18x^2 \\ &= 18x(20 - x) \end{aligned}$$

When $\frac{dV}{dx} = 0 \qquad 18x(20 - x) = 0$

$\therefore \quad 18x = 0 \quad$ or $\quad 20 - x = 0$

$\therefore \quad x = 0 \quad$ or $\quad x = 20$

x	−10	0	10	20	30
Sign of $\frac{dV}{dx}$	–	0	+	0	–
Shape	\	—	/	—	\

Hence V is a minimum when $x = 0$ and a maximum when $x = 20$.

When $x = 0$, $\quad V = 180(0)^2 - 6(0)^3$

$\qquad = 0$

$\therefore$ y-axis intercept is 0.

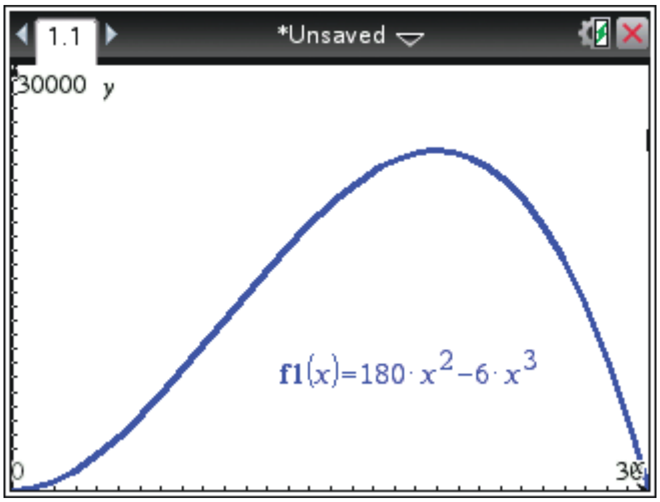
When $V = 0$, $\quad 180x^2 - 6x^3 = 0$

$\therefore \quad 6x^2(30 - x) = 0$

$\therefore \quad 6x^2 = 0$ or $30 - x = 0$

$x = 0$ or $\quad x = 30$

$\therefore$ x-axis intercepts are 0 and 30.



d From part **c**, V is a maximum when $x = 20$.

$$\therefore \quad h = 90 - 3x$$
$$= 90 - 60$$
$$= 30$$

Hence the dimensions giving the greatest volume are 20 cm × 30 cm × 40 cm.

e On a CAS calculator, with $\boldsymbol{f1 = 180x\text{^}2 - 6x\text{^}3}$ and $\boldsymbol{f1 = 20\,000}$,
TI: Press **Menu → 6:Analyze Graph → 4:Intersection**
CP: Tap **Analysis → G-Solve → Intersect**
The x-coordinates of the points of intersection are 14.817 02 and 24.402 119. Hence the values of x for which $V = 20\,000$ are 14.82 and 24.40, correct to 2 decimal places.

25 a

$$A = 8x \times y + 2\left(\frac{1}{2} \times 4x \times \sqrt{(5x)^2 - (4x)^2}\right)$$
$$= 8xy + 4x \times \sqrt{25x^2 - 16x^2}$$
$$= 8xy + 4x \times \sqrt{9x^2}$$
$$= 8xy + 4x \times 3x \text{ (only positive square root appropriate)}$$
$$= 8xy + 12x^2$$

Also

$$8x + y + y + 5x + 5x = 90$$

$$\therefore \quad 18x + 2y = 90$$

$$2y = 90 - 18x$$

$$y = 45 - 9x$$

$$\therefore \quad A = 8x(45 - 9x) + 12x^2$$

$$= 360x - 72x^2 + 12x^2$$

$$\therefore \quad A = 360x - 60x^2 \text{ as required}$$

b $A = 360x - 60x^2$, $\quad \therefore \dfrac{dA}{dx} = 360 - 120x$

When $\dfrac{dA}{dx} = 0$, $\quad 360 - 120x = 0$

$$x = 3$$

Area is a maximum at $x = 3$, as the coefficient of x^2 is negative.

When $x = 3$, $\quad y = 45 - 9x$

$$= 45 - 27$$

$$= 18$$

26 a Let r (cm) be the radius of the circle and x (cm) be the side length of the square.

$$2\pi r + 4x = 100$$

$$2\pi r = 100 - 4x$$

$$r = \frac{50 - 2x}{\pi}$$

As $r > 0$, $\quad 50 - 2x > 0$

i.e. $\quad x < 25$

Let A be the sum of the areas of the circle and the square.

$$\therefore \quad A = \pi r^2 + x^2$$

$$= \pi\left(\frac{50 - 2x}{\pi}\right)^2 + x^2$$

$$= \frac{1}{\pi}(2500 - 200x + 4x^2) + x^2$$

$$= \frac{2500}{\pi} - \frac{200}{\pi}x + \frac{4}{\pi}x^2 + x^2$$

i.e. $$A = \frac{4+\pi}{\pi}x^2 - \frac{200}{\pi}x + \frac{2500}{\pi}, x \in [0, 25]$$

$$\frac{dA}{dx} = \frac{2(4+\pi)}{\pi}x - \frac{200}{\pi}$$

When $\frac{dA}{dx} = 0$, $$\frac{2(4+\pi)}{\pi}x - \frac{200}{\pi} = 0$$

$$2(4+\pi)x = 200$$

$$x = \frac{100}{4+\pi}$$

The area is a minimum when $x = \frac{100}{4+\pi}$, as the coefficient of x^2 is positive.

When $x = \frac{100}{4+\pi}$, $$4x = \frac{400}{4+\pi} \approx 56$$

and $$\begin{aligned} 2\pi r &= 2\pi\left(\frac{50-2x}{\pi}\right) \\ &= 100 - 4x \\ &= 100 - 4\left(\frac{100}{4+\pi}\right) \\ &= 100 - \frac{400}{4+\pi} \\ &\approx 44 \end{aligned}$$

Hence the wire should be cut into a 56 cm strip to form the square, and 44 cm to form the circle.

b When $x = 0$, $$\begin{aligned} A &= \frac{4+\pi}{\pi}(0)^2 - \frac{200}{\pi}(0) + \frac{2500}{\pi} \\ &= \frac{2500}{\pi} \\ &\approx 796 \end{aligned}$$

When $x = 25$, $$A < \frac{2500}{\pi}$$

Hence the maximum area occurs when $x = 0$ and all the wire is used to form the circle.

27 $$\begin{aligned} 2(x + 2x + x) + 2x + 4y &= 36 \\ 2(4x) + 2x + 4y &= 36 \\ 10x + 4y &= 36 \\ 4y &= 36 - 10x \\ y &= 9 - \frac{5}{2}x \end{aligned}$$

Let A (m^2) be the area of the court.

$$\begin{aligned} A &= 4xy \\ &= 4x(9 - \tfrac{5}{2}x) \\ &= 36x - 10x^2 \\ \frac{dA}{dx} &= 36 - 20x \end{aligned}$$

When $\dfrac{dA}{dx} = 0$, $\quad 36 - 20x = 0$

$$\begin{aligned} 20x &= 36 \\ x &= \frac{9}{5} \end{aligned}$$

Area is a maximum when $x = \dfrac{9}{5}$, as the coefficient of x^2 is negative.

When $x = \dfrac{9}{5}$, $\quad$ length $= 4x$

$$\begin{aligned} &= 4 \times \frac{9}{5} \\ &= 7.2 \end{aligned}$$

and $\quad$ width $= y$

$$\begin{aligned} &= 9 - \frac{5}{2}x \\ &= 9 - \frac{5}{2} \times \frac{9}{5} \\ &= 4.5 \end{aligned}$$

The length is 7.2 m and the width is 4.5 m.

28 **a** $A = xy$

b $x + 2y = 16$

$$\begin{aligned} 2y &= 16 - x \\ y &= 8 - \frac{x}{2} \\ A &= \left(8 - \frac{x}{2}\right)x \end{aligned}$$

c When $A = 0$, $\quad x(8 - \dfrac{1}{2}x) = 0$

$x = 0 \quad$ or $\quad x = 16$

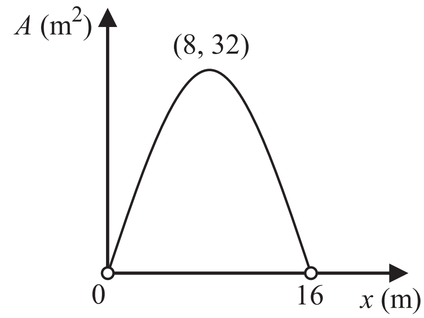
$\therefore \quad$ Domain $= \{x\colon 0 < x < 16\}$

d Turning point is at $x = \dfrac{0 + 16}{2} = 8$

When $x = 8$, $A = x(8 - \frac{1}{2}x)$

$$= 8(8 - \tfrac{1}{2}(8))$$

$$= 32$$



e Calculus could be used, but as the graph is a parabola with turning point (8, 32), the maximum is 32. Therefore, the largest area of ground which could be covered is 32 m^2.

29 $2a + h + h + 2a + 2a = 8000$

$\therefore \quad 6a + 2h = 8000$

$\therefore \quad 2h = 8000 - 6a$

$\therefore \quad h = 4000 - 3a$

Let A be the area of the shape and v be the vertical height of the triangle.

$$\begin{aligned} v &= \sqrt{(2a)^2 - a^2} \\ &= \sqrt{4a^2 - a^2} \\ &= \sqrt{3a^2} \\ &= \sqrt{3}a \end{aligned}$$

$$\begin{aligned} A &= 2ah + \frac{1}{2}(2a)v \\ &= 2a(4000 - 3a) + a \times \sqrt{3}a \\ &= 8000a - 6a^2 + \sqrt{3}a^2 \\ &= (\sqrt{3} - 6)a^2 + 8000a \end{aligned}$$

$$\frac{dA}{da} = 2(\sqrt{3} - 6)a + 8000$$

When $\dfrac{dA}{da} = 0$, $\quad 2(\sqrt{3} - 6)a + 8000 = 0$

$$\begin{aligned} \therefore \quad a &= \frac{8000}{2(6 - \sqrt{3})} \\ &= \frac{4000}{6 - \sqrt{3}} \\ &\approx 937 \end{aligned}$$

The area is a maximum when $a = 937$, as the coefficient of a^2 is negative.

When $a = \dfrac{4000}{6-\sqrt{3}}$, $\quad h = 4000 - 3a$

$$= 4000 - \frac{3 \times 4000}{6-\sqrt{3}}$$

$$\approx 4000 - 2812 = 1188$$

The maximum amount of light passes through when $a = 931$ and $h = 1188$.

30 a $\quad \pi x + y = 10$

$\therefore \quad y = 10 - \pi x$

b When $x = 0$, $\quad y = 10 - \pi(0)$

$$= 10$$

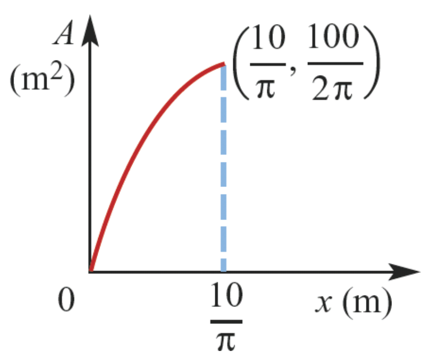
When $y = 0$, $\quad 10 - \pi x = 0$

$\therefore \quad x = \dfrac{10}{\pi}$

Hence all possible values of x are $0 \leq x \leq \dfrac{10}{\pi}$.

c
$$A = xy + \frac{\pi}{2}x^2$$
$$= x(10 - \pi x) + \frac{\pi}{2}x^2$$
$$= 10x - \pi x^2 + \frac{\pi}{2}x^2$$
$$= \frac{x}{2}(20 - \pi x)$$

d



e $$A = 10x - \frac{\pi}{2}x^2$$

$\therefore$ $$\frac{dA}{dx} = 10 - \pi x$$

When $\frac{dA}{dx} = 0, \quad 10 - \pi x = 0$

$\therefore$ $$x = \frac{10}{\pi}$$

The value of x which maximises A is $\frac{10}{\pi}$.

f When $x = \frac{10}{\pi}, \qquad y = 10 - \pi \times \frac{10}{\pi} = 0$

The capacity of the drain is a maximum when the cross-section is a semi-circle.

31 **a** $$\text{Surface area } = 2\pi xh + 2\pi x^2$$

$\therefore$ $$1000 = 2\pi xh + 2\pi x^2$$

$\therefore$ $$500 = \pi xh + \pi x^2$$

$\therefore$ $$h = \frac{500 - \pi x^2}{\pi x} = \frac{500}{\pi x} - x$$

b
$$\begin{aligned} V &= \pi x^2 h \\ &= \pi x^2\left(\frac{500 - \pi x^2}{\pi x}\right) \\ &= x(500 - \pi x^2) = 500x - \pi x^3 \end{aligned}$$

c $$\frac{dV}{dx} = 500 - 3\pi x^2$$

d $$\frac{dV}{dx} = 0$$

implies $$500 - 3\pi x^2 = 0$$

$\therefore$ $$x = \frac{\sqrt{500}}{\sqrt{3\pi}} \text{ as } x > 0$$

$\therefore$ $$x = \frac{10\sqrt{5}}{\sqrt{3\pi}} \approx 7.28$$

e $h > 0$ and $x > 0$

$$\therefore \quad \frac{500 - \pi x^2}{\pi x} > 0$$

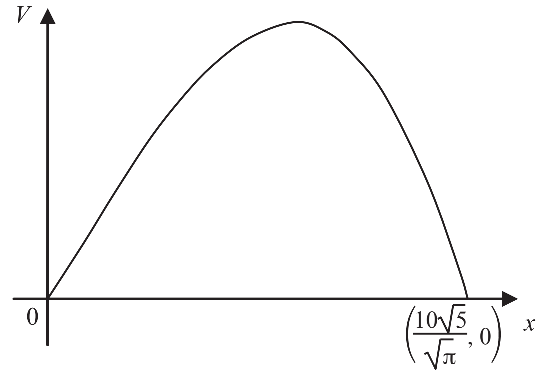
$$\therefore \quad 500 > \pi x^2$$

$$\therefore \quad \frac{500}{\pi} > x^2$$

$$\therefore \quad x < \frac{\sqrt{500}}{\sqrt{\pi}} \text{ for } x > 0$$

$$\therefore \quad x < \frac{10\sqrt{5}}{\sqrt{\pi}}$$

$$\therefore \quad \text{domain} = \left(0, \frac{10\sqrt{5}}{\sqrt{\pi}}\right)$$



f When $x = \dfrac{10\sqrt{5}}{\sqrt{3\pi}}$ (from part **d**)

$$\begin{aligned} V &= 5000 \times \frac{\sqrt{5}}{\sqrt{3\pi}} - \pi \times \left(\frac{5}{3\pi}\right)^{\frac{3}{2}} \times 1000 \\ &= \frac{\sqrt{5}}{\sqrt{3\pi}}\left(5000 - \frac{\pi \times 5000}{3\pi}\right) \\ &= \frac{\sqrt{5}}{\sqrt{3\pi}} \times \frac{10\,000}{3} \\ &= \frac{10\,000\sqrt{5}}{3\sqrt{3\pi}} \text{ cm}^3 \end{aligned}$$

The maximum volume is 2427.89 cm^3, correct to 2 decimal places.

g On a CAS calculator, with $\boldsymbol{f1 = x(500 - \pi x\text{\textasciicircum}2)}$ and $\boldsymbol{f2 = 1000}$,
TI: Press **Menu → 6:Analyze Graph → 4:Intersection**
CP: Tap **Analysis → G-Solve → Intersect**
to find $x = 2.05$ and $x = 11.46$
Corresponding values of h are $h = 75.41$ and $h = 2.42$

32 **a** Let x (cm) be the radius, h (cm) be the height, $S\,(\text{cm}^2)$ be the surface area of the can.

$$\pi x^2 h = 500$$

$$\therefore \quad h = \frac{500}{\pi x^2}$$

$$\begin{aligned} S &= 2\pi x h + 2\pi x^2 \\ &= 2\pi x\left(\frac{500}{\pi x^2}\right) + 2\pi x^2 \\ &= \frac{1000}{x} + 2\pi x^2 \\ &= 1000x^{-1} + 2\pi x^2 \end{aligned}$$

We know that $\frac{d}{dx}(x^n) = yx^{n-1}$ when n = 1, 2, 3

Now suppose this also true for $n = -1, -2, -3, \ldots$

So $\frac{d}{dx}(x^{-1}) = -x^{-2}$

$$\text{Hence } \frac{dS}{dx} = -1000x^{-2} + 4\pi x = \frac{-1000}{x^2} + 4\pi x$$

When $\frac{dS}{dx} = 0$, $\quad \frac{-1000}{x^2} + 4\pi x = 0$

$$\therefore \quad 4\pi x = \frac{1000}{x^2}$$

$$\therefore \quad x^3 = \frac{1000}{4\pi}$$

$$\therefore \quad x = \frac{10}{(4\pi)^{\frac{1}{3}}} \approx 4.3$$

x	4	4.3	5
Sign of $\frac{dS}{dx}$	$-$	0	$+$
Shape	\	—	/

The surface area is a minimum when the radius is 4.3 cm,

and $$h = \frac{500}{\pi x^2}$$

$$= \frac{500}{\pi\left(\frac{10}{(4\pi)^{\frac{1}{3}}}\right)^2}$$

$$= \frac{500}{\pi\left(\frac{100}{(4\pi)^{\frac{2}{3}}}\right)}$$

$$= \frac{500 \times (4\pi)^{\frac{2}{3}}}{100\pi}$$

$$\therefore \quad h = \frac{5(4\pi)^{\frac{2}{3}}}{\pi}$$

$$\approx 8.6$$

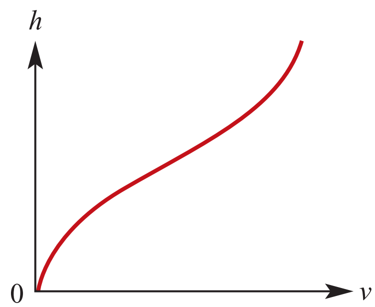
The minimum surface area occurs when the radius is approximately 4.3 cm and the height is approximately 8.6 cm.

b If the radius of the can must be no greater than 5 cm, the minimum surface area occurs when the radius is approximately 4.3 cm and the height is approximately 8.6 cm.

Chapter 19 – Revision of chapters 16–18

Solutions to Technology-free questions

1



2 **a** $x(0) = 0$ and $x(1) = 1$

Average velocity $= \dfrac{x(1) - x(0)}{1 - 0}$

$= 1$ m/s

b $x(1) = 1$ and $x(4) = 124$

Average velocity $= \dfrac{x(4) - x(1)}{4 - 1}$

$= 41$ m/s

3 **a** **i** Average rate of change

$= \dfrac{0 - 8}{3 - 1} = -4$

ii Average rate of change

$= \dfrac{5 - 8}{2 - 1} = -3$

b Average rate of change

$$= \frac{(9 - (1 + h)^2) - (9 - 1)}{1 + h - 1}$$

$$= \frac{9 - (1 + 2h + h^2) - 8}{h}$$

$$= \frac{-2h - h^2}{h}$$

$$= -2 - h$$

c -2

4 $\dfrac{f(x + h) - f(x)}{x + h - x}$

$$= \frac{\frac{1}{2}(x + h)^2 - (x + h) - (\frac{1}{2}x^2 - x)}{h}$$

$$= \frac{xh + \frac{1}{2}h^2 - h}{h}$$

$$= x + \tfrac{1}{2}h - 1$$

$\therefore f'(x) = x - 1$

5 **a** Let $f(x) = 2x^3 - x + 1$

$\therefore f'(x) = 6x^2 - 1$

b Let $f(x) = (x - 1)(x - 2) = x^2 + x - 2$

$\therefore f'(x) = 2x + 1$

c Let $f(x) = \dfrac{x^2 + 5x}{x} = x + 5$

$\therefore f'(x) = 1$

6 **a** Let $y = 3x^4 + x$

Then $\dfrac{dy}{dx} = 12x^3 + 1$

When $x = 1, \dfrac{dy}{dx} = 13$

Gradient = 13 at the point(1, 4)

b Let $y = 2x(1 - x) = 2x - x^2$

Then $\dfrac{dy}{dx} = 2 - 2x$

When $x = -2, \dfrac{dy}{dx} = 10$

Gradient = 10 at the point(-2, -12)

7 **a** $f(x) = 0$

$x - 2x^2 = 0$

$x(1 - 2x) = 0x = 0$ or $x = \dfrac{1}{2}$

b $f'(x) = 0$

$1 - 4x = 0$

$x = \frac{1}{4}$

c $f'(x) > 0$

$1 - 4x > 0$

$x < \frac{1}{4}$

d $f'(x) < 0$

$1 - 4x < 0$

$x > \frac{1}{4}$

e $f'(x) = 10$

$1 - 4x = 10$

$4x = 11$ $x = \frac{11}{4}$

8 **a** $\frac{d}{dx}(2x^{-3} - x^{-1}) = -6x^{-4} + x^{-2}$

b $\frac{d}{dz}\left(\frac{3 - z}{z^3}\right) = \frac{d}{dz}(3z^{-3} - z^{-2}) =$
$-9z^{-4} + 2z^{-3} = \frac{2z - 9}{z^4}$

9 Let $y = x^2 - 5x$
$\frac{dy}{dx} = 2x - 5$
When $x = 1, \frac{dy}{dx} = -3$
When $x = 1, y = -4$
Therefore equation of tangent:
$y + 4 = -3(x - 1)$
$y = -3x - 1.$

Normal has gradient $\frac{1}{3}$
Equation of Normal $y = \frac{1}{3}x - \frac{13}{3}$

10 $x = \frac{1}{6}t^3 - \frac{1}{2}t^2$
$v = \frac{dx}{dt} = \frac{1}{2}t^2 - t$
$a = \frac{dv}{dt} = t - 1$

a $v = 0 \Rightarrow \frac{1}{2}t^2 - t = 0$
$\Rightarrow t(\frac{1}{2}t - 1) = 0$
$\Rightarrow t = 0$ and $t = 2$

b $t = 0$, $a = -1$ cm/s^2; $t = 2$,
$a = 1$ cm/s^2

c $a = 0 \Rightarrow t = 1 \Rightarrow v = -\frac{1}{2}$ cm/s

11 $y = 2(x^3 - 4x) = 2x^3 - 8x$ $\frac{dy}{dx} = 6x^2 - 8$
$\frac{dy}{dx} = 0$
$\Rightarrow 3x^2 - 4 = 0$
$\Rightarrow x = \frac{2}{\sqrt{3}}$ or $x = -\frac{2}{\sqrt{3}}$
When $x = \frac{2}{\sqrt{3}}, y = \frac{32}{3\sqrt{3}}$
When $x = -\frac{2}{\sqrt{3}}, y = \frac{32}{3\sqrt{3}}$
Local minimum $\left(\frac{2}{\sqrt{3}}, -\frac{32}{3\sqrt{3}}\right)$
Local maximum $\left(-\frac{2}{\sqrt{3}}, \frac{32}{3\sqrt{3}}\right)$
Leading coefficient of the cubic is positive.

Solutions to multiple-choice questions

1 B $\frac{dh}{dV}$ decreases as bowl fills, so gradient must be constantly increasing.

2 A Gradient $\approx \frac{0-60}{6-0} = -10$

3 B Av. speed $= \frac{3-0}{3-0} = 1$ m/s

4 A Av. rate $= \frac{f(2)-f(0)}{2-0}$

$= \frac{13-1}{2}$

$= 6$

5 B Av. rate $= \frac{23.5-10}{12-7}$

$= 2.7$°C/h

6 A $y = 5x^2 + 1 \therefore \frac{dy}{dx} = 10x$

7 D $f(5+h) - f(5) = (5+h)^2 - 5^2$

$= 10h + h^2$

8 B Gradient = 0 at turning points $x = -1, 1.5$

9 C $V = 3t^2 + 4t + 2, \therefore V' = 6t + 4$

$\therefore V'(2) = 6(2) + 4$

$= 16 \text{ m}^3/\text{min}$

10 A $\frac{f(3+h) - f(3)}{h} = 2h^2 + 2h$

$\therefore \lim_{h\to 0} 2h^2 + 2h = 0$

11 C Curve increases for $x \in (-\infty, -2) \cup (1, \infty)$

12 B $f(x) = x^3 - x^2 - 5$

$\therefore f'(x) = 3x^2 - 2x$

$= x(3x-2)$

$\therefore f' = 0, x = 0, \frac{2}{3}$

13 A $f'(x) = \frac{0-3}{5-0} = -\frac{3}{5}$ for all x

14 C $y = 2x^3 - 3x^2, \therefore y' = 6x^2 - 6x$

$\therefore y'(1) = 6 - 6 = 0$

15 C $y = 7 + 2x - x^2, \therefore y' = 2(1-x)$

Inverted parabola, so

y max. $= y(1) = 8$

16 A $s = 28t - 16t^2, \therefore v = 28 - 32t$

$s_{\max} = s\frac{7}{8}$

$= \frac{49}{2} - \frac{49}{4}$

$= \frac{49}{4}$ m/s

17 D $f'(x) > 0$ for $x < 1$, $f'(x) < 0$ for $x > 1$

$f'(1) = 0$; only **D** fits.

18 E $f'(-2) > 0, f'(-1) = 0, f'(0) < 0$

$f(x)$ has a local max. at $x = -1$.

19 C $y = \frac{x^2}{2}(x^2 + 2x - 4)$

$= \frac{x^4}{2} + x^3 - 2x^2$

$\therefore \frac{dy}{dx} = 2x^3 + 3x^2 - 4x$

20 B $\frac{d}{dx}(5 + 3x^2) = 6x$

21 E Negative slope for $x < -1, x > 1$

22 D Rise/run $= \dfrac{(1+h)^2 - 1}{1 + h - 1}$

$= 2 + h$

23 A $y = x^2(2x - 3) = 2x^3 - 3x^2$

$y' = 6x^2 - 6x < \therefore y'(1) = 0$

24 A Rise/run $= \dfrac{b^2 - a^2}{b - a} = b + a$

25 C $f(x) = 3x^3 + 6x^2 - x + 1$

$\therefore f'(x) = 9x^2 + 12x - 1$

26 D $y + 3x = 10, \therefore y = 10 - 3x$

$A = 4x(10 - 3x)$

$\therefore A' = 40 - 24x = 0$

$\therefore 5 - 3x = 0$

27 B $y = x^2 + 3, \therefore y' = 2x$

$\therefore y'(3) = 6$

28 B $y = x^3 + 5x^2 - 8x$

$\therefore y' = 3x^2 + 10x - 8$

$= (3x - 2)(x + 4)$

x	-5	-4	0	$\frac{2}{3}$	1
y'	$+$	0	$-$	0	$+$

$x = -4$ is a local maximum.

$x = \dfrac{2}{3}$ is a local minimum.

29 B $f'(1) = \lim\limits_{h \to 0} \dfrac{f(1+h) - f'(1)}{h}$

30 C $y = x^2 + 4x - 3$

$\therefore y' = 2(x + 2)$

$y\,\text{min} = y(-2) = -7$

31 D $y = x^2, \therefore y' = 2x$

$y'(2) = 4$

$\therefore$ gradient of normal $= -\dfrac{1}{4}$

32 E $y = \dfrac{2x + 5}{x} = 2 + 5x^{-1}$

$\therefore \dfrac{dy}{dx} = -5x^{-2} = -\dfrac{5}{x^2}$

33 A $y = x^2 - 3x - 4, \therefore y' = 2x - 3$

$y' < 0, \therefore x < \dfrac{3}{2}$

34 A $\lim\limits_{x \to 0} \dfrac{x^2 - x}{x} = x - 1 = -1$

35 C Graph is discontinuous at $x = 0, 2$ since in both cases the positive and negative limits are different.

36 C Graph is discontinuous at $x = -1, 1$ since in both cases the positive and negative limits are different.

37 D

38 D

39 D

Solutions to extended-response questions

1 a When the particle returns to ground level, $y = 0$

$$\therefore \quad x - 0.01x^2 = 0$$

$$\therefore \quad x(1 - 0.01x) = 0$$

$$\therefore \quad x = 0 \quad \text{or} \quad 1 - 0.01x = 0$$

$$0.01x = 1$$

$$x = 100$$

The particle travels 100 units horizontally before returning to ground level.

b

$$y = x - 0.01x^2$$

$$\therefore \quad \frac{dy}{dx} = 1 - 0.02x$$

c

$$\frac{dy}{dx} = 0$$

$$\therefore \quad 1 - 0.02x = 0$$

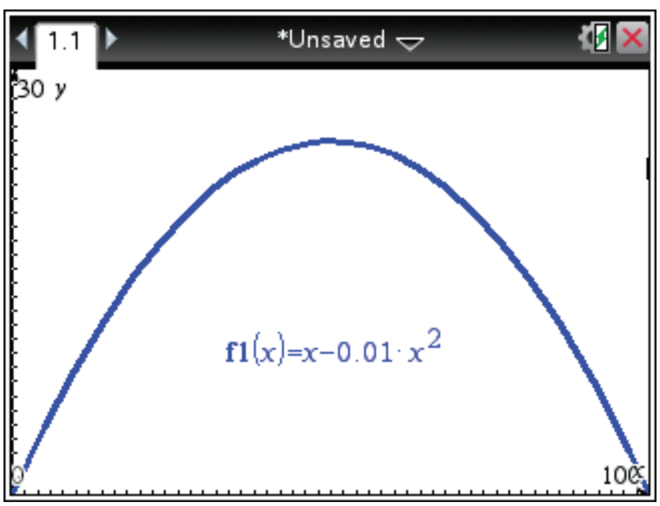
$$\therefore \quad 0.02x = 1$$

$$\therefore \quad x = 50$$

When $x = 50$,

$$\begin{aligned} y &= 50 - 0.01(50)^2 \\ &= 50 - 0.01 \times 2500 \\ &= 50 - 25 \\ &= 25 \end{aligned}$$

d



e **i** When $\frac{dy}{dx} = \frac{1}{2}$, $1 - 0.02x = \frac{1}{2}$

$\therefore \quad 0.02x = \frac{1}{2}$

$\therefore \quad x = 25$

When $x = 25$, $y = 25 - 0.01(25)^2$

$= 25 - 0.01 \times 625$

$= 25 - 6.25$

$= 18.75$

i.e. the coordinates of the point with gradient $\frac{1}{2}$ are (25, 18.75).

ii When $\frac{dy}{dx} = -\frac{1}{2}$, $1 - 0.02x = -\frac{1}{2}$

$\therefore \quad 0.02x = 1.5$

$\therefore \quad x = 75$

When $x = 75$, $y = 75 - 0.01(75)^2$

$= 75 - 0.01 \times 5625$

$= 75 - 56.25$

$= 18.75$

i.e. the coordinates of the point with gradient $-\frac{1}{2}$ are (75, 18.75).

2 **a** $y = -0.0001(x^3 - 100x^2)$

$= -0.0001x^3 + 0.01x^2$

Highest point is reached where $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -0.0003x^2 + 0.02x$

When $\frac{dy}{dx} = 0$, $-0.0003x^2 + 0.02x = 0$

$\therefore \quad x(0.02 - 0.0003x) = 0$

$\therefore \quad x = 0$ or $0.02 - 0.0003x = 0$

$\therefore \quad 0.0003x = 0.02$

$\therefore \quad x = \frac{200}{3}$

$= 66\frac{2}{3}$

When $x = 0$, $\quad y = 0$

When $x = 66\frac{2}{3}$,

$$
\begin{aligned}
y &= -0.0001x^2(x - 100) \\
&= -0.0001\left(\frac{200}{3}\right)^2\left(\frac{200}{3} - 100\right) \\
&= -0.0001 \times \frac{40\,000}{9}\left(-\frac{100}{3}\right) \\
&= -\frac{4}{9} \times -\frac{100}{3} \\
&= \frac{400}{27} \\
&= 14\frac{22}{27}
\end{aligned}
$$

i.e. the coordinates of the highest point are $\left(66\frac{2}{3}, 14\frac{22}{27}\right)$.

b **i** At $x = 20$,

$$
\begin{aligned}
\frac{dy}{dx} &= x(0.02 - 0.0003x) \\
&= 20(0.02 - 0.0003 \times 20) \\
&= 20(0.02 - 0.006) \\
&= 20 \times 0.014 \\
&= 0.28
\end{aligned}
$$

i.e. at $x = 20$, the gradient of the curve is 0.28.

ii At $x = 80$,

$$
\begin{aligned}
\frac{dy}{dx} &= x(0.02 - 0.0003x) \\
&= 80(0.02 - 0.0003 \times 80) \\
&= 80(0.02 - 0.024) \\
&= 80 \times -0.004 \\
&= -0.32
\end{aligned}
$$

i.e. at $x = 80$, the gradient of the curve is -0.32.

iii At $x = 100$,

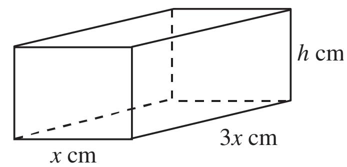
$$
\begin{aligned}
\frac{dy}{dx} &= x(0.02 - 0.0003x) \\
&= 100(0.02 - 0.0003 \times 100) \\
&= 100(0.02 - 0.03) \\
&= 100 \times -0.01 \\
&= -1
\end{aligned}
$$

i.e. at $x = 100$, the gradient of the curve is -1.

c The rollercoaster begins with a gentle upwards slope until it reaches the turning point (its highest point). On its downward trip the rollercoaster has a steeper slope and by the end of the ride it has reached a very steep downward slope.

d It would be less dangerous if the steep slope at the end were smoothed out.

3 **a** Let h = height of the block.



$$\begin{aligned} \text{Now} \quad 4(3x + x + h) &= 20 \\ \therefore \quad 4(4x + h) &= 20 \\ \therefore \quad 4x + h &= 5 \\ \therefore \quad h &= 5 - 4x \end{aligned}$$

i.e. the height of the block is $(5 - 4x)$ cm.

b $V = x \times 3x \times (5 - 4x)$

$= 3x^2(5 - 4x)$

$= 15x^2 - 12x^3$ as required

c $x > 0$ and $V > 0$

$$\begin{aligned} \therefore \quad & 15x^2 - 12x^3 > 0 \\ \Longleftrightarrow \quad & 3x^2(5 - 4x) > 0 \\ \Longleftrightarrow \quad & 5 - 4x > 0 \text{ as } 3x^2 > 0 \text{ for all } x \\ \Longleftrightarrow \quad & 5 > 4x \\ \Longleftrightarrow \quad & \frac{5}{4} > x \end{aligned}$$

Domain is $\left\{x\colon 0 < x < \frac{5}{4}\right\}$

d $\dfrac{dV}{dx} = 30x - 36x^2$

e When $\dfrac{dV}{dx} = 0$, $\quad 30x - 36x^2 = 0$

$$\begin{aligned} \therefore \quad & 6x(5 - 6x) = 0 \\ \therefore \quad & 6x = 0 \quad \text{or} \quad 5 - 6x = 0 \\ \therefore \quad & x = 0 \quad \text{or} \quad x = \frac{5}{6} \\ \therefore \quad & x = \frac{5}{6} \text{ as } x > 0 \end{aligned}$$

When $x = \frac{5}{6}$, $\quad V = 3\left(\frac{5}{6}\right)^2\left(5 - 4 \times \frac{5}{6}\right)$

$$= 3 \times \frac{25}{36}\left(5 - \frac{10}{3}\right) = \frac{25}{12} \times \frac{5}{3}$$

$$= \frac{125}{36} = 3\frac{17}{36}$$

$$\frac{dV}{dx} = 6x(5 - 6x)$$

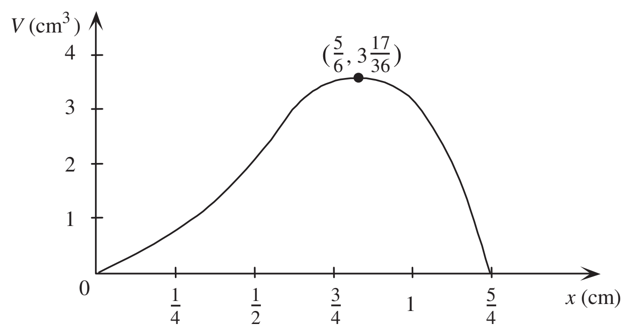
If $x < \frac{5}{6}$, e.g. $x = \frac{1}{6}$, $\frac{dV}{dx} > 0$.

If $x > \frac{5}{6}$, e.g. $x = 1$, $\frac{dV}{dx} < 0$.

$\therefore$ local maximum at $\left(\frac{5}{6}, \frac{125}{36}\right)$.

i.e. the maximum volume possible is $3\frac{17}{36}$ cm^3, for $x = \frac{5}{6}$.

f



4 a $\quad h = 30t - 5t^2$

$$\frac{dh}{dt} = 30 - 10t$$

b Maximum height is reached where $\frac{dh}{dt} = 0$

$\therefore \quad 30 - 10t = 0$

$\therefore \quad 10t = 30 \quad \therefore t = 3$

(a maximum, as it is a quadratic with negative coefficient of t^2)

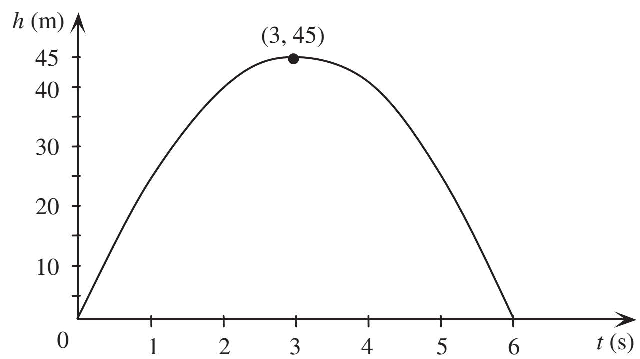
When $t = 3$, $\quad h = 30(3) - 5(3)^2$

$$= 90 - 5 \times 9$$

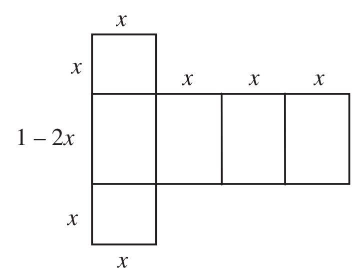
$$= 90 - 45 = 45$$

i.e. maximum height reached is 45 m after 3 seconds.

c



5 **a** Let A = surface area of the net



$$A = 4x(1 - 2x) + 2x^2$$
$$= 4x - 8x^2 + 2x^2$$
$$= 4x - 6x^2$$

b

$$V = x \times x \times (1 - 2x)$$
$$= x^2(1 - 2x)$$
$$= x^2 - 2x^3$$

c $x > 0$ and $V > 0$

$\therefore \quad x^2 - 2x^3 > 0$

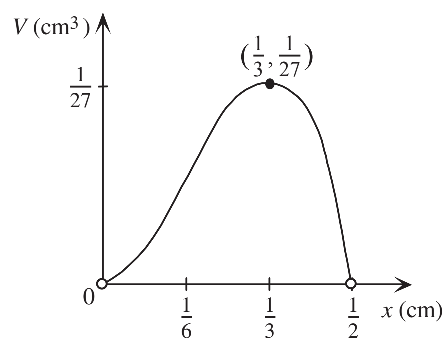
$\iff \quad x^2(1 - 2x) > 0$

$\iff \quad 1 - 2x > 0$

(as $x^2 > 0$ for all x)

$\therefore \quad x < \frac{1}{2}$

Domain $\left\{x\colon 0 < x < \frac{1}{2}\right\}$



When $x = 0$, $V = 0$

When $x = \frac{1}{2}$, $V = 0$

$\frac{dV}{dx} = 2x - 6x^2$

When $\frac{dV}{dx} = 0$, $\quad 2x - 6x^2 = 0$

$\therefore \quad 2x(1 - 3x) = 0$

$\therefore \quad x = 0$ or $x = \frac{1}{3}$

When $x = \frac{1}{3}$, $\quad V = \left(\frac{1}{3}\right)^2 - 2\left(\frac{1}{3}\right)^3 = \frac{1}{9} - \frac{2}{27} = \frac{1}{27}$

If $x < \frac{1}{3}$, e.g. $x = \frac{1}{6}$, $\frac{dV}{dx} > 0$.
If $x > \frac{1}{3}$, e.g. $x = \frac{1}{2}$, $\frac{dV}{dx} < 0$.
$\therefore$ a local maximum at $\left(\frac{1}{3}, \frac{1}{27}\right)$

d A box with dimensions $\frac{1}{3}$ cm $\times \frac{1}{3}$ cm $\times \frac{1}{3}$ cm will give a maximum volume of $\frac{1}{27}$ cm^3.

6 a i Using Pythagoras' theorem:

$$x^2 + r^2 = 1^2$$
$$\therefore \quad r^2 = 1 - x^2$$
$$\therefore \quad r = \sqrt{1 - x^2}$$

ii $h = 1 + x$

b $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi(1 - x^2)(1 + x)$$
$$= \frac{\pi}{3}(1 + x - x^2 - x^3) \text{ as required}$$

c $x > 0$ and $V > 0$
For $V > 0$, $\frac{\pi}{3}(1 - x^2)(1 + x) > 0$

$\iff \quad 1 - x^2 > 0$ as $1 + x > 0$ for all $x > 0$

$\iff \quad -1 < x < 1$

$\therefore \quad V > 0$ for $-1 < x < 1$
To satisfy $x > 0$ and $V > 0$, domain is $\{ x : 0 < x < 1 \}$.

d i $\frac{dV}{dx} = \frac{\pi}{3}(1 - 2x - 3x^2)$

ii When $\frac{dV}{dx} = 0$, $\quad \frac{\pi}{3}(1 - 2x - 3x^2) = 0$

$$\therefore \quad \frac{-\pi}{3}(3x^2 + 2x - 1) = 0$$
$$\therefore \quad \frac{-\pi}{3}(3x - 1)(x + 1) = 0$$

$\therefore \quad 3x - 1 = 0 \quad \text{or} \quad x + 1 = 0$

$\therefore \quad 3x = 1 \quad\quad x = -1$

$\therefore \quad x = \frac{1}{3}$

$\therefore \quad x = \frac{1}{3}, \text{ as } x > 0$

i.e. $\left\{x: \frac{dV}{dx} = 0\right\} = \left\{x: x = \frac{1}{3}\right\}$

iii When $x = \frac{1}{3}$,

$$V = \frac{\pi}{3}\left(1 - \left(\frac{1}{3}\right)^2\right)\left(1 + \frac{1}{3}\right)$$
$$= \frac{\pi}{3}\left(1 - \frac{1}{9}\right)\left(\frac{4}{3}\right)$$
$$= \frac{\pi}{3} \times \frac{8}{9} \times \frac{4}{3}$$
$$= \frac{32\pi}{81}$$
$$\approx 1.24$$

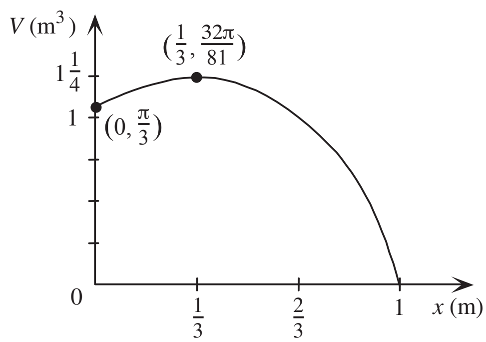
If $x < \frac{1}{3}$, e.g. $x = \frac{1}{6}$, $\frac{dV}{dx} > 0$.

If $x < \frac{1}{3}$, e.g. $x = \frac{1}{2}$, $\frac{dV}{dx} < 0$.

$\therefore$ local maximum at $\left(\frac{1}{3}, \frac{32\pi}{81}\right)$.

i.e. the maximum volume of the cone is $\frac{32\pi}{81}$ m^3 or approximately 1.24 m^3.

e



7 a When $t = 0$, $\quad P(0) = 1000 \times 2^{\frac{0}{20}}$

$$= 1000$$

On 1 January 1993, there were 1000 insects in the colony.

b When $t = 9$, $$P(9) = 1000 \times 2^{\frac{9}{20}}$$
$$= 1000 \times 2^{0.45}$$
$$\approx 1366$$
On 10 January, there were approximately 1366 insects in the colony.

c **i** When $P(t) = 4000$, $$1000 \times 2^{\frac{t}{20}} = 4000$$
$$\therefore \quad 2^{\frac{t}{20}} = 4$$
$$\therefore \quad 2^{\frac{t}{20}} = 2^2$$
$$\therefore \quad \frac{t}{20} = 2$$
$$\therefore \quad t = 40$$

ii When $P(t) = 6000$, $$1000 \times 2^{\frac{t}{20}} = 6000$$
$$\therefore \quad 2^{\frac{t}{20}} = 6$$
$$\therefore \quad \log_{10} 2^{\frac{t}{20}} = \log_{10} 6$$
$$\therefore \quad \frac{t}{20} = \frac{\log_{10} 6}{\log_{10} 2}$$
$$\therefore \quad t = \frac{20 \log_{10} 6}{\log_{10} 2}$$
$$\approx 51.70$$

d $$P(20) = 1000 \times 2^{\frac{20}{20}}$$
$$= 1000 \times 2$$
$$= 2000$$

$$P(15) = 1000 \times 2^{\frac{15}{20}}$$
$$\approx 1000 \times 1.681\,792\,831$$
$$\approx 1681.792\,831$$
Average rate of change of P with respect to time, for the interval of time $[15, 20]$
$$= \frac{P(20) - P(15)}{20 - 15}$$
$$\approx \frac{2000 - 1681.792\,831}{5}$$
$$\approx \frac{318.207\,169\,5}{5} \approx 63.64$$

e **i** Average rate of change $= \dfrac{P(15+h)-P(15)}{15+h-15}$

$$= \frac{1000 \times 2^{\frac{15+h}{20}} - 1000 \times 2^{\frac{15}{20}}}{h}$$

$$= \frac{1000 \times 2^{\frac{3}{4}} \times 2^{\frac{h}{20}} - 1000 \times 2^{\frac{3}{4}}}{h}$$

$$= \frac{1000 \times 2^{\frac{3}{4}}\left(2^{\frac{h}{20}} - 1\right)}{h}, h \neq 0$$

ii Consider h decreasing and approaching zero:

Let $h = 0.0001$

Average rate of change $\approx \dfrac{1681.792\,831(2^{0.000\,005}-1)}{0.0001}$

$\approx 58.286\,566\,86$

Let $h = 0.00001$

Average rate of change $\approx \dfrac{1681.792\,831(2^{0.000\,000\,5}-1)}{0.000\,01}$

$\approx 58.285\,894\,14$

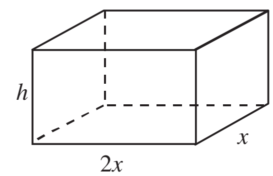
Let $h = 0.000\,001$

Average rate of change $\approx \dfrac{1681.792\,831(2^{0.000\,000\,05}-1)}{0.000\,001}$

$\approx 58.286\,566\,86$

Hence as $h \to 0$, the instantaneous rate of change is approaching 58.287 insects per day.

8 a Let A (m^2) be the total surface area of the block.



Now $\quad A = 300$

and $\quad A = 2(2xh + 2x^2 + xh)$

$\quad = 2(2x^2 + 3xh)$

$\therefore \quad 2(2x^2 + 3xh) = 300$

$\therefore \quad 2x^2 + 3xh = 150$

$\therefore \quad 3xh = 150 - 2x^2$

$\therefore \quad h = \dfrac{150 - 2x^2}{3x}$

b $V = h \times 2x \times x$

$= \dfrac{150 - 2x^2}{3x} \times 2x^2$

$= \dfrac{2}{3}x(150 - 2x^2)$

c $\quad V = 100x - \dfrac{4}{3}x^3$

$\therefore \quad \dfrac{dV}{dx} = 100 - 4x^2$

d When $V = 0$, $\quad \dfrac{2}{3}x(150 - 2x^2) = 0$

$\therefore \quad \dfrac{2}{3}x = 0 \quad$ or $\quad 150 - 2x^2 = 0$

$\therefore \quad x = 0 \quad$ or $\quad 2x^2 = 150$

$\therefore \quad x^2 = 75$

$\therefore \quad x = \pm 5\sqrt{3}$

When $x = 1$, $\quad V = \dfrac{2}{3} \times 1(150 - 2(1)^2)$

$= \dfrac{2}{3}(148) = \dfrac{296}{3} > 0$

$\therefore \quad V > 0$ for $0 < x < 5\sqrt{3}$

Note also, for $x > 0$

$$\frac{2}{3}x(150 - 2x^2) > 0$$

$$\Longleftrightarrow \quad 150 - 2x^2 > 0$$

$$\Longleftrightarrow \quad 75 > x^2$$

$$\Longleftrightarrow \quad 5\sqrt{3} > x$$

e Maximum value of V occurs when $\frac{dV}{dx} = 0$

$\therefore \quad 100 - 4x^2 = 0$

$\therefore \quad 4x^2 = 100$

$\therefore \quad x^2 = 25$

$\therefore \quad x = \pm\sqrt{25}$

$x = 5$ as $x > 0$

When $x = 5$, $V = \frac{2}{3} \times 5(150 - 2(5)^2)$

$= \frac{10}{3}(150 - 50)$

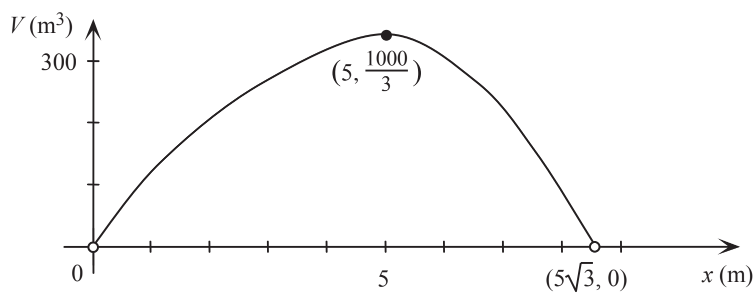
$= \frac{1000}{3} = 333\frac{1}{3}$

When $x < 5$, e.g. $x = 4$, $\frac{dV}{dx}$ and when $x > 5$, e.g. $x = 6$, $\frac{dV}{dx} < 0$

$\therefore$ a local maximum at $\left(5, \frac{1000}{3}\right)$.

i.e. when $x = 5$ m, the block has its maximum volume of $\frac{1000}{3}$ m^3 or $333\frac{1}{3}$ m^3.

f



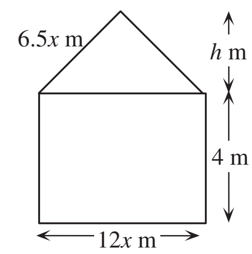
9 **a** $$12x + y + y + 6.5x + 6.5x = 70$$

$\therefore$ $$25x + 2y = 70$$

If $x = 2$, $$25(2) + 2y = 70$$

$\therefore$ $$50 + 2y = 70$$

$\therefore$ $$y = 10$$



b $$25x + 2y = 70$$

$$2y = 70 - 25x$$

$\therefore$ $y = \dfrac{70 - 25x}{2}$ as required

c **i** Using Pythagoras' theorem:

$$h^2 + (6x)^2 = (6.5x)^2$$

$\therefore$ $$h^2 + 36x^2 = 42.25x^2$$

$\therefore$ $$h^2 = 6.25x^2$$

$\therefore$ $$h = \sqrt{6.25x^2}$$

$= 2.5x$ as $x > 0$

ii Let A (m) be the area of the front face of the building.

A = area of rectangle + area of triangle

$$= 12x \times y + \frac{1}{2} \times 12x \times 2.5x$$

$$= 12xy + 15x^2$$

$= 15x^2 + 12xy$ as required

d Let V (cm^3) be the volume of the building.

$$V = A \times 40$$

$$= 40(15x^2 + 12xy)$$

$$= 40\left(15x^2 + 12x\left(\frac{70 - 25x}{2}\right)\right)$$

$$= 40(15x^2 + 6x(70 - 25x))$$

$$= 40(15x^2 + 420x - 150x^2)$$

$$= 600x(28 - 9x)$$

e **i** $$V = 600(28x - 9x^2)$$

Volume is a maximum when $\dfrac{dV}{dx} = 0$

$\therefore \quad \dfrac{dV}{dx} = 600(28 - 18x)$

$\therefore \quad 600(28 - 18x) = 0$

$\therefore \quad 28 - 18x = 0$

$\therefore \quad 18x = 28$

$\therefore \quad x = \dfrac{28}{18} = \dfrac{14}{9}$

When $x = \dfrac{14}{9}$,

$$y = \frac{70 - 25\left(\frac{14}{9}\right)}{2}$$

$$= \frac{70 - \frac{350}{9}}{2}$$

$$= \frac{630 - 350}{18}$$

$$= \frac{280}{18} = \frac{140}{9}$$

When $x < \dfrac{14}{9}$, e.g. $x = 1$, $\dfrac{dV}{dx} > 0$.

When $x > \dfrac{14}{9}$, e.g. $x = 2$, $\dfrac{dV}{dx} < 0$.

$\therefore$ a local maximum at $\left(\dfrac{14}{9}, \dfrac{140}{9}\right)$.

i.e. the volume is a maximum when $x = \dfrac{14}{9}$ and $y = \dfrac{140}{9}$.

ii When $x = \dfrac{14}{9}$,

$$V = 40\left(420\left(\frac{14}{9}\right) - 135\left(\frac{14}{9}\right)^2\right)$$

$$= 13\,066\frac{2}{3} \text{ m}^3$$

i.e. the maximum volume of the building is $13\,066\frac{2}{3}\text{m}^3$.

10 a $y = kx^2(a - x)$

At (200, 0) $\quad 0 = k \times 200^2(a - 200)$

$\therefore \quad$ either $k = 0 \quad$ or $\quad a = 200$

At (170, 8.67) $\quad 8.67 = k \times 170^2(a - 170) \quad (1)$

$\therefore \quad k \neq 0 \quad \therefore \quad a = 200 \quad (2)$

Substitute (2) into (1) $\quad 8.67 = k \times 170^2(200 - 170)$

$$\therefore \quad 8.67 = 28\,900k \times 30$$

$$\therefore \quad k = \frac{8.67}{28\,900 \times 30}$$

$$= 0.000\,01$$

$$\therefore \quad y = 0.000\,01x^2(200 - x)$$

b i $\quad y = 0.000\,01x^2(200 - x)$

$$\therefore \quad = 0.002x^2 - 0.000\,01x^3$$

At the local maximum, $\quad \dfrac{dy}{dx} = 0$

and $\quad \dfrac{dy}{dx} = 0.004x - 0.000\,03x^2$

$$\therefore \quad 0.004x - 0.000\,03x^2 = 0$$

$$\therefore \quad 0.001x(4 - 0.03x) = 0$$

$$\therefore \quad x = 0 \quad \text{or} \quad 4 - 0.03x = 0$$

$$\therefore \quad 0.03x = 4$$

$$\therefore \quad x = \frac{400}{3}$$

If $x < \dfrac{400}{3}$, e.g. $x = 100$, $\dfrac{dy}{dx} > 0$.

If $x > \dfrac{400}{3}$, e.g. $x = 150$, $\dfrac{dy}{dx} < 0$.

Therefore a local maximum when $x = \dfrac{400}{3}$.

ii When $x = \dfrac{400}{3}$, $\quad y = 0.000\,01\left(\dfrac{400}{3}\right)^2\left(200 - \dfrac{400}{3}\right)$

$$= \frac{16}{90} \times \frac{200}{3}$$

$$= \frac{320}{27}$$

c **i** When $x = 105$,

$$\begin{aligned} y &= 0.000\,01(105)^2(200 - 105) \\ &= \frac{1}{100\,000} \times 11\,025 \times 95 \\ &= \frac{104\,737\,5}{100\,000} \\ &= \frac{8379}{800} \\ &= 10\frac{379}{800} \quad (= 10.473\,75) \end{aligned}$$

ii When $x = 105$,

$$\begin{aligned} \frac{dy}{dx} &= 0.001(105)(4 - 0.03 \times 105) \\ &= \frac{105}{1000}(4 - 3.15) \\ &= \frac{105}{1000} \times \frac{85}{100} \\ &= \frac{8925}{100\,000} = \frac{357}{4000} \end{aligned}$$

d **i**

$$y - y_1 = m(x - x_1)$$

$$\therefore \quad y = \frac{357}{4000}(x - 105) + \frac{8379}{800}$$

$$\therefore \quad y = \frac{357}{4000}x - \frac{37485}{4000} + \frac{41895}{4000}$$

$$\therefore \quad y = \frac{357}{4000}x + \frac{441}{400}$$

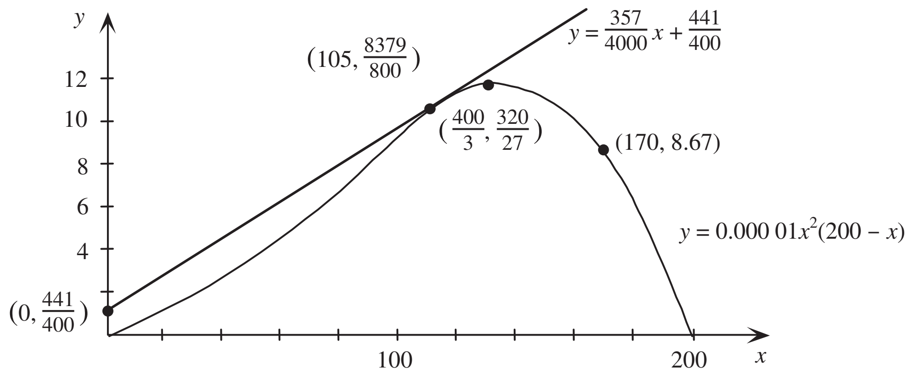
i.e. the equation of the tangent at $x = 105$ is $y = \frac{357}{4000}x + \frac{441}{400}$.

ii The y-axis intercept of the tangent is $\frac{441}{400}$.

e Average rate of change

$$\begin{aligned} &= \frac{\frac{8379}{800} - 0}{105 - 0} \\ &= \frac{8379}{800 \times 105} \\ &= 0.099\,75 \end{aligned}$$

f $y = 0.000\,01x^2(200 - x)$



11 a In the centre of the city $\quad r = 0$

and $\quad P = 10 + 40(0) - 20(0)^2$

$\quad = 10$

i.e. the population density is 10 000 people per square kilometre.

b

$$P > 0$$

$$\therefore \quad 10 + 40r - 20r^2 > 0$$

$$\therefore \quad -10(2r^2 - 4r - 1) > 0$$

When $P = 0$, $\quad 2r^2 - 4r - 1 = 0$

$$\therefore \quad r = \frac{4 \pm \sqrt{4^2 - 4(2)(-1)}}{2 \times 2}$$

$$= \frac{4 \pm \sqrt{16 + 8}}{4}$$

$$= \frac{4 \pm 2\sqrt{6}}{4}$$

$$= \frac{2 \pm \sqrt{6}}{2}$$

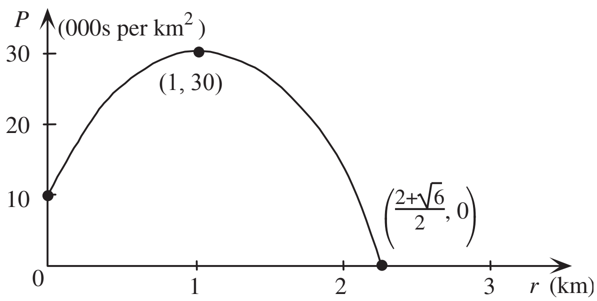
and, as $r \geq 0$ $\quad r = \dfrac{2 + \sqrt{6}}{2}$

When $r = 1$, $\quad P = 10 + 40(1) - 20(1)^2$

$\quad = 10 + 40 - 20$

$\quad = 30 > 0$

$$\therefore \quad P > 0 \text{ for } 0 \leq r \leq \frac{2 + \sqrt{6}}{2}$$

c 

d **i** $\dfrac{dP}{dr} = 40 - 40r$

ii When $r = 0.5$, $\dfrac{dP}{dr} = 40 - 40(0.5)$

$= 40 - 20$

$= 20$

When $r = 1$, $\dfrac{dP}{dr} = 40 - 40(1)$

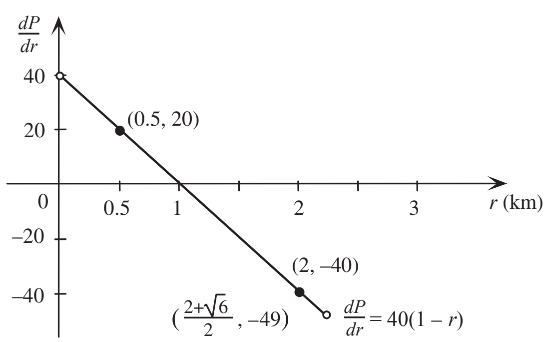
$= 40 - 40$

$= 0$

When $r = 2$, $\dfrac{dP}{dr} = 40 - 40(2)$

$= 40 - 80$

$= -40$

iii 

e The population density is greatest at a 1 km radius from the city centre.

12 **a** $y = x(a - x)$

$= ax - x^2$

b $0 < x < a$

c Maximum value of y is found where $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = a - 2x$$

$\therefore \quad a - 2x = 0$

$\therefore \quad 2x = a$

$\therefore \quad x = \frac{a}{2}$

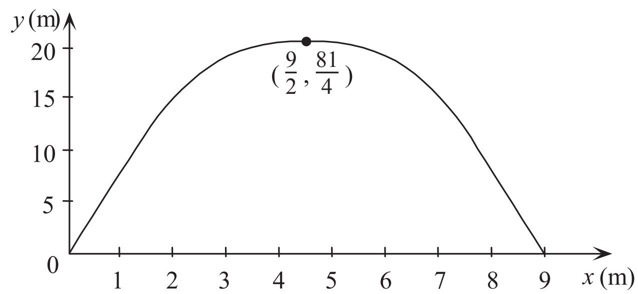
When $x = \frac{a}{2}$, $\quad y = \frac{a}{2}(a - \frac{a}{2})$

$$= \frac{a}{2} \times \frac{a}{2} = \frac{1}{4}a^2$$

So the maximum value of y is $\frac{1}{4}a^2$ when $x = \frac{a}{2}$.

d $y = \frac{1}{4}a^2$ is a maximum because the coefficient of the x^2 term is negative.

e **i** When $a = 9$, $y = x(9 - x)$



ii $0 < y \leq \frac{81}{4}$

13 **a** $V(t) = 0.6\left(20t^2 - \frac{2t^3}{3}\right)$

i When $t = 0$, $\quad V(0) = 0.6\left(20(0)^2 - \frac{2(0)^3}{3}\right)$

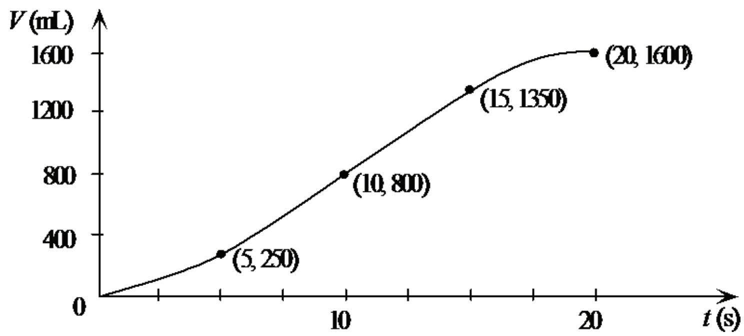
$$= 0.6(0 - 0)$$

$$= 0$$

ii When $t = 20$, $$V(20) = 0.6\left(20(20)^2 - \frac{2(20)^3}{3}\right)$$
$$= 0.6\left(8000 - \frac{16000}{3}\right)$$
$$= 0.6 \times \frac{8000}{3}$$
$$= 1600$$

b $V'(t) = 0.6(40t - 2t^2) = 1.2t(20 - t)$

c



When $V'(t) = 0$, $1.2t(20 - t) = 0$

$\therefore$ $t = 0$ or $20 - t = 0$

$t = 20$

When $t = 10$, $$V = 0.6\left(20 \times 10^2 - \frac{2 \times 10^3}{3}\right)$$
$$= 0.6\left(2000 - \frac{2000}{3}\right)$$
$$= 800$$

When $t = 5$, $$V = 0.6\left(20 \times 5^2 - \frac{2 \times 5^3}{3}\right)$$
$$= 0.6\left(500 - \frac{250}{3}\right)$$
$$= 250$$

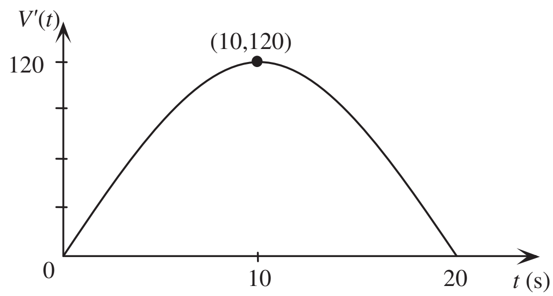
When $t = 15$, $$V = 0.6\left(20 \times 15^2 - \frac{2 \times 15^3}{3}\right)$$
$$= 0.6\left(4500 - \frac{6750}{3}\right)$$
$$= 1350$$

d $$V'(t) = 1.2t(20 - t),\ t \in [0, 20]$$
$$= 24t - 1.2t^2$$

When $t = 0$, $V'(0) = 0$

When $t = 20$, $V'(20) = 24 \times 20 - 1.2(20)^2$
$$= 480 - 480$$
$$= 0$$

When $t = 10$, $V'(10) = 24 \times 10 - 1.2(10)^2$
$$= 240 - 120$$
$$= 120$$



14 a $y = ax^3 + bx^2$

At $(1, -1)$, $-1 = a(1)^3 + b(1)^2$

$\therefore$ $a + b + 1 = 0$ (1)

b $$\frac{dy}{dx} = 3ax^2 + 2bx$$

At $(1, -1)$, $\frac{dy}{dx} = 0$

$\therefore$ $3a(1)^2 + 2b(1) = 0$

$\therefore$ $3a + 2b = 0$ (2)

$(2) - 2 \times (1)$
$$\begin{array}{r} 3a + 2b = 0 \\ -\ 2a + 2b + 2 = 0 \\ \hline a - 2 = 0 \end{array}$$

$\therefore$ $a = 2$

Substitute $a = 2$ into (1) $2 + b + 1 = 0$

$\therefore$ $b = -3$

$\therefore$ $y = 2x^3 - 3x^2$

c x-axis intercept when $y = 0$

$$\therefore \quad 2x^3 - 3x^2 = 0$$

$$\therefore \quad x^2(2x - 3) = 0$$

$$\therefore \quad x = 0 \quad \text{or} \quad x = \frac{3}{2}$$

$$\frac{dy}{dx} = 6x^2 - 6x$$

$$= 6x(x - 1)$$

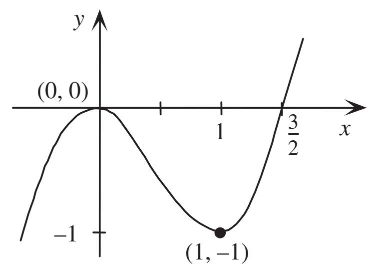
Stationary points where $\frac{dy}{dx} = 0$

$$\therefore \quad 6x(x - 1) = 0$$

$$\therefore \quad 6x = 0 \quad \text{or} \quad x - 1 = 0$$

$$\therefore \quad x = 0 \quad \text{or} \quad x = 1$$

At $x = 0, \ y = 0$
At $x = 1, \ y = -1$



	$x < 0$	0	$0 < x < 1$	1	$x > 1$
Sign of $\frac{dy}{dx}$	+	0	−	0	+
Shape	/	—	\	—	/

$\therefore$ there is a local minimum at $(1, \ -1)$ and a local maximum at $(0, 0)$.

15 a i $AD + AB + CB = 80$

$$\therefore \quad x + AB + x = 80$$

$$\therefore \quad AB = 80 - 2x$$

ii $\sin 60^\circ = \frac{h}{x}$

$$\therefore \quad h = x \sin 60^\circ$$

$$h = \frac{\sqrt{3}x}{2}$$

b Let area of trapezoid = A

$$\therefore \quad A = \text{area of rectangle} + 2(\text{area of triangle})$$

$$= \frac{\sqrt{3}}{2}x(80 - 2x) + 2\left(\frac{1}{2} \times \frac{\sqrt{3}}{2}x \times x \sin 30^\circ\right)$$

$$= \frac{80\sqrt{3}}{2}x - \sqrt{3}x^2 + \frac{\sqrt{3}}{2}x \times \frac{x}{2}$$

$$= \frac{80\sqrt{3}}{2}x - \sqrt{3}x^2 + \frac{\sqrt{3}}{4}x^2$$

$$= \frac{80\sqrt{3}}{2}x - \frac{3\sqrt{3}}{4}x^2$$

$$= \frac{\sqrt{3}}{4}x(160 - 3x)$$

(Formula for the area of a trapezium may also be used.)

c

$$A = \frac{\sqrt{3}}{4}x(160 - 3x)$$

$$= 40\sqrt{3}x - \frac{3\sqrt{3}}{4}x^2$$

$$\frac{dA}{dx} = 40\sqrt{3} - \frac{3\sqrt{3}}{4}x$$

When $\frac{dA}{dx} = 0$, $\quad 40\sqrt{3} - \frac{3\sqrt{3}}{2}x = 0$

$$\therefore \quad \frac{3\sqrt{3}}{2}x = 40\sqrt{3}$$

$$\therefore \quad x = \frac{40\sqrt{3} \times 2}{3\sqrt{3}} = \frac{80}{3}$$

The area is a maximum for $x = \frac{80}{3}$, as $A = \frac{\sqrt{3}}{4}x(160 - 3x)$ is a quadratic function with negative coefficient of x^2.

16 a Total amount of cardboard = $x^2 + 4xy + x^2 + 8x$

$$\therefore \quad 2x^2 + 4xy + 8x = 1400$$

$$\therefore \quad y = \frac{1400 - 2x^2 - 8x}{4x}$$

b

$$V = x^2y$$

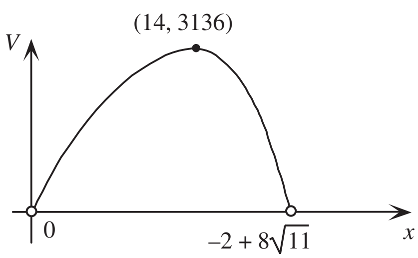
$$= x^2\left(\frac{1400 - 2x^2 - 8x}{4x}\right)$$

$$= \frac{-x^3}{2} - 2x^2 + 350x$$

c $$V = \frac{-x^3}{2} - 2x^2 + 350x$$
$$\frac{dV}{dx} = \frac{-3}{2}x^2 - 4x + 350$$

d $\frac{dV}{dx} = 0$ implies
$$\frac{-3}{2}x^2 - 4x + 350 = 0$$
$$\therefore \quad 3x^2 + 8x - 700 = 0$$
$$\therefore \quad x = \frac{-8 \pm \sqrt{64 + 8400}}{6} = \frac{-8 \pm 92}{6}$$
$$\therefore \quad x = 14, \text{ as } x \text{ is positive.}$$

e,f When $x = 14$, $V = 3136$
Maximum volume is 3136 cm^3.
From part **b**, $$V = x^2\left(\frac{1400 - 2x^2 - 8x}{4x}\right)$$
$$= \frac{x}{4}(1400 - 2x^2 - 8x)$$
defined if $x > 0$ and $V > 0$
i.e. $$-2x^2 - 8x + 1400 > 0$$
$$x^2 + 4x - 700 > 0$$
Consider $$x = \frac{-4 \pm \sqrt{16 + 2800}}{2} = -2 \pm \sqrt{704} = -2 \pm 8\sqrt{11}$$
V is defined for $0 < x < -2 + 8\sqrt{11}$.



g On a CAS calculator, with **f1 = x/4(1400 − 2x^2 − 8x)** and **f2=1000**, choose
TI: Press **Menu → 6:Analyze Graph → 4:Intersection**
CP: Tap **Analysis → G-Solve → Intersect**
From the CAS calculator, when $V = 1000$,
$$x = 22.827\ldots \quad \text{or} \quad x = 2.943\ldots$$
$$\therefore \quad y = 1.919\ldots \quad \text{or} \quad y = 115.452\ldots$$

Chapter 20 – Further differentiation and antidifferentiation

Solutions to Exercise 20A

1 a $\dfrac{d}{dx}(x-1)^{30} = 30(x-1)^{29}$

b $\dfrac{d}{dx}(x^5 - x^{10})^{20}$
$= 100(x^4 - 2x^9)(x^5 - x^{10})^{19}$

c $\dfrac{d}{dx}(x - x^3 - x^5)^4 =$
$4(1 - 3x^2 - 5x^4)((x - x^3 - x^5)^3$

d $\dfrac{d}{dx}(x^2 + 2x + 1)^4 = \dfrac{d}{dx}(x+1)^8$
$= 8(x+1)^7$

e $\dfrac{d}{dx}(x^2 + 2x)^{-2}; x \neq -2, 0$
$= -4(x+1)(x^2 + 2x)^{-3}$

f $\dfrac{d}{dx}\left(x^2 - \dfrac{2}{x}\right)^{-3}; \ x \neq 0$
$= -6(x + x^{-2})(x^2 - 2x^{-1})^{-4}$

2 a $f(x) = (2x^3 + 1)^4$
$\therefore \ f'(x) = 4(2x^3 + 1)^3(6x^2)$
$= 24x^2(2x^3 + 1)^3$

b $f'(1) = 24(3)^3 = 648$

3 a $y = \dfrac{1}{x+3}, \ \therefore \ y' = -1(x+3)^2$
$\therefore \ y'(1) = -\dfrac{1}{(1+3)^2} = -\dfrac{1}{16}$

b $y = \dfrac{1}{(x+3)^3}, \ \therefore \ y' = -\dfrac{3}{(x+3)^4}$
$\therefore \ y'(1) = -\dfrac{3}{(1+3)^4} = -\dfrac{3}{256}$

4 $f(x) = \dfrac{1}{2x+3}, \ \therefore \ f'(x) = -\dfrac{2}{(2x+3)^2}$

a $f'(0) = -\dfrac{2}{9}$

b $f'(x) = -\dfrac{2}{(2x+3)^2} = -\dfrac{2}{9}$
$\therefore \ (2x+3)^2 = 9$
$2x + 3 = \pm 3$
$2x = -6, \ 0$
$x = -3, \ 0$
$f(-3) = -\dfrac{1}{3}; \ f(0) = \dfrac{1}{3}$
Coordinates are $\left(-3, \ -\dfrac{1}{3}\right), \ \left(0, \ \dfrac{1}{3}\right)$.

5 a $y = \dfrac{1}{x}, \ \therefore \ y' = -\dfrac{1}{x^2}$
$\therefore \ y'(2) = -\dfrac{1}{4}$

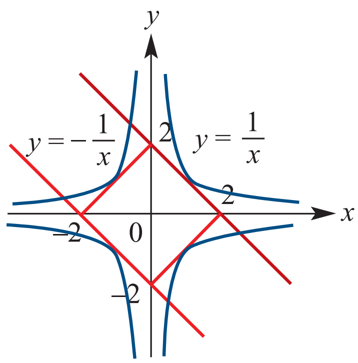
b $y = -\dfrac{1}{x}, \therefore \ y' = \dfrac{1}{x^2}$
$\therefore \ y'(2) = \dfrac{1}{4}$

c $y'(1) = -1, \ y - 1 = -(x-1)$
$y = 2 - x$

d $y'(1) = 1, \ \therefore \ y + 1 = x - 1$
$y = x - 2$

e $P: y - 1 = x + 1, \ \therefore \ y = x + 2$
$Q: y + 1 = -(x+1), \ \therefore y = -x - 2$
They intersect at $(-2, 0)$

f



Solutions to Exercise 20B

1 **a** $\dfrac{d}{dx}(x^{\frac{1}{3}}) = \dfrac{1}{3}x^{-\frac{2}{3}}$

b $\dfrac{d}{dx}x^{\frac{3}{2}} = \dfrac{3}{2}x^{\frac{1}{2}}; x > 0$

c $\dfrac{d}{dx}\left(x^{\frac{5}{2}} - x^{\frac{3}{2}}\right) = \dfrac{5}{2}x^{\frac{3}{2}} - \dfrac{3}{2}x^{\frac{1}{2}} > 0$

d $\dfrac{d}{dx}\left(2x^{\frac{1}{2}} - 3x^{\frac{5}{3}}\right) = x^{-\frac{1}{2}} - 5x^{\frac{2}{3}};\ x = 0$

e $\dfrac{d}{dx}x^{-\frac{5}{6}} = -\dfrac{5}{6}x^{-\frac{11}{6}}; x > 0$

f $\dfrac{d}{dx}\left(x^{-\frac{1}{2}} - 4\right) = -\dfrac{1}{2}x^{-\frac{3}{2}}; x > 0$

2 **a** $\dfrac{d}{dx}\sqrt{1 + x^2} = x(1 + x^2)^{-\frac{1}{2}}$

b $\dfrac{d}{dx}(x + x^2)^{\frac{1}{3}} = \dfrac{1}{3}(1 + 2x)(x + x^2)^{-\frac{2}{3}}$

c $\dfrac{d}{dx}(1 + x^2)^{-\frac{1}{2}} = 2x\left(-\dfrac{1}{2}\right)(1 + x^2)^{-\frac{3}{2}}$

$= -x(1 + x^2)^{-\frac{3}{2}}$

d $\dfrac{d}{dx}(1 + x)^{\frac{1}{3}} = \dfrac{1}{3}(1 + x)^{-\frac{2}{3}}$

3 $y = x^{\frac{1}{3}},\ \therefore\ y' = \dfrac{1}{3}x^{-\frac{2}{3}}$

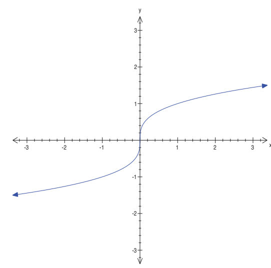
a **i** $y'\left(\dfrac{1}{8}\right) = \dfrac{1}{3}\left(\dfrac{1}{8}\right)^{-\frac{2}{3}} = \dfrac{1}{3}(8)^{\frac{2}{3}} = \dfrac{4}{3}$

ii $y'\left(-\dfrac{1}{8}\right) = \dfrac{1}{3}\left(-\dfrac{1}{8}\right)^{-\frac{2}{3}} =$
$\dfrac{1}{3}(-8)^{\frac{2}{3}} = \dfrac{4}{3}$

iii $y'(1) = \dfrac{1}{3}(1)^{-\frac{2}{3}} = \dfrac{1}{3}(1)^{\frac{2}{3}} = \dfrac{1}{3}$

iv $y'(-1) = \dfrac{1}{3}(-1)^{-\frac{2}{3}} = \dfrac{1}{3}(-1)^{\frac{2}{3}} = \dfrac{1}{3}$

b Graph has rotational symmetry around $(0, 0)$.

4 

a $x^{\frac{1}{2}} < x^{\frac{1}{3}}$

$\therefore\ \left(x^{\frac{1}{2}}\right)^6 < \left(x^{\frac{1}{3}}\right)^6; x > 0$

$x^3 < x^2$

$x^3 - x^2 < 0$

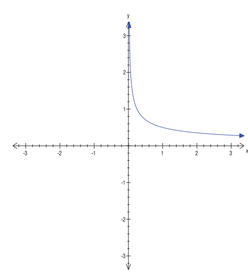
$x^2(x - 1) < 0$

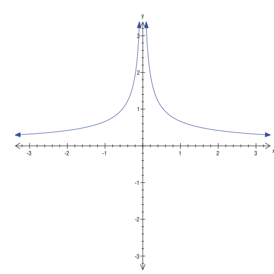
$x^2 > 0,\ \therefore\ x - 1 < 0$

$x < 1$

$\{x :, 0 < x < 1\}$

b $\dfrac{d}{dx}x^{\frac{1}{2}} = \dfrac{1}{2}x^{-\frac{1}{2}}$ $\qquad \dfrac{d}{dx}x^{\frac{1}{3}} = \dfrac{1}{3}x^{-\frac{2}{3}}$





As in **a**:
$\left(\dfrac{1}{2}x^{-\frac{1}{2}}\right)^6 > \left(\dfrac{1}{3}x^{-\frac{2}{3}}\right)^6; x > 0$

$\therefore\ x > \left(\dfrac{2}{3}\right)^6$

$\{x \colon x > \frac{64}{729}\}$

5 a

$$\frac{d}{dx}(2 - 5\sqrt{x})^2 = 2\left(-\frac{5}{2}x^{-\frac{1}{2}}\right)(2 - 5\sqrt{x})$$
$$= -5x^{-\frac{1}{2}}(2 - 5\sqrt{x})$$

b $$\frac{d}{dx}(3\sqrt{x} + 2)^2 = 2\left(\frac{3}{2}x^{-\frac{1}{2}}\right)(3\sqrt{x} + 2)$$
$$= 3x^{-\frac{1}{2}}(3\sqrt{x} + 2)$$

c $$\frac{d}{dx}\left(\frac{2 + \sqrt{x}}{x^2}\right) = \frac{d}{dx}(2x^{-2} + x^{-\frac{3}{2}})$$
$$= -4x^{-3} - \frac{3}{2}x^{-\frac{5}{2}}$$

d $$\frac{d}{dx}\left(\frac{x^2 + 2}{\sqrt{x}}\right) = \frac{d}{dx}\left(x^{\frac{3}{2}} + 2x^{-\frac{1}{2}}\right)$$
$$= \frac{3}{2}x^{\frac{1}{2}} - x^{-\frac{3}{2}}$$

e $$\frac{d}{dx}(3\sqrt{/x})(x^2 + 2) = \frac{d}{dx}\left(3x^{\frac{5}{2}} + 6x^{\frac{1}{2}}\right)$$
$$= \frac{15}{2}x^{\frac{3}{2}} + 3x^{-\frac{1}{2}}$$

Solutions to Exercise 20C

1 a $\int 3x^{-2}dx = -\frac{3}{x} + c$

b $\int 2x^{-4} + 6xdx = -\frac{2}{3}x^{-3} + 3x^2 + c$

c $\int \sqrt{x}(2 + x)dx = \int 2x^{\frac{1}{2}} + x^{\frac{3}{2}}dx$

$= \frac{4}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + c$

d $\int 3x^{\frac{1}{3}} - 5x^{\frac{5}{4}}dx = \frac{9}{4}x^{\frac{4}{3}} - \frac{20}{9}x^{\frac{9}{4}} + c$

e $\int \frac{3z^4 + 2z}{z^3}dz = \int 3z + 2z^{-2}dz$

$= \frac{3}{2}z^2 - \frac{2}{z} + c$

f $\int 3x^{\frac{3}{4}} - 7x^{\frac{1}{2}}dx = \frac{12}{7}x^{\frac{7}{4}} - \frac{14}{3}x^{\frac{3}{2}} + c$

2 a $\frac{dy}{dx} = x^{\frac{1}{2}} + x$

$\therefore \quad y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 + c$

$y(4) = \frac{2}{3}(8) + \frac{16}{2} + c = 6$

$\therefore \quad c = 6 - 8 - \frac{16}{3} = -\frac{22}{3}$

$\therefore \quad y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 - \frac{22}{3}$

b $\frac{dy}{dx} = \frac{1}{x^3}, \therefore y = -\frac{1}{2x^2} + c$

$y(1) = c - \frac{1}{2} = 1, \therefore c = \frac{3}{2}$

$\therefore \quad y = \frac{3}{2} - \frac{1}{2x^2}$

c $\frac{dy}{dx} = 3x + \frac{1}{x^2}, \therefore y = \frac{3}{2}x^2 - \frac{1}{x} + c$

$y(1) = \frac{3}{2} - 1 + c = 5, \therefore c = \frac{9}{2}$

$\therefore \quad y = \frac{3}{2}x^2 - \frac{1}{x} + \frac{9}{2}$

3

$f'(x) = 3x^2 - \frac{1}{x^2}, \therefore f(x) = x^3 + \frac{1}{x} + c$

$f(2) = 8 + \frac{1}{2} + c = 0, \therefore c = -\frac{17}{2}$

$\therefore f(x) = x^3 + \frac{1}{x} - \frac{17}{2}$

4 $\frac{ds}{dt} = 3t - \frac{8}{t^2}, \therefore s = \frac{3}{2}t^2 + \frac{8}{t} + c$

$s(1) = \frac{3}{2} + 8 + c = \frac{3}{2}, \therefore c = -8$

$\therefore \quad s = \frac{3}{2}t^2 + \frac{8}{t} - 8$

5 $\frac{dy}{dx} = \frac{a}{x^2} + 1, \therefore y = x - \frac{a}{x} + c$

$y'(1) = a + 1 = 3, \therefore a = 2$

$y(1) = -1 + c = 3, \therefore c = 4$

$\therefore \quad y = x - \frac{2}{x} + 4$

$y(2) = 2 - 1 + 4 = 5$

6 $\frac{dy}{dx} = ax, \therefore y = \frac{a}{2}x^2 + c$

a

Tangent at $(1, 2)$: $\quad y - 2 = a(x - 1)$

$\therefore \quad y = ax + (2 - a)$

Tangent passes through $(0, 0)$,

$\therefore a = 2$

b $$y = x^2 + c$$

$$y(1) = 2, \therefore c = 1$$

$$\therefore \quad y = x^2 + 1$$

7 $$\frac{dy}{dx} = x^2, \therefore y = \frac{1}{3}x^3 + c$$

$$y(-1) = -\frac{1}{3} + c = 2, \therefore c = \frac{7}{3}$$

$$\therefore \quad y = \frac{x^2 + 7}{3}$$

Solutions to Exercise 20D

1 a $f(x) = x^3 + 2x + 1$

$\therefore f'(x) = 3x^2 + 2$

$\therefore f''(x) = 6x$

b $f(x) = 3x + 2$

$\therefore f'(x) = 3$

$\therefore f''(x) = 0$

c $f(x) = (3x + 1)^4$

$\therefore f'(x) = 12(3x + 1)^3$

$\therefore f''(x) = 108(3x + 1)^2$

d $f(x) = x^{\frac{1}{2}} + 3x^3; x > 0$

$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}} + 9x^2; x > 0$

$\therefore f'(x) = -\frac{1}{4}x^{-\frac{3}{2}} + 18x; x > 0$

e $f(x) = (x^6 + 1)^3$

$\therefore f'(x) = 18x^5(x^6 + 1)^2$

$= 18x^{17} + 36x^{11} + 18x^5$

$\therefore f''(x) = 306x^{16} + 396x^{10} + 90x^4$

f $f(x) = 5x^2 + 6x^{-1} + 3x^{\frac{3}{2}}$

$\therefore f'(x) = 10x - 6x^{-2} + 9x^{\frac{1}{2}}$

$\therefore f''(x) = 10 + 12x^{-3} + \frac{9}{4}x^{-\frac{1}{2}}$

2 a $y = 3x^3 + 4x + 1$

$\therefore \frac{dx}{dy} = 9x^2 + 4$

$\frac{d^2y}{dx^2} = 18x$

b $y = 6$

$\therefore \frac{dy}{dx} = \frac{d^2y}{dx^2} = 0$

c $y = 6x^2 + 3x + 1$

$\therefore \frac{dy}{dx} = 12x + 3$

$\frac{d^2y}{dx^2} = 12$

d $y = (6x + 1)^4$

$\therefore \frac{dy}{dx} = 24(6x + 1)^3$

$\therefore \frac{d^2y}{dx^2} = 432(6x + 1)^2$

e $y = (5x + 2)^4$

$\therefore \frac{dy}{dx} = 20(5x + 2)^3$

$\therefore \frac{d^2y}{dx^2} = 300(5x + 2)^2$

f $y = x^3 + 2x^2 + 3x^{-1}$

$\therefore \frac{dy}{dx} = 3x^2 + 4x - 3x^{-2}$

$\therefore \frac{d^2y}{dx^2} = 6x + 4 + 6x^{-3}$

3 $h = 20t - 4.9\,t^2$ m

$\therefore v = 20 - 9.8\,t$ m/s

$\therefore a = -9.8$ m/s^2

4 $x(t) = 4t - 3t^3$ m

$\therefore v(t) = 4 - 9t^2$ m/s

$\therefore a(t) = -18t$ m/s^2

a **i** $x(2) = 8 - 24 = -16$ m

ii $v(0) = 4$ m/s

iii $v(0.5) = 4 - \frac{9}{4} = \frac{7}{4}$ m/s

iv $v(2) = 4 - 36 = -32$ m/s

b $a(t) = 0, \therefore \ t = 0$

c Av. $v = \dfrac{x(2) - x(0)}{2 - 0}$

$$= \frac{-16 - 0}{2} = -8 \text{ m/s}$$

Solutions to Exercise 20E

1 $y = 4x + \frac{1}{x}, \therefore\ y' = 4 - \frac{1}{x^2}$

a $y' = 0, \ \therefore\ 4x^2 = 1$

$$x = \pm\frac{1}{2}$$

$y\left(-\frac{1}{2}\right) = -4; y\left(\frac{1}{2}\right) = 4$

Turning pts at $\left(-\frac{1}{2}, -4\right)$ and $\left(\frac{1}{2}, 4\right)$.

b $y'(2) = 4 - \frac{1}{4} = \frac{15}{4}$

$y(2) = 8 + \frac{1}{2} = \frac{17}{2}$

Tangent equation:

$y - \frac{17}{2} = \frac{15}{4}(x - 2)$

$\therefore\ y = \frac{15}{4}x + 1$ **OR** $4y - 15x = 1$

2 $y = \frac{x^2 - 1}{x} = x - \frac{1}{x}$

$\therefore\ y' = 1 + \frac{1}{x^2} = 5$

$\therefore\ x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

3 $y = \frac{2x - 4}{x^2} = 2x^{-1} - 4x^{-2}$

$\therefore\ y' = -2x^2 + 8x^{-3}$

$y = 0, \therefore\ 2x^{-1} = 4x^{-2}; x \neq 0$

$2x = 4; \ \therefore\ x = 2$

$\therefore\ y'(2) = -\frac{2}{2^2} + \frac{8}{2^3} = \frac{1}{2}$

4 $y = x - 5 + \frac{4}{x} = \frac{(x - 1)(x - 4)}{x}$

a Axis intercepts at $(1, 0), (4, 0)$.

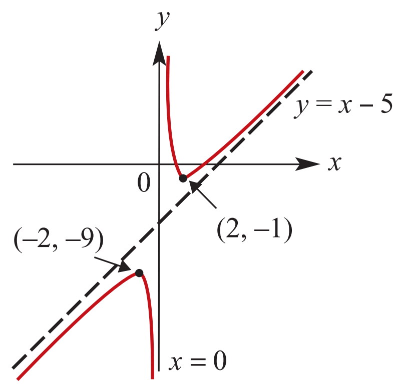
b Vertical asymptote at $x = 0$

Oblique asymptote at $y = x - 5$

c $y'(x) = 1 - \frac{4}{x^2} = 0, \ x = \pm 2$

$y(-2) = -9; y(2) = -1$

Turning pts at $(-2, -9), (2, -1)$



5 $y = x + \frac{4}{x^2}; x > 0$

$\therefore\ y' = 1 - \frac{8}{x^3}; y > 0$

$y'(1) < 0; y'(2) = 0; y'(3) > 0$

local and absolute minimum at $x = 2$:

$y(2) = 2 + 1 = 3$

6 $y = x + \frac{4}{x}; x > 0$

$= \frac{x^2 + 4}{x}; x > 0$

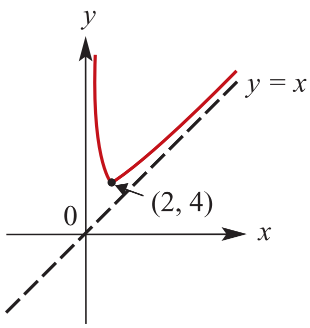
$x^2 + 4 > 0; x \in R, \therefore$ no axis intercepts

Asymptotes at $x = 0$ and $y = x$

$y' = 1 - \frac{4}{x^2} = 0; x > 0$

$x^2 = 4, \therefore\ x = 2$

$y(2) = 4$; Local and absolute minimum is 4.



7 **a** $y = x + \frac{1}{x}; x \neq 0 \quad = \frac{x^2 + 1}{c}; x \neq 0$

$x^2 + 1 > 0; x \in R, \therefore$ no axis intercepts

Asymptotes at $x = 0$ and $y = x$

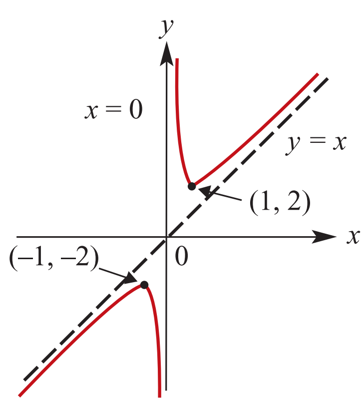
$$y' = 1 - \frac{1}{x^2} = 0; x \neq 0$$

$$x^2 = 1, x = \pm 1$$

$y(-1) = -2; y(1) = 2$

x	-2	-1	-0.5	0	0.5	1	2
y'	$+$	0	$-$	N	$-$	0	$+$

Local maximum $(-1, -2)$,
local minimum $(1, 2)$.



b $y = \frac{1}{x^2} - x; x \neq 0$

$$= \frac{1 - x^3}{x^2}; x \neq 0$$

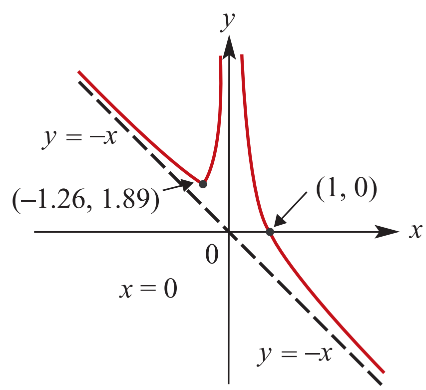
Axis intercept at $(1, 0)$.

Asymptotes at $x = 0$ and $y = -x$

$$y' = -1 - \frac{2}{x^3} = 0; x \neq 0$$

$$x^3 = -2, \therefore \; x = -2^{\frac{1}{3}}$$

Local minimum $\left(-2^{\frac{1}{3}}, \left(\frac{3}{4}\right)^{\frac{1}{3}}\right) \approx$
$(-1.26, 1.89)$



c $y = x + 1 + \frac{1}{x + 3}; x \neq -3$

$$= \frac{(x + 1)(x + 3) + 1}{x + 3}; x \neq -3$$

$$= \frac{(x + 2)^2}{x + 3}; x \neq -3$$

Axis intercepts at $(0, \frac{4}{3})$ and $(-2, 0)$.

Asymptotes at $x = -3$ and $y = x + 1$

$$y' = 1 - \frac{1}{(x + 3)^2} = 0; x \neq -3$$

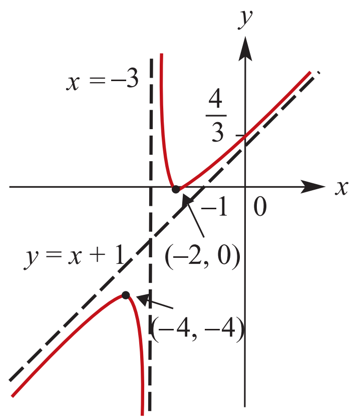
$$(x + 3)^3 = 1$$

$$x^2 + 6x + 8 = 0$$

$$(x + 4)(x + 2) = 0, \; \therefore \; x = -4, 2$$

x	-5	-4	-3.5	-3	-2.5	-2	0
y'	$+$	0	$-$	N	$-$	0	$+$

Local minimum $(-2, 0)$, maximum
$(-4, -4)$.



d $y = x^3 + \dfrac{243}{x}; x \neq 0$

$= \dfrac{x^4 + 243}{x}; x \neq 0$

No axis intercept: $x^4 + 243 > 0; x \in R$

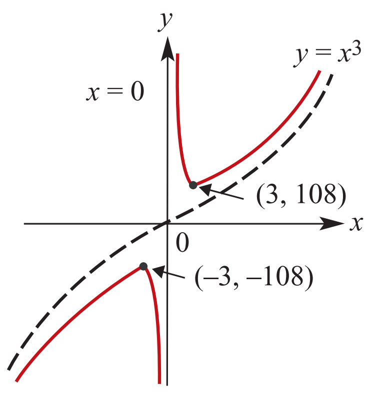
Asymptotes at $x = 0$.

$y' = 3x^2 - \dfrac{243}{x^2} = 0; x \neq 0$

$3x^4 = 243, \therefore\ x = \pm 3$

x	−4	−3	−1	0	1	3	4
y'	+	0	−	N	−	0	+

Local maximum (−3, −108), minimum (3, 108).



e $y = x - 5 + \dfrac{1}{x}; x \neq 0$

$= \dfrac{x^2 - 5x + 1}{x}; x \neq 0$

$y = 0, \therefore\ x = \dfrac{1}{2}(5 \pm \sqrt{21})$

Axis intercepts: (0.21, 0), (4.79, 0).

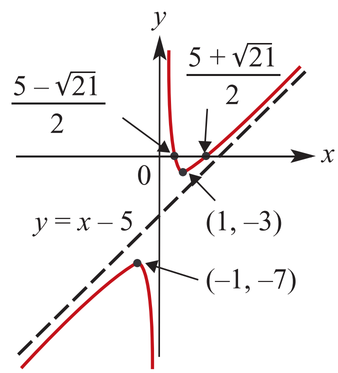
Asymptotes at $x = 0$ and $y = x - 5$

$y' = 1 - \dfrac{1}{x^2} = 0; x \neq 0$

$x^2 = 1, \therefore\ x = \pm 1$

x	−2	−1	−0.5	0	0.5	1	2
y'	+	0	−	N	−	0	+

Local maximum (−1, −7), minimum (1, −3).



f $y = \dfrac{x^2 - 4}{x + 2}; x \neq -2$

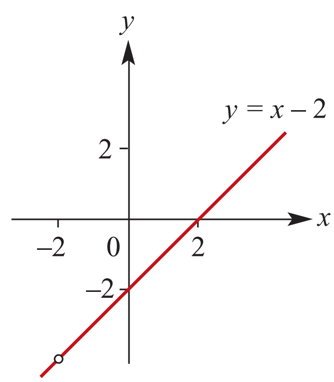
$= \dfrac{(x + 2)(x - 2)}{x + 2}; x \neq -2$

$= x - 2; x \neq -2$

Axis intercepts at (2, 0) and (0, −2).

No asymptotes, no turning points.

The graph is a straight line with equation $y = x - 2$ with a hole at (−2, −4).



Solutions to Technology-free questions

1 **a** $\dfrac{d}{dx}x^{\frac{1}{2}} = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2x^{\frac{1}{2}}}$

b $\dfrac{d}{dx}x^{\frac{1}{3}} = \dfrac{1}{3}x^{-\frac{2}{3}} = \dfrac{1}{3x^{\frac{2}{3}}}$

c $\dfrac{d}{dx} - 2x^{-\frac{1}{3}} = \dfrac{2}{3}x^{-\frac{4}{3}} = \dfrac{2}{3x^{\frac{4}{3}}}$

d $\dfrac{d}{dx}x^{\frac{4}{3}} = \dfrac{4}{3}x^{\frac{1}{3}}$

e $\dfrac{d}{dx}x^{-\frac{1}{3}} = -\dfrac{1}{3}x^{-\frac{4}{3}} = -\dfrac{1}{3x^{\frac{4}{3}}}$

f $\dfrac{d}{dx}x^{-\frac{1}{3}} + 2x^{\frac{3}{5}} = -\dfrac{1}{3}x^{-\frac{4}{3}} + \dfrac{6}{5}x^{-\frac{2}{5}}$

$= -\dfrac{1}{3x^{\frac{4}{3}}} + \dfrac{6}{5x^{\frac{2}{5}}}$

2 **a** $\dfrac{d}{dx}(2x+3)^2 = 4(2x+3) = 8x+12$

b $\dfrac{d}{dx}2(3x+4)^4 = 24(3x+4)^3$

c $\dfrac{d}{dx}(3-2x)^{-\frac{1}{2}} = (3-2x)^{\frac{3}{2}}$

$= \dfrac{1}{(3-2x)^{\frac{3}{2}}}$

d $\dfrac{d}{dx}\dfrac{1}{3+2x} = -\dfrac{2}{(3+2x)^2}$

e $\dfrac{d}{dx}(2x-1)^{-\frac{2}{3}} = -\dfrac{4}{3}(2x-1)^{-\frac{5}{3}}$

$= -\dfrac{4}{3(2x-1)^{\frac{5}{3}}}$

f $\dfrac{d}{dx}3(2+x^2)^{-\frac{1}{2}} = -3x(2+x^2)^{\frac{3}{2}}$

$= -\dfrac{3x}{(2+x^2)^{-\frac{3}{2}}}$

g $\dfrac{d}{dx}\left(2x^2 - \dfrac{3}{x^2}\right)^{\frac{1}{3}} =$

$\dfrac{1}{3}\left(4x + \dfrac{6}{x^3}\right)\left(2x^2 - \dfrac{3}{x^2}\right)^{-\frac{2}{3}}$

3 **a** $\dfrac{-1}{x^2} + c$

b $\dfrac{2x^{\frac{5}{2}}}{5} - \dfrac{4x^{\frac{3}{2}}}{3} + c$

c $\dfrac{3x^2}{2} + 2x + c$

d $\dfrac{-6x-1}{2x^2} + c$

e $\dfrac{5x^2}{2} - \dfrac{4x^{\frac{3}{2}}}{3} + c$

f $\dfrac{20x^{\frac{7}{4}}}{7} - \dfrac{3x^{\frac{4}{3}}}{2} + c$

g $2x - \dfrac{2x^{\frac{3}{2}}}{3} + c$

h $-\dfrac{3x+1}{x^2+c} + c$

4 $s = \dfrac{1}{2}t^2 + 3t + \dfrac{1}{t} + \dfrac{3}{2}$

5 **a** $y = \sqrt{x}, \therefore\ y' = \dfrac{1}{2}x^{-\frac{1}{2}}$

$y'(9) = \dfrac{1}{2}(9)^{-\frac{1}{2}} = \dfrac{1}{6}$

b $y = \dfrac{1}{2x+1}, \therefore\ y' = -\dfrac{2}{(2x+1)^2}$

$y'(0) = -\dfrac{2}{1^2} = -2$

c $y = \dfrac{2}{x^2},\ \therefore\ y' = -\dfrac{4}{x^3}$

$y'(4) = -\dfrac{4}{4^3} = -\dfrac{1}{16}$

d $y = 3 + \dfrac{2}{x}, \therefore\ y' = -\dfrac{2}{x^2}$

$y'(1) = -\dfrac{2}{1^2} = -2$

e $y = \sqrt{x+1},\ \therefore\ y' = \dfrac{1}{2}(x+1)^{-\frac{1}{2}}$

$y'(8) = \dfrac{1}{2}\left(9^{-\frac{1}{2}}\right) = \dfrac{1}{6}$

f $y = (x^2 - 7x - 8)^3$

$\therefore\ y' = 3(2x-7)(x^2 - 7x - 8)^2$

$y'(8) = 3(16-9)(0)^2 = 0$

6 $y = \dfrac{1}{x},\ \therefore\ y' = -\dfrac{1}{x^2}$ Gradient is −4, so

$y'(x) = -4$

$-\dfrac{1}{x^2} = -4$, so $x^2 = \dfrac{1}{4}$

$x = \pm\dfrac{1}{2}$

$y\left(\dfrac{1}{2}\right) = 2, y\left(-\dfrac{1}{2}\right) = -2$

7 $y = \sqrt{x},\ \therefore\ y' = \dfrac{1}{2}x^{-\frac{1}{2}}$

$y'(x) = 2,\ \therefore\ \dfrac{1}{2\sqrt{x}} = 2$

$\sqrt{x} = \dfrac{1}{4}$

$x = \dfrac{1}{16}$

$\therefore y = \dfrac{1}{4}$

Gradient is 2 at $\left(\dfrac{1}{16},\ \dfrac{1}{4}\right)$.

Solutions to multiple-choice questions

1 B $f(x) = \dfrac{4x^4 - 12x^2}{3x}$

$= \dfrac{4}{3}x^3 - 4x$

$\therefore\ f'(x) = 4x^2 - 4$

2 D $f(x) = 2x^{\frac{p}{q}}$

$\therefore\ f'(x) = \left(\dfrac{2p}{q}\right)x^{\frac{p}{q}-1}$

3 A $f(x) = 4 + \dfrac{4}{2-x}; x \neq 2$

$\therefore\ f'(x) = \dfrac{4}{(2-x)^2} > 0; x \neq 2$

4 A $x = -t^3 + 7t^2 - 14t + 6$ cm

$\therefore\ v = -3t^2 + 14t - 14$ cm/s

$\therefore\ a = -6t + 14$ cm/s^2

$a(3) = 14 - 18 = -4$ cm/s^2

5 A $\dfrac{dy}{dx} = \left(\dfrac{dy}{df}\right)f'(x) = 3x^2 f'(x)$

6 E $f(x) = x + \dfrac{1}{x},\ \therefore\ f'(x) = 1 - \dfrac{1}{x^2}$

$f'(x) = 0,\ \therefore\ x^2 = 1$

$x = \pm 1$

Local minimum where $x = 1$

$\therefore\ a = 1$

7 A $f(x) = x^{\frac{1}{5}},\ \therefore\ f'(x) = \dfrac{1}{5}x^{-\frac{4}{5}}$

A Gradient undefined for $x = 0$ **X**

B Curve passes through the origin ✓

C Curve passes through (1, 1), (−1, −1) ✓

D $f'(x) > 0; x \in R$ ✓

D $x > 0$, gradient is decreasing ✓

8 B $f(x) = x^{\frac{3}{4}} < \therefore\ f'(x) = \dfrac{1}{4}x^{-\frac{1}{4}}$

A Maximal domain $= R^+ \cup \{0\}$ ✓

B $f(x) < x$ for all $x > 1$ **X**

C Curve passes through (1, 1) ✓

D $f'(x) > 0; x \in R$ ✓

E $x > 0$, gradient is decreasing ✓

9 A $\dfrac{d}{dx}(5x^2 + 2x)^n$

$= n(10x + 2)(5x^2 + 2x)^{n-1}$

10 D $y = \dfrac{k}{2(x^2+1)}$

$\therefore\ y' = -kx(x^2 + 1)^{-2}$

$y'(1) = 1,\ \therefore\ -\dfrac{k}{4} = 1$

$k = -4$

Solutions to extended-response questions

1 **a** The volume of the cylinder $= \pi r^2 h = 400$

Therefore $$h = \frac{400}{\pi r^2}$$

b The surface area is
$$\begin{aligned} A &= 2\pi rh + 2\pi r^2 \\ &= 2\pi r \times \frac{400}{\pi r^2} + 2\pi r^2 \\ &= \frac{800}{r} + 2\pi r^2 \text{ as required} \end{aligned}$$

c $$\frac{dA}{dr} = \frac{-800}{r^2} + 4\pi r = 4\pi r - \frac{800}{r^2}$$

d $\frac{dA}{dr} = 0$ implies $4\pi r - \frac{800}{r^2} = 0$

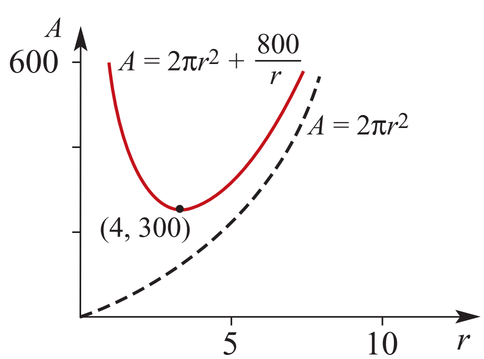
Therefore $$r^3 = \frac{200}{\pi}$$

and $$r = \left(\frac{200}{\pi}\right)^{\frac{1}{3}} \approx 3.99$$

e Minimum surface area $= 120(5\pi)^{\frac{1}{3}}$

$= 301\text{ cm}^2$, correct to 3 significant figures

f



2 **a** Area is 16 cm^2. Therefore $xy = 16$ and $y = \frac{16}{x}$.

b The perimeter is given by $P = 2(x + y) = 2x + \frac{32}{x}$ as required.

c The minimum occurs when $\dfrac{dP}{dx} = 0$,

$\therefore \dfrac{dP}{dx} = 2 - \dfrac{32}{x^2}$ and $\dfrac{dP}{dx} = 0$ implies $2 - \dfrac{32}{x^2} = 0$

therefore $2 = \dfrac{32}{x^2}$

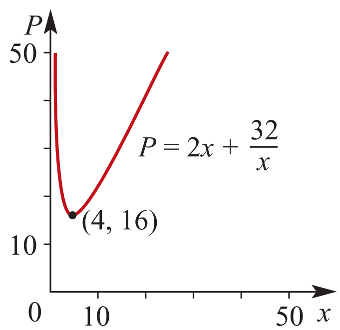
$$x^2 = 16$$

and $x = \pm 4$

But $x > 0$ and so $x = 4$

When $x = 4$, $P = 2 \times 4 + \dfrac{32}{4} = 16$

The minimum value of P is 16.



3 $OC = x$ cm, $CZ = 5$ cm and $AX = 7$ cm

a $OA \times OC = 120$, therefore OA $= \dfrac{120}{x}$

b
$$OX = OA + AX$$
$$= \frac{120}{x} + 7$$

c
$$OZ = OC + CZ$$
$$= x + 5$$

d
$$A = (x+5)\left(\frac{120}{x} + 7\right) = 155 + 7x + \frac{600}{x}$$

e
$$\frac{dA}{dx} = 7 - \frac{600}{x^2}$$

$\dfrac{dA}{dx} = 0$ implies $x^2 = \dfrac{600}{7}$

and $x = \dfrac{10\sqrt{42}}{7} \approx 9.26$ cm$(x > 0)$

4 **a** For $y = \sqrt{x+2}$, the axis intercepts have coordinates $A(-2, 0)$ and $B(0, \sqrt{2})$.

b By the chain rule, $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x+2}}$.

c **i** When $x = -1$, $\dfrac{dy}{dx} = \dfrac{1}{2}$.

ii When $x = -1$, $y = 1$, and the equation of the tangent at this point is

$$y - 1 = \frac{1}{2}(x+1)$$

This implies $\quad y = \dfrac{1}{2}x + \dfrac{3}{2}$ or $2y - x = 3$

iii The tangent meets the x-axis at $(-3, 0)$ and the y-axis at $\left(0, \dfrac{3}{2}\right)$. Let these intercepts be the points C and D respectively.

Distance $CD = \sqrt{\dfrac{9}{4} + 9} = \dfrac{3\sqrt{5}}{2}$

d $\dfrac{1}{2\sqrt{x+2}} < 1$ implies $\quad \dfrac{1}{2} < \sqrt{x+2}$

Square both sides $\quad x + 2 > \dfrac{1}{4}$

and hence $\quad x > -\dfrac{7}{4}$

5 The volume $V = 2x^2h$

a Since the volume is 36 cm^3, $h = \dfrac{36}{2x^2}$

$$= \frac{18}{x^2}$$

b The surface area is given by $A = 2x^2 + 6xh$

$$= 2x^2 + 6x \times \frac{18}{x^2}$$

$$= 2x^2 + \frac{108}{x} \text{ as required}$$

c

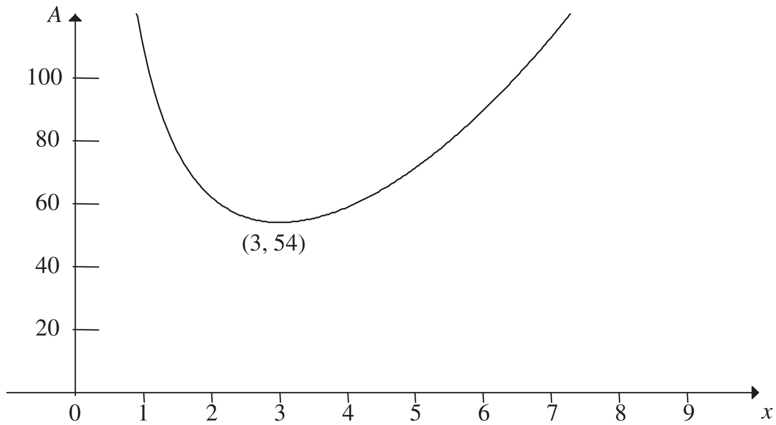
$$\frac{dA}{dx} = 4x - \frac{108}{x^2}$$

and $\dfrac{dA}{dx} = 0$ implies $\quad 4x^3 = 108$

Hence $\quad x^3 = 27$

and $\quad x = 3$

The minimum surface area occurs for $x = 3$ and $h = 2$.

d 

6 **a** The height of the prism is y cm and the volume 1500 cm^3.

$$\text{Area of triangle } ABC = \frac{1}{2} \times 3x \times 4x$$
$$= 6x^2$$

Therefore $6x^2 y = 1500$

and
$$y = \frac{1500}{6x^2}$$
$$= \frac{250}{x^2}$$

b $AB = 5x$ by Pythagoras' theorem.
Therefore the surface area is given by

$$S = 5xy + 3xy + 4xy + 12x^2$$
$$= 12xy + 12x^2$$
$$= 12x^2 + \frac{3000}{x}$$

c $\frac{dS}{dx} = 24x - \frac{3000}{x^2}$

d $\frac{dS}{dx} = 0$ implies $x^3 = 125$

and hence $x = 5$

When $x = 5$, $S_{\text{min}} = 12 \times 25 + 600$

$$= 900$$

The minimum surface area is 900 cm^2.

Chapter 21 – Integration

Solutions to Exercise 21A

1 $f(2) = 6, f(3) = 11, f(4) = 18, f(5) = 27$
Therefore area $= 6 \times 1 + 11 \times 1 + 18 \times 1 + 27 \times 1 = 62$

2 $f(1) = 4, f(2) = 12, f(3) = 24, f(4) = 40$
Therefore area $= 4 \times 1 + 12 \times 1 + 24 \times 1 + 40 \times 1 = 80$

3 $f(0.5) = 12.1875, f(1) = 20, f(1.5) = 23.9375, f(2) = 24, f(2.5) = 21.6875, f(3) = 20$
Therefore area $= 12.1875 \times 0.5 + 20 \times 0.5 + 23.9375 \times 0.5 + 24 \times 0.5 + 21.6875 \times 0.5 + 20 \times 0.5 = 60.90625$

4 $f(0) = 0, f(1) = 5, f(2) = 14, f(3) = 27, f(4) = 44$
Area $= \frac{1}{2}(f(0) + f(1)) + \frac{1}{2}(f(1) + f(2)) + \frac{1}{2}(f(2) + f(3)) + \frac{1}{2}(f(3) + f(4)) = 68$

5 **a** $A \approx 5 + 3.5 + 2.5 + 2.2$
$= 13.2$

b $A \approx 3.5 + 2.5 + 2.2 + 2$
$= 10.2$

c $A \cong \frac{1}{2}(5 + 2) + (3.5 + 2.5 + 2.2)$
$= 11.7$

6 Trapezoidal estimate:

$$\int_0^3 x(3 - x)\, dx$$

a $= \dfrac{1.25 + 2 + 2.25 + 2 + 1.25}{2}$

$= \dfrac{35}{8}$

b A trapezoidal approximation can be performed with a CAS calculator giving a value of 4.536.

7 $y = f(x)$:

x	0	1	2	3	4	5
y	3	3.5	3.7	3.8	3.9	3.9

x	6	7	8	9	10
y	4	4	3.7	3.3	2.9

a Left-endpoint: add all except 2.9 = 36.8

b Trapezoidal: $\dfrac{3 + 2.9}{2} +$ middle 8 = 36.75

8 $\displaystyle\int_0^1 \frac{1}{1 + x^2} dx \approx$ (strip width = 0.25)

$$\frac{\left(1 + \frac{1}{2}\right)}{8} + \frac{\left(\frac{16}{16} + \frac{4}{5} + \frac{16}{25}\right)}{4} \cong 0.783$$

$\therefore\ \pi \approx 4 \times 0.783 = 3.13$

9 **a** $\displaystyle\int_0^2 2^x dx \approx$ (strip width 0.5)

$$\frac{1 + 4}{4} + \frac{\sqrt{2} + 2 + 2\sqrt{2}}{2} \cong 4.371$$

b $\int_0^{0.9} \frac{1}{\sqrt{1-x^2}} dx \approx$ 1.128 (strip width 0.1)

10

D	0	3	6	9	12	15
S	1	2	3	4	5	5

D	18	21	24	27	30
S	6	4	4	2	2

Trapezium: $\frac{3}{2}(1 + 2) +$ 3(middle 9) = 109.5 m^2

Solutions to Exercise 21B

1 **a** $\int_1^2 x^2 dx = \frac{1}{3}(2^3 - 1^3) = \frac{7}{3}$

b $\int_{-1}^3 x^3 dx = \frac{1}{4}(3^4 - 1^4) = 20$

c $\int_0^1 (x^3 - x) dx = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

d $\int_{-1}^2 (x+1)^2 dx = \frac{1}{3}(3^3 - 0) = 9$

e $\int_1^2 x^3 dx = \frac{1}{4}(2^4 - 1^4) = \frac{15}{4}$

f

$$\int_1^4 x + 2x^2 dx = \frac{1}{2}(4^2 - 1^2) + \frac{2}{3}(4^3 - 1^3)$$

$$= \frac{15}{2} + 42 = 49.5$$

g $\int_0^2 x^3 + 2x^2 + x + 2 dx$

$$= \frac{2^4}{4} + \frac{2}{3}2^3 + \frac{1}{2}2^2 + 2(2)$$

$$= 4 + \frac{16}{3} + 4 + 2$$

$$= \frac{46}{3}$$

h $\int_1^4 2x + 5 dx = (4^2 - 1^2) + 5(4 - 1)$

$$= 30$$

2 $\int_0^2 (x+1)\, dx = \left[\frac{x^2}{2} + x\right]_0^2$

$$= 4 - 0$$

$$= 4$$

3 $\int_0^3 (x^2)\, dx = \left[\frac{x^3}{3}\right]_0^3$

$$= 9 - 0$$

$$= 9$$

4 $\int_{-1}^1 (1 - x^2)\, dx = \left[x - \frac{x^3}{3}\right]_{-1}^1$

$$= \frac{2}{3} - (-\frac{2}{3})$$

$$= \frac{4}{3}$$

5 $\int_0^2 (4x - x^3)\, dx = \left[2x^2 - \frac{x^4}{4}\right]_0^2$

$$= 4 - 0$$

$$= 4$$

Solutions to Exercise 21C

1 $\displaystyle\int_0^4 (x^2 - 4x)\,dx = \left[\frac{x^3}{3} - 2x^2\right]_0^4$

$$= -\frac{32}{3} - 0$$

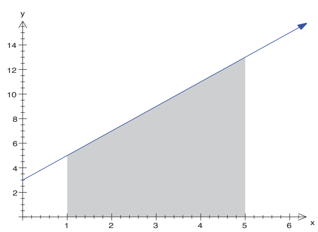
$$= -\frac{32}{3}$$

2 $\displaystyle\int_{-3}^3 (x^2 - 9)\,dx = \left[\frac{x^3}{3} - 9x\right]_{-3}^3$

$$= -36 - 0$$

$$= -36$$

3



$$A = \frac{1}{2}(5 - 1)(5 + 13) = 36$$

$$\int_1^5 2x + 3\,dx = (5^2 - 1^2) + 3(5 - 1) = 36$$

4 $y = x(x - 1)(3 - x)$

$$= -x^3 + 4x^2 - 3x$$

$$A = \int_1^3 y\,dx - \int_0^1 y\,dx$$

$$I = -\frac{x^4}{4} + \frac{4}{3}x^3 - \frac{3x^2}{2}$$

$$\therefore A = \left[\frac{9}{4} - \left(-\frac{5}{12}\right)\right] - \left[-\frac{5}{12} - 0\right]$$

$$= \frac{37}{12} \cong 3.08$$

5 $\displaystyle\int_1^5 h(x)dx = 4$

a $\displaystyle\int_1^5 2h(x)dx = 2\int_1^5 h(x)dx = 8$

b $\displaystyle\int_1^5 h(x) + 3dx = 4 + \int_1^5 3dx$

$$= 4 + 3(5 - 1) = 16$$

c $\displaystyle\int_5^1 h(x)dx = -\int_1^5 h(x)dx = -4$

6 $\displaystyle\int_2^5 f(x)dx = 12$

a $\displaystyle\int_5^2 f(x)dx = -\int_2^5 f(x)dx = -12$

b $\displaystyle\int_2^5 3f(x)dx = 3\int_2^5 f(x)dx = 36$

c $\displaystyle\int_2^4 f(x)dx + \int_4^5 f(x)dx + \int_2^4 4dx$

$$= \int_2^5 f(x)dx + \int_2^4 4dx$$

$$= 12 + 4(4 - 2) = 20$$

7 **a** $\displaystyle\int_1^3 6x\,dx = 3(3^2 - 1^2) = 24$

$$\int_3^4 6x\,dx = 3(4^2 - 3^2) = 21$$

$$\int_1^4 6x\,dx = 3(4^2 - 1^2) = 45$$

b $\int_1^3 6 - 2x\,dx = 6(3-1) - (3^2 - 1^2)$

$= 4$

$\int_3^4 6 - 2x\,dx = 6(4-3) - (4^2 - 3^2)$

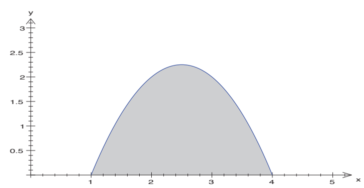
$= -1$

$\int_1^4 6 - 2x\,dx = 6(4-1) - (4^2 - 1^2)$

$= 3$

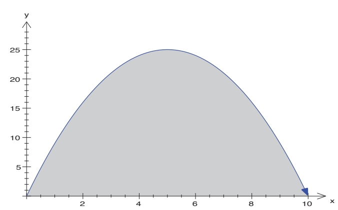
(1) + (2) = (3) in both cases

8 $y = 5x - x^2 - 4$



$A = \int_1^4 5x - x^2 - 4dx = 4.5$

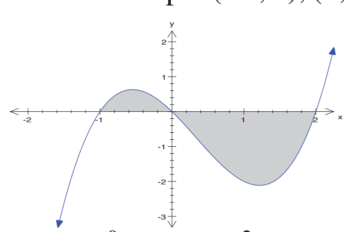
9 $y = x(10 - x)$



$A = \int_0^{10} x(10 - x)dx = \frac{500}{3}$

10 $y = x(x-2)(x+1)$

Axis intercepts $(-1, 0), (0, 0), (2, 0)$



$A = \int_{-1}^0 ydx - \int_0^2 ydx = \frac{37}{12}$

11 a

$$\int_1^2 \frac{(2+x)(2-x)}{x^2}dx = \int \frac{4-x^2}{x^2}dx$$

$$= \int_1^2 \left(\frac{4}{x^2} - 1\right)dx$$

$$= \left[-\frac{4}{x} - x\right]_1^2$$

$$= -\left(\frac{4}{2} + 2\right) + \left(\frac{4}{1} + 1\right)$$

$$= -4 + 5 = 1$$

b

$$\int_1^4 2x - 3x^{\frac{1}{2}}dx = \left[x^2 - 2x^{\frac{3}{2}}\right]_1^4$$

$$= (4^2 - 1^2) - 2\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$$

$$= 15 - 14 = 1$$

c

$$\int_1^3 \frac{4x^2+9}{x^2}dx = \int_1^3 4 + \frac{9}{x^2}dx$$

$$= \left[4x - \frac{9}{x}\right]_1^3$$

$$= 4(3-1) - \left(\frac{9}{3} - \frac{9}{1}\right) = 14$$

d

$$\int_1^4 6x - 3x^{\frac{1}{2}}dx = \left[3x^2 - 2x^{\frac{3}{2}}\right]_1^4$$

$$= 3(4^2 - 1^2) - 2\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$$

$$= 3(15) - 2(8 - 1) = 31$$

e $\int_1^4 \frac{x^2-1}{x^2}dx = \int_1^4 1 - \frac{1}{x^2}dx$

$$= \left[x + \frac{1}{x}\right]_1^4$$

$$= (4-1) + \left(\frac{1}{4} - \frac{1}{1}\right)$$

$$= 3 - \frac{3}{4} = \frac{9}{4}$$

f $\int_1^4 \frac{2x - 3x^{\frac{1}{2}}}{x} dx = \int_1^4 2 - 3x^{-\frac{1}{2}} dx$

$= \left[2x - 6x^{\frac{1}{2}}\right]$

$= 2(4 - 1) - 6(\sqrt{4} - \sqrt{1})$

$= 6 - 6 = 0$

12 a $A = -\int_0^2 x^2 - 2x\,dx = \frac{4}{3}$

b $A = -\int_3^4 (4 - x)(3 - x)dx = \frac{1}{6}$

c $A = \int_{-2}^7 (x + 2)(7 - x)dx = 121.5$

d $A = \int_2^3 x^2 - 5x + 6\,dx = \frac{1}{6}$

e $A = \int_{-\sqrt{3}}^{\sqrt{3}} 3 - x^2 dx = 4\sqrt{3} \approx 6.93$

f $A = -\int_0^6 x^3 - 6x^2 dx = 108$

All areas are measured in square units.

Solutions to Technology-free questions

1 **a** $\int_1^2 2x\,dx = [x^2]_1^2 = 2^2 - 1^2 = 3$

b $\int_1^2 2\,dx = [2x]_2^5 = 10 - 4 = 6$

c
$$\int_3^5 3x^2 + 2x\,dx = [x^3 + x^2]_3^5$$
$$= (5^3 - 3^3) + (5^2 - 3^2)$$
$$= 114$$

d
$$\int_1^5 x^2 + 2x\,dx = \left[\frac{x^3}{3} + x^2\right]_1^5$$
$$= \frac{1}{3}(5^3 - 1^3) + (5^2 - 1^2)$$
$$= \frac{196}{3}$$

e $\int_{-3}^{-2} 5dx = 5(-2 + 3) = 5$

2 **a**
$$\int_1^4 \sqrt{x}\,dx = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4$$
$$= \frac{2}{3}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$$
$$= \frac{14}{3}$$

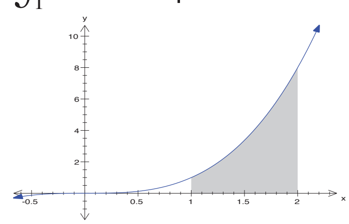
b
$$\int_1^4 x^3 - 2x\,dx = \left[\frac{1}{4}x^4 - x^2\right]_1^4$$
$$= \frac{1}{4}(4^4 - 1^4) - (4^2 - 1^2)$$
$$= \frac{195}{4}$$

c $\int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = -\left(\frac{1}{2} - \frac{1}{1}\right) = \frac{1}{2}$

d
$$\int_1^4 \frac{2}{x^3}\,dx = \left[-\frac{1}{x^2}\right]_1^4 = -\left(\frac{1}{4^2} - \frac{1}{1^2}\right)$$
$$= \frac{15}{16}$$

e
$$\int_0^1 \sqrt{x}(x + 2)\,dx = \int_0^1 x^{\frac{3}{2}} + x^{\frac{1}{2}}\,dx$$
$$= \left[\frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}}\right]_0^1$$
$$= \frac{2}{5} + \frac{2}{3} = \frac{16}{15}$$

3 $\int_1^2 x^3dx = \frac{1}{4}(2^4 - 1^4) = \frac{15}{4}$



4
$$A = \int_2^1 (1 - x)(2 + x)\,dx$$
$$= \int_2^1 2 - x - x^2\,dx$$
$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^1$$
$$= \left[2 - \frac{1}{2} - \frac{1}{3}\right] - \left[-3 - 2 + \frac{8}{3}\right]$$
$$= \frac{7}{6} + \frac{10}{3} = 4.5$$

5 $y = x(x-3)(x+2) = x^3 - x^2 - 6x$

$$A = \int_{-2}^{0} y\,dx - \int_{0}^{3} y\,dx$$

$$I = \frac{x^4}{4} - \frac{x^3}{3} - 3x^2$$

$$A = \left(\frac{1}{4}(0-16) - \frac{1}{3}(0+8) - 3(0-4)\right)$$

$$-\left(\frac{1}{4}(81) - \frac{1}{3}(27) - 3(9)\right)$$

$$= \left(-4 - \frac{8}{3} + 12\right) - \left(\frac{81}{4} - 9 - 27\right)$$

$$= \frac{16}{3} + \frac{63}{4} = \frac{253}{12}$$

6 **a** $B = (1, 3), C = (3, 3)$

b $ABCD$ area $= 3(2) = 6$

c $a = \int_{1}^{3} 4x - x^2 dx - 6$

$$= \left[2x^2 - \frac{x^3}{3}\right]_1^3 - 6$$

$$= 2(3^2 - 1^2) - \frac{1}{3}(3^3 - 1^3) - 6$$

$$= 16 - \frac{26}{3} - 6 = \frac{4}{3}$$

Solutions to multiple-choice questions

1 C $\int x^3 + 3x\,dx = \frac{x^4}{4} + \frac{3x^2}{2} + c$

2 D $\int \sqrt{x} + x\,dx = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 + c$

3 A $3\int x^{-4}dx = -x^{-3} + c$

4 D $\frac{dy}{dx} = 2x + 5, \therefore y = x^2 + 5x + c$

$y(0) = 1, \therefore c = 1$

5 B $f'(x) = 5x^4 - 9x^2$

$\therefore f(x) = x^5 - 3x^3 + c$

$f(1) = 2, \therefore 1 - 3 + c = 2$

$\therefore c = 4$

6 B $\frac{dy}{dx} = \frac{4}{x^3}, \therefore y = -\frac{2}{x^2} + c$

$y(1) = 0, \therefore c - 2 = 0$

$c = 2$

7 D $F'(x) = f(x)$

$\therefore \int_3^5 f(x)dx = F(5) - F(3)$

8 B $\int_0^2 3x^2 - 2x\,dx = 2^3 - 2^2$

$= 4$

9 C $\int_0^2 3f(x) + 2dx$

$= 3\int_0^2 f(x)dx + \int_0^2 2dx$

$= 3\int_0^2 f(x)dx + 4$

10 A $k\int_0^3 (x-3)^2dx = 36$

$\therefore -\frac{(-3)^3}{3} = \frac{36}{k}$

$k = \frac{36}{9} = 4$

Solutions to extended-response questions

1 $$\frac{dy}{dx} = \frac{9}{32}(x^2 - 4x)$$
$$= \frac{9}{32}x^2 - \frac{9}{8}x$$

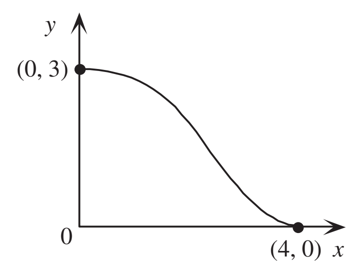
a $$y = \int \frac{dy}{dx}\,dx$$
$$= \frac{3}{32}x^3 - \frac{9}{16}x^2 + c$$

The coordinates of the graph that represent the highest part of the slide are (0, 3). Substituting these values into the equation of the curve determines c.

$$3 = \frac{3}{32}(0)^3 - \frac{9}{16}(0)^2 + c$$
$$\therefore \quad 3 = 0 - 0 + c$$
$$\therefore \quad c = 3$$
$$\therefore \quad y = \frac{3}{32}x^3 - \frac{9}{16}x^2 + 3 \qquad \text{for } x \in [0, 4]$$

b Domain = $\{x\colon 0 \le x \le 4\}$

Range = $\{y\colon 0 \le y \le 3\}$



At the stationary points, $\frac{dy}{dx} = 0$

$$\therefore \quad \frac{9}{32}(x^2 - 4x) = 0$$
$$\therefore \quad \frac{9}{32}x(x - 4) = 0$$
$$\therefore \quad x = 0 \quad \text{or} \quad x = 4$$

At $x = 0$, $$y = \frac{3}{32}(0)^3 - \frac{9}{16}(0)^2 + 3$$
$$= 3$$

At $x = 4$, $$y = \frac{3}{32}(4)^3 - \frac{9}{16}(4)^2 + 3$$
$$= \frac{3 \times 64}{32} - \frac{9 \times 16}{16} + 3$$
$$= 6 - 9 + 3$$
$$= 0$$

c If the slope of the slide exceeds 45°, then the gradient of the curve will be less than −1.

$$\therefore \quad \frac{dy}{dx} < -1$$

$$\therefore \quad \frac{9}{32}(x^2 - 4x) < -1$$

$$\therefore \quad x^2 - 4x + \frac{32}{9} < 0$$

Consider $\quad x^2 - 4x + \frac{32}{9} = 0$

$$\therefore \quad 9x^2 - 36x + 32 = 0$$

$$\therefore \quad (3x - 4)(3x - 8) = 0$$

$$\therefore \quad x = \frac{4}{3} \quad \text{or} \quad x = \frac{8}{3}$$

The gradient of y at $x = \frac{4}{3}$ and $x = \frac{8}{3}$ is −1, and $x^2 - 4x + \frac{32}{9} < 0$ for $\frac{4}{3} < x < \frac{8}{3}$.

The slope of the slide exceeds 45° for $\frac{4}{3} < x < \frac{8}{3}$.

2 a $\text{Area}_{OABC} = 9 \times 3 = 27$ square units

b $\quad y = k(x - 4)^2$

When $x = 9$, $\quad y = 3$

$$\therefore \quad 3 = k(9 - 4)^2$$

$$\therefore \quad 3 = 25k$$

$$\therefore \quad k = \frac{3}{25}$$

$$\therefore \quad y = \frac{3}{25}(x - 4)^2$$

c $\int_0^9 y\,dx = \int_0^9 \frac{3}{25}(x-4)^2 dx$

$$= \frac{3}{25}\int_0^9 x^2 - 8x + 16dx$$

$$= \frac{3}{25}\left[\frac{1}{3}x^3 - 4x^2 + 16x\right]_0^9$$

$$= \frac{3}{25}\left[\left(\frac{1}{3}(9)^3 - 4(9)^2 + 16(9)\right) - \left(\frac{1}{3}(0)^3 - 4(0)^2 + 16(0)\right)\right]$$

$$= \frac{3}{25}(9)(\frac{9\times 9}{3} - 4\times 9 + 16)$$

$$= \frac{27}{25}(27 - 36 + 16)$$

$$= \frac{27\times 7}{25}$$

$$= \frac{189}{25}$$

$$= 7\frac{14}{25}$$

The total area of the region enclosed between the curve and the x-axis for $x \in [0, 9]$ is $7\frac{14}{25}$ square units.

d Area of shaded region $= \text{Area}_{OABC} - \int_0^9 y\,dx$

$$= 27 - 7\frac{14}{25} = \frac{486}{25}$$

The area of the cross-section of the pool is $\frac{486}{25}$ or $19\frac{11}{25}$ square units.

3 a i Let V (litres) be the volume of water in the container.

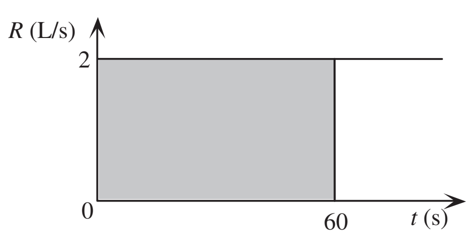
$\therefore \quad V = Rt, \quad$ and $R = 2$ L/s, $t = 60$ s

$\therefore \quad V = 2\times 60$

$\therefore \quad = 120$

After 1 minute, 120 1itres of water has flowed into the container.

ii



b **i** $V = \int_0^{60} R\,dt$

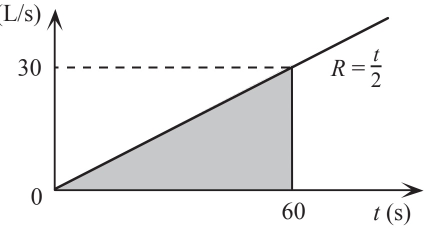
$= \int_0^{60} \frac{t}{2}dt$

$= \left[\frac{t^2}{4}\right]_0^{60}$

$= \frac{3600}{4} - 0$

$= 900$

After 1 minute, 900 1itres of water has flowed into the container.

ii 

iii $V = \int_0^{60} \frac{t}{2}dt$

$= \frac{(60a)^2}{4} = 900a^2$

$900a^2$ litres have flowed into the container after a minutes.

c **i** $V = \int_0^{60} \frac{t^2}{10}dt$

$= \left[\frac{t^3}{30}\right]_0^{60}$

$\therefore \quad V = 7200$

The area of the shaded region is 7200 square units.

ii The area of the shaded region represents the volume of water (in litres) that has flowed into the container after 60 seconds.

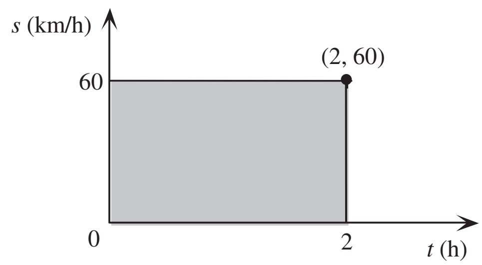
iii $V = \frac{t^3}{30}$ and $V = 10\,000$

$\therefore \quad t^3 = 30V = 30 \times 10\,000$

$\therefore \quad t = \sqrt[3]{300\,000} \approx 66.94$

10 000 litres had flowed into the container after about 67 seconds.

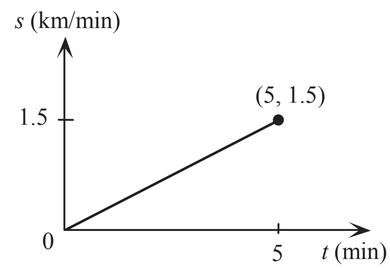
4 a i



ii See shaded area above which indicates the total distance travelled by the car after 2 hours.

b i Let acceleration = $a(\text{km/min}^2) = 0.3$

$\therefore \qquad s = \int a\,dt = at + c$



When $t = 0$, $\qquad s = 0$

$\therefore \qquad 0 = a(0) + c$

$\therefore \qquad c = 0$

$\therefore \qquad s = at$

$\therefore \qquad s = 0.3t$

When $t = 5$, $\qquad s = 1.5$

ii Let d denote the distance travelled (km) at time t (minutes).

$$d = \int_0^5 s\,dt$$

$$= \int_0^5 0.3t\,dt = [0.15t^2]_0^5$$

$$= 0.15(5)^2$$

$$= 3.75$$

The car has travelled 3.75 km after 5 minutes.

c i acceleration $= \dfrac{dV}{dt}$

$$= \frac{d}{dt}(20t - 3t^2)$$

$= 20 - 6t$, the acceleration of the particle at time t in m/s^2.

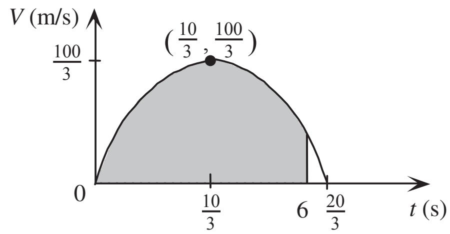
ii $V = 0$ implies $\quad t(20 - 3t) = 0$

i.e. $\quad t = 0 \quad$ or $t = \dfrac{20}{3}$

When $\dfrac{dV}{dt} = 0, \quad 20 - 6t = 0$

$\therefore \quad 6t = 20$

$\therefore \quad t = \dfrac{10}{3}$



When $t = \dfrac{10}{3}$,

$$V = 20\left(\frac{10}{3}\right) - 3\left(\frac{10}{3}\right)^2$$
$$= \frac{20 \times 10}{3} - \frac{3 \times 10 \times 10}{3 \times 3}$$
$$= \frac{200 - 100}{3}$$
$$= \frac{100}{3}$$

iii distance travelled $= \displaystyle\int_0^6 V\,dt$

$$= \int_0^6 20t - 3t^2 dt$$
$$= [10t^2 - t^3]_0^6$$
$$= 10(6)^2 - (6)^3$$
$$= 360 - 216$$
$$= 144$$

The particle has travelled 144 m after 6 seconds (shaded area of graph).

5 a i When $x = 10$,

$$y = \frac{10^2}{1000}(50 - 10)$$
$$= \frac{100 \times 40}{1000}$$
$$= 4$$

The height of the mound 10 m from the edge is 4 m.

ii When $x = 40$,
$$y = \frac{40^2}{1000}(50 - 40)$$
$$= \frac{1600 \times 10}{1000}$$
$$= 16$$

The height of the mound 40 m from the edge is 16 m.

b
$$y = \frac{x^2}{1000}(50 - x)$$
$$= \frac{1}{20}x^2 - \frac{1}{1000}x^3$$

Gradient is given by
$$\frac{dy}{dx} = \frac{2}{20}x - \frac{3}{1000}x^2$$
$$= \frac{x}{10} - \frac{3x^2}{1000}$$

i When $x = 10$,
$$\frac{dy}{dx} = \frac{10}{10} - \frac{3(10)^2}{1000}$$
$$= 1 - \frac{3}{10}$$
$$= 0.7$$

The gradient of the boundary curve when 10 m from the edge is 0.7.

ii When $x = 40$,
$$\frac{dy}{dx} = \frac{40}{10} - \frac{3(40)^2}{1000}$$
$$= 4 - \frac{48}{10} = -0.8$$

The gradient of the boundary curve when 40 m from the edge is −0.8.

c The height of the mound is a maximum when $\frac{dy}{dx} = 0$.
$$\frac{dy}{dx} = \frac{x}{10} - \frac{3x^2}{1000}$$
$$= \frac{x}{1000}(100 - 3x)$$

When $\frac{dy}{dx} = 0$,
$$\frac{x}{1000}(100 - 3x) = 0$$
$$\therefore \quad \frac{x}{1000} = 0 \quad \text{or} \quad 100 - 3x = 0$$
$$\therefore \quad x = 0 \quad \text{or} \quad 3x = 100$$
$$x = \frac{100}{3}$$

When $x = \frac{100}{3}$, $\quad y = \frac{\left(\frac{100}{3}\right)^2}{1000}\left(50 - \frac{100}{3}\right)$

$$= \frac{100 \times 100}{9 \times 1000}\left(\frac{150 - 100}{3}\right)$$

$$= \frac{10 \times 50}{9 \times 3}$$

$$= \frac{500}{27} \approx 18.52$$

i The height of the mound is a maximum when $x = \frac{100}{3}$.

ii The maximum height of the mound is $\frac{500}{27}$ m, or approximately 18.52 m.

d $\int_0^{50} y\,dx = \int_0^{50} \frac{x^2}{1000}(50 - x)dx$

$$= \int_0^{50} \frac{1}{20}x^2 - \frac{1}{1000}x^3 dx$$

$$= \left[\frac{1}{60}x^3 - \frac{1}{4000}x^4\right]_0^{50}$$

$$= \frac{(50)^3}{60} - \frac{(50)^4}{4000}$$

$$= \frac{50 \times 50 \times 50}{60} - \frac{50 \times 50 \times 50 \times 50}{4000}$$

$$= \frac{1}{6}(12\,500 - 9375)$$

$$= \frac{3125}{6} = 520\frac{5}{6}$$

The cross-sectional area of the mound is $520\frac{5}{6}$ m^2.

e **i** $\quad y = \frac{x^2}{1000}(50 - x)$

When $y = 12$, $\quad 12 = \frac{x^2}{1000}(50 - x)$

i.e. $\quad 12000 = 50x^2 - x^3$

It is known that when $x = 20, y = 12$, and therefore by the factor theorem $(x - 20)$

is a factor of $x^3 - 50x^2 + 1200$.

$$\therefore \quad x^3 - 50x^2 + 1200 = 0$$

$$\therefore \quad (x - 20)(x^2 - 30x - 600) = 0$$

$$\therefore \; x = 20 \quad \text{or} \quad x^2 - 30x - 600 = 0$$

$x^2 - 30x - 600 = 0$ implies

$$\begin{aligned} x &= \frac{30 \pm \sqrt{900 + 2400}}{2} \\ &= \frac{30 \pm \sqrt{3300}}{2} \\ &= \frac{30 \pm 10\sqrt{33}}{2} \\ &= 15 \pm 5\sqrt{33} \end{aligned}$$

The required value is $x = 15 + 5\sqrt{33}$, as $x \geq 0$.
Hence B has coordinates $(15 + 5\sqrt{33}, 12)$.

ii The top of the mound can be represented by the curve $y = \dfrac{x^2}{1000}(50 - x) - 12$, a translation of the curve $y = \dfrac{x^2}{1000}(50 - x)$ in the negative direction of the y-axis by 12 units.

Take $p = 15 + 5\sqrt{33}, q = 20$ and

$$\begin{aligned} R &= \int_{20}^{15+5\sqrt{33}} 12\, dx \\ &= 12[x]_{20}^{15+5\sqrt{33}} \\ &= 12[15 + 5\sqrt{33} - 20] \\ &= 12 \times (5\sqrt{33} - 5) \qquad = 60\sqrt{33} - 60 \end{aligned}$$

6 a i

$$f'(x) = 6x + 3$$

At (1, 6)

$$\begin{aligned} f'(1) &= 6(1) + 3 \\ &= 9 \end{aligned}$$

The gradient of the curve at (1, 6) is 9.

ii The tangent to the curve is a straight line of the form $(y - y_1) = m(x - x_1)$, where $m = 9$ and $(x_1, y_1) = (1, 6)$.

$$y - 6 = 9(x - 1)$$

$$\therefore \quad y - 6 = 9x - 9$$

$\therefore \quad y = 9x - 3$ is the equation of the tangent.

iii $y = f(x)$

$$f'(x) = 6x + 3$$

$$\therefore \quad f(x) = \int f'(x)dx$$

$$= 3x^2 + 3x + c$$

The curve passes through the point with coordinates (1, 6).

$\therefore \quad f(1) = 3(1)^2 + 3(1) + c = 6$

$\therefore \quad 3 + 3 + c = 6$

$\therefore \quad c = 0$

$\therefore \quad f(x) = 3x^2 + 3x$ is the equation of the curve.

b **i** The gradient of the tangent is $f'(2) = 6(2) + k = 12 + k$.

ii Let $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (2, 10)$

The gradient of the tangent is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{10 - 0}{2 - 0} = 5$$

The gradient of the tangent is 5.

$f'(x) = 6x + k$

When $x = 2$, $f'(2) = 6(2) + k$

$= 12 + k$

But from above, $f'(2) = 5$

$\therefore \quad 12 + k = 5$

$\therefore \quad k = -7$

iii $f'(x) = 6x - 7$

$f(x) = 3x^2 - 7x + c$

The curve passes through the point with coordinates (2, 10).

$\therefore \quad f(2) = 3(2)^2 - 7(2) + c = 10$

$\therefore \quad 12 - 14 + c = 10$

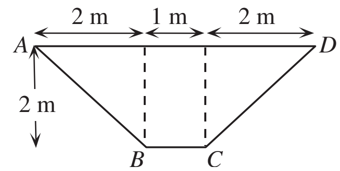
$\therefore \quad c = 12$

$\therefore f(x) = 3x^2 - 7x + 12$ is the equation of the curve.

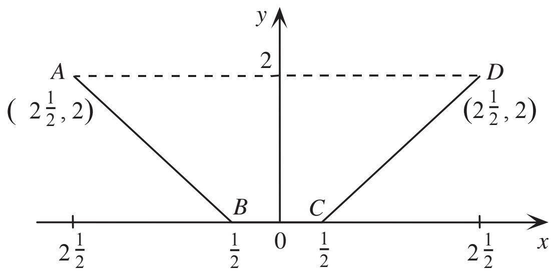
7 a Area $= \frac{1}{2}(2 \times 2) + 1 \times 2 + \frac{1}{2}(2 \times 2)$

$= 2 + 2 + 2$

$= 6$ square metres



b



i CD passes through $\left(\frac{1}{2}, 0\right)$ and $\left(2\frac{1}{2}, 2\right)$.

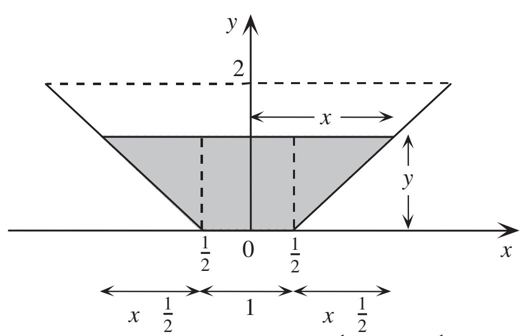
Let $y - y_1 = m(x - x_1)$ be the equation of the line CD

where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, $(x_1, y_1) = \left(\frac{1}{2}, 0\right)$ and $(x_2, y_2) = \left(2\frac{1}{2}, 2\right)$.

$\therefore \quad y - 0 = \dfrac{2 - 0}{2\frac{1}{2} - \frac{1}{2}}\left(x - \frac{1}{2}\right)$

$\therefore \quad y = x - \dfrac{1}{2}$

ii



Area of shaded region $= \frac{1}{2}y\left(x - \frac{1}{2}\right) + 1 \times y + \frac{1}{2}y\left(x - \frac{1}{2}\right)$

$= y\left(x - \frac{1}{2}\right) + y$

$= xy - \frac{1}{2}y + y$

$= xy + \frac{1}{2}y$

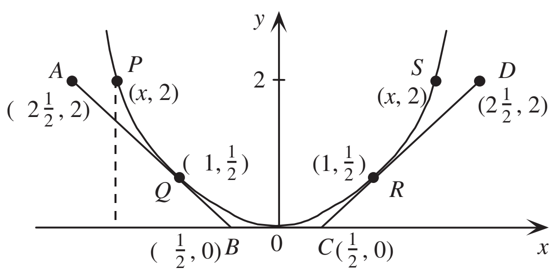
or by considering the trapezium

$$\text{Area} = \frac{y}{2}(1 + 2x)$$
$$= \frac{1}{2}y + xy$$

But $y = x - \frac{1}{2}$ $\qquad \therefore \text{Area} = \frac{1}{2}\left(x - \frac{1}{2}\right) + x\left(x - \frac{1}{2}\right)$

$$= \frac{1}{2}x - \frac{1}{4} + x^2 - \frac{1}{2}x$$
$$= \left(x^2 - \frac{1}{4}\right)\text{m}^2$$

c i



The equation of the parabola can be expressed as $y = ax^2$.
The point R is on both the line CD and the parabola.

When $x = 1$, $\qquad y = \frac{1}{2}$

$\therefore \qquad \frac{1}{2} = a \times 1$

$\therefore \qquad a = \frac{1}{2}$

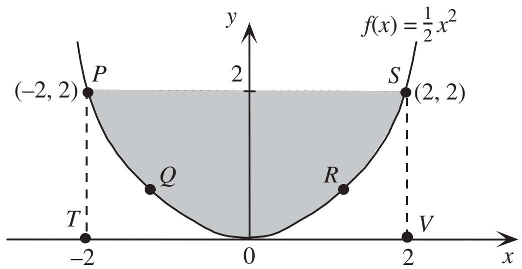
and $\qquad y = \frac{1}{2}x^2$

For the coordinates of P and S consider

$$\frac{1}{2}x^2 = 2$$
$$x^2 = 4$$
$$x = \pm 2$$

The coordinates of S are (2, 2) and the coordinates of P are (−2, 2).

ii



$$\text{Area of shaded region} = \text{Area}_{PTVS} - \int_{-2}^{2} f(x)\,dx$$

$$= 4 \times 2 - \int_{-2}^{2} \frac{1}{2}x^2 dx$$

$$= 8 - \frac{1}{2}\left[\frac{1}{3}x^3\right]_{-2}^{2}$$

$$= 8 - \frac{1}{6}[(2)^3 - (-2)^3]$$

$$\therefore \quad \text{Area} = 8 - \frac{1}{6}(8 + 8)$$

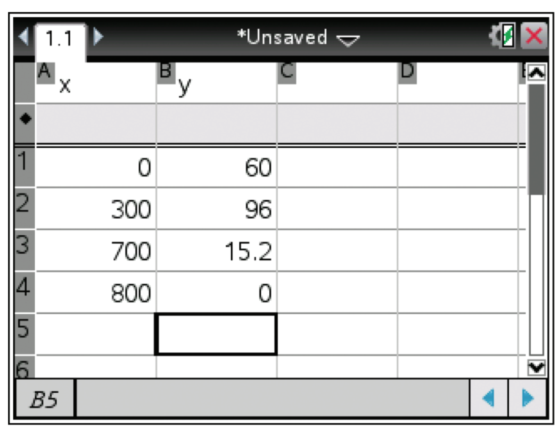
$$= 8 - \frac{8}{3}$$

$$= \frac{16}{3}$$

$$= 5\frac{1}{3}$$

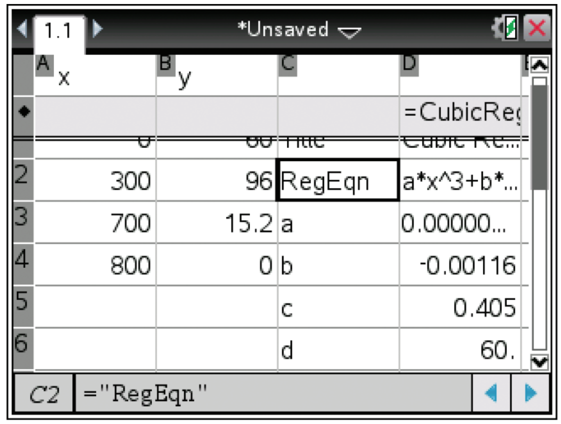
Area of the shaded region is $5\frac{1}{3}$ square metres.

8 a TI: Enter the coordinate points into a new Lists & Spreadsheet application as shown right. Press **Menu→4:Statistics→1:Stat Calculations→7:Cubic Regression** and set **X List** to ***x*** and **Y List** to ***y*** then ENTER.



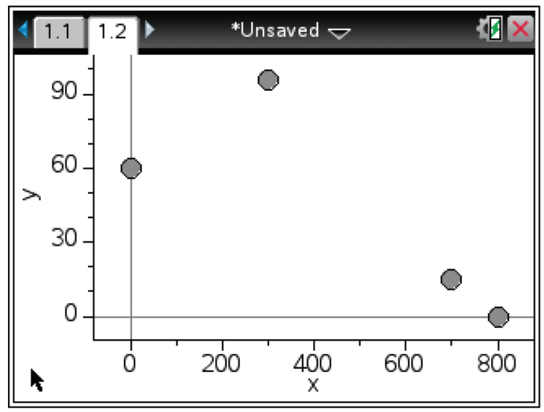
	A x	B y	C	D
◆				
1	0	60		
2	300	96		
3	700	15.2		
4	800	0		
5				
6				

CP: Enter the coordinate points into a new Statistics application. Tap **Calc→Cubic Reg** and set **XList** to **listl** and **YList** to **list2** then EXE.

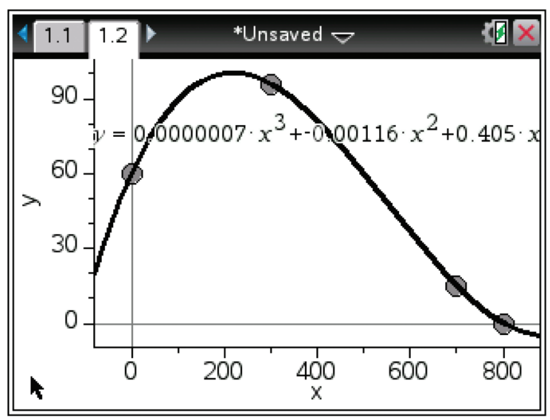


$y = (7 \times 10^{-7})x^3 - 0.001\ 16x^2 + 0.405x + 60$

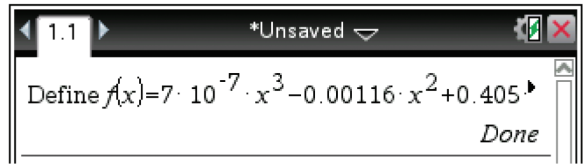
b **TI:** Open a Data & Statistics application. Add the variable ***x*** to the horizontal and the variable ***y*** to the vertical. Press **Menu→4:Analyze→6:Regression→ 5:Show Cubic**. Now Press **Menu→4:Analyze→A:Graph Trace** and move the cursor to the maximum to find the maximum height of 100 metres, to the nearest metre.



CP: After completing part **a**. the graph will be shown on the screen. Tap **Analysis→Trace** and move the cursor to the maximum to find the maximum height of 100 metres, to the nearest metre.

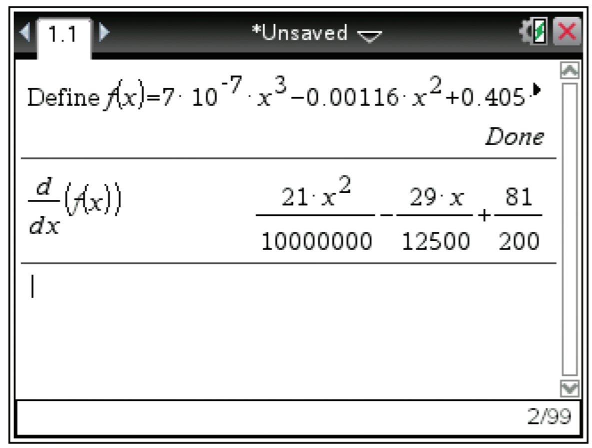


c **i** **TI:** In a Calculator application press **Menu→1:Actions→1:Define**. Type $\mathbf{f(x) = 7 \times 10^{\wedge -}7x^{\wedge}3 -}$ $\mathbf{.00116x^{\wedge}2 + .405x + 60}$ then ENTER. Using the mathematical template tool select the derivative template and complete as follows:

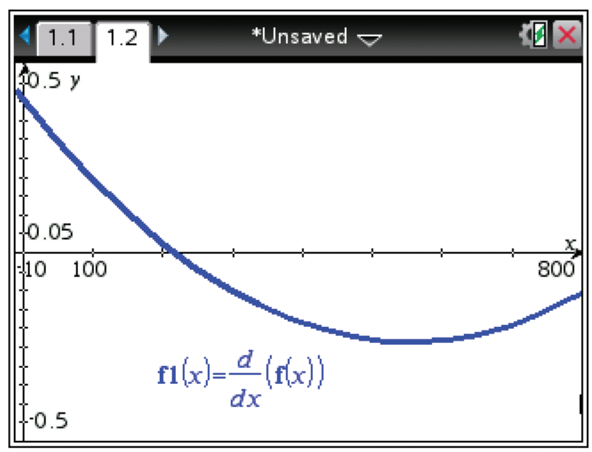


$$\frac{d}{dx}(f(x))$$

In a Graphs application input $\frac{d}{dx}(f(x))$ into $f1(x) =$ then ENTER

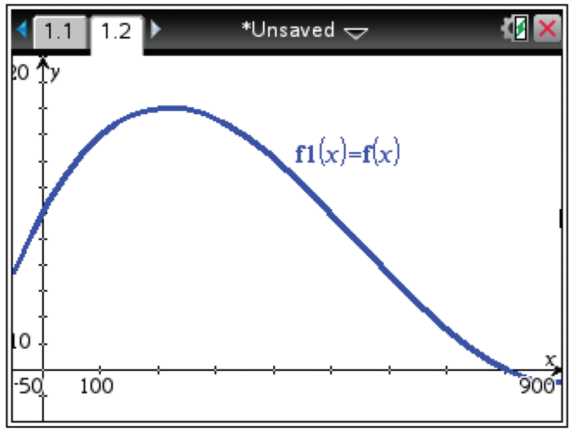


CP: In the Main application tap **Action→Command→Define** and type $\mathbf{f(x) = 7 \times 10^{\wedge -}7x^{\wedge}3 -}$ $\mathbf{.00116x^{\wedge}2 + .4\ 05x + 60}$ then EXE. In the Graph & Table application type $\frac{d}{dx}(f(x))$ into yl then EXE.

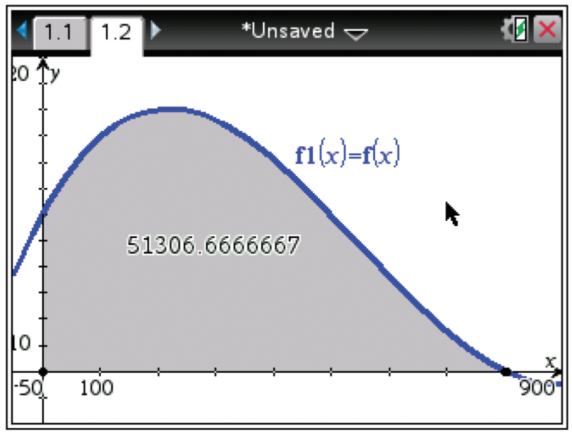


ii The gradient is greatest when $x = 0$, and is 0.405, at the point with coordinates $(0, 60)$.

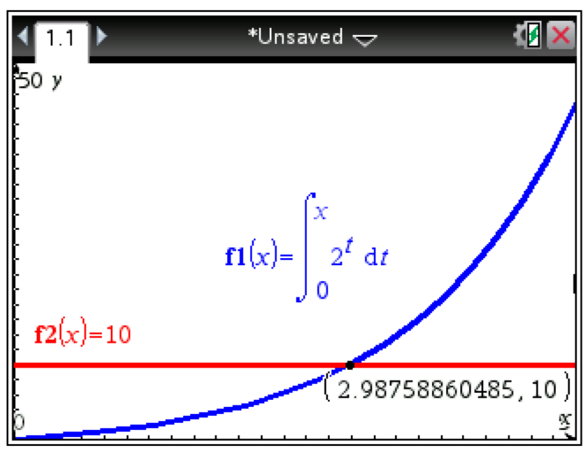
d Sketch the graph of $f1(x) = f(x)$.
TI: Press **Menu→6:Analyze Graph→7:Integral**. Enter 0 as the lower limit and 800 as the upper limit.



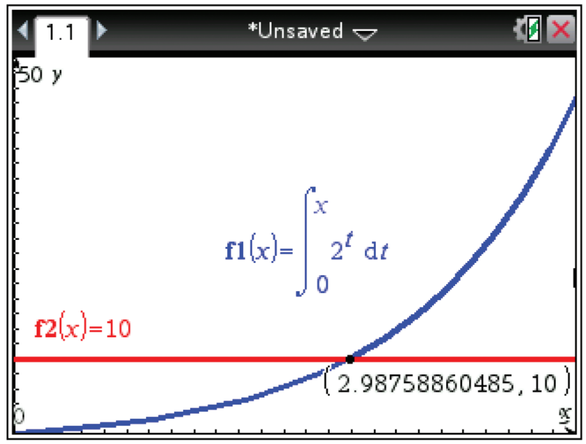
CP: Tap **Analysis→G-Solve→$\int$dx**.
Enter 0 as the lower limit and 800 as the upper limit.
The cross-sectional area is 51 307 m^2, to the closest square metre.



9 a **TI:** Type $\int_0^x 2^t dt$ into $f1(x)$ using the mathematical template tool.
CP: Type $\int_0^x 2^t dt$ into $y1$ using the 2D template tool.



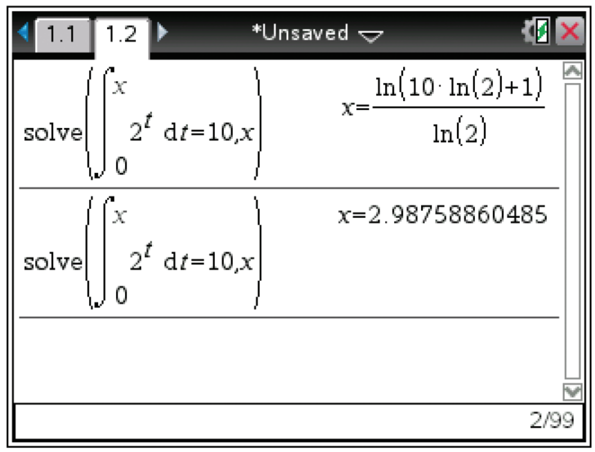
b Sketch the graph of $f2 = 10$.
TI: Press **Menu→6:Analyze Graph→4:Intersection**
CP: Tap **Analysis→ G-Solve→Intersect**
to find that $\int_0^x f(t)dt = 10$ is satisfied by $x = 2.988$, correct to 3 decimal places.



Altematively, in the Calculator screen type **solve** $\left(\int_0^x 2^t dt = 10, x\right)$ to give

$$x = \frac{\ln(10\ln(2) + 1)}{\ln(2)} = 2.988,$$

correct to 3 decimal places.



Chapter 22 – Revision of chapters 20–21

Solutions to Technology-free questions

1 a Left-endpoint estimate $f(0) = 1, f(1) = \frac{1}{2}, f(2) = \frac{1}{3}, f(3) = \frac{1}{4}$

Left-endpoint estimate$=1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$

b Right-endpoint estimate

$f(1) = \frac{1}{2}, f(2) = \frac{1}{3}, f(3) = \frac{1}{4}, f(4) = \frac{1}{5}$

Right-endpoint estimate$=$
$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60}$

c Trapezoidal estimate

Trapezoidal estimate$=\frac{1}{2}(1 + \frac{1}{2}) + \frac{1}{2}(\frac{1}{2} + \frac{1}{3}) + \frac{1}{2}(\frac{1}{3} + \frac{1}{4}) + \frac{1}{2}(\frac{1}{4} + \frac{1}{5}) = \frac{101}{60}$

2 a Let $y = 3x^{\frac{3}{2}}$

$\frac{dy}{dx} = 3 \times \frac{3}{2}x^{\frac{1}{2}} = \frac{9}{2}x^{\frac{1}{2}}$

b Let $y = \sqrt[5]{x} = x^{\frac{1}{5}}$

$\frac{dy}{dx} = \frac{1}{5}x^{-\frac{4}{5}}$

c Let $y = -\frac{2}{x^{\frac{5}{3}}} = -2x^{-\frac{5}{3}}$

$\frac{dy}{dx} = -2 \times -\frac{5}{3}x^{-\frac{8}{3}} = \frac{10}{3}x^{-\frac{8}{3}}$

d Let $y = 6x^{\frac{5}{3}}$

$\frac{dy}{dx} = 6 \times \frac{5}{3}x^{\frac{2}{3}} = 10x^{\frac{2}{3}}$

e Let $y = x^{-\frac{1}{5}}$

$\frac{dy}{dx} = -\frac{1}{5}x^{-\frac{6}{5}}$

f Let $y = x^{-\frac{2}{3}} - 2x^{\frac{3}{2}}$

$\frac{dy}{dx} = -\frac{2}{3}x^{-\frac{5}{3}} - 3x^{\frac{1}{2}}$

3 a Let $y = (3x + 5)^2$ Let $u = 3x + 5$.

Then $y = u^2$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dx}\\ &= 2u \times 3\\ &= 6u\\ &= 6(3x + 5)\end{aligned}$$

b Let $y = -(2x + 7)^4$

Let $u = 2x + 7$. Then $y = -u^4$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dx}\\ &= -4u^3 \times 2\\ &= -8u^3\\ &= -8(2x + 7)^3\end{aligned}$$

c Let $y = (5 - 2x)^{-\frac{1}{3}}$ Let $u = 5 - 2x$.

Then $y = u^{-\frac{1}{3}}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dx}\\ &= -\frac{1}{3}u^{-\frac{4}{3}} \times (-2)\\ &= \frac{2}{3}u^{-\frac{4}{3}}\\ &= \frac{2}{3}(5 - 2x)^{-\frac{4}{3}}\end{aligned}$$

d Let $y = \frac{4}{5 + 3x} = 4(5 + 3x)^{-1}$

Let $u = 5 + 3x$. Then $y = u^{-1}$

$$\frac{dy}{dx} = \frac{dy}{du}\ \frac{du}{dx}$$
$$= -4u^{-2} \times (3)$$
$$= -12u^{-2}$$
$$= -12(5 + 3x)^{-2}$$

e Let $y = \dfrac{1}{(x-1)^{\frac{2}{3}}}$

Let $u = x - 1$. Then $y = \dfrac{1}{u^{\frac{2}{3}}} = u^{-\frac{2}{3}}$

$$\frac{dy}{dx} = \frac{dy}{du}\ \frac{du}{dx}$$
$$= -\frac{2}{3}u^{-\frac{5}{3}} \times 1$$
$$= -\frac{2}{3}u^{-\frac{5}{3}}$$
$$= -\frac{2}{3}(x-1)^{-\frac{5}{3}}$$

f Let $y = \dfrac{3}{\sqrt{2 + 3x^2}} = -(2 + 3x^2)^{-\frac{1}{2}}$

Let $u = (2 + 3x^2)$. Then $y = 3u^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{dy}{du}\ \frac{du}{dx}$$
$$= -\frac{3}{2}u^{-\frac{3}{2}} \times 6x$$
$$= 9xu^{-\frac{3}{2}}$$
$$= 9x(2 + 3x^2)^{-\frac{3}{2}}$$

g Let $y = \left(2x^3 - \dfrac{5}{x}\right)^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{1}{3}\left(2x^3 - \frac{5}{x}\right)^{-\frac{2}{3}}\left(6x^2 + \frac{5}{x^2}\right)$$

4 $\dfrac{dx}{dt} = t + 4 - \dfrac{3}{t^2}$

$\therefore x = \dfrac{t^2}{2} + 4t + \dfrac{3}{t} + c$

When $x = 6, t = 1$

$\therefore 6 = \dfrac{1}{2} + 4 + 3 + c$

$\therefore c = -\dfrac{3}{2}$

$\therefore x = \dfrac{t^2}{2} + 4t + \dfrac{3}{t} - \dfrac{3}{2}$

5 **a** $y = x^{\frac{1}{3}}$

$\dfrac{dy}{dx} = \dfrac{1}{3}x^{-\frac{2}{3}}$

$\therefore$ gradient of tangent at $x = 27$ is $\dfrac{1}{27}$

b $y = \dfrac{1}{3x+1}$

$\dfrac{dy}{dx} = -\dfrac{3}{(3x+1)^2}$

$\therefore$ gradient of tangent at $x = 0$ is -3

c $y = \dfrac{2}{x^3}$

$\dfrac{dy}{dx} = -\dfrac{6}{x^4}$

$\therefore$ gradient of tangent at $x = 2$ is $-\dfrac{3}{8}$

6 $y = \dfrac{1}{x^2}$

$\dfrac{dy}{dx} = -\dfrac{2}{x^3}$

When $\dfrac{dy}{dx} = 4, -\dfrac{2}{x^3} = 4$

Therefore $y = \dfrac{1}{x^2}$ has gradient 4 at the point $\left(-\left(\dfrac{1}{2}\right)^{\frac{1}{3}}, 4^{\frac{1}{3}}\right)$

7 $y = \sqrt[3]{x} = x^{\frac{1}{3}}$

$\dfrac{dy}{dx} = \dfrac{1}{3}x^{-\frac{2}{3}}$

When $\dfrac{dy}{dx} = 2, \dfrac{1}{3}x^{-\frac{2}{3}} = 2$

Therefore $y = \sqrt[3]{x}$ has gradient 2 at the points $\left(\left(\dfrac{1}{6}\right)^{\frac{3}{2}}, \left(\dfrac{1}{6}\right)^{\frac{1}{2}}\right)$ and $\left(-\left(\dfrac{1}{6}\right)^{\frac{3}{2}}, -\left(\dfrac{1}{6}\right)^{\frac{1}{2}}\right)$

8 a $\int 3x^2 + 1\ dx = x^3 + x + c$

b
$$\int (t+1)(2-3t)\ dt = \int -3t^2 - t + 2\ dt$$
$$= -t^3 - \frac{t^2}{2} + 2t + c$$

c
$$\int \sqrt{x}\ dx = \int x^{\frac{1}{2}}\ dx$$
$$= \frac{2}{3}x^{\frac{3}{2}} + c$$

d
$$\int 2x^{\frac{3}{2}} + x^{\frac{1}{3}}\ dx = \frac{4}{5}x^{\frac{5}{2}} + \frac{3}{4}x^{\frac{4}{3}} + c$$

9 a
$$\int_1^3 x^{-2}\ dx = \Big[-x^{-1}\Big]_1^3$$
$$= -\frac{1}{3} + 1$$
$$= \frac{2}{3}$$

b
$$\int_{-3}^{-2} (1 - x^{-2})\ dx = \Big[x + x^{-1}\Big]_{-3}^{-2}$$
$$= (-2 - \frac{1}{2}) - (-3 - \frac{1}{3})$$
$$= \frac{5}{6}$$

10 a
$$\int_1^2 (-x^2 + 3x - 2)\ dx = \Big[-\frac{x^3}{3} + \frac{3x^2}{2} - 2x\Big]_1^2$$
$$= (-\frac{8}{3} + 6 - 4) - (-\frac{1}{3} + \frac{3}{2} - 2)$$
$$= \frac{1}{6}$$

b Two regions - one positive and one negative
$$\int_0^1 (x^3 - 3x^2 + 2x)\ dx = \Big[\frac{x^4}{4} - x^3 + x^2\Big]_0^1$$
$$= (\frac{1}{4} - 1 + 1) - (0)$$
$$= \frac{1}{4}$$
$$\int_1^2 (x^3 - 3x^2 + 2x)\ dx = \Big[\frac{x^4}{4} - x^3 + x^2\Big]_1^2$$
$$= (4 - 8 + 4) - (\frac{1}{4} - 1 + 1)$$
$$= -\frac{1}{4}$$

Total area $= \frac{1}{2}$

Solutions to multiple-choice questions

1 E $f(x) = (9x^2 + 4)^{\frac{1}{2}}$

$$\therefore f'(x) = \frac{1}{2}(18x)(9x^2 + 4)^{-\frac{1}{2}}$$
$$= 9x(9x^2 + 4)^{-\frac{1}{2}}$$

2 C $f(x) = (3x^2 - 7)^4$

$$\therefore f'(x) = 4(6x)(3x^2 - 7)^3$$
$$= 24x(3x^2 - 7)^3$$

3 E $\dfrac{d}{dx}\dfrac{2}{3 + x} = -\dfrac{2}{(3 + x)^2}$

4 D $\dfrac{d}{dx}\dfrac{x - 1}{\sqrt{x}} = \dfrac{d}{dx}(x^{\frac{1}{2}} - x^{\frac{1}{2}})$

$$= \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$$
$$= \frac{x + 1}{2x\sqrt{x}}$$

5 A $\int 3x^2 + 6dx = x^3 + 6x + c$

6 A $\displaystyle\int_1^3 x - 2\mathrm{d}x = \left[\frac{1}{2}x^2 - 2x\right]_1^3$

$$= \frac{1}{2}(3^2 - 1^2) - 2(3 - 1) = 0$$
$$= 0$$

7 E $g'(x) = f(x)$

$$\therefore g(x) = \int f(x)\,dx$$
$$\therefore \int_1^3 f(x)\,dx = g(3) - g(1)$$

8 C

$$f(x) = x(x + 2) = x^2 + 2x$$
$$A = \int_0^1 x^2 + 2xdx - \int_{-2}^0 x^2 + 2xdx$$
$$= \int_0^1 x^2 + 2xdx + \int_0^{-2} x^2 + 2xdx$$

9 E

$$\int_1^4 5f(x) + 2\,dx = 5\int_1^4 f(x)\,dx + \int_1^4 2dx$$
$$= 5\int_1^4 f(x)\,dx + 6$$

10 C $\displaystyle\int_0^1 k(1 - x^2)\,dx = 40$

$$\therefore (1 - 0) - \frac{1}{3}(1^3 - 0) = \frac{40}{k}$$
$$\frac{40}{k} = \frac{2}{3}$$
$$k = 60$$

11 B $y = 5x - x^2 <\therefore y(1) = y(4) = 4$

Rectangle has area = 4(3) = 12

$$A = \int_1^4 5x - x^2\,\mathrm{d}x - 12$$
$$= \left[\frac{5}{2}x^2 - \frac{1}{3}x^3\right]_1^4$$
$$= \frac{5}{2}(4^2 - 1^2) - \frac{1}{3}(4^3 - 1^3) - 12$$
$$= \frac{75}{12} - \frac{63}{3} - 12$$
$$= 4.5$$

12 C $\displaystyle\int 4x^2(2x + 1)dx = \int 8x^3 + 4x^2dx$

$$= 2x^4 + \frac{4}{3}x^3 + c$$

13 C $\displaystyle A = \int_0^3 f(x)\,dx - \int_3^7 f(x)\,dx$

Chapter 25 – Mathematical Methods Units 1 and 2 Revision of Chapters 1-22 Solutions

Solutions to Technology-free questions

1
$$\begin{aligned} 2x + 3(4 - x) &= 8 \\ 2x + 12 - 3x &= 8 \\ -x &= -4 \\ x &= 4. \end{aligned}$$

2
$$\begin{aligned} \frac{at + b}{ct + d} &= 2 \\ at + b &= 2ct + 2d \\ (a - 2c)t &= 2d - b \\ t &= \frac{2d - b}{a - 2c}. \end{aligned}$$

3
$$\begin{aligned} \frac{4x}{3} - 4 &\le 2x - 3 \\ -4 + 3 &\le 2x - \frac{4x}{3} \\ -1 &\le \frac{2x}{3} \\ -3 &\le 2x \\ x &\ge \frac{3}{2}. \end{aligned}$$

4 **a** For $x - y$ to be as small as possible choose the smallest possible value of x and the largest possible value of y.
Thus take $x = -4$ and $y = 8$.
Hence the smallest value of $x - y$ is -12.

b The largest possible value of $\frac{x}{y}$ is achieved by making x as large as possible and y as small as possible. Thus take $x = 6$ and $y = 2$.
Hence the largest value of $\frac{x}{y}$ is 3.

c The largest possible value of $x^2 + y^2$ is obtained by choosing values of the largest magnitude for both x and y.

Thus take $x = 6$ and $y = 8$.

Hence the largest possible value of $x^2 + y^2$ is 100.

5 Let x be the number of the first type book and y be the number of the other type of book.

There is a total of 20 books. So

$x + y = 20 \qquad (1)$

There is total cost of \$720. So

$72x + 24y = 720 \qquad (2)$

Multiply equation (1) by 24.

$24x + 24y = 480 \qquad (3)$

Subtract equation (3) from equation (2).

$48x = 240$

$x = 5.$

Hence $x = 5$ and $y = 15$.

There were 5 of one type of book and 15 of the other.

6
$$\begin{aligned}\frac{1-5x}{3} &\geq -12\\ 1 - 5x &\geq -36\\ -5x &\geq -37\\ x &\leq \frac{37}{5}.\end{aligned}$$

7 $a = \dfrac{y^2 - xz}{10}$

When $x = -5, y = 7$ and $z = 6$,

$$\begin{aligned}a &= \frac{7^2 + 5 \times 6}{10}\\ &= \frac{79}{10}.\end{aligned}$$

8 **a** Midpoint $M(xy)$: $x = \dfrac{8+a}{2}$ and $y = \dfrac{14+b}{2}$

b If (5, 10) is the midpoint,

$\dfrac{8+a}{2} = 5$ and $\dfrac{14+b}{2} = 10$. Hence $a = 2$ and $b = 6$.

9 a The line passes through $A(-2, 6)$ and $B(10, 15)$.

Using the form $y - y_1 = m(x - x_1)$,

$$m = \frac{15 - 6}{10 - (-2)}$$

$$= \frac{9}{12}$$

$$= \frac{3}{4}$$

The equation is thus,

$$y - 6 = \frac{3}{4}(x + 2)$$

Simplifying,

$$4y - 24 = 3x + 6$$

$$4y - 3x = 30.$$

b When $x = 0, y = \dfrac{15}{2}$

When $y = 0,\ x = -10$

By Pythagoras's theorem,

$$\text{the length of } PQ = \sqrt{(-10 - 0)^2 + \left(0 - \frac{15}{2}\right)^2}$$

$$= \sqrt{100 + \left(-\frac{225}{4}\right)}$$

$$= \sqrt{\frac{625}{4}}$$

$$= \frac{25}{2}.$$

10 a $A = (-7, 6)$ and $B = (11, -5)$.

The midpoint $M(x, y)$ of AB has coordinates,

$$x = \frac{-7 + 11}{2} \text{ and } y = \frac{6 + (-5)}{2}$$

The midpoint is $M(2, \frac{1}{2})$.

b The distance between A and B $= \sqrt{(11 - (-7)) + (-5 - 6)}$

$$= \sqrt{18^2 + 11^2}$$

$$= \sqrt{324 + 121}$$

$$= \sqrt{445}$$

c The equation of AB.

Gradient, $m = \dfrac{-5-6}{11-(-7)} = -\dfrac{11}{18}$.

Using the form $y - y_1 = m(x - x_1)$.

$y - 6 = -\dfrac{11}{18}(x + 7)$

Simplifying,

$18y - 108 = -11x - 77.$

$18y + 11x = 31.$

d The gradient of a line perpendicular to line AB is $\dfrac{18}{11}$.

The midpoint of AB is $M\left(2, \dfrac{1}{2}\right)$.

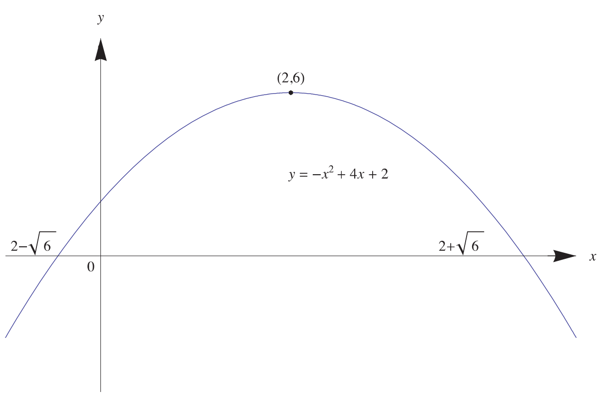
Using the form $y - y_1 = m(x - x_1)$.

$y - \dfrac{1}{2} = \dfrac{18}{11}(x - 2)$

$22y - 11 = 36x - 72$

$22y - 36x + 61 = 0.$

11



12 A parabola has turning point $(2, -6)$.

It has equation of the form $y = k(x - 2)^2 - 6$.

It passes through the point $(6, 12)$.

Hence,

$12 = k(4)^2 - 6$

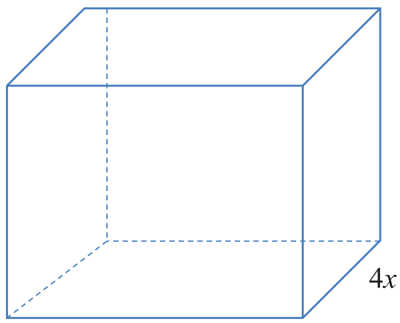
$18 = 16k$

$k = \dfrac{9}{8}.$

Hence the equation is $y = \dfrac{9}{8}(x - 2)^2 - 6$.

13 Let $P(x) = ax^3 + 4x^2 + 3$. It has remainder 3 when divided by $x - 2$.
The remainder theorem gives us that:
$P(2) = 3$.
That is, $3 = 8a + 16 + 3$.
Hence $a = -2$.

14



a The length of the wire is 6000 cm.
We have: $4 \times 5x + 4 \times 4x + 4w = 6000$

$$5x + 4x + w = 1500$$

$$w = 1500 - 9x.$$

b Let V cm^3 be the volume of cuboid.

$$\begin{aligned} V &= 5x \times 4x \times x \\ &= 20x^2(1500 - 9x) \end{aligned}$$

c We have $0 \leq x \leq \dfrac{500}{3}$ since $w = 1500 - 9x > 0$.

d If $x = 100$,

$$\begin{aligned} V &= 20 \times 100^2(1500 - 9 \times 100) \\ &= 200\,000 \times 600 \\ &= 120\,000\,000 \text{ cm}^3 \end{aligned}$$

15 **a** Probability of both red $= \dfrac{4}{9} \times \dfrac{4}{9} = \dfrac{16}{81}$.

b Probability of both red $= \dfrac{4}{9} \times \dfrac{7}{17} = \dfrac{28}{153}$.

16

	Box 1			
Box 2		1	3	5
	2	3	5	7
	4	5	7	9
	6	7	9	11

Sample space = $\{3, 5, 7, 9, 11\}$

The outcomes are not equally likely.

Pr(divisible by 3) = $\frac{1}{3}$.

17 There are six letters and three vowels.

a The probability that the letter withdrawn is a vowel = $\frac{1}{2}$.

b The probability that the letter is a vowel is $\frac{1}{3}$.

18 This can be done simply by considering the cases.

SSF or SFF are the only two possibilities.

$$\begin{aligned}\text{The probability of fruit on Wednesday} &= 0.4 \times 0.6 + 0.6 \times 0.3\\ &= 0.24 + 0.18\\ &= 0.42.\end{aligned}$$

19 Solve $\cos(3x) = \frac{1}{2}$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$3x = \ldots, -\frac{7\pi}{3}, -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \ldots$$

$$x = -\frac{\pi}{9}, \frac{\pi}{9}.$$

20 The graph of $y = ax^3 + bx + c$ has intercepts (0,6) and $(-2, 0)$ and has a stationary point where $x = 1$.

$\frac{dy}{dx} = 3ax^2 + b$.

a The graph passes through (0,6). Therefore $6 = c$.

b The graph passes through $(-2, 0)$. Therefore

$$0 = -8a - 2b + 6 \qquad (1)$$

There is a stationary point where $x = 1$. Therefore

$$0 = 3a + b \qquad (2)$$

c Multiply (2) by 2.

$$0 = 6a + 2b \qquad (3)$$

Add equations (1) and (3).

$$0 = -2a + 6$$

Therefore $a = 3$. Substitute in (2) to find $b = -9$.

21 $y = x^4$ and so $\dfrac{dy}{dx} = 4x^3$.

The gradient of the line $y = -32x + a$ is -32.

$4x^3 = -32$ implies $x^3 = -8$.

Hence $x = -2$

For $y = x^4$, when $x = -2$, $y = 16$.

Therefore for the tangent $y = -32x + a$,

$16 = -32 \times (-2) + a$.

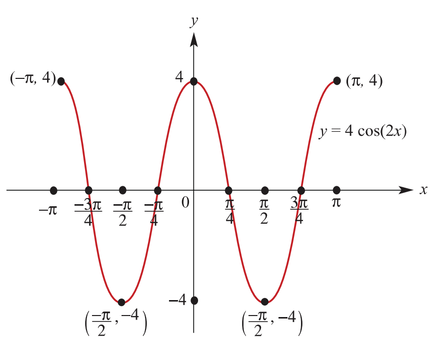
Hence $a = -48$.

22 $f\colon [-\pi, \pi] \to R, f(x) = 4\cos(2x)$.

a Period $= \dfrac{2\pi}{2} = \pi$

Amplitude $= 4$

b



23 **a** The first ball can be any ball except 1. The probability of a 3, 5 or 7 is $\dfrac{3}{4}$.

There are 3 balls left and the probability of obtaining the white ball is $\dfrac{1}{3}$.

The probability of white on the second $= \dfrac{3}{4} \times \dfrac{1}{3} = \dfrac{1}{4}$.

b The sum of 8 can be obtained from the following ordered pairs: $(1, 7), (7, 1), (3, 5), (5, 3)$.

The probability of each of these pairs $= \dfrac{1}{4} \times \dfrac{1}{3} = \dfrac{1}{12}$.

Therefore the probability of obtaining a sum of 8 $= 4 \times \dfrac{1}{12} = \dfrac{1}{3}$

c We can see that for a sum of 8 we must only consider the pairs (1, 7), (7, 1), (3, 5), (5,3). The probability that the second is 1 is $\frac{1}{4}$.

24 The line $y = x + 1$ cuts the circle $x^2 + y^2 + 2x - 4y + 1 = 0$ at the points A and B.
To find the points of intersection:

$y = x + 1 \qquad (1)$

$x^2 + y^2 + 2x - 4y + 1 = 0 \qquad (2)$

Substitute from (1) into (2)

$$x^2 + (x + 1)^2 + 2x - 4(x + 1) + 1 = 0$$

$$x^2 + x^2 + 2x + 1 + 2x - 4x - 4 + 1 = 0$$

$$2x^2 - 2 = 0$$

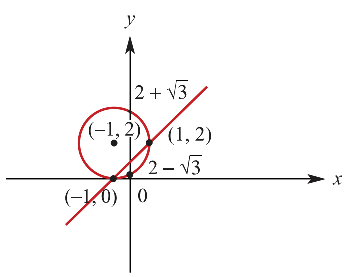
$$2(x^2 - 1) = 0$$

$$x = 1 \text{ or } x = -1$$

The points of intersection are $A(1, 2)$ and $B(-1, 0)$.

a The midpoint of $AB = (0, 1)$.

b



25 **a** $4^x - 5 \times 2^x - 24 = 0$.
Let $a = 2^x$.
The equation becomes.

$$a^2 - 5a - 24 = 0$$

$$(a - 8)(a + 3) = 0$$

$$a = 8 \text{ or } a = -3.$$

Now $2^x > 0$ for all x and so there are no solutions of $2^x = -3$.
$2^x = 8$ implies $x = 3$.

b $2^{5-3x} = -4^{x^2} = 0$

$$2^{5-3x} = 2^{2x^2}$$

We note that if $2^a = 2^b$ then $a = b$.

Hence,

$5 - 3x = 2x^2$

$2x^2 + 3x - 5 = 0$

$(2x + 5)(x - 1) = 0.$

So, $x = -\frac{5}{2}$ or $x = 1$.

26 $\frac{dy}{dx} = -4x + k$, where k is a constant. Stationary point at $(1, 5)$.

$\frac{dy}{dx} = 0$ when $x = 1$.

$$0 = -4 + k$$

$$k = 4$$

Thus,

$$\frac{dy}{dx} = -4x + 4$$

Integrating with respect to x.

$$y = -2x^2 + 4x + c.$$

When $x = 1,\ y = 5$.

Hence,

$$5 = -2 + 4 + c$$

$$c = 3.$$

The equation is $y = -2x^2 + 4x + 3$.

27 $y = ax^3 - 2x^2 - x + 7$ has a gradient of 4, when $x = -1$.

$\frac{dy}{dx} = 3ax^2 - 4x - 1.$

Therefore,

$4 = 3a + 4 - 1$

$a = \frac{1}{3}.$

28 Let $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

Hence $x' = 3x$ and $y' = -2y$.

Thus $x = \frac{x'}{3}$ and $y = -\frac{y'}{2}$.

Therefore $y = x^2$ maps to $-\frac{y}{2} = \frac{x^2}{9}$.

That is $y = -\frac{2x^2}{9}$.

29 Let $P(x) = 3x^2 + x + 10$.

$P(-b) = 3b^2 - b + 10$ and $P(2b) = 12b^2 + 2b + 10$.

By the remainder theorem,

$$3b^2 - b + 10 = 12b^2 + 2b + 10$$

$$9b^2 + 3b = 0$$

$$3b(3b + 1) = 0.$$

Hence, $b = -\frac{1}{3}$.

30 $y = x^3$ and $y = x^3 + x^2 + 6x + 9$

The curves meet where

$x^3 = x^3 + x^2 + 6x + 9$.

That is,

$0 = (x + 3)^2$

Thus $x = -3$ and $y = -27$.

For the first curve,

$\frac{dy}{dx} = 3x^2$

and the second,

$\frac{dy}{dx} = 3x^2 + 2x + 6$.

When $x = -3$, $\frac{dy}{dx} = 27$ for the first curve.

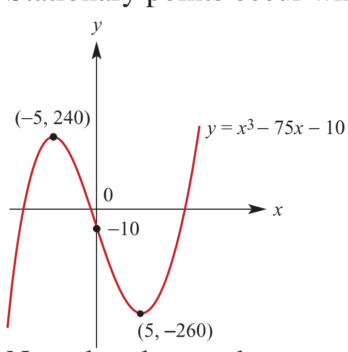
When $x = -3$, $\frac{dy}{dx} = 27$ for the second curve.

There is a common tangent to the two curves.

31 **a** $y = x^3 - 75x - 10$

$\frac{dy}{dx} = 3x^2 - 75 = 3(x^2 - 25)$.

Stationary points occur when $x = 5$ or $x \neq -5$



Note that the graph crosses the y-axis at -10.

b $x^3 - 75x - 10 = p$ has more than one real solution when the line with equation $y = p$ crosses the graph of $y = x^3 - 75x - 10$ more than once.

This is true when $-260 \le p \le 240$.

32 **a** Maximal domain $R \backslash \{3\}$.

b Maximal domain $R \backslash \{2\}$.

c Maximal domain $(-\infty,\ 2]$

d Maximal domain $[4, \infty)$

e Maximal domain $(-\infty,\ 5)$

Solutions to multiple-choice questions

1 **B** Period $= 2\pi \div \dfrac{1}{4} = 8\pi$
Amplitude $= 5$

2 **A** $f(x) = x^2 + 2x$
Average rate of change for the interval $[0, 3] = \dfrac{f(3) - f(0)}{3 - 0}$

$$= \frac{15 - 0}{3}$$

$$= 5$$

3 **D** $f\colon [1, 4) \to R, f(x) = (x - 2)^2 + 3.$
End points: $f(1) = 4$ and $f(4) = 7$.
The minimum value $= f(2) = 3$.
The range $= [3, 7)$.

4 **D** A function g with domain R has the following properties:

- $g'(x) = 3x^2 - 4x$
- the graph of $y = g(x)$ passes through the point $(1, 0)$

Taking the anti-derivative of g with respect to x:
$g(x) = x^3 - 2x + c$.
Also $g(1) = 0$,
so $0 = -1 + c$
$c = 1$.
Hence $g(x) = x^3 - 2x + 1$.

5 **D** Simultaneous equations

$$(m - 2)x + y = 0 \qquad (1)$$

$$2x + (m - 3)y = 0 \qquad (2)$$

Gradient of line (1) $= 2 - m$
Gradient of line (2) $= \dfrac{2}{3 - m}$
Infinitely many solutions implies $2 - m = \dfrac{2}{3 - m}$.

$$\text{Hence } (2-m)(3-m) = 2$$
$$6 - 5m + m^2 = 2$$
$$m^2 - 5m + 4 = 0$$
$$(m-1)(m-4) = 0$$
$$m = 4 \text{ or } m = 1.$$

Both lines go through the origin and so there are infinitely many solutions for $m = 4$ or $m = 1$.

6 C $f(x) = 2\log_e(3x)$. If $f(5x) = \log_e(y)$
First $f(5x) = 2\log_e(15x)$.
Hence $2\log_e(15x) = \log_e(y)$
Thus $y = 15x^2 = 225x^2$.

7 C A bag contains 2 white balls and 4 black balls. Three balls are drawn from the bag without replacement.
Probability of black on the first $= \frac{2}{3}$.
There are now 2 white balls and 3 black balls.
Probability black on the second $= \frac{3}{5}$.
There are now 2 white balls and 2 black balls.
Probability of black on the third $= \frac{1}{2}$.
Probability of three black $= \frac{2}{3} \times \frac{3}{5} \times \frac{1}{2} = \frac{1}{5}$

8 A $f: R \to R, f(x) = \frac{1}{3}x^3 - 2x^2 + 1$
$f'(x) = x^2 - 4x = x(x-4)$
$f'(x) < 0$ if and only if $x(x-4) < 0$.
This happens when the factors have different signs:
So either,
$x < 0$ and $x > 4$ or $x > 0$ and $x < 4$.
Only the second of these is possible.
Hence $0 < x < 4$.
This can also be seen by looking at the graph of $y = f'(x)$.

9 B $f(x) = \sqrt{2x+1}$ is defined for $2x + 1 \geq 0$.
That is the maximal domain of f is $x \geq -\frac{1}{2}$.
In interval notation $[-\frac{1}{2}, \infty)$.

10 A In algebraic notation, 11 is four times 9 more than x is written as $11 = 4(x + 9)$

11 B Time taken by the car = $\dfrac{120}{a}$ hours.

Time taken by the train = $\dfrac{120}{a-4}$.

Time taken by the train = time taken by the car +1.

Therefore,

$$\frac{120}{a-4} = \frac{120}{a} + 41$$

Multiplying both sides of the equation by $a(a-4)$ we have,

$$120a = 120(a-4) + a(a-4)$$

$$120a = 120a - 480 + a^2 - 4a$$

$$0 = a^2 - 4a - 480$$

$$0 = (a-24)(a+20)$$

Therefore $a = 24$ or $a = -20$.

But we assume positive speed.

12 A The parabola that passes through the point $(-3, 12)$ and has its vertex at $(-2, 8)$ has equation of the form:

$y = k(x+2)^2 + 8$.

It passes through the point $(-3, 12)$.

Hence,

$$12 = k(-1)^2 + 8$$

$k = 4$.

The equation is

$y = 4(x+2)^2 + 8$.

13 A $f: [-3, 5) \to R,\ f(x) = 5 - 2x$.

The graph of f is a straight line with negative gradient.

$f(-3) = 11$ and $f(5) = -5$.

The range is $(-5, 11]$

14 E $f: [-3, 2) \to R,\ f(x) = 2x^2 + 7$.

The graph is a parabola with a minimum at $(0, 7)$.

$f(-3) = 25$ and $f(2) = 15$.

The range is $[7, 25]$

15 B For (A), $y = 11x(x-1)$. It has a turning point at $x = \dfrac{1}{2}$. It is increasing on the given interval. Therefore, one-to-one.

For (B), $y = \sqrt{11 - x^2}$. The function has the same value for 1 and -1 for example.

It is not one-to-one.

16 C The function with rule $f(x) = mx + 2,\ m > 0$, has an inverse function with rule $f^{-1}(x) = ax + b,\ a,\ b \in R$.

We consider

$x = mf^{-1}(x) + 2.$

$f^{-1}(x) = \dfrac{x}{m} - \dfrac{2}{m}$

Here $\dfrac{1}{m} > 0$ and $-\dfrac{2}{m} < 0$.

Hence $a > 0$ and $b < 0$.

17 B $x + 1$ is a factor of $x^2 + ax + b$, then $-a + b + 7$ equals?

By the factor theorem,

$1 - a + b = 0.$

Thus $-a + b = -1$

$-a + b + 7 = -1 + 7 = 6$

18 A The choices are all cubic functions of the form $y = a(x + h)^3 + b$ where $a < 0$.

The graph shows a stationary point of inflexion at $(-1, 2)$

Hence it is of the form $y = a(x + 1)^3 + 2$.

It passes through the origin.

19 C $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

This implies, $x' = 5x + 7$ and $y' = 3y + 1$.

$x = \dfrac{x' - 7}{5}$ and $y = \dfrac{y' - 1}{3}$.

Hence $y = x^2$ is mapped to

$\dfrac{y' - 1}{3} = \dfrac{(x' - 7)^2}{25}$

$y' = \dfrac{3(x' - 7)^2}{25} + 1$

$25y' = 3(x' - 7)^2 + 25$

20 A The transformation $T : R^2 \to R^2$ maps the curve with equation $y = 5^x$ to the curve with equation $y = 5^{(2x+4)} - 3$

$y' = 5^{(2x'+4)} - 3$

$y' + 3 = 5^{(2x'+4)}$

Hence take $y' + 3 = y$ and $2x' + 4 = x$.
Thus,
$y' = y - 3$ and $x' = \dfrac{x-4}{2}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

This can be written as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

21 B $f : R \to R,\ f(x) = x$.
$f(x) - f(-x) = x - (-x) = 2x$.

22 B The tangent at the point $(1, 5)$ on the graph of the curve $y = f(x)$ has equation $y = 6 + x$.
The tangent at the point $(3, 6)$ on the curve
$y = f(x - 2) + 1$ has equation??
The transformation is ‘2 to the right’ and ‘1 up’.
So the tangent at the point (3, 6) on the curve
$y = f(x - 2) + 1$ is a translation of $y = 6 + x$.
It transforms to:
$y - 1 = 6 + x - 2$.
That is,
$y = 5 + x$.

23 The graph of the of the derivative function f' of the cubic function with rule $y = f(x)$ crosses the x axis at $(1, 0)$ and $(-3, 0)$. The maximum value of the derivative function is 12.
$f'(x) = k(x - 1)(x + 3)$.
The maximum value of the derivative function is 12. This occurs when $x = 2$.
This tells us that $k < 0$ as the quadratic has a maximum.
The turning points of the cubic occur at $x = 1$ and $x = -3$.
For a local maximum we look where the gradient changes from positive to negative going from left to right.
$f'(x) = k(x - 1)(x + 3)$.
For $x < 1,\ f'(x) > 0$ (Remember $k < 0$)
For $x > 1,\ f'(x) < 0$.
Hence there is a local maximum where $x = 1$.

24 **D** The random variable X has the following probability distribution

X	0	2	4
$\Pr(X = x)$	a	b	0.1

The mean of X is 2.

We have,

$a + b + 0.1 = 1$ (Probability distribution)

That is, $a + b = 0.9$ (1)

and

$2b + 0.4 = 2$ (2)

From (2), $b = 0.8$.

From (1), $a = 0.1$.

25 **A** Let $f'(x) = 5g'(x) + 4$ and $f(1) = 5$ and $g(x) = x^2 f(x)$.

We have $f(x) = 5g(x) + 4x + c$ and $f(1) = 5$ and $g(1) = f(1) = 5$

So $5 = 25 + 4 + c$.

Hence $c = -24$.

Finally, $f(x) = 5g(x) + 4x - 24$

26 **C** $25^x - 7 \times 5^x + 12 = 0$.

Let $a = 5^x$

$a^2 - 7a + 12 = 0$

$(a - 3)(a - 4) = 0$

Hence $a = 3$ or $a = 4$.

That is $5^x = 3$ or $5^x = 4$.

Therefore,

$x = \log_5 3$ or $x = \log_5 4$

27 **B** A particle moves in a straight line so that its position s m relative to O at a time t seconds ($t > 0$), is given by $s = 4t^3 - 5t - 10$.

The velocity $\dfrac{ds}{dt} = v = 12t^2 - 5$.

The acceleration $= \dfrac{dv}{dt} = 24$.

When $t = 1$, the acceleration is 24 m/s^2.

28 **A** The average rate of change of the function $y = 2x^4 + x^3 - 1$ between $x = -1$ and $x = 1$ is equal to $\dfrac{2 - 0}{2} = 1$.

29 **E** A function $f: R \to R$ is such that

- $f'(x) = 0$ where $x = 3$
- $f'(x) = 0$ where $x = 5$
- $f'(x) > 0$ where $3 < x < 5$
- $f'(x) < 0$ where $x > 5$
- $f'(x) < 0$ where $x < 3$

Stationary points when $x = 3$ and $x = 5$.
Immediately to the left of 3, $f'(x) < 0$ and immediately to the right of 3, $f'(x) > 0$.
Therefore there is a local minimum at $x = 3$.
Immediately to the left of 5, $f'(x) > 0$ and immediately to the right of 5, $f'(x) < 0$.
Therefore there is a local maximum at $x = 5$.

30 **C** The number of pets X a family has is a random variable with the following probability distribution.

x	0	1	2	3
$\Pr(X = x)$	0.3	0.2	0.4	0.1

The probability that two families have the same number of pets is equal to
$0.3 \times 0.3 + 0.2 \times 0.2 + 0.4 \times 0.4 + 0.1 \times 0.1 = 0.09 + 0.04 + 0.16 + 0.01 = 0.30$

Solutions to extended-response questions

1 **a** **i** Gradient of $AB = \dfrac{16 - b^2 - 16}{b}$

$= -b$

ii $f'(x) = -2x$

The tangent at point $(x, f(x))$ has gradient $-2x$

$-2x = -b$

$x = \dfrac{b}{2}$

The tangent at the point where $x = \dfrac{b}{2}$ has gradient $-b$.

b **i** Area of a trapezium $= \dfrac{h}{2}(a + b)$

$$S(b) = \frac{b}{2}(16 - b^2 + 16)$$

$$= \frac{b}{2}(32 - b^2)$$

ii
$$\frac{b}{2}(32 - b^2) = 28$$

$$32b - b^3 = 56$$

$$b^3 - 32b + 56 = 0$$

$$(b - 2)(b^2 + 2b - 28) = 0$$

$$b = 2 \text{ or } (b + 1)^2 = 29$$

Thus $S(2) = 28$.

The other solutions of the equation are not in the interval $(0, 4)$.

The area of the trapezium is 28 when $b = 2$.

2 $f(x) = (\sqrt{x} - 2)^2(\sqrt{x} + 1)^2$

a To find the x-intercept

$f(x) = 0$ implies $\sqrt{x} - 2 = 0$ or $\sqrt{x} + 1 = 0$

Thus $\sqrt{x} - 2 = 0$ which implies $x = 4$.

Therefore $a = 4$.

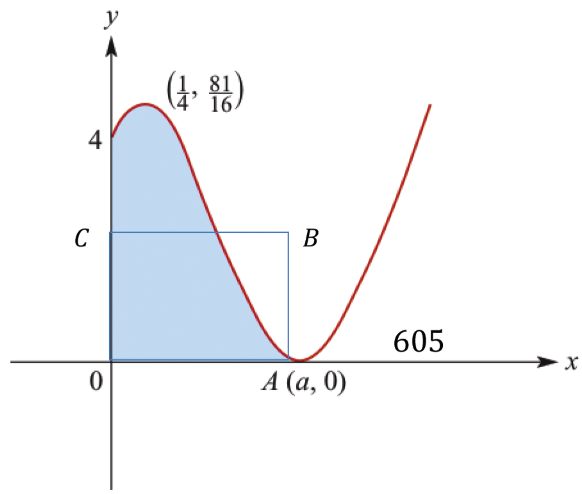
b From the graph there is a local maximum at $x = \dfrac{1}{4}$ and a local minimum at $A(4, 0)$.

The graph has negative gradient for the interval $\left(\dfrac{1}{4}, 4\right)$

c $OABC$ is a rectangle with $OA = BC = 4$.
Let $OC = AB = x$.

$$4x = \frac{136}{15}$$
$$x = \frac{34}{15}$$

That is, $\quad OC = \frac{34}{15}$



3 a $v = 4t - 6$
Find an expression for x the position at time t by finding the antiderivative.
$x = 2t^2 - 6t + c$
When $t = 0$, $x = 0$ and therefore $c = 0$.
$x = 2t^2 - 6t$

b $x(3) = 18 - 18$
$= 0$
The body has returned to O after 3 seconds.

c The average velocity $= \dfrac{\text{change in position}}{\text{change in time}} = 0$ cm/s

d Body reverses direction when $v = 0$.

$$4t - 6 = 0$$
$$t = \frac{3}{2}$$

$x\left(\frac{3}{2}\right) = 2 \times \frac{9}{4} - 6 \times \frac{3}{2} = -\frac{9}{2}$.
The particle returns to O at $t = 3$.
The total distance travelled = 9 centimetres.

e Average speed $= \dfrac{9}{3} = 3$ centimetres/second.

4 $f(x) = -x^3 + ax^2$.
$f'(x) = -3x^2 + 2ax$

a **i** f has negative gradient for $f'(x) < 0$

$$-3x^2 + 2ax < 0$$

$$-x(3x - 2a) < 0$$

$$x < 0 \text{ or } x > \frac{2a}{3}$$

ii f has positive gradient for $f'(x) > 0$

$$0 < x < \frac{2a}{3}$$

b When $x = a$,

$$f'(a) = -3a^2 + 2a^2 \qquad \text{and} \qquad f(a) = 0$$

$$= -a^2$$

The equation of the tangent at the point $(a, f(a))$

$$y - f(a) = f'(a)(x - a)$$

Thus, $y = -a^2(x - a)$

c The gradient of the normal $= -\dfrac{1}{f'(a)}$

$$= \frac{1}{a^2}$$

The equation of the normal is

$$y = \frac{1}{a^2}(x - a)$$

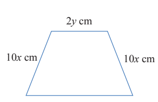
d Area $= \displaystyle\int_0^a (-x^3 + ax^2)dx$

$$= \left[\frac{-x^4}{4} + \frac{ax^3}{3}\right]_0^a$$

$$= \frac{-a^4}{4} + \frac{a^4}{3}$$

$$= \frac{a^4}{12}$$

5



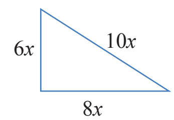
a $16x + 2y + 2y + 10x + 10x = 52$

$$36x + 4y = 52$$

$$4y = 52 - 36x$$

$$y = 13 - 9x$$

b Using Pythagoras' theorem.



Height $s = 6x$ cm

$$A(x) = \frac{6x}{2}(2y + 16x + 2y) \qquad \text{(Using the formula for the area of a trapezium)}$$

$$= 3x(4y + 16x)$$

$$= 3x(52 - 36x + 16x) \qquad \text{(Substituting for } y \text{ from part } \mathbf{a}\text{)}$$

$$= 156x - 60x^2$$

c Finding the derivative:

$A'(x) = 156 - 120x$

$A'(x) = 0$ implies $x = \frac{13}{10}$. A maximum occurs at this value as

$A(x) = 156x - 60x^2$ is a quadratic with negative coefficient of x^2.

Substituting for x in $y = 13 - 9x$ gives $y = \frac{13}{10}$.

6 **a** Total area of the two squares $= x^2 + y^2$, $\quad x \le y$

Total length of fencing $= 2x + 3y$

Given that the length of the fencing is 5200 m

$$x + 3y = 5200$$

$$3y = 5200 - 2x$$

$$y = \frac{5200 - 2x}{3}$$

Therefore the total area, $\quad A = x^2 + \left(\frac{5200 - 2x}{3}\right)^2$

b

$$A = x^2 + \frac{5200^2}{9} - \frac{20\,800x}{9} + \frac{4x^2}{9}$$

$$= \frac{13x^2}{9} - \frac{20\,800x}{9} + \frac{5200^2}{9}$$

$$A'(x) = \frac{26x}{9} - \frac{20\,800}{9}$$

$A'(x) = 0$ implies $x = 800$

The parabola has positive coefficient of x^2 and

therefore a minimum when $x = 800$.

When $x = 800$ substituting in $y = \frac{5200 - 2x}{3}$ gives $y = 1200$.

c $x \geq 0$ and $x \leq y$

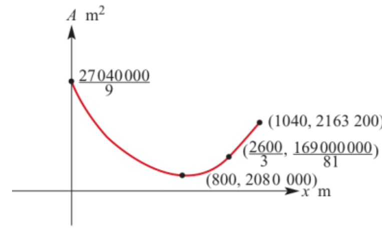
Substitute $y = \dfrac{5200 - 2x}{3}$

$$\frac{5200 - 2x}{3} \geq x$$

$$5200 - 2x \geq 3x$$

$$5200 \geq 5x$$

$$x \leq 1040$$



7 **a**

(0, 1)	(0, 3)	(0, 5)	(0, 7)	(0, 9)	(0, 11)
(2, 1)	(2, 3)	(2, 5)	(2, 7)	(2, 9)	(2, 11)
(4, 1)	(4, 3)	(4, 5)	(4, 7)	(4, 9)	(4, 11)
(6, 1)	(6, 3)	(6, 5)	(6, 7)	(6, 9)	(6, 11)
(8, 1)	(8, 3)	(8, 5)	(8, 7)	(8, 9)	(8, 11)
(10, 1)	(10, 3)	(10, 5)	(10, 7)	(10, 9)	(10,11)

36 outcomes

b Table showing sums

	0	2	4	6	8	10
1	1	3	5	7	9	11
3	3	5	7	9	11	13
5	5	7	9	11	13	15
7	7	9	11	13	15	17
9	9	11	13	15	17	19
11	11	13	15	17	19	21

Let X be the sum of the results.

i $\Pr(X = 1) = \dfrac{1}{36}$

ii $\Pr(X = 13) = \dfrac{5}{36}$

iii $\Pr(X = 9) = \dfrac{5}{36}$

c $$\Pr(X = 15 | X > 7) = \frac{\Pr(X = 15)}{\Pr(X > 7)} = \frac{2}{13}$$

8 **a** $AB = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & -2 \\ -5 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} -2 & 5 & 5 \\ 0 & 6 & 2 \\ 0 & 0 & -4 \end{bmatrix}$

b $\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & -2 \\ -5 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \\ 2 \end{bmatrix}$

c Multiply both sides of the equation by A

$$\begin{bmatrix} 2 & -1 & 1 \\ 4 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & -2 \\ -5 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ -7 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 5 & 5 \\ 0 & 6 & 2 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 23 \\ 18 \\ -12 \end{bmatrix}$$

d

$$-2x + 5y + 5z = 23 \qquad \boxed{1}$$

$$6y + 2z = 18 \qquad \boxed{2}$$

$$-4z = -12 \qquad \boxed{3}$$

From $\boxed{3}$, $z = 3$

From $\boxed{2}$, $y = 2$

From $\boxed{1}$, $x = 1$

9 **a** For function to be defined $x - 2a \geq 0$. That is $x \geq 2a$.

b $\sqrt{x - 2a} = x$

$x - 2a = x^2$ Squaring both sides

$x^2 - x + 2a = 0$

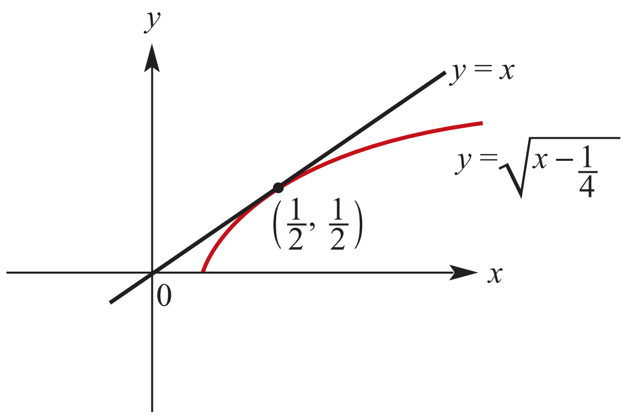
$x^2 - x + \frac{1}{4} = -2a + \frac{1}{4}$ Completing the square

$\left(x - \frac{1}{2}\right)^2 = \frac{-8a + 1}{4}$

$x = \frac{1}{2} + \frac{\sqrt{1 - 8a}}{2}$ or $x = \frac{1}{2} - \frac{\sqrt{1 - 8a}}{2}$

c The equation $f(x) = x$ has one solution for $a = \frac{1}{8}$

d



10 a Probability that Frederick goes to the library on each of the next three nights

$= 0.7 \times 0.7 \times 0.7$

$= 0.343$

b The possible sequences for 3 days for exactly two days going to the library:

LSLL LLSL LLLS

Probability of exactly two nights

$= 0.3 \times 0.6 \times 0.7 + 0.7 \times 0.3 \times 0.6 + 0.7 \times 0.7 \times 0.3$

$= 0.126 + 0.126 + 0.147$

$= 0.399$

11 a Probability that sticks are brought from Platypus for the next three years

$= 0.75 \times 0.75 \times 0.75$

$= 0.4219$ (correct to four decimal places.)

b The possible sequences for three years for exactly two years buying from Platypus

PNPP PPNP PPPN

Probability of buying from Platypus for exactly two of the three years

$= 0.25 \times 0.2 \times 0.75 + 0.75 \times 0.25 \times 0.2 + 0.75 \times 0.75 \times 0.25$

$= 0.2156$ (correct to four decimal places.)

c Probability that they will buy from Platypus in the third year is 0.6125

12 a The line has negative gradient.

The range $= [-mb + 3, \ -ma + 3]$

b Interchanging x and y and solving for y.

$$x = -my + 3$$
$$my = -x + 3$$
$$y = -\frac{1}{m}x + \frac{3}{m}$$

The equation of the inverse function is $f^{-1}(x) = \frac{-1}{m}x + \frac{3}{m}$

c The coordinates of the midpoint are found by using the midpoint of the line segment joining (x_1, y_1) to $(x_2,\ y_2)$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

The midpoint of $AB = \left(\frac{a + b}{2}, \frac{-ma - mb + 6}{2}\right)$

d Gradient line through $AB = -m$

Gradient of a line perpendicular to $AB = \frac{1}{m}$

$$y - \left(\frac{-ma - mb + 6}{2}\right) = \frac{1}{m}\left(x - \left(\frac{a + b}{2}\right)\right)$$
$$2my + m^2a + m^2b - 6m = 2x - (a + b)$$
$$2my - 2x = -m^2(a + b) + 6m - (a + b)$$

e The transformation is defined by $(x, y) \to (x - 3, y + 5)$.

If $(x, y) \to (x', y')$ then

$x' = x - 3$ and $y' = y + 5$

Hence, $x = x' + 3$ and $y = y' - 5$

Substituting in $y = -mx + 3$ gives

$$y' - 5 = -m(x' + 3) + 3$$
$$y' = -mx' - 3m + 8$$

The equation of the image is $y = -mx - 3m + 8$

Considering the end points:

$(a, -ma + 3) \to (a - 3,\ -ma + 8)$

and

$(b, -mb + 3) \to (b - 3, -mb + 8)$.

f The transformation is defined by $(x, y) \to (-x, y)$.

If $(x, y) \to (x', y')$ then

$x' = -x$ and $y' = y$

The line is transformed to $y' = mx' + 3$.

That is, $y = mx + 3$

Considering the end points:
$(a, -ma + 3) \to (-a, -ma + 3)$
and
$(b, -mb + 3) \to (-b, -mb + 3)$.

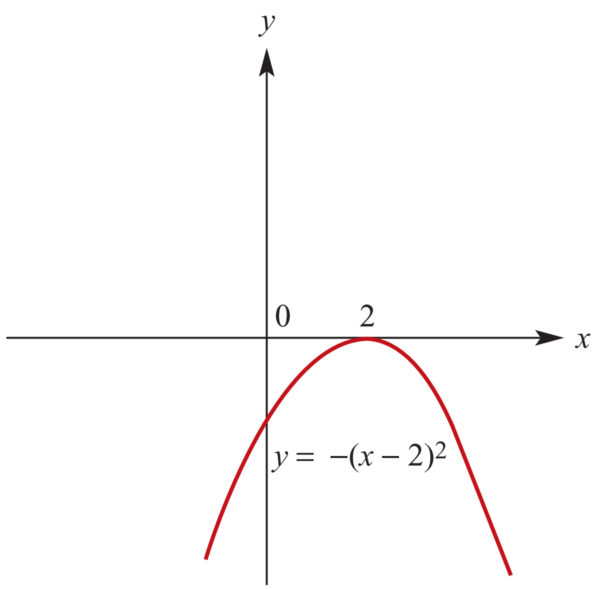
g If $a = 0$ the midpoint of AB has coordinates $\left(\frac{b}{2}, \frac{-mb+6}{2}\right)$

Thus $\frac{b}{2} = 6$ and $\frac{-mb+6}{2} = -4$

Hence $b = 12$ and $m = \frac{7}{6}$

13 a $f(x) = (p-1)x^2 + 4x + (p-4)$

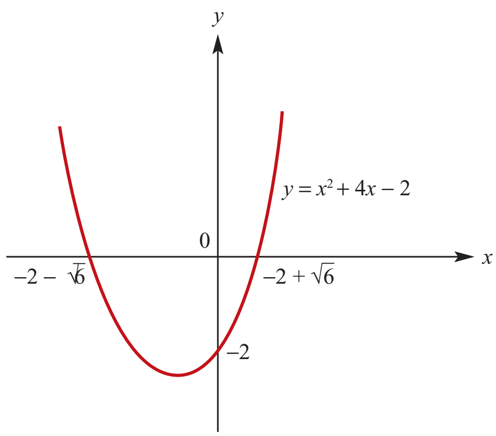
i When $p = 0$, $f(x) = -x^2 + 4x - 4 = -(x^2 - 4x + 4) = -(x-2)^2$



ii When $p = 2$, $f(x) = x^2 + 4x - 2$

For the x axis intercepts:

$$x^2 + 4x - 2 = 0$$
$$x^2 + 4x + 4 = 6$$
$$(x+2)^2 = 6$$
$$x = -2 + \sqrt{6} \text{ or } x = -2 - \sqrt{6}$$



b $f'(x) = 2(p - 1)x + 4$

$f'(x) = 0$ implies $x = \dfrac{2}{1 - p}$

and $f\left(\dfrac{2}{1-p}\right) = (p - 1) \times \dfrac{4}{(1 - p)^2} + \dfrac{8}{1 - p} + (p - 4)$

$$= \frac{4}{(p - 1)} + \frac{8}{1 - p} + (p - 4)$$

$$= \frac{-4}{(p - 1)} + (p - 4)$$

$$= \frac{5p - p^2}{p - 1}$$

The coordinates of the turning point are $\left(\dfrac{2}{1 - p}, \dfrac{5p - p^2}{p - 1}\right)$

c The turning point lies on the x axis when the y coordinate is zero.

That is, when $5p - p^2 = 0$.

$p = 0$ or $p = 5$

d The discriminant of the quadratic $(p - 1)x^2 + 4x + (p - 4)$ is

$$16 - 4(p - 1)(p - 4) = 16 - 4[p^2 - 5p + 4]$$
$$= -4p^2 + 20p$$

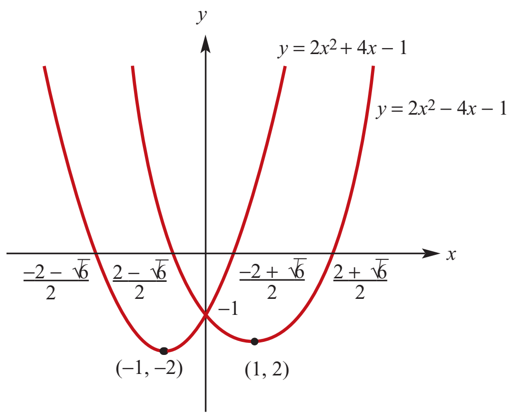
There are two solutions when the discriminant is positive.

That is, when $-4p^2 + 20p > 0$

Equivalently when $p^2 - 5p < 0$

Thus $0 < p < 5$ and $p \neq 1$.

e The question should ask to sketch the graph of $y = g(x)$ and the graph of the reflection in the y-axis.



14 $h(t) = 2.3\cos(kt)$

a High tide occurs every 12 hours

$$\frac{2\pi}{k} = 12$$

$$k = \frac{\pi}{6}$$

b
$$\begin{aligned} h(1.5) &= 2.3\cos\left(\frac{\pi}{6} \times 1.5\right) \\ &= 2.3\cos\left(\frac{\pi}{4}\right) \\ &= 2.3 \times \frac{1}{\sqrt{2}} \end{aligned}$$

This is measured in metres

Thus the height of the road above mean sea level is:

$$\begin{aligned} 2.3 \times \frac{1}{\sqrt{2}} \text{ metres } &= 1.15\sqrt{2} \times 100 \text{ cm} \\ &= 115\sqrt{2} \text{ cm} \end{aligned}$$

c
$$\begin{aligned} h(1.5) &= 2.3\cos\left(\frac{\pi}{6}\right) \\ &= 2.3 \times \frac{\sqrt{3}}{2} \end{aligned}$$

Thus the height of the raised footpath above mean sea level is:

$$\begin{aligned} 2.3 \times \frac{\sqrt{3}}{2} \text{ metres } &= 1.15\sqrt{3} \times 100 \text{ cm} \\ &= 115\sqrt{3} \text{ cm} \end{aligned}$$

Chapter 1 – Functions and relations

Solutions to Exercise 1A

1 a $\{8, 11\}$

b $\{8, 11\}$

c $\{1, 3, 8, 11, 18, 22, 23, 24, 25, 30\}$

d $\{3, 8, 11, 18, 22, 23, 24, 25, 30, 32\}$

e $\{3, 8, 11, 18, 22, 23, 24, 25, 30, 32\}$

f $\{1, 8, 11, 25, 30\}$

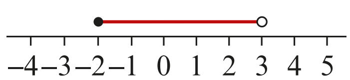
2 a $\{3, 18, 22, 23, 24\}$

b $\{25, 30, 32\}$

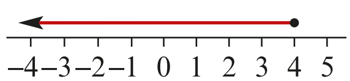
c $\{3, 18, 22, 23, 24\}$

d $\{1, 25, 30\}$

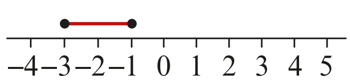
3 a



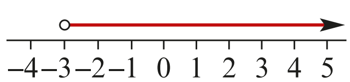
b



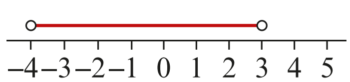
c



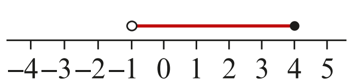
d



e



f



4 a $X \cap Y = \{7, 9\}$

b $X \cap Y \cap Z = \{7, 9\}$

c $X \cup Y = \{2, 3, 5, 7, 9, 11, 15, 19, 23\}$

d $X \backslash Y = \{2, 3, 5, 11\}$

e $Z \backslash Y = \{2\}$

f $X \cap Z = \{2, 7, 9\}$

g $[-2, 8] \cap X = \{2, 3, 5, 7\}$

h $(-3, 8] \cap Y = \{7\}$

i $(2, \infty) \cap Y = \{7, 9, 15, 19, 23\}$

j $(3, \infty) \cup Y = (3, \infty)$

5 a $X \cap Y = \{a, e\}$

b $X \cup Y = \{a, b, c, d, e, i, o, u\}$

c $X \backslash Y = \{b, c, d\}$

d $Y \backslash X = \{i, o, u\}$

6 a $B \cap C = \{6\}$

b $B \backslash C = \{2, 4, 8, 10\}$

c $A \backslash B = \{1, 3, 5, 7, 9\}$

d $A \backslash B = \{1, 3, 5, 7, 9\}$
$A \backslash C = \{2, 4, 5, 7, 8, 10\}$
$(A \backslash B) \cup (A \backslash C) = \{1, 2,$
$3, 4, 5, 7, 8, 9, 10\}$

e $B \cap C = \{6\}$
$A \backslash (B \cap C) = \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$

f $A \backslash B = \{1, 3, 5, 7, 9\}$
$A \backslash C = \{2, 4, 5, 7, 8, 10\}$

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$(A \backslash B) \cap (A \backslash C) = \{5, 7\}$

g $B \cup C = \{1, 2, 3, 4, 6, 8, 9, 10\}$
$A \backslash (B \cup C) = \{5, 7\}$

h $A \cap B \cap C = \{6\}$

7 a $[-3, 1)$

b $(-4, 5]$

c $(-\sqrt{2}, 0)$

d $(-\frac{1}{\sqrt{2}}, \sqrt{3})$

e $(-\infty, -3)$

f $(0, \infty)$

g $(-\infty, 0)$

h $[-2, \infty)$

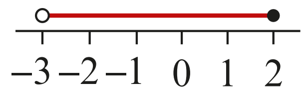
8 a $(-2, 3)$

b $[-4, 1)$

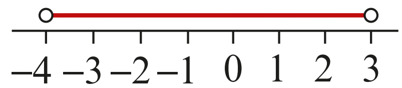
c $[-1, 5]$

d $(-3, 2]$

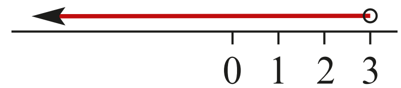
9 a



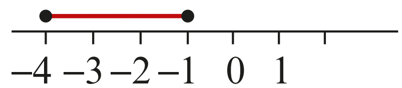
b



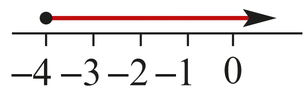
c



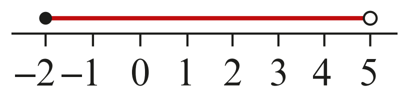
d



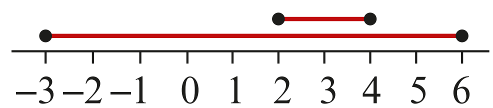
e



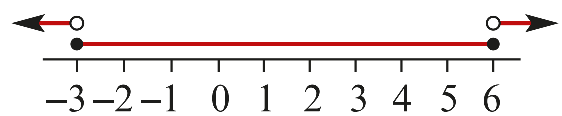
f



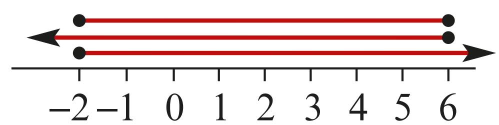
10 a



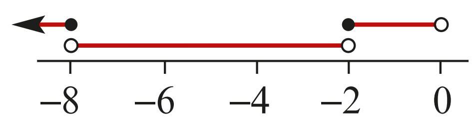
b



c



d



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Solutions to Exercise 1B

1 a Domain = $\mathbb{R}$

range = $[-2, \infty)$

b Domain = $(-\infty, 2]$

range = $\mathbb{R}$

c Domain = $(-2, 3)$

range = $[0, 9)$

d Domain = $(-3, 1)$

range = $(-6, 2)$

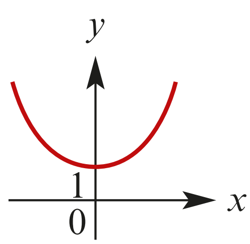
e Domain = $[-4, 0]$

range = $[0, 4]$

f Domain = $\mathbb{R}$

range = $(-\infty, 2)$

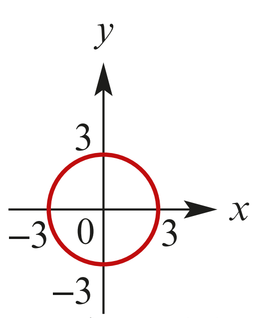
2 a



Domain = $\mathbb{R}$

range = $[1, \infty)$

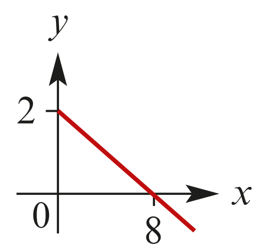
b



Domain = $[-3, 3]$

range = $[-3, 3]$

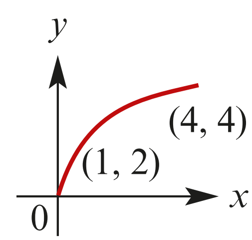
c



Domain = $\mathbb{R}^+ \cup \{0\}$

range = $(-\infty, 2]$

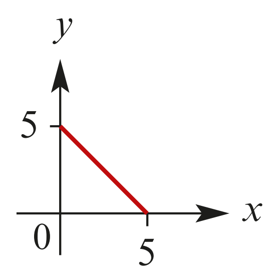
d



Domain = $[0, \infty)$

range = $[0, \infty)$

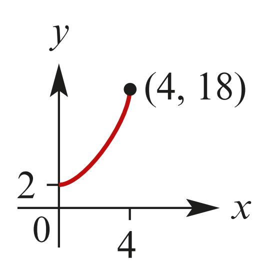
e



Domain = $[0, 5]$

range = $[0, 5]$

f



Domain = $[0, 4]$

range = $[2, 18]$

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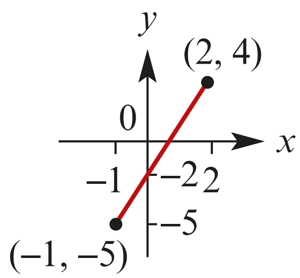
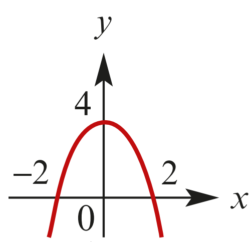
g 

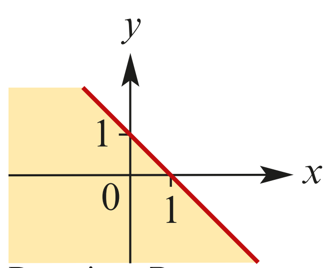
Domain = $[-1, 2]$

range = $[-5, 4]$

h 

Domain = $\mathbb{R}$

range = $(-\infty, 4]$

i 

Domain = $\mathbb{R}$

Range = $\mathbb{R}$

3 **a** not a function

Domain = $\{-1, 1, 2, 3\}$

range = $\{1, 2, 3, 4\}$

b function

Domain = $\{-2, -1, 0, 1, 2\}$

range = $\{-4, -1, 0, 3, 5\}$

c not a function

Domain = $\{-2, -1, 2, 4\}$

range = $\{-2, 1, 2, 4, 6\}$

d function

Domain = $\{-1, 0, 1, 2, 3\}$

range = $\{4\}$

4 **a** function Domain = $\mathbb{R}$

range = $\{4\}$

b not a function

Domain = $\{2\}$

range = $\mathbb{Z}$

c function

Domain = $\mathbb{R}$

range = $\mathbb{R}$

d not a function

Domain = $\mathbb{R}$

range = $\mathbb{R}$

e not a function

Domain = $[-4, 4]$

range = $[-4, 4]$

5 $f(x) = 2x^2 + 4x;$

$g(x) = 2x^3 + 2x - 6$

a $f(-1) = 2(-1)^2 + 4(-1) = -2$

$f(2) = 2(2)^2 + 4(2) = 16$

$f(-3) = 2(-3)^2 + 4(-3) = 6$

$f(2a) = 2(2a)^2 + 4(2a) = 8a^2 + 8a$

b

$$g(-1) = 2(-1)^3 + 2(-1) - 6 = -10$$

$$g(2) = 2(2)^3 + 2(2) - 6 = 14$$

$$g(3) = 2(3)^3 + 2(3) - 6 = 54$$

$$\begin{aligned} g(a-1) &= 2(a-1)^3 + 2(a-1) - 6 \\ &= 2(a^3 - 3a^2 + 3a - 1) + 2a - 8 \\ &= 2a^3 - 6a^2 + 8a - 10 \end{aligned}$$

6 $g(x) = 3x^2 - 2$

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a $g(-2) = 3(-2)^2 - 2 = 10$

$g(4) = 3(4)^2 - 2 = 46$

b **i** $g(-2) = 3(-2)^2 - 2 = 12x^2 - 2$

ii $g(x-2)^2 = 3(x-2)^2 - 2 = 3x^2 - 12x + 10$

iii $g(x+2)^2 = 3(x+2)^2 - 2 = 3x^2 + 12x + 10$

iv $g(x^2) = 3(x^2)^2 - 2 = 3x^4 - 2$

7 $f(x) = 2x - 3$

a $f(3) = 2(3) - 3 = 3$

b $f(x) = 11$

$11 = 2x - 3x = 7$

c $f(x) = 4x$

$4x = 2x - 3$

$2x = -3$

$x = \dfrac{-3}{2}$

d $f(x) > x$

$2x - 3 > x$

$x > 3$

8 $g(x) = 6x + 7 \quad h(x) = 3x - 2$

a $6x + 7 = 3x - 2$

$3x = -9$

$x = -3$

b $6x + 7 > 3x - 2$

$3x > -9$

$x > -3$

c $3x - 2 = 0$

$x = \dfrac{2}{3}$

9 **a** $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x + 3$

b $3y + 4x = 12$

$3y = 12 - 4x$

$y = 4 - \dfrac{4x}{3}$

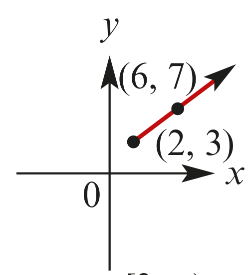
$f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \dfrac{-4x}{3} + 4$

c $f : [0, \infty) \to \mathbb{R}$ where $f(x) = 2x - 3$

d $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2 - 9$

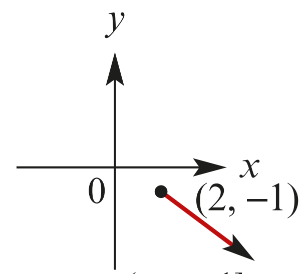
e $f : [0, 2] \to \mathbb{R}$ where $f(x) = 5x - 3$

10 **a**



range = $[3, \infty)$

b



range = $(-\infty, -1]$

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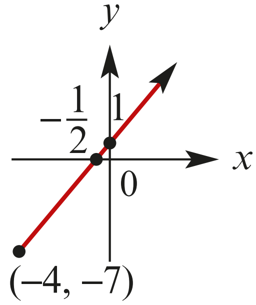
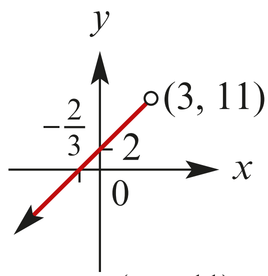
c



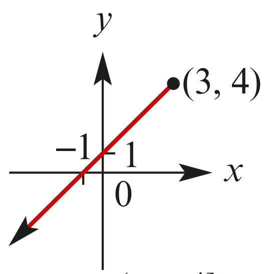
range $= [-7, \infty)$

d



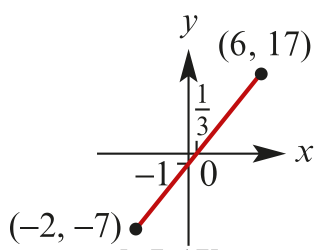
range $= (-\infty, 11)$

e



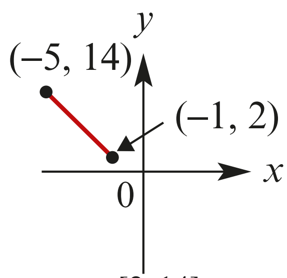
range $= (-\infty, 4]$

f



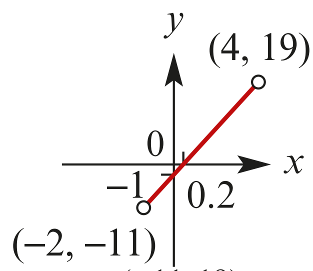
range $= [-7, 17]$

g



range $= [2, 14]$

h



range $= (-11, 19)$

11 $f(x) = 2x^2 - 6x + 1; \quad g(x) = 3 - 2x$

a $f(2) = 2(2)^2 - 6(2) + 1 = -3$

$f(-3) = 2(-3)^2 - 6(-3) + 1 = 37$

$f(-2) = 2(-2)^2 - 6(-2) + 1 = 21$

b $g(-2) = 3 - 2(-2) = 7$

$g(1) = 3 - 2(1) = 1$

$g(-3) = 3 - 2(-3) = 9$

c **i** $f(a) = 2a^2 - 6a + 1$

ii $f(a+2) = 2(a+2)^2 - 6(a+2) + 1$
$= 2a^2 + 2a - 3$

iii $g(-a) = 3 + 2a$

iv $g(2a) = 3 - 4a$

v $f(5-a) = 2(5-a)^2 - 6(5-a) + 1$
$= 2a^2 - 14a + 21$

vi $f(2a) = 8a^2 - 12a + 1$

vii $g(a) + f(a) = (2a^2 - 6a + 1) + (3 - 2a)$
$= 2a^2 - 8a + 4$

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viii $g(a) - f(a) = (3 - 2a)$

$$\begin{aligned} &\quad - (2a^2 - 6a + 1) \\ &= -2a^2 + 4a + 2 \end{aligned}$$

12 $f(x) = 3x^2 + x - 2$

a

$$f(x) = 0$$

$$3x^2 + x - 2 = 0$$

using the quadratic formula

$$x = \frac{-(1) \pm \sqrt{(1)^2 - 4(3)(-2)}}{2(3)}$$

$$x = \frac{-1 \pm \sqrt{25}}{6}$$

$$x = -1, \frac{2}{3}$$

in set notation

$$\left\{-1, \frac{2}{3}\right\}$$

b

$$f(x) = x$$

$$3x^2 + x - 2 = x$$

$$3x^2 = 2$$

$$x^2 = \frac{2}{3}$$

$$x = \pm\sqrt{2/3}$$

in set notation

$$\left\{-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right\}$$

c

$$f(x) = -2$$

$$3x^2 + x - 2 = -2$$

$$3x^2 + x = 0$$

$$x(3x + 1) = 0$$

$\therefore$ *either* $x = 0$ *or* $3x + 1 = 0$

$$x = 0, \frac{-1}{3}$$

in set notation

$$\left\{0, \frac{-1}{3}\right\}$$

d $f(x) > 0$

$3x^2 + x - 2 > x$

from (a), the x-intercepts are $-1, \dfrac{2}{3}$

as the coefficient of $x^2 > 0$

the shape of the graph $y = f(x)$ is

$\therefore$ $f(x) > 0$ for

$x \in (-\infty, -1) \cup (\frac{2}{3}, \infty)$

e

$$f(x) > x$$

$$3x^2 + x - 2 > x$$

$$3x^2 - 2 > 0$$

$$x^2 > \frac{2}{3}$$

$$x \in \left(-\infty, \frac{-\sqrt{2}}{\sqrt{3}}\right) \cup \left(\frac{\sqrt{2}}{\sqrt{3}}, \infty\right)$$

f $f(x) \leq -2$

$3x^2 + x - 2 \leq -2$

from (c), the x-intercepts are $-\dfrac{1}{3}, 0$

as as the coefficient of $x^2 > 0$

the shape of the graph $y = f(x)$ is

$\therefore$ $f(x) \leq -2$ for $x \in \left[\dfrac{-1}{3}, 0\right]$

13 $f(x) = x^2 + x$

a $f(-2) = (-2)^2 + (-2) = 2$

b $f(2) = (2)^2 + (2) = 6$

c $f(-a) = (-a)^2 + (-a) = a^2 - a$

d $f(a) + f(-a) = (a^2 + a) + (a^2 - a)$

$$= 2a^2$$

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e $f(a) - f(-a) = (a^2 + a) - (a^2 - a)$

$= 2a$

f $f(a^2) = (a^2)^2 + (a^2) = a^4 + a^2$

14 $g(x) = 3x - 2$

a $g(x) = 4$

$3x - 2 = 4$

$x = 2$

b $g(x) > 4$

$3x - 2 > 4$

$x > 2$

in set notation

$\{x : x > 2\}$

c $g(x) = a$

$3x - 2 = a$

$x = \frac{a + 2}{3}$

d $g(-x) = 6$

$-3x - 2 = 6$

$x = \frac{-8}{3}$

e $g(2x) = 4$

$6x - 2 = 4$

$x = 1$

f $\frac{1}{g(x)} = 6$

$1 = 6g(x)$

$1 = 6(3x - 2)$

$1 = 18x - 12$

$18x = 13$

$x = \frac{13}{18}$

15 **a** $f(x) = kx - 1$

$3 = 3k - 1$

$k = \frac{4}{3}$

b $f(x) = x^2 - k$

$3 = 9 - k$

$k = 6$

c $f(x) = x^2 + kx + 1$

$3 = 9 + 3k + 1$

$k = \frac{-7}{3}$

d $f(x) = \frac{k}{x}$

$3 = \frac{k}{3}$

$k = 9$

e $f(x) = kx^2$

$3 = 9k$

$k = \frac{1}{3}$

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f $f(x) = 1 - kx^2$

$$3 = 1 - 9k$$

$$9k = -2$$

$$k = \frac{-2}{9}$$

16 a $5x - 4 = 2$

$$x = \frac{6}{5}$$

b $\frac{1}{x} = 5$

$$x = \frac{1}{5}$$

c $\frac{1}{x^2} = 9$

$$x = \pm\frac{1}{3}$$

d

$$x = \frac{1}{x} = 2$$

$$x^2 - 2x + 1 = 0$$

$$(x - 1)^2 = 0$$

$$x = 1$$

e $(x + 1)(x - 2) = 2$

$\therefore$ either $x + 1 = 0$ or $x - 2 = 0$

$x = -1 \quad x = 2$

$\therefore \quad x = -1, 2$

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Solutions to Exercise 1C

1 **a** The functions which are one - to - one are **b** and **c**

2 **a** The functions which are one - to - one are **b**,**d** and **f**

3 **a** The graphs of functions are **i**, **iii**, **iv**, **vi**, **vii**, and **viii**.

b The graphs of one - to - one functions are **iii**, and **vii**.

4 $y^2 = x + 2, x \geq -2$

$y = \pm\sqrt{x+2}$

two possible functions f and g are

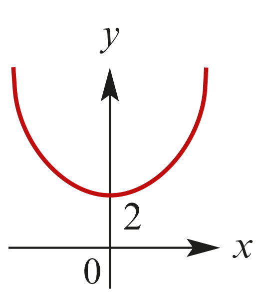
$f : [-2, \infty) \to \mathbb{R} \quad f(x) = \sqrt{x+2}$

range of $f : [0, \infty) = \mathbb{R}^+ \cup \{0\}$

$g : [-2, \infty) \to \mathbb{R}\; g(x) = -\sqrt{x-2}$

range of $g : (-\infty, 0] = \mathbb{R}^- \cup \{0\}$

5 **a**



b two possible functions are the right half

$g_1 : [0, \infty) \to \mathbb{R} \quad g_1(x) = x^2 + 2$

and the left half

$g_2 : (-\infty, 0) \to \mathbb{R} \quad g_2(x) = x^2 + 2$

6 **a** Domain: $\mathbb{R}$ range: $\mathbb{R}$

b Domain: $\mathbb{R}^+ \cup \{0\}$ range: $\mathbb{R}^+ \cup \{0\}$

c Domain: $\mathbb{R}$ range: $[-2, \infty)$

d Domain: $[-4, 4]$ range: $[0, 4]$

e Domain: $\mathbb{R}\backslash\{0\}$ range: $\mathbb{R}\backslash\{0\}$

f Domain: $\mathbb{R}$ range: $(-\infty, 4]$

g Domain $[3, \infty)$ range: $[0, \infty)$

7 **a** Domain: $\mathbb{R}$ range: $\mathbb{R}$

b Domain: $\mathbb{R}$ range: $[-2, \infty)$

c Domain [-3, 3] range: $[0, 3]$

d Domain: $\mathbb{R}\backslash\{1\}$ range: $\mathbb{R}\backslash\{0\}$

8 **a** $\mathbb{R}\backslash\{3\}$

b $(-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$

c $\mathbb{R}$

d $[4, 11]$

e $\mathbb{R}\backslash\{-1\}$

f $h(x) = \sqrt{(x+1)(x-2)}$

Domain : $(-\infty, -1] \cup [2, \infty)$

g $\mathbb{R}\backslash\{-1, 2\}$

h Domain: $(-\infty, -2) \cup [1, \infty)$

i $f(x) = \sqrt{x(1-3x)}$ Domain : $\left[0, \dfrac{1}{3}\right]$

j $[-5, 5]$

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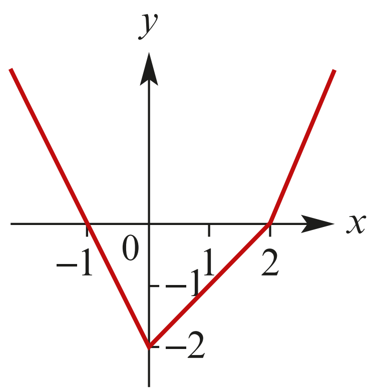
k $[3, 12]$

9 a



b $[-2, \infty)$

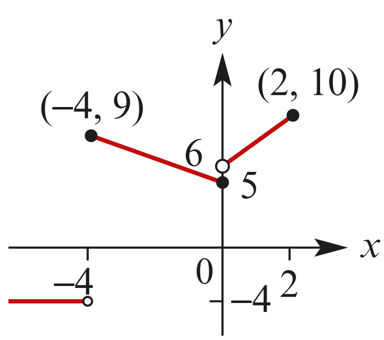
10 Domain: $(-3, 0] \cup [1, 3)$

range: $[-2, 3)$

11 Domain: $[-5, 4]$

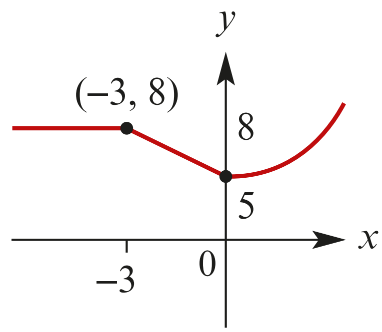
range: $[-4, 0) \cup [2, 5]$

12 a



b Domain $= (-\infty, 2]$
range $= [5, 10] \cup \{-4\}$

13 a



b range $= [5, \infty\}$

14 $f(x) = \begin{cases} \dfrac{1}{x}, & x > 3 \\ 2x, & x \le 3 \end{cases}$

a $f(-4) = 2(-4) = -8$

b $f(0) = 2(0) = 0$

c $f(4) = \dfrac{1}{(4)} = \dfrac{1}{4}$

d $f(a+3) = \begin{cases} \dfrac{1}{a+3}, & a > 0 \\ 2a+6, & a \le 0 \end{cases}$

e $f(2a) = \begin{cases} \dfrac{1}{2a}, & a > \dfrac{3}{2} \\ 4a, & a \le \dfrac{3}{2} \end{cases}$

f $f(a-3) = \begin{cases} \dfrac{1}{a-3}, & a > 6 \\ 2a-6, & a \le 6 \end{cases}$

15 a $f(0) = 4$

b $f(3) = \sqrt{(3)-1} = \sqrt{2}$

c $f(8) = \sqrt{(8)-1} = \sqrt{7}$

d $f(a+1) = \begin{cases} \sqrt{a}, & a \ge 0 \\ 4, & a < 0 \end{cases}$

e $f(a-1) =$

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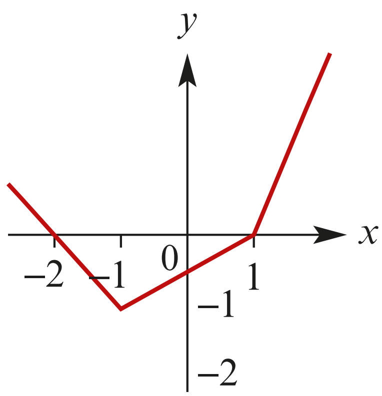
$$\begin{cases} \sqrt{a-2}, & a-1>1 \Rightarrow a \geq 2 \\ 4, & a-0<1 \Rightarrow a<2 \end{cases}$$

16 a



range = $[-1, \infty)$

17 $$f(x) = \begin{cases} ax+b, & x<-2 \\ cx+d, & -2 \leq x \leq 3 \\ ex+f, & x>3 \end{cases}$$

using the points given

$$f(x) = \begin{cases} -x-4, & x<-2 \\ \frac{1}{2}x-1, & -2 \leq x \leq 3 \\ -\frac{1}{2}x+2, & x>3 \end{cases}$$

18 a Even

b Odd

c Neither

d Even

e Odd

f Neither

19 a Even

b Even

c Odd

d Odd

e Neither

f Even

g Neither

h Neither

i Even

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Solutions to Exercise 1D

1 a $(f+g)(x) = 3x + x + 2$

$= 4x + 2$

Domain: $\mathbb{R}$

$(fg)(x) = 3x(x+2)$

$= 3x^2 + 6x$

Domain: $\mathbb{R}$

b $(f+g)(x) = 1 - x^2 + x^2 = 1$

Domain: $(0, 2]$

(from Domain $(g) \cap$ Domain (f))

$(fg)(x) = (1 - x^2)x^2$

$= x^2 - x^4$

Domain: $(0, 2]$

(from Domain $(g) \cap$ Domain(f))

c $(f+g)(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}} = \dfrac{x+1}{\sqrt{x}}$

Domain: $[1, \infty)$ (from g)

$(fg)(x) = \dfrac{\sqrt{x}}{\sqrt{x}}$

$= 1$

Domain: $[1, \infty)$ (from g)

d $(f+g)(x) = x^2 + \sqrt{4-x}$

Domain: $[0, 4]$ (from g)

$(fg)(x) = x^2\sqrt{4-x}$

Domain: $[0, 4]$ (from g)

2 a functions f and h are even, g and k are odd

b $(f+h)(x) = x^2 + 1 + \dfrac{1}{x^2}, x \in \mathbb{R}\backslash\{0\}$

it is even

$(fh)(x) = 1 + \dfrac{1}{x^2}, x \in \mathbb{R}\backslash\{0\}$

it is even

$(g+k)(x) = x + \dfrac{1}{x}, x \in \mathbb{R}\backslash\{0\}$

it is odd

$(gk)(x) = 1, x \in \mathbb{R}\backslash\{0\}$

it is even

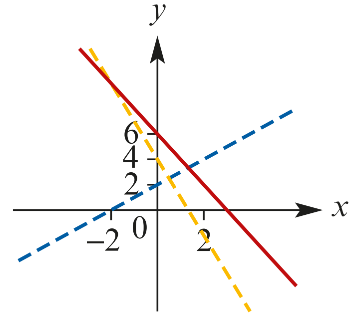
$(f+g)(x) = x^2 + x + 1, x \in \mathbb{R}$

it is neither odd nor even

$(fg)(x) = x^3 + x, x \in \mathbb{R}$

it is odd

3



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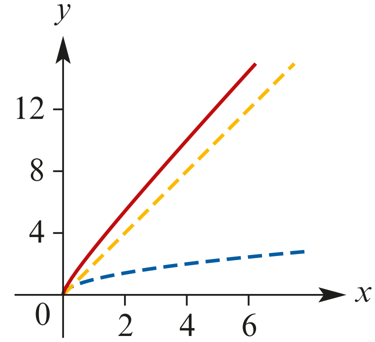

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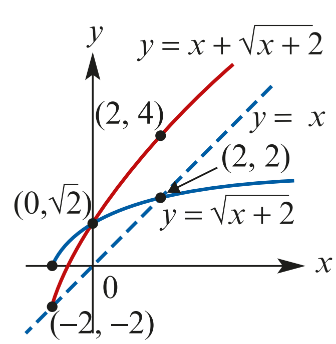

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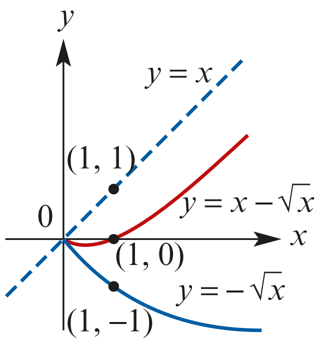

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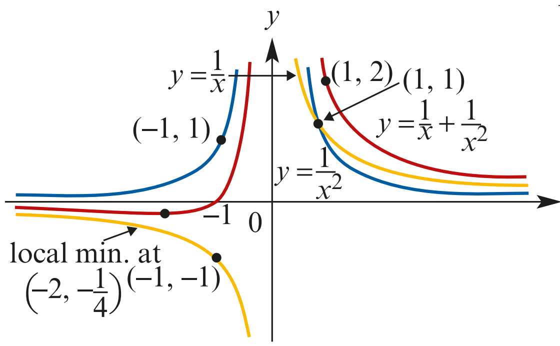
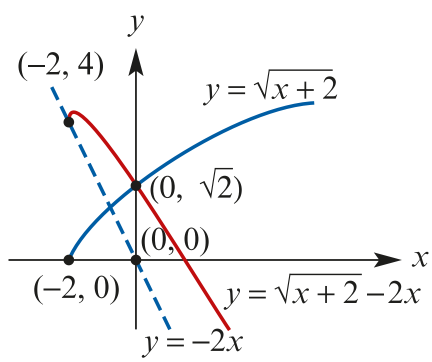


8 a



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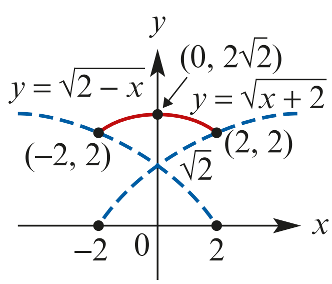

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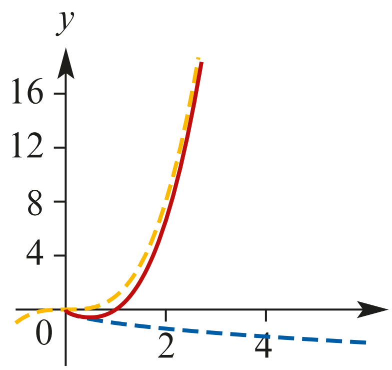

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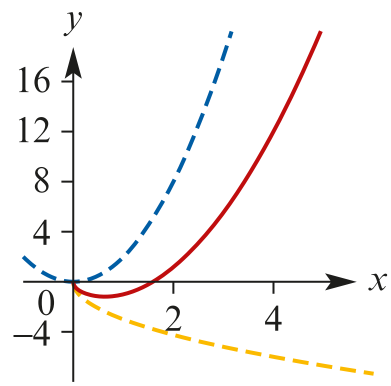

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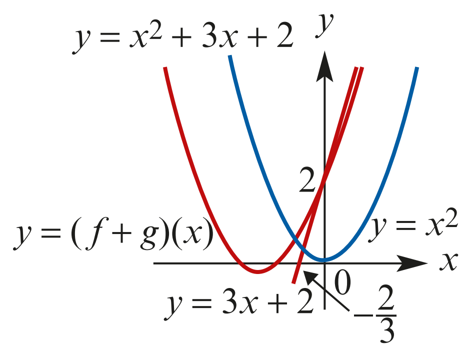
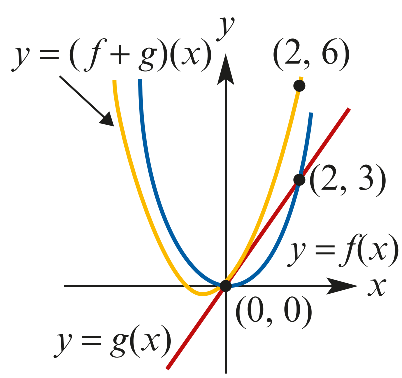


12 a



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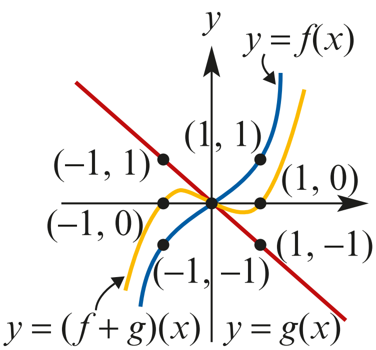
b



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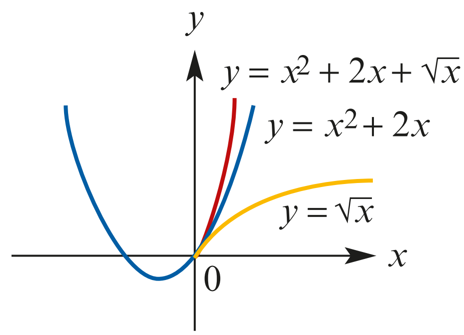
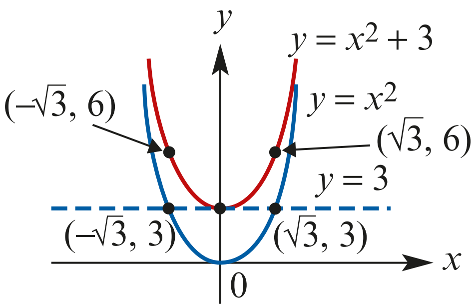


13 a



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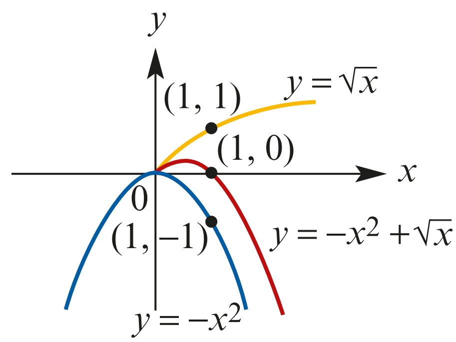

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Solutions to Exercise 1E

1 **a** $f(g(x)) = 2(2x) - 1 = 4x - 1$
$g(f(x)) = 2(2x - 1) = 4x - 2$

b $f(g(x)) = 4(2x + 1) + 1 = 8x + 5$
$g(f(x)) = 2(4x + 1) + 1 = 8x + 3$

c $f(g(x)) = 2(2x - 3) - 1 = 4x - 7$
$g(f(x)) = 2(2x - 1) - 3 = 4x - 5$

d $f(g(x)) = 2(x^2) - 1 = 2x^2 - 1$
$g(f(x)) = (2x - 1)^2 = 4x^2 - 4x + 1$

e $$\begin{aligned} f(g(x)) &= 2(x - 5)^2 + 1 \\ &= 2x^2 - 20x + 51 \\ g(f(x)) &= (2x^2 + 1) - 5 \\ &= 2x^2 - 4 \end{aligned}$$

f $f(g(x)) = 2(x^2) + 1 = 2x^2 + 1$
$g(f(x)) = (2x + 1)^2$

2 **a** $f \circ h(x) = 2(3x + 2) - 1 = 6x + 3$

b $h(f(x)) = 3(2x - 1) + 2 = 6x - 1$

c $f \circ h(2) = 6(2) + 3 = 15$

d $h \circ f(2) = 6(2) - 1 = 11$

e $f(h(3)) = 6(3) + 3 = 21$

f $h(f(-1)) = 6(-1) - 1 = -7$

g $f \circ h(0) = 6(0) + 3 = 3$

3 **a** $$\begin{aligned} f \circ h(x) &= (3x + 1)^2 + 2(3x + 1) \\ &= 9x^2 + 12x + 3 \end{aligned}$$

b $h \circ f(x) = 3(x^2 + 2x) + 1 = 3x^2 + 6x + 1$

c $f \circ h(3) = 9(3)^2 + 12(3) + 3 = 120$

d $h \circ f(3) = 3(3)^2 + 6(3) + 1 = 46$

e $f \circ h(0) = 9(0)^2 + 12(0) + 3 = 3$

f $h \circ f(0) = 3(0)^2 + 6(0) + 1 = 1$

4 **a** $h \circ g : \mathbb{R}^+ \to \mathbb{R}, h \circ g(x) = \dfrac{1}{(3x + 2)^2}$

b $g \circ h : \mathbb{R}\backslash\{0\}, g \circ h(x) = \dfrac{3}{x^2} + 2$

c $h \circ g(1) = \dfrac{1}{(3(1) + 2)^2} = \dfrac{1}{25}$

d $g \circ h(1) = \dfrac{3}{(1)^2} + 2 = 5$

5 **a** $\text{range}(f) = [-4, \infty)$
$\text{range}(g) = \mathbb{R}^+ \cup \{0\}$

b $f \circ g : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}, f \circ g(x) = x - 4$
$\text{range}(\text{f} \circ \text{g}) = [-4, \infty)$

c $g \circ f$ does not exist because the range of f is not a subset of the Domain of g

6 **a** $f(g(x)) = \dfrac{1}{2}(2x) = x$
$f \circ g : \mathbb{R}\backslash\left\{\dfrac{1}{2}\right\} \to \mathbb{R}, f \circ g(x) = x$
Range: $\mathbb{R}\backslash\left\{\dfrac{1}{2}\right\}$

b $g \circ f : \mathbb{R}\backslash\{0\} \to R, g \circ f(x) = x$
Range: $\mathbb{R}\backslash\{0\}$

7 **a** the range of is $[-2, \infty)$, which is not a subset of the Domain of g, $\therefore g \circ f$ does not exist.

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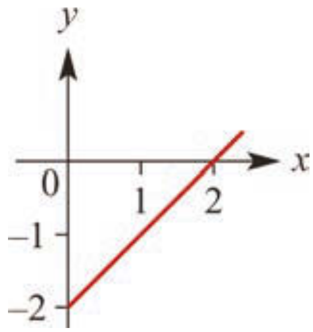
b $f \circ g : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}, f \circ g(x) = x - 2$



8 **a** the range of g is $[-1, \infty)$, which is not a subset of the Domain of $f((-\infty, 3])$,
$\therefore \quad f \circ g$ does not exist.

b the range of g^* needs to be $[-1, 3]$ at most.
$g^* : [-2, 2] \to \mathbb{R}, g^*(x) = x^2 - 1$
$f \circ g^* : [-2, 2] \to \mathbb{R}, f \circ g^*(x) = 4 - x^2$

9 **a** The range of g is $\mathbb{R}$, which is not a subset of the Domain of f,
$f \circ g$ does not exist.

b the range of g needs to be $\mathbb{R}^+$ at most.
$\therefore \quad$ let $g_1 : \{x : x < 3\} \to \mathbb{R}$,
$g_1(x) = 3 - x$
then $f \circ g_1 : \{x : x < 3\} \to \mathbb{R}$,
$f \circ g_1(x) = \dfrac{1}{\sqrt{3 - x}}$

10 **a** the Domain of f is R, the range of g is $\mathbb{R}^+ \cup \{0\}$
$\therefore \quad f \circ g$ exists.

b Range of f is $\mathbb{R}^+ \cup \{0\}$
Domain of g is $(-\infty, 3]$
The range of f is not a subset of the Domain of g
$\therefore \quad g \circ f$ does not exist.

11 **a** S is the maximal Domain of f,
$\therefore S = [-2, 2]$

b Range of $f = [0, 2]$
range of $g = [1, \infty)$

c $f \circ g$ is not defined as the range of g is not a subset of the Domain of f.
$g \circ f$ is defined as the range of f is a subset of the Domain of g.

12 For both $f \circ g$ and $g \circ f$ to exist, the range of g must be a subset of the Domain of $f$and the range must be a subset of the Domain of g.

Domain of$f : [2, \infty]$; Range of $f : (-\infty, a - 2]$

Domain of $g : (-\infty, 1]$; range of $g : [a, \infty)$

So $a \geq 2$ from $f \circ g$

& $a - 2 \leq 1$ from $g \circ f$

$\therefore \quad 2 \leq a \leq 3$

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Solutions to Exercise 1F

1 a Let $y = f^{-1}(x)$ then

$$x = 2y + 3$$
$$y = \frac{x-3}{2}$$
$$f^{-1}(x) = \frac{x-3}{2}$$

b Let $y = f^{-1}(x)$ then

$$x = 4 - 3y$$
$$y = \frac{4-x}{3}$$
$$f^{-1}(x) = \frac{4-x}{3}$$

c Let $y = f^{-1}(x)$ then

$$x = 4y + 3$$
$$y = \frac{x-3}{4}$$
$$f^{-1}(x) = \frac{x-3}{4}$$

d Let $y = f^{-1}(x)$ then

$$x = \frac{3y-2}{4}$$
$$3y = 4x + 2$$
$$f^{-1}(x) = y = \frac{4x+2}{3}$$

2 a Let $y = f^{-1}(x)$ then

$$x = y - 4$$
$$f^{-1}(x) = y = x + 4$$

b Let $y = f^{-1}(x)$ then

$$x = 2y$$
$$f^{-1}(x) = y = \frac{x}{2}$$

c Let $y = f^{-1}(x)$ then

$$x = \frac{3}{4}y$$
$$f^{-1}(x) = y = \frac{4}{3}x$$

3 a Let $y = f^{-1}(x)$ then

$$x = 2y - 4$$
$$f^{-1}(x) = y = \frac{x+4}{2}$$

Domain (f^{-1}) = range $(f) = [-8, 8]$

$\therefore \quad f^{-1} : [-8, 8] \to R, f^{-1}(x) = \frac{x+4}{2}$

range (f^{-1}) = Domain $(f) = [-2, 6]$

b let $g^{-1}(x) = y$ then

$$x = \frac{1}{9-y}$$
$$9 - y = \frac{1}{x}$$
$$g^{-1}(x) = y = 9 - \frac{1}{x}$$

Domain (g^{-1}) = range $(g) = \mathbb{R}^-$

$\therefore \quad g^{-1} : R^- \to \mathbb{R}, g^{-1}(x) = 9 - \frac{1}{x}$

range (g^{-1}) = Domain $(g) = (9, \infty)$

c Let $h^{-1}(x) = y$. Then

$$x = y^2 + 2$$
$$y^2 = x - 2$$
$$y = \pm\sqrt{x-2}$$

but range (h^{-1}) = Domain (h)

$=\mathbb{R}^+ \cup (0\}$

$\therefore \quad h^{-1}(x) = y = \sqrt{x-2}$

Domain (h^{-1}) = range $(h) = [2, \infty)$

$\therefore \quad h^{-1}[2, \infty) \to \mathbb{R}, h^{-1}(x) = \sqrt{x-2}$

range $(h^{-1}) = [0, \infty)$

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d Let $f^{-1}(x) = y$. Then

$$x = 5y - 2$$

$$f^{-1}(x) = y = \frac{x+2}{5}$$

Domain (f^{-1}) = range (f) =

$[-17, 28]$

$\therefore\ f^{-1}[-17, 28] \to R, f^{-1}(x) = \dfrac{x+2}{5}$

range (f^{-1}) = Domain (f) = $[-3, 6]$

e Let $g^{-1}(x) = y$. Then $x = y^2 - 1$

$$y^2 = x + 1$$

$$y = \pm\sqrt{x+1}$$

but range (g^{-1}) = Domain (g) =

$(1, \infty)$

$\therefore\ g^{-1}(x) = \sqrt{x+1}$

Domain (g^{-1}) = range (g) = $(0, \infty)$

$\therefore\ g^{-1}(0, \infty) \to R, g^{-1}(x) = \sqrt{x+1}$

range$(g^{-1}) = (1, \infty)$

f Let $h^{-1}(x) = y$. Then $x = \sqrt{y}$

$h^{-1}(x) = y = x^2$

Domain (h^{-1}) = range(h) = R^+

$\therefore\ h^{-1}: R^+ \to R, h^{-1}(x) = x^2$

range (h^{-1}) = Domain (h) = R^+

4 a Interchange x and y

$x = y^2 + 2y$

Completing the square:

$(y+1)^2 - x - 1 = 0$

$$y + 1 = \pm\sqrt{1+x}$$

$$y = -1 \pm \sqrt{1+x}$$

but range (g^{-1})=Domain (g) =

$[-1, \infty)$

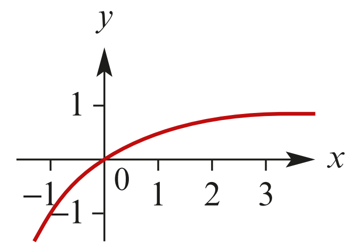
$\therefore\ g^{-1}(x) = y = \sqrt{1+x} - 1$

Domain (g^{-1}) =range (g) = $[-1, \infty)$

$g^{-1}[-1, \infty) \to \mathbb{R}, g^{-1}(x) = \sqrt{1+x} - 1$

range $(g^{-1}) = [-1, \infty)$

b



5 Let $f : \mathbb{R} \to \mathbb{R}, f(x) = \dfrac{1}{x} - 3$

Let $y = f^{-1}(x)$. Then we can write

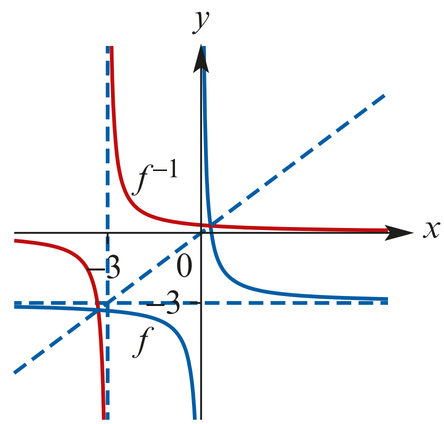
$x = \dfrac{1}{y} - 3$

Hence $y = \dfrac{1}{x+3}$.

That is $f^{-1}(x) = \dfrac{1}{x+3}$.

The Domain of f^{-1} is $\mathbb{R}\backslash\{-3\}$

$f^{-1}: \mathbb{R} \setminus \{-3\} \to \mathbb{R}, f^{-1}(x) = \dfrac{1}{x+3}$



6 a to find f^{-1} (2), use $f(x) = 2$

$$2 = 3 - 2x$$

$$f^{-1}(2) = x = \frac{1}{2}$$

Domain f^{-1} = range$(f) = [-3, 3]$

7 a Let $f^{-1}(x) = y$

$$x = 2y$$

$$f^{-1}(x) = y = \frac{x}{2}$$

Domain f^{-1} = range$(f) = [-2, 6]$

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range $f^{-1} = \text{Domain}(f) = [-1, 3]$

$\therefore \quad f^{-1}[-2, 6] \to R, f^{-1}(x) = \dfrac{x}{2}$

b Let $f^{-1}(x) = y$

$$x = 2y^2 - 4$$

$$y^2 = \frac{(x+4)}{2}$$

$$y = \pm\sqrt{\frac{(x+4)}{2}}$$

but range $f^{-1} = \text{Domain}(f) =$ $[-0, \infty)$

$\therefore \quad f^{-1}(x) = y = \sqrt{\dfrac{(x+4)}{2}}$

Domain $f^{-1} = \text{range}(f) = [-4, \infty)$

$\therefore \quad f^{-1}[-4, \infty) \to R, f^{-1}(x) = \sqrt{\dfrac{(x+4)}{2}}$

range $f^{-1} = [0, \infty)$

c $\{(4, 2), (6, 1), (8, 3), (11, 5)\}$

Domain $= \{4, 6, 8, 11\}$

range $= \{1, 2, 3, 5\}$

d Let $h^{-1}(x) = y$.Then

$$x = \sqrt{-y}$$

$$h^{-1}(x) = y = -x^2$$

Domain $h^{-1} = \text{range}(h) = \mathbb{R}^+$

$\therefore \quad h^{-1}\colon \mathbb{R}^+ \to \mathbb{R}, h^{-1}(x) = -x^2$

range $(h^{-1}) = \mathbb{R}^-$

e Let $f^{-1}(x) = y$. Then

$$x = y^3 + 1$$

$$f^{-1}(x) = y = (x-1)^{\frac{1}{3}}$$

Domain $(f^{-1}) = \text{range}(f) = \mathbb{R}$

$f^{-1} : R \to \mathbb{R}, f^{-1}(x) = (x-1)^3$

range $(f^{-1}) = \mathbb{R}$

f Let $g^{-1}(x) = y$. Then

$$x = (y+1)^2$$

$$y = \pm\sqrt{x} - 1$$

but range $(g^{-1}) = \text{Domain}(g)$

$= (-1, 3)$

$\therefore \quad g^{-1}(x) = y = \sqrt{x} - 1$

Domain $g^{-1} = \text{range}(g) = (0, 16)$

$\therefore \quad g^{-1} : (0, 16) \to \mathbb{R}, g^{-1}(x) = \sqrt{x} - 1$

range $(g^{-1}) = (-1, 3)$

g Let $g^{-1}(x) = y$. Then

$$x = \sqrt{y-1}$$

$$g^{-1}(x) = y = x^2 + 1$$

Domain $g^{-1} = \text{range}(g) = [0, \infty)$

$\therefore g^{-1} : [0, \infty) \to \mathbb{R}, g^{-1}(x) = x^2 + 1$

range $g^{-1} = [1, \infty)$

h Let $h^{-1}(x) = y$. Then

$$x = \sqrt{4 - y^2}$$

$$y^2 = 4 - x^2$$

$$y = \pm\sqrt{4 - x^2}$$

but range $(h^{-1}) = \text{Domain } (h) = [0, 2]$

$\therefore h^{-1}(x) = y = \sqrt{4 - x^2}$

Domain $(h^{-1}) = \text{range } (h) = [0, 2]$

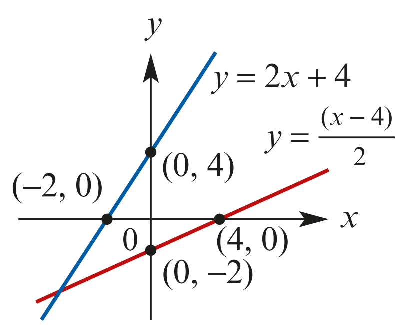
$\therefore h^{-1}\colon [0, 2] \to \mathbb{R}, h^{-1}(x) = \sqrt{4 - x^2}$

range $(h^{-1}) = [0, 2]$

8 a $x = 2y + 4$

$y = \dfrac{x-4}{2}$

implied Domain: $\mathbb{R}$ and range: $\mathbb{R}$



b $\quad x = \dfrac{3 - f^{-1}(x)}{2}$

$f^{-1}(x) = 3 - 2x$

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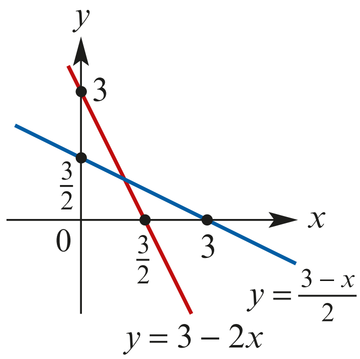
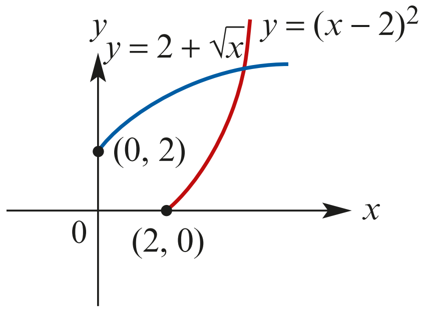
implied Domain: $\mathbb{R}$
and range: $\mathbb{R}$



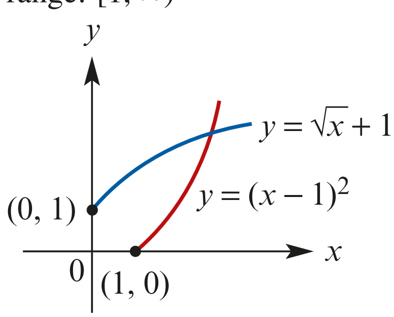
c $$x = (f^{-1}(x) - 2)^2$$

$\pm\sqrt{x} + 2 = f^{-1}(x)$
but range (f^{-1}) =dom$(f) = [2, \infty)$
$\therefore f^{-1}(x) = \sqrt{x} + 2$
Domain: $[0, \infty)$
range: $[2, \infty)$

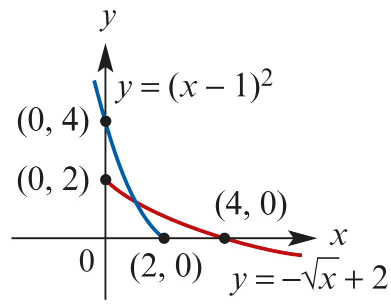


d $$x = (f^{-1}(x) - 1)^2$$

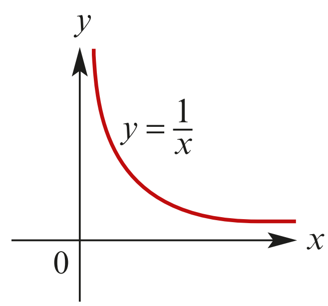
$f^{-1}(x) = \sqrt{x} + 1$
Domain: $[0, \infty)$
range: $[1, \infty)$



e similar to (c)
but $f^{-1}(x) = -\sqrt{x} + 2$
Domain: $[0, \infty)$
range: $(-\infty, 2]$

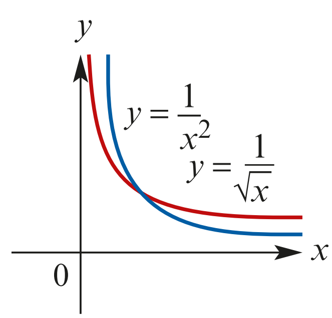


f $f^{-1}(x) = \frac{1}{x}$
Domain: R^+
range : R^+



g $$x = \frac{1}{(f^{-1}(x))^2}$$

$f^{-1}(x) = \pm\frac{1}{\sqrt{x}}$
but range $f^{-1}(x)$ =Domain$(f) = \mathbb{R}^+$
$\therefore f^{-1}(x) = \frac{1}{\sqrt{x}}$
Domain: $\mathbb{R}^+$
range: $\mathbb{R}^+$



h $$x = \frac{1}{2}(h^{-1}(x) - 4)$$

$h^{-1}(x) = 2x + 4$

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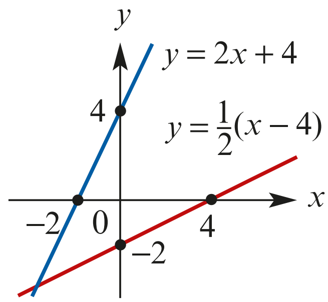
implied Domain: $\mathbb{R}$
and range: $\mathbb{R}$



9 a $x = \sqrt{f^{-1}(x)} + 2$

$(x-2)^2 = f^{-1}(x)$

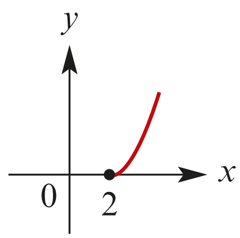
$f^{-1}(x) = x^2 - 4x + 4$

$f^{-1}(x) = (x-2)^2$

Therefore,

$f^{-1}: [2, \infty) \to \mathbb{R},$

$f^{-1}(x) = (x-2)^2$



b $x = \dfrac{1}{f^{-1}(x) - 3}$

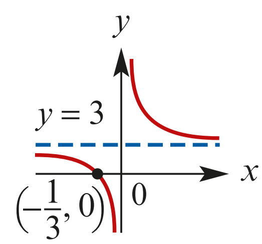
$f^{-1}(x) - 3 = \dfrac{1}{x}$

$f^{-1}(x) = \dfrac{1}{x} + 3$

Therefore,

$f^{-1}: \mathbb{R} \setminus \{0\} \to \mathbb{R},$

$f^{-1}(x) = \dfrac{1}{x} + 3$



c $x = \sqrt{f^{-1}(x) - 2} + 4$

$f^{-1}(x) - 2 = (x-4)^2$

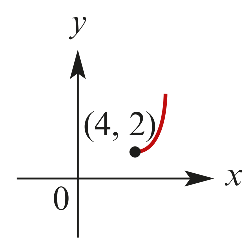
$f^{-1}(x) = x^2 - 8x + 18$

$f^{-1}(x) = (x-4)^2 + 2$

Therefore,

$f^{-1}: [4, 4 + \sqrt{6}) \to \mathbb{R},$

$f^{-1}(x) = (x-4)^2 + 2$



d $x = \dfrac{3}{f^{-1}(x) - 2} + 1$

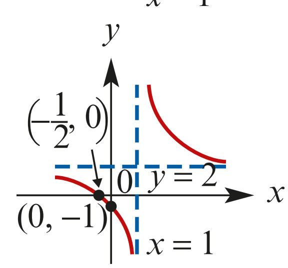
$f^{-1}(x) - 2 = \dfrac{3}{x-1}$

$f^{-1}(x) = \dfrac{3}{x-1} + 2$

Therefore,

$f^{-1}: \mathbb{R} \setminus \{1\} \to \mathbb{R},$

$f^{-1}(x) = \dfrac{3}{x-1} + 2$



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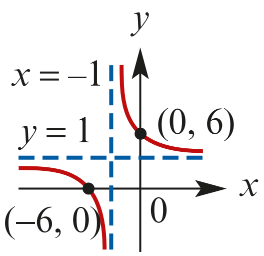
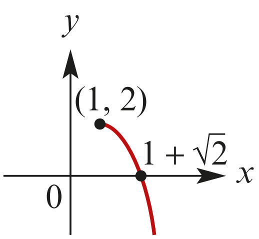
e
$$x = \frac{5}{f^{-1}(x) - 1} - 1$$
$$f^{-1}(x) - 1 = \frac{5}{x+1}$$
$$f^{-1}(x) = \frac{5}{x+1} + 1$$
Therefore,
$f^{-1} \colon \mathbb{R} \setminus \{-1\} \to \mathbb{R}$,
$$f^{-1}(x) = \frac{5}{x+1} + 1$$



f
$$x = \sqrt{2 - f^{-1}(x)} + 1$$
$$(x-1)^2 = 2 - f^{-1}(x)$$
$$f^{-1}(x) = 2 - (x-1)^2$$
$$f^{-1}(x) = -x^2 + 2x + 1$$
Therefore,
$f^{-1} \colon [1, \infty) \to \mathbb{R}$,
$$f^{-1}(x) = 2 - (x-1)^2$$



10 a
$$f(x) = 1 + \frac{2}{x-1}$$
$$x = 1 + \frac{2}{f^{-1}(x) - 1}$$
$$x - 1 = \frac{2}{f^{-1}(x) - 1}$$
$$f^{-1}(x) - 1 = \frac{2}{x-1}$$
$$f^{-1}(x) = 1 + \frac{2}{x-1}$$
$$f^{-1}(x) = \frac{x+1}{x-1}$$

b
$$f(x) = \sqrt{x-2}$$
$$x = \sqrt{f^{-1}(x) - 2}$$
$$x^2 = f^{-1}(x) - 2$$
$$f^{-1}(x) = x^2 + 2$$

c
$$f(x) = \frac{2x+3}{3x-2}$$
$$= \frac{\frac{2}{3}(3x-2) + \frac{4}{3} + 3}{3x-2}$$
$$= \frac{2}{3} + \frac{\frac{13}{3}}{3x-2}$$
$$= \frac{2}{3} + \frac{13}{9x-6}$$
$$x = \frac{2}{3} + \frac{13}{9f^{-1}(x) - 6}$$
$$x - \frac{2}{3} = \frac{13}{9f^{-1}(x) - 6}$$
$$9f^{-1}(x) - 6 = \frac{13}{x - \frac{2}{3}}$$
$$3f^{-1}(x) - 2 = \frac{13}{3x-2}$$
$$3f^{-1}(x) = \frac{13 + 6x - 4}{3x-2}$$
$$f^{-1}(x) = \frac{2x+3}{3x-2}$$

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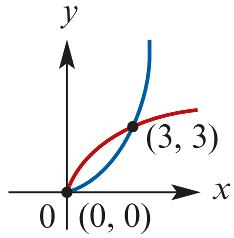
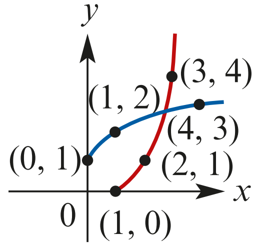
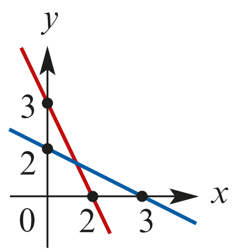
11 **a**



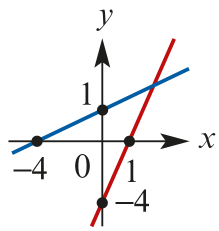
b



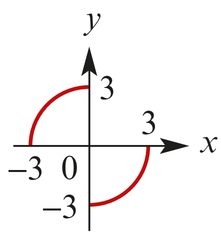
c



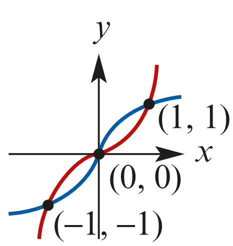
d



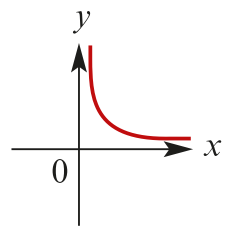
e



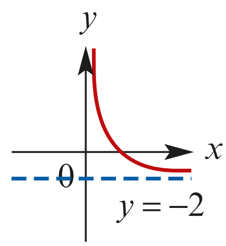
f



g



h



12 **a** C

b B

c D

d A

13 **a** $3 - x \geq 0$

$x \leq 3$

$\therefore \quad A = (-\infty, 3]$

b minimum b is at the turning point

$i.e.\ b = 0$

let $g^{-1}(x) = y$

$x = 1 - y^2$

$y = \pm\sqrt{1 - x}$

,but range (g^{-1}) =Domain $(g) = [0, 2]$

$\therefore y = \sqrt{1 - x}$

Domain (g^{-1}) =range$(g) = [-3, 1]$

$\therefore \; g^{-1} : [-3, 1] \to R, g^{-1}(x) = \sqrt{1 - x}$

14 $b = -2, \quad g^{-1}(x) = -2 + \sqrt{x + 4}$

15 $a = 3, \quad f^{-1}(x) = 3 - \sqrt{x + 9}$

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16 a
$$x = \frac{3}{g^{-1}(x)}$$
$$g^{-1}(x) = \frac{3}{x}$$
Domain $= \mathbb{R}\backslash\{0\}$

b
$$x = \sqrt[3]{g^{-1}(x) + 2} - 4$$
$$(x+4)^3 = g^{-1}(x) + 2$$
$$g^{-1}(x) = (x+4)^3 - 2$$
Domain $= \mathbb{R}$

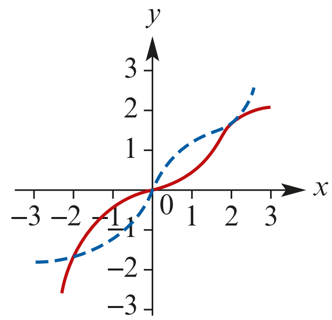
c
$$x = 2 - \sqrt{h^{-1}(x)}$$
$$\sqrt{h^{-1}(x)} = 2 - x$$
$$h^{-1}(x) = (x-2)^2$$
Domain (h^{-1}) = range $(h) = (-\infty,\ 2]$

d
$$x = \frac{3}{f^{-1}(x)} + 1$$
$$f^{-1}(x) = \frac{3}{x-1}$$
Domain $= \mathbb{R}\backslash\{1\}$

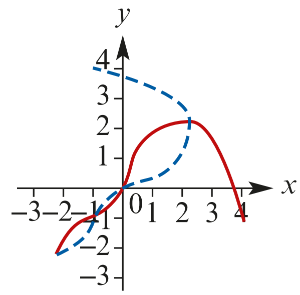
e
$$x = 5 - \frac{2}{(h^{-1}(x) - 6)^3}$$
$$\frac{2}{5-x} = (h^{-1}(x) - 6)^3$$
$$h^{-1}(x) = \sqrt[3]{\frac{2}{5-x}} + 6$$
Domain $= \mathbb{R}\backslash\{5\}$

f
$$x = \frac{1}{(g^{-1}(x) - 1)^{\frac{3}{4}}} + 2$$
$$(g^{-1}(x) - 1)^{3/4} = \frac{1}{x-2}$$
$$g^{-1}(x) = \frac{1}{(x-2)^{\frac{4}{3}}} + 1$$
Domain $= (2, \infty)$

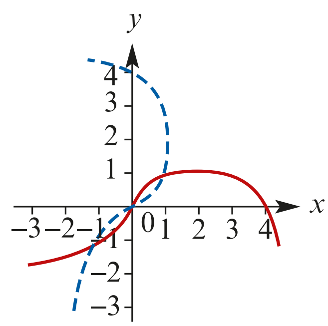
17 a Inverse is a function



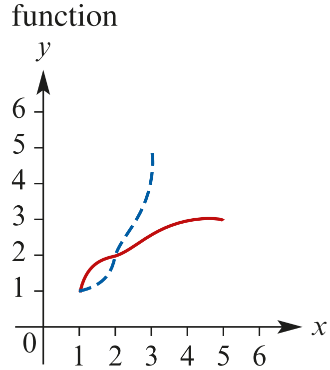
b Inverse is not a function



c Inverse is not a function



d Inverse is a function



e Inverse is not a function

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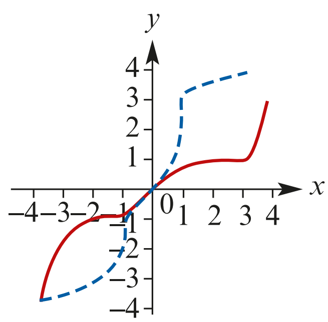


18 a $$f(x) = \frac{x+3}{2x-1}$$

$$\text{Domain} = \mathbb{R}\backslash\left\{\frac{1}{2}\right\}$$

$$f(x) = \frac{\frac{1}{2}(2x-1) + \frac{7}{2}}{2x-1}$$

$$= \frac{1}{2} + \frac{7}{2(2x-1)}$$

$$\text{range} = \mathbb{R}\backslash\left\{\frac{1}{2}\right\}$$

Since range(f) =Domain(f)

$f \circ f$ is defined.

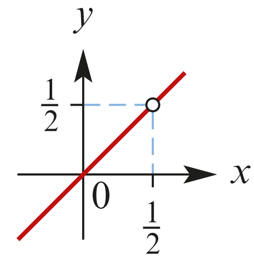
b $$f \circ f(x) = \frac{1}{2} + \frac{\frac{7}{2}}{2\frac{(2x+3)}{(2x-1)} - 1}$$

$$= \frac{1}{2} + \frac{7}{2(\frac{2x+6-2x+1}{2x-1})}$$

$$= \frac{1}{2} + \frac{7(2x-1)}{14}$$

$$= \frac{1}{2} + x - \frac{1}{2}$$

$$f \circ f(x) = x,\ x \in \mathbb{R}\backslash\left\{\frac{1}{2}\right\}$$



c Since $f \circ f(x) = x$ and $f^{-1} \circ f(x) = x$

$$f^{-1} = f = \frac{x+3}{2x-1},\ x \in \mathbb{R}\backslash\left\{\frac{1}{2}\right\}$$

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Solutions to Exercise 1G

1 a Maximal Domain = $\mathbb{R} \setminus \{0\}$;
Range = $\mathbb{R}^+$

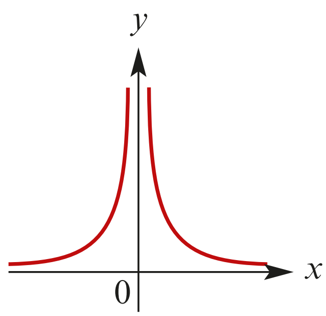
b **i** $\dfrac{1}{16}$

ii $\dfrac{1}{16}$

iii 16

iv 16

c



2 a Odd

b Even

c Odd

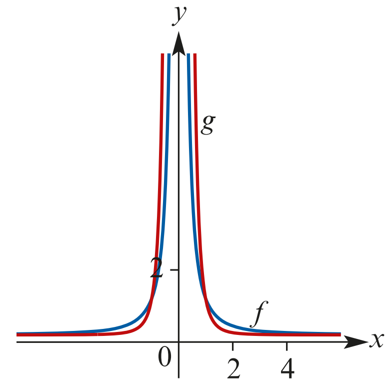
d Odd

e Even

f Odd

3 a
$$f(x) = g(x)$$
$$x^{-2} = x^{-4}$$
$$x^2 = 1$$
$$x = 1 \text{ or } x = -1$$

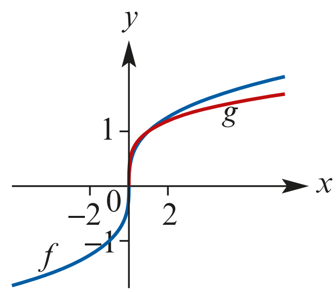
b



4 a
$$f(x) = g(x)$$
$x = 0$ is one solution.

Now assume$x \neq 0$
$$x^{\frac{1}{3}} = x^{\frac{1}{4}}$$
$$x^{\frac{1}{3}-\frac{1}{4}} = 1$$
$$x^{\frac{1}{12}} = 1$$
$$\therefore x = 1 \text{ or } x = 0$$

b



5 a
$$x = \left(f^{-1}(x)\right)^7$$
$$f^{-1}(x) = x^{\frac{1}{7}}$$
Domain of f^{-1} = range of $f = \mathbb{R}$

$f^{-1} : \mathbb{R} \to \mathbb{R},\ f^{-1}(x) = x^{\frac{1}{7}}$

b
$$x = \left(f^{-1}(x)\right)^6$$
$$f^{-1}(x) = x^{\frac{1}{6}}$$

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Domain of f^{-1} = range of $f = [0, \infty)$

$f^{-1}: [0, \infty) \to \mathbb{R}, f^{-1}(x) = -x^{\frac{1}{6}}$

c $\quad x = 27\left(f^{-1}(x)\right)^3$

$$\frac{x}{27} = \left(f^{-1}(x)\right)^3$$

$$f^{-1}(x) = \left(\frac{x}{27}\right)^{\frac{1}{3}} = \frac{1}{3}x^{\frac{1}{3}}$$

Domain of f^{-1} = range of $f = \mathbb{R}$

$f^{-1}: \mathbb{R} \to \mathbb{R}, f^{-1}(x) = \frac{1}{3}x^{\frac{1}{3}}$

d $\quad x = 16\left(f^{-1}(x)\right)^4$

$$\frac{x}{16} = \left(f^{-1}(x)\right)^4$$

$$f^{-1}(x) = \left(\frac{x}{16}\right)^{\frac{1}{4}} = \frac{1}{2}x^{\frac{1}{4}}$$

Domain of f^{-1} = range of $f = (16, \infty)$

$f^{-1}: (16, \infty) \to \mathbb{R}, f^{-1}(x) = \frac{1}{2}x^{\frac{1}{4}}$

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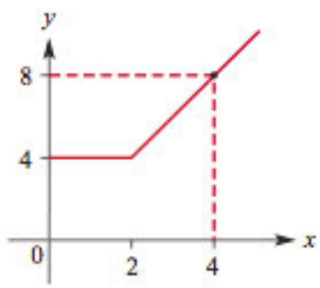
Solutions to Exercise 1H

1 For $0 \le x \le 2$, the cost is \$ 4. For $x > 2$, the cost is \$4 + \$2 for each extra km over 2 km, i.e \$$2(x-2)$. Hence:

$$f(x) = \begin{cases} 4 & \text{if } 0 \le x \le 2 \\ 4 + 2(x-2) & \text{if } x > 2 \end{cases}$$

$$= \begin{cases} 4 & \text{if } 0 \le x \le 2 \\ 2x & \text{if } x > 2 \end{cases}$$



2 The box has length $(36 - 2x)$ cm, width $(20 - 2x)$ cm and height x cm. So the volume V cm^3 is given by

$$V = x(20 - 2x)(36 - 2x)$$
$$= 4x(10 - x)(18 - x)$$

where $x > 0$ and $x < 10$ for a box to exist.
The Domain is $[0, 10]$.

3 a Perimeter $= 2x + 2y = 160$, so $y = 80 - x$. The area can be found by subtracting a rectangle of dimensions 12 by $(y - 20)$ from a rectangle of dimensions x by y:

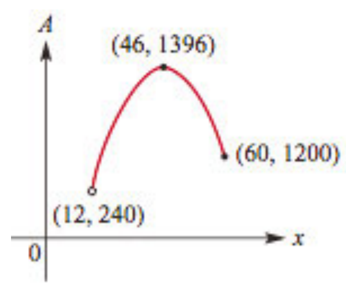
$$A = xy - 12(y - 20)$$
$$= x(80 - x) - 12(60 - x)$$
$$= -x^2 + 80x + 12x - 720$$
$$= -x^2 + 92\text{x} - 720$$

b $x > 12$; also $y > 20$ implies $80 - x > 20$ so that $x < 60$.
The Domain is $[12, 60]$.

c The function is a quadratic with (non-included) endpoints where $x = 12, 60$. When $x = 12, A = 240$; when $x = 60, A = 1200$.
Endpoints are $(12, 240)$ and $(60, 1200)$.
There is a turning point where

$$x = -\frac{b}{2a}$$
$$= -\frac{92}{-2} = 46$$

Then $A = 1396$.
The graph is shown here.



d The maximum area is 1396 m^2 and it occurs for $x = 46$ and $y = 80 - 46 = 34$.

4 a i $S = 2x^2 + 2 \leftrightarrow 2x \leftrightarrow h + 2 \leftrightarrow x \leftrightarrow h$
$= 2x^2 + 6xh$

ii $V = 2x^2h$ where $h = \dfrac{V}{2x^2}$

$$S = 2x^2 + 6x \leftrightarrow \frac{V}{2x^2}$$
$$= 2x^2 + \frac{3V}{x}$$

b $x > 0$, so maximal Domain is $(0, \infty)$.

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c $V = 1000$ so $S = 2x^2 + \dfrac{3000}{x}$.
A sketch using a CAS calculator shows that there is an endpoint maximum where $x = 2$. Then $S = 1508\ \text{m}^2$.

5 Let x be the with of the rectangle and y be the length of the rectangle.
The diagonal has length $2a$.
$\therefore x^2 + y^2 = 4a^2$
$\therefore y^2 = 4a^2 - x^2$
$\therefore y = \sqrt{4a^2 - x^2}$
$\therefore$ Area $= xy = x(\sqrt{4a^2 - x^2})$
The Domain is clearly [0,2a].

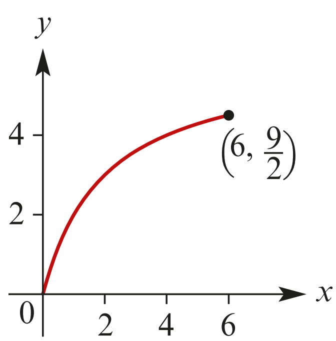
6 The coordinates of C are $\left(a, \dfrac{6}{a+2}\right)$

a Area $= a \times \dfrac{6}{a+2} = \dfrac{6a}{a+2}$

b Domain $= [0, 6]$; Range $= \left[0, \dfrac{9}{2}\right]$

c Maximum value$= \left[0, \dfrac{9}{2}\right]$

d



7 **a** Distance is speed by time, so during the first 45 minutes, the man runs a distance of $\dfrac{2}{60}t = \dfrac{1}{30}t$ km; after 45 minutes, he has run $\dfrac{3}{2}$ km and thereafter adds a distance of $\dfrac{4}{60}t = \dfrac{1}{15}t$ during the next 30 minutes. Hence:

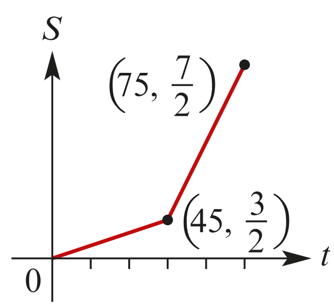
$$S(t) = \begin{cases} \dfrac{1}{30}t & \text{if } 0 \le t \le 45 \\ \dfrac{3}{2} + \dfrac{1}{15}(t - 45) & \text{if } 45 < t \le 75 \end{cases}$$

$$= \begin{cases} \dfrac{1}{30}t & \text{if } 0 \le t \le 45 \\ \dfrac{1}{15}t - \dfrac{3}{2} & \text{if } 45 < t \le 75 \end{cases}$$

$$a = \frac{1}{30}, \quad b = \frac{1}{15}, \quad c = 45,$$

$$d = -\frac{3}{2}, \quad e = 75$$

b The graph comprises two line segments as shown here.



c The range is $\left[0, \dfrac{7}{2}\right]$.

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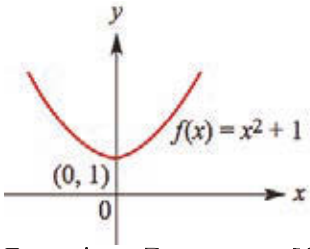
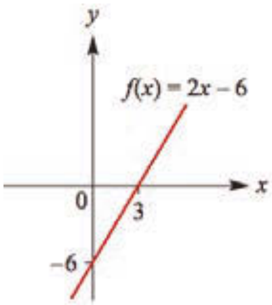
Solutions to technology-free questions

1 a



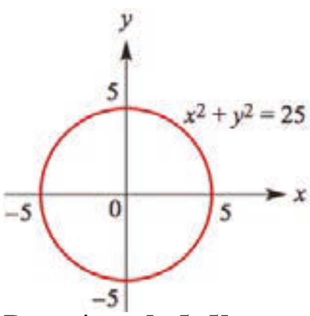
Domain = $\mathbb{R}$, range = $[1, \infty)$

b



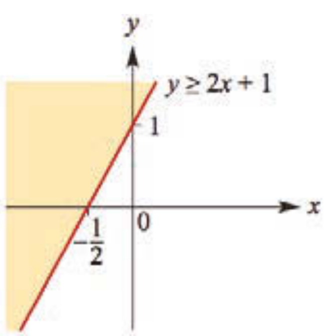
Domain = $\mathbb{R}$, range = $\mathbb{R}$

c



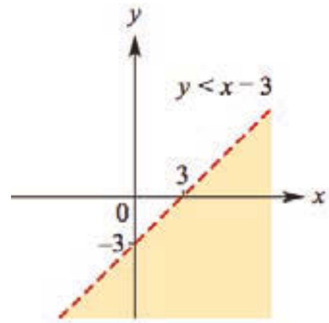
Domain = $[-5, 5]$, range = $[-5, 5]$

d



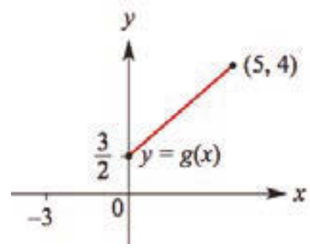
Domain = $\mathbb{R}$, range = $\mathbb{R}$

e



Domain = $\mathbb{R}$, range = $\mathbb{R}$

2 a



b range = $[1.5, 4]$

c Interchange x and y and solve for y:

$$x = \frac{y+3}{2}$$

$$y + 3 = 2x$$

$$y = 2x - 3$$

$$g^{-1} : [1.5, 4] \to \mathbb{R}, g^{-1}(x) = 2x - 3$$

Domain = $[1.5, 4]$, range = $[0, 5]$

d

$$g(x) = 4$$

$$\frac{x+3}{2} = 4$$

$$x + 3 = 8$$

$$x = 5$$

$$\{x : g(x) = 4\} = \{5\}$$

e If $g^{-1}(x) = 4$, then $x = g(4) = 3.5$.
$\{x : g^{-1}(x) = 4\} = \{3.5\}$
(Alternatively, solve the equation $2x - 3 = 4$ for x.)

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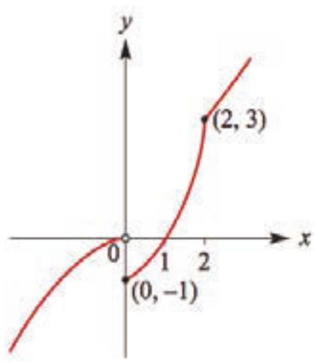
3 **a**
$$\begin{aligned} 5x + 1 &= 2 \\ 5x &= 1 \\ x &= \frac{1}{5} \\ \{x : g(x) = 2\} &= \left\{\frac{1}{5}\right\} \end{aligned}$$

b If $g^{-1}(x) = 2$, then $x = g(2) = 11$.
$\{x : g(x) = 2\} = \{11\}$

c
$$\begin{aligned} \frac{1}{5x+1} &= 2 \\ 5x + 1 &= \frac{1}{2} \\ 5x &= -\frac{1}{2} \\ x &= -\frac{1}{10} \\ \left\{x : \frac{1}{g(x)} = 2\right\} &= \left\{-\frac{1}{10}\right\} \end{aligned}$$

4



5 **a** $2x - 6 \neq 0$, so $x \neq 3$
Domain $= R\backslash\{3\}$

b
$$\begin{aligned} x^2 - 5 &> 0 \\ \left(x - \sqrt{5}\right)\left(x + \sqrt{5}\right) &> 0 \\ x < -\sqrt{5} \text{ or } x &> \sqrt{5} \end{aligned}$$
Domain $= R\backslash\left[-\sqrt{5}, \sqrt{5}\right]$

c $(x - 1)(x + 2) \neq 0$, so $x \neq 1, -2$
Domain $= \mathbb{R}\backslash\{1, -2\}$

d
$$\begin{aligned} 25 - x^2 &\geq 0 \\ (5 - x)(5 + x) &\geq 0 \\ -5 \leq x &\leq 5 \end{aligned}$$
Domain $= [-5, 5]$

e $x - 5 \geq 0$ and $15 - x \geq 0$
$5 \leq x \leq 15$
Domain $= [5, 15]$

f $3x - 6 \neq 0$, so $x \neq 2$
Domain $= \mathbb{R}\backslash\{2\}$

6
$$\begin{aligned} (f + g)(x) &= (x + 2)^2 + x - 3 \\ &= x^2 + 4x + 4 + x - 3 \\ &= x^2 + 5x + 1 \\ (fg)(x) &= (x - 3)(x + 2)^2 \end{aligned}$$

7
$$\begin{aligned} (f + g)(x) &= (x - 1)^2 + 2x \\ &= x^2 + 1 \end{aligned}$$
$$(f + g)\colon [1, 5] \to \mathbb{R}, (f + g)(x) = x^2 + 1$$
$$(fg)(x) = 2x(x - 1)^2$$
$$(fg)\colon [1, 5] \to \mathbb{R}, (fg)(x) = 2x(x - 1)^2$$

8 $f(3) = 8$, so range of f is $[8, \infty)$ (the graph of $y = f(x)$ is increasing for $x \geq 3$).
Hence Domain of f^{-1} is $[8, \infty)$ and the range is $[3, \infty)$.

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Interchange x and y and solve for y:

$$x = y^2 - 1$$

$$y^2 = x + 1$$

$$y = \sqrt{x+1} \text{ (as } y > 0)$$

$$f^{-1} : [8, \infty) \to \mathbb{R}, f^{-1}(x) = \sqrt{x+1}$$

9 a $(f + g)(x) = -x^2 + 2x + 3$

b $(fg)(x) = -x^2(2x + 3)$

c
$$(f+g)(x) = 0$$
$$\{x: (f+g)(x) = 0\} = \{-1, 3\}$$
$$-x^2 + 2x + 3 = 0$$
$$-(x^2 - 2x - 3) = 0$$
$$-(x+1)(x-3) = 0$$
$$x = -1, 3$$

10 $f(2) = 2$, so range of f is $(-\infty, 2]$ (the graph of $y = f(x)$ is a straight line with endpoint at (2, 2)).
Interchange x and y and solve for y:

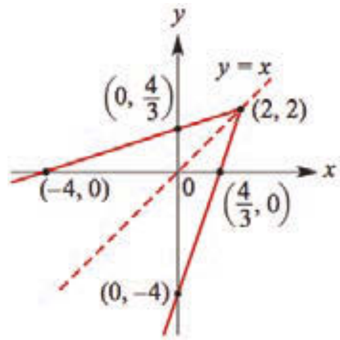
$$x = 3y - 4$$

$$3y = x + 4$$

$$y = \frac{x+4}{3}$$

$$f^{-1} : (-\infty, 2] \to R, f^{-1}(x) = \frac{x+4}{3}$$

The graphs are straight lines, reflections of each other in the line $y = x$, each with endpoint $(2, 2)$.
The graph of $y = f(x)$ has axes intercepts $\left(\frac{4}{3}, 0\right)$, $(0, -4)$. The graph of $y = f^{-1}(x)$ has axes intercepts $(-4, 0)$, $\left(0, \frac{4}{3}\right)$.



11 a
$$x = 8\left(f^{-1}(x)\right)^3$$
$$\frac{x}{8} = \left(f^{-1}(x)\right)^3$$
$$f^{-1}(x) = \left(\frac{x}{8}\right)^{\frac{1}{3}} = \frac{1}{2}x^{\frac{1}{3}}$$
Domain of f^{-1} = range of $f = \mathbb{R}$

b
$$x = 32\left(f^{-1}(x)\right)^5$$
$$\frac{x}{32} = \left(f^{-1}(x)\right)^5$$
$$f^{-1}(x) = \left(\frac{x}{32}\right)^{\frac{1}{5}} = \frac{1}{2}x^{\frac{1}{5}}$$
Domain of f^{-1} = range of $f = (-\infty, 0]$

c
$$x = 64\left(f^{-1}(x)\right)^6$$
$$\frac{x}{64} = \left(f^{-1}(x)\right)^6$$
$$f^{-1}(x) = \left(\frac{x}{64}\right)^{\frac{1}{6}} = \frac{1}{2}x^{\frac{1}{6}}$$
Domain of f^{-1} = range of $f = [0, \infty]$

d
$$x = 10\,000\left(f^{-1}(x)\right)^4$$
$$\frac{x}{10\,000} = \left(f^{-1}(x)\right)^4$$
$$f^{-1}(x) = \left(\frac{x}{10\,000}\right)^{\frac{1}{4}} = \frac{1}{10}x^{\frac{1}{4}}$$
Domain of f^{-1} = range of $f = (10\,000, \infty)$

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12 a $f \circ g(x) = f(-x^3)$
$= -2x^3 + 3$

b $g \circ f(x) = g(2x + 3)$
$= -(2x + 3)^3$

c $g \circ g(x) = g(-x^3)$i
$= (-x^3)^3$
$= -x^9$

d $f \circ f(x) = f(2x + 3)$
$= 2(2x + 3) + 3$
$= 4x + 9$

e $f \circ (f + g)(x) = f(f + g(x))$
$= f(-x^3 + 2x + 3)$
$= 2(-x^3 + 2x + 3) + 3$
$= -2x^3 + 4x + 9$

f $f \circ (f - g)(x) = f(f - g(x))$
$= f(2x + 3 + x^3)$
$= 2(2x + 3 + x^3) + 3$
$= 2x^3 + 4x + 9$

g $f \circ (f \cdot g)(x) = f(f \cdot g(x))$
$= f(-2x^4 - 3x^3)$
$= 2(-2x^4 - 3x^3) + 3$
$= -4x^4 - 6x^3 + 3$

13 $x \geq -1$ or $x \leq -9$

14 $h^{-1}(x) = \left(\dfrac{x - 64}{2}\right)^{\frac{1}{5}}$

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Solutions to multiple-choice questions

1 E $6 - 2x \geq 0$

$6 \geq 2x$

$3 \geq x$

$\therefore\ (-\infty, 3]$

2 B $f : [-1, 3) \to R, f(x) = -x^2$

$f(3) = -9;$ maximum 0 at $x = 0$

$\therefore (-9, 0].$

3 E $f(x) = 3x^2 + 2x$

$f(2a) = 3(2a)^2 + 2(2a)$

$f(2a) = 12a^2 + 4a$

4 C $f(x) = 2x - 3$

let$f(x) = 2(f^{-1}(x)) - 3$

$f(x) + 3 = 2(f^{-1}(x))$

$f^{-1}(x) = \dfrac{f(x) + 3}{2}$

$f^{-1}(x) = \dfrac{x + 3}{2}$

$f^{-1}(x) = \dfrac{1}{2}x + \dfrac{3}{2}$

5 E $f : (a, b] \to R, f(x) = 10 - x, a < b$

The minimum is:

$f(b) = 10 - b$

The maximum is:

$f(a) = 10 - \mathrm{a}$

$\therefore\ [10 - b, 10 - a)$

6 C As a is a negative real number:

$f(a + 3) = -(a + 3) + 6$

$f(a + 3) = -a + 3$

7 D $f(x) = (x + 3)^2 - 6$ Graph must be one to one to have an inverse function.

Turning point of function is at $(-3, -6)$

Domain must be a sub set of either: $(-\infty, -3]$ or $[-3, \infty)$

$\therefore [6, \infty)$

8 B An inverse only exists if the function is one to one.

$g : [-4, 4] \to R, g(x) = \sqrt{16 - x^2}$

Is not one to one for the specified Domain.

9 B The asymptote is at $x = -2$ therefore the asymptote of the inverse is at $y = -2$.

10 C $f(x) = \dfrac{2x + 1}{x - 1} = 2 + \dfrac{3}{x - 1}$

Therefore asymptotes $x = 1$ and $y = 2$.

11 B $f(x) = 3x^2$ and $g(x) = 2x + 1$

$\therefore f(g(x)) = 3(2x + 1)^2$

$f(g(x)) = 12x^2 + 12x + 3$

$\therefore f(g(a)) = 12a^2 + 12a + 3$

12 E $f(x) = x^2 + 2x - 6 = (x + 1)^2 - 7$

$\therefore$ vertex has coordinates $(-1, -7)$

$f(-2) = (-2)^2 + 2(-2) - 6 =$

$4 - 4 - 6 = -6$

$f(4) = (4)^2 + 2(4) - 6 = 18$

$\therefore$ range $= [-7, 18)$

13 C If $a > b$ then $a^{\frac{1}{5}} > b^{\frac{1}{5}}$

14 C Maximal Domain

$= (-1, \infty) \cap (-\infty, 4] = (-1, 4]$

15 A Domain of f^{-1} = Range of $f = (\sqrt{7}, \infty)$

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$x = \sqrt{2f^{-1}(x) + 3}$

$\therefore f^{-1}(x) = \frac{1}{2}(x^2 - 3)$

16 **B** $5 - x = -2 \Rightarrow x = 7$

$5 - x = 3 \Rightarrow x = 2$ $\therefore$ Domain of $f = (2, 7]$

17 **A** $g : R\{3\} \to \mathbb{R}, g(x) = \frac{1}{x-3} + 2$

Let $x = \frac{1}{g^{-1}(x) - 3} + 2$

$g^{-1}(x) - 3 = \frac{1}{x-2}$

$g^{-1}(x) = \frac{1}{x-2} + 3$

$x \neq 2$

$\therefore$ dom $g^{-1}(x) = R\backslash\{2\}$

18 **B**

19 **B** Asymptotes of $y(x)$ occur at

$x + 1 = 0$

$\therefore x = -1$

And at $y = -2$

$\therefore$ Asymptotes of $y^{-1}(x)$ occur at:

$y = -1$ and $x = -2$

20 **C** Asymptotes of $\frac{-2}{(x+3)^4} - 5$ occur when $x + 3 = 0$

$\therefore x = -3$

And when $y = -5$

21 **A** $f : [0, \infty) \to R,\ f(x) = (x-2)^2\ f(x)$ does not have an inverse function as it is not a one to one function.

22 **D** Note that the graph of $y = \frac{1}{x^4}$ will be like that of $y = \frac{1}{x^2}$, but 'steeper'. Looking at the alternatives, D stands out: its Domain runs from negative to positive numbers with 0 removed. for numbers close to 0, the value of y will be very large. As $x \to 0, f(x) \to \infty$. Its range is actually $[1, \infty)$. (Checking each of the remaining alternatives shows that the range is correct in each case.)

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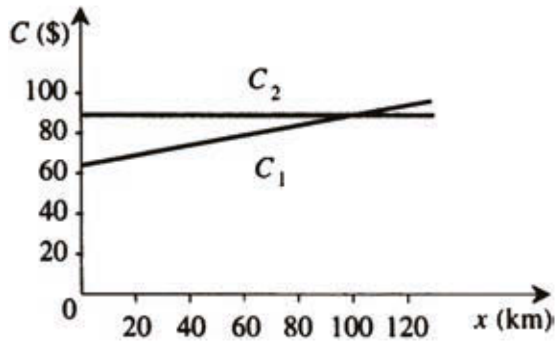
Solutions to extended-response questions

1 a $C_1(x) = 0.25x + 64$

$C_2(x) = 89$

b



c From the graph or using the inequality

$0.25x + 64 > 89$

$0.25x > 25$

$x > 100$

Method 2 is cheaper than Method 1 if the distance travelled is greater than 100 km.

From this it can be seen that Method 2 is cheaper than Method 1 if the distance travelled is more than 100 km.

2 a Area of each face $= x^2$

$\therefore$ the total surface area, $S = 6x^2$

b The volume, $V = x^3$

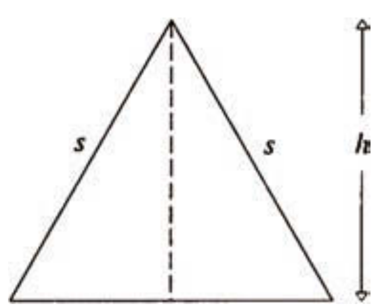
$\therefore \quad x = V^{\frac{1}{3}}$

and $S = 6V^{\frac{2}{3}}$

3 a The triangle is equilateral.

Area $A = \frac{1}{2}s^2 \sin 60°$ (Area of triangle $= \frac{1}{2}bc \sin A$)

$= \frac{\sqrt{3}}{4}s^2 \ldots \langle 1 \rangle$



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b By Pythagoras' Theorem, $h^2 = s^2 - \dfrac{s^2}{4}$

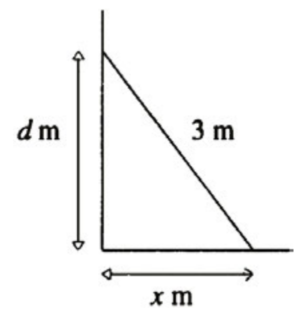
$$= \frac{3s^2}{4}$$

$$\therefore \quad h = \frac{\sqrt{3}s}{2} \quad \text{and} \quad s = \frac{2h}{\sqrt{3}}$$

by $\langle 1 \rangle A = \dfrac{\sqrt{3}}{4}\left(\dfrac{2h}{\sqrt{3}}\right)^2 = \dfrac{\sqrt{3}}{4} \times \dfrac{4h^2}{3} = \dfrac{\sqrt{3}h^2}{3}$

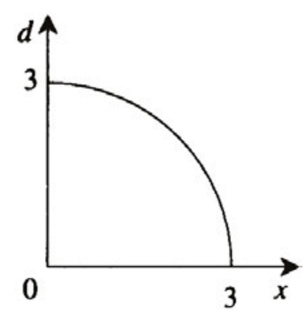
4 **a** By Pythagoras' Theorem $d^2 = 9 - x^2$

$$\therefore \quad d = \sqrt{9 - x^2}$$



b maximal Domain = [0, 3]

The range of the function is [0, 3]



5 Let d km be the distance travelled.

The time taken for journey travelling at 80 km per hour

$= \dfrac{d}{2} \div 80 = \dfrac{d}{160}$

The time taken for journey travelling at x km per hour

$= \dfrac{d}{2} \div x = \dfrac{d}{2x}$

$\therefore$ Total time taken $= \dfrac{d}{160} + \dfrac{d}{2x} = \dfrac{d}{2}\left(\dfrac{1}{80} + \dfrac{1}{x}\right) = \dfrac{d}{2}\left(\dfrac{x+80}{80x}\right)$

Average speed $= \dfrac{\text{distance travelled}}{\text{total time taken}}$

$$\therefore \quad S(x) = d \div \frac{d}{2}\left(\frac{x+80}{80x}\right)$$

$$= d \times \frac{2}{d} \times \frac{80x}{x+80}$$

$$= \frac{160x}{x+80}$$

Domain of S is $[0, \infty)$

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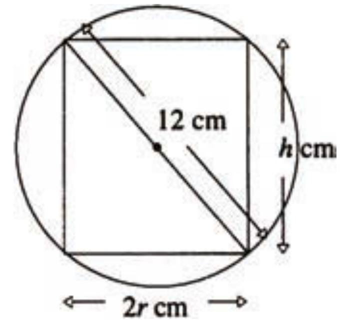
6 Volume of cylinder $= \pi r^2 h$

a The diameter has length 12 cm.
By Pythagoras' Theorem



$$12^2 = h^2 + 4r^2 \ldots \langle 1 \rangle$$

$$\therefore \quad r^2 = \frac{12^2 - h^2}{4}$$

$$V_1(h) = \frac{\pi}{4}\left(144 - h^2\right)h$$

$$= \pi\left(36 - \frac{h^2}{4}\right)h$$

As $V_1 > 0, h > 0$ and $r > 0$ Domain of $V_1 = (0, 12)$

b by $\langle 1 \rangle$

$$h^2 = 144 - 4r^2$$

$$\therefore \quad h = \sqrt{144 - 4r^2} = 2\sqrt{36 - r^2}$$

$$\therefore \quad V_2(r) = \pi r^2 \times 2\sqrt{36 - r^2}$$

$$= 2\pi r^2 \sqrt{36 - r^2}$$

Domain of $V_2 = (0, 6)$

7 a

	Domain	range
f	R	R
g	R	R

ran f = dom g g of exists,

$g \circ f(x) = g(x + 1) = 2 + (1 + x)^3$

b $g \circ f$ is a one-to-one function

$\therefore$ $(g \circ f)^{-1}$ is defined,

Solve the equation $g \circ f(x) = 10$

$$2 + (1 + x)^3 = 10$$

$$\therefore \quad (1 + x)^3 = 8$$

$$\therefore \quad 1 + x = 2$$

$$\therefore \quad x = 1$$

$\therefore$ $(g \circ f)^{-1}$ is defined, $(g \circ f)^{-1}(10) = 1$

8 $f(x) = \begin{cases} x^2 - 4 & \text{if } x \in (-\infty, 2) \\ x & \text{if } x \in [2, \infty) \end{cases}$

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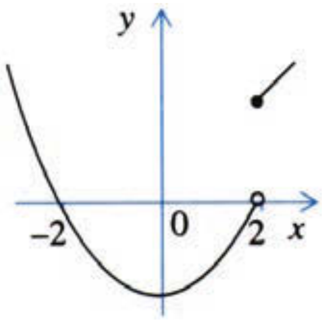
a



b **i** $f(-1) = 1 - 4 = -3$ as $-1 \in (-\infty, 2)$

ii $f(3) = 3$ as $3 \in [2, \infty)$

c $S = (-\infty, 0]$ as f is one to one for this interval. and $-1 \in S$.

d $h(x) = 2x$, then $f(h(x)) = f(2x)$

$$f(2x) = \begin{cases} (2x)^2 - 4 & \text{if } 2x \in (-\infty, 2) \\ 2x & \text{if } 2x \in [2, \infty) \end{cases}$$

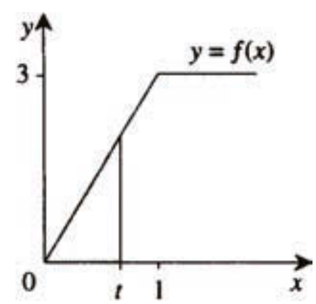
Therefore $f \text{ o } h(x) = \begin{cases} 4x^2 - 4 & \text{if } x < 1 \\ 2x & \text{if } x \geq 1 \end{cases}$

Now $h \text{ o } f(x) = h\left(\begin{cases} x^2 - 4 & \text{if } x \in (-\infty, 2) \\ x & \text{if } x \in [2, \infty) \end{cases}\right)$

$$h \text{ o } f(x) = \begin{cases} 2x^2 - 8 & \text{if } x < 2 \\ 2x & \text{if } x \geq 2 \end{cases}$$

9 For $0 \leq t \leq 1$

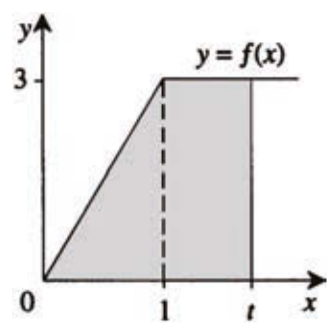
$$\text{Area} = \frac{1}{2} \times t \times 3t$$
$$= \frac{3}{2}t^2$$



For $t > 1$

Area = area of triangle△ + area of rectangle□

$$= \frac{1}{2} \times 1 \times 3 + 3(t-1)$$
$$= \frac{3}{2} + 3t - 3$$
$$= 3t - \frac{3}{2}$$



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$$A(t) = \begin{cases} \frac{3}{2}t^2 & \text{for } 0 \le t \le 1 \\ 3t - \frac{3}{2} & \text{for } t > 1 \end{cases}$$

Domain of $A = [0, \infty)$

Range of $A = [0, \infty)$

10 **a** Let $x = \dfrac{ay + b}{cy + d}$

$\therefore x(cy + d) = ay + b$

and $xcy - ay = b - xd$

$y(xc - a) = b - xd$

$\therefore y = \dfrac{b - xd}{xc - a}$

Hence $f^{-1} : \mathbb{R} \setminus \left\{\dfrac{a}{c}\right\} \to \mathbb{R}, \; f^{-1}(x) = \dfrac{b - xd}{xc - a}$

For the range of f note: $f(x) = \dfrac{ax + b}{cx + d} = \dfrac{a}{c} + \dfrac{cb - da}{c(cx + d)}$ (by division)

$\therefore$ range of $f = \mathbb{R} \setminus \left\{\dfrac{a}{c}\right\}$

and Domain of f^{-1} = range of $f = \mathbb{R} \setminus \left\{\dfrac{a}{c}\right\}$

range of f^{-1} = Domain of $f = \mathbb{R} \setminus \left\{\dfrac{-d}{c}\right\}$

b **i** For $f(x) = \dfrac{3x + 2}{3x + 1}$

$a = 3, b = 2, c = 3, d = 1$

and $f^{-1}(x) = \dfrac{2 - x}{3x - 3}$; Domain of $f^{-1} = \mathbb{R} \setminus \{1\}$

ii For $f(x) = \dfrac{3x + 2}{2x - 3}$

$a = 3, b = 2, c = 2, d = -3$

and $f^{-1}(x) = \dfrac{3x + 2}{2x - 3}$; Domain of $f^{-1} = \mathbb{R} \setminus \left\{\dfrac{3}{2}\right\}$

iii For $f(x) = \dfrac{x - 1}{-x - 1}$

$f^{-1}(x) = \dfrac{x - 1}{-x - 1} = \dfrac{1 - x}{x + 1}$; Domain of $f^{-1} = \mathbb{R} \setminus \{-1\}$

iv For $f(x) = \dfrac{-x + 1}{x + 1}$

$f^{-1}(x) = \dfrac{1 - x}{x + 1}$; Domain of $f^{-1} = \mathbb{R} \setminus \{-1\}$

c If $f^{-1} = f$ then Domain of f^{-1} = Domain of f

$\therefore \dfrac{a}{c} = \dfrac{-d}{c}$ (we will assume $c \neq 0$)

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$\therefore a = -d$

As $f(x) = \dfrac{ax+b}{cx+d}$

and $f^{-1}(x) = \dfrac{b-xd}{xc-a}$

If $a = -d$ $f^{-1}(x) = \dfrac{ax+b}{cx+d} = f(x)$

$\therefore$ For $c \neq 0$ $f^{-1} = f \Leftrightarrow a = -d$

11 a i $YB = r$ cm (sides of square)

ii $ZB = r$ cm (sides of square)

iii $AZ = (x - r)$ cm

iv $CY = (3 - r)$ cm

b $CY = CX = 3 - r$ (tangents from a point)

$AX = AZ = x - r$ (tangents from a point)

Therefore $AC = AX + XC = x - r + 3 - r = x + 3 - 2r$

Using Pythagoras' Theorem for triangle ABC

$$x^2 + 9 = (x + 3 - 2r)^2$$

i.e. $x^2 + 9 = (x+3)^2 - 4r(x+3) + 4r^2$

$$\therefore \quad x^2 + 9 = x^2 + 6x + 9 - 4rx - 12r + 4r^2$$

$$\therefore \quad 0 = 6x - 4rx - 12r + 4r^2$$

$$\therefore \quad 0 = 2r^2 - 2r(x+3) + 3x$$

$$\therefore \quad r = \frac{2(x+3) \pm \sqrt{4(x+3)^2 - 24x}}{4}$$

$$= \frac{2x + 6 \pm \sqrt{4(x^2 + 6x + 9) - 24x}}{4}$$

$$= \frac{2x + 6 \pm \sqrt{4x^2 + 36}}{4}$$

$$= \frac{x + 3 \pm \sqrt{x^2 + 9}}{2}$$

But $r < \dfrac{x+3}{2}$

$\therefore r = \dfrac{x + 3 - \sqrt{x^2+9}}{2}$

When $x = 4$,

c i

$$r = \frac{7 - \sqrt{25}}{2}$$

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i.e. $r = 1$

ii When $r = \frac{1}{2}$

$$\frac{1}{2} = \frac{(x+3) - \sqrt{x^2+9}}{2}$$

$$\therefore \quad -2 - x = -\sqrt{x^2+9}$$

$$\therefore 4 + 4x + x^2 = x^2 + 9$$

$$\therefore \quad 4x = 5$$

$x = \frac{5}{4}$ (Note this must be tested because of squaring)

12 $f(x) = \dfrac{px+q}{x+r}$ $x \in R\backslash\{-r, r\}$ for $x \in R\backslash\{-r, r\}$

a $f(x) = f(-x)$
implies

$$\frac{px+q}{x+r} = \frac{-px+q}{-x+r}$$

$$\therefore \quad (-x+r)(px+q) = (-px+q)(x+r)$$

$$\therefore -px^2 - qx + pxr + qr = -px^2 - pxr + qx + qr$$

$$\therefore \quad 2pxr = 2qx$$

$$\therefore \quad pr = q$$

$$\therefore \quad f(x) = \frac{px+pr}{x+r}$$

$$\therefore \quad f(x) = p$$

b $f(-x) = -f(x)$
implies

$$\frac{-px+q}{-x+r} = \frac{-px-q}{x+r}$$

$$\therefore -px^2 + qx - prx + qr = px^2 + qx - pxr - qr$$

$$\therefore \quad 2px^2 - 2qr = 0$$

$$\text{i.e. } px^2 = qr$$

$\therefore \quad p = \frac{qr}{x^2}$ since $x \neq 0$.

Substitute for p in $f(x) = \dfrac{Px+q}{x+r}$:

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$$f(x) = \frac{\frac{qr}{x} + q}{x + r}$$
$$= \frac{qr + qx}{x(x + r)}$$
$$= \frac{q(x + r}{x(x + r)}$$
$$= \frac{q}{x} \text{ (make that } x \pm -r)$$

c **i** If $p = 3$, $q = 8$ and $r = -3$

$$f(x) = \frac{3x + 8}{x - 3}$$

Consider $x = \dfrac{3y + 8}{y - 3}$

$$yx - 3x = 3y + 8$$
$$\therefore yx - 3y = 3x + 8$$
$$\therefore y(x - 3) = 3x + 8$$
$$\therefore \quad y = \frac{3x + 8}{x - 3}$$

$f(x)$ Hence $f^{-1}(x) = \dfrac{3x + 8}{x - 3}$

Domain of $f^{-1} = R\backslash\{3\}$

ii
$$x = \frac{3x + 8}{x - 3}$$
$$3x + 8 = x^2 - 3x$$
$$0 = x^2 - 6x - 8$$
$$\therefore \quad x = \frac{6 \pm \sqrt{36 + 32}}{2}$$
$$= \frac{6 \pm 2\sqrt{9 + 8}}{2}$$
$$= 3 \pm \sqrt{17}$$

13 **a** $f : R\backslash\{1\} \to R, f(x) = \dfrac{x + 1}{x - 1}$

Note: For this function $f = f^{-1}$ from Question 10.

i
$$f(2) = \frac{2 + 1}{2 - 1} = 3$$
$$f(f(2)) = f(3) = \frac{3 + 1}{3 - 1} = 2$$
$$f(f(f(2))) = f(2) = 3$$

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ii $f(f(x)) = x$ for all x

b $f : R\backslash\{-1\} \to R,$

$$f(x) = \frac{x-3}{x+1}$$

$$f(f(x)) = f\left(\frac{x-3}{x+1}\right)$$

$$= \frac{\dfrac{x-3}{x+1} - 3}{\dfrac{x-3}{x+1} + 1}$$

$$= \frac{x-3-3x-3}{x-3+x+1}$$

$$= \frac{-x-3}{x-1}$$

$$f(f(f(x))) = f\left(\frac{-x-3}{x-1}\right)$$

$$= \frac{\dfrac{-x-3}{x-1} - 3}{\dfrac{-x-3}{x-1} + 1}$$

$$= \frac{-x-3-3x+3}{-x-3+x-1}$$

$$= \frac{-4x}{-4}$$

$$= x$$

$$i.e.:\ f(fx) = f^{-1}(x)$$

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Chapter 2 – Coordinate geometry and matrices

Solutions to Exercise 2A

1 a $3x - 4 = 2x + 6$

$x = 10$

b $8x - 4 = 3x + 1$

$5x = 5$

$x = 1$

c $3(2 - x) - 4(3 - 2x) = 14$

$6 - 3x - 12 + 8x = 14$

$5x - 20 = 0$

$x = 4$

d $\frac{3x}{4} - 4 = 17$

$\frac{x}{4} = 7$

$x = 28$

e $6 - 3y = 5y - 62$

$8y = 68$

$y = \frac{17}{2}$

f $\frac{2}{3x - 1} = \frac{3}{7}$

$14 = 9x - 3$

$x = \frac{17}{9}$

g $\frac{2x - 1}{3} = \frac{x + 1}{4}$

$8x - 4 = 3x + 3$

$5x = 7$

$x = \frac{7}{5}$

h $\frac{2(x - 1)}{3} - \frac{(x + 4)}{2} = \frac{5}{6}$

$4x - 4 - 3x - 12 = 5$

$x = 21$

i $4y - \frac{3y + 4}{2} + \frac{1}{3} = \frac{5(4 - y)}{3}$

$24y - 9y - 12 + 2 = 40 - 10y$

$25y = 50$

$y = 2$

j $\frac{x + 1}{2x - 1} = \frac{3}{4}$

$4x + 4 = 6x - 3$

$2x = 7$

$x = \frac{7}{2}$

2 a $x - 4 = y \ldots (1)$

$4y - 2x = 8 \ldots (2)$

$(2) + 2 \times (1) \Rightarrow 4y - 8 = 8 + 2y$

$2y = 16$

$y = 8$

$\Rightarrow x = 12$

b $9x + 4y = 13 \ldots (1)$

$2x + y = 2 \ldots (2)$

$(1) - 4 \times (2) \Rightarrow x = 5$

$\Rightarrow 10 + y = 2$

$y = -8$

c $7x - 3y = 18 \ldots (1)$

$22x + 5y = 11 \ldots (2)$

$5 \times (1) + 3 \times (2) \Rightarrow 41x = 123$

$x = 3$

$\Rightarrow 6 + 5y = 11$
$y = 1$

d $5x + 3y = 13 \ldots (1)$
$7x + 2y = 16 \ldots (2)$
$3 \times (2) - 2 \times (1) \Rightarrow 11x = 22$
$x = 2$
$\Rightarrow 10 + 3y = 13$
$y = 1$

e $19x + 17y = 0 \ldots (1)$
$2x - y = 53 \ldots (2)$
From (1)$y = \frac{-19}{17}x$
$\Rightarrow 2x + \frac{19}{17}x = 53$
$53x = 17 * 53$
$x = 17$
$\Rightarrow 34 - y = 53$
$y = -19$

f $\frac{x}{5} + \frac{y}{2} = 5 \ldots (1)$
$x - y = 4 \ldots (2)$
$(2) + 2 \times (1) \Rightarrow \frac{7x}{5} = 14$
$x = 10$
$\Rightarrow 10 - y = 4$
$y = 6$

3 $l = w + 4 \ldots (1)$
$2(l - 5) + 2(w - 2) = 18 \ldots (2)$
Substitute from (1) into (2)
$w - 1 + w - 2 = 9$
$w = 6$ cm
$\Rightarrow l = 10$ cm

4 Let g represent the number of goals scored, and $t$the number of throws.
$t_0 = t_j \ldots (1)$
$g_0 = 2g_j \ldots (2)$
$t_j + 2g_j = 11 \ldots (3)$
$t_j + 2g_0 = 19 \ldots (4)$
Subtract Equation(3) from Equation(4)
$\Rightarrow 2g_0 - 2g_j = 8$
Substitute from Equation (2)
$4g_j - 2g_j = 8$
$g_j = 4$
$g_0 = 8$
John scored 4 goals and David scored 8.

5 **a** $w = 800 + 20n$

b $w = 800 + 20(30)$
$\$w = \$1400$

c $1620 = 800 + 20n$
$20n = 820$
$n = 41$ units

6 **a** $V = 250 + 15t$

b $V = 250 + 15(60)$
$V = 1150$L

c $5000 = 250 + 15t$
$t = \frac{4750}{15} = \frac{950}{3}$ min
$t = 5$h 16 min 40s

7 **a** $V = 10000 - 10t$

b $V = 10000 - 10(60)$
$= 9400$ L

c $0 = 10000 - 10t$
$t = 1000$ min
$t = 16$ h 40 min

8 $\frac{x}{240} + \frac{x}{320} = \frac{35}{60}$

$$\frac{7x}{12} = 80 \times \frac{7}{12}$$
$x = 80$ km

9 $\frac{x}{48} + \frac{x}{4.8} = 24 - 2$

$11x = 22 \times 48$

$x = 96$ km

10 **a** $C = 100 + 25t$

b **i** $C = 100 + 252$

$\$C = \$150$

ii $C = 100 + 25(2.5)$

$\$C = \$162.50$

c **i** $375 = 100 + 25t$

$t = 11$ h

ii $400 = 100 + 25t$

$t = 12$ h

Solutions to Exercise 2B

1 a $ax + n = m$

$$x = \frac{m - n}{a}$$

b $ax + b = bx$

$$x = \frac{b}{b - a}$$

c $\frac{ax}{b} + c = 0$

$$x = \frac{-bc}{a}$$

d $px = qx + 5$

$$x = \frac{5}{p - q}$$

e $mx + n = nx - m$

$$(m - n)x = -(n + m)$$

$$x = \frac{n + m}{n - m}$$

f $\frac{1}{x + a} = \frac{b}{x}$

$$x = b(x + a)$$

$$(1 - b)x = ba$$

$$x = \frac{ba}{1 - b}$$

g $\frac{b}{x - a} = \frac{2b}{x + a}$

$$bx + ab = 2bx - 2ab$$

$$bx = 3ab$$

$$x = 3a,\ b \neq 0$$

h $\frac{x}{m} + n = \frac{x}{n} + m$

$$nx + n^2m = mx + m^2n$$

$$(n - m)x = nm(m - n)$$

$$x = -mn$$

i $-b(ax + b) = a(bx - a)$

$$-bax - b^2 = abx - a^2$$

$$2abx = a^2 - b^2$$

$$x = \frac{a^2 - b^2}{2ab}$$

j $p^2(1 - x) - 2pqx = q^2(1 + x)$

$$p^2 - (p^2 + 2pq)x = q^2 + q^2x$$

$$p^2 - q^2 = (p + q)^2x$$

$$x = \frac{p - q}{p + q}$$

k $bx - ab = ax + 2b$

$$(b - a)x = 3ab$$

$$x = \frac{3ab}{b - a}$$

l $\frac{x}{a - b} + \frac{2x}{a + b} = \frac{1}{a^2 - b^2}$

$$x(a + b) + 2x(a - b) = 1$$

$$x(a + b + 2a - 2b) = 1$$

$$x = \frac{1}{3a - b}$$

m $\frac{p - qx}{t} + p = \frac{qx - t}{p}$

$$p^2 - qpx + p^2t = qtx - t^2$$

$$qtx + qpx = p^2 + p^2t + t^2$$

$$x = \frac{p^2 + p^2t + t^2}{q(t + p)}$$

n $\frac{1}{x + a} + \frac{1}{x + 2a} = \frac{2}{x + 3a}$

$$(x + 2a)(x + 3a) + (x + a)(x + 3a)$$
$$= 2(x + a)(x + 2a)$$

$$2x^2 + 9ax + 9a^2 = 2x^2 + 6ax + 4a^2$$
$$3ax = -5a^2$$
$$x = \frac{-5a}{3}$$

2 a $ax + y = c \ldots (1)$

$x + by = d \ldots (2)$

$(1) - a \times (2)$

$\Rightarrow y(1 - ab) = c - ad$

$y = \dfrac{c - ad}{1 - ab}$

Equation (2)$-b\times$ Equation (1)

$\Rightarrow x(1 - ab) = d - bc$

$x = \dfrac{d - bc}{1 - ab}$

b $ax - by = a^2 \ldots (1)$

$bx - ay = b^2 \ldots (2)$

$b\times$ Equation(1)$-a\times$ Equation (2)

$\Rightarrow (-b^2 + a^2)y = a^2b - b^2a$

$y = \dfrac{ab(a - b)}{a^2 - b^2}$

$y = \dfrac{ab}{a + b}$

$a\times$ Equation(1) $-b\times$ Equation (2)

$\Rightarrow (a^2 - b^2)x = a^3 - b^3$

$x = \dfrac{a^3 - b^3}{a^2 - b^2}$

$x = \dfrac{a^2 + ab + b^2}{a + b}$

c $ax + by = t \ldots (1)$

$ax - by = s \ldots (2)$

$(1) + (2) \Rightarrow 2ax = t + s$

$x = \dfrac{t + s}{2a}$

Equation (1) - Equation (2)

$\Rightarrow 2by = t - s$

$y = \dfrac{t - s}{2b}$

d $ax + by = a^2 + 2ab - b^2 \ldots (1)$

$bx + ay = a^2 + b^2 \ldots (2)$

$a \times (1) - b \times (2)$

$\Rightarrow (a^2 - b^2)x$

$= a^3 + 2a^2b - ab^2 - a^2b - b^3$

$x = \dfrac{a^3 + a^2b - ab^2 - b^3}{a^2 - b^2}$

$x = \dfrac{(a + b)(a^2 - b^2)}{a^2 - b^2}$

$x = a + b$

Substitute into (2)

$\Rightarrow b(a + b) + ay = a^2 + b^2$

$ay = a^2 + b^2 - ba - b^2$

$y = a - b$

e $(a + b)x + cy = bc \ldots (1)$

$(b + c)y + ax = -ab \ldots (2)$

$a \times (1) - (a + b) \times (2)$

$\Rightarrow (ac - ab - b^2 - ac - bc)y$

$= abc + a^2b + b^2 \quad y = \dfrac{ab(c + a + b)}{-b(a + b + c)}$

$y = -a$

Substitute into (2)

$\Rightarrow (-ab - ac) + ax = -ab$

$x = -b + b + c$

$x = c$

f $3(x - a) - 2(y + a) = 5 - 4a \ldots (1)$

$\Rightarrow 3x - 2y = 5 + a$

$2(x + a) + 3(y - a) = 4a - 1 \ldots (2)$

$\Rightarrow 2x + 3y = 5a - 1$

$3 \times (1) + 2 \times (2)$

$\Rightarrow 13x = 15 + 3a + 10a - 2$

$13x = 13 + 13a$

$x = 1 + a$

Substitute into (1)

$\Rightarrow 3 + 3a - 2yt = 5 + a$

$y = -1 + a$

$y = a - 1$

3 **a** $s = a(2a + 1)$

$s = 2a^2 + a$

b $h = a(2 + h)$

$h = 2a + ah$

$(1 - a)h = 2a$

$h = \dfrac{2a}{1 - a}$

$s = a\left(\dfrac{2a}{1 - a}\right)$

$s = \dfrac{2a^2}{1 - a}$

$h = \dfrac{1}{1 + a}$

c $as = a + \dfrac{1}{1 + a}$

$s = 1 + \dfrac{1}{a + a^2}$

$s = 1 + \dfrac{1}{a + a^2}$

$s = \dfrac{a^2 + a + 1}{a^2 + a}$

d $ah = a + h$

$(a - 1)h = a$

$h = \dfrac{a}{a - 1}$

$as = s + \dfrac{a}{a - 1}$

$(a - 1)s = \dfrac{a}{a - 1}$

$s = \dfrac{a}{(a - 1)^2}$

e $s = (3a^2)^2 + a(3a^2)$

$s = 9a^4 + 3a^3$

$s = 3a^3(3a + 1)$

f $as = a + 2(a - s)$

$as = a + 2a - 2s$

$(a + 2)s = 3a$

$s = \dfrac{3a}{a + 2}$

g $s = 2 + a\left(a - \dfrac{1}{a}\right) + \left(a - \dfrac{1}{a}\right)^2$

$s = 2 + a^2 - 1 + a^2 - 2 + \dfrac{1}{a^2}$

$s = 2a^2 - 1 + \dfrac{1}{a^2}$

h $3s - ah = a^2 \ldots (1)$

$as + 2h = 3a \ldots (2)$

$2\times$ Eq(1)$+a\times$ Eq(2)$\Rightarrow (6 + a^2)s = 5a^2$

$s = \dfrac{5a^2}{6 + a^2}$

4 $ax + by = p \ldots (1)$

$bx - ay = q \ldots (2)$

$a \times (1) + b \times (2) \Rightarrow (a^2 + b^2)x = pa + bq$

$x = \dfrac{ap + bq}{a^2 + b^2}$

$b \times (1) - a \times (2) \Rightarrow (b^2 + a^2)y = bp - aq$

$y = \dfrac{bp - aq}{a^2 + b^2}$

5 $bx + ay = ab \ldots (1)$

$ax + by = ab \ldots (2)$

$a \times (1) - b \times (2)$

$\Rightarrow (a^2 - b^2)y = ab(a - b)$

$y = \dfrac{ab}{a + b}$

$b \times (1) - a \times (2)$

$\Rightarrow (b^2 - a^2)x = ab(b - a)$

$x = \dfrac{ab}{a + b}$

Solutions to Exercise 2C

1 **a** $\sqrt{205}$

b $\left(1, -\frac{1}{2}\right)$

c $-\frac{13}{6}$

d $13x + 6y = 10$

e $13x + 6y = 43$

f $13y - 6x = -\frac{25}{2}$

2 **a** $\left(3, \frac{15}{2}\right)$

b $\left(\frac{-5}{2}, -2\right)$

c $\left(\frac{3}{2}, \frac{1}{2}\right)$

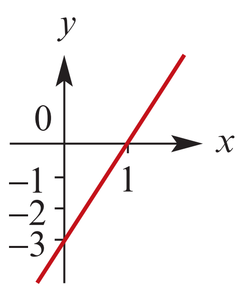
3 **a** $(4, 7)$

b $(5, -2)$

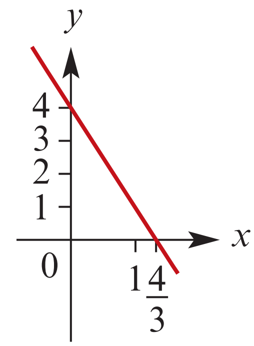
c $(2, 19)$

d $(-2, -9)$

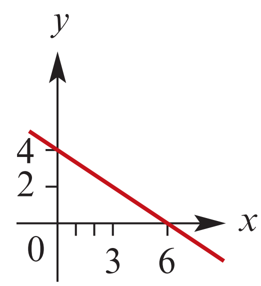
4 **a**



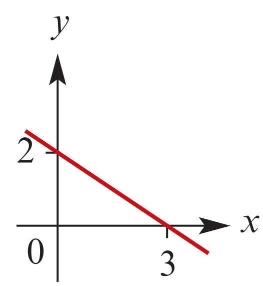
b



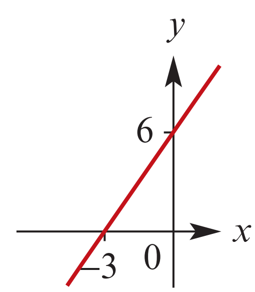
c



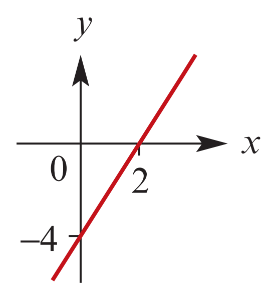
d



e



f



5 **a** $y - 2 = 2(x - 4)$

$y = 2x - 6$

b $y - 4 = -3(x + 3)$

$y = -3x - 5$

$m = \frac{4}{3}$

c $y - 3 = \frac{4}{3}(x - 1)$

$y = \frac{4}{3}x + \frac{5}{3}$ or $3y - 4x = 5$

d $m = 2$

$y - 5 = 2(x - 2)$

$y = 2x + 1$

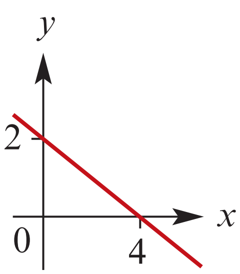
6 **a** $\dfrac{\frac{x}{-3} + \frac{y}{2}}{\frac{y}{2} - \frac{x}{3}} = 1$

b $\frac{x}{4} + \frac{y}{6} = 1$

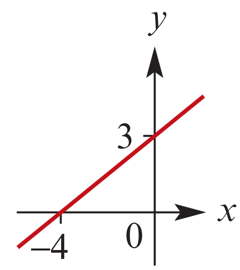
c $\dfrac{\frac{x}{-4} + \frac{y}{-3} = 1}{\left(-\frac{x}{4} - \frac{y}{3} = 1\right)}$

d $\dfrac{\frac{x}{6} + \frac{y}{-2} = 1}{\frac{x}{6} - \frac{y}{2} = 1}$

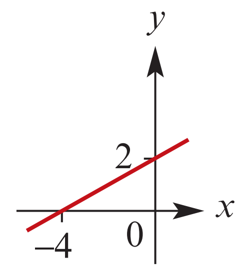
7 **a** $\frac{x}{4} + \frac{y}{2} = 1$



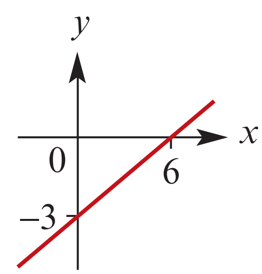
b $\frac{y}{3} - \frac{x}{4} = 1$



c $\frac{y}{2} - \frac{x}{4} = 1$



d $\frac{x}{6} - \frac{y}{3} = 1$



8 (600,35) (800,46)

$m = \frac{11}{200}$

$C - 35 = \frac{11}{200}(n - 600)$

$C = \frac{11}{200}n + 2$

$C(1000) = \frac{11}{200}(1000) + 2$

$\$C = \$57$

9 **a** (120,775) (160,975)

$$m = 5$$
$$C - 775 = 5(n - 120)$$
$$C = 5n + 175$$

b *yes*

c \$175

10 a $d = \sqrt{1^2 + 2^2}$
$= \sqrt{5} \approx 2.236$

b $d = \sqrt{1^2 + 1^2}$
$= \sqrt{2} \approx 1.414$

c $d = \sqrt{5^2 + 2^2}$
$= \sqrt{29} \approx 5.385$

d $d = \sqrt{2^2 + 18^2}$
$= \sqrt{326}$
$= 2\sqrt{82} \approx 18.111$

e $d = \sqrt{4^2 + 2^2}$
$= \sqrt{20}$
$= 2\sqrt{5} \approx 4.472$

f $d = \sqrt{3^2 + 4^2}$
$= 5$

11 a i $y - 6 = 2(x + 1)$
$y = 2x + 4$

ii $y - 6 = \frac{-1}{2}(x - 1)$
$y = \frac{-1}{2}x + \frac{13}{2}$ or $2y + x = 13$

b i $y - 3 = -2(x - 2)$
$y = -2x + 7$

ii $y - 3 = \frac{1}{2}(x - 2)$
$y = \frac{1}{2}x + 2$
or $2y - x = 4$

12 $(3, 3)$, $m = \frac{-1}{-3/6} = 2$
$y - 3 = 2(x - 3)$
$y = 2x - 3$

13 $5 = \sqrt{3^2 + (y + 1)^2}$
$25 = 9 + (y + 1)^2$
$y + 1 = \pm 4$
$y = -1 \pm 4$
$y = -5, 3$

14 $10 = \sqrt{8^2 + (y - 6)^2}$
$100 = 64 + (y - 6)^2$
$y - 6 = \pm 6$
$y = 6 \pm 6$
$y = 0, 12$

15 $26 = \sqrt{10^2 + (y - 8)^2}$
$676 = 100 + (y - 6)^2$
$y - 8 = \pm 576$
$y = 8 \pm 24$
$y = -16, 32$

16 **a** **i** $y - 3 = \frac{-2}{5}(x + 1)$

$y = \frac{-2}{5}x + \frac{13}{5}$ or $5y + 2x = 13$

ii $y - 3 = \frac{-4}{5}(x + 1)$

$y = \frac{-4}{5}x + \frac{11}{5}$ or $5y + 4x = 11$

b **i** $y - 3 = \frac{5}{2}(x + 1)$

$y = \frac{5}{2}x + \frac{11}{2}$ or $2y - 5x = 11$

ii $y - 3 = \frac{5}{4}(x + 1)$

$y = \frac{5}{4}x + \frac{17}{4}$ or $4y - 5x = 17$

17 **a** $m = \frac{6 - 1}{4 + 4} = \frac{5}{8}$

$\theta = \tan^{-1}\left(\frac{5}{8}\right) = 32.01°$

b $m = \frac{-1}{2}$

$\theta = \tan^{-1}\left(\frac{-1}{2}\right) = 153.43°$

c $m = \frac{3}{2}$

$\theta = \tan^{-1}\left(\frac{3}{2}\right) = 56.31°$

d $m = \frac{-10}{6}$

$\theta = \tan^{-1}\left(\frac{-5}{3}\right) = 120.96°$

18 $m_1 = 2$

$m_2 = -3$

$\theta_1 = 63.43°$

$\theta_2 = 108.43°$

$\alpha = \theta_2 - \theta_1 = 45°$

19 $\sqrt{(-2 - a)^2 + (-2)^2} = 2\sqrt{5^2 + 1^2}$

square both sides

$4 + 4a + a^2 + 4 = 4(26)$

$a^2 + 4a - 96 = 0$

$(a + 2)^2 - 100 = 0$

$a + 2 = \pm 10$

$a = -12, 8$

20 **a** $\frac{5 - 7}{7 - 1} = \frac{-1}{3}$

$m = 3$

midpoint $= (4, 6)$

$y - 6 = 3(x - 4)$

$y = 3x - 6$

b $\overrightarrow{BC}$ *has* $m = 1$

$1y = x - 2$

$2y = 3x - 6$

$x = 2, y = 0$

point of intersection: (2,0)

21 $k = h + 1 \ldots (1)$

$\sqrt{h^2 + (2-k)^2} = 5 \ldots (2)$

$\Rightarrow h^2 + (k-2)^2 = 25$

Substitute in (1)

$\Rightarrow h^2 + (h-1)^2 = 25$

$2h^2 - 2h + 1 = 25$

$2h^2 - 2h - 24 = 0$

$h^2 - h - 12 = 0$

$(h+3)(h-4) = 0$

$h = -3, 4$

Substitute in (1)

$\Rightarrow k = -2, 5$

$(h,\ k) = (-3,\ -2)$ or$(4,5)$

22 $P = (3,0)$, $Q = (0,2)$

a $\overrightarrow{QR} : y - 2 = \frac{1}{2}x$

$$y = \frac{1}{2}x + 2$$

if $x = 2a$,

$$y = a + 2$$

$$R = (2a, a+2)$$

b $\overrightarrow{PR}$: *has* $m = \dfrac{a+2}{2a-3}$

but $m = -2$

$$\therefore \quad -2 = \frac{a+2}{2a-3}$$

$$6 - 4a = a + 2$$

$$4 = 5a$$

$$a = \frac{4}{5}$$

23 a AB has gradient $-3m$

$$-3m = \frac{1-4}{1+1} = \frac{-3}{2}$$

$$m = \frac{1}{2}$$

b $AC : 1y = \frac{3}{2}x - \frac{1}{2}$

$$BC : y - 4 = \frac{1}{2}x + \frac{1}{2}$$

$$2y = \frac{1}{2}x + \frac{9}{2}$$

$$1 - 20 = x - 5$$

$$x = 5$$

$$\Rightarrow 2y = \frac{5}{2} + \frac{9}{2} = 7$$

$$C = (5, 7)$$

c $AC = \sqrt{4^2 + 6^2} = \sqrt{52} = 2\sqrt{13}$

$AB = \sqrt{2^2 + 3^2} = \sqrt{13}$

$AC = 2AB$

QED

24 a BC has gradient $= \dfrac{-1}{\text{grad}(AB)} = \dfrac{1}{3}$

$$y - 8 = \frac{1}{3}(x - 2)$$

$BC : y = \frac{1}{3}x + \frac{22}{3}$ or $3y - x = 22$

b $1y = \frac{1}{3}x + \frac{22}{3}$

$$2y = x - 2$$

$$2 - 10 = \frac{2x}{3} + \frac{28}{3}$$

$$x = 14$$

$$\Rightarrow 2y = 12$$

$$C = (14, 12)$$

c *because it is a rectangle*

$D = C - (B - A)$

$= (14, 12) - ((2, 8) - (4, 2))$

$= (14, 12) - (-2, 6)$

$= (16, 6)$

d Area $= AB \times BC$

$= \sqrt{2^2 + 6^2} \times \sqrt{12^2 + 4^2}$

$= \sqrt{40} \times \sqrt{160}$

$= 2\sqrt{10} \times 4\sqrt{10}$

$= 80$ square units

25 **a** (2,3)

b BD has $m = -5$

$y - 3 = -5(x - 2)$

$y = -5x + 13$

c **i** BC has $m = \dfrac{1}{\text{grad}(AC)} = \dfrac{3}{2}$

$y - 1 = \dfrac{3}{2}(x - 5)$

$y = \dfrac{3}{2}x - \dfrac{13}{2}$ or $2y = 3x - 13$

ii $1y = \dfrac{3}{2}x - \dfrac{13}{2}$

$2y = -5x + 13$

$1 - 20 = \dfrac{13x}{2} - \dfrac{39}{2}$

$x = 3$

$\Rightarrow 2y = -2$

$B = (3, -2)$

iii $D = A + (C - B)$

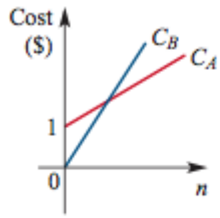
$D = (-1, 5) + (5, 1) - (3, -2)$

$D = (1, 8)$

Solutions to Exercise 2D

1 a $C_A = 0.4n + 1; C_B = 0.6n$

b The graphs are straight lines as shown here.



c $C_A = C_B$

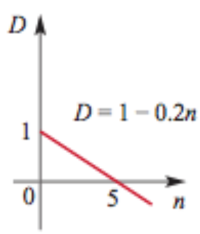
$0.4n + 1 = 0.6n$

$0.2n = 1$

$n = 5$

The charge is the same for 5 km.

d $D = C_A\ C_B = 1\ 0.2n$, so the graph is a straight line with intercept (0, 1) and gradient −0.2.



D gives the difference in charges of the two firms in terms of the distance travelled.

2 a Since the journey lasts 4 hours, $4 - T$ hours are spent on country roads.

b **i** 90 km/h for T hours gives a distance of $90T$ km.

ii 70 km/h for $4 - T$ hours gives a distance of $70(4 - T)$ km.

c **i** $90T + 70(4 - T) = 300$

$20T = 20$

$T = 1$

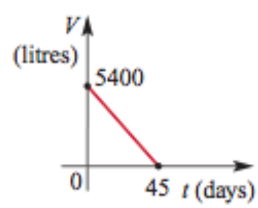
ii 90 km on the freeway, $70 \times 3 = 210$ km.

3 Let $L = at + b$, since a constant rate of decrease means the relation is linear.

a $t = 20,\ L = 3000$: $20a + b$
$= 3000 \quad \ldots 1$
$t = 35,\ L = 1200$: $35a + b$
$= 1200 \quad \ldots 2$
$2\ -1$: $15a = -1800$, so $a = -120$
Sub in 1: $b = 3000 - 20(-120)$
$= 5400$
$L = -120t + 5400$

b 5400 litres (at $t = 0$ when it was filled)

c The tank will be empty when
$-120t + 5400 = 0 \Rightarrow t = 45$



d From **c**, the domain is [0,45]

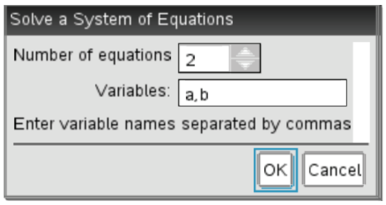
e 45 days

f The coefficient of t, i.e. −120, in the linear relation represents the rate of ‘increase’. So the water is decreasing, or leaving the tank, at 120 litres/day

Graphic calculator techniques for

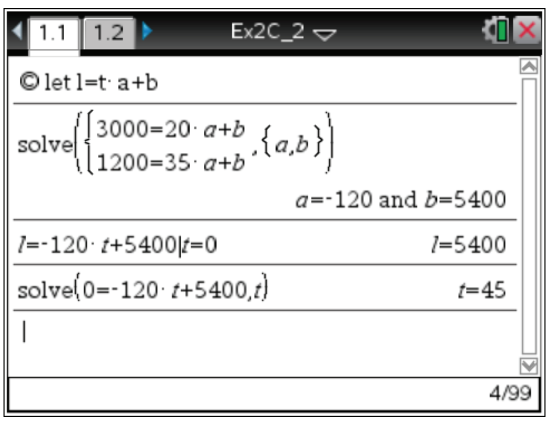
question

In a **Calculator** page insert the linear equation template (b>**Algebra**>**Solve System of Equations**>**Solve System of Equations**) and complete the dialogue box as shown.



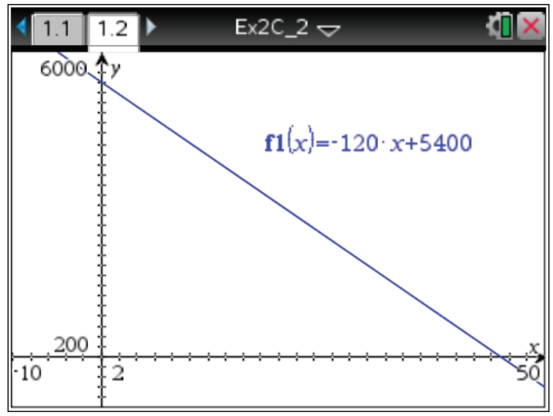
Defining the rule(b>**Actions**>**Define**) allows you to use the rule elsewhere by just typing in *l*.

To find when the tank will be empty use the solve command (b>**Algebra**>**Solve**) with the equation equal to zero.

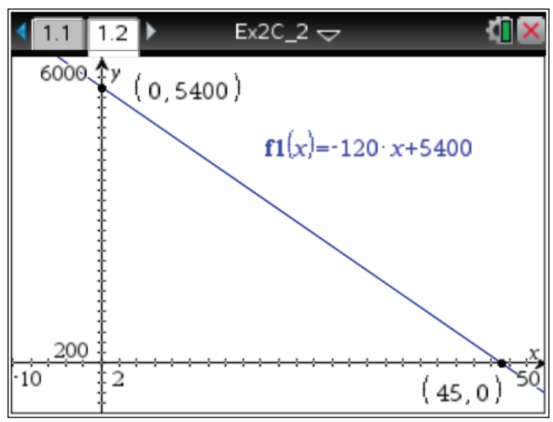


Insert a **Graphs** page (/ + **I**) and type in the rule $f1(x) = -120x + 5400$

Use b>**Window/Zoom**>**Window Settings** to set an appropriate window.

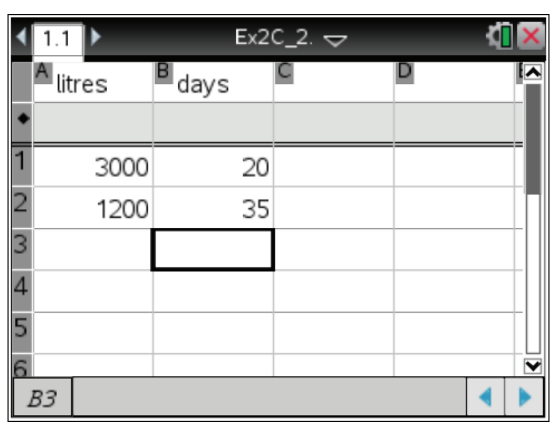


Note: the default digit display for **Graphs** is float3. You can increase this either in the settings or by placing the cursor over the designated value and pressing the + key. Hence 5.4E+3 now displays as 5400.

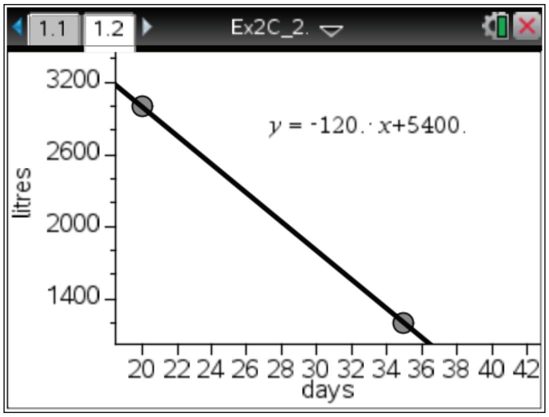


Alternative method as is current example

In a Lists & Spreadsheet page enter the list data as shown.



Insert a **Data & Statistics** page(/ + *I*) and plot the data as shown.
A linear regression can be obtained using b>**Analyze>Regression>Show Linear (mx+b)**



4 a $y = \frac{22.5}{10}x$
$= \frac{9}{4}x$

b $OA = \sqrt{10^2 + 22.5^2}$
$= 24.622$ km(24622 m)

c $y - 9 = \frac{9 - 22.5}{23 - 10}(x - 23)$
$y = -\frac{27}{26}x + \frac{855}{26}$

d The midpoint of AB has coordinates $\left(\frac{33}{2}, \frac{61}{4}\right)$ and the perpendicular has gradient $\frac{26}{27}$, so its equation is:
$y - \frac{61}{4} = \frac{26}{27}\left(x - \frac{33}{2}\right)$
As the port has x-coordinate 52, substitute to find the y-coordinate:
$y - \frac{61}{4} = \frac{26}{27}\left(52 - \frac{33}{2}\right)$
$y = \frac{5339}{108}$

5 a i grad $AB = \frac{2 - 4}{2\ 1.5} = -4$

ii grad $AD =$ grad $BC = \frac{6 - 4}{6 - 1.5}$
$= \frac{4}{9}$

b i $y - 4 = \frac{4}{9}(x - 1.5)$
$y = \frac{4}{9}x + \frac{10}{3}$

ii $y - 6 =$ d$(x - 6)$
$y = -4x + 30$

c Notice that A is '0.5 across and 2 down' from B, so D is '0.5 across and 2 down' from C, so its coordinates are (6.5, 4).
As B and D have the same y-coordinate, the equation of BD is $y = 4$.
The equation of AC is given by:
$y - 6 = \frac{6 - 2}{6 - 2}(x - 6)$
$y = x$

d The diagonals intersect at (4, 4).

6 a M is the midpoint of AC
Coordinates of $M\left(\frac{5 + 9}{2}, \frac{0 + 10}{2}\right) =$
$M(7, 5)$
Coordinates of $N\left(\frac{13 + 9}{2}, \frac{0 + 10}{2}\right) =$
$M(11, 5)$

b i Gradient of $AC = \frac{10}{4} = \frac{5}{2}$
∴ Equation of lineAC is
$y - 0 = \frac{5}{2}(x - 5)$

ii Gradient of $BC = \frac{10}{-4} = -\frac{5}{2}$

$\therefore$ Equation of lineBC is

$y - 0 = -\frac{5}{2}(x - 13)$

iii Gradient of $MN = 0$

$\therefore$ Equation of line MN is

$y = 5$

c Line perpendicular to AC has gradient$-\frac{2}{5}$

$\therefore$ equation of line passing through M perpendicular to AC has equation

$y - 5 = -\frac{2}{5}(x - 7)$

Line perpendicular to BC has gradient$\frac{2}{5}$

$\therefore$ equation of line passing through N perpendicular to BC has equation

$y - 5 = \frac{2}{5}(x - 11)$

Intersect when

$$-\frac{2}{5}(x - 7) = \frac{2}{5}(x - 11)$$

$$7 - x = x - 11$$

$$18 = 2x$$

$$x = 9$$

When $x = 9, y = \frac{21}{5}$

Solutions to Exercise 2E

1 a $\begin{bmatrix}1 & 3\\2 & -1\end{bmatrix}+\begin{bmatrix}2 & -1\\-2 & 3\end{bmatrix}=\begin{bmatrix}3 & 2\\0 & 2\end{bmatrix}$

b $\begin{bmatrix}4\\-2\end{bmatrix}-\begin{bmatrix}6\\-2\end{bmatrix}=\begin{bmatrix}-2\\0\end{bmatrix}$

2 $2\mathbf{A}+\mathbf{X}=\mathbf{B}$

$$\mathbf{X}=\mathbf{B}-2\mathbf{A}$$

$$\mathbf{X}=\mathbf{B}-2\mathbf{A}$$

$$\mathbf{X}=\begin{bmatrix}1 & -4\\-3 & 6\end{bmatrix}-\begin{bmatrix}2 & 6\\-4 & 8\end{bmatrix}$$

$$=\begin{bmatrix}-1 & -10\\1 & -2\end{bmatrix}$$

3 $\begin{bmatrix}3 & 1\\2 & 5\end{bmatrix}\begin{bmatrix}2\\5\end{bmatrix}=\begin{bmatrix}10\\24\end{bmatrix}$

4 a $\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}+\begin{bmatrix}-2 & -2\\3 & 2\end{bmatrix}=\begin{bmatrix}0 & -1\\6 & 4\end{bmatrix}$

b $\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}-2 & -2\\3 & 2\end{bmatrix}=\begin{bmatrix}-1 & -2\\0 & -2\end{bmatrix}$

c $\begin{bmatrix}-2 & -2\\3 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}=\begin{bmatrix}-10 & -6\\12 & 7\end{bmatrix}$

d $\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}-\begin{bmatrix}-2 & -2\\3 & 2\end{bmatrix}=\begin{bmatrix}4 & 3\\0 & 0\end{bmatrix}$

e $k\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}=\begin{bmatrix}2k & k\\3k & 2k\end{bmatrix}$

f

$$2\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}+3\begin{bmatrix}-2 & -2\\3 & 2\end{bmatrix}=\begin{bmatrix}4 & 2\\6 & 4\end{bmatrix}+\begin{bmatrix}-6 & -6\\9 & 6\end{bmatrix}$$

$$=\begin{bmatrix}-2 & -4\\15 & 10\end{bmatrix}$$

g

$$\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}-2\begin{bmatrix}-2 & -2\\3 & 2\end{bmatrix}=\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}-\begin{bmatrix}-4 & -4\\6 & 4\end{bmatrix}$$

$$=\begin{bmatrix}6 & 5\\-3 & -2\end{bmatrix}$$

5 a $2\begin{bmatrix}3 & 4\\-3 & -3\end{bmatrix}=\begin{bmatrix}6 & 8\\-6 & -6\end{bmatrix}$

b $3\begin{bmatrix}0 & -4\\5 & 1\end{bmatrix}=\begin{bmatrix}0 & -12\\15 & 3\end{bmatrix}$

c

$$2\begin{bmatrix}3 & 4\\-3 & -3\end{bmatrix}+3\begin{bmatrix}0 & -4\\5 & 1\end{bmatrix}=\begin{bmatrix}6 & 8\\-6 & -6\end{bmatrix}+\begin{bmatrix}0 & -12\\15 & 3\end{bmatrix}$$

$$=\begin{bmatrix}6 & -4\\9 & -3\end{bmatrix}$$

d

$$3\begin{bmatrix}0 & -4\\5 & 1\end{bmatrix}-2\begin{bmatrix}3 & 4\\-3 & -3\end{bmatrix}=\begin{bmatrix}0 & -12\\15 & 3\end{bmatrix}-\begin{bmatrix}6 & 8\\-6 & -6\end{bmatrix}$$

$$=\begin{bmatrix}-6 & -20\\21 & 9\end{bmatrix}$$

6 a $\begin{bmatrix}1 & 0\\0 & -4\end{bmatrix}+\begin{bmatrix}-1 & -4\\5 & 0\end{bmatrix}=\begin{bmatrix}0 & -4\\5 & -4\end{bmatrix}$

b

$$\begin{bmatrix}1 & 0\\0 & -4\end{bmatrix}+3\begin{bmatrix}-1 & -4\\5 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & -4\end{bmatrix}+\begin{bmatrix}-3 & -12\\15 & 0\end{bmatrix}$$

$$=\begin{bmatrix}-2 & -12\\15 & -4\end{bmatrix}$$

c $2\begin{bmatrix}1 & 0\\0 & -4\end{bmatrix} - \begin{bmatrix}-1 & -4\\5 & 0\end{bmatrix} + \begin{bmatrix}0 & -4\\1 & 1\end{bmatrix}$

$= \begin{bmatrix}2 & 0\\0 & -8\end{bmatrix} - \begin{bmatrix}-1 & -4\\5 & 0\end{bmatrix} + \begin{bmatrix}0 & -4\\1 & 1\end{bmatrix}$

$= \begin{bmatrix}3 & 0\\-4 & -7\end{bmatrix}$

7 $\mathbf{2A} - \mathbf{3X} = \mathbf{B}$

Therefore, $\mathbf{X} = \frac{1}{3}(\mathbf{2A} - \mathbf{B})$

Therefore, $\mathbf{X} = \begin{bmatrix}2 & 3\\-\frac{4}{3} & -3\end{bmatrix}$

$3\mathbf{A} + \mathbf{2Y} = 2\,\mathbf{B}$

Therefore, $\mathbf{Y} = \frac{1}{2}(2\mathbf{B} - 3\mathbf{A})$

Therefore $\mathbf{Y} = \begin{bmatrix}-\frac{9}{2} & -8\\\frac{5}{2} & 7\end{bmatrix}$

8 $\mathbf{AX} = \begin{bmatrix}4\\-5\end{bmatrix}$

$\mathbf{BX} = \begin{bmatrix}6\\-1\end{bmatrix}$

$\mathbf{IX}\begin{bmatrix}2\\-1\end{bmatrix}$

$\mathbf{AI} = \begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix}$

$\mathbf{IB} = \begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}$

$\mathbf{AB} = \begin{bmatrix}3 & -2\\-3 & 3\end{bmatrix}$

$\mathbf{BA} = \begin{bmatrix}3 & -6\\-1 & 3\end{bmatrix}$

$\mathbf{A}^2 = \begin{bmatrix}3 & -8\\-4 & 11\end{bmatrix}$

$\mathbf{B}^2 = \begin{bmatrix}9 & 0\\0 & 1\end{bmatrix}$

Solutions to Exercise 2F

1 a $3x + 2y = 6 \ldots (1)$

$x - y = 7 \ldots (2)$

Multiply (2) by 2 and add to (1)

$5x = 20$

$x = 4$

$\therefore y = -3$

b $2x + 6y = 0 \ldots (1)$

$y - x = 2 \ldots (2)$

Multiply (2) by 2 and add to (1)

$8y = 4$

$y = \dfrac{1}{2}$

$\therefore x = -\dfrac{3}{2}$

c $4x - 2y = 7 \ldots (1)$

$5x + 7y = 1 \ldots (2)$

Multiply (2) by 4, (1)by 5 and subtract

$38y = -31$

$y = -\dfrac{31}{38}$

$\therefore x = \dfrac{51}{38}$

d $2x - y = 6 \ldots (1)$

$4x - 7y = 5 \ldots (2)$

Multiply (1) by 2,and subtract

$-5y = -7$

$y = \dfrac{7}{5}$

$\therefore x = \dfrac{37}{10}$

2 a one solution

b infinitely many solutions

c no solutions

3 The two corresponding lines are parallel but not equal, and have no intersection.

4 $x - y = 6$

$y = x - 6$

Let$x = \lambda, y = \lambda - 6$

$y = \lambda - 6$

5 Lines are parallel. The gradients are the same and the lines have a common point.

$$3x + my = 5 \ldots (1)$$

$$(m + 2)x + 5y = m \ldots (2)$$

$$\text{Gradient of } (1) = -\frac{3}{m}$$

$$\text{Gradient of } (2) = -\frac{m+2}{5}$$

For the lines to coincide:

$$\frac{3}{m} = \frac{m+2}{5}$$

$$m^2 + 2m - 15 = 0$$

$$(m - 3)(m + 5) = 0$$

$$m = 3 \text{ or } m = -5$$

a If $m = 3$ the equations are

$3x + 3y = 5 \ldots (1)$

$5x + 5y = 5 \ldots (2)$

The lines don't coincide.

If $m = -5$ the equations are

$3x - 5y = 5 \ldots (1)$

$-3x + 5y = -5 \ldots (2)$

The lines coincide.

b A unique solution if $m \in \mathbb{R}\backslash\{3, -5\}$
No solution if $m = 3$

Alternative method

$$\begin{bmatrix} 3 & m \\ m+2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ m \end{bmatrix}$$

$$\begin{bmatrix} 3 & m \\ m+2 & 4 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 & m \\ m+2 & 5 \end{bmatrix} = 15 - m^2 - 2m$$

$$m^2 + 2m - 15 = 0$$

$$(m+5)(m-3) = 0$$

$$m = -5, 3$$

substitute $m = -5$

$$3x - 5y = 5$$

$$-3x + 5y = -5$$

infinite solutions
Substitute $m = 3$
$3x + 3y = 5$
$5x + 5y = 3$
No solutions
$\therefore$ **a** $m = -5$ **b** $m = 3$

6 $m = 9$

7 **a** $mx + 2y = 8 \ldots (1)$
$4x - (2 - m)y = 2m \ldots (2)$
$4 \times (1) - m \times (2)$
$\Rightarrow (8 + m(m-2))y = 32 - 2m^2$

$$y = \frac{32 - 2m^2}{m^2 - 2m + 8}$$

$$y = \frac{2(m+4)(m-4)}{(m-4)(m+2)}$$

$$y = \frac{2(m+4)}{(m+2)}$$

$m \neq 4, -2$
Substitute in (1)

$$\Rightarrow mx + 4\frac{(m-4)(m+2)}{(m-4)(m+2)} = 8$$

$$x = \frac{8(m-4)(m+2) - 4(m+4)(m-4)}{m(m-4)(m+2)}$$

$$x = \frac{4(m-4)(2m+4-m-4)}{m(m-4)(m+2)}$$

$$x = \frac{4(m-4)m}{m(m-4)(m+2)}$$

$$x = \frac{4}{m+2}, \quad m \neq -2, 0, 4,$$

b Values to test: $m = -2, 0, 4$
$m - 2$
$-2x + 2y = 8 \ldots (1)$
$4x - 4y = -4 \ldots (2)$
No solutions
$m = 0$
$10 + 2y = 8$
$y = 4$
$2 - 4x - 2y = 0$
$4x - 8 = 0$
$x = 2$ *unique solution*
(*Note*: *this value for m would not have appeared if equation* 2 *had been used to find x*.)
$m = 4$
$4x + 2y = 8 \ldots (1)$
$4x + 2y = 8 \ldots (2)$
infinite solutions
$\therefore$

i $m = -2$

ii $m = 4$

8 **a** $2x - 3y = 4 \ldots (1)$
$x + ky = 2 \ldots (2)$
$(1) - 2 \times (2)$
$\Rightarrow (3 - 2k)y = 0$
$y = 0,\ k \neq \frac{-3}{2}$
$\Rightarrow\ 2x = 4$

$x = 2$

b $k = \frac{-3}{2}$

9 $x + 5y = 4 \ldots (1)$

$2x + by = c \ldots (2)$

Gradient of (1) $= -\frac{1}{5}$

Gradient of (2) $= -\frac{2}{b}$

For the lines to be parallel or coincide:

$\frac{1}{5} = \frac{2}{b}$

$b = 10$

a A unique solution for $b \in \mathbb{R} \setminus \{10\}$

b If $b = 10$ and $c = 8$ the corresponding lines coincide and there are infinitely many solutions.

c If $b = 10$ and $c \neq 8$ the corresponding lines are parallel and there are no solutions

Solutions to Exercise 2G

1 **a**
$$2x + 3y - z = 12 \ldots (1)$$
$$2y + z = 7 \ldots (2)$$
$$2y - z = 5 \ldots (3)$$
Add (2) and (3)
$$4y = 12$$
$$y = 3$$
$$\therefore z = 1$$
Substitute in (1) to find x
$x = 2$

b
$$x + 2y + 3z = 13 \ldots (1)$$
$$-x - y + 2z = 2 \ldots (2)$$
$$-x + 3y + 4z = 26 \ldots (3)$$
Add (1) and (2)
$$y + 5z = 15 \ldots (4)$$
Subtract (2) from(3)
$$4y + 2z = 24$$
$$2y + z = 12 \ldots (5)$$
Multiply (4) by 2 and

subtract from (4)
$$-9z = -18$$
$$z = 2$$
$$\therefore y = 5$$
$$\therefore x = -3$$

c
$$x + y = 5 \ldots (1)$$
$$y + z = 7 \ldots (2)$$
$$z + x = 12 \ldots (3)$$
Subtract (2) from (3)
$$x - y = 5 \ldots (4)$$
Add (1) and(4)
$$\therefore x = 5$$
$$\therefore y = 0$$
$$\therefore z = 7$$

d
$$x - y - z = 0 \ldots (1)$$
$$5x + 20z = 50 \ldots (2)$$
$$10y - 20z = 30 \ldots (3)$$
Simplify (2) and (3)
$$x + 4z = 10 \ldots (4)$$
$$y - 2z = 3 \ldots (5)$$
Subtract (5) from (4)
$$(x - y) + 6z = 7 \ldots (6)$$
Subtract (1) from (6)
$$7z = 7$$
$$\therefore z = 1$$
$$\therefore x = 6$$
$$\therefore y = 5$$

2 **a** $y - 4z = -2$
$y = 4z - 2$

b $z = \lambda$
$y = 4\lambda - 2$
$\therefore x + 8\lambda - 4 - 3\lambda = 4$
$x = 8 - 5\lambda$

3 a $-y + 5z = 15 \quad (2) + (1)$
$-y + 5z = 15 \quad (3) - (2)$

b They are the same

c $z = \lambda$
$-y + 5\lambda = 15$
$y = 5\lambda - 15$

d $x + 10\lambda - 30 + 3\lambda = 13$
$x = 43 - 13\lambda$

4 a (1) +(2)
$\Rightarrow 2z = 10$
$z = 5$
Substitute into (1)
$\Rightarrow x - y + 5 = 4$
let $y = \lambda$
$x = \lambda - 1$

b Let $z = \lambda$
Substitute in (2)
$\Rightarrow x = 3 + \lambda$
Substitute into (1)
$\Rightarrow 6 + 2\lambda - y + \lambda = 6$
$y = 3\lambda$

c $(1) + 2 \times (2)$
$6x + 3z = 14$
Let $z = \lambda$
$6x = 14 - 3\lambda$
$x = \dfrac{14 - 3\lambda}{6}$
Substitute into (2)
$\dfrac{14 - 3\lambda}{6} + y + \lambda = 4$
$y = 4 - \dfrac{14 + 3\lambda}{6}$
$y = \dfrac{10 - 3\lambda}{6}$

5 $x + y + z + w = 4 \ldots (1)$
$x + 3y + 3z = 2 \ldots (2)$
$x + y + 2z - w = 6 \ldots (3)$
(3) − (1)
$\Rightarrow z - 2w = 2$
Let $z = t,\ t \in \mathbb{R}$
$w = \dfrac{t - 2}{2} = \dfrac{1}{2}t - 1$
(2) − (3) gives
$2y + z + w = -4$
$2y + t + \dfrac{1}{2}t - 1 = -4$
$2y = -3 - \dfrac{3}{2}t$
$y = -\dfrac{3}{2} - \dfrac{3}{4}t$
Substitute into (1)
$x - \dfrac{3}{2} - \dfrac{3}{4}t + t + \dfrac{1}{2}t - 1 = 4$
$x = 6\dfrac{1}{2} - \dfrac{3}{4}t = \dfrac{26 - 3t}{4}$
when $w = 6$, $\dfrac{t - 2}{2} = 6$ so $t = z = 14$
$y = \dfrac{-3(14 + 2)}{4} = -12$
$x = \dfrac{26 - 42}{4} = -4$

6 a $3x - y + z = 4 \ldots (1)$
$x + 2y - z = 2 \ldots (2)$
$-x + y - z = -2 \ldots (3)$
$(2) - (3) \Rightarrow 2x + y = 4$
$(3) + (1) \Rightarrow 2x = 2$
$x = 1$
$y = 2$
Substitute into (3)
$\Rightarrow -1 + 2 - z = -2$
$z = 3$

b $x - y - z = 0 \ldots (1)$

$3y + 3z = -5 \ldots (2)$

$3 \times (1) + (2)$

$\Rightarrow 3x = -5$

$x = \dfrac{-5}{3}$

$3y = -5 - 3z$

Let $z = \lambda$

$y = \dfrac{-5 - 3\lambda}{3}$

c $12x - y + z = 0$

$2y + 2z = 2$

Let $z = \lambda$

$y = 2 - 2\lambda$

$2x - 2 + 2\lambda + \lambda = 0$

$2x = 2 - 3\lambda$

$x = \dfrac{2 - 3\lambda}{2}$

Solutions to technology-free questions

1 a $3x - 2 = 4x + 6$

$4x - 3x = -2 - 6$

$x = -8$

b $\dfrac{x+1}{2x-1} = \dfrac{4}{3}$

$3(x+1) = 4(2x-1)$

$3x + 3 = 8x - 4$

$8x - 3x = 3 + 4$

$5x = 7$

$x = \dfrac{7}{5}$

c $\dfrac{3x}{5} - 7 = 11$

$\dfrac{3x}{5} = 18$

$3x = 90$

$x = 30$

d $\dfrac{2x+1}{5} = \dfrac{x-1}{2}$

$2(2x+1) = 5(x-1)$

$4x + 2 = 5x - 5$

$x = 7$

2 a $y = x + 4 \quad \ldots 1$

$5y + 2x = 6 \quad \ldots 2$

Substitute 1 into 2:

$5(x+4) + 2x = 6$

$5x + 20 + 2x = 6$

$7x = -14$

$x = -2$

Substitute into 1:

$y = -2 + 4 = 2$

b $\dfrac{x}{4} - \dfrac{y}{3} = 2 \quad \ldots 1$

$y - x = 5 \quad \ldots 2$

Multiply 1 by 12:

$3x - 4y = 24 \quad \ldots 3$

Multiply 2 by 3:

$3y - 3x = 15 \quad \ldots 4$

3 + 4 gives $-y = 39$, so $y = -39$

Substitute into 2:

$-39 - x = 5$, so $x = -44$

3 a $\dfrac{n+m}{b}$

b $\dfrac{b}{c+b}$

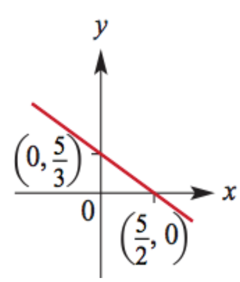
c d

d $\dfrac{6}{q-p}$

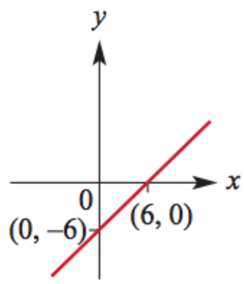
e $\dfrac{m+n}{m-n}$

f $\dfrac{a^2}{a-1}$

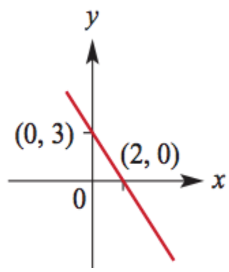
4 a intercepts $\left(\frac{5}{2}, 0\right), \left(0, \frac{5}{3}\right)$



b intercepts $(6, 0), (0, -6)$



c intercepts $(2, 0)$, $(0, 3)$



5 a $y - 3 = -2(x - 1)$
$y = -2x + 5$

b $m = \dfrac{8-4}{3-1} = 2$

$y - 4 = 2(x - 1)$

$y = 2x + 2$

c $y = -2x + 6$ has gradient $m_1 = -2$; for the gradient m_2 of a perpendicular line: $m_1 m_2 = -1$

$m_2 = \dfrac{-1}{-2} = \dfrac{1}{2}$

$y - 1 = \dfrac{1}{2}(x - 1)$

$y = \dfrac{1}{2}x + \dfrac{1}{2}$

d $y = 6 - 2x$ has gradient -2; a parallel line has the same gradient:
$y - 1 = -2(x - 1)$
$y = -2x + 3$

6 distance $= \sqrt{(2-(-1))^2 + (4-6)^2}$
$= \sqrt{3^2 + (-2)^2}$
$= \sqrt{9+4}$
$= \sqrt{13}$

7 Midpoint $= \left(\dfrac{4+(-2)}{2}, \dfrac{6+8}{2}\right) = (1, 7)$

8 a Let (x, y) be the coordinates of Y.

$\left(\dfrac{x+(-6)}{2}, \dfrac{y+2}{2}\right) = (8, 3)$

$\therefore \dfrac{x-6)}{2} = 8$ and $\dfrac{y+2}{2} = 3$

$\therefore x = 22$ and $y = 4$

b Let (x, y) be the coordinates of Y.

$\left(\dfrac{x+(-1)}{2}, \dfrac{y+(-4)}{2}\right) = (2, -8)$

$\therefore \dfrac{x-1)}{2} = 2$ and $\dfrac{y-4}{2} = -8$

$\therefore x = 5$ and $y = -12$

9 a $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & 4 \end{bmatrix}$

b $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 14 & 4 \end{bmatrix}$

c $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

d $\begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$

e $3\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$

f $\begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 8 & 5 \end{bmatrix}$

g $\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix} - \begin{bmatrix}4 & 0\\1 & 2\end{bmatrix} = \begin{bmatrix}-2 & 1\\2 & 0\end{bmatrix}$

h $k\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix} = \begin{bmatrix}2k & k\\3k & 2k\end{bmatrix}$

i

$$2\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix} + 3\begin{bmatrix}4 & 0\\1 & 2\end{bmatrix} = \begin{bmatrix}4 & 2\\6 & 4\end{bmatrix} + \begin{bmatrix}12 & 0\\3 & 6\end{bmatrix}$$

$$= \begin{bmatrix}16 & 2\\9 & 10\end{bmatrix}$$

j $\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix} - 2\begin{bmatrix}4 & 0\\1 & 2\end{bmatrix} = \begin{bmatrix}2 & 1\\3 & 2\end{bmatrix} - \begin{bmatrix}8 & 0\\2 & 4\end{bmatrix}$

$$= \begin{bmatrix}-6 & 1\\1 & -2\end{bmatrix}$$

10 $\sqrt{(10-5)^2 + (y-12)^2} = 13$

$$25 + (y-12)^2 = 169$$

$$(y-12)^2 = 144$$

$$y - 12 = \pm 12$$

$$y = 0 \text{ or } y = 24$$

11 $mx - 4y = m + 3 \ldots (1)$

$4x + (m+10)y = -2 \ldots (2)$

Gradient of (1) $= \dfrac{m}{4}$

Gradient of (2) $= -\dfrac{4}{m+10}$

Infinitely many or no solutions when the gradients are the same.

$$-\frac{m}{4} = \frac{4}{m+10}$$

$m^2 + 10m + 16 = 0$

$(m+8)(m+2) = 0$

$$m = -8 \text{ or } m = -2$$

a Checking back in the equations there are infinitely many solutions when $m = -2$.
Equation (1) becomes $-2x - 4y = 1$
Equation (2) becomes $4x + 8y = -2$

b There is a unique solution for $m \in \mathbb{R}\backslash\{-2, -8\}$

12 **a** $2x - 3y + z = 6 \ldots (1)$

$-2x + 3y + z = 8 \ldots (2)$

Add (1) and (2)

$$2z = 14$$

$$z = 7$$

Substitute in (1)

$2x - 3y = -1$

$$\therefore y = \frac{2x+1}{3}$$

Let $x = \lambda$

The solution is $(\lambda, \dfrac{2\lambda+1}{3}, 7)$ where $\lambda \in \mathbb{R}$

b $x - z + y = 6 \ldots (1)$

$2x + z = 4 \ldots (2)$

Let $z = \lambda$

Then $x = \dfrac{4-\lambda}{2}$

Substitute in (1)

$\dfrac{4-\lambda}{2} - \lambda + y = 6$

$$\therefore y = \frac{8+3\lambda}{2}$$

The solution is $(\dfrac{4-\lambda}{2}, \dfrac{3\lambda+8}{2}, \lambda)$ where $\lambda \in \mathbb{R}$

Solutions to multiple-choice questions

1 E $y = mx + c$

The m (gradient) value is $-\frac{1}{2}$,

It passes through the point (1, 4)

$4 = -\frac{1}{2}1 + c$

$\therefore\ c = \frac{9}{2}$

$\therefore\ y = -\frac{1}{2}x + \frac{9}{2}$

2 E $y = -2x + 4$

Point $(a, 3)$

$3 = -2a + 4$

$a = \frac{1}{2}$

3 D Line passes through the points $(-2, 0)$ and $(0, -1)$

$\therefore\ \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{-1 - 0}{0 - -2}$

m (gradient) $= -\frac{1}{2}$

Perpendicular line $= -\frac{1}{m}$

$\therefore\ -\frac{1}{-\frac{1}{2}} = 2$

4 C Midpoint at $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$

$= \left(-\frac{2}{2}, \frac{17}{2}\right)$

$= (-1, 8.5)$

5 B $2ax - 10by = 22$

$+ \quad + \quad +$

$4ax + 10by = 2$

$\therefore\ 6ax = 24$

$\therefore\ x = \frac{4}{a}$

Do not need to solve for y as there is only one possible option.

6 A Line passes through the points $(3, -2)$ and $(-1, 10)$

$\therefore\ \frac{y_2 - y_1,}{x_2 - x_1}$

$= \frac{10 - -2}{-1 - 3}$

$= \frac{12}{-4}$

$= -3$

7 A Eqn 1: $y = 2x + 3$

Eqn 2: $y = \frac{ax}{3} + \frac{4}{3}$

To be parallel gradients must be the same.

$\therefore\ \frac{a}{3} = 2$

$\therefore\ a = 6$

8 C $y = mx + c$

$m = \frac{10 - -2}{3 - -1}$

$m = 3$

Passes through the point (3, 10)

$\therefore\ 10 = 9 + c$

$\therefore\ c = 1$

$\therefore\ y = 3x + 1$

9 B Distance between x points

$= |x_2 - x_1|$

$= |5 - 1|$

$= 4$

Distance between y points

$= y_2 - y_1$

$= |-2 - 4|$

$= 6$

Using Pythagoras

$\sqrt{4^2 + 6^2}$

$= \sqrt{52}$

$= 2\sqrt{13}$

10 C $y = mx + c$

Passes through points $(4, 0)$ and $(0, -3)$

$$m = \frac{-3-0}{0-4}$$

$$m = \frac{3}{4}$$

Y intercept $= -3$

$\therefore$ $c = -3$

$\therefore$ $f(x) = \frac{3}{4} - 3$

11 D $bx + 3y = 0 \ldots (1)$

$4x + (b+1)y = 0 \ldots (2)$

Gradient of (1) $= -\frac{b}{3}$

Gradient of (2) $= -\frac{4}{b+1}$

Infinitely many solutions when the gradients are the same.

$$\frac{b}{3} = \frac{4}{b+1}$$

$$b^2 + b - 12 = 0$$

$$(b+4)(b-3) = 0$$

$$b = -4 \text{ or } b = 3$$

12 A

$(a-1)x + 5y = 7 \ldots (1)$

$3x + (a-3)y = 0 \ldots (2)$

Gradient of (1) $= -\frac{a-1}{5}$

Gradient of (2) $= -\frac{3}{a-3}$

Infinitely many or no solutions when the gradients are the same.

$$\frac{a-1}{5} = \frac{3}{a-3}$$

$$a^2 - 4a - 12 = 0$$

$$(a-6)(a+2) = 0$$

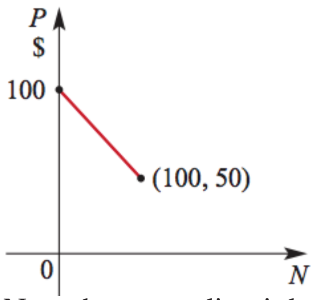
$$a = -2 \text{ or } a = 6$$

13 D $\left(\frac{0+4}{2}, \frac{d-6}{2}\right) = \left(2, \frac{d-6}{2}\right)$

14 C Gradient of line segment joining $(3, 0)$ and $(0, -6)$ is $\frac{6}{3} = 2$ Gradient of line perpendivular to this is $-\frac{1}{2}$

Solutions to extended-response questions

1 a Graph is a straight line passing through (100, 50) and (50, 75).



Note that extending it back to the P axis shows that the intercept is (0, 100); this is confirmed in part **b** below.

b Relationship is linear: $P = aN + b$

$$P = 50, N = 100: \quad 50 = 100a + b \quad \ldots 1$$

$$P = 75, N = 50: \quad 75 = 50a + b \quad \ldots 3$$

$$1 - 2: \quad 50a = -25$$

$$a = -\frac{1}{2}$$

which implies $b = 100$

Hence $P = -\frac{1}{2}N + 100$.

c **i** $N = 88: \quad P = -\frac{1}{2} \times 88 + 100$

$$= 56$$

So the price would be \$56.

ii $P = 60: \quad 60 = -\frac{1}{2}N + 100$

$$\frac{1}{2}N = 40$$

$$N = 80$$

So the number of jackets would be 80.

2 a The rule is of the form $p = at + b$

When $t = 3$, $p = 12000$ and when $t = 8$, $p = 19240$

Therefore the equations

$12000 = 3a + b1$

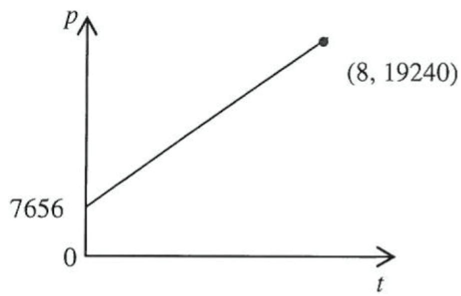
and $19240 = 8a + b2$ are satisfied.

Subtract 1 from 2 to give $5a = 7240$.

Hence $a = 1448$

Substitute in 1 to find that $b = 7656$

Therefore $p = 1448t + 7656$

b 

The p axis intercept gives the initial population.

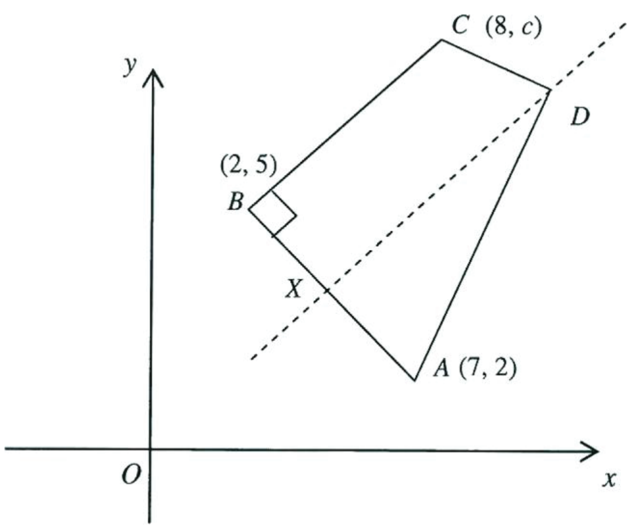
c When $t = 10$,

$p = 14480 + 7656$

$= 22136$

d The average rate of growth is the gradient. The growth rate is 1448 people per year.

3



a Midpoint of $AB = \left(\frac{7+2}{2}, \frac{5+2}{2}\right) = \left(\frac{9}{2}, \frac{7}{2}\right)$

Gradient of $AB = \frac{5-2}{2-7} = -\frac{3}{5}$

Therefore equation of perpendicular bisector of AB is

$y - \frac{7}{2} = \frac{5}{3}\left(x - \frac{9}{2}\right)$

Therefore $y = \frac{5}{3}x - 4$

b Solving the equations $y = 4x - 26$ and $y = \frac{5}{3}x - 4$ simultaneously for x and y will give the coordinates of D

Consider $4x - 26 = \frac{5}{3}x - 4$

$\frac{7x}{3} = 22$

$x = \frac{66}{7}$

Substitute $x = \frac{66}{7}$ in the equation $y = 4x - 26$ to give $y = \frac{82}{7}$

Coordinates of D are $\left(\frac{66}{7}, \frac{82}{7}\right)$

c Line BC is perpendicular to line AB. Therefore gradient of BC is $\frac{5}{3}$

d $B(2, 5)$ and $C(8, c)$. The gradient of BC can also be written as $\frac{5-c}{-6}$

Therefore $\frac{5-c}{-6} = \frac{5}{3}$

Solving for c gives $c = 15$

e The area will be found by calculating the area of triangle DXA and trapezium $BCDX$. Let X be the midpoint of AB. From the above the coordinates of X are $\left(\frac{9}{2}, \frac{7}{2}\right)$

$$\text{Distance } XD = \sqrt{\left(\frac{66}{7} - \frac{9}{2}\right)^2 + \left(\frac{82}{7} - \frac{7}{2}\right)^2}$$

$$= \sqrt{\frac{8993}{98}}$$

$$= \frac{23\sqrt{34}}{14}$$

$$\text{Distance } XA = \text{distance } XB = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{3}{2}\right)^2}$$

$$= \sqrt{\frac{17}{2}}$$

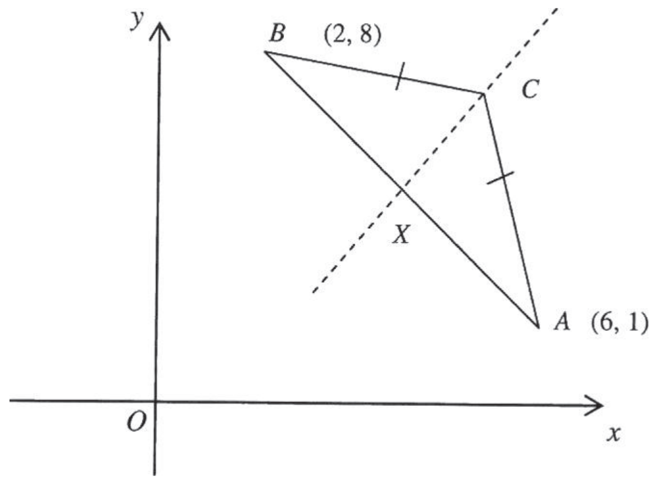
Area = area of triangle DXA + area of trapezium $BCDX$.

$= \frac{1}{2}XA \times XB + \frac{1}{2}BX(BC + XD)$

$= \frac{1}{2}AX(BX + 2XD)$

$= \frac{629}{14}$

4



a Midpoint of $BC = \left(4, \frac{9}{2}\right)$

Gradient of $AB = \frac{8-1}{2-6} = -\frac{7}{4}$

Therefore gradient of perpendicular bisector $= \frac{4}{7}$

The equation of the perpendicular bisector is

$y - \frac{9}{2} = \frac{4}{7}(x - 4)$

Therefore $y = \frac{4}{7}x + \frac{31}{14}$

b The perpendicular bisector passes through C as the triangle is isosceles.

When $x = 3.5$, $y = \frac{4}{7} \times 3.5 + \frac{31}{14} = \frac{59}{14}$

The coordinates of C are $\left(\frac{7}{2}, \frac{59}{14}\right)$

c The length of $AB = \sqrt{(6-2)^2 + (1-8)^2} = \sqrt{65}$

d The area of the triangle $= \frac{1}{2} \times \sqrt{65} \times XC$ where X is the midpoint of AB

$XC = \sqrt{\left(\frac{7}{2} - 4\right)^2 + \left(\frac{59}{14} - \frac{9}{2}\right)^2} = \frac{\sqrt{65}}{14}$

Therefore area $= \frac{1}{2} \times \sqrt{65} \times \frac{\sqrt{65}}{14} = \frac{65}{28}$ square units.

5 $A(-4, 6)$ and $B(6, -7)$

a Midpoint $= \left(\frac{-4+6}{2}, \frac{6+-2}{2}\right) = \left(1, -\frac{1}{2}\right)$

b/c The length $AB = \sqrt{(-7-6)^2 + (6 - -4)^2} = \sqrt{269}$ = the distance between A and B

d gradient of $AB = \dfrac{6 - -7}{-4 - 6}$

$= -\dfrac{13}{10}$

The equation of AB is $y - 6 = -\dfrac{13}{10}(x + 4)$

Rearranging gives

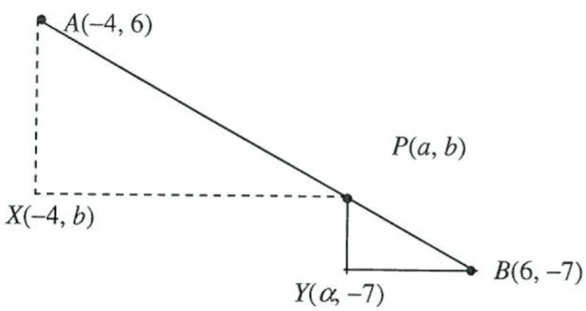
$y = -\dfrac{13}{10}x + \dfrac{4}{5}$

e The perpendicular bisector has gradient $\dfrac{10}{13}$

The equation is $y + \dfrac{1}{2} = \dfrac{10}{13}(x - 1)$

Therefore $y = \dfrac{10}{13}x - \dfrac{33}{26}$

f



Triangles AXP and PYB are similar with scale factor 3.

$AX : PY = 3 : 1$

Therefore $\dfrac{6 - b}{b + 7} = \dfrac{3}{1}$

Therefore $b = -\dfrac{15}{4}$

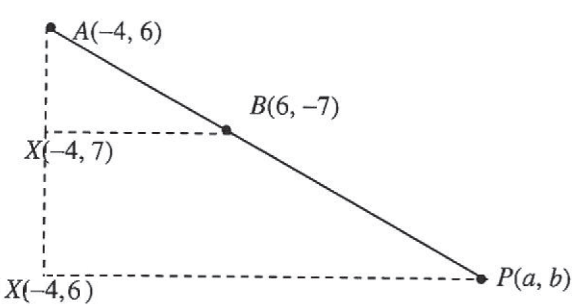
Also $XP : YB = 3$

$\dfrac{a + 4}{6 - a} = 3$

$a = \dfrac{7}{2}$

coordinates of P are $\left(\dfrac{7}{2}, -\dfrac{15}{4}\right)$

g



Triangles AXB and AYP are similar with scale factor 3.

Therefore

$$\frac{a+4}{10} = \frac{3}{1}$$

$a = 26$

Also $\dfrac{b-6}{-7-6} = 3$

$b = -33$

The coordinates of P are $(26, -33)$

6 **a** 25% of 500 = 125

125 litres of acid is required to produce 500 litres of a 25% acid solution.

b Let x denote the amount of 30% solution.

Let y denote the amount of 18% solution.

$\therefore \quad x + y = 500 \quad 1$

$0.3x + 0.18y = 125 \quad 3$

From 1 $y = 500 - x$. Substitute in 2

$\therefore \quad 0.3x + 0.18(500 - x) = 125$

$\therefore \quad (0.3 - 0.18)x + 90 = 125$

$\therefore \quad 0.12x = 35$

$\therefore \quad x = \dfrac{875}{3}$

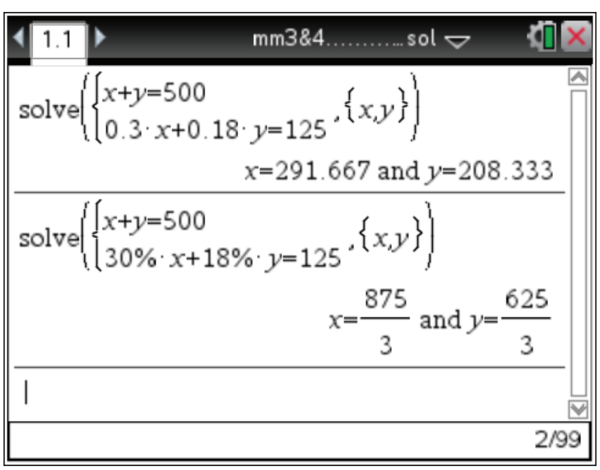
Substitute in 1 $y = 500 - \dfrac{875}{3} = \dfrac{625}{3}$

$\dfrac{875}{3}$ litres of the 30% solution and $\dfrac{625}{3}$ litres of the 18% solution are required.

Graphical Calculator techniques for Question 6.

In a **Calculator** page use **Algebra>Solve System of Equations>Solve System of Equations**.

For exact answers enter decimal inputs as fractions such as 30/100 or 30 as shown.



Chapter 3 – Transformations

Solutions to Exercise 3A

1 a $(-2, 5) \to (-2+1, 5-2) = (-1, 3)$

b $(-2, 5) \to (-2-3, 5+5) = (-5, 10)$

c $(-2, 5) \to (-2-1, 5-6) = (-3, -1)$

d $(-2, 5) \to (-2-3, 5+2) = (-5, 7)$

e $(-2, 5) \to (-2-1, 5+1) = (-3, 6)$

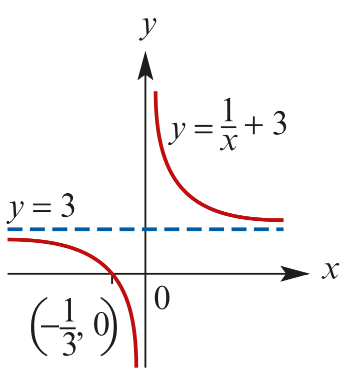
2 a $y = \dfrac{1}{x-2} - 3$

b $y = \dfrac{1}{x+2} + 3$

c $y = \dfrac{1}{x - \frac{1}{2}} + 4 = \dfrac{2}{2x-1} + 4$

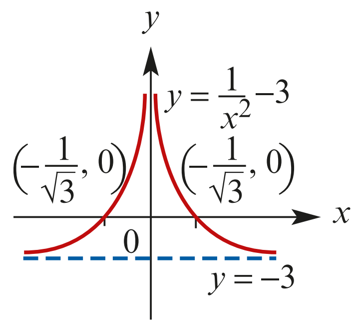
3 a Domain $= \mathbb{R} \setminus \{0\}$

Range $= \mathbb{R} \setminus \{3\}$



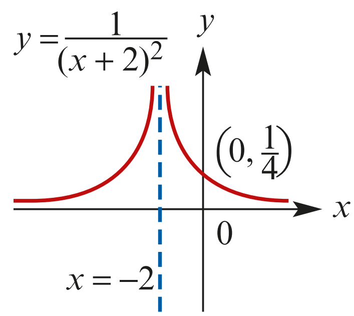
b Domain $= \mathbb{R} \setminus \{0\}$

Range $= (-3, \infty)$



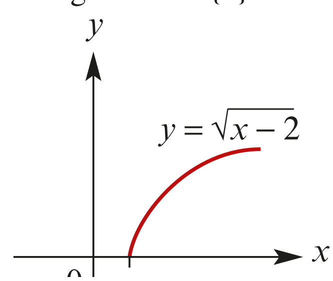
c Domain $= \mathbb{R} \setminus \{-2\}$

Range $= \mathbb{R}^+$



d Domain $= [2, \infty)$

Range $= \mathbb{R}^+ \cup \{0\}$



e Domain $= \mathbb{R} \setminus \{1\}$

Range $= \mathbb{R} \setminus \{0\}$

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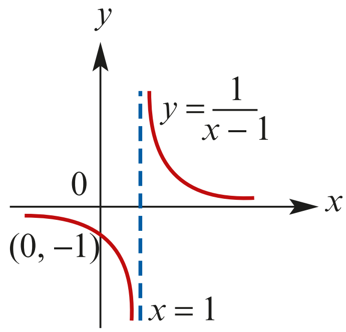
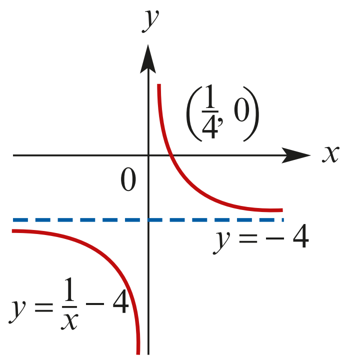


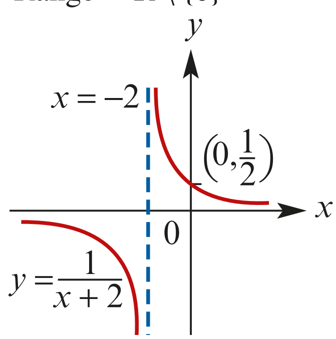
f Domain = $\mathbb{R} \setminus \{0\}$

Range = $\mathbb{R} \setminus \{-4\}$



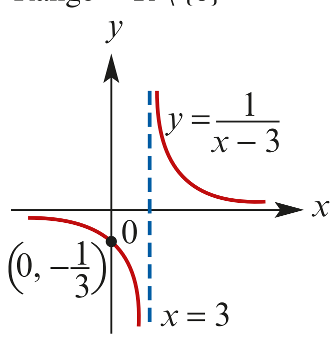
g Domain = $\mathbb{R} \setminus \{-2\}$

Range = $\mathbb{R} \setminus \{0\}$



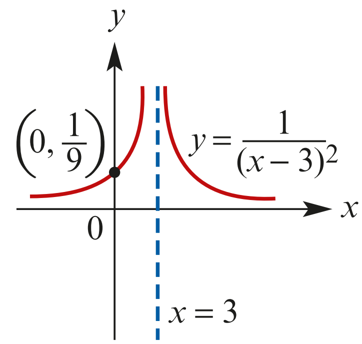
h Domain = $\mathbb{R} \setminus \{3\}$

Range = $\mathbb{R} \setminus \{0\}$



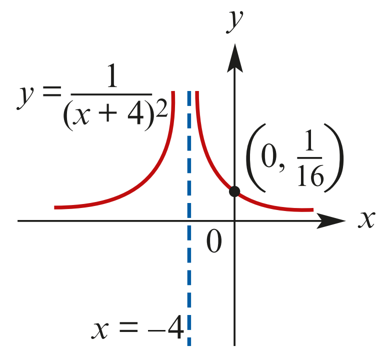
i Domain = $\mathbb{R} \setminus \{3\}$

Range = $\mathbb{R}^+$



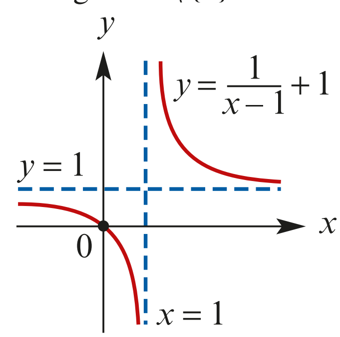
j Domain = $\mathbb{R} \setminus \{-4\}$

Range = $\mathbb{R}^+$



k Domain = $\mathbb{R} \setminus \{1\}$

Range = $\mathbb{R} \setminus \{1\}$



l Domain = $\mathbb{R} \setminus \{2\}$

Range = $\mathbb{R} \setminus \{2\}$

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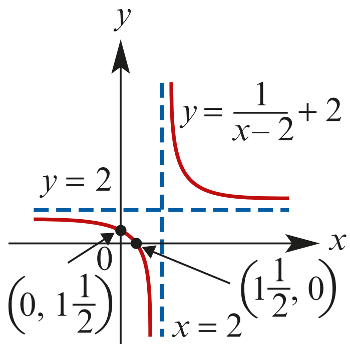
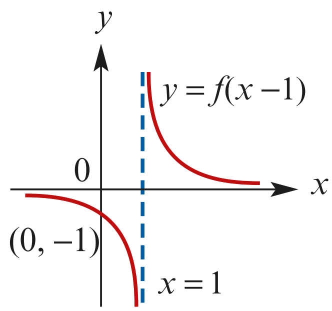


4 a



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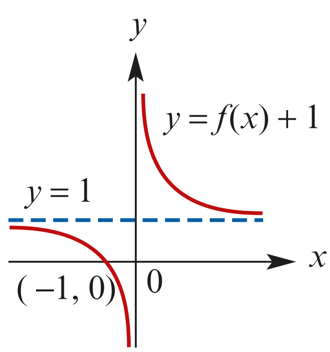

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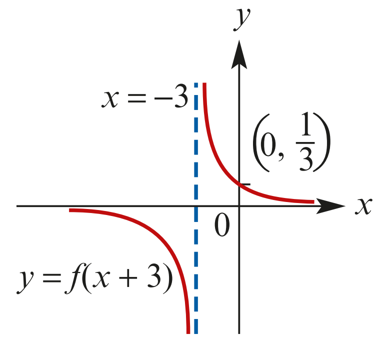

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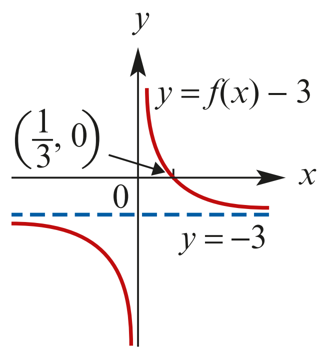

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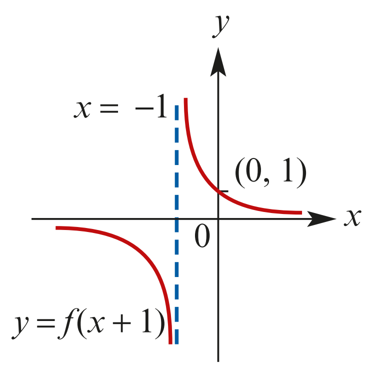

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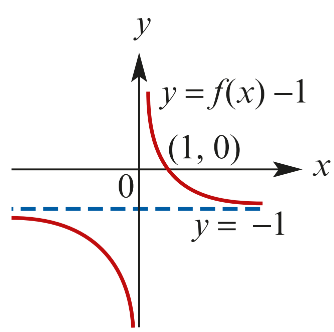

5 a Translation of 5 to the left;
$(x, y) \to (x - 5, y)$

b Translation of 2 up; $(x, y) \to (x, y + 2)$

c Translation of 4 up; $(x, y) \to (x, y + 4)$

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d Translation $(x, y) \to (x, y + 3)$

e Translation $(x, y) \to (x - 3, y)$

6 a i $y - 1 = (x - 7)^{\frac{1}{4}}$; $y = (x - 7)^{\frac{1}{4}} + 1$

ii $y + 6 = (x + 2)^{\frac{1}{4}}$; $y = (x + 2)^{\frac{1}{4}} - 6$

iii $y + 3 = (x - 2)^{\frac{1}{4}}$; $y = (x - 2)^{\frac{1}{4}} - 3$

iv $y - 4 = (x + 1)^{\frac{1}{4}}$; $y = (x + 1)^{\frac{1}{4}} + 4$

b i $y = \sqrt[3]{(x - 7)} + 1$

ii $y = \sqrt[3]{(x + 2)} - 6$

iii $y = \sqrt[3]{(x - 2)} - 3$

iv $y = \sqrt[3]{(x + 1)} + 4$

c i $y = \dfrac{1}{(x - 7)^3} + 1$

ii $y = \dfrac{1}{(x + 2)^3} - 6$

iii $y = \dfrac{1}{(x - 2)^3} - 3$

iv $y = \dfrac{1}{(x + 1)^3} + 4$

d i $y = \dfrac{1}{(x - 7)^4} + 1$

ii $y = \dfrac{1}{(x + 2)^4} - 6$

iii $y = \dfrac{1}{(x - 2)^4} - 3$

iv $y = \dfrac{1}{(x + 1)^4} + 4$

7 a $x' = x - 3$ and $y' = y + 2$

$\therefore x = x' + 3$ and $y = y' - 2$

$\therefore y = (x - 2)^2 + 3$ maps to

$y' - 2 = (x' + 3 - 2)^2 + 3$

The image is $y = (x + 1)^2 + 5$

b $x' = x + 3$ and $y' = y - 3$

$\therefore x = x' - 3$ and $y = y' + 3$

$\therefore y = 2(x + 3)^2 + 3$ maps to

$y' + 3 = 2(x' - 3 + 3)^2 + 3$

The image is $y = 2x^2$

c $x' = x + 4$ and $y' = y - 2$

$\therefore x = x' - 4$ and $y = y' + 2$

$\therefore y = \dfrac{1}{(x - 2)^2} + 3$ maps to

$y' + 2 = \dfrac{1}{(x' - 4 - 2)^2} + 3$

The image is $y = \dfrac{1}{(x - 6)^2} + 1$

d $x' = x - 1$ and $y' = y + 1$

$\therefore x = x' + 1$ and $y = y' - 1$

$\therefore y = (x + 2)^3 + 1$ maps to

$y' - 1 = (x' + 1 + 2)^3 + 1$

The image is $y = (x + 3)^3 + 2$

e $x' = x - 1$ and $y' = y + 1$

$\therefore x = x' + 1$ and $y = y' - 1$

$\therefore y = \sqrt[3]{x - 3} + 2$ maps to

$y' - 1 = \sqrt[3]{x' + 1 - 3} + 2$

The image is $y = \sqrt[3]{x - 2} + 3$

8 a Write

$y = \dfrac{1}{x^2}$ and $y' = \dfrac{1}{(x' - 2)^2} + 3$

Therefore, choose:

$y = y' - 3$ and $x = x' - 2$

$\therefore y' = y + 3$ and $x' = x + 2$

That is, $(x, y) \to (x + 2, y + 3)$

b Write

$y = \dfrac{1}{x}$ and $y' = \dfrac{1}{(x' + 2)} - 3$

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Therefore, choose:
$y = y' + 3$ and $x = x' + 2$
$\therefore y' = y - 3$ and $x' = x - 2$
That is, $(x, y) \to (x - 2, y - 3)$

c Write
$y = \sqrt{x}$ and $y' = \sqrt{x' + 4} + 2$
Therefore, choose:
$y = y' - 2$ and $x = x' + 4$
$\therefore y' = y + 2$ and $x' = x - 4$
That is, $(x, y) \to (x - 4, y + 2)$

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Solutions to Exercise 3B

1 a $x' = x, y' = 3y$

$\therefore y = \dfrac{1}{x}$ maps to $\dfrac{y'}{3} = \dfrac{1}{x'}$

The image is $y = \dfrac{3}{x}$

b

c $x' = 3x, y' = y$

$\therefore y = \dfrac{1}{x}$ maps to $y' = \dfrac{1}{\frac{x'}{3}}$

The image is $y = \dfrac{3}{x}$

2 a $x' = x, y' = 2y$

$\therefore y = \dfrac{1}{x^2}$ maps to $\dfrac{y'}{2} = \dfrac{1}{(x')^2}$

The image is $y = \dfrac{2}{x^2}$

b

c $x' = 2x, y' = y$

$\therefore y = \dfrac{1}{x^2}$ maps to $y' = \dfrac{1}{\left(\frac{x'}{2}\right)^2}$

The image is $y = \dfrac{4}{x^2}$

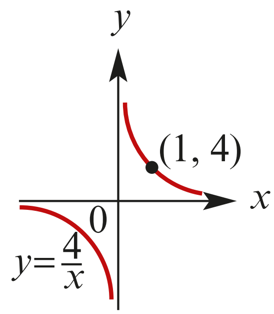
3 a $y = 2\sqrt{x}$

b $y = \sqrt{\dfrac{x}{2}}$

4 a $y = 2x^3$

b $y = \dfrac{x^3}{8}$

5 a



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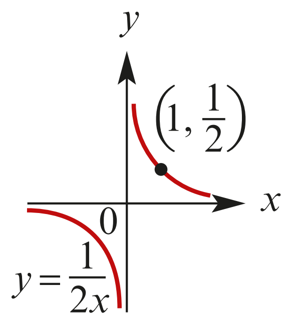

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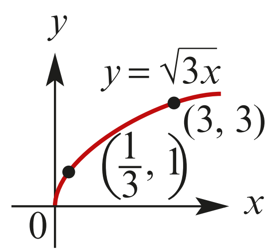

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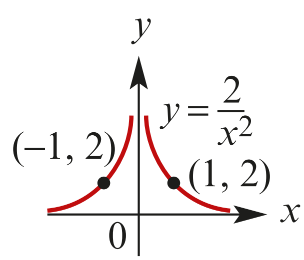

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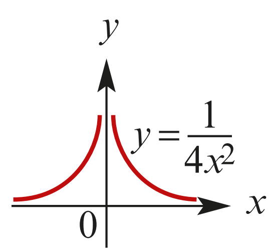

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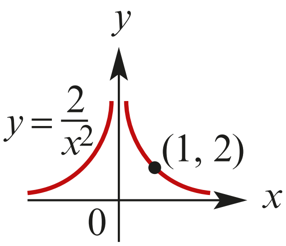

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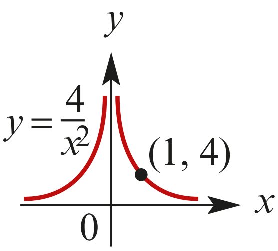

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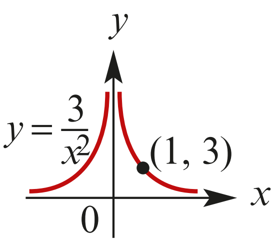

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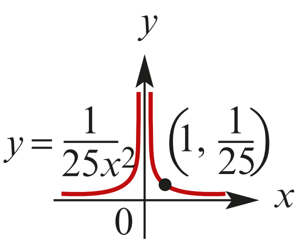

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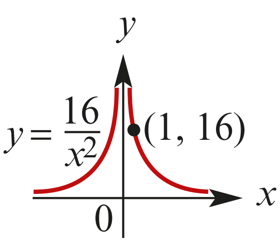

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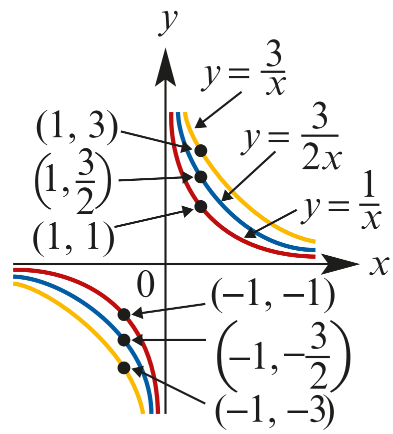

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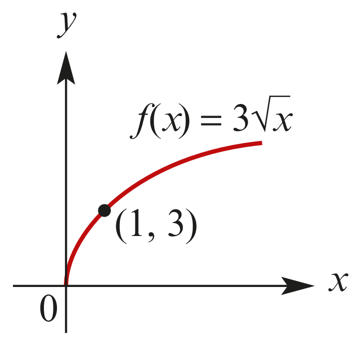

9 **a** Dilation factor $\frac{1}{5}$ from the y-axis

b Dilation factor $\sqrt{5}$ from the x-axis

10 **a** Let $y = f(x) = \frac{1}{x^2}$ and

$y' = f_1(x') = \frac{5}{(x')^2}$

Then rewrite as $y = \frac{1}{x^2}$ and

$\frac{y'}{5} = \frac{1}{(x')^2}$.

Choose $\frac{y'}{5} = y$ and $x = x'$.

One transformation is $y' = 5y$ and

$x' = x$

A dilation of factor 5 from the x-axis.

b Let $y = f(x) = \sqrt{x}$ and

$y' = f_1(x') = 4\sqrt{x'}$

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Then rewrite as $y = \sqrt{x}$ and $\frac{y'}{4} = \sqrt{x'}$.
Choose $\frac{y'}{4} = y$ and $x = x'$.
One transformation is $y' = 4y$ and $x' = x$
A dilation of factor 4 from the x-axis.

c Let $y = f(x) = \sqrt{x}$ and $y' = f_1(x') = \sqrt{5x'}$
Then rewrite as $y = \sqrt{x}$ and $y' = \sqrt{5x'}$.
Choose $y' = y$ and $x = 5x'$.
One transformation is $y' = y$ and $x' = \frac{1}{5}x$
A dilation of factor $\frac{1}{5}$ from the y-axis.

d Let $y = f(x) = \sqrt{\frac{x}{3}}$ and $y' = f_1(x') = \sqrt{x'}$
Then rewrite as $y = \sqrt{\frac{x}{3}}$ and $y' = \sqrt{x'}$.
Choose $y' = y$ and $\frac{x}{3} = x'$.
One transformation is $y' = y$ and $x' = \frac{1}{3}x$
A dilation of factor $\frac{1}{3}$ from the y-axis.

e Let $y = f(x) = \frac{1}{4x^2}$ and $y' = f_1(x') = \frac{1}{(x')^2}$
Then rewrite as $y = \frac{1}{(2x)^2}$ and $y' = \frac{1}{(x')^2}$.
Choose $y' = y$ and $2x = x'$.
One transformation is $y' = 5y$ and $x' = 2x$
A dilation of factor 2 from the y-axis.

11 a i $y = 4x^2$

ii $y = \frac{2}{3}x^2$

iii $y = (2x)^2 = 4x^2$

iv $y = \left(\frac{x}{5}\right)^2 = \frac{1}{25}x^2$

b i $y = \frac{4}{x^2}$

ii $y = \frac{2}{3x^2}$

iii $y = \frac{1}{4x^2}$

iv $y = \frac{25}{x^2}$

c i $y = 4\sqrt[3]{x}$

ii $y = \frac{2}{3} \times \sqrt[3]{x}$

iii $y = \sqrt[3]{2x}$

iv $y = \sqrt[3]{\frac{x}{5}}$

d i $y = \frac{4}{x^3}$

ii $y = \frac{2}{3x^3}$

iii $y = \frac{1}{8x^3}$

iv $y = \frac{125}{x^3}$

e i $y = \frac{4}{x^4}$

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ii $y = \dfrac{2}{3x^4}$

iii $y = \dfrac{1}{16x^4}$

iv $y = \dfrac{625}{x^4}$

f **i** $y = 4\sqrt[4]{x}$

ii $y = \dfrac{2}{3} \times \sqrt[4]{x}$

iii $y = \sqrt[4]{2x}$

iv $y = \sqrt[4]{\dfrac{x}{5}}$

g **i** $y = 4x^{\frac{1}{5}}$

ii $y = \dfrac{2}{3}x^{\frac{1}{5}}$

iii $y = (2x)^{\frac{1}{5}}$

iv $y = \left(\dfrac{x}{5}\right)^{\frac{1}{5}}$

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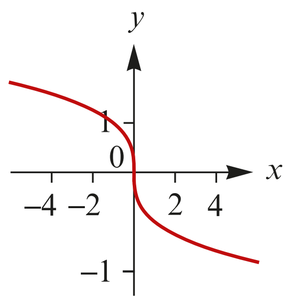
Solutions to Exercise 3C

1 **a** $y = -(x-1)^2$

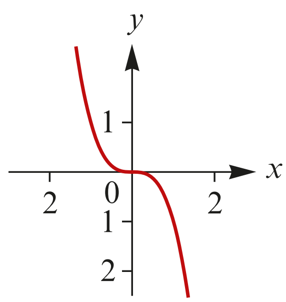
b $y = (x+1)^2$

2 **a**



Domain = $\mathbb{R}$

b



Domain = $\mathbb{R}$

3 Reflection in the y-axis

4 **a** **i** $y = -x^3$

ii $y = -x^3$

b **i** $y = -\sqrt[3]{x}$

ii $y = -\sqrt[3]{x}$

c **i** $y = \dfrac{-1}{x^3}$

ii $y = \dfrac{-1}{x^3}$

d **i** $y = \dfrac{-1}{x^4}$

ii $y = \dfrac{1}{x^4}$

e **i** $y = -x^{\frac{1}{3}}$

ii $y = -x^{\frac{1}{3}}$

f **i** $y = -x^{\frac{1}{5}}$

ii $y = -x^{\frac{1}{5}}$

g **i** $y = -x^{\frac{1}{4}}$

ii $y = (-x)^{\frac{1}{4}}$

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Solutions to Exercise 3D

1 Part a will be done with the method.

a **i** The mapping is
$(x, y) \to (x + 2, 2y - 3) = (x', y')$
Hence $x' = x + 2$ and $y' = 2y - 3$
This implies $x = x' - 2$ and
$y = \dfrac{y' + 3}{2}$
$\therefore y = x^2$ maps to
$\dfrac{y' + 3}{2} = (x' - 2)^2$
That is, $y = 2(x - 2)^2 - 3$

ii The mapping is
$(x, y) \to (3x - 2, y - 4) = (x', y')$
Hence $x' = 3x - 2$ and $y' = y - 4$
This implies $x = \dfrac{x' + 2}{3}$ and
$y = y' + 4$
$\therefore y = x^2$ maps to
$y' + 4 = \left(\dfrac{x' + 2}{3}\right)^2$
That is, $y = \left(\dfrac{x + 2}{3}\right)^2 - 4$

iii The mapping is
$(x, y) \to (3x - 2, y - 4) = (x', y')$
Hence $x' = -x$ and $y' = 2y$
This implies $x = -x'$ and $y = \dfrac{y'}{2}$
$\therefore y = x^2$ maps to $y = 2x^2$

b **i** $y = 2\sqrt[3]{x - 2} - 3$

ii $y = \sqrt[3]{\dfrac{x + 2}{3}} - 4$

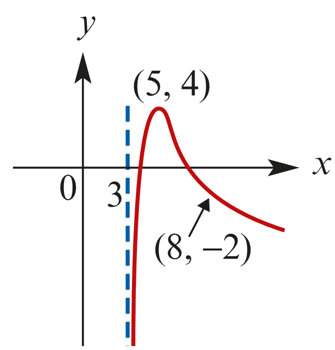
iii $y = -2\sqrt[3]{x}$

c **i** $y = \dfrac{2}{(x - 2)^2} - 3$

ii $y = \dfrac{9}{(x + 2)^2} - 4$

iii $y = \dfrac{2}{x^2}$

2



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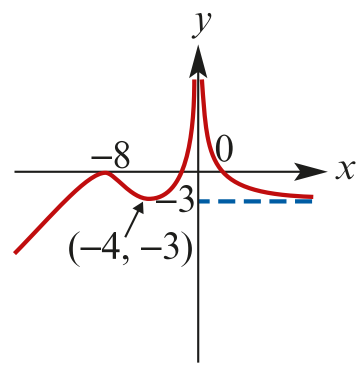

4 **a** **i** $y = -2(x - 3)^2 - 4$

ii $y = -(2(x - 3)^2) - 4$
$y = -2(x - 3)^2 + 4$

iii $y = 2(-(x - 3)^2) - 4$
$y = -2(x - 3)^2 - 4$

iv $y = 2(-(x - 3)^2 - 4)$
$y = -2(x - 3)^2 - 8$

v $y = -2((x - 3)^2 - 4)$
$y = -2(x - 3)^2 + 8$

vi $y = 2(-((x - 3)^2 - 4))$
$y = -2(x - 3)^2 + 8$

b **i** $y = -2\sqrt[3]{x - 3} - 4$

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ii $y = -2\sqrt[3]{x-3} + 4$

iii $y = -2\sqrt[3]{x-3} - 4$

iv $y = -2\sqrt[3]{x-3} - 8$

v $y = -2\sqrt[3]{x-3} + 8$

vi $y = -2\sqrt[3]{x-3} + 8$

c i $y = \dfrac{-2}{(x-3)^2} - 4$

ii $y = \dfrac{-2}{(x-3)^2} + 4$

iii $y = \dfrac{-2}{(x-3)^2} - 4$

iv $y = \dfrac{-2}{(x-3)^2} - 8$

v $y = \dfrac{-2}{(x-3)^2} + 8$

vi $y = \dfrac{-2}{(x-3)^2} + 8$

d i $y = -2(x-3)^4 - 4$

ii $y = -2(x-3)^4 + 4$

iii $y = -2(x-3)^4 - 4$

iv $y = -2(x-3)^4 - 8$

v $y = -2(x-3)^4 + 8$

vi $y = -2(x-3)^4 + 8$

e i $y = \dfrac{-2}{(x-3)^3} - 4$

ii $y = \dfrac{-2}{(x-3)^3} + 4$

iii $y = \dfrac{-2}{(x-3)^3} - 4$

iv $y = \dfrac{-2}{(x-3)^3} - 8$

v $y = \dfrac{-2}{(x-3)^3} + 8$

vi $y = \dfrac{-2}{(x-3)^3} + 8$

f i $y = \dfrac{-2}{(x-3)^4} - 4$

ii $y = \dfrac{-2}{(x-3)^4} + 4$

iii $y = \dfrac{-2}{(x-3)^4} - 4$

iv $y = \dfrac{-2}{(x-3)^4} - 8$

v $y = \dfrac{-2}{(x-3)^4} + 8$

vi $y = \dfrac{-2}{(x-3)^4} + 8$

g i $y = \dfrac{-2}{(x-3)^2} - 4$

ii $y = \dfrac{-2}{(x-3)^2} + 4$

iii $y = \dfrac{-2}{(x-3)^2} - 4$

iv $y = \dfrac{-2}{(x-3)^2} - 8$

v $y = \dfrac{-2}{(x-3)^2} + 8$

vi $y = \dfrac{-2}{(x-3)^2} + 8$

5 $y = -\sqrt{\dfrac{x+12}{3}}$

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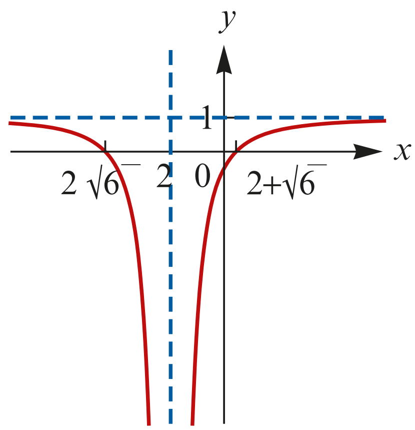
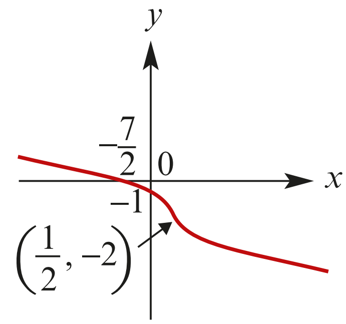
6 a



b $y = 1 - \dfrac{6}{(x+2)^2}$

7 a



b $y = (1 - 2x)^{\frac{1}{3}} - 2$

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Solutions to Exercise 3E

1 a i Dilation of factor 2 from the x-axis, then translation 1 unit to the right and 3 units up

ii Reflection in the x-axis, then translation 1 unit to the left and 2 units up

iii Dilation of factor $\frac{1}{2}$ from the y-axis, then translation $\frac{1}{2}$ unit to the left and 2 units down

b i Dilation of factor 2 from the x-axis, then translation 3 units to the left

ii Translation 3 units to the left and 2 units up

iii Translation 3 units to the right and 2 units down

c i Translation 3 units to the left and 2 units up

ii Dilation of factor $\frac{1}{3}$ from the y-axis and dilation of factor 2 from the x-axis

iii Reflection in the x-axis, then translation 2 units up

2 a Translation 1 unit to the left and 6 units down

b Dilation of factor $\frac{1}{2}$ from the x-axis, then translation $\frac{3}{2}$ units up and 1 unit to the left

c Translation 1 unit to the left and 6 units up

d Dilation of factor $\frac{1}{2}$ from the x-axis, then translation $\frac{5}{2}$ units up and 1 unit to the left

e Dilation of factor 2 from the y-axis, then translation of 1 unit to the left and 6 units down

3 a Dilation of factor $\frac{1}{5}$ from the x-axis, then translation $\frac{7}{5}$ units up and 3 units to the left

b Dilation of factor 3 from the y-axis, then translation 2 units to the right and 5 units down

c Reflection in the x-axis, dilation of factor $\frac{1}{3}$ from the x-axis, translation $\frac{7}{3}$ units up, dilation of factor 3 from the y-axis, translation 1 unit to the right

d Reflection in the y-axis, translation 4 units to the right, dilation of factor $\frac{1}{2}$ from the x-axis

e Reflection in the y-axis, translation 4 units to the right, reflection in the x-axis, dilation of factor $\frac{1}{2}$ from the x-axis, translation $\frac{15}{2}$ units up

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4 **a** Dilation of factor 2 from the x-axis, then translation 1 unit to the right and 3 units up

b Dilation of factor 2 from the x-axis, then translation 4 units to the left and 7 units down

c Reflection in the y-axis and dilation of factor 4 from the x-axis (in either order), then translation 1 unit to the right and 5 units down

d Reflection in the x-axis, then translation 1 unit to the left and 2 units up

e Reflection in the y-axis and dilation of factor 2 from the x-axis (in either order), then translation 3 units up

f Translation 3 units to the left and 4 units down, then reflection in either axis and dilation of factor $\frac{1}{2}$ from the x-axis (in either order)

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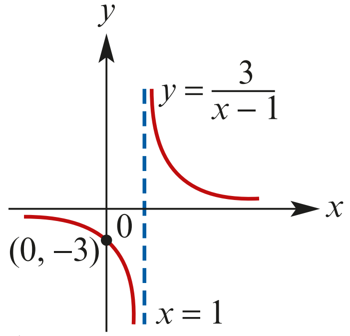
Solutions to Exercise 3F

1 a $f(x) = \dfrac{3}{x-1}$



Asymptotes

$y = 0$

$x - 1 = 0$

$x = 1$

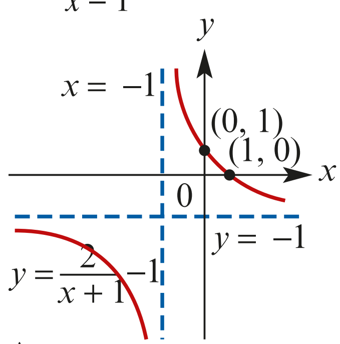
Axis intercepts

$y = \dfrac{3}{0-1} = -3$

$0 = \dfrac{3}{x-1}$ $\therefore$ no x- axis intercept.

Range $\mathbb{R}\backslash\{0\}$

b $y = \dfrac{2}{x-1} - 1$



Asymptotes

$y = 0 - 1 = -1$

$x + 1 = 0$

$x = -1$

Axis intercepts

$0 = \dfrac{2}{x+1} - 1$

$x + 1 = 2$

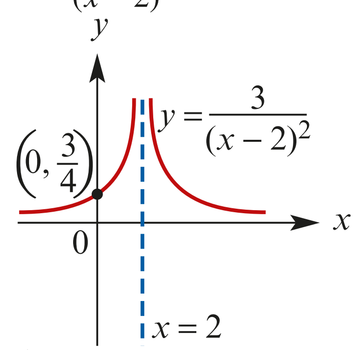
$x = 1$

$y = \dfrac{2}{0+1} - 1$

$y = 1$

Range $\mathbb{R}\backslash\{-1\}$

c $y = \dfrac{3}{(x-2)^2}$



Asymptotes

$y = 0$

$x - 2 = 0$

$x = 2$

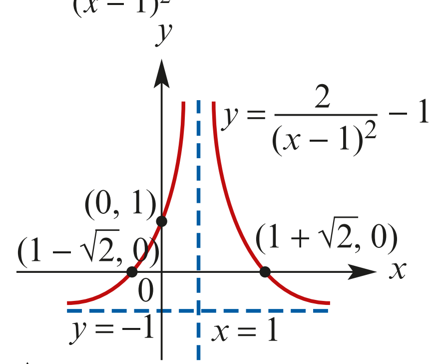
Axis intercepts

$0 = \dfrac{3}{(x-2)^2}$ $\therefore$ no x- axis intercept.

$y = \dfrac{3}{(-2)^2} = \dfrac{3}{4}$

Range $= \mathbb{R}^+$

d $y = \dfrac{2}{(x-1)^2} - 1$



Asymptotes

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$y = 0 - 1 = -1$

$x - 1 = 0$

$x = 1$

Axis intercepts

$$0 = \frac{2}{(x-1)^2} - 1$$

$$x - 1 = \pm\sqrt{2}$$

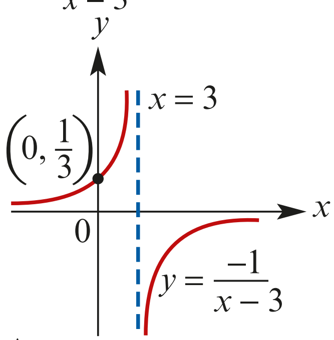
$$x = 1 \pm \sqrt{2}$$

$$y = \frac{2}{(0-1)^2} - 1$$

$$y = 1$$

Range $= (-1, \infty)$

e $y = \dfrac{-1}{x-3}$



Asymptotes

$y = 0$

$x - 3 = 0$

$x = 3$

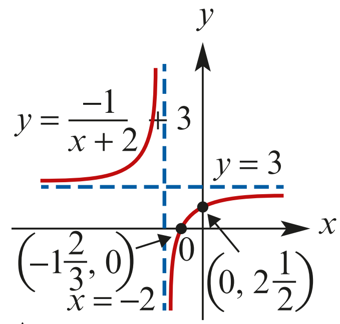
Axis intercepts

$0 = \dfrac{-1}{x-3}$ $\therefore$ no x- axis intercept.

$y = \dfrac{-1}{0-3} = \dfrac{1}{3}$

Range $= \mathbb{R}\backslash\{0\}$

f $y = \dfrac{-1}{x+2} + 3$



Asymptotes

$y = 0 + 3 = 3$

$x + 2 = 0$

$x = -2$

Axis intercepts

$y = \dfrac{-1}{0+2} = 3 = \dfrac{5}{2}$

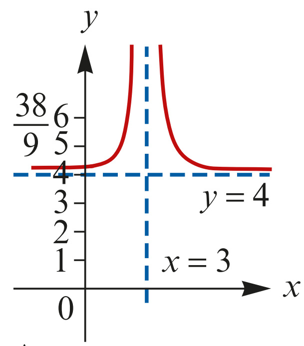
$0 = \dfrac{-1}{x+2} + 3$

$3x + 6 = 1$

$x = \dfrac{-5}{3}$

Range $= \mathbb{R}\backslash\{3\}$

g $y = \dfrac{2}{(x-3)^2} + 4$



Asymptotes

$x - 3 = 0$

$x = 3$

$y = 0 + 4 = 4$

Range $= (4, \infty)$

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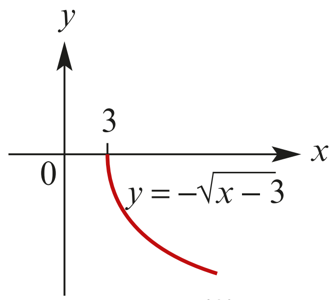
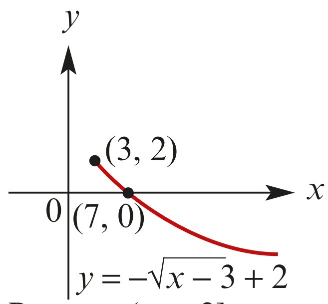


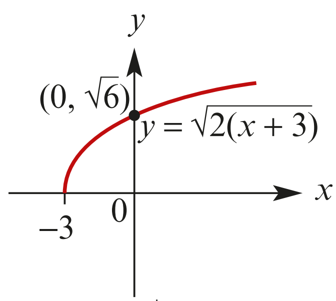
Range = $\mathbb{R}^- \cup \{0\}$

b



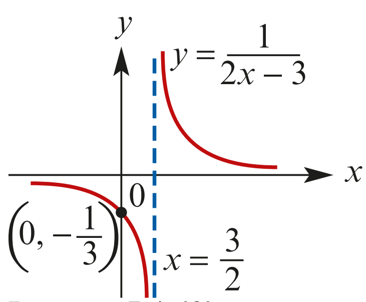
Range = $(-\infty, 2]$

c



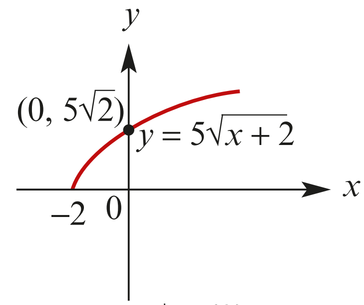
Range = $\mathbb{R}^+ \cup \{0\}$

d



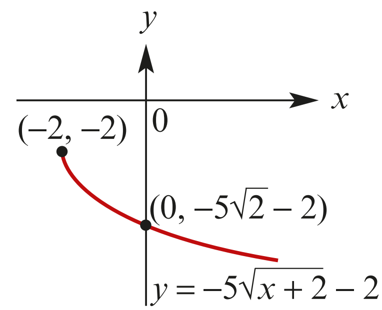
Range = $\mathbb{R} \setminus \{0\}$

e



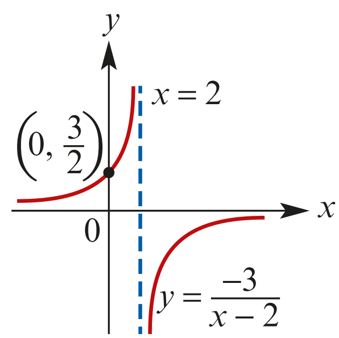
Range = $\mathbb{R}^+ \cup \{0\}$

f



Range = $(-\infty, -2]$

g



Range = $\mathbb{R} \setminus \{0\}$

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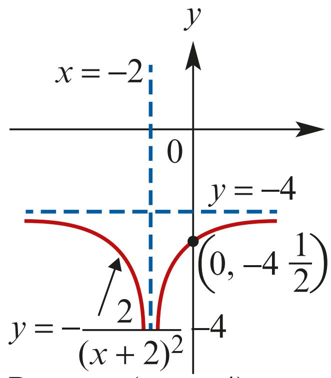
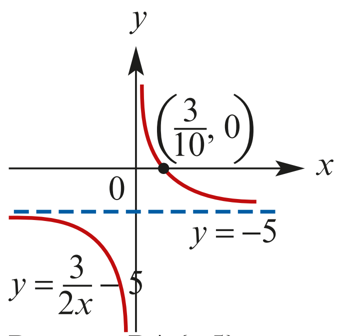
h



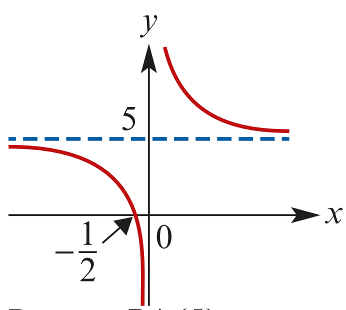
Range $= (-\infty, -4)$

i



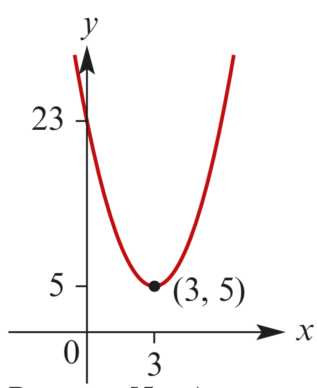
Range $= \mathbb{R} \setminus \{-5\}$

j



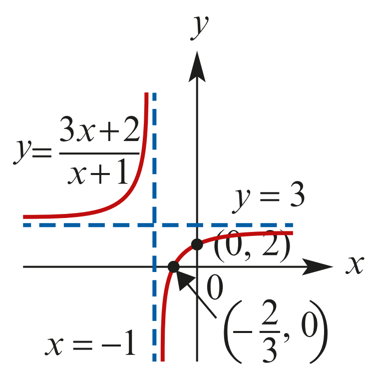
Range $= \mathbb{R} \setminus \{5\}$

k



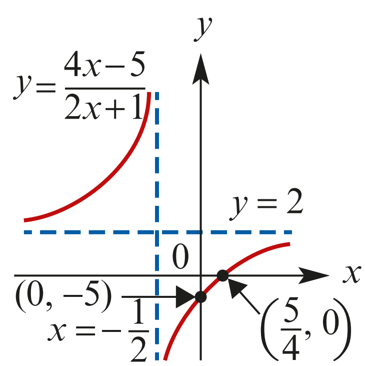
Range $= [5, \infty)$

3



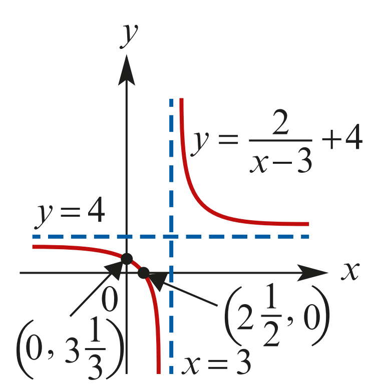
$$\frac{3x+2}{x+1} = \frac{3x+3-1}{x+1}$$
$$= 3(x+1) - \frac{1}{x+1}$$
$$= 3 - \frac{1}{x+1}$$

4



$$\frac{4x-5}{2x+1} = \frac{4x+2-7}{2x+1}$$
$$= \frac{2(2x+1)}{2x+1} - \frac{7}{2x+1}$$
$$= 2 - \frac{7}{2x+1}$$

5 a



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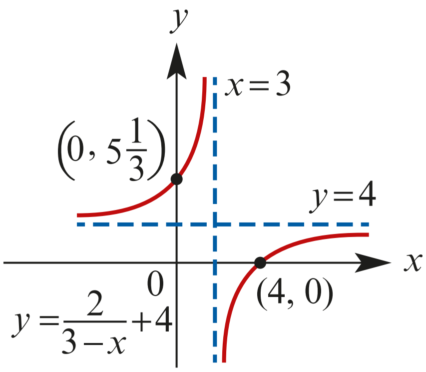
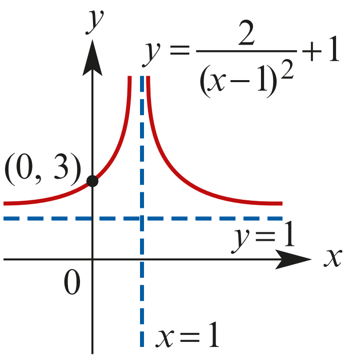
Range = $\mathbb{R}\backslash\{4\}$

b



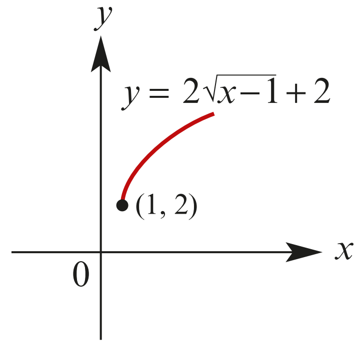
Range = $R\backslash\{4\}$

c



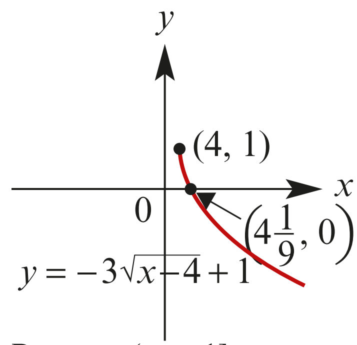
Range = $(1, \infty)$

d



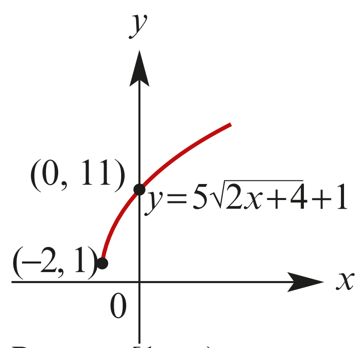
Range = $[2, \infty)$

e



Range = $(-\infty, 1]$

f



Range = $[1, \infty)$

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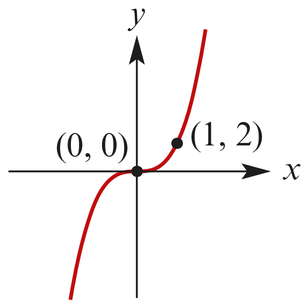
Solutions to Exercise 3G

1 a



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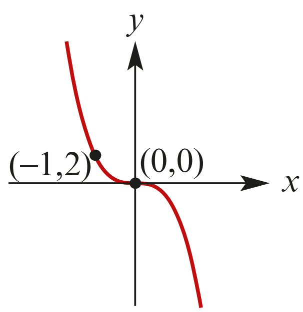

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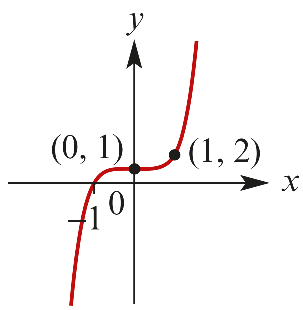
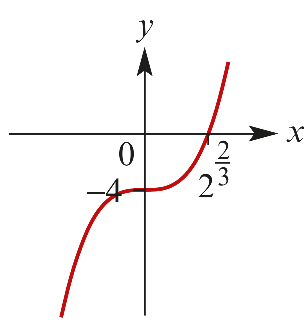


d



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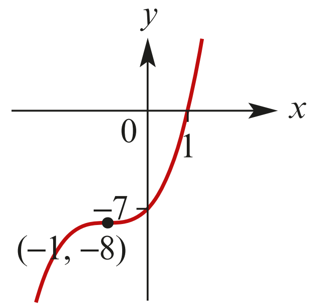
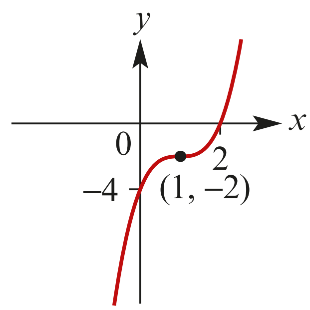
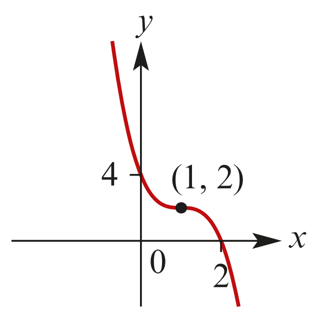


f



g



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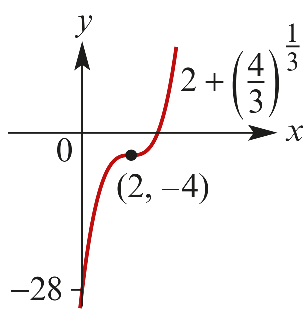
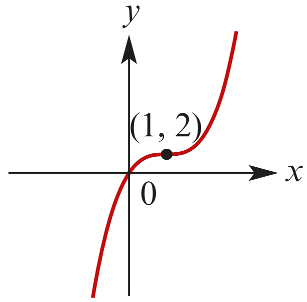
h



i



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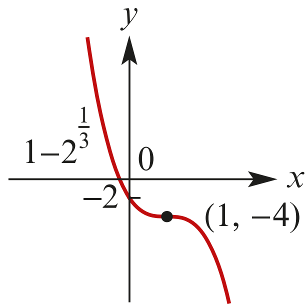
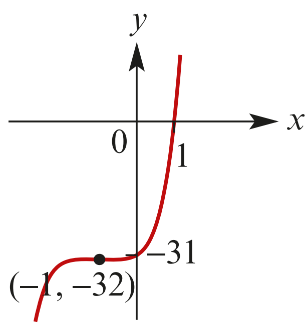
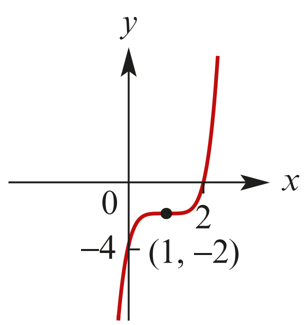


k



l



2 $h = 0$ and $k = 4$
$y = ax^3 + 4$
When $x = 1, y = 1$
$\therefore 1 = a + 4$
$\therefore a = -3$
$\therefore y = -3x^3 + 4$

3 **a** $y = 3x^3$

b $y = (x + 1)^3 + 1$

c $y = -(x - 2)^3 - 3$

d $y = 2(x + 1)^3 - 2$

e $y = \dfrac{x^3}{27}$

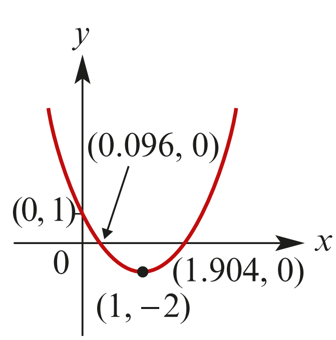
4 **a** $y = \dfrac{(3 - x)^3}{27} + 1$

b Dilation of factor 3 from the x-axis, reflection in the x-axis, then translation 1 unit to the left and 4 units up

5 $y = \dfrac{(x + 2)^4}{16} - 1$

6 Dilation of factor 3 from the x-axis, reflection in the x-axis, then translation 1 unit to the right and 5 units up

7 **a**



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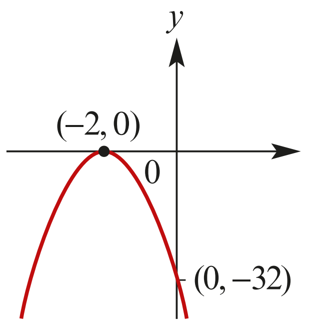
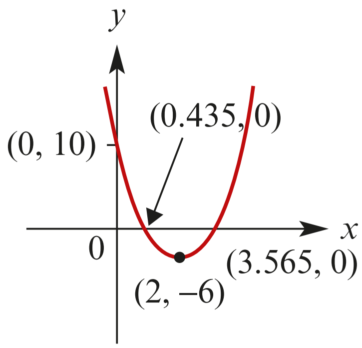
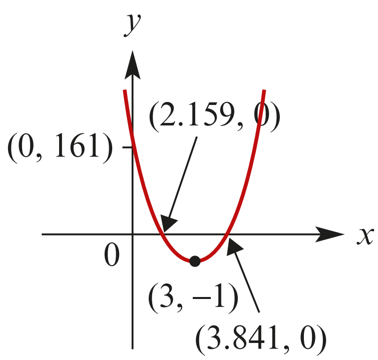
b



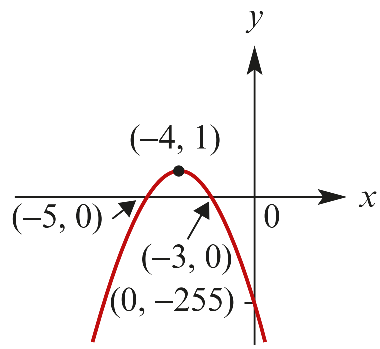
c



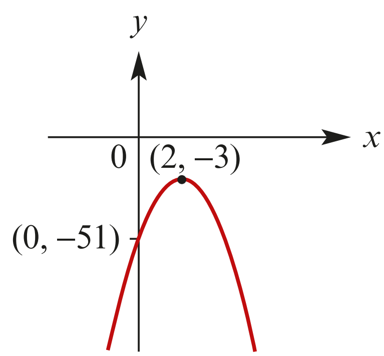
d



e



f



8 $h = -2$ and $k = 3$

Passes through $(0, -6)$

$y = a(x + 2)^4 + 3$

$-6 = 16a + 3$

$a = -\dfrac{9}{16}$

$y = -\dfrac{9}{16}(x + 2)^4 + 3$

9 $h = 1$ and $k = 7$

Passes through $(0, 23)$

$y = a(x - 1)^4 + 7$

$23 = a + 7$

$a = 16$

$y = 16(x - 1)^4 + 7$

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Solutions to Exercise 3H

1
$$4 = a + b \ldots (1)$$
$$1 = \frac{a}{3} + b \ldots (2)$$
Equation (1) − Equation (2)
$$3 = \frac{2a}{3}$$
$$a = \frac{9}{2}$$
From (1)
$$b = -\frac{1}{2}$$

2 **Asymptotes**: $x = 1,\ y = 2$

$x + b = 0$ $\quad$ $y = B$

$1 + b = 0$ $\quad$ $B = 2$

$b = -1$

Point: $(0, 1)$

$$1 = \frac{A}{-1} + 2$$
$$A = 1$$

3
$$1 = A + B \ldots (1)$$
$$6 = 3A + B \ldots (2)$$
Equation (2) − Equation (1)
$$5 = 2A$$
$$A = \frac{5}{2}$$
From (1)
$$B = -\frac{3}{2}$$

4
$$y = A\sqrt{x} + B$$
$$5 = A\sqrt{1} + \mathrm{B} \ldots (1)$$
$$= A + B$$
$$11 = A\sqrt{16} + \mathrm{B} \ldots (2)$$
$$= 4\mathrm{A} + \mathrm{B}$$
Equation (2) − Equation (1)
$$\Rightarrow 3A = 6$$
$$A = 2$$
From (1)
$$\Rightarrow 5 = 2 + B$$
$$B = 3$$

5
$$y = \frac{A}{x^2} + B$$
$$1 = \frac{A}{1^2} + B \ldots (1)$$
$$= A + B$$
$$7 = \frac{A}{0.5^2} + B \ldots (2)$$
$$= 4A + B$$
Equation (2) − Equation (1)
$$\Rightarrow 3A = 6$$
$$A = 2$$
From (1)
$$1 = 2 + B$$
$$B = -1$$

6 $y = \dfrac{A}{(x + b)^2} + B$

Asymptotes

$x = -2 \quad y = -3$

$x + b = 0$ $\quad$ $y = 0 + B$

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$-2 + b = 0 \qquad B = -3$

$b = 2$

Point: $(0, -1)$

$-1 = \frac{A}{2^2} - 3$

$A = 8$

7

$$y = \frac{a}{x^3} + b$$

$$-1 = \frac{a}{1^3} + b = a + b \ldots (1)$$

$$\frac{3}{4} = \frac{a}{2^3} + b = \frac{1}{8}a + b \ldots (2)$$

Equation (2) − Equation (1)

$$\Rightarrow \quad \frac{-7}{8}a = \frac{7}{4}$$

$$a = -2$$

From (1)

$$\Rightarrow -1 = -2 + b$$

$$b = 1$$

8

$$y = ax^{\frac{1}{3}} + b$$

$$-8 = a + b \ldots (1)$$

$$4 = -a + b \ldots (2)$$

Equation (2) + Equation (1)

$$2b = -4$$

$$b = -2$$

From (2)

$$a = -6$$

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Solutions to Exercise 3I

1 a $\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}1\\-6\end{bmatrix}$

$\therefore (1,-2)\rightarrow(1,-6)$

b $\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}2\\-2\end{bmatrix}$

$\therefore (1,-2)\rightarrow(2,-2)$

c $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$

$\therefore (1,-2)\rightarrow(1,2)$

d $\begin{bmatrix}11 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}-1\\-2\end{bmatrix}$

$\therefore (1,-2)\rightarrow(-1,-2)$

e $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}$

$\therefore (1,-2)\rightarrow(-2,1)$

2 a $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$

b $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

c $\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$

d $\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$

e $\begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}$

3 $\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$

$\begin{bmatrix}a\\c\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$

$\therefore a=2$ and $c=-1$

$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-2\\5\end{bmatrix}$

$\begin{bmatrix}b\\d\end{bmatrix}=\begin{bmatrix}-2\\5\end{bmatrix}$

$\therefore b=-2$ and $d=5$

$\begin{bmatrix}2 & -2\\-1 & 5\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix}=\begin{bmatrix}-6\\7\end{bmatrix}$

4 $\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$

$\begin{bmatrix}a\\c\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$

$\therefore a=2$ and $c=-1$

$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}4\\-2\end{bmatrix}$

$\begin{bmatrix}b\\d\end{bmatrix}=\begin{bmatrix}4\\-2\end{bmatrix}$

$\therefore b=4$ and $d=-2$

$\begin{bmatrix}2 & 4\\-1 & -2\end{bmatrix}\begin{bmatrix}-3\\5\end{bmatrix}=\begin{bmatrix}14\\-7\end{bmatrix}$

5 $\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & -2\end{bmatrix}$

$\begin{bmatrix}1 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}-3\\2\end{bmatrix}=\begin{bmatrix}-3\\-4\end{bmatrix}$

6 $\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}x'\\y'\end{bmatrix}$

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$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\left(\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x' - 2 \\ y' - 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' - 2 \\ \frac{y' - 1}{3} \end{bmatrix}$$

$$x = x' - 2, \quad y = \frac{y' - 1}{3}$$

7 $\begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x^{\frac{1}{4}} \\ y^{\frac{1}{2}} \end{bmatrix}$$

$x = -x^{\frac{1}{4}}, \; y = y^{\frac{1}{2}}$

but $y = x^2 + x + 2$

$$\frac{y'}{2} = \frac{(x')^2}{16} - \frac{x'}{4} + 2$$

$$y' = \frac{(x')^2}{8} - \frac{x'}{2} + 4$$

8 $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{2} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$x = x', \; y = \frac{-y'}{2}$

but $y = x^3 + 2x$

$$\frac{-y'}{2} = (x')^3 + 2x'$$

$$y' = -2(x')^3 - 4x'$$

9 $\begin{bmatrix} 0 & 3 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$x = \frac{-y'}{2}, \; y = \frac{x'}{3}$

but $y = 2x + 3$

$$\frac{x'}{3} = -y' + 3$$

$$y' = 3 - \frac{x'}{3}$$

10 $\begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{3} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{-y'}{2} \\ \frac{x'}{2} \end{bmatrix}$$

$x = \frac{-y'}{3}, \; y = \frac{x'}{2}$

but $y = -2x + 4$

$$\frac{x'}{2} = \frac{2y'}{3} + 4$$

$$\frac{2y'}{3} = \frac{x'}{2} - 4$$

$$y' = \frac{3x'}{4} - 6$$

11 $\begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{3} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{3} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-1}{3}y' + 1 \\ x' - 2 \end{bmatrix}$$

$x = \frac{-1}{3}y' + 1, \; y = x' - 2$

but $y = -2x + 6$

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$x' - 2 = \frac{2y'}{3} - 2 + 6$

$\frac{2y'}{3} = x' - 6$

$y' = \frac{3x'}{2} - 9$

12 $\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{-1}{3} & 0 \end{bmatrix} + \left(\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -3 \\ 2 \end{bmatrix}\right)$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-1}{3} & 0 \end{bmatrix}\left(\begin{bmatrix} x' + 3 \\ y' - 2 \end{bmatrix}\right)$

$x = y' - 2,\ y = \frac{-1}{3}(x' + 3)$

but $y = -2x + 6$

$-\frac{1}{3}(x' + 3) = -2y' + 4 + 6$

$-2y' = \frac{-x'}{3} - 1 - 10$

$y' = \frac{x'}{6} + \frac{11}{2}$

13 $\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\left(\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix}\right)$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x' + 1}{4} \\ \frac{y' - 4}{2} \end{bmatrix}$

$x = \frac{x' + 1}{4},\ y = \frac{y' - 4}{2}$

but $y = -2x^3 + 6x$

$\frac{y' - 4}{2} = -2\left(\frac{x' + 1}{4}\right)^3 + 6\left(\frac{x' + 1}{4}\right)$

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14 $\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} & 0 \\ 0 & \frac{-1}{2} \end{bmatrix}\left(\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-(x'+1)}{3} \\ \frac{-(y'-2)}{2} \end{bmatrix}$

$x = \frac{-x'-1}{3},\ y = \frac{2-y'}{2}$

but $y = -2x^3 + 6x^2 + 2$

$\frac{2-y'}{2} = -2\left(\frac{-(x+1)}{3}\right)^3$

$+ 6\left(\frac{-(x+1)}{3}\right)^2 + 2$

$\frac{2-y'}{2} = 2\left(\frac{(x'+1)}{3}\right)^3 + 6\left(\frac{(x'+1)}{3}\right)^2 + 2$

15 $\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$\therefore x' = 4x + 3$ and $y' = -2y - 1$

$\therefore x = \frac{x'-3}{4}$ and $y = -\frac{y'+1}{2}$

$\therefore y = x^2$ maps to $-\frac{y'+1}{2} = \left(\frac{x'-3}{4}\right)^2$

The image is: $y = -\frac{1}{8}(x-3)^2 - 1$

16 $\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$\therefore x' = 3x - 2$ and $y' = -y - 1$

$\therefore x = \frac{x'+2}{3}$ and $y = -y' - 1$

$\therefore y = \frac{1}{x^2}$ maps to $-y' - 1 = \frac{1}{\left(\frac{x'+2}{3}\right)^2}$

The image is: $y = -\frac{9}{(x+2)^2} - 1$

17 $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$\therefore x' = 2x + 3$ and $y' = y + 4$

$\therefore x = \frac{x'-3}{2}$ and $y = y' - 4$

$\therefore y = 3(x-2)^2 - 4$ maps to

$y' - 4 = 3\left(\frac{x'-3}{2} - 2\right)^2 - 4$

The image is: $y = \frac{3}{4}(x-7)^2$

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Solutions to Exercise 3J

1 a 1

b $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

c 2

d $\begin{bmatrix} 1 & 1 \\ -\frac{3}{2} & -1 \end{bmatrix}$

2 a $det = -3 + 4 = 1$

$inverse = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$

b $det = 12 + 2 = 14$

$inverse = \begin{bmatrix} \frac{2}{7} & \frac{-1}{14} \\ \frac{1}{7} & \frac{3}{14} \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}$

c $det = k$

$inverse = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$

3 a $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix}$, $\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

b

$AB = \begin{bmatrix} 2+3 & 0+1 \\ 0-3 & 0-1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -3 & -1 \end{bmatrix}$

$\det(AB) = -5 + 3 = -2$

$(AB)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -3 & -5 \end{bmatrix}$

c $A^{-1}B^{-1} = \frac{1}{2}\begin{bmatrix} 1-3 & 0+1 \\ 0+6 & 0-2 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} -2 & 1 \\ 6 & -2 \end{bmatrix}$

$B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} 1+0 & 1+0 \\ -3+0 & -3-2 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -3 & -5 \end{bmatrix}$

$(AB)^{-1} = B^{-1}A^{-1} \neq A^{-1}B^{-1}$

4 a $\det(A) = 4 - 6 = -2$

$A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix}$

b $AX = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$

$IX = A^{-1}\begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$

$X = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$

$X = \frac{1}{2}\begin{bmatrix} 0 & 14 \\ 2 & 16 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 1 & -8 \end{bmatrix}$

c $YA = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$

$YI = \frac{1}{2}\begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix}$

$Y = \frac{1}{2}\begin{bmatrix} -3+8 & 9-16 \\ -1+12 & 3-24 \end{bmatrix}$

$Y = \frac{1}{2}\begin{bmatrix} 5 & -7 \\ 11 & -21 \end{bmatrix}$

5 a $AX + B = C$

$AX = C - B$

$X = A^{-1}(C - B)$

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$\det A = 18 - 2 = 16$

$$A^{-1} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

$$X = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}\left(\begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}\right)$$

$$X = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

$$X = \frac{1}{16}\begin{bmatrix} (6 \times -1) + (-2 \times 0) & (6 \times 5) + (-2 \times 4) \\ (-1 \times -1) + (3 \times 0) & (-1 \times 5) + (3 \times 4) \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} -6 & 22 \\ 1 & 7 \end{bmatrix}$$

b $YA = \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$

$$Y = \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix} \times \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

$$Y = \frac{1}{16}\begin{bmatrix} -11 & 17 \\ -4 & 12 \end{bmatrix}$$

6 $\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$\therefore x = -\frac{1}{3}x'$ and $y = y'$

$\therefore y = x^2 + 2x$ maps to $y' = \frac{1}{9}(x')^2 - \frac{2}{3}x'$

The image is $y = \frac{x^2}{9} - \frac{2x}{3}$

7 $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{9}\begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$\therefore x = \frac{1}{3}x'$ and $y = -\frac{1}{3}y'$

$\therefore y = x^3 + 4$ maps to $-\frac{1}{3}y' = \frac{1}{27}(x')^3 + 4$

The image is $y = -\frac{x^3}{9} - 12$

8 $\begin{bmatrix} 0 & 4 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 0 & -4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$\therefore x = -\frac{1}{2}y'$ and $y = \frac{1}{4}x'$

$\therefore y = 5x + 3$ maps to $\frac{1}{4}x' = -5(\frac{1}{2}y') + 3$

The image is $x + 10y = 12$

9 $\begin{bmatrix} 0 & 5 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{15}\begin{bmatrix} 0 & -5 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$\therefore x = \frac{1}{3}y'$ and $y = -\frac{1}{5}x'$

$\therefore y = -2x + 7$ maps to

$-\frac{1}{5}x' = -2(\frac{1}{3}y') + 7$

The image is $3x - 10y = 105$

10 $\mathbf{A}(\mathbf{X} + \mathbf{B}) = \mathbf{X}'$

Therefore

$\mathbf{X} = \mathbf{A}^{-1}\mathbf{X}' - \mathbf{B}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$\therefore x = -\frac{1}{4}y' + 2$

and $y = -\frac{1}{2}x' - 2$

Hence $y = -2x + 8$ is mapped to

$-\frac{1}{2}x' - 2 = -2\left(-\frac{1}{4}y' + 2\right) + 8$

The image is $x + y = -12$

11 $\begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

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$$\begin{bmatrix} x \\ y \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{-1}{3} & 0 \end{bmatrix} + \left(\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -3 \\ 2 \end{bmatrix}\right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-1}{3} & 0 \end{bmatrix}\left(\begin{bmatrix} x' + 3 \\ y' - 2 \end{bmatrix}\right)$$

$$x = y' - 2, \; y = \frac{-1}{3}(x' + 3)$$

but $y = -2x + 6$

$$-\frac{1}{3}(x' + 3) = -2y' + 4 + 6$$

$$-2y' = \frac{-x'}{3} - 1 - 10$$

$$y' = \frac{x'}{6} + \frac{11}{2}$$

12 $\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\left(\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix}\right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x' + 1}{4} \\ \frac{y' - 4}{2} \end{bmatrix}$$

$$x = \frac{x' + 1}{4}, \; y = \frac{y' - 4}{2}$$

but $y = -2x^3 + 6x$

$$\frac{y' - 4}{2} = -2\left(\frac{x' + 1}{4}\right)^3 + 6\left(\frac{x' + 1}{4}\right)$$

13 $\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} & 0 \\ 0 & \frac{-1}{2} \end{bmatrix}\left(\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-(x' + 1)}{3} \\ \frac{-(y' - 2)}{2} \end{bmatrix}$$

$$x = \frac{-x' - 1}{3}, \; y = \frac{2 - y'}{2}$$

but $y = -2x^3 + 6x^2 + 2$

$$\frac{2 - y'}{2} = -2\left(\frac{-(x + 1)}{3}\right)^3 + 6\left(\frac{-(x + 1)}{3}\right)^2 + 2$$

$$\frac{2 - y'}{2} = 2\left(\frac{(x' + 1)}{3}\right)^3 + 6\left(\frac{(x' + 1)}{3}\right)^2 + 2$$

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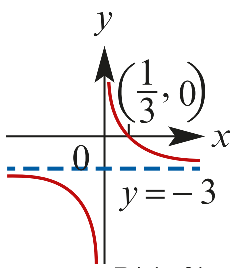
Solutions to technology-free questions

1 a $y = \frac{1}{x} - 3, \ x \neq 0$

no y intercept

$y = 0$: $x = \frac{1}{3}$

asymptotes: $x = 0$ & $y = -3$

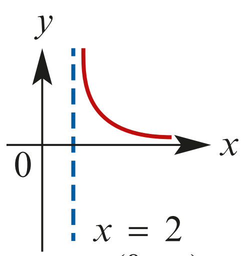


range = $\mathbb{R} \setminus \{-3\}$

b $y = \frac{1}{x-2}, \ x > 2$

no intercepts

asymptotes: $x = 2$ & $y = 0$



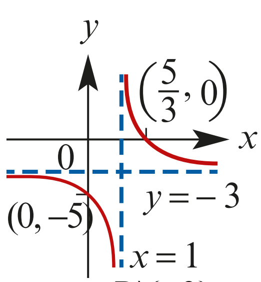
range = $(0, \infty)$

c $y = \frac{2}{x-1} - 3, \ x \neq 1$

$x = 0$: $y = -5$

$y = 0$: $\frac{2}{x-1} = 3 \Rightarrow x = \frac{5}{3}$

asymptotes: $x = 1$ & $y = -3$

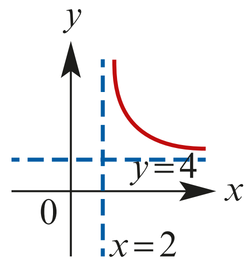


range = $\mathbb{R} \setminus \{-3\}$

d $y = -\frac{3}{2-x} + 4, \ x > 2$

no intercepts

asymptotes: $x = 2$ & $y = 4$



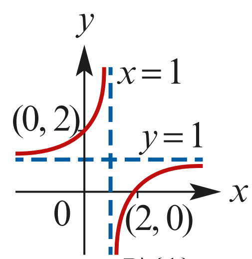
range = $(4, \infty)$

e $y = 1 - \frac{1}{x-1}, \ x \neq 1$

$x = 0$: $y = 2$

$y = 0$: $\frac{1}{x-1} = 1 \Rightarrow x = 2$

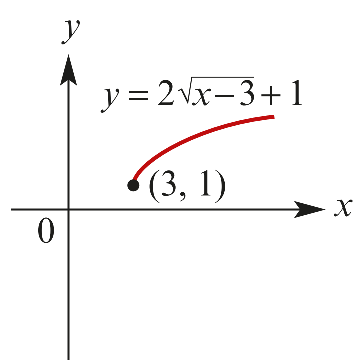
asymptotes: $x = 1$ & $y = 1$



range = $R \setminus \{1\}$

2 a $y = 2\sqrt{x-3} + 1$

$x \geq 3$; $y \geq 1$; endpoint $(3, 1)$



b $y = \frac{3}{(x-2)^2} - 1$

$x = 0$: $y = -\frac{1}{4}$

$y = 0$: $\frac{3}{(x-2)^2} = 1$

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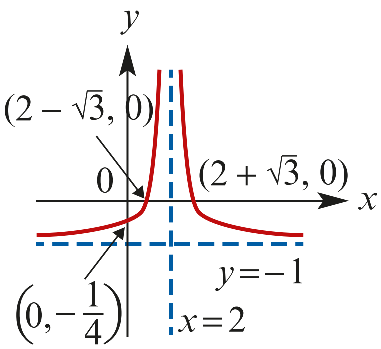
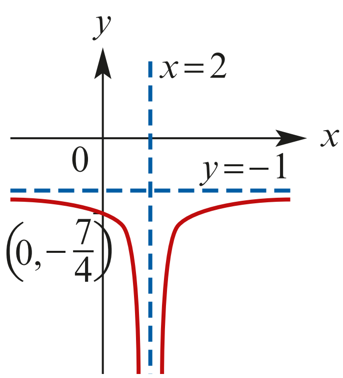
$(x-2)^2 = 3 \Rightarrow \ x = 2 \pm \sqrt{3}$

asymptotes: $x = 2$ & $y = -1$

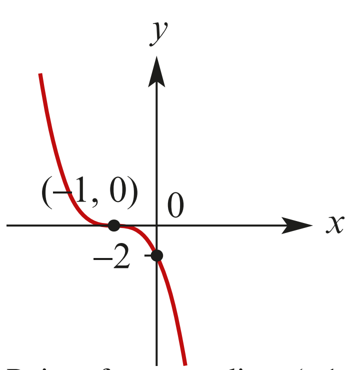


c $y = \dfrac{-3}{(x-2)^2} - 1$

This is a reflection in the line $y = -1$ of the graph in part **b** above. There are no x intercepts, the y intercept is at $y = -\frac{7}{4}$ and the asymptotes are the same.



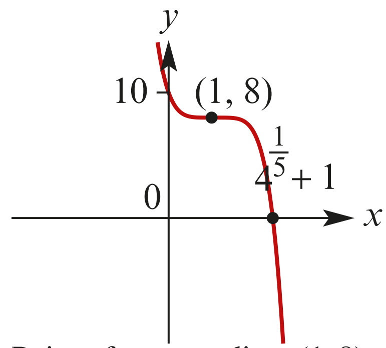
3 a



Point of zero gradient $(-1, 0)$;

Axis intercepts $(-1, 0)$, $(0, -2)$

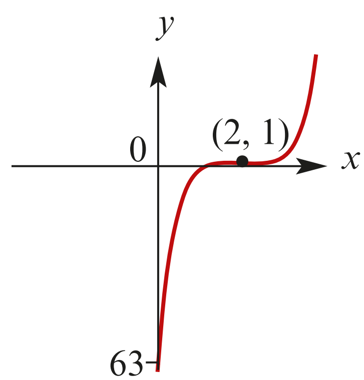
b



Point of zero gradient $(1, 8)$;

Axis intercepts $(4^{\frac{1}{5}}, 0)$, $(0, 10)$

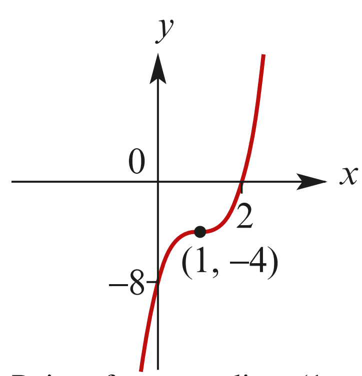
c



Point of zero gradient $(2, 1)$;

Axis intercepts $((-\frac{1}{2})^{\frac{1}{5}} + 2, 0)$, $(0, -63)$

d



Point of zero gradient $(1, -4)$;

Axis intercepts $(2, 0)$, $(0, -8)$

4 $y = a\sqrt{x} + b$

$(1, 6)$ and $(16, 12)$ lie on the curve

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$6 = a + b \ldots (1)$

$12 = 4a + b \ldots (2)$

Subtract (1) from (2)

$6 = 3a$

$a = 2$

$\therefore b = 4$

5 $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ -1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \therefore x' = x - 4$

and $y' = -2y - 1 \therefore x = x' + 4$

and $y = -\dfrac{y' + 1}{2} \therefore$ the image of

$y = \sqrt{x}$ under this transformation is

$-\dfrac{y' + 1}{2} = \sqrt{x' + 4}$

The image is $y = -2\sqrt{x + 4} - 1$

6 $\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$\therefore x' = 3x - 4$ and $y' = -y - \dfrac{1}{2}$

$\therefore x = \dfrac{x' + 4}{3}$ and $y = -y' - \dfrac{1}{2}$

$\therefore$ the image of $y = 2\sqrt{x - 4} + 3$ under this transformation is

$-y' - \dfrac{1}{2} = 2\sqrt{\dfrac{x' + 4}{3} - 4} + 3$

The image is $y = -2\sqrt{\dfrac{x - 8}{3}} - \dfrac{7}{2}$

7 $(1, 3)$: $3 = a + b \quad \ldots (1)$

$(3,7)$: $7 = \dfrac{a}{3} + b \quad \ldots (2)$

Subtract (1) from (2):

$\dfrac{a}{3} - a = 4$

$-\dfrac{2a}{3} = 4$

$a = -6$

Substitute into (1): $b = 9$

8 **a** $(x, y) \rightarrow (-x, y) \rightarrow (-2x, y) \rightarrow (-2x + 4, y + 6) = (x', y')$

$\therefore x' = -2x + 4$ and $y' = y + 6$

$\therefore x = \dfrac{4 - x'}{2}$ and $y = y' - 6$.

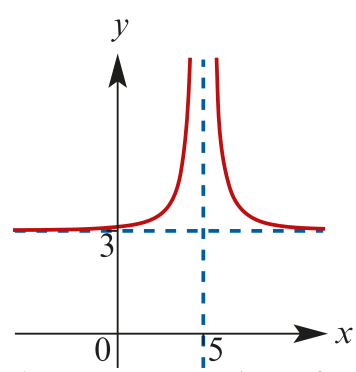
$\therefore y = -x^2$ maps to

$y' - 6 = -\left(\dfrac{4 - x'}{2}\right)^2$ That is to

$y = -\left(\dfrac{x - 4}{2}\right)^2 + 6$

b Reflection in the x-axis, dilation of factor 4 from the x-axis, then translate1 unit to the left and 6 units up

9 Dilation of factor 3 from the x-axis, then translation 5 units to the right and 3 units up



Asymptotes $x = 5$, $y = 3$; Intercept $\left(0, \dfrac{78}{25}\right)$

10 Dilation of factor $\frac{1}{2}$ from the x-axis, then translation $\frac{3}{2}$ units up

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11 Dilation of factor $\frac{1}{2}$ from the x-axis, then translation 3 units to the left and 2 units down

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Solutions to multiple-choice questions

1 B $(3,-4) \to (3,-1) \to (3,1)$

2 B $y = x^3 + 4 \to y = x^3 + 1 \to y = (x-2)^3 + 1$

3 B

4 E $y = x^2 \to y = -x^2 \to y = -(x+4)^2 - 3$

5 D Asymptotes at $x = -3$ and $y = -2$
$\therefore b = 3$ and $c = -2$

6 A Let $y = x^{\frac{1}{3}}$ Reflection in the y-axis: $y = -x^{\frac{1}{3}}$ Dilation by a factor of 5 units from the x-axis: $y = -5x^{\frac{1}{3}}$

7 D $x' = 3x - 2$ and $y' = -y - 1$
$\therefore x = \dfrac{x'+2}{3}$ and $y = -y' - 1$

The image is $-y' - 2 = \sqrt[3]{\dfrac{x'+2}{3}}$

$\therefore y = -\sqrt[3]{\dfrac{x'+2}{3}} - 2$

8 A Rearranging $\dfrac{y+4}{3} = \dfrac{1}{2x+1}$

Choose $x = 2x' + 1$ and $y = \dfrac{y'+4}{3}$

$\therefore x' = \dfrac{1}{2}x - \dfrac{1}{2}$ and $y' = 3y - 4$

9 A Rearrange $-\dfrac{y-3}{5} = \dfrac{1}{2x-1}$
Choose $x' = 2x - 1$ and

$y' = -\dfrac{y-3}{5} = -\dfrac{y}{5} + \dfrac{3}{5}$

10 A $g(f(x)) = (3x-2)^2 - 4(3x-2) + 2$.
Therefore $x = 3x' - 2$
$\therefore x' = \dfrac{x+2}{3} = \dfrac{x}{3} - \dfrac{2}{3}$

.

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Solutions to extended-response questions

1 a $\mathbb{R}\backslash\{-2\}$

b
- dilation of factor 24 from the x-axis
- translation of 2 in the negative direction of the x-axis
- translation of 6 in the negative direction of the y-axis

c $f(0) = \dfrac{24}{2} - 6 = 12 - 6 = 6$

$\therefore$ y axis intercept is 6: $(0, 6)$

$f(x) = 0$ implies

$$\frac{24}{x+2} - 6 = 0$$

$$\therefore 24 = 6(x + 2)$$

$$\therefore 24 = 6x + 12$$

$$\therefore x = 2$$

$y = f(x)$ intercepts with the x axis at $(2, 0)$

d $g:(-2, \infty) \to R,\ g(x) = \dfrac{24}{x+2} - 6$

Consider $x = \dfrac{24}{y+2} - 6$

i.e. $(y + 2)x = 24 - 6(y + 2)$

$\therefore yx + 6y = 24 - 12 - 2x$

$y(x + 6) = 12 - 2x$

$\therefore y = \dfrac{12 - 2x}{x + 6}$

$\therefore g^{-1}(x) = \dfrac{12 - 2x}{x + 6} = -2 + \dfrac{24}{x + 6}$

e $\therefore$ domain of g^{-1} = range of $g = (-6, \infty)$

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f,g

For $g(x) = x$

$$\frac{24}{x+2} - 6 = x$$

i.e. $\dfrac{12-2x}{x+6} = x$

$\therefore 12 - 2x = x^2 + 6x$

$\therefore x^2 + 8x - 12 = 0$

$$\therefore x = \frac{-8 \pm \sqrt{64+48}}{2}$$

$$= \frac{-8 \pm \sqrt{112}}{2}$$

$$= \frac{-8 \pm 4\sqrt{7}}{2}$$

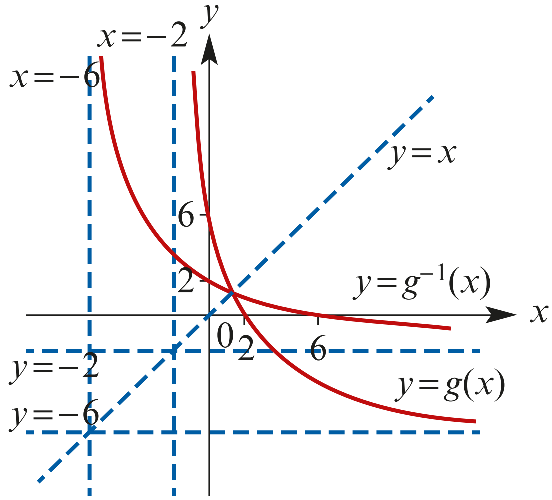
$$= -4 \pm 2\sqrt{7}$$

But $x \in (-6, \infty)$

Therefore $x = -4 + 2\sqrt{7}$

The graphs of $y = g(x)$ and $y = g^{-1}(x)$ intersect where $y = x$

$\therefore$ they intersect at $x = -4 + 2\sqrt{7}$



2 $f : D \to R,\ f(x) = 4 - 2\sqrt{2x+6}$

a $2x + 6 \geq 0$

i.e. $x \geq -3$

$\therefore$ domain is $[-3, \infty)$

b
- dilation of factor $\dfrac{1}{2}$ from the y axis
- dilation of factor 2 from the x axis
- reflection in the x axis
- translation 3 units in the negative direction of the x axis
- translation 4 units in the positive direction of the y axis

c $f(0) = 4 - 2\sqrt{6}$

$\therefore y = f(x)$ cuts the y axis at $(0, 4 - 2\sqrt{6})$

When $4 - 2\sqrt{2x+6} = 0$

$4 = 2\sqrt{2x+6}$

i.e. $2 = \sqrt{2x+6}$

$\therefore 4 = 2x + 6$

$\therefore x = -1$

$\therefore y = f(x)$ cuts the x axis at $(-1, 0)$

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d Consider $x = 4 - 2\sqrt{2y+6}$
Then $2\sqrt{2y+6} = 4 - x$
Squaring both sides yields

$$4(2y+6) = (4-x)^2$$
$$\therefore 8y + 24 = 16 - 8x + x^2$$
$$\therefore y = \frac{1}{8}(x^2 - 8x - 8)$$
$$= \frac{1}{8}(x^2 - 8x + 16 - 24)$$
$$= \frac{1}{8}(x-4)^2 - 3$$

i.e. $f^{-1}(x) = \frac{1}{8}(x-4)^2 - 3$

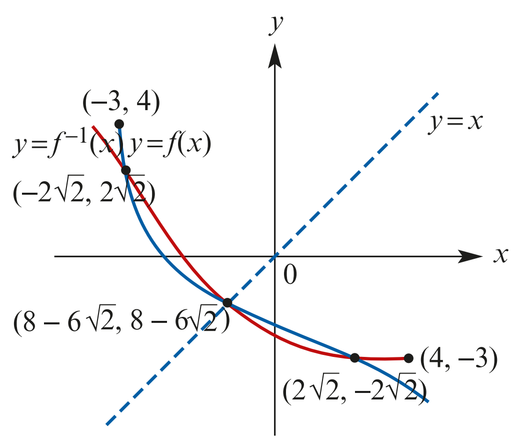
e The domain of f^{-1} = range of $f = (-\infty, 4]$

f,g

$$f(x) = x$$
$$4 - 2\sqrt{2x+6} = x$$
$$\text{implies } 2\sqrt{2x+6} = 4 - x$$
$$\therefore 4(2x+6) = 16 - 8x + x^2$$
$$\therefore 8x + 24 = 16 - 8x + x^2$$
$$\therefore x^2 - 16x - 8 = 0$$
$$\therefore x = \frac{16 \pm \sqrt{256+32}}{2} = \frac{16 \pm \sqrt{288}}{2} = \frac{16 \pm 12\sqrt{2}}{2}$$
$$x = 8 \pm 6\sqrt{2}$$

and the required solution is $x = 8 - 6\sqrt{2}$
The curves intersect at two other points
Consider $4 - 2\sqrt{2x+6} = \frac{1}{8}(x^2 - 8x - 8)$
Use a CAS calculator to find the other solutions.
It can be shown that they intersect on the line

$$y = -x$$
$$4 - 2\sqrt{2x+6} = -x$$
$$-2\sqrt{2x+6} = -x - 4$$
$$4(2x+6) = 16 + 8x + x^2$$
$$\therefore x^2 = 8$$
$$x = \pm 2\sqrt{2}$$



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3 a i $(x, y) \to (x,\ ky)$, so $(25, 625) \to (25, 15)$

$\therefore k = \dfrac{15}{625} = \dfrac{3}{125}$

Dilation of factor $\dfrac{3}{125}$ from the x axis

ii $(x, y) \to (x, -y)$

iii $(x, y) \to (x + 25, y + 15)$

iv $(x, y) \to \left(x + 25, \dfrac{-3}{125}y + 15\right)$

b i $y = \dfrac{-3}{125}(x - 25)^2 + 15$

ii $(x, y) \to (x + 50, y)$

iii $y = \dfrac{-3}{125}(x - 75)^2 + 15$

c i Dilation factor from the x axis

$(x, y) \to (x,\ ky)$

$\left(\dfrac{m}{2}, \dfrac{m^2}{4}\right) \to \left(\dfrac{m}{2}, n\right)$

$\therefore k = \dfrac{n}{\frac{m^2}{4}}$

$= \dfrac{4n}{m^2}$

reflection in x axis $(x, y) \to (x, -y)$

translation $(x, y) \to \left(x + \dfrac{m}{2}, y + n\right)$

overall $(x, y) \to \left(x + \dfrac{m}{2}, \dfrac{-4n}{m^2}y + n\right)$

ii $y = \dfrac{-4n}{m^2}\left(x - \dfrac{m}{2}\right)^2 + n$

iii $y = \dfrac{-4n}{m^2}\left(x - \dfrac{3m}{2}\right)^2 + n$

4 a $\mathbb{R}\setminus\left\{\dfrac{4}{3}\right\}$

b $a = \dfrac{4}{3}$

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c Consider

$$x = \frac{3}{(3y-4)^2} + 6$$

$$x - 6 = \frac{3}{(3y-4)^2}$$

$$\frac{x-6}{3} = \frac{1}{(3y-4)^2}$$

$$\therefore (3y-4)^2 = \frac{3}{x-6}$$

$$\therefore y = \frac{1}{3}\sqrt{\frac{3}{x-6}} + \frac{4}{3} \quad \text{as range of } f^{-1} = \text{domain of } f = \left(\frac{4}{3}, \infty\right)$$

d Consider $\dfrac{3}{(3x-4)^2} + 6 = x$ as $y = f(x)$ and $y = f^{-1}(x)$ intersect on the line $y = x$

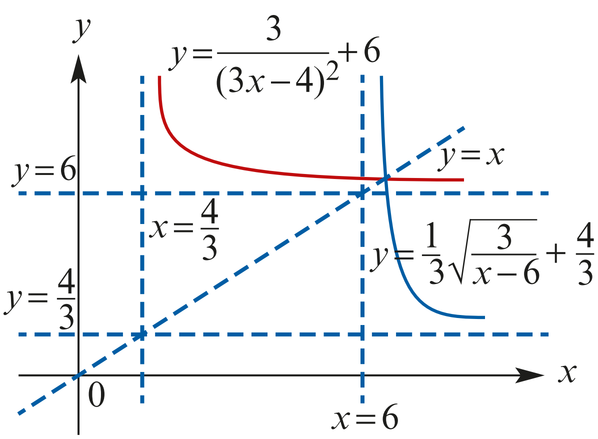
$$\frac{3}{(3x-4)^2} = x - 6$$

$3 = (x-6)(3x-4)^2$

$x = 6.015$

(Solve the equation with the 'solve' command of a CAS calculator.)

e



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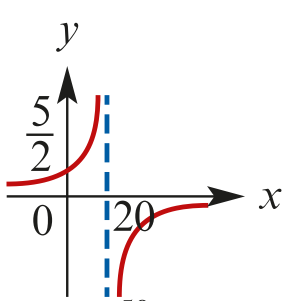

$g(x) = \dfrac{50x}{20-x}$

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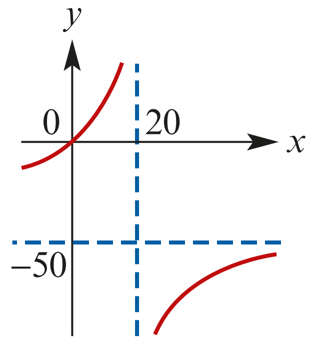
a **i** $\dfrac{1000}{20-x} - 50$

$$= \frac{1000 - 50(20 - x)}{20 - x}$$

$$= \frac{1000 - 1000 + 50x}{20 - x}$$

$$= \frac{50x}{20 - x} = g(x)$$

ii



iii $20f(x) - 50 = 20\left(\dfrac{50}{20-x}\right) - 50$

$$= \frac{1000 - 50(20 - x)}{20 - x}$$

$$= \frac{1000 - 1000 + 50x}{20 - x}$$

$$= \frac{50x}{20 - x} = g(x)$$

b Consider $x = \dfrac{50y}{20 - y}$

$(20 - y)x = 50y$

$\therefore 20x = 50y + yx$

$\therefore 20x = y(50 + x)$

$y = \dfrac{20x}{50 + x}$

$\therefore g{-}1(x) = \dfrac{20x}{50 + x}$

6 **a** **i** $(x, y) \to (x + 3, y + 5) \to (y + 5, x + 3)$

(x, y) maps to a unique point (x', y')

Hence $x' = y + 5$ and $y' = x + 3$

Hence $y = x' - 5$ and $x = y' - 3$

Therefore the graph of $y = f(x)$ maps to the graph of $x'\ 5 = f(y' - 3)$

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The inverse function exists and therefore

$y' = f^{-1}(x' - 5) + 3$

ii $(x, y) \to (y, x) \to (y + 3, x + 5)$

(x, y) maps to a unique point (x', y')

Hence $x' = y + 3$ and $y' = x + 5$

Hence $y = x' - 3$ and $x = y' - 5$

Therefore the graph of $y = f(x)$ maps to the graph of $x' - 3 = f(y' - 5)$

The inverse function exists and therefore

$y' = f^{-1}(x' - 3) + 5$

iii $(x, y) \to (5x, 3y) \to (3y, 5x)$

(x, y) maps to a unique point (x', y')

Hence $x' = 3y$ and $y' = 5x$

Hence $y = \frac{x'}{3}$ and $x = \frac{y'}{5}$

Therefore the graph of $y = f(x)$ maps to the graph of $\frac{x'}{3} = f\left(\frac{y'}{5}\right)$

The inverse function exists and therefore

$y' = 5f^{-1}\left(\frac{x'}{3}\right)$

iv $(x, y) \to (y, x) \to (5y, 3x)$

(x, y) maps to a unique point (x', y')

Hence $x' = 5y$ and $y' = 3x$

Hence $y = \frac{x'}{5}$ and $x = \frac{y'}{3}$

Therefore the graph of $y = f(x)$ maps to the graph of $\frac{x'}{5} = f\left(\frac{y'}{3}\right)$

The inverse function exists and therefore

$y' = 3f^{-1}\left(\frac{x'}{5}\right)$

b $x' = ay + b$ and $y' = cx + d$

Therefore

Therefore $y = \frac{x' - b}{a}$ and $x = \frac{y' - d}{c}$

The graph of $y = f(x)$ maps to the graph of $\frac{x' - b}{a} = f\left(\frac{y' - d}{c}\right)$

Therefore as the inverse function exists $y' = cf^{-1}\left(\frac{x' - b}{a}\right) + d$

From

$x' = ay + b$ and $y' = cx + d$:

the graph of $y = f(x)$ has undergone the following sequence of transformations:

A reflection in the line $y = x$, then a dilation of factor c from the x axis and factor a from the y axis, and a translation of b units in the positive direction of the x axis and

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d units in the positive direction of the y axis.

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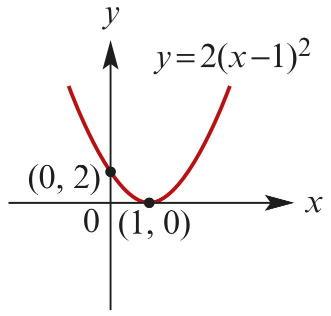
Chapter 4 – Polynomial functions

Solutions to Exercise 4A

1 a



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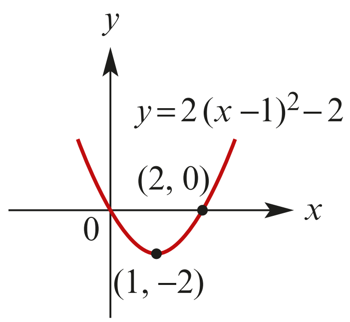

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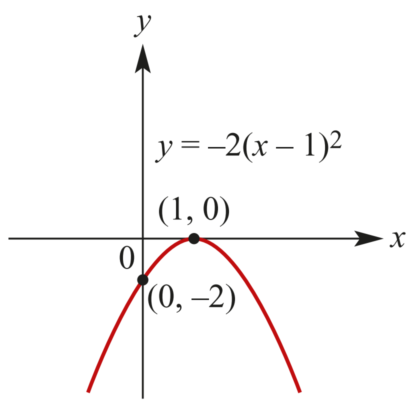

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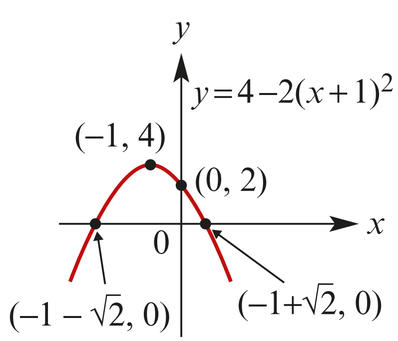

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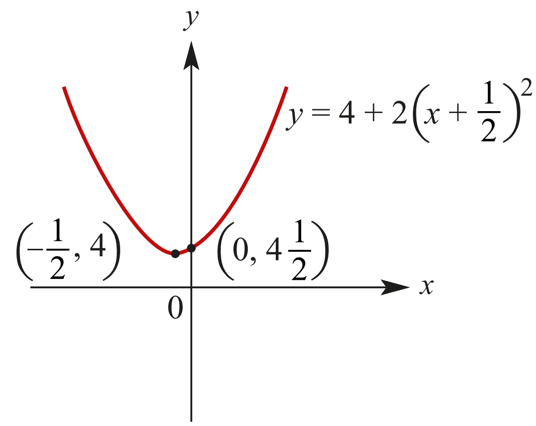

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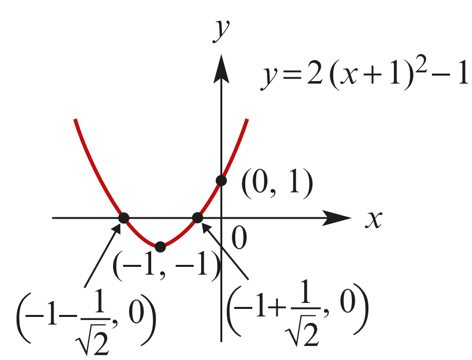

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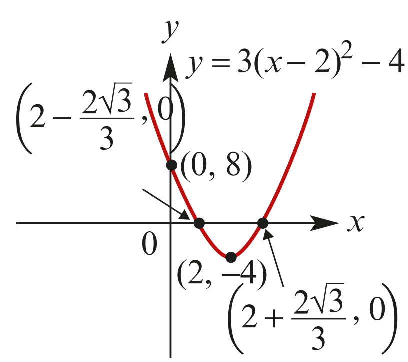

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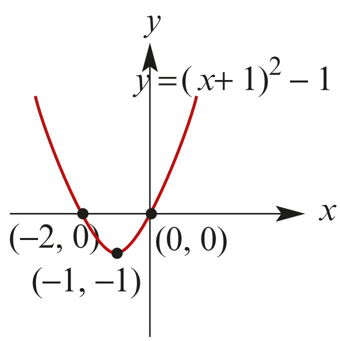
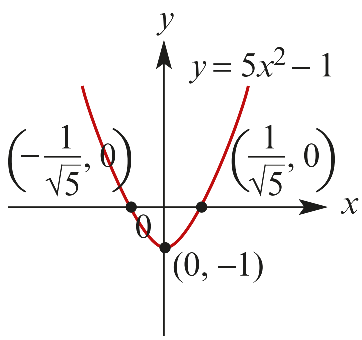
h



i



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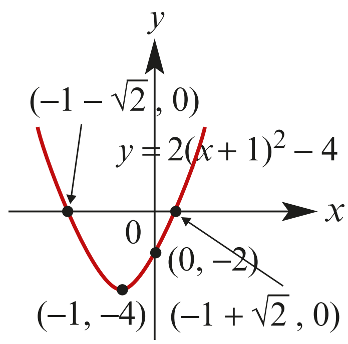

2 **a** $f(x) = x^2 + 3x - 2$

$= x^2 + 3x + 2.25 - 2.25 - 2$

$= (x + 1.5)^2 - 4.25$

Minimum $= -4.25$ and the range is $[-4.25, \infty)$

b $f(x) = x^2 - 6x + 8$

$= x^2 - 6x + 9 - 9 + 8$

$= (x - 3)^2 - 1$

Minimum $= -1$ and the range is $[-1, \infty)$

c $f(x) = 2x^2 + 8x - 6$

$= 2(x^2 + 4x - 3)$

$= 2(x^2 + 4x + 4) - 14$

$= 2(x + 2)^2 - 14$

Minimum $= -14$ and the range is $[-14, \infty)$

d $f(x) = 4x^2 + 8x - 7$

$= 4(x^2 + 2x) - 7$

$= 4(x^2 + 2x + 1) - 4 - 7$

$= 4(x + 1)^2 - 11$

Minimum $= -11$ and the range is $[-11, \infty)$

e $f(x) = 2x^2 - 5x$

$= 2\left(x^2 - \frac{5}{2}x\right)$

$= 2\left(x^2 - \frac{5}{2}x + \frac{25}{16}\right) - \frac{25}{8}$

$= 2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8}$

Minimum $= \frac{-25}{8}$ and the range is $\left[\frac{-25}{8}, \infty\right)$

f $f(x) = -3x^2 - 2x + 7$

$= -3\left(x^2 - \frac{2}{3}x\right) + 7$

$= -3\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) + \frac{1}{3} + 7$

$= -3\left(x + \frac{1}{3}\right)^2 + \frac{22}{3}$

maximum $= \frac{22}{3}$ and the range is $\left(-\infty, \frac{22}{3}\right]$

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g $f(x) = -2x^2 + 9x + 11$

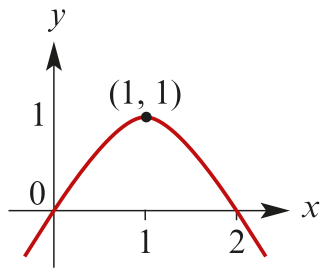
$$= -2\left(x^2 - \frac{9}{2}x\right) + 11$$

$$= -2\left(x^2 - \frac{9}{2}x + \frac{81}{16}\right) + \frac{81}{8} + 11$$

$$= -2\left(x - \frac{9}{4}\right)^2 + \frac{169}{8}$$

maximum $= \frac{169}{8}$ and the range is $\left(-\infty, \frac{169}{8}\right]$

3 a



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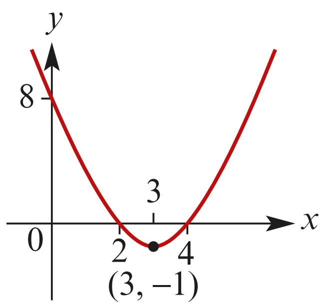

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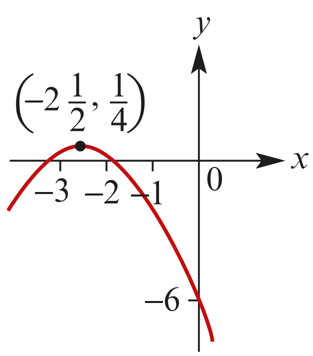

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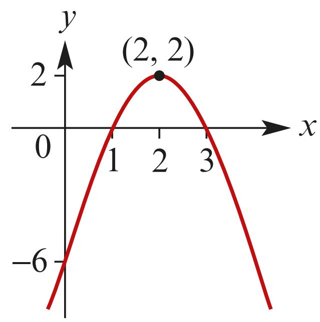

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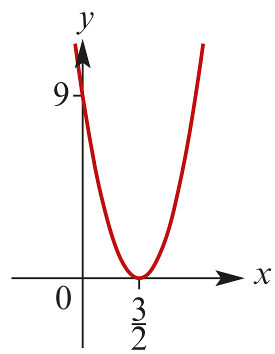

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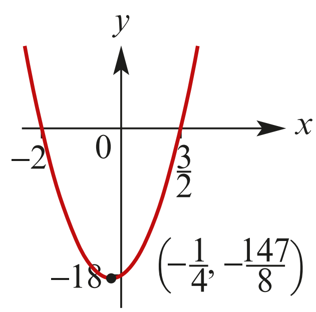
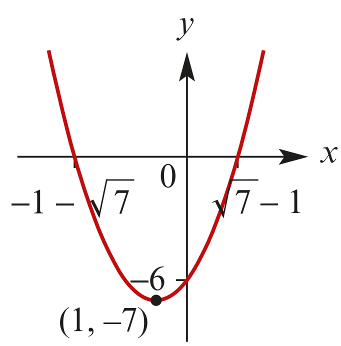


4 a $y = (x + 1)^2 - 7$



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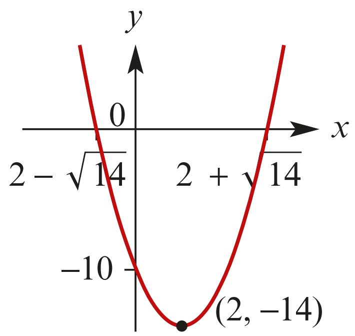
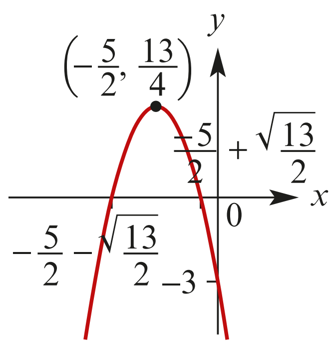
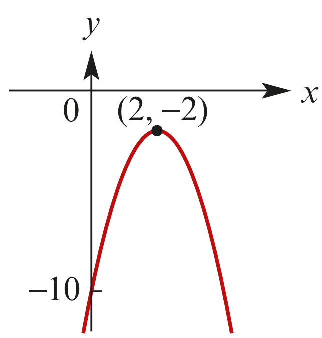
b $y = (x - 2)^2 - 14$



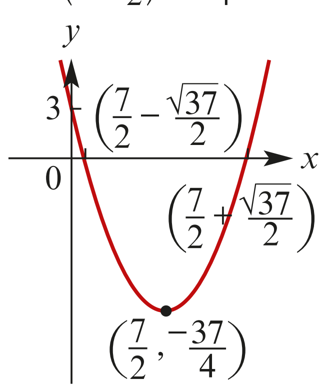
c $y = \frac{13}{4} - \left(x + \frac{5}{2}\right)^2$



d $y = -2(x - 2)^2 - 2$



e $y = \left(x - \frac{7}{2}\right)^2 - \frac{37}{4}$



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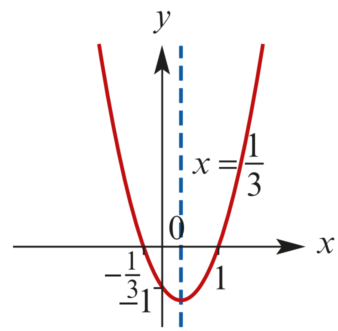

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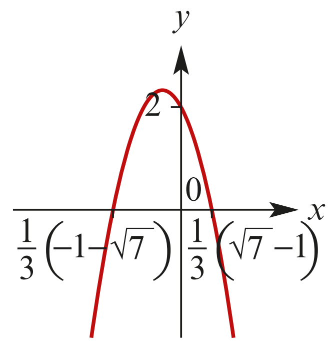
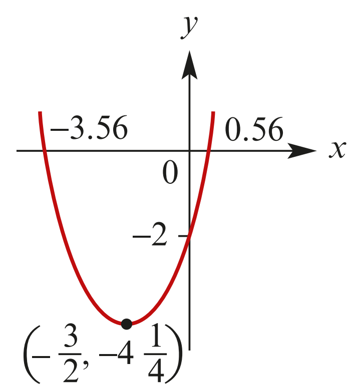


7 **a**



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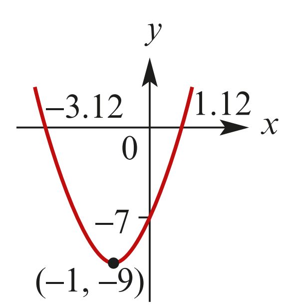

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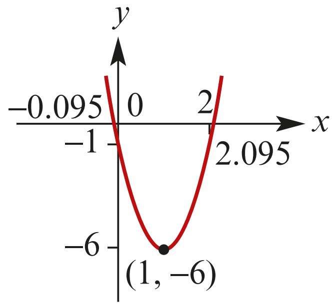
c



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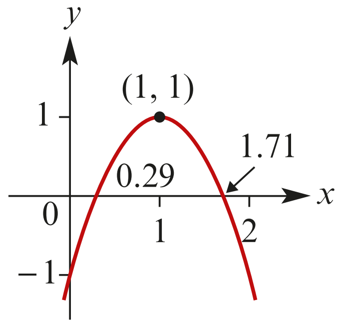

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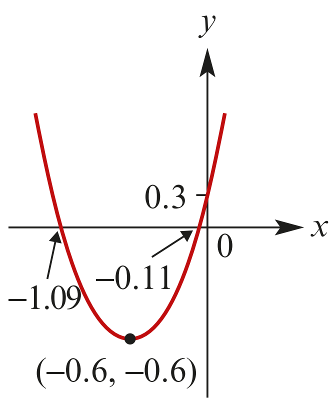

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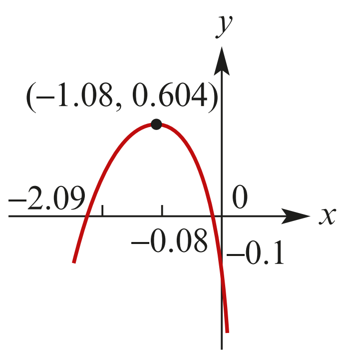

8 a *B*

b *D*
by looking at turning point coordinates

9 a *C* x-axis intercepts

b *B* turning point x-value

c *D* turning point coordinates

d *A* turning point x-value

10 a $b^2 - 4ac = 25 - 8$

> 0

$\therefore$ it crosses the x-axis

b $b^2 - 4ac = 4 - 4 \times -4 - 1$

$= 4 - 16$

< 0

$\therefore$ it does not intersect the x-axis

c $b^2 - 4ac = 36 - 36 = 0$

$\therefore$ it touches the x-axis

d $b^2 - 4ac = 9 - 4 \times 8 \times -2$

$= 9 + 64$

> 0

$\therefore$ it crosses the x-axis

e $b^2 - 4ac = 4 - 60 < 0$

$\therefore$ it does not intersect the x-axis

f $b^2 - 4ac = 1 - 4$

< 0

$\therefore$ it does not intersect the x-axis

11 $mx^2 - 2mx + 3 = 0$

$$b^2 - 4ac = 4m^2 - 12m$$

$$= 4m(m - 3)$$

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a $4m(m-3) > 0$

$m < 0$ or $m > 3$

b $4m(m-3) = 0$

$m = 3$

($m = 0$, is not a solution as it gives $3 = 0$)

12 $\Delta = 36m^2 - 16(4m+1)$

$= 36m^2 - 64m - 16$

$= 4(9m^2 - 16m - 4)$

$= 4(9m+2)(m-2)$

Perfect square if $\Delta = 0$

$\therefore m = -\frac{2}{9}$ or $m = 2$

13 $\Delta = 4a^2 - 4(a+2)(a-3)$

$= 4a^2 - 4(a^2 - a - 6)$

$= 4a + 24$

No solutions if $\Delta < 0$

$\therefore a < -6$

14 $\Delta = (a+1)^2 - 4(a-2)$

$= a^2 + 2a + 1 - 4a + 8$

$= a^2 - 2a + 9$

$= (a-1)^2 + 8$

$\therefore \Delta > 0$ for all values of a

15 **a** $(k+1)x^2 - 2x - k = 0$

$b^2 - 4ac = 4 + k(k+1)$

need to show

$4 + k(k+1) > 0$

i.e. $k(k+1) > -4$

$LHS = k^2 + k$

$= k^2 + k + \frac{1}{4} - \frac{1}{4}$

$= (k + \frac{1}{2})^2 - \frac{1}{4}$

> -4

which is what is required

16 $\Delta = 4k^2 + 20k$

$= 4k(k+5)$

a $\Delta > 0 \Leftrightarrow k \in (-\infty, -5) \cup (0, \infty)$.

b $\Delta = 0 \Leftrightarrow k = 0$ or $k = -5$

17 $\Delta = 4k^2 - 4(k+2)(k-3)$

$= 4k^2 - 4(k^2 - k - 6)$

$= 4(k+6)$

a Two solutions if $k > -6$

b One solution if $k = -6$

18 **a** $ax^2 - (a+b)x + b = 0$

$(a+b)^2 - 4ab = a^2 + 2ab + b^2 - 4ab$

$= (a-b)^2 \geq 0$

$\therefore$ the equation always has at least one solution

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Solutions to Exercise 4B

1 $y = k(x+3)(x+2)$
When $x = 1, y = -24$
$\therefore -24 = k(4)(3)$
$\therefore k = -2$
$\therefore y = -2(x+3)(x+2)$

2 $y = k(x+3)(2x+3)$
When $x = 1, y = 20$
$\therefore 20 = k(4)(5)$
$\therefore k = 1$
$\therefore y = (x+3)(2x+3)$

3 $y = a(x+2)^2 + 4$
When $x = 4, y = 58$
$\therefore 58 = 36a + 4$
$\therefore a = \frac{54}{36} = \frac{3}{2}$
$\therefore y = \frac{3}{2}(x+2)^2 + 4$

4 $y = a(x+2)^2 - 3$
When $x = -3, y = -5$
$\therefore -5 = a - 3$
$\therefore a = -2$
$\therefore y = -2(x+2)^2 - 3$

5 Passes through $(1, 19)$, $(0, 18)$ and $(-1, 7)$
The equation has form $y = ax^2 + bx + 18$

$$19 = a + b + 18 \ldots (1)$$
$$7 = a - b + 18 \ldots (2)$$

Equation (1) − Equation (2)

$$12 = 2b$$
$$\therefore b = 6$$
$$\therefore a = -5$$
$$\therefore y = -5x^2 + 6x + 18$$

6 Passes through $(2, -14)$, $(0, 10)$ and $(-4, 10)$
The equation has form $y = ax^2 + bx + 10$

$$-14 = 4a + 2b + 10 \ldots (1)$$
$$10 = 16a - 4b + 10 \ldots (2)$$

$2 \times$ Equation (1) + Equation (2)

$$-18 = 24a + 30$$
$$\therefore a = -2$$
$$\therefore b = -8$$
$$\therefore y = -2x^2 - 8x + 10$$

7 **a** $y = ax^2 + bx + c$

$c = 4$(y-intercept)

$b = 0$(x-value at turning point)

$y = ax^2 + 4$

$x = 5, y = 0$

$0 = 25a + 4$

$a = \frac{-4}{25}$

$y = \frac{-4}{25}x^2 + 4$

b $y = a(x+h)^2 + k$

$y = ax_2$

$x = 2, y = -4$

$-4 = 4a$

$a = -1$

$y = -x^2$

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c $y = a(x + b)(x + c)$

$y = a(x + 2)(x + 0)$

$y = ax^2 + 2ax$

$x = 1, y = 3$

$3 = a + 2a$

$a = 1$

$y = x^2 + 2x$

d $y = a(x + b)(x + c)$

$y = a(x + 0)(x - 2)$

$y = ax^2 - 2ax$

$x = -1, y = -3$

$-3 = a + 2a$

$a = -1$

$y = -x^2 + 2x$

e $y = a(x + b)(x + c)$

$y = a(x - 1)(x - 4)$

$y = ax^2 - 5ax + 4a$

$4a = 4(y\text{-}intercept)$

$a = 1$

$y = x^2 - 5x + 4$

f $y = a(x + b)(x + c)$

$y = a(x + 1)(x - 5)$

$y = ax^2 - 4ax - 5a$

$-5a = -5$ (y-intercept)

$a = 1$

$y = x^2 - 4x - 5$

g $y = a(x + h)^2 + k$

$y = a(x - 1)^2 - 2$

$x = -1, y = 2$

$2 = 4a - 2$

$a = 1$

$y = (x - 1)^2 - 2 = x^2 - 2x - 1$

h $y = a(x + h)^2 + k$

$y = a(x - 2)^2 + 2$

$x = 0, y = 6$

$6 = 4a + 2$

$a = 1$

$y = (x - 2)^2 + 2 = x^2 - 4x + 6$

8 left hand curve

$-y = ax^2 + x + c$

$c = -5 \qquad C$

$x = 4, y = 1 \qquad B$

$1 = 16a + 4 - 5$

$a = \dfrac{1}{8}$

$y = \dfrac{1}{8}x^2 + x - 5$

right hand curve

$y = ax^2 + x + c$

$c = 1 \qquad D$

$y = ax^2 + x + 1$

$x = 4, y = 3 \qquad A$

$3 = 16a + 4 + 1$

$16a = -2$

$a = -\dfrac{1}{8}$

$y = -\dfrac{1}{8}x^2 + x + 1$

9 $f(x) = A(x + b)^2 + B$

$= A(x + 2)^2 + 4(\text{vertex})$

$f(0) = 8$

$8 = 4a + 4$

$A = 1, b = 2, B = 4$

$f(x) = (x + 2)^2 + 4$

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Solutions to Exercise 4C

1 a $P(1) = 3$

b $P(-1) = -5$

c $P(2) = 7$

d $P(-2) = -21$

e $P(\frac{1}{2}) = \frac{17}{8}$

f $P(-\frac{1}{2}) = -\frac{9}{8}$

2 a $P(0) = 6$

b $P(1) = 6$

c $P(2) = 18$

d $P(-1) = 12$

e $P(a) = a^3 + 3a^2 - 4a + 6$

f $P(2a) = 8a^3 + 12a^2 - 8a + 6$

3 a

$$\begin{aligned} P(2) &= 0 \\ 8 + 12 - 2a - 30 &= 0 \\ -2a &= 10 \\ a &= -5 \end{aligned}$$

b

$$\begin{aligned} P(3) &= 68 \\ 27 + 9a + 15 - 14 &= 68 \\ 9a &= 40 \\ a &= \frac{40}{9} \end{aligned}$$

c

$$\begin{aligned} P(1) &= 6 \\ 1 - 1 - 2 + c &= 6 \\ c &= 8 \end{aligned}$$

d

$$\begin{aligned} P(-1) = P(2) &= 0 \\ 2 + 5 + a - b + 12 &= 0 \\ a - b &= -19 \ldots (1) \\ 128 - 40 + 4a + 2b + 12 &= 0 \\ 4a + 2b &= -100 \\ 2a + b &= -50 \ldots (2) \end{aligned}$$

Equation (1) + Equation(2)

$$\begin{aligned} 3a &= -69 \\ a &= -23 \\ \therefore b &= -4 \end{aligned}$$

e

$$\begin{aligned} P(3) = P(1) &= 0 \\ 3^5 - 2 \times 3^4 + 27a + 9b + 36 - 36 &= 0 \\ 81 + 27a + 9b &= 0 \\ 3a + b &= -9 \ldots (1) \\ 1 - 2 + a + b + 12 - 36 &= 0 \\ a + b &= 25 \ldots (2) \end{aligned}$$

Equation (1) − Equation(2)

$$\begin{aligned} 2a &= -34 \\ a &= -17 \\ \therefore b &= 42 \end{aligned}$$

4 a $2x^3 - x^2 + 2x + 2$

b $2x^3 + 5x$

c $2x^3 - x^2 + 4x - 2$

d $6x^3 - 3x^2 + 9x$

e $-2x^4 + 5x^3 - 5x^2 + 6x$

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f $4x - x^3$

g $2x^3 + 4x + 2$

h $2x^5 + 3x^4 + x^3 + 6x^2$

5 a $x^3 - 5x^2 + 10x - 8$

b $x^3 - 7x^2 + 13x - 15$

c $2x^3 - x^2 - 7x - 4$

d $x^2 + (b + 2)x^2 + (2b + c)x + 2c$

e $2x^3 - 9x^2 - 2x + 3$

6 a $(x + 1)(x^2 + bx + c) =$
$x^3 + (b + 1)x^2 + (c + b)x + c$

b $x^3 - x^2 - 6x - 4 =$
$x^3 + (b + 1)x^2 + (c + b)x + c$
for all x. $\therefore (b + 1) = -1, c = -4$ and
$c + b = -6$
$\therefore b = -2$ and $c = -4$

c $x^3 - x^2 - 6x - 4 = (x + 1)(x^2 - 2x - 4)$
$\therefore x^3 - x^2 - 6x - 4 =$
$(x + 1)(x + \sqrt{5} - 1)(x - \sqrt{5} - 1)$

7 a $2x^3 - 18x^2 + 54x - 49 =$
$a(x^3 - 9x^2 + 27x - 27) + b$
Equating coefficients
$a = 2$ and $-27a + b = -49$
$\therefore a = 2$ and $b = 5$

b $-2x^3 + 18x^2 - 54x + 52 =$
$a(x^3 + 3cx^2 + 3c^2x + c^3) + b$
Equating coefficients
$a = -2$ and $3ca = 18$ and $52 = ac^3 + b$
$\therefore a = -2, c = -3$ and $b = -2$

c $x^3 - 5x^2 - 2x + 24 =$
$a(x^3 + 3cx^2 + 3c^2x + c^3) + b$
Equating coefficients: For x^3: $a = 1$
For x^2: $-5 = 3c$
For x: $-2 = 3c^2$ which is impossible

8 $A(x + 3) + B(x + 2) = 4x + 9$

$(A + B)x + (3A + 2B) = 4x + 9$
by equating coefficients

(1) $A + B = 4$

(2) $3A + 2B = 9$

(2) + 2(1) $\Rightarrow$ $A = 1$

(1) $\Rightarrow$ $B = 3$

9 a $x^2 - 4x + 10 = Ax^2 + 2ABx + AB^2 + C$
by equating coefficients

(1) $A = 1$

$2AB = 4$

(2) $AB^2 + C = 10$

(1) $\Rightarrow$ $= 2B = 4$

$B = -2$

$\Rightarrow 2 \Rightarrow$ $4 + C = 10$

$C = 6$

b $4x^2 - 12x + 14 = Ax^2 + 2AB + C$
by equating coefficients

$A = 4$

(1) $2AB = -12$

(2) $AB^2 + C = 14$

$\Rightarrow$ (1) $\Rightarrow$ $8B = -12$

$B = \dfrac{-3}{2}$

(2) $\Rightarrow$ $4 \times \dfrac{9}{4} + C = 14$

$C = 5$

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c $x^3 - 9x^2 + 27x - 22 = A(x + B)^3 + C$

$(x - 3)^3 + 5 = A(x + B)^3 + C$

$A = 1$

$B = -3$

$C = 5$

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Solutions to Exercise 4D

1 a

$$\begin{array}{r l}
 & x^2 - 5x + 6 \\
x+4 & \overline{)\,x^3 - x^2 - 14x + 24} \\
 & \underline{x^3 + 4x^2} \\
 & \quad -5x^2 - 14x \\
 & \quad \underline{-5x^2 - 20x} \\
 & \qquad\qquad 6x + 24 \\
 & \qquad\qquad \underline{6x + 24} \\
 & \qquad\qquad\qquad 0
\end{array}$$

b

$$\begin{array}{r l}
 & 2x^2 + 7x - 4 \\
x-3 & \overline{)\,2x^3 + x^2 - 25x + 12} \\
 & \underline{2x^3 - 6x^2} \\
 & \quad 7x^2 - 25x \\
 & \quad \underline{7x^2 - 21x} \\
 & \qquad\qquad -4x + 12 \\
 & \qquad\qquad \underline{-4x + 12} \\
 & \qquad\qquad\qquad 0
\end{array}$$

2 a

$$\begin{array}{r l}
 & x^2 - 4x - 3 + \dfrac{34}{x+3} \\
x+3 & \overline{)\,x^3 - x^2 - 15x + 25} \\
 & \underline{x^3 + 3x^2} \\
 & \quad -4x^2 - 15x \\
 & \quad \underline{-4x^2 - 12} \\
 & \qquad\qquad -3x + 25 \\
 & \qquad\qquad \underline{-3x - 9} \\
 & \qquad\qquad\qquad 34
\end{array}$$

b

$$\begin{array}{r l}
 & 2x^2 + 6x + 14 + \dfrac{54}{x-3} \\
x-3 & \overline{)\,2x^3 + 0x^2 - 4x + 12} \\
 & \underline{2x^3 - 6x^2} \\
 & \quad 6x^2 - 4x \\
 & \quad \underline{6x^2 - 18x} \\
 & \qquad\qquad 14x + 12 \\
 & \qquad\qquad \underline{14x - 42} \\
 & \qquad\qquad\qquad 54
\end{array}$$

3 a

$$\begin{array}{r l}
 & x^2 - \dfrac{5}{2}x - \dfrac{15}{4} + \dfrac{145}{4(2x+3)} \\
2x+3 & \overline{)\,2x^3 - 2x^2 - 15x + 25} \\
 & \underline{2x^3 + 3x^2} \\
 & \quad -5x^2 - 15x \\
 & \quad \underline{-5x^2 - \dfrac{15}{2}x} \\
 & \qquad\qquad -\dfrac{15}{2}x + 25 \\
 & \qquad\qquad \underline{-\dfrac{15}{2}x - \dfrac{45}{4}} \\
 & \qquad\qquad\qquad \dfrac{145}{4}
\end{array}$$

b

$$\begin{array}{r l}
 & 2x^2 + 6x + 7 + \dfrac{33}{2x-3} \\
2x-3 & \overline{)\,4x^3 + 6x^2 - 4x + 12} \\
 & \underline{4x^3 - 6x^2} \\
 & \quad 12x^2 - 4x \\
 & \quad \underline{12x^2 - 18x} \\
 & \qquad\qquad 14x + 12 \\
 & \qquad\qquad \underline{14x - 21} \\
 & \qquad\qquad\qquad 33
\end{array}$$

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4 a

$$\begin{array}{r} 2x^2 - x + 12 \\ x-3\,\overline{)\,2x^3 - 7x^2 + 15x - 3} \\ \underline{2x^3 - 6x^2 \qquad\qquad} \\ -x^2 + 15x \qquad \\ \underline{-x^2 + 3x \qquad} \\ 12x - 3 \\ \underline{12x - 36} \\ 33 \end{array}$$

$$\frac{2x^3 - 7x^2 + 15x - 3}{x-3} = 2x^2 - x + 12 + \frac{33}{x-3}$$

b

$$\begin{array}{r} 5x^4 + 8x^3 - 8x^2 + 6x - 6 \\ x+1\,\overline{)\,5x^5 + 13x^4 - 2x^2 - 6} \\ \underline{5x^5 + 5x^4 \qquad\qquad\qquad} \\ 8x^4 + 0x^3 \qquad\qquad \\ \underline{8x^4 + 8x^3 \qquad\qquad} \\ -8x^3 - 2x^2 \qquad \\ \underline{-8x^3 - 8x^2 \qquad} \\ 6x^2 + 0x \\ \underline{6x^2 + 6x} \\ -6x - 6 \end{array}$$

$$\frac{5x^5 + 13x^4 - 2x^2 - 6}{x+1} = 5x^4 + 8x^3 - 8x^2 + 6x - 6$$

5 a

$$\begin{array}{r} x^2 - 9x + 27 \\ x^2-2\,\overline{)\,x^4 - 9x^3 + 25x^2 - 8x - 2} \\ \underline{x^4 - 2x^2 \qquad\qquad\qquad} \\ -9x^3 - 8x \qquad \\ \underline{-9x^3 + 18x \qquad} \\ 27x^2 - 2 \\ \underline{27x^2 - 54} \\ -26x + 52 \end{array}$$

$$\frac{x^4 - 9x^3 + 25x^2 - 8x - 2}{x^2 - 2} = x^2 - 9x + 27 - 26\left(\frac{x-2}{x^2-2}\right)$$

b

c

$$\begin{array}{r} x^2 + x + 2 \\ x^2-1\,\overline{)\,x^4 + x^3 + x^2 - x - 2} \\ \underline{x^4 + 0x^3 - x^2 \qquad\qquad} \\ x^3 + 2x^2 - x \qquad \\ \underline{x^3 + 0x^2 - x \qquad} \\ 2x^2 + 0x - 2 \\ \underline{2x^2 - 2} \\ 0 \end{array}$$

6 a remainder $= P(-2)$
$= (-2)^3 + 3(-2) - 2 = -16$

b $P(x) = (1 - 2a)x^2 + 5ax$
$\quad + (a-1)(a-8)$

$P(2) = 0$

$P(1) \neq 0$

$P(2) = 4 - 8a + 10a + a^2 - 9a + 8$
$= a^2 - 7a + 12$

$(a-3)(a-4) = 0$

$a = 3, 4$

$P(1) = 1 - 2a + 5a + a^2 - 9a + 8$
$= a^2 - 6a + 9$
$= (a-3)^2$

$P(1) \neq 0, \quad \therefore a \neq 3, \quad \therefore a = 4$

7 a $f(x) = 6x^3 + 5x^2 - 17x - 6$

$f(2) = 6 \times 8 + 5 \times 4 - 17 \times 2 - 6$
$= 48 + 20 - 34 - 6$
$= 28$

b $f(-2) = (6 \times -8) + (5 \times 4)$
$\quad - (17 \times -2) - 6$
$= -48 + 20 + 34 - 6$
$= 0$

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c $f(x) = (x + 2)(6x^2 - 7x - 3)$

$= (x + 2)(3x + 1)(2x - 3)$

8 a $P(-1) = -1 + (k - 1) - (k - 9) - 7$

$= -1 + k - 1 - k + 9 - 7$

$= 0$

$\therefore$ for any value of k, $P(x)$ is divisible by $x + 1$

b $P(2) = 8 + 4(k - 1) + 2(k - 9) - 7$

$P(2) = 12$

$1 + 4k - 4 + 2k - 18 = 12$

$6k - 21 = 12$

$6k = 33$

$k = \dfrac{11}{2}$

9 $f(x) = 2x^3 + ax^2 - bx + 3$

a $f(-3) = 0 = -54 + 9a + 3b + 3$

$9a + 3b = 51$

$3a + b = 17 \ldots (1)$

$f(2) = 15 = 16 + 4a - 2b + 3$

$4a - 2b = -4$

$2a - b = -2 \ldots (2)$

$(1) + (2)$

$\Rightarrow 5a = 15$

$a = 3$

Sub in (1) $\Rightarrow$ $b = 8$

b $f(x) = (x + 3)(2x^2 - 3x + 1)$

$= (x + 3)(2x - 1)(x - 1)$

$\therefore$ the other two factors are $(2x - 1)$ & $(x - 1)$

10 a $f(x) = 4x^3 + ax^2 - 5x + b$

$f\left(\dfrac{3}{2}\right) = -8 = 4 \times \left(\dfrac{27}{8}\right)$

$+ a \times \left(\dfrac{9}{4}\right) - 5 \times \left(\dfrac{3}{2}\right) + b$

$-8 = \dfrac{27}{2} + \dfrac{9}{4}a - \dfrac{15}{2} + b$

$-32 = 54 + 9a - 30 + 4b$

$9a + 4b = -56 \ldots (1)$

$f(3) = 10 = 4 \times 27 + a \times 9 - 5 \times 3 + b$

$10 = 108 + 9a - 15 + b$

2 $\quad 9a + b = -83$

$1 - 2 \Rightarrow \quad 3b = 27$

$b = 9$

Sub in 2 $\Rightarrow$ $9a + 9 = -83$

$a = \dfrac{-92}{9}$

11 $P(2) = (3)^4$

$= 81$

12 $P(x) = x^5 - 3x^4 + 2x^3 - 2x^2 + 3x + 1$

a $P1 = -1 - 3 + 2 - 2 + 3 + 1$

$= 2 \neq 0$

$\therefore$ $(x - 1)$ is not a factor

$P(-1) = -1 - 3 - 2 - 2 - 3 + 1$

$= -10 \neq 0$

$\therefore$ $(x + 1)$ is not a factor

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b $P(x) = x^5 - 3x^4 + 2x^3 - 2x^2 + 3x + 1$

$$\begin{array}{r l}
 & x^3 - 3x^2 + 3x - 5 \\ \cline{2-2}
x^2 - 1 \big) & x^5 - 3x^4 + 2x^3 - 2x^2 + 3x + 1 \\
 & x^5 - x^3 \\ \cline{2-2}
 & -3x^4 + 3x^3 - 2x^2 \\
 & -3x^4 + 3x^2 \\ \cline{2-2}
 & 3x^3 - 5x^2 + 3x \\
 & 3x^3 - 3x \\ \cline{2-2}
 & -5x^2 + 6x + 1 \\
 & -5x^2 + 5 \\ \cline{2-2}
 & 6x - 4
\end{array}$$

$P(x) = (x^3 - 3x^2 + 3x - 5)(x^2 - 1) + 6x - 4$

$\therefore$ the remainder when $(x^3 - 3x^2 + 3x - 5)$ is divided by $(x^2 - 1)$ is $6x - 4$

13 $P(-1) = -2 - 5 + 4 + 3 =$

$\therefore (x + 1)$ is factor

$2x^3 - 5x^2 - 4x + 3 = (x + 1)(2x^2 - 7x + 3)$

$= (x + 1)(2x - 1)(x - 3)$

14 a

$$\begin{aligned}
P(x) &= x^4 + x^3 - x^2 - 3x - 6 \\
P(\sqrt{3}) &= 9 + 3\sqrt{3} - 3 - 3\sqrt{3} - 6 \\
&= 0 \\
P(\sqrt{-3}) &= 9 - 3\sqrt{3} - 3 + 3\sqrt{3} - 6 \\
&= 0
\end{aligned}$$

b the quadratic factor is

$(x + \sqrt{3})(x - \sqrt{3})$

$= (x^2 - 3)$

$\therefore\ P(x) = (x^2 - 3)(x^2 + x + 2)$

$\therefore$ an other factor is $(x^2 + x + 2)$

15 a $(2a + 3b)(4a^2 - 6ab + 9b^2$

b $(4 - a)(a^2 + 4a + 16)$

c $(5x + 4y)(25x^2 - 20xy + 16y^2)$

d $2a(a^2 + 3b^2)$

16 a $(2x - 1)(2x + 3)(3x + 2)$

b $(2x - 1)(2x^2 + 3)$

17 a $(2x - 3)(2x^2 + 3x + 6)$

b $(2x - 3)(2x - 1)(2x + 1)$

18 a $x = -4, 2, 3$

b $x = 0, 2$

c $x = \dfrac{1}{2}, 2$

d $x = -2, 2$

e $x = 0, -2, 2$

f $x = 0, -3, 3$

g $x = 1, -2, \dfrac{-1}{4}, \dfrac{1}{3}$

h $x = 1, -2$

i $x = 1, -2, \dfrac{1}{3}, \dfrac{3}{2}$

19 Use a CAS calculator to solve $y = 0$ in each case to obtain the x-exis intercepts.

a $(-1, 0), (0, 0), (2, 0)$

b $(-2, 0), (0, 6), (1, 0), (3, 0)$

c $(-1, 0), (0, 6), (2, 0), (3, 0)$

d $\left(\dfrac{-1}{2}, 0\right), (0, 2), (1, 0), (2, 0)$

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e $(-2, 0), (-1, 0), (0, -2), (1, 0)$

f $(-1, 0), \left(\frac{-2}{3}, 0\right), (0, -6), (3, 0)$

g $(-4, 0), (0, -16), \left(-\frac{2}{5}, 0\right), (2, 0)$

h $\left(\frac{-1}{2}, 0\right), (0, 1), \left(\frac{1}{3}, 0\right), (1, 0)$

i $(-2, 0), \left(-\frac{-3}{2}, 0\right), (0, -30), (5, 0)$

20 $16p - 10 + q = 0 \ldots (1)$

$16 - 16 - 4p - 2q - 8 = 0 \ldots (2)$

$\Rightarrow \quad 2p + q + 4 = 0$

$(1) - (2) \Rightarrow \quad 14p - 14 = 0$

$p = 1$

Sub in 1 $\Rightarrow \quad 16 - 10 + q = 0$

$q = -6$

21 $f(x) = x^4 - x^3 + 5x^2 + 4x - 36$

$f(-1) = 1 + 1 + 5 - 4 - 36$

$= -33$

22 a $(x - 9)(x - 13)(x + 11)$

b $(x + 11)(x - 9)(x - 11)$

c $(x + 11)(2x - 9)(x - 11)$

d $(x + 11)(2x - 13)(2x - 9)$

23 a $(x - 1)(x + 1)(x - 7)(x + 6)$

b $(x - 3)(x + 4)(x^2 + 3x + 9)$

24 a $(x - 9)(x - 5)(2x^2 + 3x + 9)$

b $(x + 5)(x + 9)(x^2 - x + 9)$

c $(x - 3)(x + 5)(x^2 + x + 9)$

d $(x - 4)(x - 3)(x + 5)(x + 6)$

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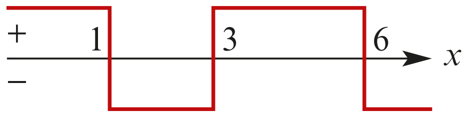
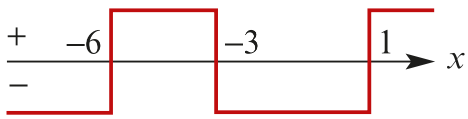
Solutions to Exercise 4E

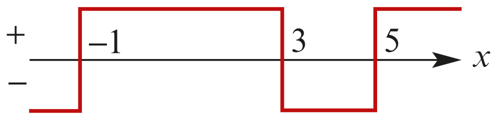
1 a



b

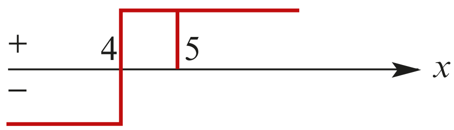


c

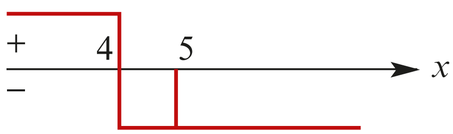


d

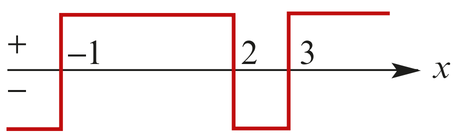
e



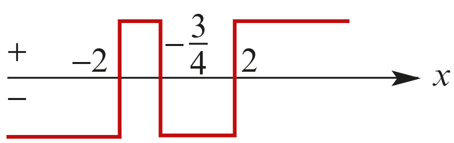
f



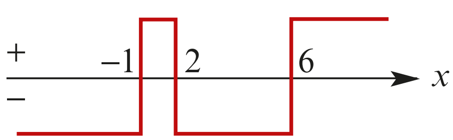
2 a



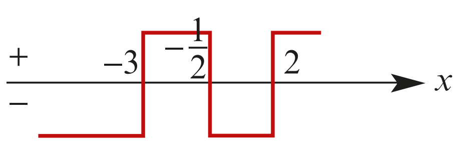
b



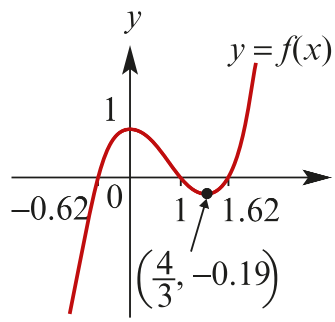
c



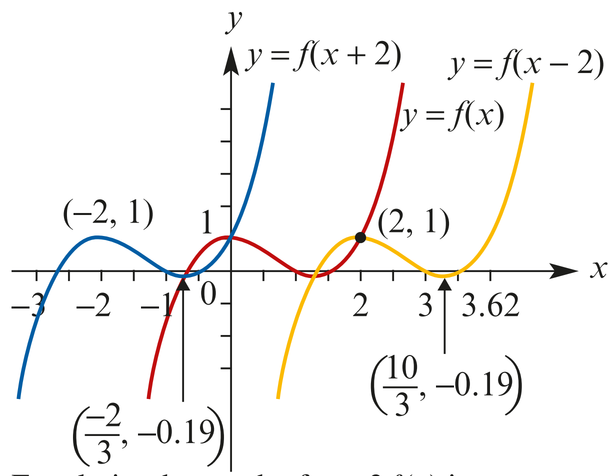
d



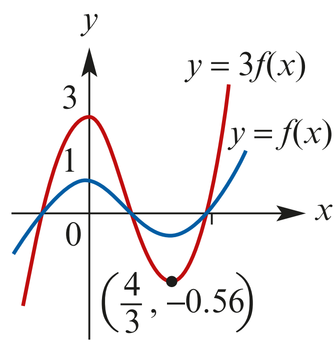
3 a



b



For clarity the graph of $y = 3f(x)$ is shown on separate axes:



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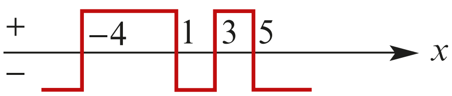
Solutions to Exercise 4F

1 a



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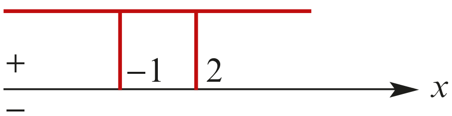
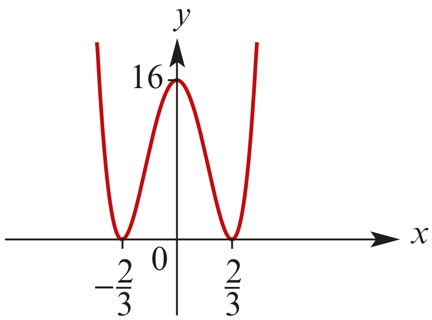


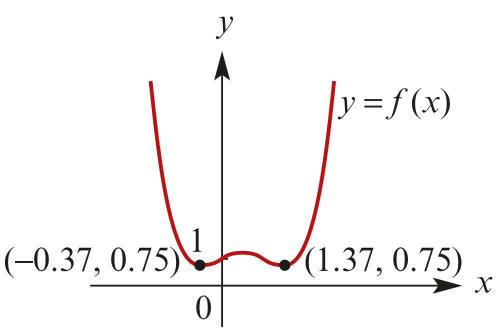
2 $(0, 16)$

$(\frac{2}{3}, 0)$

$(-\frac{2}{3}, 0)$



3 a



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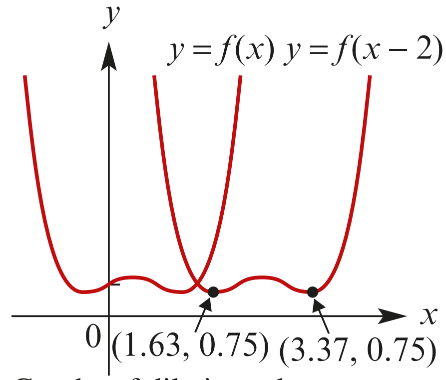
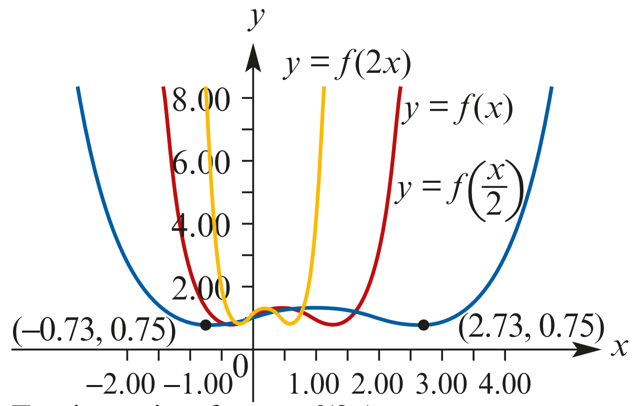
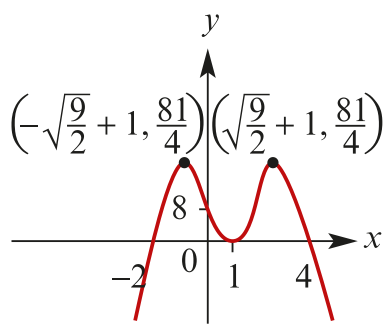


Graphs of dilations shown on separate axes for clarity:



Turning points for $y = f(2x)$ are at $(-0.18, 0.75)$ and $(0.68, 0.75)$

4 a



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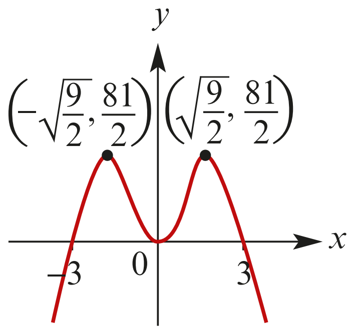
b



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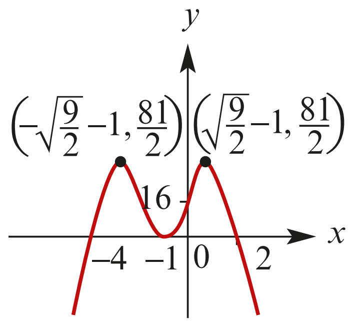

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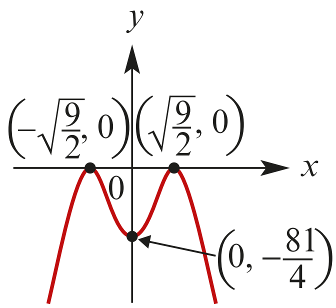

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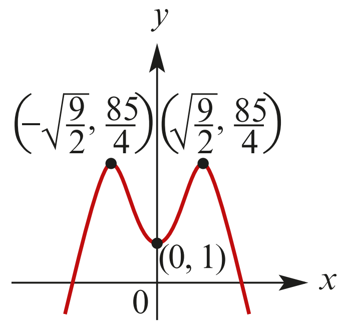

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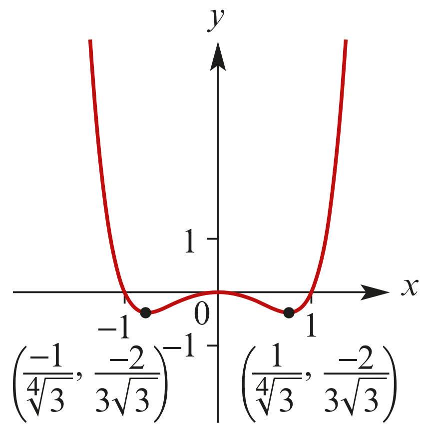
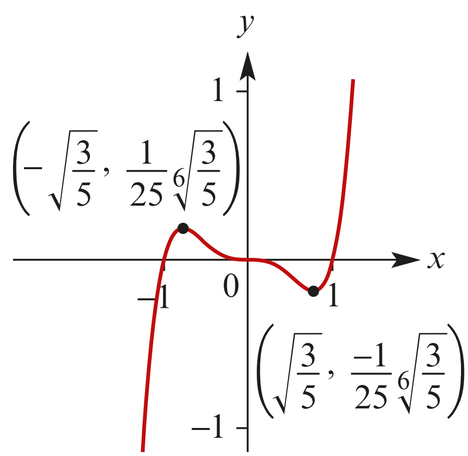


6



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Solutions to Exercise 4G

1 **a** $y = a(x - 5)^3 - 2$

When $x = 4, y = 0$

$0 = -a - 2$

$a = -2$

b $y = a(x - 1)(x + 1(x + 2)$

When $x = 3, y = 120$

$120 = a(2)(4)(5)$

$a = 3$

c $y = ax^3 + bx$

$(2, -20)$ and $(-1, 20)$ lie on the graph

$$-20 = 8a + 2b$$

$$-10 = 4a + b \ldots (1)$$

$$20 = -a - b \ldots (2)$$

Add (1)(2)

$$10 = 3a$$

$$a = \frac{10}{3}$$

$$b = -\frac{70}{3}$$

2 We know that the y-intercept is 5.

Consider $f(x) = ax^3 + bx^2 + cx + 5$

$f(-1) = 14, \therefore -a + b - c + 5 = 14$

$-a + b - c = 9 \ldots (1)$

$f(1) = 0, \therefore a + b + c + 5 = 0$

$a + b + c = -5 \ldots (2)$

$f(2) = -19, \therefore 8a + 4b + c + 5 = -19$

$8a + 4b + 2c = -24$

Add (1) and (2)

$$2b = 4$$

$$b = 2$$

$$\therefore a + c = -7$$

$$\text{and } 4a + c = -16$$

$$\therefore 3a = -9$$

$$\therefore a = -3 \text{ and } c = -4$$

3 Note: A CAS calculator can be used for all questions in this exercise, but should be used for questions 5 and 6.

$$y = a(x - b)(x - c)(x - d)$$

$$b = -5, c = -2, d = 6$$

$$y = a(x + 5)(x + 2)(x - 6)$$

$$x = 0, y = -11$$

$$-11 = -60a$$

$$a = \frac{11}{60}$$

$$y = \frac{11}{60}(x + 5)(x + 2)(x - 6)$$

4 $y = a(x - b)(x - c)^2$

$y = a(x + 1)(x - 3)^2$

$x = 0, y = 5$

$5 = 9a$

$a = \frac{5}{9}$

$y = \frac{5}{9}(x + 1)(x - 3)^2$

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5 a $y = ax^3 + bx^2 + cx + d$

$(0, 1) \Rightarrow d = 1$

$y = ax^3 + bx^2 + cx + 1$

$(1, 3) \Rightarrow 3 = a + b + c + 1$

$a + b + c = 2 \ldots (1)$

$(-1, -1) \Rightarrow -1 = -a + b - c + 1$

$-a + b - c = 2 \quad \ldots (2)$

$(1) + (2) \Rightarrow 2b = 0$

$b = 0$

$(2, 11) \Rightarrow 11 = 8a + 2c + 1$

$4a + c = 5 \ldots (3)$

$(3) + (2) \Rightarrow 3a = 3$

$a = 1, c = 1$

$y = x^3 + x + 1$

b $y = ax^3 + bx^2 + cx + d$

$(0, 1) = d = 1$

$(1, 1) = 1 = a + b + c + 1$

$a + b + c = 0 \ldots (1)$

$(-1, 1) = 1 = -a + b - c + 1$

$-a + b - c = 0 \ldots (2)$

$(1) + (2) \Rightarrow b = 0$

$(2, 7) \Rightarrow 7 = 8a + 2c + 1$

$4a + c = 3 \ldots (3)$

$3 + 2 \Rightarrow 3a = 3$

$a = 1$

Sub in $\Rightarrow (1) \Rightarrow c = -1$

$y = x^3 - x + 1$

c $y = ax^3 + bx^2 + cx + d$

$(0, -2) \Rightarrow d = -2$

$(1, 0) \Rightarrow 0 = a + b + c - 2$

$a + b + c = 2 \quad \ldots (1)$

$(-1, -6) \Rightarrow -6 = -a + b - c - 2$

$-a + b - c = -4 \ldots (2)$

$(2, 12) \Rightarrow 12 = 8a + 4b + 2c - 2$

$4a + 2b + c = 7 \ldots (3)$

$(1) + (2) : 2b = -2$

$b = -1$

Sub in $\Rightarrow (3) \Rightarrow 4a + c = 9$

Sub in $\Rightarrow (2) \Rightarrow -a - c = -3$

$(3) + (2) \Rightarrow 3a = 6$

$a = 2$

Sub in $\Rightarrow (3) \Rightarrow c = 1$

$y = 2x^3 - x^2 + x - 2$

6 a $y = a(x - b)(x - c)(x - d)$

$y = a(2x + 1)(x - 1)(x - 2)$

$2 = 2a$

$a = 1$

$y = (2x + 1)(x - 1)(x - 2)$

b $y = ax^3 + bx^2 + cx$

$(1, 0.75) \Rightarrow \frac{3}{4} = a + b + c \ldots (1)$

$(2, 3) \Rightarrow 3 = 8a + 4b + 2c \ldots (2)$

$(-2, -3) \Rightarrow -3 = -8a + 4b - 2c \ldots (3)$

$(2) + (3) \Rightarrow 8b = 0$

$b = 0$

$(2) - 2(1) \Rightarrow \frac{6}{4} = 6a$

$a = \frac{1}{4}$

Sub in $(2) \Rightarrow 3 = 2 + 2c$

$c = \frac{1}{2}$

$y = \frac{1}{4}x^3 + \frac{1}{2}x = \frac{1}{4}x(x^2 + 2)$

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c $y = a(x - b)(x - c)^2$

$y = a(x + 1)x^2$

$2 = a(2)1^2$

$a = 1$

$y = x^2(x + 1)$

d $y = a(x - b)(x - c)(x - d)$

$y = a(x + 2)(x + 1)(x - 1)$

$x = 0, y = -2$

$y = (x + 2)(x + 1)(x - 1)$

e $y = a(x - b)(x - c)^2$

$y = a(x + 2)(x - 3)^2$

$x = 0, y = 18$

$y = (x + 2)(x - 3)^2$

7 **a** $y = -2x^3 - 25x^2 + 48x + 135$

b $y = 2x^3 - 30x^2 + 40x + 13$

8 **a** $y = -2x^4 + 22x^3 - 10x^2 - 37x + 40$

b $y = x^4 - x^3 + x^2 + 2x + 8$

c $y = \frac{31}{36}x^4 + \frac{5}{4}x^3 - \frac{157}{36}x^2 - \frac{5}{4}x + \frac{11}{2}$

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Solutions to Exercise 4H

1 a $kx^2 + x + k = 0$

$$x = \frac{-1 \pm \sqrt{1 - 4k^2}}{2k},$$

$k \in \left[, -\frac{1}{2}, \frac{1}{2}\right] \backslash \{0\}$ since $k^2 \le \frac{1}{4}$.

(Note: If $k = 0, x = 0$)

b $x^3 - 7ax^2 + 12a^2x = 0$

$\Rightarrow x(x^2 - 7ax + 12a^2) = 0$

$x(x - 3a)(x - 4a) = 0$

$x = 0, 3a, 4a$

c $x(x^3 - a) = 0$

$x = 0, (a)^{\frac{1}{3}}$

d $x^2 - kx + k = 0$

$x = \dfrac{k \pm \sqrt{k^2 - 4k}}{2}, k \le 0$ a $k \ge, 4,$

since $k^2 - 4k \ge 0$

e $x(x^2 - a) = 0$

$x = 0, \pm\sqrt{a}, a \ge 0$

f $x^4 - a^4 = 0$

$(x^2 + a^2)(x^2 - a^2) = 0$

$(x^2 + a^2)(x - a)(x + a) = 0$

$x = -a, a$

g $(x - a)^2(x - b) = 0$

$x = a, b$

h $(x - a)^4(a - x^3)(x^2 - a) = 0$

$x = a, (a)^{\frac{1}{3}}, \pm\sqrt{a}$ *if* $a \ge 0$

2 a $ax^3 + b = 2c$

$$x^3 = \frac{2c - b}{a}$$

$$x = \left(\frac{2c - b}{a}\right)^{\frac{1}{3}}$$

b $ax^3 - b = c$

$$x^3 = \frac{b + c}{a}$$

$$x = \left(\frac{b + c}{a}\right)^{\frac{1}{3}}$$

c $a - bx^2 = c$

$$x^2 = \frac{a - c}{b}$$

$$x = \left(\frac{a - c}{b}\right)^{\frac{1}{2}}$$

d $x^{\frac{1}{3}} = a$

$x = a^3$

e $(x)^{\frac{1}{n}} + c = a$

$(x)^{\frac{1}{n}} = a - c$

$x = (a - c)^n$

f $a(x - 2b)^3 = c$

$$(x - 2b)^3 = \frac{c}{a}$$

$$x - 2b = \left(\frac{c}{a}\right)^{\frac{1}{3}}$$

$$x = 2b + \left(\frac{c}{a}\right)^{\frac{1}{3}}$$

g $ax^{\frac{1}{3}} = b$

$$x = \left(\frac{b}{a}\right)^3$$

h $x^3 = c + d$

$x = (c + d)^{\frac{1}{3}}$

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3 a $x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, 1$

$y = 0, 1$

$Pts.\ (0, 0)\ \&\ (1, 1)$

b $2x^2 = x$

$2x^2 - x = 0$

$x(2x-1) = 0$

$x = 0, \frac{1}{2}$

$Pts.\ (0, 0) \quad \left(\frac{1}{2}, \frac{1}{2}\right)$

c $y = x^2 - x,$

$y = 2x + 1$

$\Rightarrow x^2 - x = 2x + 1$

$x^2 - 3x - 1 = 0$

$$x = \frac{3 \pm \sqrt{9+4}}{2}$$

$$x = \frac{3 \pm \sqrt{13}}{2}$$

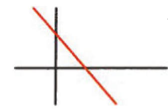
$y = 2x + 1 = 4 \pm$ V13

$$co - ords = \left(\frac{3 - \sqrt{13}}{2}, 4 - \sqrt{13}\right),$$

$$\left(\frac{3 + \sqrt{13}}{2}, 4 + \sqrt{13}\right)$$

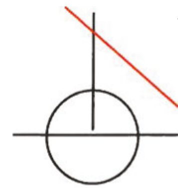
4 a $y = 16 - x$

$y^2 = (16 - x)^2$



$y^2 + x^2 = 178$

$y^2 = 178 - x^2$



$178 - x^2 = (16 - x)^2$

$178 - x^2 = 256 - 32x + x^2$

$2x^2 - 32x + 78 = 0$

$x^2 - 16x + 39 = 0$

$(x-3)(x-13) = 0$

$x = 3, 13$

$Pts.\ (3, 13), (13, 3)$

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b

$$\begin{aligned} y^2 &= 125 - x^2 \\ y &= 15 - x \\ \Rightarrow \quad y^2 &= 225 - 30x + x^2 \\ x^2 - 30x + 225 &= 125 - x^2 \\ 2x^2 - 30x + 100 &= 0 \\ x^2 - 15x + 50 &= 0 \\ (x-5)(x-10) &= 0 \\ x &= 5, 10 \end{aligned}$$

$Pts.$ (5, 10), (10, 5)

c

$$\begin{aligned} y^2 &= 185 - x^2 \\ y &= x - 3 \\ y^2 &= x^2 - 6x + 9 \\ x^2 - 6x + 9 &= 185 - x^2 \\ 2x^2 - 6x - 176 &= 0 \\ x^2 - 3x - 88 &= 0 \\ (x+8)(x-11) &= 0 \\ x &= -8, 11 \end{aligned}$$

$Pts.$ (−8, −11), (11, 8)

d

$$\begin{aligned} y^2 &= 97 - x^2 \\ y &= 13 - x \\ y^2 &= 169 - 26x + x^2 \\ x^2 - 26x + 169 &= 97 - x^2 \\ 2x^2 - 26 + 72 &= 0 \\ x^2 - 13x + 36 &= 0 \\ (x-9)(x-4) &= 0 \\ x &= 4, 9 \end{aligned}$$

$Pts.$ (4, 9), (9, 4)

e

$$\begin{aligned} y^2 &= 106 - x^2 \\ y &= x - 4 \\ y^2 &= x^2 - 8x + 16 \\ x^2 - 8x + 16 &= 106 - x^2 \\ 2x^2 - 8x - 90 &= 0 \\ x^2 - 4x - 45 &= 0 \\ (x+5)(x-9) &= 0 \\ x &= -5, 9 \end{aligned}$$

$Pts.$ (−5, −9), (9, 5)

5 a

$$\begin{aligned} y &= 28 - x \ldots (1) \\ xy &= 187 \ldots (2) \\ \Rightarrow \quad x(28 - x) &= 187 \\ -x^2 + 28x &= 187 \\ x^2 - 28x + 187 &= 0 \\ x &= \frac{28 \pm \sqrt{784 - 748}}{2} \\ x &= \frac{28 \pm 6}{2} \\ x &= 11, 17 \\ \Rightarrow \quad pts &= (11, 17), (17, 11) \end{aligned}$$

b

$$\begin{aligned} y &= 51 - x \\ x(51 - x) &= 518 \\ x^2 - 51x + 518 &= 0 \\ x &= \frac{51 \pm \sqrt{2601 - 2072}}{2} \\ x &= \frac{51 \pm \sqrt{529}}{2} \\ x &= \frac{51 \pm 23}{2} \\ x &= 14, 37 \end{aligned}$$

$\Rightarrow pts = (14, 37), (37, 14)$

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c

$$\begin{aligned}
y &= x - 5\\
xy &= 126\\
x^2 - 5x &= 126\\
x^2 - 5x - 126 &= 0\\
x &= \frac{5 \pm \sqrt{25 - 504}}{2}\\
x &= \frac{5 \pm 23}{2}\\
x &= -9, 14\\
\Rightarrow \quad pts &= (-9, -14), (14, 9)
\end{aligned}$$

6

$$\begin{aligned}
y^2 &= 25 - (x - 5)^2\\
&= 25 - x^2 + 10x - 25\\
y^2 &= -x^2 + 10x \ldots (1)\\
y &= 2x\\
y^2 &= 4x^2 \ldots (2)\\
4x^2 &= -x^2 + 10x\\
x^2 - 2x &= 0\\
x &= 0, 2\\
pts &= (0, 0), (2, 4)
\end{aligned}$$

7 $x = \dfrac{1}{x - 2} + 3$

$$\begin{aligned}
x(x - 2) &= 1 + 3(x - 2)\\
x^2 - 2x &= 1 + 3x - 6\\
x^2 - 5x + 5 &= 1\\
x &= \frac{5 \pm \sqrt{25 - 20}}{2}\\
x &= \frac{5 \pm \sqrt{5}}{2}\\
pts &= \left(\frac{5 + \sqrt{5}}{2}, \frac{5 + \sqrt{5}}{2}\right),\\
&\left(\frac{5 - \sqrt{5}}{2}, \frac{5 - \sqrt{5}}{2}\right)
\end{aligned}$$

8 a

$$\begin{aligned}
\frac{y}{4} - \frac{x}{5} &= 1 \ldots (1)\\
\Rightarrow \quad y &= \frac{4}{5}x + 4\\
x^2 + 4x + y^2 &= 12 \ldots (2)\\
\Rightarrow \quad y^2 &= 12 - 4x - x^2\\
\Rightarrow \quad y^2 &= \frac{16}{25}x^2 + \frac{32}{5}x + 16\\
16\left(\frac{1}{25}x^2 + \frac{2}{5}x + 1\right) &= 12 - 4x - x^2\\
16x^2 + 160x + 400 &= 300 - 100x - 25x^2\\
41x^2 + 260x + 100 &= 0\\
x &= \frac{-260 \pm \sqrt{67600 - 16400}}{82}\\
x &= \frac{-130 \pm 80\sqrt{2}}{41}
\end{aligned}$$

Sub in (1)

$$\left(\frac{-130 - 80\sqrt{2}}{41}, \frac{60 - 64\sqrt{2}}{41}\right),$$

$$\left(\frac{-130 + 80\sqrt{2}}{41}, \frac{60 + 64\sqrt{2}}{41}\right)$$

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9

$$-x = \frac{1}{x+2} - 3$$
$$-x^2 - 2x = 1 - 3x - 6$$
$$x^2 - x - 5 = 0$$
$$x = \frac{1 \pm \sqrt{1+20}}{2}$$
$$x = \frac{1 \pm \sqrt{21}}{2}$$
$$pts = \left(\frac{1+\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}\right),$$
$$\left(\frac{1-\sqrt{21}}{2}, \frac{\sqrt{21}-1}{2}\right)$$

10

$$y = \frac{9}{4}x + 1$$
$$y^2 = \frac{81}{16}x^2 + \frac{9}{2}x + 1$$
$$9x = \frac{81}{16}x^2 + \frac{9}{2}x + 1$$
$$\frac{81}{16}x^2 - \frac{9}{2}x + 1 = 0$$
$$x = \frac{\frac{9}{2} \pm \sqrt{\frac{81}{4} - \frac{81}{4}}}{\frac{81}{8}}$$
$$x = \frac{9}{2} \times \frac{8}{81}$$
$$x = \frac{4}{9}$$
$$\text{co ords} = \left(\frac{4}{9}, 2\right)$$

11

$$y^2 = 9 - x^2$$
$$y = 2x + 3\sqrt{5}$$
$$y^2 = 4x^2 + 12\sqrt{5x} + 45$$
$$9 - x^2 = 4x^2 + 12\sqrt{5x} + 45$$
$$5x^2 + 12\sqrt{5x} + 36 = 0$$
$$x = \frac{-12\sqrt{5} \pm \sqrt{144 \times 5 - 20 \times 36}}{10}$$
$$x = \frac{-12\sqrt{5}}{10}$$
$$x = \frac{-6\sqrt{5}}{5}$$
$$y = \frac{-12\sqrt{5}}{10} + 3\sqrt{5}$$
$$y = \frac{3\sqrt{5}}{5}$$
$$pts = \left(\frac{-6\sqrt{5}}{5}, \frac{3\sqrt{5}}{5}\right)$$

12

$$\frac{1}{4}x + 1 = -\frac{1}{x}$$
$$\frac{1}{4}x^2 + x + 1 = 0$$
$$x = \frac{-1 \pm \sqrt{1-1}}{\frac{1}{2}}$$
$$x = -2$$
$$pt = \left(-2, \frac{1}{2}\right)$$

13

$$x - 1 = \frac{2}{x-2}$$
$$(x-1)(x-2) = 2$$
$$x^2 - 3x + 2 = 2$$
$$x(x-3) = 0$$
$$x = 0, 3$$
$$pts = (0, -1), (3, 2)$$

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14 **a**

$$5x - 4y = 7$$
$$4y = 5x - 7$$
$$y = \frac{5x-7}{4}$$
$$xy = 6$$
$$x\left(\frac{5x-7}{4}\right) = 6$$
$$5x^2 - 7x - 24 = 0$$
$$x = \frac{7 \pm \sqrt{49+480}}{10}$$
$$x = \frac{7 \pm \sqrt{529}}{10}$$
$$x = \frac{-8}{5}, 3$$
$$pts = \left(\frac{-8}{5}, \frac{-15}{4}\right), (3, 2)$$

b

$$y = \frac{37-2x}{3}$$
$$xy = 45$$
$$37x - 2x^2 = 135$$
$$2x^2 - 37x + 135 = 0$$
$$x = \frac{37 \pm \sqrt{1369-1080}}{4}$$
$$x = 5, 13.5$$
$$pts = (5, 9), \left(13.5, \frac{10}{3}\right)$$

c

$$5x - 3y = 18$$
$$y = \frac{5x-18}{3}$$
$$xy = 24$$
$$5x^2 - 18x = 72$$
$$x = \frac{18 \pm \sqrt{324+1440}}{10}$$
$$x = \frac{18 \pm 42}{10}$$
$$x = -\frac{12}{5}, 6$$
$$pts = \left(-\frac{12}{5}, -10\right), (6, 4)$$

15 $x^2 + ax + b$ *div* by $x + c$

$$(-c)^2 + a(-c) + b = 0$$
$$c^2 - ac + b = 0$$

16

$$x + 2 = \frac{160}{x}$$
$$x^2 + 2x - 160 = 0$$
$$x = \frac{-2 \pm \sqrt{4+640}}{2}$$
$$x = -1 \pm \sqrt{161}$$
$$pts = \left(-1 - \sqrt{161}, 1 - \sqrt{161}\right),$$
$$\left(\sqrt{161} - 1, \sqrt{161} + 1\right)$$

17 $y = -7x + 14, \; y = 5x + 12$

18 $m < -7$ or $m > 1$

19 $c = -8$ or $c = 4$

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20 **a** $mx = \frac{1}{x} + 5$

$mx^2 - 5x - 1 = 0$

$x = \dfrac{5 \pm \sqrt{25 + 4m}}{2m}, m \pm 0$

Note that if $m = 0, x = -\frac{1}{5}$.

b $25 + 4m = 0$

$m = \dfrac{-25}{4}$

$x = \dfrac{5}{\frac{-25}{2}}$

$x = \dfrac{-2}{5}$

$pt\left(\dfrac{-2}{5}, \dfrac{5}{2}\right)$

c $25 + m < 0$

$m < \dfrac{-25}{4}$

21 $y = 3x + 3, \; y = -x + 3$

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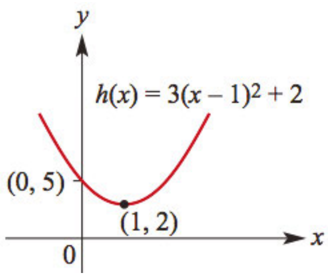
Solutions to technology-free questions

1 **a** $h(x) = 3(x-1)^2 + 2$

$x = 0$: $y = 3(-1)^2 + 2 = 5$

$y = 0$: no solutions

TP $(1, 2)$; no x int; y int $(0, 5)$



b $h(x) = (x-1)^2 - 9$

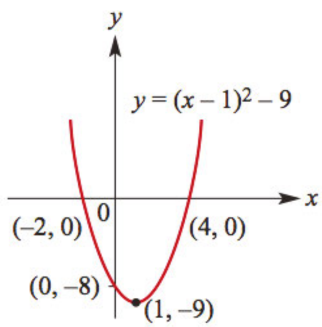
$x = 0$: $y = (-1)^2 - 9 = -8$

$y = 0$: $(x-1)^2 - 9 = 0$

$x - 1 = \pm 3$, so $x = -2, 4$

TP$(1, -9)$; x int $(-2, 0)$, $(4, 0)$;

y int $(0, -8)$



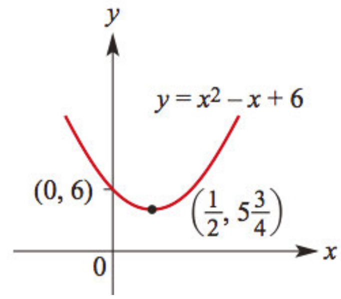
c $f(x) = x^2 - x + 6$

$x = 0$: $y = 6$

$y = 0$: no solutions $(b^2 - 4ac < 0)$

$x^2 - x + 6 = \left(x - \frac{1}{2}\right)^2 + 5\frac{3}{4}$

TP$\left(\frac{1}{2}, 5\frac{3}{4}\right)$; no x int; y int $(0, 6)$



d $f(x) = x^2 - x - 6$

$x = 0$: $y = -6$

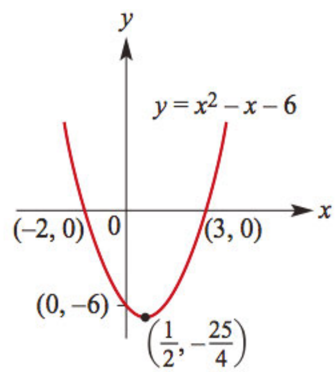
$y = 0$: $x^2 - x - 6 = 0$

$(x+2)(x-3) = 0$, so $x = -2, 3$

$x^2 - x - 6 = \left(x - \frac{1}{2}\right)^2 - 6\frac{1}{4}$

TP$\left(\frac{1}{2}, -6\frac{1}{4}\right)$; x int $(-2, 0)$, $(3, 0)$;

y int $(0, -6)$



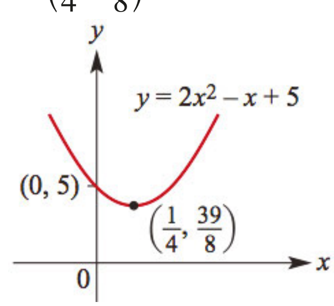
e $f(x) = 2x^2 - x + 5$

$x = 0$: $y = 5$

$y = 0$: no solutions $(b^2 - 4ac < 0)$

$2x^2 - x + 5 = 2\left(x - \frac{1}{4}\right)^2 + 4\frac{7}{8}$

TP$\left(\frac{1}{4}, 4\frac{7}{8}\right)$; no x int; y int $(0, 5)$



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f $h(x) = 2x^2 - x - 1$

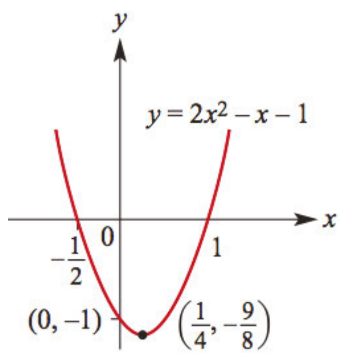
$x = 0$: $y = -1$

$y = 0$: $2x^2 - x - 1 = 0$

$(2x + 1)(x - 1) = 0$, so $x = -\dfrac{1}{2}, 1$

$2x^2 - x - 1 = 2\left(x - \dfrac{1}{4}\right)^2 - 1\dfrac{1}{8}$

TP$\left(\dfrac{1}{4}, -1\dfrac{1}{8}\right)$; x int $\left(-\dfrac{1}{2}, 0\right)$, $(1, 0)$;
y int $(0, -1)$



2 $(1, 1): 1 = a + b$ 1

$(2, 5): 5 = 4a + b$ 2

Subtract 2 from 1:

$3a = 4$

$a = \dfrac{4}{3}$

Substitute into 1: $b = -\dfrac{1}{3}$

3 $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{2 \pm \sqrt{4 - 4(3)(-10)}}{6}$$
$$= \frac{2 \pm \sqrt{124}}{6} = \frac{1}{3}\left(1 \pm \sqrt{31}\right)$$

4 a $f(x) = 2(x - 1)^3 - 16$

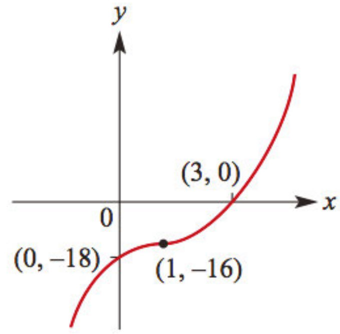
$x = 0$: $y = 2(-1)^3 - 16 = -18$

$y = 0$: $2(x - 1)^3 - 16 = 0$

$(x - 1)^3 = 8, x - 1 = 2$, so $x = 3$

zero gradient: $(1, -16)$

x int $(3, 0)$; y int $(0, -18)$



b $g(x) = -(x + 1)^3 + 8$

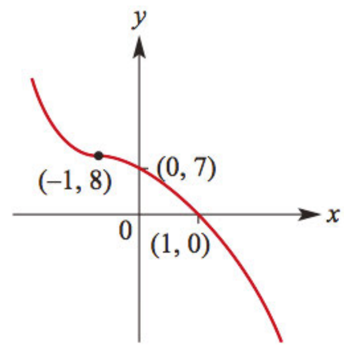
$x = 0$: $y = -(1)^3 + 8 = 7$

$y = 0$: $-(x + 1)^3 + 8 = 0$

$(x + 1)^3 = 8, x + 1 = 2$, so $x = 1$

zero gradient: $(-1, 8)$

x int $(1, 0)$; y int $(0, 7)$



c $h(x) = -(x + 2)^3 - 1$

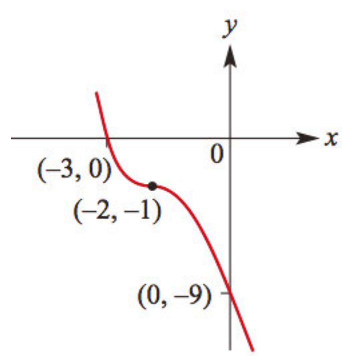
$x = 0$: $y = -(2)^3 - 1 = -9$

$y = 0$: $-(x + 2)^3 - 1 = 0$

$(x + 2)^3 = -1, x + 2 = -1$, so $x = -3$

zero gradient: $(-2, -1)$

x int $(-3, 0)$; y int $(0, -9)$



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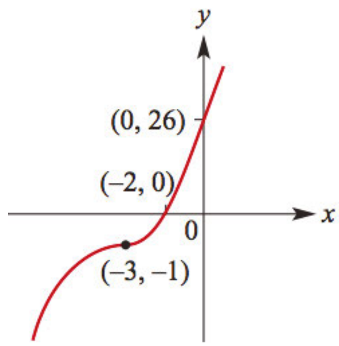
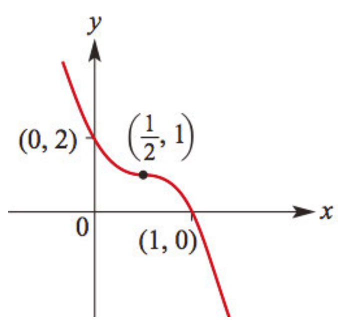
d $f(x) = (x+3)^3 - 1$
$x = 0$: $y = (3)^3 - 1 = 26$
$y = 0$: $(x+3)^3 - 1 = 0$
$(x+3)^3 = 1$, $x + 3 = 1$, so $x = -2$
zero gradient: $(-3, -1)$
x int $(-2, 0)$; y int $(0, 26)$



e $f(x) = 1 - (2x-1)^3$
$x = 0$: $y = 1 - (-1)^3 = 2$
$y = 0$: $1 - (2x-1)^3 = 0$
$(2x-1)^3 = 1$, $2x - 1 = 1$, so $x = 1$
zero gradient: $\left(\frac{1}{2}, 1\right)$
x int $(1, 0)$; y int $(0, 2)$



5 **a** $(x+2)^2 - 4$

b $3(x+1)^2 - 3$

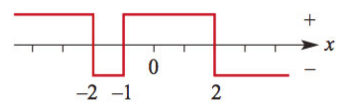
c $(x-2)^2 + 2$

d $2\left(x - \frac{3}{2}\right)^2 - \frac{17}{2}$

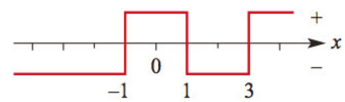
e $2\left(x - \frac{7}{4}\right)^2 - \frac{81}{8}$

f $-\left(x - \frac{3}{2}\right)^2 - \frac{7}{4}$

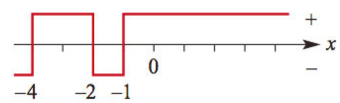
6 **a**



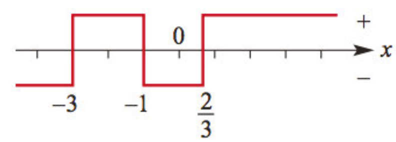
b



c



d



7 **a** $P(x) = x^3 + 3x^2 - 4x + 2$
$P(-1) = (-1)^3 + 3(-1)^2 - 4(-1) + 2$
$= 8$

b $P(x) = x^3 - 3x^2 - x + 6$
$P(2) = 2^3 - 3 \times 2^2 - 2 + 6$
$= 0$

c $P(x) = 2x^3 + 3x^2 - 3x - 2$
$P(-2) = 2(-2)^3 + 3(-2)^2 - 3(-2) - 2$
$= 0$

8 From the x intercepts, the rule must be
$y = a(x+3)(x+2)(x-7)$
$x = 0$: $y = a(3)(2)(-7) = -42a$
But the y intercept is $(0, -42)$ and hence
$-42a = -42$, so $a = 1$.
Thus $y = (x+3)(x+2)(x-7)$.

9 **a** $(x-2)(x+1)(x+3)$

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b $(x-1)(x+1)(x-3)$

c $(x-1)(x+1)(x-3)(x+2)$

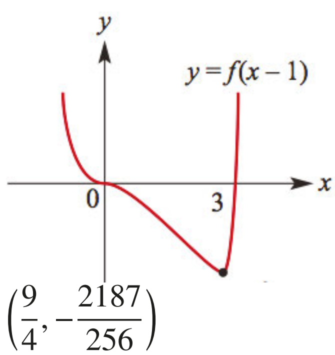
d $\frac{1}{4}(x-1)(2x+3+\sqrt{13})(2x+3-\sqrt{13})$

10 $x^2+4 = 1\times(x^2-2x+2)+2x+2$

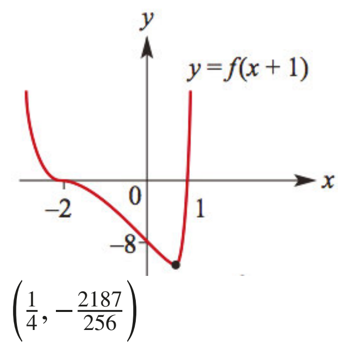
11 $a=-6$

12 $f(x)=(x+1)^3(x-2)$ Note: The tp on the diagrams are incorrect

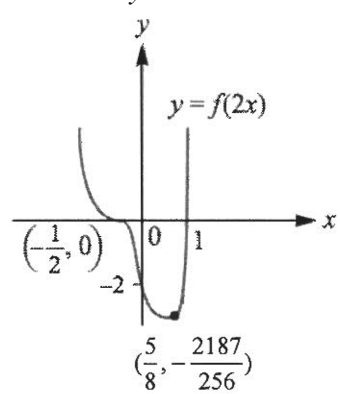
a $y=f(x-1)$
Translate the given graph 1 unit right.
The new intercepts are $(0,0)$, $(3,0)$.
The new minimum is at $(\frac{9}{4}, -\frac{2187}{256})$ since the y value does not change.



b $y=f(x+1)$
Translate the given graph 1 unit left.
The new x intercepts are $(-2,0)$, $(1,0)$.
$x=0$: $y=f(1)=2^3(-2)=-16$, so the new y intercept is $(0,-16)$.
The new minimum is at $\left(\frac{1}{4}, -\frac{2187}{256}\right)$ since the y value does not change.



c $y=f(2x)$
Dilate the given graph $\frac{1}{2}$ unit from the y axis.
The new x intercepts are $(1,0)$, $(-\frac{1}{2},0)$.
The new y intercept stays at $(0,-2)$.
The new minimum is at $(\frac{5}{8}, -\frac{2187}{256})$ since the y value does not change.



d $y=f(x)+2$
Translate the given graph 2 units up.
This makes the origin an intercept.
A second x intercept is between $\frac{5}{4}$ and 2.
The minimum has the same x value of $\frac{5}{4}$ and y value of $-\frac{2187}{256}+2=-\frac{1675}{256}$.
The new minimum is at $\left(\frac{5}{4}, -\frac{1675}{256}\right)$.

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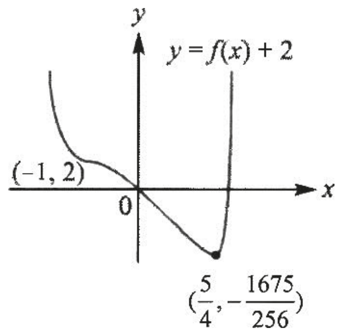

13 $k = \pm 8$

14 $(4, -5),\ (3, 9)$

15 $a = 3,\ b = \frac{5}{6},\ c = -\frac{13}{12}$

16 $64x^3 + 144x^2 + 108x + 27$

17 $a = 1,\ b = -1,\ c = 4$

18 $-2 < p < 6$

19 The rule of the cubic function is of the form $y = ax^3 + bx^2 + cx + d$. Since its graph passes through $(0, 6)$, $d = 6$. Write the equation as $y - 6 = ax^3 + bx^2 + cx$. Use the remaining points to form three simultaneous equations in a, b, and c.

$(1, 1)$: $-5 = a + b + c$ 1

$(2, 4)$: $-2 = 8a + 4b + 2c$ 2

$(3, 9)$: $3 = 27a + 9b + 3c$ 3

$2{-}21 : 6a + 2b = 8$ or equivalently

$3a + b = 4$ 4

$2 - 31 : 24a + 6b = 18$ or equivalently

$4a + b = 3$ 5

$5 - 4$ gives $a = -1$.

Substitution into 4 gives $b = 7$.

Substitution into 1 gives $c = -11$.

Hence $a = -1, b = 7, c = -11, d = 6$ and so $y = -x^3 + 7x^2 - 11x + 6$.

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Solutions to multiple-choice questions

1 E $= 5x^2 - 10x - 2$
$= 5x^2 - 2x - 2$
$= 5x - 2x + 1 - 1 - 2$
$= 5(x - 2)^2 - 1 - 2$
$= 5(x - 2)^2 - 5 - 2$
$= 5(x - 2)^2 - 7$

2 D There are 2 real roots when the determinant > 0
$b^2 - 4ac > 0$
$36 + 12m > 0$
$12m > -36$
$m > -3$

3 E $x^3 + 27$
$= x^3 + 3^3$
$a^3 + b^3 = (ax + b)(ax^2 - abx + b^2)$
Where $a = 1$ and $b = 3$
$(x + 3)(x^2 - 3x + 9)$

4 C The equation is a cubic.
From null factor theorem:
The only possible options are B and C
Sub in an x value to determine if the graph has a positive or negative y value:
When $x = 2$
Option C: $y = 16 \times -6$
Option D: $y = 4 \times 4$
Therefore it must be option C

5 E $x - 1$ is a factor
$\therefore$ $1^3 + 3(1)^2 - 2a + 1 = 0$
$-2a = -5$
$a = \frac{5}{2}$

6 A Check by expanding:
For option A,
$(3x + 2y)(2x - 4y)$
$= 6x^2 - 12xy + 4xy - 8y^2$
$= 6x^2 - 8xy - 8y^2$

7 C Looking at the part of the graph shown, we can see that at $x = 1$, the graph is also showing a turning point. Therefore we can see that the answer must be either D or C, as the x-intercept points in the other graphs either show points of inflection (i.e. $f(x) = (x - 1)^3$), or an intercept where the graph doesn't change direction (i.e. $f(x) = x^2(x - 1)$. Then substitute values into the equations to check which one of C or D it is. Looking at C, you can see that for all values of x greater than zero other than 1, the function will be equal to a number less than zero. Looking at D, you can see that for all values of x greater than zero other than 1, the function will be equal to a number greater than zero.

8 E Expand the outer set of brackets to get the function into turning point for m. So $p(x) = 3((x - 2)^2 + 4)$ becomes $p(x) = 3(x - 2)^2 + 12$. Therefore the graph is shifted right 2 and up 12 from the origin. The answer is

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9 C From the graph there is a intercept at $x = c$ and turning point at $(b, 0)$. So the polynomial must have functions $(x - c)$ and $(x - b)^2$
Now $(x - b)^2$ is the same as $(b - x)^2$.
$y = (x - c)(b - x)^2$ fits.
(Note: that option D gives a reflection in the x-axis of the graph given.)

10 C We can see immediately by looking at the equation that the function will touch the x-axis when $x = b$, and when $x = -c$. The remaining factor of the function is $(x^2 + a)$ and we know that is a positive real number. When we attempt to solve for x, we get the following: $x^2 = -a$.
Knowing that a is a positive real number, we realise that the solutions are not real numbers and hence are not roots.

11 C

12 B

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Solutions to extended-response questions

1 **a** The graph passes through the point (15, 20)

$\therefore \quad 20 = k \times 15^3 \times (20 - 15)$

$\therefore \quad k = \dfrac{4}{15^3}$

$\therefore \quad k = \dfrac{4}{3375} \approx 0.0019$

$\therefore \quad R = \dfrac{4t^3}{3375}(20 - t)$

b When $t = 10 \quad R = \dfrac{4 \times 10^3}{3375} \times 10$

$= \dfrac{4 \times 10^4}{3375}$

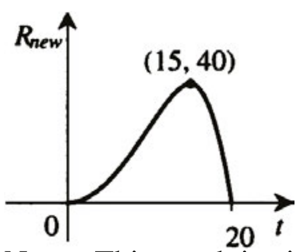
$= \dfrac{4 \times 80}{27}$

$= \dfrac{320}{27}$

The rate of flow $= \dfrac{320}{27}$ mL/ min when $t = 10$

$\left(\dfrac{320}{27} \approx 11.852\right)$

c **i**



Note: This graph is given by a dilation of factor 2 from the t-axis

ii When $t = 10$

$R_{new} = 2 \times \dfrac{4}{3375} \times 10^3 \times 10$

$= \dfrac{640}{27} - $ mL/min

The rate of flow $= \dfrac{640}{27}$ mL/min when $t = 10\left(\dfrac{640}{27} \approx 23.704\right)$

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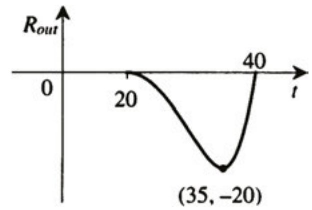
d **i** The hint gives that R_{out} is obtained by a translation of 20 units to the right.



$\therefore\ (t, R) \to (t + 20, R)$

$\therefore\ t' = t + 20$ and $R' = R$

$\therefore\ R = kt^3(20 - t)$ is transformed to

$$R' = k(t' + 20)^3(20 - (t' - 20))$$

$$= k(t' - 20)^3(40 - t')$$

A reflection in the x-axis give

$R_{out} = -k(t - 20)^3(40 - t)$

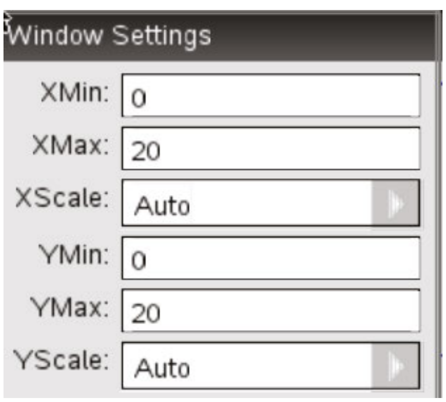
ii When $t = 30$, $R_{out} = \dfrac{-320}{27}$ mL/ min$\left(-\dfrac{320}{27} \approx -11.852\right)$

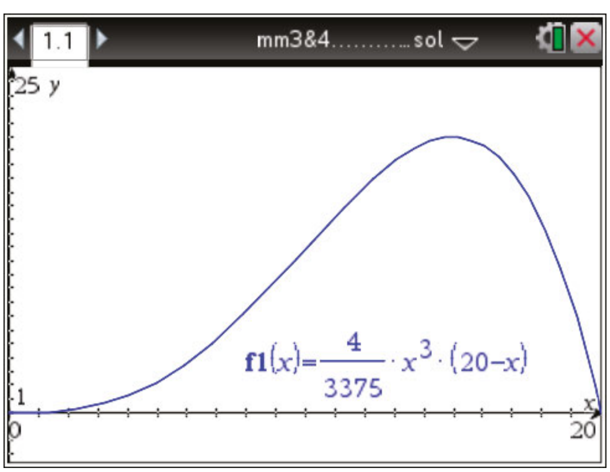
Note: the simplest way to obtain this is to move $\left(10, \dfrac{320}{7}\right) \to \left(30, \dfrac{-320}{7}\right)$ with this transformation

The rate of flow out is $\dfrac{320}{27}$ mL/ min

Calculator technique for question:

a In a Graphs page enter the rule: $f1(x) = 4/3375x^3(20 - x)$.
Suitable window settings are:





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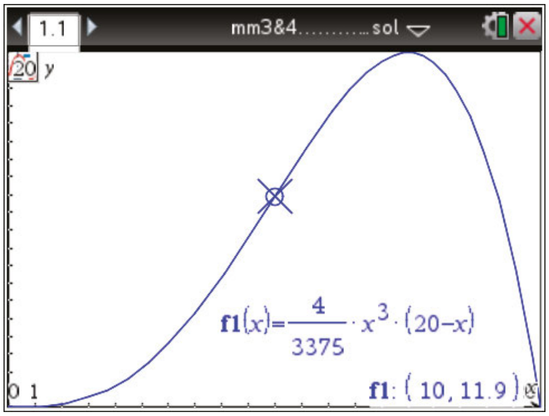
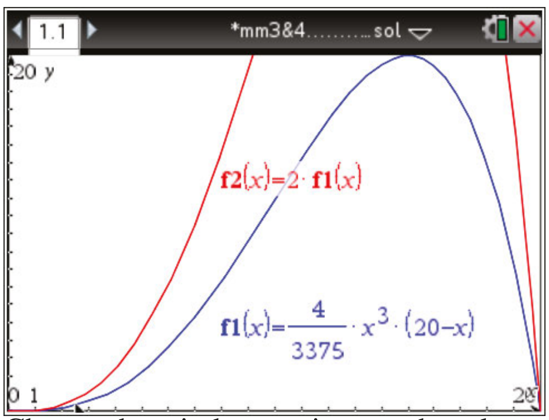
b The rate of flow when $t = 10$ is obtained by using Graph Trace from the Trace menu and typing in 10. Press.

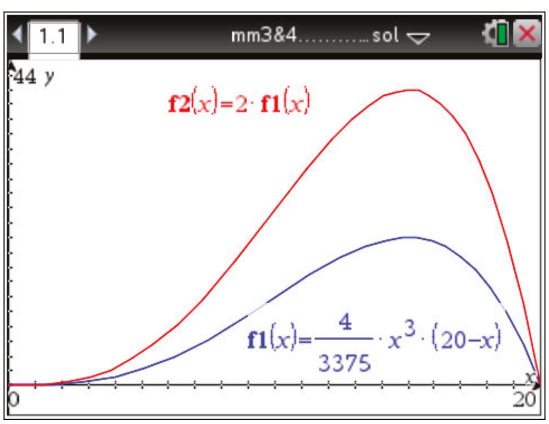
Hint: press d to exit the Graph Trace tool.



c The new function is obtained by entering $f2(x) = 2f1(x)$ in the function entry line (press e or /+G to show the function entry line if required). Press to plot the new graph.



Change the window settings to show key points of both graphs. Hint: use b>Window/Zoom>ZoomFit



In order to see the graphs by R against t and Rout against t a hybrid function must be entered as shown,

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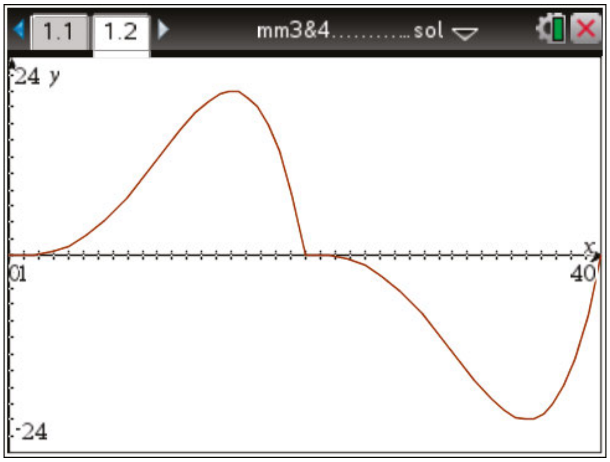
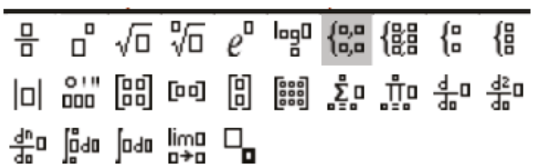


$$f3(x) = \begin{cases} \left(\dfrac{4}{3375}\right)x^3(20 - x) & \text{for } 0 \geq x \geq 20 \\ -\left(\dfrac{4}{3375}\right)(x - 20)^3(40 - x) & \text{for } 20 \geq x \geq 40 \end{cases}$$

Insert a new Graphs page (/ + I) From the math templates palette (t)select the piecewise template.



The graph is as shown. For this choose Xmin = 0 and Xmax = 40. Adjust values.

2 a i When $t = 0, V = 4 \times 9^3 = 2916$

The volume is 2916 m^3

ii When t = 9, V = 0

b The volume is 0 m^3

c

$$512 = 4(9 - t)^3$$

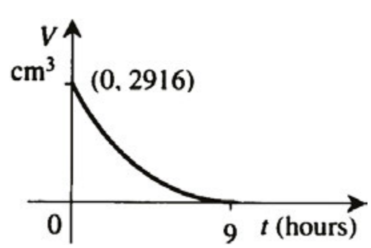
$$128 = (9 - t)^3$$

$$128^{\frac{1}{3}} = 9 - t$$

$$t = 9 - 128^3$$

$$= 9 - 4 \times 2^{\frac{1}{3}} \approx 3.9603$$

After 3.96 hr the volume is 512m^3



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3 a i $V = \frac{1}{3}\pi \times 4 \times (18 - 2)$

$= \frac{\pi}{3} \times 4 \times 16$

$= \frac{64\pi}{3}$

Volume is $= \frac{64\pi}{3}$ cm^3 when $x = 2$

ii $V = \frac{1}{3}\pi \times 3^2 \times (18 - 3)$

$= \pi \times 45$

$= 45\pi$

Volume is 45πcm^3 when$x = 3$.

iii $V = \frac{1}{3}\pi \times 4^2 \times (18 - 4)$

$= \frac{\pi}{3} \times 16 \times 14$

$= \frac{224\pi}{3}$

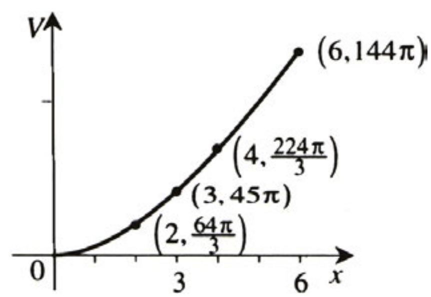
Volume is $= \frac{224\pi}{3}$ cm^3 when $x = 4$.

b When the bowl is full, depth is 6 cm.

When $x = 6, V = \frac{1}{3}\pi \times 36 \times 12$

$= 144\pi$

The volume of water is 144 πcm^3 when the bowl is full.



c If $V = \frac{325\pi}{3}$, $\quad \frac{325\pi}{3} = \frac{1}{3}\pi x^2(18 - x)$

which implies $325 = x^2(18 - x)$

and $\therefore\ x^3 - 18x^2 + 325 = 0$

Let $P(x) = x^3 - 18x^2 + 325$

$P(5) = 5^3 - 18 \times 5^2 + 325 = 0$

which, by the Factor Theorem, implies that $x - 5$ is a factor.

$\therefore$ $P(x) = (x - 5)(x^2 - 13x + 65)$

$x^2 - 13x - 65 = 0$ implies $x = \frac{13 \pm \sqrt{169 + 4 \times 65}}{2}$

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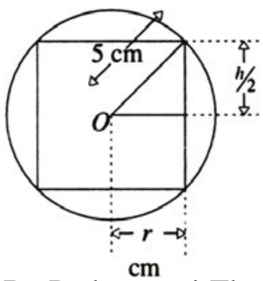
$= \dfrac{13 \pm \sqrt{429}}{2}$

but these two values of x lie outside the domain of $V = (0, 6)$

$\therefore$ $x = 5$ is the only solution.

i.e. the depth of the water when $V = \dfrac{325\pi}{3}$ is 5 cm.

4 a



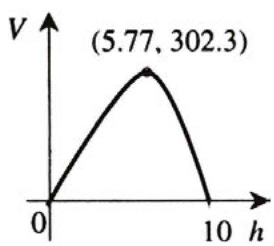
By Pythagoras' Theorem

$r^2 + \left(\dfrac{h}{2}\right)^2 = 25$

$\therefore\ r^2 = 25 - \left(\dfrac{h}{2}\right)^2$

i.e. $r = \dfrac{1}{2}\sqrt{100 - h^2}$

b



$\therefore$ Volume of cylinder $= \dfrac{1}{2}\pi r^2 h$

$= \pi \times \dfrac{1}{4}(100 - h^2)h$

$= \dfrac{1}{4}\pi h(100 - h^2)$

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c $V = \frac{1}{4}\pi h(100 - h^2)$

When $h = 6$

$V = \frac{1}{4} \times \pi \times 6(100 - 36)$

$= 96\pi$

The volume of the cylinder is $96\pi \text{cm}^3$

d When $V = 48\pi$

$$48\pi = \frac{1}{4}\pi h(100 - h^2)$$

$$\therefore 192 = 100h - h^3$$

$$\therefore h^3 - 100h + 192 = 0$$

$$\text{Let } P(h) = h^3 - 100h + 192$$

$$P(2) = 2^3 - 100 \times 2 + 192$$

$$= 0$$

$\therefore h - 2$is a factor

$\therefore P(h) = (h - 2)(h^2 + 2h - 96)$

$P(h) = 0$ implies $h = 2$ or $h^2 + 2h - 96 = 0$

$$\therefore h = \frac{-2 \pm \sqrt{4 + 4 \times 96}}{2}$$

$$= \frac{-2 \pm \sqrt{388}}{2}$$

$$= -1 \pm \sqrt{97}$$

But $h > 0, \therefore$ the only solutions are $h = 2$ and $h = -1 + \sqrt{97}$

When $h = 2$

$r = \frac{1}{2}\sqrt{100 - 4}$

$= \frac{1}{2}\sqrt{96}$

$= 2\sqrt{6}$

When $h = -1 + \sqrt{97} \approx 8.849$

$r \approx \frac{1}{2}\sqrt{100 - 78.30}$

≈ 2.33

When the volume of the cylinder is 48π cm^3 the height is 2 cm and the radius $2\sqrt{6} \approx 4.899$ cm.

OR the height is $(-1 + \sqrt{97}) \approx 8.849$ and the radius is ≈ 2.33 cm.

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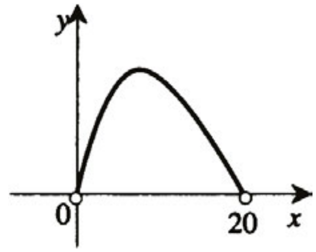
5 **a** $V = (84 - 2x)(40 - 2x)x$

b $84 - 2x > 0$ and $40 - 2x > 0$ and $x > 0$

$\therefore$ $x < 42$ and $x < 20$ and $x < 0$

$\therefore$ maximal domain $= (0, 20)$

c



d **i** $5\,760 \text{ cm}^3$

ii $12\,096 \text{ cm}^3$

iii $13\,056 \text{ cm}^3$

iv $12\,800 \text{ cm}^3$

e Use **Intersection** from the **Analyze Graph** menu

$x = 13.50$ or $x = 4.18$ (answers given correct to two decimal places)

f $13\,098.71 \text{ cm}^3$ (use **Maximum** from the **Analyze Graph** menu)

6 **a** **i** $A = 2x(16 - x^2)$

ii $0 < x < 4$

b **i** $A = 6(16 - 9)$

$= 42$

ii $x = 0.82$ or $x = 3.53$ (use **Intersection** from the **Analyze Graph** menu)

c **i** $V = xA$

$= 2x^2(16 - x^2)$

ii $x = 2.06$ or $x = 3.43$ (use **Intersection** from the **Analyze Graph** menu)

7 **a** $A = yx + \frac{\pi}{2}x^2$

b **i** $100 = y + \pi x$

$\therefore\ y = 100 - \pi x$

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ii $A = (100 - \pi x)x + \dfrac{\pi}{2}x^2$

$= 100x - \pi x^2 + \dfrac{\pi}{2}x^2$

$= 100x - \dfrac{\pi}{2}x^2$

iii $\left(0, \dfrac{100}{\pi}\right)$ as $x > 0$ and $y > 0$ which implies $100 - \pi x > 0$

c $x = 12.425$

Intersection from the Analyze Graph menu has been used.

d **i** $V = \dfrac{x}{50}\left(\dfrac{\pi}{2}x^2 + yx\right)$

$= \dfrac{x}{50}\left(100x - \dfrac{\pi}{2}x^2\right)$

$= \dfrac{x^2}{50}\left(100 - \dfrac{\pi}{2}x\right) \quad x \in \left(0, \dfrac{100}{\pi}\right)$

ii $V = 248.5 \text{ m}^3$ using $x = 12.425$ when $A = 1000$

iii Using Intersection from the Analyze Graph menu gives $x = 18.84$

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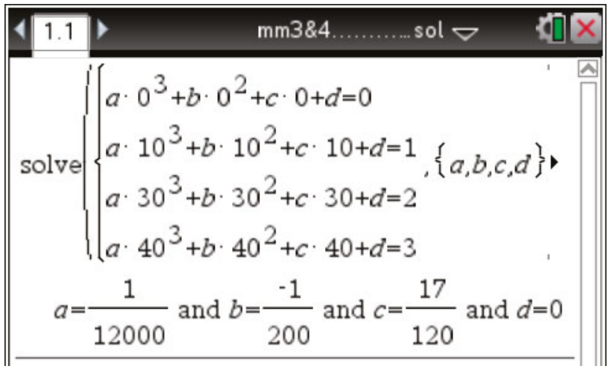
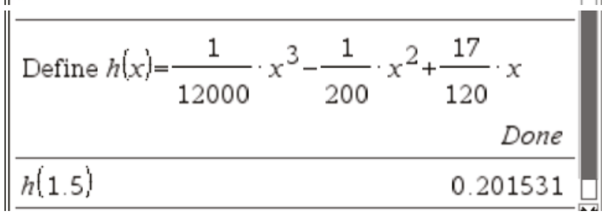
8a In a **Calculator** page solve the system of equations using b**>Algebra>Solve System of Equations>Solve System of Equations**.

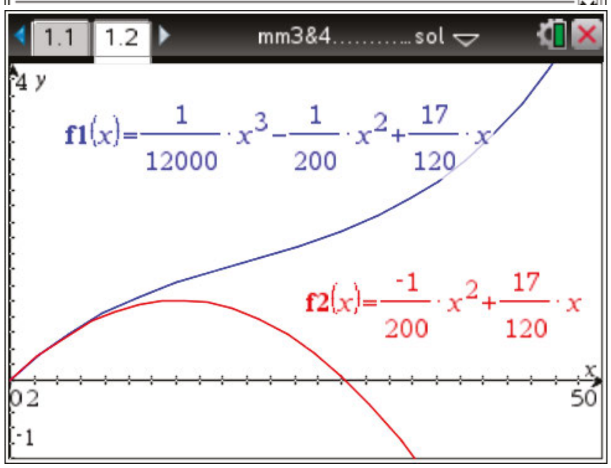


b Define the function $h(x)$
Find the height when $x = 1.5$ m



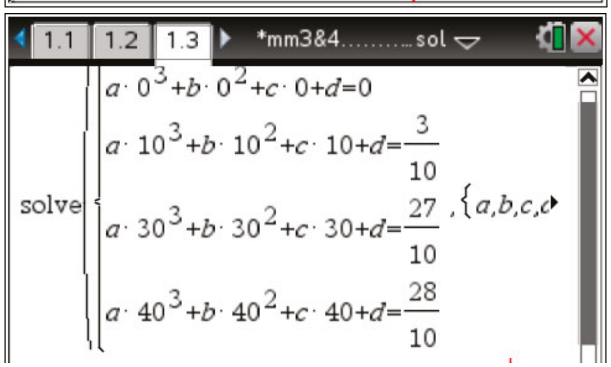
c In a **Graphs** page, enter the two functions

The coefficient of x^3, although small, is clearly influential.



d Solve the system of equations.

Hint: to obtain exact (fraction) answers the decimal values in the system of equations can be written as fractions as shown.



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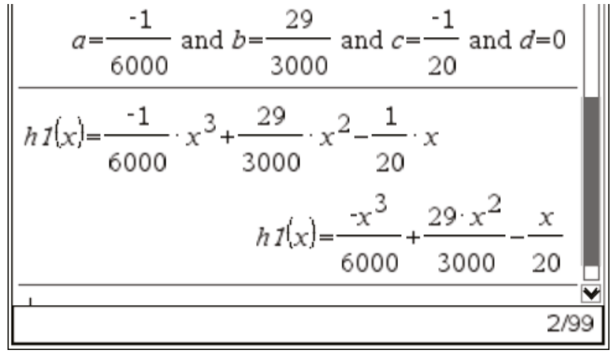
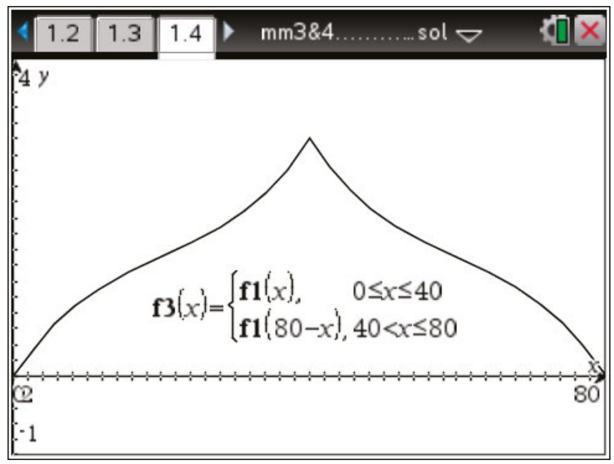
Alternatively use **b>Number>Approximate to Fraction** and edit the tolerance to 5.E-5)



e (i) in a **Graphs** page enter the hybrid function using the piecewise template from the **Math Template** palette (t)



The result is as shown.

e (ii) The second section of the graph is formed by a reflection of the graph of $y = f1(x), x \in (0, 40)$ in the line $x = 40$

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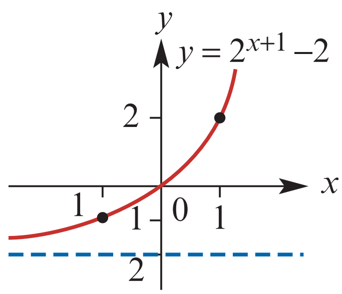
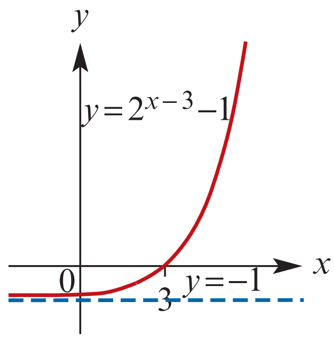
Chapter 5 – Exponential and logarithmic functions

Solutions to Exercise 5A

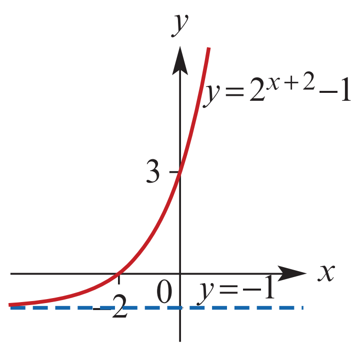
1 a Range = $(-2, \infty)$



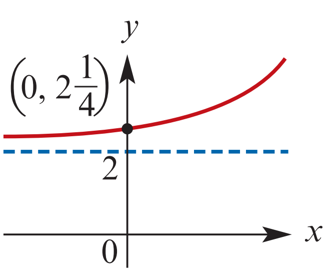
b Range = $(-1, \infty)$



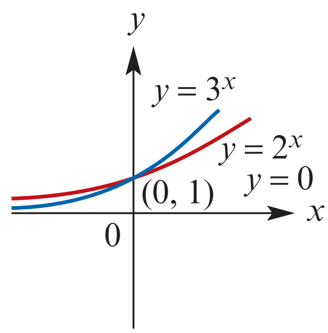
c Range = $(-1, \infty)$



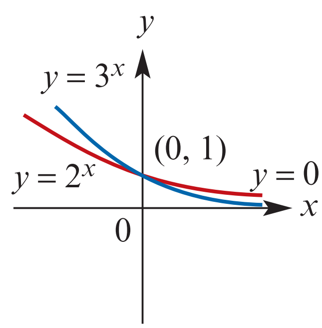
d Range = $(2, \infty)$



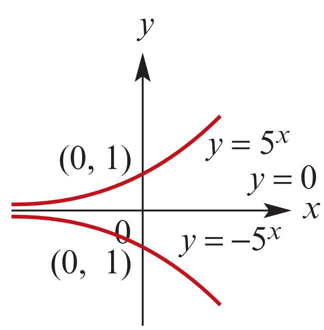
2 a



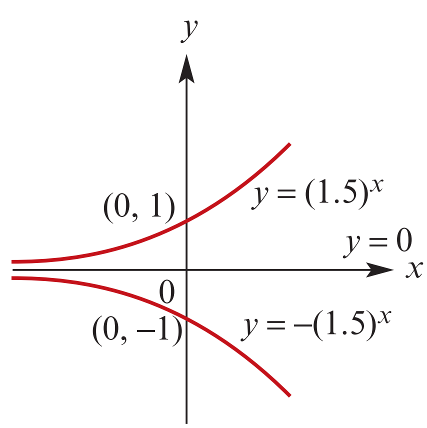
b



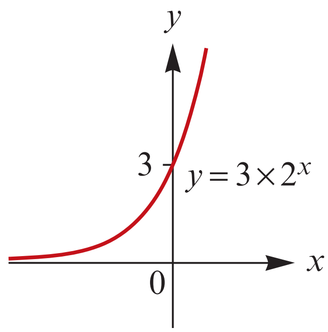
c



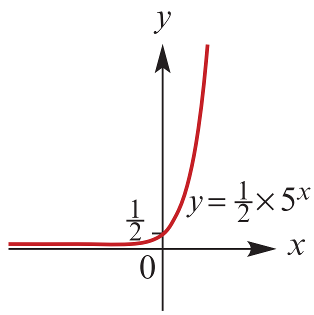
d



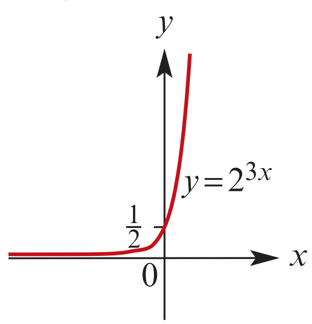
3 a Range = $(0, \infty)$



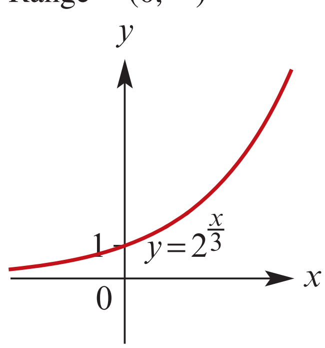
b Range = $(0, \infty)$



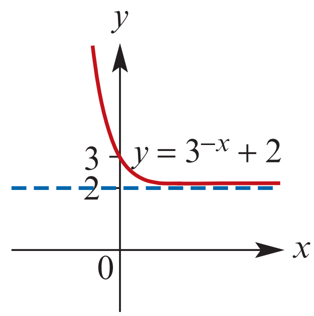
c Range = $(0, \infty)$



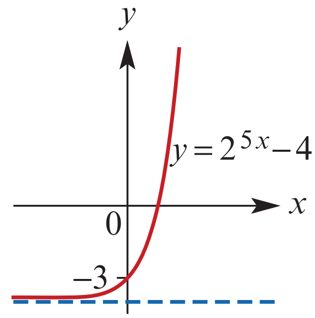
d Range = $(0, \infty)$



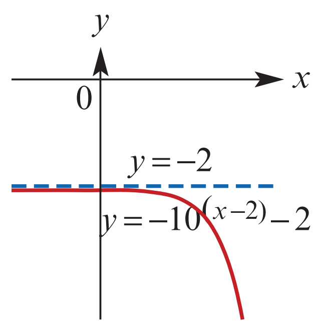
4 a Range = $(2, \infty)$



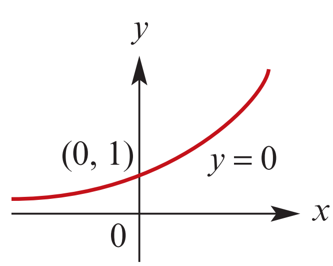
b Range = $(-4, \infty)$



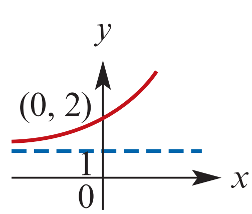
c Range = $(-\infty, 2)$



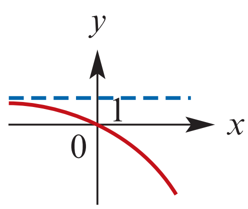
5 a Range = $\mathbb{R}^+$



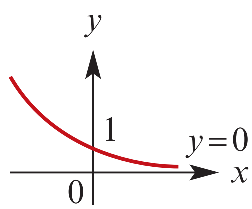
b Range = $(1, \infty)$



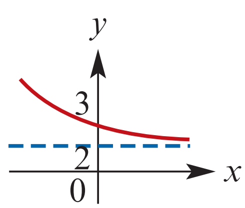
c Range = $(-\infty, 1)$



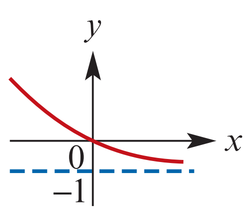
d Range = $\mathbb{R}^+$



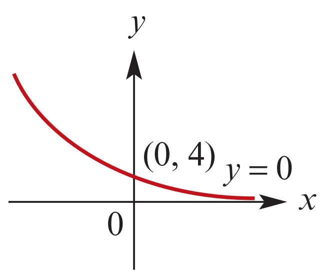
e Range = $(2, \infty)$



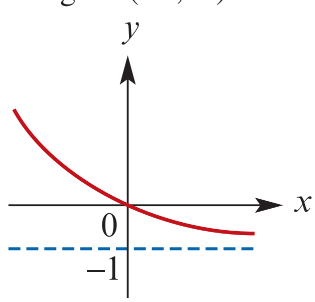
f Range = $(-1, \infty)$



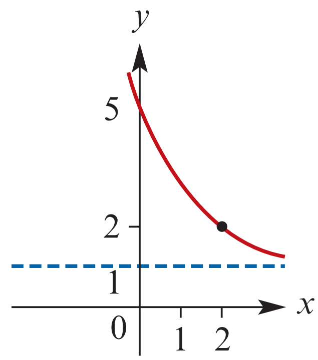
6 a Range = $\mathbb{R}^+$



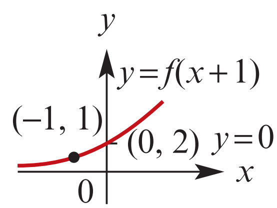
b Range = $(-1, \infty)$



c Range = $(1, \infty)$



7 a



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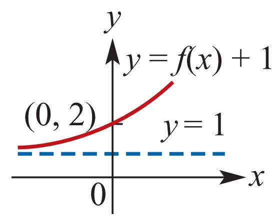

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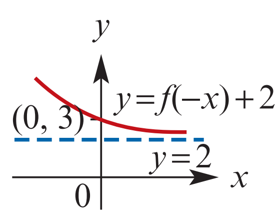

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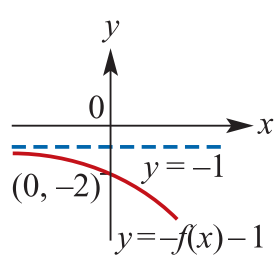

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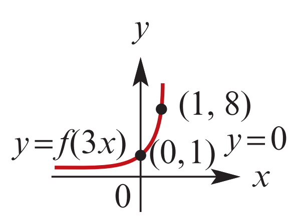

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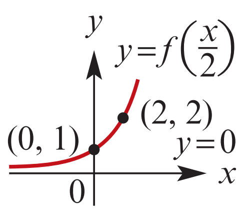

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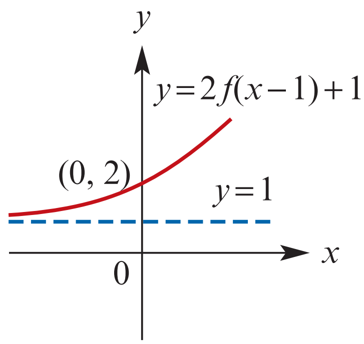

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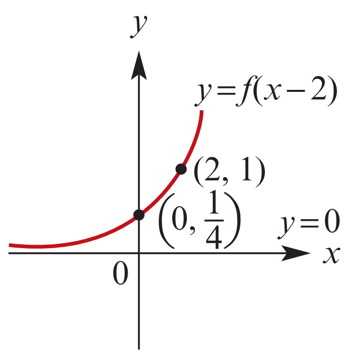
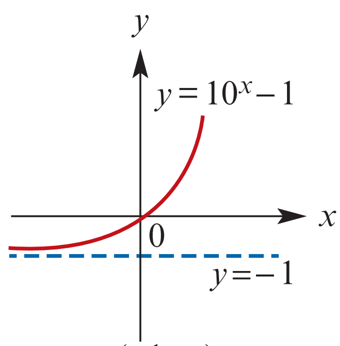
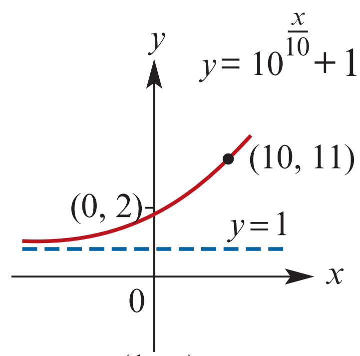


8 a



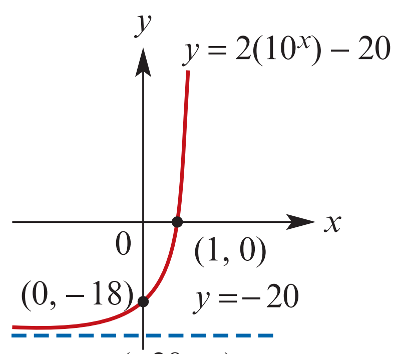
range = $(-1, \infty)$

b



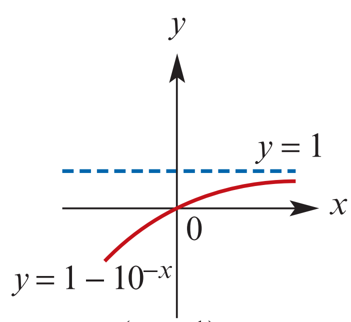
range = $(1, \infty)$

c



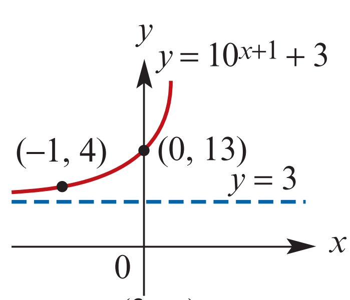
range = $(-20, \infty)$

d



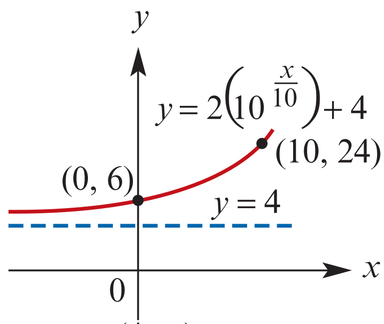
range = $(-\infty, 1)$

e



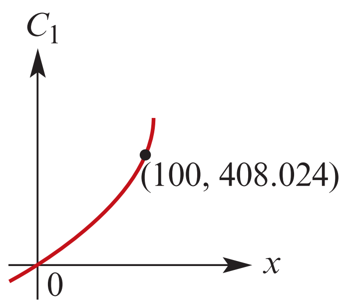
range = $(3, \infty)$

f



range = $(4, \infty)$

9 a



b $C_1 = 10000((1.0004)^x - 1)$

i $C_1 = 10000((1.0004)^{100} - 1)$

$= 10000(1.040802 - 1)$

$= \$408.02$

ii $C_1 = 10000((1.0004)^{300} - 1)$

$= 10000(1.127470 - 1)$

$= \$1,274.70$

c $1000 = 10000((1.0004)^x - 1)$

$(1.0004)^x = 1.1$

Use the 'solve' command of a CAS calculator to solve for x. This gives $x = 238.32\ldots\ x = 239$ days (you must round up in this case)

d i

ii to find $C_2 < C_1$

find $C_2 = C_1$ then round up using the CAS calculator at

$C_2 = C_1, x = 301.16$

$\therefore$ for $C_2 < C_1$

minimum $x = 302$ days

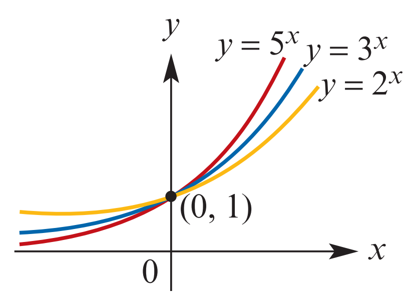
10 $y = 100(1.02)^x$

what is x when $y = 200$?

$2 = (1.02)^x$

Use the 'solve' command of a CAS calculator to solve for x. This gives $x = 35.003$. So your money has not quite doubled after 35 days; it will take 36 days.

11 a i

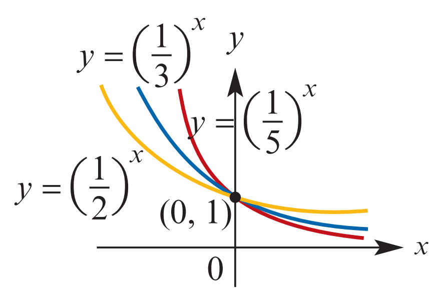


ii $x < 0$

iii $x > 0$

iv $x = 0$ (read off graph)

b i

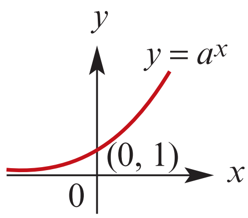


ii $x > 0$

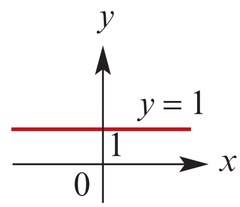
iii $x < 0$

iv $x = 0$ (read off graph)

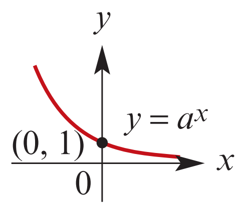
c **i**



ii

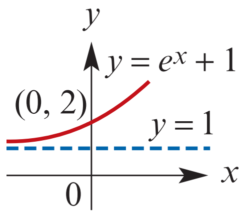


iii



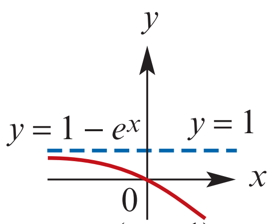
Solutions to Exercise 5B

1 a



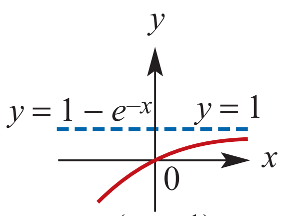
range = $(1, \infty)$

b



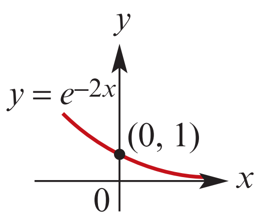
range = $(-\infty, 1)$

c



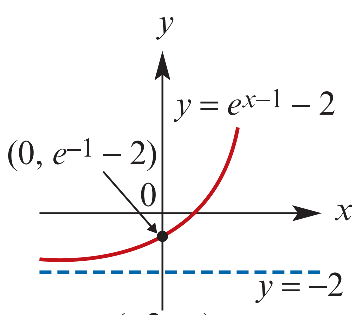
range = $(-\infty, 1)$

d



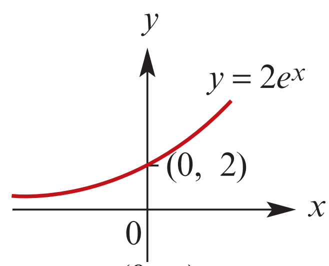
range = $(0, \infty)$

e



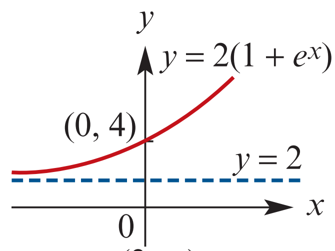
range = $(-2, \infty)$

f



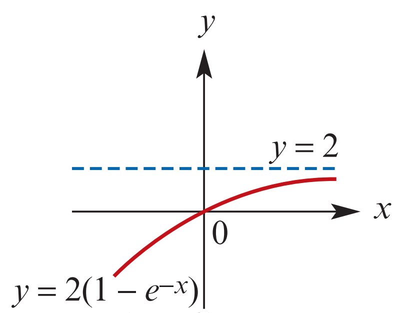
range = $(0, \infty)$

g



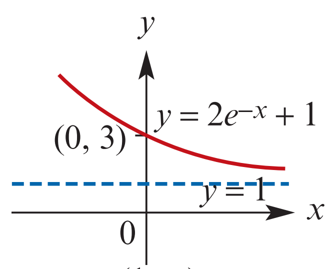
range = $(2, \infty)$

h



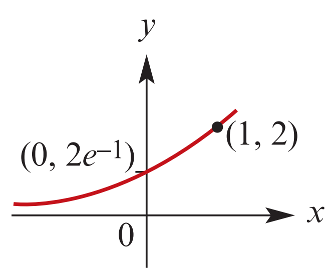
range = $(-\infty, 2)$

i



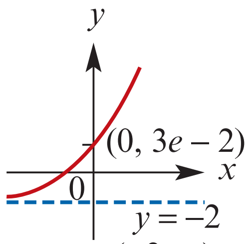
range = $(1, \infty)$

j



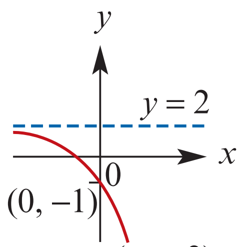
range = $(0, \infty)$

k



range = $(-2, \infty)$

l



range = $(-\infty, 2)$

2 **a** Translation 2 units to the left and 3 units down

b Dilation of factor 3 from the x-axis, then translation 1 unit to the left and 4 units down

c Dilation of factor 5 from the x-axis and factor $\frac{1}{2}$ from the y-axis, then translation $\frac{1}{2}$ unit to the left

d Reflection in the x-axis, then translation 1 unit to the right and 2 units up

e Dilation of factor 2 from the x-axis, reflection in the x-axis, then translation 2 units to the left and 3 units up

f Dilation of factor 4 from the x-axis and factor $\frac{1}{2}$ from the y-axis, then translation 1 unit down

3 **a** $y = -2e^{x-3} - 4$

b $y = 4 - e^{2x-3}$

c $y = -2e^{x-3} - 4$

d $y = -2e^{x-3} - 8$

e $y = -2e^{x-3} + 8$

f $y = -2e^{x-3} + 8$

4 **a** Translation 2 units to the right and 3 units up

b Translation 1 unit to the right and 4 units up, then dilation of factor $\frac{1}{3}$ from the x-axis

c Translation $\frac{1}{2}$ unit to the right, then dilation of factor $\frac{1}{5}$ from the x-axis and factor 2 from the y-axis

d Translation 1 unit to the left and 2 units down, then reflection in the x-axis

e Translation 2 units to the right and 3 units down, then dilation of factor $\frac{1}{2}$ from the x-axis and reflection in the x-axis

f Translation 1 unit up, then dilation of factor $\frac{1}{4}$ from the x-axis and factor 2 from the y-axis

5 **a** $x = 1.146$ or $x = -1.841$

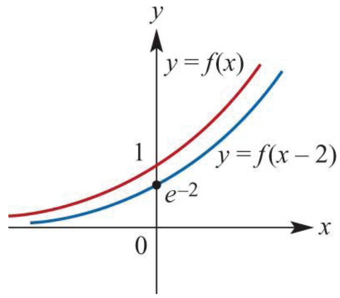
b $x = -0.443$

c $x = -0.703$

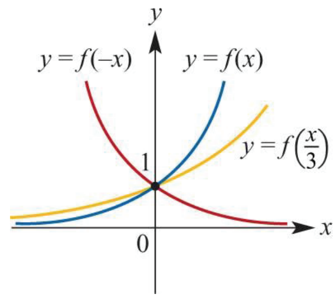
d $x = 1.857$ or $x = 4.536$

6

a b i



ii iii



Solutions to Exercise 5C

1 a $3x^2y^2 + 2x^4y^6 = 6x^6y^9$

b $\dfrac{12x^8}{4x^2} = 3x^6$

c $\dfrac{18x^2y^3}{3x^4y} = 6x^{-2}y^2$

$= \dfrac{6y^2}{x^2}$

d $(4x^4y^2)^2 + 2(x^2y)^4$

$= 16x^8y^4 + 2x^8y^4$

$= 18x^8y^4$

e $(4x^0)^2$

$= 4^2$

$= 16$

f $\dfrac{15(x^5y^{-2})^4}{3(x^4y)^{-2}} = 5x^{20}y^{-8}x^8y^2$

$= 5x^{28}y^{-6}$

$= \dfrac{5x^{28}}{y^6}$

g $\dfrac{3(2x^2y^3)^4}{2x^3y^2} = \dfrac{3 * 16x^8y^{12}}{2x^3y^2}$

$= 24x^5y^{10}$

h $(8x^3y^6)^{\frac{1}{3}} = 2xy^2$

i $\dfrac{x^2 + y^2}{x^{-2} + y^{-2}} = \dfrac{x^2 + y^2}{\dfrac{1}{x^2} + \dfrac{1}{y^2}}$

$= \dfrac{x^2 + y^2}{\dfrac{y^2 + x^2}{x^2y^2}}$

$= x^2y^2$

2 a $3^x = 81$

$3^x = 3^4$

$x = 4$

b $81^x = 9$

$81^x = 81^{\frac{1}{2}}$

$x = \dfrac{1}{2}$

c $4^x = 256$

$4^x = 4^4$

$x = 4$

d $625^x = 5$

$625^x = 625^{\frac{1}{4}}$

$x = \dfrac{1}{4}$

e $32^x = 8$

$2^{5x} = 2^3$

$5x = 3$

$x = \dfrac{3}{5}$

f $5^x = 125$

$5^x = 5^3$

$x = 3$

g $16^x = 1024$

$2^{4x} = 2^{10}$

$x = \dfrac{5}{2}$

h $2^{-x} = \frac{1}{64}$

$2^{-x} = 2^{-6}$

$x = 6$

i $5^{-x} = \frac{1}{625}$

$5^{-x} = 5^{-4}$

$x = 4$

3 a $5^{2n} \times 25^{2n-1} = 625$

$5^{2n} \times 5^{4n-2} = 5^{4}$

$5^{6n-2} = 5^{4}$

$6n - 2 = 4$

$n = 1$

b $4^{2n-2} = 1$

$4^{2n-2} = 4^{0}$

$2n - 1 = 0$

$n = 1$

c $4^{2n-1} = \frac{1}{256}$

$4^{2n-1} = 4^{-4}$

$2n - 2 = -4$

$n = \frac{-3}{2}$

d $\frac{3^{n-2}}{9^{2-n}} = 27$

$3^{n-2} \times 3^{-2n-4} = 3^{3}$

$3^{3n-6} = 3^{3}$

$3n - 6 = 3$

$n = 3$

e $2^{2n-2} \times 4^{-3n} = 64$

$2^{2n-2} \times 2^{-6n} = 2^{6}$

$2^{-4n-2} = 2^{6}$

$-4n - 2 = 6$

$-4n = 8$

$n = -2$

f $2^{n-4} = 8^{4-n}$

$2^{n-4} = 2^{12-3n}$

$n - 4 = 12 - 3n$

$n = 4$

g $27^{n-2} = 9^{3n+2}$

$3^{3n-6} = 32^{6n+4}$

$3n - 6 = 6n + 4$

$3n = -10$

$n = \frac{-10}{3}$

h $8^{6n+2} = 8^{4n-1}$

$6n + 2 = 4n - 1$

$2n = -3$

$n = \frac{-3}{2}$

i $125^{4-n} = 5^{6-2n}$

$5^{12-3n} = 5^{6-2n}$

$12 - 3n = 6 - 2n$

$n = 6$

j $2^{n-1} \times 4^{2n+1} = 16$

$$2^{n-1} \times 2^{4n+2} = 2^4$$

$$2^{5n+1} = 2^4$$

$$5n + 1 = 4$$

$$5n = 3$$

$$n = \frac{3}{5}$$

k $(27 \times 3^n)^n = 27^n \times 3^{\frac{1}{4}}$

$$(3^3 \times 3^n)^n = 3^{3n} \times 3^{\frac{1}{4}}$$

$$(3^{3+n})^n = 3^{3n+\frac{1}{4}}$$

$$3^{3n+n^2} = 3^{3n+\frac{1}{4}}$$

$$3n + n^2 = 3n + \frac{1}{4}$$

$$n^2 = \frac{1}{4}$$

$$n^2 = \pm\frac{1}{2}$$

4 **a** $3^{2x} - 2(3^x) - 3 = 0$

$$\Rightarrow (3^x - 3)(3^x + 1) = 0$$

$$3^x = 3; \quad -1$$

$$\therefore x = 1;$$

$$x = 1$$

b $5^{2x} - 23(5^x) - 50 = 0$

$$\Rightarrow (5^x - 25)(5^x + 2) = 0$$

$$5^x = 25; \quad -2$$

$$\therefore x = 2;$$

$$x = 2$$

c $5^{2x} - 10(5^x) + 25 = 0$

$$(5^x - 5)^2 = 0$$

$$5^x = 5$$

$$x = 1$$

d $2^{2x} - 6(2^x) + 8 = 0$

$$\Rightarrow (2^x - 2)(2^x - 4) = 0$$

$$2^x = 2, 4$$

$$x = 1, 2$$

e $8(3^x) - 6 = 2(3^{2x})$

$$3^{2x} - 4(3^x) + 3 = 0$$

$$(3^x - 3)(x - 1) = 0$$

$$3^x = 3, 1$$

$$x = 1, 0$$

f $2^{2x} - 20(2^x) + 64 = 0$

$$(2^x - 16)(2^x - 4) = 0$$

$$2^x = 16, 4$$

$$x = 4, 2$$

g $4^{2x} - 5(4^x) + 4 = 0$

$$(4^x - 4)(4^x - 1) = 0$$

$$4^x = 4, 1$$

$$x = 1, 0$$

h $3(3^{2x}) - 28(3^x) + 9 = 0$

$$(3(3^x) - 1)(3^x - 9) = 0$$

$$3^x = \frac{1}{3}, 9$$

$$x = -1, 2$$

i $7(7^{2x}) - 8(7^x) + 1 = 0$

$$(7(7^x) - 1)(7^x - 1) = 0$$

$$7^x = \frac{1}{7}, 1$$

$$x = -1, 0$$

Solutions to Exercise 5D

1 **a** 3

b −4

c −3

d 6

e 6

f −7

2 Note: the natural logarithm function $\log_e x$ is often written $\ln x$; this notation is used here.

a $\ln 6$

b $\ln 4$

c $\ln 10^6 = 6 \ln 10$

d $\ln 7$

e $\ln \dfrac{1}{3 \times 4 \times 5} = \ln \dfrac{1}{60} = -\ln 60$

f $\ln(uv \times uv^2 \times uv^3) = \ln u^3 v^6$

$$= 3 \ln uv^2$$

g $7 \ln x = \ln x^7$

h $\ln\left(\dfrac{(x+y)(x-y)}{(x^2-y^2)}\right)$

$$= \ln 1$$

$$= 0$$

3 **a** $x = 10^2 = 100$

b $\log_2 x = 4$

$$x = 2^4 = 16$$

c $x - 5 = e^0 = 1$

$$x = 6$$

d $x = 2^6 = 64$

e $\ln(x + 5) = 3$

$$x + 5 = e^3$$

$$x = e^3 - 5 \approx 15.086$$

f $2x = e^0 = 1$

$$x = \frac{1}{2}$$

g $2x + 3 = e^0 = 1$

$$2x = -2$$

$$x = -1$$

h $x = 10^{-3}$

$$= \frac{1}{1000}$$

i $\log_2(x - 4) = 5$

$$x - 4 = 2^5 - 32$$

$$x = 36$$

4 **a** $\log_{10} x = \log_{10} 15$

$$x = 15$$

b $\ln x = \ln 5$

$$x = 5$$

c $\ln x = \ln\left(8^{\frac{2}{3}}\right)$

$$= \ln 4$$

$$x = 4$$

d $\ln(2x^2 - x) = 0, x > 0$

$2x^2 - x = 1$

$2x^2 - x - 1 = 0$

$(2x + 1)(x - 1) = 0$

$x = -\frac{1}{2}, 1$

since $x > 0, x = 1$

e $\ln x^2 - \ln(x - 1) = \ln(x + 3)$

$\ln \frac{x^2}{x - 1} = \ln(x + 3)$

$x^2 = (x + 3)(x - 1)$

$x^2 = x^2 + 2x - 3$

$2x = 3$

$x = \frac{3}{2}$

5 a $\log_{10}(3 \times 9) = \log_{10} 27$

b $\log_2\left(\frac{24}{6}\right) = \log_2 4 = 2$

c $\frac{1}{2}(\log_{10} a - \log_{10} b)$

$= \frac{1}{2}\left(\log_{10} \frac{a}{b}\right)$

$= \log_{10} \sqrt{\frac{a}{b}}$

d $1 + \log_{10} a - \log_{10}(b)^{\frac{1}{3}}$

$= \log_{10} 10 + \log_{10}\left(\frac{a}{b^{\frac{1}{3}}}\right)$

$= \log_{10}\left(\frac{10a}{b^{\frac{1}{3}}}\right)$

e $\log_{10 \sqrt{36}} - \log_{10}(27)^{\frac{1}{3}} - \log_{10}(64)^{\frac{2}{3}}$

$= \log_{10} 6 - \log_{10} 3 - \log_{10} 16$

$= \log_{10}\left(\frac{6}{3 \times 16}\right)$

$= \log_{10}\left(\frac{1}{8}\right)$

6 a $\log_{10} 10 = 1$

b $\log_{10} 5 + \log_{10} 8 - \log_{10} 4$

$= \log_{10} 10$

$= 1$

c $\log_2 \sqrt{2} + \log_2 1 + \log_2 4$

$= \log_2 4\sqrt{2}$

$= 2\frac{1}{2}$

$= \frac{5}{2}$

d $\log_{10} 25 + \log_{10} 4 + \log_{10} 10$

$= \log_{10} 1000$

$= 3$

e $\log_{10} 16 - \log_{10} 16$

$= 0$

7 a $\log_3\left(\frac{1}{3^x}\right) = \log_3(3^{-x})$

$= -x \log_3 3$

$= -x$

b $\log_2 x - \log_2 y^2 + \log_2(xy^2)$

$= \log_2(x^2)$

$= 2\log_2 x$

c $\ln(x^2 - y^2) - \ln(x - y) - \ln(x + y)$

$$= \ln\left(\frac{x^2 - y^2}{(x - y)(x + y)}\right)$$

$$= \ln 1 = 0$$

8 a $\ln(x^2 - 2x + 8) = \ln x^2$

$$x^2 - 2x + 8 = x^2$$

$$2x = 8$$

$$x = 4$$

b

$$\ln(5x) - \ln(3 - 2x) = \ln e$$

$$\ln(5x) = \ln(e(3 - 2x))$$

$$3e - 2ex = 5x$$

$$(5 + 2e)x = 3e$$

$$x = \frac{3e}{5 + 2e} \approx 0.7814$$

9 a $\ln x + \ln(3x + 1) = \ln e$

$$\ln(3x^2 + x) = \ln e$$

$$3x^2 + x - e = 0$$

$$x = \frac{-1 \pm \sqrt{1 + 12e}}{6}$$

$$\text{but } x > \frac{-1}{3}$$

$$x = \frac{-1 + \sqrt{1 + 12e}}{6}$$

$$\approx 0.7997$$

b $8e^{-x} - e^x - 2 = 0$

$$8 - e^{2x} - 2e^x = 0$$

$$(e^x)^2 + 2e^x - 8 = 0$$

$$(e^x + 4)(e^x - 2) = 0$$

$$e^x = -4, 2$$

But $e^x > 0$, so:

$$e^x = 2$$

$$x = \ln 2 \approx 0.6931$$

10 a $\log_x 81 = 4$

$$x^4 = 81$$

$$x = 3$$

b $\log_x \frac{1}{32} = 5$

$$x^5 = \frac{1}{32}$$

$$x = \frac{1}{2}$$

11 $\ln x^2 + \ln 4 = \ln(9x - 2)$

$$4x^2 = 9x - 2$$

$$4x^2 - 9x + 2 = 0$$

$$(4x - 1)(x - 2) = 0$$

$$x = \frac{1}{4}, 2$$

12

$$\log_a N = \frac{1}{2}(\log_a 24 - \log_a 0.375 - \log_a 729)$$

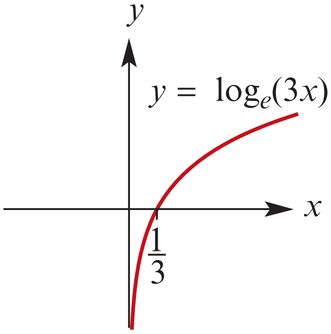
$$= \frac{1}{2}\left(\log_a \frac{64}{729}\right)$$

$$= \log_a \frac{8}{27}$$

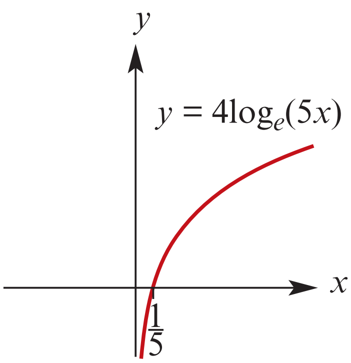
$$N = \frac{8}{27}$$

Solutions to Exercise 5E

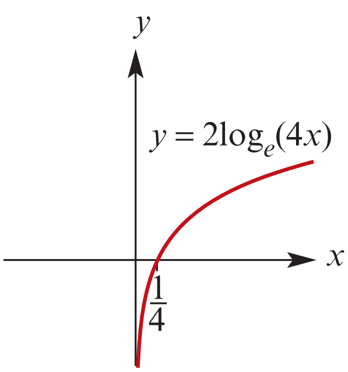
1 **a**



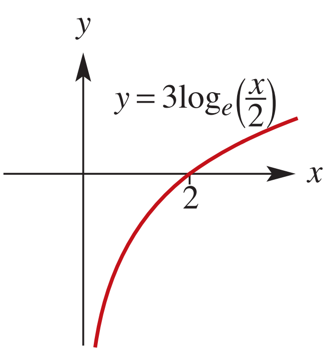
b



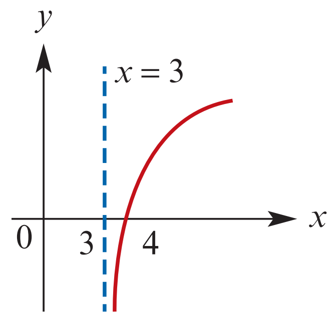
c



d

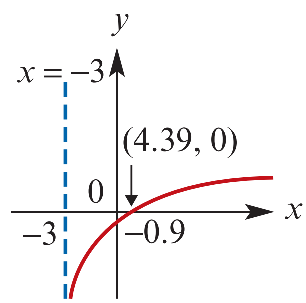


2 **a**



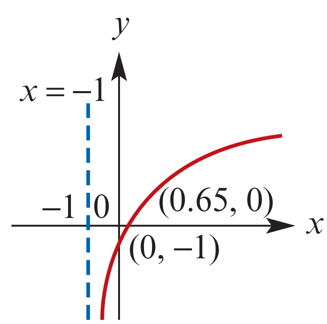
domain = $(3, \infty)$, range = $\mathbb{R}$

b



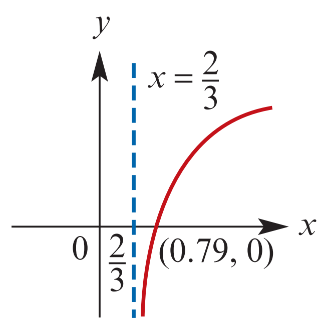
domain = $(-3, \infty)$, range = $\mathbb{R}$

c



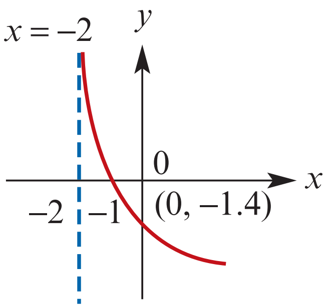
domain = $(-1, \infty)$, range = $\mathbb{R}$

d



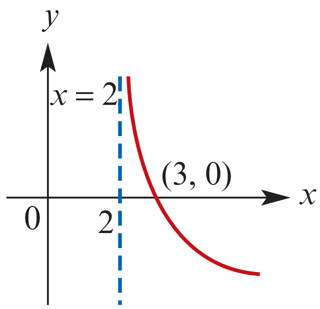
domain = $\left(\frac{2}{3}, \infty\right)$, range = $\mathbb{R}$

e



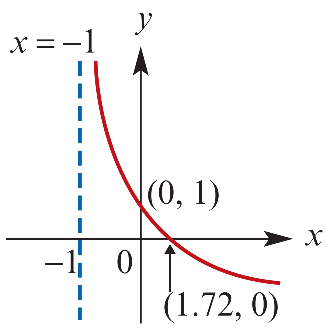
domain = $(-2, \infty)$, range = $\mathbb{R}$

f



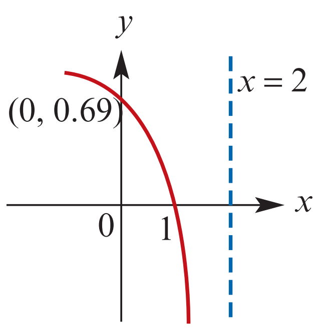
domain = $(2, \infty)$, range = $\mathbb{R}$

g



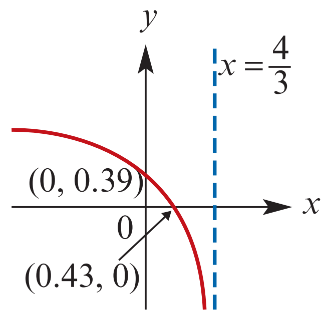
domain = $(-1, \infty)$, range = $\mathbb{R}$

h



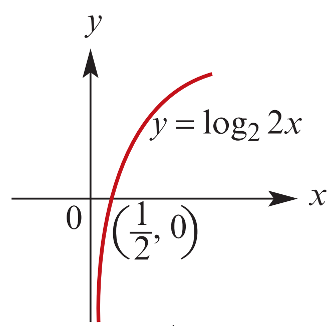
domain = $(-\infty, 2)$, range = $\mathbb{R}$

i



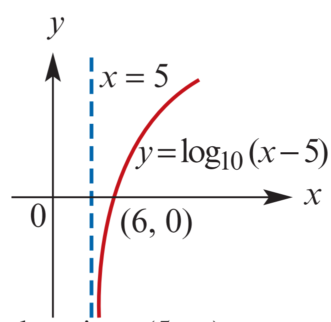
domain = $\left(-\infty, \frac{4}{3}\right)$, range = R

3 a



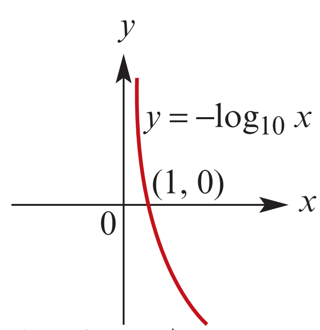
domain = $\mathbb{R}^+$

b



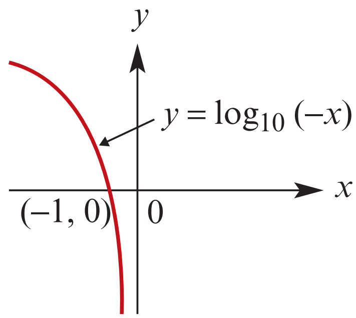
domain = $(5, \infty)$

c



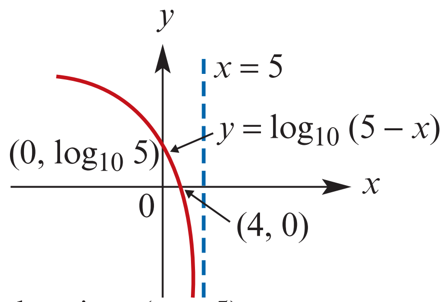
domain = $\mathbb{R}^+$

d



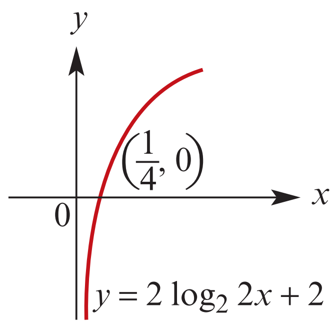
domain = $\mathbb{R}^-$

e



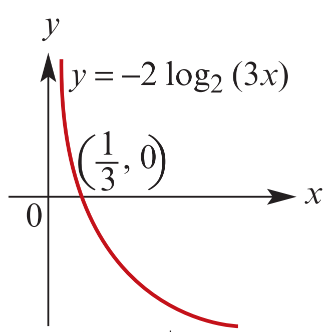
domain = $(-\infty, 5)$

f



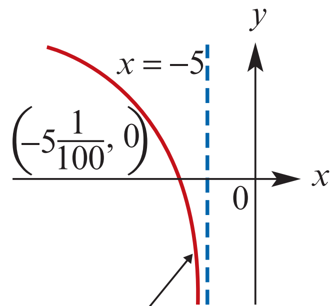
domain = R^+

g



domain = $\mathbb{R}^+$

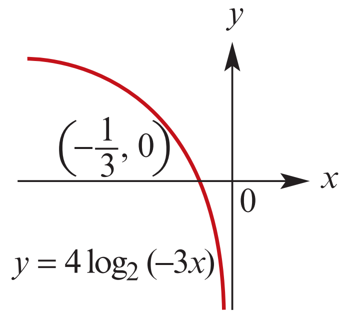
h



$y = \log_{10}(-x - 5) + 2$

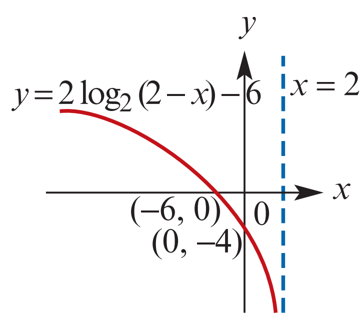
domain = $(-\infty, -5)$

i



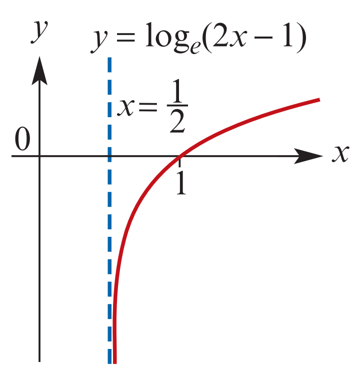
domain = $\mathbb{R}^-$

j



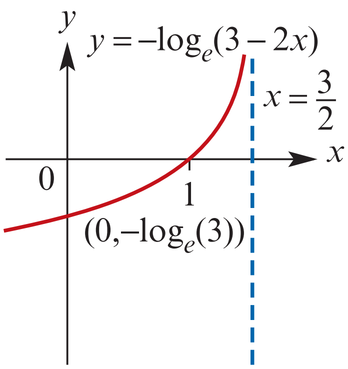
domain = $(-\infty, 2)$

k



Domain = $(\frac{1}{2}, \infty)$

l



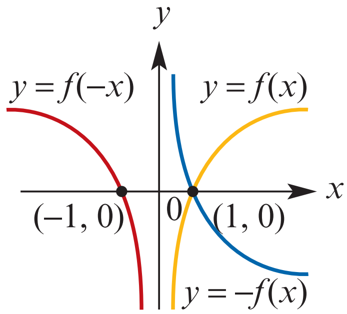
Domain = $(-\infty, \frac{3}{2})$

4 **a** $x = 1.557$

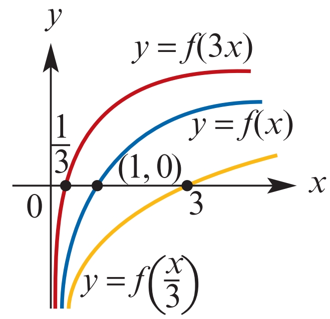
b $x = 1.189$

5 b

i ii



iii iv



6 A dilation of factor $\log_e 3$ from the y-axis

7 A dilation of factor $\dfrac{1}{\log_e 2}$ from the y-axis

Solutions to Exercise 5F

1
$$a + b = 5 \ldots (1)$$
$$ae^4 + b = 11 \ldots (2)$$
$$(2) - (1)$$
$$a(e^4 - 1) = 6$$
$$a = \frac{6}{e^4 - 1}$$
$$\therefore b = 5 - \frac{6}{e^4 - 1}$$
$$b = \frac{5e^4 - 11}{e^4 - 1}$$

2
$$a \log_e(5 + b) = 0 \ldots (1)$$
$$a \log_e(10 + b) = 2 \ldots (2)$$
From (1)
$$\log_e(5 + b) = 0$$
$$5 + b = e^0$$
$$b = -4$$
From (2)
$$\therefore a \log_e 6 = 2$$
$$\therefore a = \frac{2}{\log_e 6}$$

3
$$y = ae^x + b$$
$$x \to -\infty, y \to 4$$
$$4 = b$$
$$x = 0, y = 6$$
$$6 = a + b$$
$$= a + 4$$
$$a = 2$$

4
$$y = ae^x + b$$
$$x = 0, y = 0$$
$$0 = a + b$$
$$a = -b$$
$$x = 1, y = 14$$
$$14 = ae + b$$
$$= (e - 1)a$$
$$a = \frac{14}{e - 1} \approx 8.148$$
$$b = \frac{-14}{e - 1} \approx -8.148$$

5
$$y = ae^{-bx}$$
$$x = 3, y = 50$$
$$50 = ae^{-3b} \ldots (1)$$
$$x = 6, y = 10$$
$$10 = ae^{-6b} \ldots (2)$$
$$\frac{1}{2} \Rightarrow 5 = e^{3b}$$
$$3b = \ln 5$$
$$b = \frac{1}{3} \ln 5$$
$$b = \ln(5)^{\frac{1}{3}}$$
$$y = a \times 5^{\frac{-x}{3}}$$
$$\therefore a = 250$$

6 $f(x) = ae^{-x} + b$

$x \to \infty, f(x) \to 500$

$500 = b$

$x = 0, f(x) = 700$

$700 = a + 500$

$a = 200$

Sub in equation $\Rightarrow 50 = a \times \frac{1}{5}$

$a = 250$

7 $y = a\log_2 x + b$

$x = 8, y = 10$

$10 = 3a + b \ldots (1)$

$x = 32, y = 14$

$14 = 5a + b \ldots (2)$

$(2) - (1) \Rightarrow a = 2$

Sub in (1) $\Rightarrow 10 = 6 + b$

$b = 4$

8 $y = a\log_2(x - b)$

$x \to 5, y \to -\infty$

$b = 5$

$x = 7,\ y = 3$

$3 = a\log_2(7 - 5)$

$a = 3$

9 $y = ae^{bx}$

$x = 3,\ y = 10$

$x = 6,\ y = 50$

$10 = ae^{3b} \ldots (1)$

$50 = ae^{6b} \ldots (2)$

$\frac{(2)}{(1)} \Rightarrow 5 = e^{3b}$

$b = \ln(5)^{\frac{1}{3}}$

$y = a \times 5^{\frac{x}{3}}$

Sub in (1) $\Rightarrow 10 = a \times 5$

$a = 2$

10 $y = a\log_2(x - b)$

$x = 5, y = 2$

$2 = a\log_2(5 - b) \ldots (1)$

$x = 7, y = 4$

$4 = a\log_2(7 - b) \ldots (2)$

$(2) \div (1) \Rightarrow$

$2\log_2(5 - b) = \log_2(7 - b)$

$(5 - b)^2 = 7 - b$

$b^2 - 10b + 25 = -b + 7$

$b^2 - 9b + 18 = 0$

$(b - 6)(b - 3) = 0$

$b = 3$ *or* 6

since $\log_2(x)$ is only defined for $x > 0$ and $\log_2(5 - b)$ is one of the points, $b = 6$ is impossible

$\therefore\ b = 3$

Sub in (1) $\Rightarrow 2 = a\log_2(5 - 3)$

$a = 2$

11 $y = a\ln(x - b) + c$

vertical asymptote $x = 1, \therefore b = 1$

$$y = a\ln(x-1) + c$$

$$x = 3, y = 10$$

$$10 = a\ln 2 + c \ldots (1)$$

$$x = 5, y = 12$$

$$12 = a\ln 4 + c \ldots (2)$$

$$= 2a\ln 2 + c$$

$(2) - (1) \Rightarrow a\ln 2 = 2$

$$a = \frac{2}{\ln 2} \approx 2.885$$

$$y = 2\log_2(x-1) + c$$

Sub in (1) $\Rightarrow 10 = 2\log_2 2 + c$

$$c = 8$$

12

$$f(x) = a\ln(-x) + b$$

$$x = -2, \quad f(-2) = 6$$

$$6 = a\ln 2 + b \ldots (1)$$

$$x = -4, \quad f(-4) = 8$$

$$8 = a\ln(4) + b \ldots (2)$$

$$8 = 2a\ln 2 + b$$

$(2) - (1) \Rightarrow 2 = a\ln 2$

$$a = \frac{2}{\ln 2} \approx 2.885$$

Sub in 1 $\Rightarrow 6 = 2 + b$

$$b = 4$$

Solutions to Exercise 5G

1 a $\log_2 8 = k\log_2 7 + 2$

$$3 - 2 = k\log_2 7$$

$$1 = k\log 27$$

$$k = \frac{1}{\log_2 7}$$

b $\log_2 7 - x\log_2 7 = 4$

$$(1 - x)\log 27 = 4$$

$$1 - x = \frac{4}{\log_2 7}$$

$$x = 1 - \frac{4}{\log_2 7}$$

$$x = \frac{\log_2(7) - 4}{\log_2 7}$$

c $\log_e 7 - x\log_e 14 = 1$

$$\log_e 7 - 1 = x\log_e 14$$

$$x = \frac{\log_e 7 - 1}{\log_e 14}$$

2 a 2.58

b −0.32

c 2.18

d 1.16

e −2.32

f −0.68

g −2.15

h −1.38

i 2.89

j −1.7

k −4.42

l 5.76

m −6.21

n 2.38

o 2.80

3 a $x < 2.81$

b $x > 1.63$

c $x < -0.68$

d $x \le 3.89$

e $x \ge 0.57$

4 a $x = \log_2 5$

b $2x - 1 = \log_3 8$

$$2x = \log_3(8) + 1$$

$$x = \frac{\log_3(8) + 1}{2}$$

c $3x + 1 = \log_7 20$

$$3x = \log_7(20) - 1$$

$$x = \frac{\log_7(20) - 1}{3}$$

d $x = \log_3 7$

e $x = \log_3 6$

f $x = \log_5 6$

g Let $a = 3^x$

$$a^2 - 9a + 8 = 0$$
$$(a-8)(a-1) = 0$$
$$\therefore a = 8 \text{ or } a = 1$$
$$\therefore 3^x = 8 \text{ or } 3^x = 1$$
$$\therefore x = \log_3 8 \text{ or } x = 0$$

h Let $a = 5^x$

$$a^2 - 4a - 5 = 0$$
$$(a-5)(a+1) = 0$$
$$\therefore a = 5 \text{ or } a = -1$$
$$\therefore 5^x = 5 \text{ or } 5^x = -1$$
$$\therefore x = 1$$

5 a $7^x > 52 \Leftrightarrow x > \log_7 52$

b $3^{2x-1} < 40 \Leftrightarrow 2x - 1 < \log_3 40$

$$\Leftrightarrow 2x < \log_3(40) + 1$$
$$\Leftrightarrow x < \frac{1}{2}(\log_3(40) + 1)$$
$$= \frac{1}{2}(\log_3(120))$$

c $4^{3x+1} \geq 5 \Leftrightarrow 3x + 1 \geq \log_4 5$

$$\Leftrightarrow 3x \geq \log_4(5) - 1$$
$$\Leftrightarrow x \geq \frac{1}{3}\log_4(\frac{5}{4})$$
$$= \frac{1}{6}\log_2(\frac{5}{4})$$

d $3^{x-5} \leq 30 \Leftrightarrow x - 5 \leq \log_3 30$

$$\Leftrightarrow 3x \leq \log_3(30) + 5$$
$$\Leftrightarrow x < \frac{1}{3}(\log_3(30) + 5)$$
$$= \log_3(7290)$$

e $3^x < 106 \Leftrightarrow x < \log_3 106$

f $5^x < 0.6 \Leftrightarrow x \leq \log_5 0.6$

6 a $a\log_2 7 = 3 - \log_6 14$

$$a\log_2 7 = \log_6 216 - \log_6 14$$
$$a\log_2 7 = \log_6\left(\frac{108}{7}\right)$$
$$a = \frac{\log_6\left(\frac{108}{7}\right)}{\log_2 7}$$
$$a = \frac{\ln\left(\frac{108}{7}\right)}{\ln 6} \times \frac{\ln 2}{\ln 7}$$
$$a = \frac{2.73622}{1.791759} \times \frac{0.69314}{194591}$$
$$a = 1.5271138 \times 0.356207$$
$$a = 0.544$$

b $\log_3 18 = \log_{11} k$

$$\log_{11} k = \frac{\ln 18}{\ln 3}$$
$$= 2.6309$$
$$k = 11^{2.6309}$$
$$k = 549.3$$

7 $\log_r p = q \Rightarrow p = r^q \quad (1)$

$\log_q(r) = p \Rightarrow r = q^p \quad (2)$

Raise both sides of (2) to the power q:

$r^q = (q^p)^q$

$p = q^{pq}$ (from (1))

Change to logarithm form:

$\log_q p = pq$

8 $u = \log_9 x$

a $x = 9^u$

b $\log_9(3x) = \log_9(3 \times 9^u)$

$$= \log_9 9^u + \log_9 3$$
$$= u + \frac{1}{2}$$

c $$x = 9^u$$
$$\Rightarrow \log_x x = \log_x 9^u$$
$$\Rightarrow 1 = \mathrm{u} \log_x 9$$
$$\Rightarrow \frac{1}{\mathrm{u}} = \frac{1}{2}\log_x 81$$
$$\log_x 81 = \frac{2}{u}$$

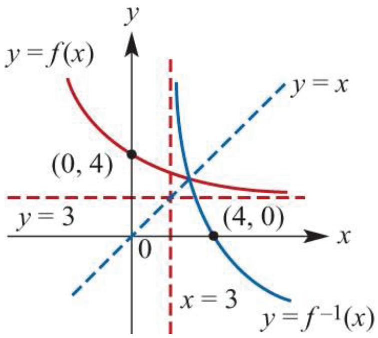
9 $$\log_5 x = 16 \log_x 5$$
$$\Rightarrow \frac{\ln x}{\ln 5} = \frac{16 \ln 5}{\ln x}$$
$$(\ln x)^2 = 16(\ln 5)^2$$
$$\ln x = \pm 4 \ln 5$$
$$x = e^{\pm \ln 625}$$
$$x = 625, \frac{1}{625}$$

10 $q^p = 25 \Rightarrow p = \log_q 25$
$$\log_5 q = \frac{\log_q q}{\log_q 5} = \frac{1}{\log_q 5}$$
$$= \frac{2}{\log_q 25} = \frac{2}{p}$$

Solutions to Exercise 5H

1 $f^{-1}: (-2, \infty) \to \mathbb{R},\ f^{-1}(x) = \log_e(x + 2)$

2



to find $f^{-1}(x)$,

$$f(x) = e^{-x} + 3$$

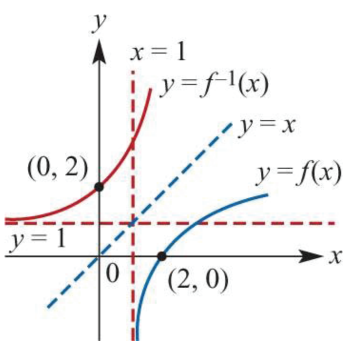
$$\therefore \quad x = e^{-f^{-1}(x)} + 3$$

$$x - 3 = e^{-f^{-1}(x)}$$

$$-f^{-1}(x) = \ln(x - 3)$$

$$f^{-1}(x) = -\ln(x - 3)$$

3



to find $f^{-1}(x)$,

$$f(x) = \ln(x - 1)$$

$$x = \ln(f^{-1}(x) - 1)$$

$$e = f^{-1}(x) - 1$$

$$f^{-1}(x) = e^x + 1$$

4 $x = e^{\frac{y+4}{3}}$

5 a

$$\text{domain}(f) = \mathbb{R}^+$$

$$\text{range}(f) = \mathbb{R}$$

$$\therefore \quad \text{domain}(f^{-1}) = \mathbb{R}$$

$$\text{range}(f^{-1}) = \mathbb{R}^+$$

$$f(x) = \ln 2x$$

$$x = \ln(2f^{-1}(x))$$

$$2f^{-1}(x) = \mathrm{e}^x$$

$$f^{-1}(x) = \frac{1}{2}e^x$$

b

$$\text{domain} f = \mathrm{R}^+$$

$$\text{range} f = \mathrm{R}$$

$$\therefore \quad \text{domain} f^{-1} = \mathrm{R}$$

$$\text{range} f^{-1} = \mathrm{R}^+$$

$$f(x) = 3\ln(2x) + 1$$

$$x = 3\ln(2f^{-1}(x)) + 1$$

$$\frac{x-1}{3} = \ln(2f^{-1}(x))$$

$$2f^{-1}(x) = e^{\frac{x-1}{3}}$$

$$f^{-1}(x) = \frac{1}{2}e^{\frac{x-1}{3}}$$

c

$$\text{domain}(f) = \mathrm{R}, \text{range(f)} = (2, \infty)$$

$$\therefore \quad \text{domain}(f^{-1}) = (2, \infty), \text{range}(\mathrm{f}^{-1}) = \mathrm{R}$$

$$f(x) = e^x + 2$$

$$\therefore \quad x = e^{f^{-1}(x)} + 2$$

$$x - 2 = e^{f^{-1}(x)}$$

$$f^{-1}(x) = \ln(x - 2)$$

d $\text{domain}(f) = \mathbb{R}, \text{range(f)} = \mathbb{R}^+$

$$\therefore \quad \text{domain}(\text{f}^{-1}) = \text{R}^+, \text{range}(\text{f}^{-1}) = \mathbb{R}$$

$$f(x) = e^{x+2}$$

$$\therefore \quad x = e^{f^{-1}(x)+2}$$

$$\ln x = f^{-1}(x) + 2$$

$$f^{-1}(x) = \ln x - 2$$

e $\text{domain}(f) = \left(-\frac{1}{2}, \infty\right),$

$$\text{range}(f) = \mathbb{R}$$

$$\therefore \quad \text{domain}(\text{f}^{-1}) = \mathbb{R},$$

$$\text{range}(f^{-1}) = \left(-\frac{1}{2}, \infty\right)$$

$$f(x) = \ln(2x + 1)$$

$$x = \ln(2f^{-1}(x) + 1)$$

$$e^x = 2f^{-1}(x) + 1$$

$$f^{-1}(x) = \frac{e^x - 1}{2}$$

f $\text{domain(f)} = \left(-\frac{2}{3}, \infty\right),$

$$\text{range(f)} = \text{R}$$

$$\therefore \quad \text{domain}(\text{f}^{-1}) = \mathbb{R},$$

$$\text{range}(f^{-1}) = \left(-\frac{2}{3}, \infty\right)$$

$$f(x) = 4\ln(3x + 2)$$

$$x = 4\ln(3f^{-1}(x) + 2)$$

$$e^{\frac{x}{4}} = 3f^{-1}(x) + 2$$

$$f^{-1}(x) = \frac{e^{\frac{x}{4}} - 2}{3}$$

g

$$\text{domain}(f) = (-1, \infty), \text{range(f)} = \text{R}$$

$$\therefore \quad \text{domain}(\text{f}^{-1}) = \mathbb{R}, \text{range}(\text{f}^{-1}) = (-1, \infty)$$

$$f(x) = \log_{10}(x + 1)$$

$$f(x) = \log_{10}(f^{-1}(x) + 1)$$

$$f^{-1}(x) + 1 = 10^x$$

$$f^{-1}(x) = 10^x - 1$$

h $\text{domain(f)} = \text{R}, \text{range(f)} = \mathbb{R}^+$

$$\therefore \quad \text{domain}(\text{f}^{-1}) = \mathbb{R}^+, \text{range}(\text{f}^{-1}) = \mathbb{R}$$

$$f(x) = 2e^{x-1}$$

$$x = 2e^{(f^{-1}(x)-1)}$$

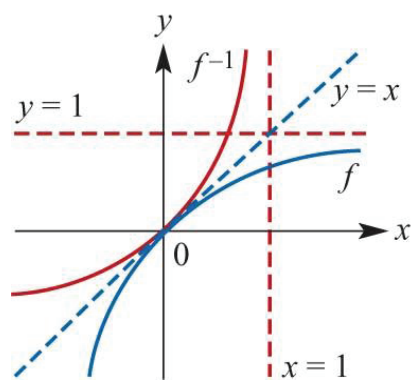
$$\frac{x}{2} = e^{(f^{-1}(x)-1)}$$

$$f^{-1}(x) - 1 = \ln\left(\frac{x}{2}\right)$$

$$f^{-1}(x) = \ln\left(\frac{x}{2}\right) + 1$$

6

a c



b $\text{range(f)} = (-\infty, 1)$

$$\therefore \quad \text{domain}(\text{f}^{-1}) = (-\infty, 1)$$

$$f(x) = 1 - e^{-x}$$

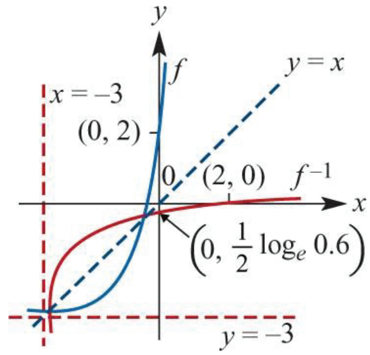
$$x = 1 - e^{-f^{-1}(x)}$$

$$-f^{-1}(x) = \ln(1 - x)$$

$$f^{-1}(x) = -\ln(1 - x)$$

7

a c

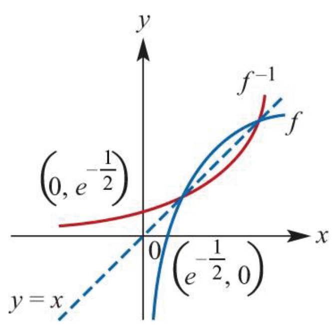


b
$$f(x) = 5e^{2x} - 3$$
$$x = 5e^{2f^{-1}(x)} - 3$$
$$\frac{x+3}{5} = e^{2f^{-1}(x)}$$
$$2f^{-1}(x) = \ln\left(\frac{x+3}{5}\right)$$
$$f^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+3}{5}\right)$$
domain $= (-3, \infty)$

$\therefore \quad f^{-1} : (-3, \infty) \to \mathbb{R}$,
$f^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+3}{5}\right)$

8

a c



b
$$f(x) = 2\ln x + 1$$
$$x = 2\ln f^{-1}(x) + 1$$
$$\frac{x-1}{2} = \ln f^{-1}(x)$$
$$f^{-1}(x) = e^{\left(\frac{x-1}{2}\right)}$$
range (f^{-1}) = domain$(f) = \mathbb{R}^+$

9 $t = \dfrac{-1}{k}\log_e\left(\dfrac{P-b}{A}\right)$

10 a $\dfrac{y-5}{2} = \ln x$
$$x = e^{\left(\frac{y-5}{2}\right)}$$

b $\dfrac{P}{\text{A}} = e^{-6x}$
$$-6x = \ln\left(\frac{P}{A}\right)$$
$$x = -\frac{1}{6}\ln\left(\frac{P}{A}\right)$$

c $\dfrac{y}{a} = x^n$
$$n = \log_x\left(\frac{y}{a}\right) = \frac{\ln\left(\frac{y}{a}\right)}{\ln x}$$

d $10^x = \dfrac{y}{5}$
$$x = \log_{10}\left(\frac{y}{5}\right)$$

e $\ln(2x) = \dfrac{5-y}{3}$
$$2x = e^{\left(\frac{5-y}{3}\right)}$$
$$x = \frac{1}{2}e^{\left(\frac{5-y}{3}\right)}$$

f $\dfrac{y}{6} = x^{2n}$

$2n = \log_x\left(\dfrac{y}{6}\right)$

$n = \dfrac{1}{2}\log_x\left(\dfrac{y}{6}\right) = \dfrac{1}{2}\left(\dfrac{\ln\left(\frac{y}{6}\right)}{\ln x}\right)$

$y = \ln(2x - 1)$

g $2x - 1 = e^y$

$x = \dfrac{e^y + 1}{2}$

h $y = 5(1 - e^{-x})$

$e^{-x} = 1 - \dfrac{y}{5}$

$-x = \ln\left(\dfrac{5 - y}{5}\right)$

$x = -\ln\left(\dfrac{5 - y}{5}\right) = \ln\left(\dfrac{5}{5 - y}\right)$

11 a $f(x) = 2e^x - 4$

$\dfrac{x + 4}{2} = e^{f^{-1}(x)}$

$f^{-1}(x) = \ln\left(\dfrac{x + 4}{2}\right)$

b using the CAS calculator
(0.895, 0.895), (−3.962, −3.962)

12 a $f(x) = 2\ln(x + 3) + 4$

$x = 2\ln(f^{-1}(x) + 3) + 4$

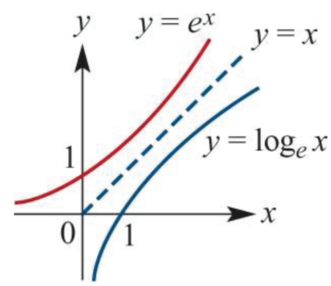
$\left(\dfrac{x - 4}{2}\right) = \ln(f^{-1}(x) + 3)$

$f^{-1}(x) + 3 = e^{\left(\frac{x-4}{2}\right)}$

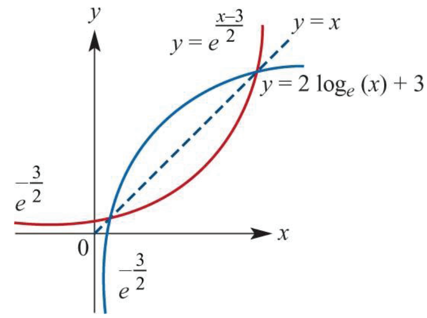
$f^{-1}(x) = e^{\left(\frac{x-4}{2}\right)} - 3$

b using the CAS calculator
(8.964, 8.964), (−2.969, −2.969)

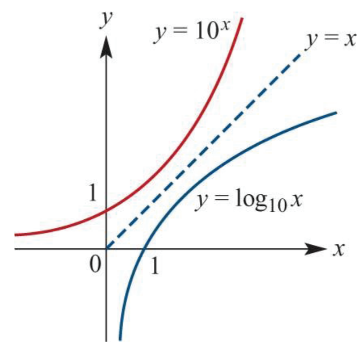
13 a i



ii



iii



b $f(x)$ and $g(x)$ are inverse functions

Solutions to Exercise 5I

1 **a** $N = 1000 \times 2^{\frac{t}{15}}$

b 50 minutes

2 $d = d_o 10^{mt}$

When $t = 1$, $d = 52$ cm

When $t = 3$, $d = 80$

Consider the equations

$52 = d_o 10^m \quad \ldots(1)$

$80 = d_o 10^{3m} \ldots(2)$

Divide (2) by (1)

$$\frac{80}{52} = 10^{2m}$$

$$2m = \log_{10}\left(\frac{20}{13}\right)$$

$$m = \frac{1}{2}\log_{10}\left(\frac{20}{13}\right) \approx 0.094$$

Substitute in (1)

$$52 = d_0 10^{\frac{1}{2}\log_{10}\left(\frac{20}{13}\right)}$$

$$\therefore \quad 52 = d_0 10^{\log_{10}\left(\frac{20}{13}\right)^{\frac{1}{2}}}$$

Hence $52 = \left(\frac{20}{13}\right)^2 1/2 d_0$

and $d_0 = \left(\frac{13}{20}\right)^{\frac{1}{2}} \times 52$

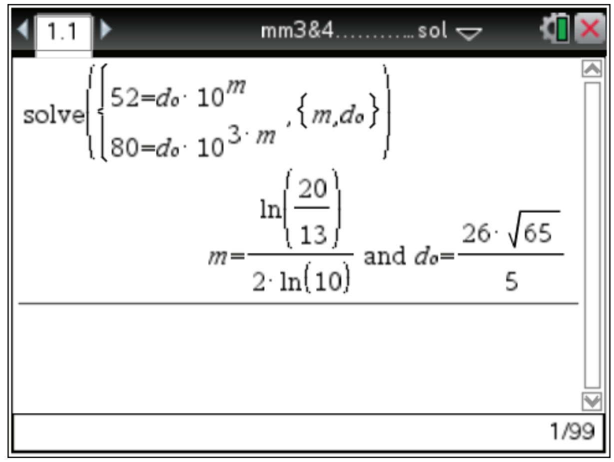
$\therefore \quad m \approx 0.094$ and $d_0 \approx 41.9237$

Graphic calculator techniques for question

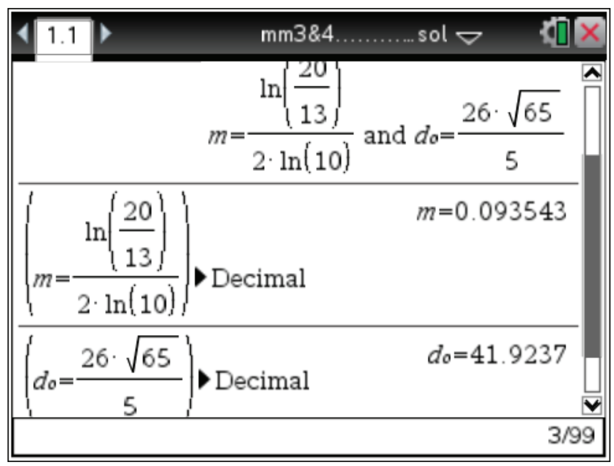
In a **Calculator** page use:

b>**Algebra>Solve System of Equations>Solve System of Equations** and enter as shown opposite.

Hint: do can be entered using a template from t, otherwise just use d0.



Approximate the solutions using b>**Number>Convert to Decimal.**



3 **a** $N = N_0 e^{kt}$

i When $t = 0, N = 20\,000$

$\therefore \quad 20\,000 = N_0 e^0$

i.e. $\quad N_0 = 20\,000$

ii $N = 20\,000e^{kt}$

When $t = 1$, $N = 20\,000$and20% of $20\,000 = 16\,000$

$\therefore \quad 16\,000 = 20\,000 \times e^k$

$\therefore \quad e^k = 0.8$

$\therefore \quad k = \log_e(0.8) \approx -0.223$

b When $N = 5000$

$$5000 = 20\,000e^{\log_e(08)t}$$

$$\therefore \quad 0.25 = 0.8^t$$

$$\text{and } t = \frac{\log_e(0.25)}{\log_e(0.8)}$$

$$\approx 6.2126$$

It takes about 6.2 years for there to be 5000 people infected.

4 $M = M_0e^{-kt}$

When $t = 0, M = 10$

When $t = 140, M = 5$

a $$10 = M_0e^0$$

$$\therefore \quad M_0 = 10$$

Also $5 = 10e^{-140k}$

$$0.5 = e^{-140k}$$

$$k = \frac{-1}{140}\log_e(0.5)$$

$$= \frac{1}{140}\log_e(2) \approx 0.00495$$

$$= 4.95 \times 10^{-3}$$

b When $t = 70$ $M = 10e^{\frac{-1}{140}\log_e(2)\times 70}$

$$= 10e^{-05\log_e 2}$$

$$= 10 \times 2^{-0.5}$$

$$\approx 7.0711$$

The mass is 7.07 g after 70 days.

c When $M = 2$ $\quad 2 = 10e^{\frac{-1}{140}(\log_e 2)t}$

$$\therefore \quad 0.2 = 2^{\frac{t}{140}}$$

$$\therefore \quad t = -140\frac{\log_e(0.2)}{\log_e(2)} \approx 325.07$$

After 325 days the mass remaining is 2g.

5 **a** $A(t = 1690) = \frac{1}{2}A_0$

$$A_0e^{-1690k} = \frac{1}{2}A_0$$

$$2 = e^{1690k}$$

$$\log_e 2 = 1690k$$

$$k = \frac{\log_e 2}{1690}$$

b $A = 0.2A_0$

$$A_0e^{-\frac{\log_e 2}{1690}t} = \frac{1}{5}A_0$$

$$5 = e^{\frac{\log_e 2}{1690}t}$$

$$\log_e 5 = \frac{\log_e 2}{1690}t$$

$$t = 1690\frac{\log_e 5}{\log_e 2}$$

$$= 3924$$

6 $A = A_0e^{kt}$

When $t = 0, A = 20$

$\therefore A_0 = 20$

Half life is 24 000 years.

$\therefore 10 = 20e^{24000k}$

$\therefore k = \frac{1}{24000}\log_e(\frac{1}{2})$

When does 20% remain?

$20e^{kt} = 4$

$e^{kt} = \frac{1}{5}$

$t = \frac{1}{k}\log_e(\frac{1}{5})$

$t \approx 55726$ years

7 $A = A_0e^{kt}$

$A = \frac{1}{2}A_0$ when $t = 5730$

$\therefore \frac{1}{2} = e^{5730k}$

$\therefore k = \frac{1}{5730}\log_e(\frac{1}{2})$

When does 40% remain?

$e^{kt} = 0.4$

$e^{kt} = \frac{2}{5}$

$t = \frac{1}{k}\log_e(\frac{2}{5})$

$t \approx 7575$ years

8 $P = P_0e^{kt}$

When $t = 0, P = 10000$

$\therefore P_0 = 10000$

$A = 15000$ when $t = 13$

$\therefore \frac{3}{2} = e^{13k}$

$\therefore k = \frac{1}{13}\log_e(\frac{3}{2})$

$P = 10000e^{kt}$

a When $t = 16$

$P = 10000e^{16k}$

$\therefore P = 16471$

b $30000 = 10000e^{kt}$

$\log_e 3 = kt$

$t = \frac{1}{k}\log_e(3)$

$t \approx 35$

9 $C = C_0(1.12)^n$

$M = M_0(0.94)^n$

$M_0 = 5C_0$

$\therefore M = 5C_0(0.94)^n$

$C > 5M \Leftrightarrow (1.12)^n > 25(0.94)^n$

This happens after approximately 18.4 years.

10 $P(h) = 1000 \times 10^{-0.05428h}$

a 607 millibars

b 6.389 km

11 $P = 500000(1.1)^n$

$4000000 = 500000(1.1)^n$

$8 = (1.1)^n$

$n \approx 21.82$

12 $T = T_0e^{-kt}$

When $t = 0, T = 100$

$\therefore T_0 = 100$

$T = 40$ when $t = 5$

$\therefore \frac{2}{5} = e^{-5k}$

$\therefore k = -\frac{1}{5}\log_e(\frac{2}{5})$

$T = 100e^{-kt}$

When $t = 15$

$T = 100e^{-15k}$

$\therefore T = 6.4$

13 $k = 0.349,\ N_0 = 50.25$

14 **a** $k = \log_e\left(\frac{5}{4}\right)$

b 7.21 hours

15 **a** $a = 1000,\ b = 15^{\frac{1}{5}}$

b 3 hours

c 13 hours

d 664 690

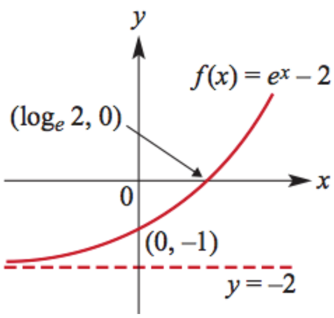
Solutions to Technology-free questions

1 **a** $y = e^x - 2$

$x = 0$: $y = -1$

$y = 0$: $e^x = 2 \Rightarrow x = \log_e 2$

asymptote: $y = -2$

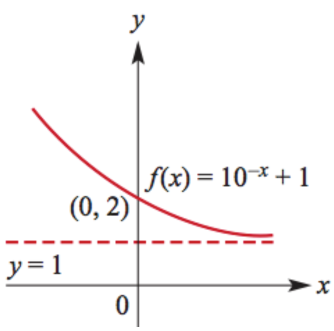


b $y = 10^{-x} + 1$

$x = 0$: $y = 2$

no x intercepts as $y > 1$

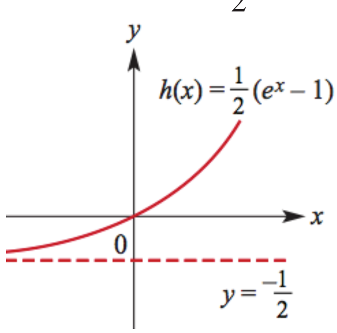
asymptote: $y = 1$



c $y = \frac{1}{2}(e^x - 1)$

$x = 0$: $y = 0$

asymptote: $y = -\frac{1}{2}$

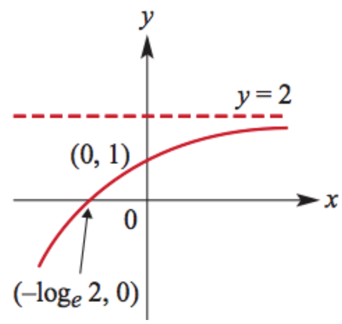


d $y = 2 - e^{-x}$

$x = 0$: $y = 1$

$y = 0$: $e^{-x} = 2 \Rightarrow x = -\log_e 2$

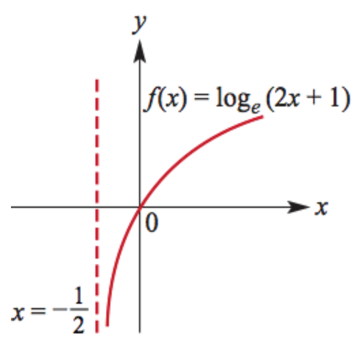
asymptote: $y = 2$



e $y = \log_e(2x + 1)$

$x = 0$: $y = 0$

asymptote: $x = -\frac{1}{2}$

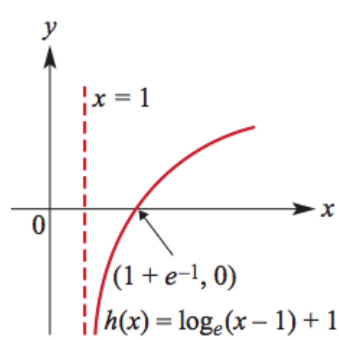


f $y = \log_e(x - 1) + 1$

no y intercepts as $x > 1$

$y = 0$: $\log_e(x-1) = -1 \Rightarrow x = 1 + e^{-1}$

asymptote: $x = 1$

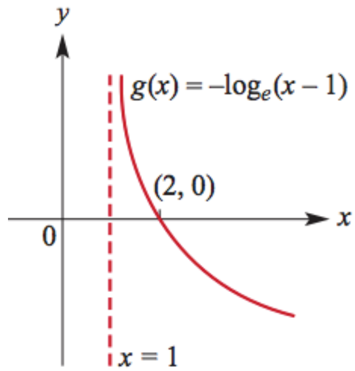


g $y = -\log_e(x-1)$

no y intercepts as $x > 1$

$y = 0$: $-\log_e(x-1) = 0 \Rightarrow x = 2$

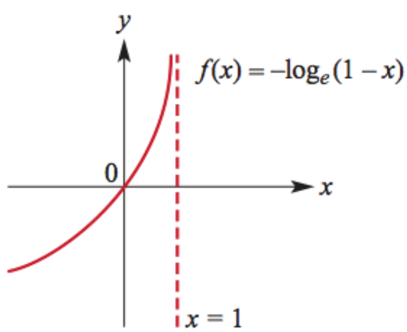
asymptote: $x = 1$



h $y = -\log_e(1-x)$

$x = 0$: $y = 0$

asymptote: $x = 1$



2 **a** $f(x) = e^{2x} - 1$

domain $= \mathbb{R}$, range $= (-1, \infty)$

The domain of f^{-1} is $(-1, \infty)$.

Interchange x and y and solve for y:

$$x = e^{2y} - 1$$

$$e^{2y} = x + 1$$

$$2y = \log_e(x+1)$$

$$y = \frac{1}{2}\log_e(x+1)$$

$f^{-1}: (-1, \infty) \to \mathbb{R}$, $f^{-1}(x) = \frac{1}{2}\log_e(x+1)$

b $f(x) = 3\log_e(x-2)$

domain $= (2, \infty)$, range $= \mathbb{R}$

The domain of f^{-1} is $\mathbb{R}$.

Interchange x and y and solve for y:

$$x = 3\log_e(y-2)$$

$$\log_e(y-2) = \frac{x}{3}$$

$$y = e^{\frac{x}{3}} + 2$$

$f^{-1}: \mathbb{R} \to \mathbb{R}$, $f^{-1}(x) = e^{\frac{x}{3}} + 2$

c $f(x) = \log_{10}(x+1)$

domain $= (-1, \infty)$, range $= \mathbb{R}$

The domain of f^{-1} is $\mathbb{R}$.

Interchange x and y and solve for y:

$$x = \log_{10}(y+1)$$

$$y = 10^x - 1$$

$f^{-1}: \mathbb{R} \to \mathbb{R}$, $f^{-1}(x) = 10^x - 1$

d $f(x) = 2^x + 1$

domain $= \mathbb{R}^+$, range $= (2, \infty)$

The domain of f^{-1} is $(2, \infty)$.

Interchange x and y and solve for y:

$$x = 2^y + 1$$

$$2^y = x - 1$$

$$y = \log_2(x-1)$$

$f^{-1}: (2, \infty) \to \mathbb{R}$, $f^{-1}(x) = \log_2(x-1)$

3 **a**
$$\begin{aligned}\log_e y &= \log_e(x) + 2\\ &= \log_e(x) + \log_e(e^2)\\ &= \log_e(e^2 x)\\ y &= e^2 x\end{aligned}$$

b
$$\begin{aligned}\log_{10} y &= \log_{10} x + 1\\ &= \log_{10} x + \log_{10} 10\\ &= \log_{10} 10x\\ y &= 10x\end{aligned}$$

c $\log_2 y = 3\log_2 x + 4$

$= \log_2 x^3 + \log_2 2^4$

$= \log_2 16x^3$

$y = 16x^3$

d $\log_{10} y = -1 + 5\log_{10} x$

$= -\log_{10} 10 + \log_{10} x^5$

$= \log_{10} \dfrac{x^5}{10}$

$y = \dfrac{x^5}{10}$

e $\log_e y = 3 - \log_e x$

$= \log_e e^3 - \log_e x$

$= \log_e \dfrac{e^3}{x}$

$y = \dfrac{e^3}{x}$

f $\log_e y = 2x - 3$

$y = e^{2x-3}$

4 a $3^x = 11$

$x = \log_3 11$

$x = \dfrac{\log_e 11}{\log_e 3}$ by change of base

(Alternatively, take logarithms to base e of both sides and simplify, as in part **c** below.)

b $2^x = 0.8$

$x = \log_2(0.8)$

$= \dfrac{\log_e(0.8)}{\log_e 2}$ by change of base

c
$$2^x = 3^{x+1}$$
$$\log_e 2^x = \log_e 3^{x+1}$$
$$x\log_e 2 = (x+1)\log_e 3$$
$$x\log_e 2 - x\log_e 3 = \log_e 3$$
$$x(\log_e 2 - \log_e 3) = \log_e 3$$
$$x = \frac{\log_e 3}{\log_e 2 - \log_e 3}$$
$$= \frac{\log_e 3}{\log_e\left(\frac{2}{3}\right)}$$

5 a
$$2^{2x} - 2^x - 2 = 0$$
$$(2^x)^2 - 2^x - 2 = 0$$
$$(2^x - 2)(2^x + 1) = 0$$
$$2^x = 2, -1$$

But $2^x > 0$ for all real x, so the only solution is given by $2^x = 2$, i.e. $x = 1$.

b $\log_e(3x - 1) = 0$

$3x - 1 = 1$

$3x = 2$

$x = \dfrac{2}{3}$

c $\log_{10}(2x) + 1 = 0$

$\log_{10}(2x) = -1$

$2x = 10^{-1}$

$= \dfrac{1}{10}$

$x = \dfrac{1}{20}$

d
$$\begin{aligned}10^{2x} - 7 \times 10^x + 12 &= 0\\ (10^x)^2 - 7 \times 10^x + 12 &= 0\\ (10^x - 3)(10^x - 4) &= 0\\ 10^x &= 3, 4\\ x &= \log_{10} 3, \log_{10} 4\end{aligned}$$

6 $y = 3\log_2(x + 1) + 2$

$x = 0$: $y = 3\log_2 1 + 2 = 2$
y intercept: $(0, 2)$, so $b = 2$.
$y = 0$: $3\log_2(x + 1) + 2 = 0$
Solving for x:
$$\begin{aligned}3\log_2(x + 1) &= -2\\ \log_2(x + 1) &= -\frac{2}{3}\\ x + 1 &= 2^{-\frac{2}{3}}\\ x &= 2^{-\frac{2}{3}} - 1\end{aligned}$$
x intercept: $\left(2^{-\frac{2}{3}} - 1, 0\right)$ so $a = 2^{-\frac{2}{3}} - 1$.

7 $f(k) = 5\log_{10}(k + 1) = 6$, so solving for k:
$$\begin{aligned}5\log 10(k + 1) &= 6\\ \log 10(k + 1) &= \frac{6}{5}\\ k + 1 &= 10^{\frac{6}{5}}\\ k &= 10^{\frac{6}{5}} - 1\end{aligned}$$

8
$$\begin{aligned}4e^{3x} &= 287\\ e^{3x} &= \frac{287}{4}\\ 3x &= \log_e\left(\frac{287}{4}\right)\\ x &= \frac{1}{3}\log_e\left(\frac{287}{4}\right)\end{aligned}$$

9
$$\begin{aligned}3\log_a x &= 3 + \log_a 8\\ &= 3 + \log_a 2^3\\ &= 3 + 3\log_a 2\\ &= 3(1 + \log_a 2)\\ \log_a x &= 1 + \log_a 2\\ &= \log_a a + \log_a 2\\ &= \log_a 2a\\ x &= 2a\end{aligned}$$

10 The range of f is the range of a complete log function, which is $\mathbb{R}$. So the domain of f^{-1} is $\mathbb{R}$.

11 $y = f(x) = e^{2x} - 3ke^x + 5$
$(0, 0)$: $1 - 3k + 5 = 0$, so $k = 2$
Hence $y = e^{2x} - 6e^x + 5$.
$x \to -\infty$, $e^{2x} - 6e^x + 5 \to 0 + 0 + 5 = 5$, so the horizontal asymptote is $y = 5$ and therefore $b = 5$.
Now find when $y = 0$, i.e.
$$\begin{aligned}e^{2x} - 6e^x + 5 &= 0\\ (e^x - 1)(e^x - 5) &= 0\\ e^x &= 1, 5\\ x &= 0, \log_e 5\end{aligned}$$
$x = 0$ corresponds to the intercept $(0, 0)$, so $x = \log_e 5$ corresponds to the intercept $(a, 0)$. Thus $a = \log_e 5$.

12 **a** $f^{-1}(x) = \frac{1}{3}\log_e(x + 4)$,
dom $f^{-1} = (-4, \infty)$

b $\dfrac{1}{3x + 4} - 4$

13 **a** $f(-x) = f(x)$

b $2(e^u + e^{-u})$

c 0

d $e^{2u} + e^{-2u}$

e $g(-x) = -g(x)$

f $2e^x$, $2e^{-x}$, $e^{2x} - e^{-2x}$

Solutions to multiple-choice questions

1 C $4\log_b x^2 = \log_b 16 + 8$

$= \log_b 2^4 + 8$

$= 4\log_b 2 + 8$

$$4\log_b \frac{(x)^2}{2} = 8$$

$$\log_b \frac{x^2}{2} = 2$$

$$\frac{x^2}{2} = b^2$$

$$x^2 = 2b^2$$

$$x = \pm\sqrt{2}b$$

2 D $\log_e 4e^{3x}$

$= \log_e 4 + \log_e e^{3x}$

$= \log_e 4 + 3x$

3 B $3\log_3(x-4)$

$= x - 4$

4 E The Functions g and h here the same domain of $R\backslash\{-1\}$, so $B = C$. It follows that either option **D** or **E** Must be true.
Now range $(g) = R\backslash\{0\}$.
Using a CAS calculator to plot the graph of h shows that range $(h) \neq R\backslash\{0\}$.

5 A As $x = 5$

$\log_{10}(5k-3) = 2$

$$5k - 3 = 10^2$$

$$5k = 103$$

$$k = \frac{103}{5}$$

6 C $3^{4\log_3 x + \log_3 4x}$

$= 3^{\log_3 x^4 + \log_3 4x}$

$= 3^{\log_3 4x^5}$

$= 4x^5$

7 B Using the 'solve' command CAS calculator gives $x = 0.2755\ldots$, so $x \approx 0.28$.

8 A The graph is translated 3 units in the negative direction of the y axis

$\therefore\ b = -3$

When $x = 0, y = 0$

$\therefore\ 0 = ae^0 - 3$

$0 = a - 3$

$a = 3$

9 C $f: R^+ \to R, f(x) = \log_5 x$

$(5, 0)$

$0 \neq \log_5 5$

$0 \neq 1$

The graph does not pass through the point (5,0).

10 D $3\log_2 x - 7\log_2(x-1) = 2 + \log_2 y$

$$\log_2 \frac{x^3}{(x-1)^7} = 2 + \log_2 y$$

$$\log_2 \frac{x^3}{(x-1)^7} - \log_2 y = 2$$

$$\log_2 \frac{x^3}{y(x-1)^7} = 2$$

$$\frac{x^3}{y(x-1)^7} = 2^2$$

$$y = \frac{x^3}{4(x-1)^7}$$

11 A

12	C	**15**	B
13	C	**16**	D
14	D		

Solutions to extended-response questions

1 The temperature, T°C, of a liquid x minutes after it begins to cool is given by $T = 90(0.98)^x$

a When $x = 10$

$$T = 90(0.98)^{10} = 73.5366$$

b When $T = 27$

$$27 = 90(0.98)^x$$

$$\frac{27}{90} = 10.98^x$$

$$0.3 = 0.98^x$$

$$\therefore \quad \log_e(0.3) = x\log_e(0.98)$$

$$\therefore \quad x = \frac{\log_e(0.3)}{\log_e(0.98)} = 59.5946$$

2 Let P denote the population of the village in years after 1800.

$P = 240(1.06)^n$

When $n = 0$, $P = 240$

a When $n = 20$

$P = 240(1.06)^{20} = 769.71$

At the beginning of 1820 the population is approximately 770.

b If $P = 2500$

$$2500 = 240(1.06)^n$$

$$\frac{2500}{240} = (1.06)^n$$

i.e. $\dfrac{125}{12} = (1.06)^n$

Taking logarithms of both sides

$$\log_e\left(\frac{125}{12}\right) = n\log(1.06)$$

$$\therefore \quad n = \frac{\log_e\left(\frac{125}{12}\right)}{\log_e(1.06)} = 40.217$$

The population will reach 2500 in the year 1840.

3 $V = ke^{-\lambda t}$

a as $V = 22\,497$ when $t = 0$

$k = 22\,497$

After one year the value of the car is \$18 000

$\therefore$ Take logarithms, base e of both sides.

$\log_e 18\,000 = \log_e(22\,497)\lambda$

$$\therefore \quad \lambda = \log_e\left(\frac{22\,497}{18\,000}\right)$$

$$\approx 0.223$$

$$\approx 0.22 \text{ (correct to two decimal places)}$$

b $V = 22\,497e^{-0.22\times 3}$

when $t = 3$

$V = 22\,497e^{-0.22\times 3}$

$= 11\,627.60$

The value is \$11 627.6 after 3 years. (This is obtained by taking $\lambda = 0.22$)

4 \$$M$ is the value of a particular house in a certain area t years after January lst 1988.

a It is given that $M = Ae^{-pt}$

and when $t = 0$, $M = \$65\,000$

$\therefore$ $A = 65\,000$

Furthermore when $t = 1$, $M = 61\,000$

$\therefore \quad 61\,000 = 65\,000e^{-p}$

$$\therefore \quad \frac{61}{65} = e^{-p}$$

$$\text{and} \quad -p = \log_e\left(\frac{61}{65}\right)$$

$$\text{i.e.} \quad p = \log_e\left(\frac{65}{61}\right)$$

$$p = 0.635$$

$\therefore$ $A = 65\,000$ and $p = 0.064$ to two significant figures.

b $M = 65\,000e^{-pt}$

When $t = 5$

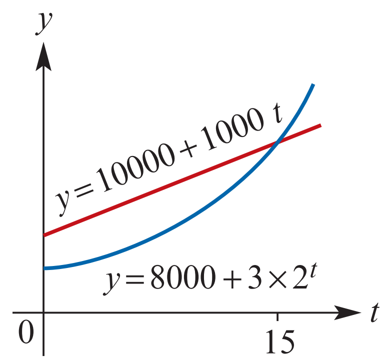
$M = 65\,000e^{-5p}$

$= 47\,199.687$

To the nearest hundred the value is \$47 200

5 **a** $N_A(t) = 10\,000 + 1\,000t$

$N_C(t) = 8\,000 + 3 \times 2^t$



b **i** Using **intersect** from the **CALC** menu the point of intersection of the two graphs has coordinates (12.21, 22209.62)

ii $t = 12.21$ i.e. on Jan 13

iii 22 210

c **i**

$$10\,000 + 100t = 8000 + 3 \times 2^t$$

$$\therefore \quad 2000 + 1000t = 3 \times 2^t$$

$$\therefore \quad \frac{2000 + 1000t}{3} = 2^t$$

$$\therefore \quad \log_{10}\left(\frac{2000 + 1000t}{3}\right) = t\log_{10} 2$$

$$\therefore \quad \log_{10} 1000 + \log_{10}\left(\frac{2+t}{3}\right) = t\log_{10} 2$$

$$\therefore \quad t = \frac{1}{\log_{10} 2}\left(3 + \log_{10}\left(\frac{2+t}{3}\right)\right)$$

ii (12.21, 12.21) is found by

d

$$N_c(15) = N_A(15)$$

$$\therefore \quad 8000 + c \times 2^{15} = 10\,000 + 1000 \times 15$$

$$\therefore \quad c \times 2^{15} = 17\,000$$

$$\therefore \quad c = 0.52$$

6 $n = A(1 - e^{-Bt})$

a **i** When $t = 2, n = 10\,000$ and when $t = 4, n = 15\,000$

$$10\,000 = A(1 - e^{-2B}) \qquad (1)$$

$$\text{and } 15\,000 = A(1 - e^{-4B}) \qquad (2)$$

Divide (2) by (1)

$$\frac{3}{2} = \frac{A(1 - e^{-4B})}{A(1 - e^{-2B})}$$

$$\therefore \quad 3(1 - e^{-2B}) = 2(1 - e^{-4B})$$

$$\therefore \quad 3 - 3e^{-2B} = 2 - 2e^{-4B}$$

$$\therefore \quad 1 + 2e^{-4B} - 3e^{-2B} = 0$$

ii Let $a = e^{-2B}$

Then $1 + 2a^2 - 3a = 0$

i.e. $2a^2 - 3a + 1 = 0$

iii $\therefore (2a - 1)(a - 1) = 0$

$\therefore a = \frac{1}{2}$ or $a = 1$

iv $\therefore e^{-2B} = \frac{1}{2}$ or $e^{-2B} = 1$

$\therefore -2B = \log_e\left(\frac{1}{2}\right)$ or $-2B = 0$

$\therefore B = \frac{1}{2}\log_e 2$ or $B = 0$, and then $A \in R^+$ and $n = 0$ for any A.

v Substitute in (1)

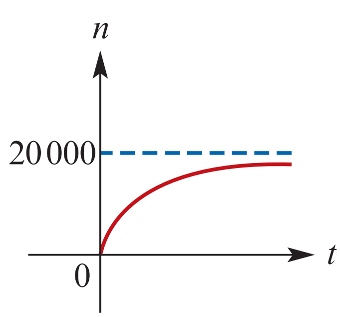
$$10\,000 = A\left(1 - e^{-\log_e 2}\right)$$

$$10\,000 = A\left(1 - e^{\log_e \frac{1}{2}}\right)$$

$$10\,000 = A\left(\frac{1}{2}\right)$$

$$\therefore \quad A = 20\,000$$

b



c $18\,000 = 20\,000\left(1 - e^{\left(-\frac{1}{2}\log_e 2\right)}\right)$

$18\,000 = 20\,000\left(1 - 2^{-\frac{1}{2}}\right)$

$$\therefore \quad \frac{9}{10} = 1 - 2^{-\frac{1}{2}}$$

$$\therefore \quad 2^{-\frac{1}{2}} = 0.1$$

$$-\frac{t}{2}\log_e 2 = \log_e 0.1$$

$$\therefore \quad t = \frac{2\log_e 10}{\log_e 2} \approx 6.644$$

After 6.65 hours the population is 18 000

7 $P = 75(10^{-0.15h})$

a When $h = 0, P = 75$

The barometric pressure is 75 cm of mercury when $h = 0$.

b When $h = 10, P = 75 \times 10^{-1.5} = 2.3717$

The barometric pressure is 2.37 cm when $h = 10$.

c When $P = 60$

$$60 = 75 \times 10^{-0.15h}$$

$$\therefore \quad 0.8 = 10^{-0.15h}$$

$$\therefore \quad \log_{10}(0.8) = -0.15h$$

$$\therefore \quad h = \frac{-1}{0.15}\log_{10}(0.8)$$

$$= 0.646 \text{ km}$$

The barometric pressure is 60 cm of mercury then $h = 0.646$.

8 $A = A_0 e^{kt}$

When $t = 1$, $a = 60.7$

When $t = 6$, $a = 5$

Consider the equations

$$60.7 = A_0 e^{kt} \qquad (1)$$

$$5 = A_0 e^{kt} \qquad (2)$$

Divide 2 by 1

$$\frac{50}{607} = e^{5k}$$

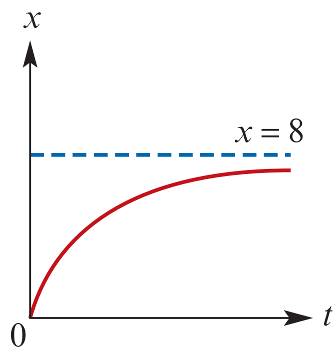
$$\therefore \quad k = \frac{1}{5}\log_e\left(\frac{50}{607}\right) \approx -0.4993 \approx -0.5$$

Substitute in (1)

$$60.7 = A_0\left(\frac{50}{607}\right)^{\frac{1}{5}}$$

$$\therefore \quad A_0 = 60.7 \times \left(\frac{607}{50}\right)^{\frac{1}{5}} \approx 100.007 \approx 100$$

9 **a**



Note: When $t = 0, x = 8(1 - 1) = 0$
As $t \to \infty, e^{-0.2t} \to 0 \therefore x \to 8$

b **i** When $t = 0, x = 8(1 - 1) = 0$ Amount reacted after 0 min is 0 gram

ii When $t = 2, x = 8(1 - e^{-04}) \approx 2.64$ Amount reacted after 2 min is $\approx$ 2.64 gram

iii When $t = 10, x = 8(1 - e^{-2}) \approx 6.92$ Amount reacted after 10 min is $\approx$ 6.92 gram

c When $x = 7, 7 = 8(1 - e^{-02\times t})$

$$0.875 = 1 - e^{-0.2t}$$

$$e^{-0.2t} = 0.125$$

$$-0.2t = \log_e(0.125)$$

$$t = -5\log_e(0.125)$$

$$= 5\log_e 8$$

$$\approx 10.397$$

After 10.4 minutes there is 7 g of the substance which has reacted.

10 $T - T_s = (T_0 - T_s)e^{-kt}$

$$T_s = 15^\circ$$

$$T_0 = 96^\circ$$

a When $t = 5, T = 40$

$$\therefore \quad 40 - 15 = (96 - 15)e^{-5k}$$

$$25 = 81e^{-5k}$$

$$e^{-5k} = \frac{25}{81}$$

$$-5k = \log_e \frac{25}{81}$$

$$k = -\frac{1}{5}\log_e \frac{25}{81}$$

$$\approx 0.235$$

b When $t = 10$

$$T - 15 = (96 - 15)e^{\frac{1}{5}\left(\log_e \frac{25}{81}\right)\times 10}$$

$$\text{i.e. } T - 15 = 81 \times \left(\frac{25}{81}\right)^2$$

$$T = 22.716$$

The temperature of the egg is 22.7°C when $t = 10$.

c When $T = 30$

$$30 - 15 = (96 - 15)e^{\frac{1}{5}\left(\log_e \frac{25}{81}\right)}$$

$$\frac{15}{81} = \left(\frac{25}{81}\right)^{\frac{t}{5}}$$

$$\text{i.e. } \frac{5}{27} = \left(\frac{25}{81}\right)^{\frac{t}{5}}$$

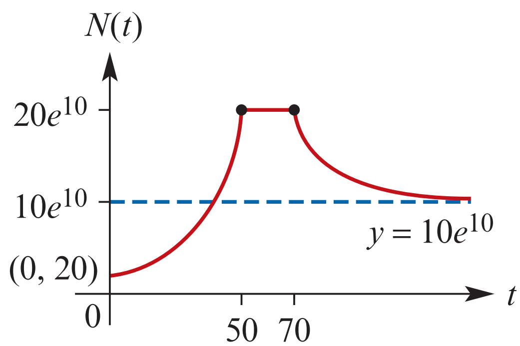
$$\therefore \quad \frac{\log_e\left(\frac{5}{27}\right)}{\log_e\left(\frac{25}{81}\right)} = \frac{t}{5}$$

$$\therefore \quad t \approx 7.17$$

The egg reaches a temperature of 30°C after 7.17 minutes.

11 $N(t) = \begin{cases} 20e^{0.2t} & 0 \le t \le 50 \\ 20e^{10} & 50 < t \le 70 \\ 10e^{10}(e^{70-t} + 1) & t > 70 \end{cases}$

a



b **i** $N(10) = 20e^{0.2\times10} \quad (0 \le t \le 50)$

$= 20e^2$

≈ 147.78

ii $N(40) = 20e^{0.2\times40} \quad (0 \le t \le 50)$

$= 20e^8$

$\approx 59\,619.16$

iii $N(60) = 20e^{10} \quad (50 < t \le 70)$

$\approx 440\,529.32$

iv $N(80) = 10e^{10}(e^{70-80} + 1)(t > 70)$

$= 10\mathrm{e}^{10}(e^{-10} + 1)$

$= 10(1 + e^{10})$

$\approx 220\,274.66$

c **i** Considering the graph

$N = 2968$ for $0 \le t \le 50$

$\therefore \quad 2968 = 20.e^{0.2t}$

$148.4 = e^{0.2t}$

$\therefore \quad t = 5\log_e(148.4)$

$= 24.99955$

After 25 days the population is 2968.

ii For $N = 21\,932$, $0 \le t \le 50$. This can be seen from the graph above.

$$21\,932 = 20.e^{0.2\mathrm{t}}$$

$$1096.6 = e^{0.2t}$$

$$\therefore \quad t = 5\log_e(1096.6)$$

$$\approx 34.9998$$

After 35 days the population is 21932.

Chapter 6 – Circular functions

Solutions to Exercise 6A

1 a $50^\circ = \dfrac{50}{180}\pi$
$= \dfrac{5\pi}{18}$

b $136^\circ = \dfrac{136}{180}\pi$
$= \dfrac{34\pi}{45}$

c $250^\circ = \dfrac{250}{180}\pi$
$= \dfrac{25\pi}{18}$

d $340^\circ = \dfrac{340}{180}\pi$
$= \dfrac{17\pi}{9}$

e $420^\circ = \dfrac{420}{180}\pi$
$= \dfrac{7\pi}{3}$

f $490^\circ = \dfrac{490}{180}\pi$
$= \dfrac{49\pi}{18}$

2 a $\dfrac{\pi}{3} = \dfrac{180^\circ}{3} = 60^\circ$

b $\dfrac{5\pi}{6} = 180^\circ \times \dfrac{5}{6} = 150^\circ$

c $\dfrac{4\pi}{3} = 180^\circ \times \dfrac{4}{3} = 240^\circ$

d $\dfrac{7\pi}{9} = 180^\circ \times \dfrac{7}{9} = 140^\circ$

e $3.5\pi = \dfrac{7\pi}{2} = \dfrac{7}{2} \times 180^\circ = 630^\circ$

f $\dfrac{7\pi}{5} = \dfrac{7}{5} \times 180^\circ = 252^\circ$

3 a $0.8 = \dfrac{180^\circ}{\pi} \times 0.8 = 45.84^\circ$

b $1.64 = \dfrac{180^\circ}{\pi} \times 1.64 = 93.97^\circ$

c $2.5 = \dfrac{180^\circ}{\pi} \times 2.5 = 143.24^\circ$

d $3.96 = \dfrac{180^\circ}{\pi} \times 3.96 = 226.89^\circ$

e $4.18 = \dfrac{180^\circ}{\pi} \times 4.18 = 239.50^\circ$

f $5.95 = \dfrac{180^\circ}{\pi} \times 5.95 = 340.91^\circ$

4 a $37^\circ = \dfrac{\pi}{180^\circ} \times 37^\circ = 0.65$

b $74^\circ = \dfrac{\pi}{180^\circ} \times 74^\circ = 1.29$

c $115^\circ = \dfrac{\pi}{180^\circ} \times 115^\circ = 2.01$

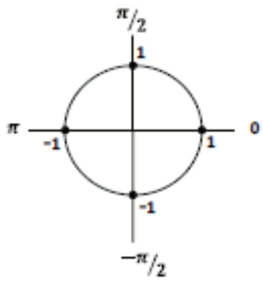
d $122.25^\circ = \dfrac{\pi}{180^\circ} \times 122.25^\circ = 2.13$

e $340^\circ = \dfrac{\pi}{180^\circ} \times 340^\circ = 5.93$

f $132.5^\circ = \dfrac{\pi}{180^\circ} \times 132.5^\circ = 2.31$

Solutions to Exercise 6B

1



a $\sin 3\pi = 0$

b $\cos\left(-\dfrac{5\pi}{2}\right) = 0$

c $\sin\left(\dfrac{7\pi}{2}\right) = -1$

d $\cos 3\pi = -1$

e $\sin(-4\pi) = 0$

f $\tan -\pi = 0$

g $\tan 2\pi = 0$

h $\tan -2\pi = 0$

i $\cos(23\pi) = \cos\pi = -1$

j $\cos\left(\dfrac{49\pi}{2}\right) = \cos\left(\dfrac{\pi}{2}\right) = 0$

k $\cos(35\pi) = \cos\pi = -1$

l $\cos\left(\dfrac{-45\pi}{2}\right) = \cos\left(\dfrac{-\pi}{2}\right) = 0$

m $\tan(24\pi) = \tan(0) = 0$

n $\cos(20\pi) = \cos(0) = 1$

2 **a** 0.99

b 0.52

c −0.87

d 0.92

e −0.67

f −0.23

g −0.99

h 0.44

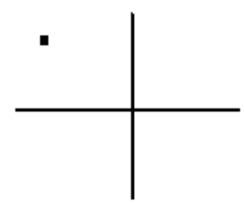
i −34.23

j −2.57

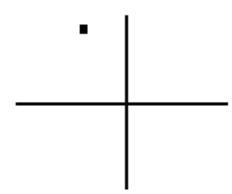
k 0.95

l 0.75

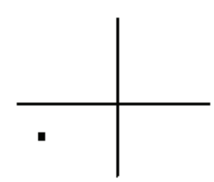
3 **a** $\sin\left(\dfrac{3\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$



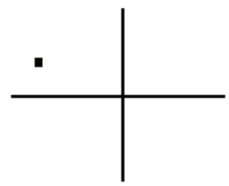
b $\cos\left(\dfrac{2\pi}{3}\right) = \dfrac{-1}{2}$



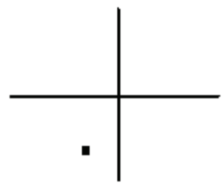
c $\cos\left(\dfrac{7\pi}{6}\right) = \dfrac{-\sqrt{3}}{2}$



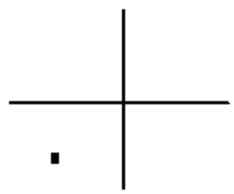
d $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$



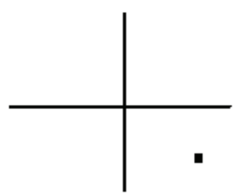
e $\cos\left(\frac{4\pi}{3}\right) = \frac{-1}{2}$



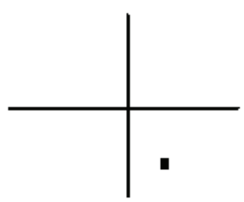
f $\sin\left(\frac{5\pi}{4}\right) = \frac{-1}{\sqrt{2}}$



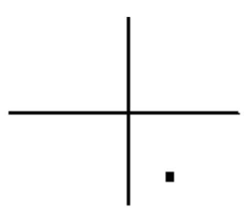
g $\sin\left(\frac{7\pi}{4}\right) = \frac{-1}{\sqrt{2}}$



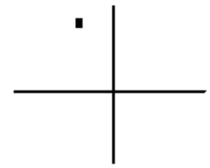
h $\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$



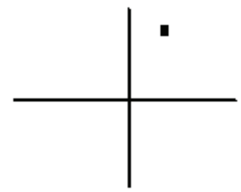
i $\cos\left(\frac{11\pi}{3}\right) = \frac{1}{2}$



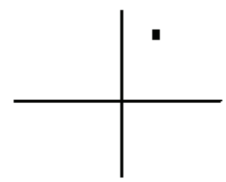
j $\sin\left(\frac{200\pi}{3}\right) = \frac{\sqrt{3}}{2}$



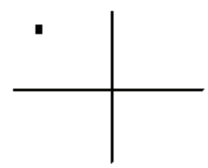
k $\cos\left(\frac{-11\pi}{3}\right) = \frac{1}{2}$



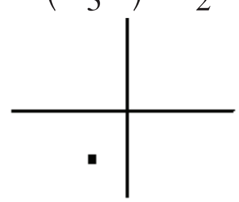
l $\sin\left(\frac{25\pi}{3}\right) = \frac{\sqrt{3}}{2}$



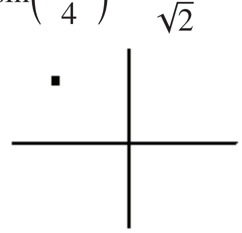
m $\sin\left(\frac{-13\pi}{4}\right) = \frac{1}{\sqrt{2}}$



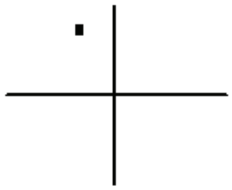
n $\cos\left(\frac{-20\pi}{3}\right) = \frac{-1}{2}$



o $\sin\left(\frac{67\pi}{4}\right) = \frac{1}{\sqrt{2}}$



p $\cos\left(\frac{68\pi}{3}\right) = \frac{-1}{2}$



q $\tan\left(\frac{11\pi}{3}\right) = -\sqrt{3}$

r $\tan\left(\frac{200\pi}{3}\right) = -\sqrt{3}$

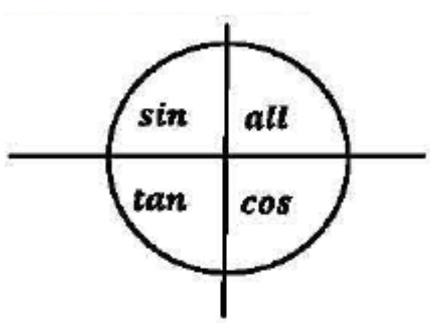
s $\tan\left(\frac{-11\pi}{6}\right) = \frac{1}{\sqrt{3}}$

t $\tan\left(\frac{25\pi}{3}\right) = \sqrt{3}$

u $\tan\left(-\frac{13\pi}{4}\right) = -1$

v $\tan\left(-\frac{25\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

4



a $\sin(\pi - \theta) = \sin\theta = 0.52$

b $\cos(\pi + x) = -\cos x = -0.68$

c $\sin(2\pi + \theta) = \sin\theta = 0.52$

d $\tan(\pi + \alpha) = \tan(\alpha) = 0.4$

e $\sin(\pi + \theta) = -\sin\theta = -0.52$

f $\cos(2\pi - x) = \cos(-x) = \cos x = 0.68$

g $\tan(2\pi - \alpha) = \tan(-\alpha)$

$= -\tan\alpha$

$= -0.4$

h $\cos(\pi - x) = -\cos x = -0.68$

i $\sin(-\theta) = -\sin\theta = -0.52$

j $\cos(-x) = \cos x = 0.68$

k $\tan(-\alpha) = -\tan(\alpha) = -0.4$

5 **a** 0.4

b −0.7

c 0.4

d 1.2

e −0.4

f 0.7

g −1.2

h −0.7

i −0.4

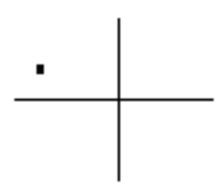
j 0.7

k −1.2

6 **a** $\sin(150°) = \frac{1}{2}$

$\cos(150°) = \frac{-\sqrt{3}}{2}$

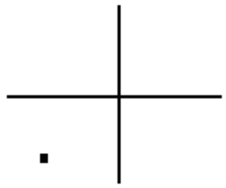
$\tan(150°) = \frac{-1}{\sqrt{3}}$



b $\sin(225°) = \dfrac{-1}{\sqrt{2}}$

$\cos(225°) = \dfrac{-1}{\sqrt{2}}$

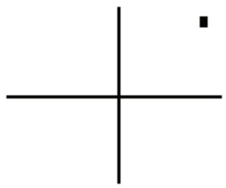
$\tan(225°) = 1$



c $\sin(405°) = \dfrac{1}{\sqrt{2}}$

$\cos(405°) = \dfrac{1}{\sqrt{2}}$

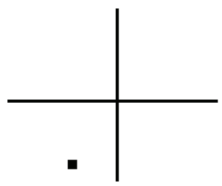
$\tan(405°) = 1$



d $\sin(-120°) = \dfrac{-\sqrt{3}}{2}$

$\cos(-120°) = \dfrac{-1}{2}$

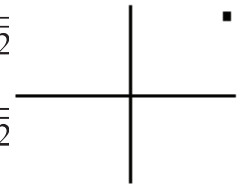
$\tan(-120°) = \sqrt{3}$



e $\sin(-315°) = \dfrac{1}{\sqrt{2}}$

$\cos(-315°) = \dfrac{1}{\sqrt{2}}$

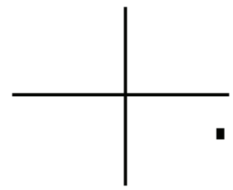
$\tan(-315°) = 1$



f $\sin(-30°) = \dfrac{-1}{2}$

$\cos(-30°) = \dfrac{\sqrt{3}}{2}$

$\tan(-30°) = \dfrac{-1}{\sqrt{3}}$



(ensure calculator is in radians not degrees)

Solutions to Exercise 6C

1 $\sin x = 0.3, \cos x = 0.6, \tan x = 0.7$

a $\cos(-\alpha) = \cos\alpha = 0.6$

b $\sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha = 0.6$

c $\tan(-\theta) = -\tan\theta = -0.7$

d $\cos\left(\frac{\pi}{2} - x\right) = \sin x = 0.3$

e $\sin(-x) = -\sin x = -0.3$

f $\tan\left(\frac{\pi}{2} - \theta\right) = \text{cotan}(\theta) = \frac{10}{7}$

g $\cos\left(\frac{\pi}{2} + x\right) = -\sin x = -0.3$

h $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha = 0.6$

i $\sin\left(\frac{3\pi}{2} + \alpha\right) = \cos(\pi + \alpha)$

$= -\cos\alpha = -0.6$

j $\cos\left(\frac{3\pi}{2} - x\right) = \cos\left(\frac{-\pi}{2} - x\right)$

$= \cos\left(\frac{\pi}{2} + x\right)$

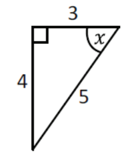
$= -\sin x$

$= -0.3$

k $\tan\left(\frac{3\pi}{2} - \theta\right) = \frac{1}{\tan\theta} = \frac{10}{7}$

l $\cos\left(\frac{5\pi}{2} - \theta\right) = \sin x = 0.3$

2 **a** $\cos x = \frac{3}{5}$



$\frac{3\pi}{2} \le x \le 2\pi$

Method 1

$$\cos^2 x + \sin^2 x = 1$$

$$\frac{9}{25} + \sin^2 x = 1$$

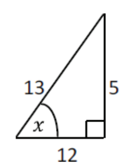
$$\sin^2 x = \frac{16}{25}$$

$$\sin x = \frac{\pm 4}{5}$$

$$\frac{3\pi}{2} \le x \le 2\pi, \sin x = \frac{-4}{5}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{-4}{3}$$

b $\sin x = \frac{5}{13}$



$\frac{\pi}{2} < x < \pi$

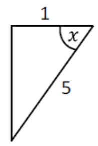
Method 2

from the triangle,

$\cos x = \frac{-12}{13}$ SOH **CAH** TOA

$\tan x = \frac{\sin x}{\cos x} = \frac{-5}{12}$

c $\cos x = \frac{1}{5}$



$$\frac{3\pi}{2} < x < 2\pi$$

$$\cos^2 x = \frac{1}{25}$$

$$\therefore \sin^2 x = 1 - \cos^2 x$$

$$\sin^2 x = \frac{24}{25}$$

$$\sin x = \frac{\pm 2\sqrt{6}}{5}$$

$$\textit{since } \frac{3\pi}{2} < x < 2\pi,\ \sin x = \frac{-2\sqrt{6}}{5}$$

$$\tan x = \frac{\sin x}{\cos x} = -2\sqrt{6}$$

d $\sin x = -\frac{12}{13}$

$$\therefore \cos^2 x = 1 - \sin^2 x$$

$$= \frac{25}{169}$$

$$\text{Since } \pi < x < \frac{3\pi}{2}$$

$$\cos x = -\frac{5}{13}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$= \frac{12}{5}$$

e $\cos x = \frac{4}{5}$

$$\therefore \cos^2 x = 1 - \sin^2 x$$

$$= \frac{9}{25}$$

$$\text{Since } \frac{3\pi}{2} < x < 2\pi$$

$$\sin x = -\frac{3}{5}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$= -\frac{3}{4}$$

f $\sin x = -\frac{12}{13}$

$$\therefore \cos^2 x = 1 - \sin^2 x$$

$$= \frac{25}{169}$$

$$\text{Since } \pi < x < \frac{3\pi}{2}$$

$$\cos x = -\frac{5}{13}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$= \frac{12}{5}$$

g $\cos x = \frac{8}{10}$

$$\therefore \cos^2 x = 1 - \sin^2 x$$

$$= \frac{36}{100}$$

$$\text{Since } \frac{3\pi}{2} < x < 2\pi$$

$$\sin x = -\frac{3}{5}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$= -\frac{3}{4}$$

Solutions to Exercise 6D

1 **a** 2π, 3

b $\frac{2\pi}{3}$, 5

c π, $\frac{1}{2}$

d 6π, 2

e $\frac{\pi}{2}$, 3

f 2π, $\frac{1}{2}$

g 4π, 3

h 3π, 2

2 **a** Dilation of factor 4 from the x-axis, dilation of factor $\frac{1}{3}$ from the y-axis; Amplitude = 4; Period = $\frac{2\pi}{3}$

b Dilation of factor 5 from the x-axis, dilation of factor 3 from the y-axis; Amplitude = 5; Period = 6π

c Dilation of factor 6 from the x-axis, dilation of factor 2 from the y-axis; Amplitude = 6; Period = 4π

d Dilation of factor 4 from the x-axis, dilation of factor $\frac{1}{5}$ from the y-axis; Amplitude = 4; Period = $\frac{2\pi}{5}$

3 **a** Dilation of factor 2 from the x-axis, dilation of factor $\frac{1}{3}$ from the y-axis; Amplitude = 2; Period = $\frac{2\pi}{3}$

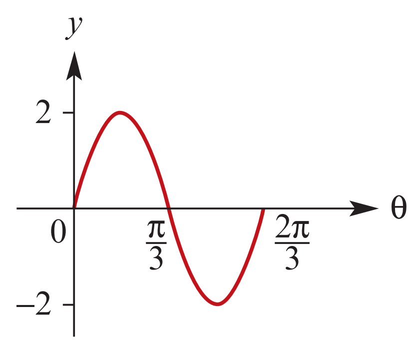
b Dilation of factor 3 from the x-axis, dilation of factor 4 from the y-axis; Amplitude = 3; Period = 8π

c Dilation of factor 6 from the x-axis, dilation of factor 5 from the y-axis; Amplitude = 6; Period = 10π

d Dilation of factor 3 from the x-axis, dilation of factor $\frac{1}{7}$ from the y-axis; Amplitude = 3; Period = $\frac{2\pi}{7}$

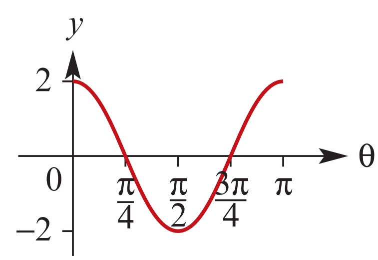
4 **a** Amplitude = 2

Period = $\frac{2\pi}{3}$



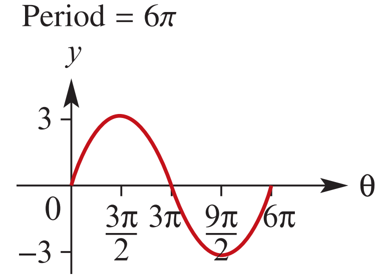
b Amplitude = 2

Period = π



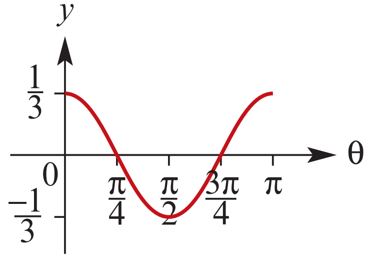
c Amplitude = 3

Period = 6π



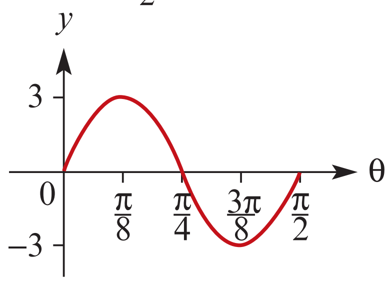
d Period = π

Amplitude = $\frac{1}{3}$



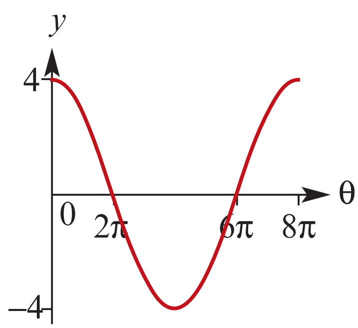
e Amplitude = 3

Period = $\frac{\pi}{2}$



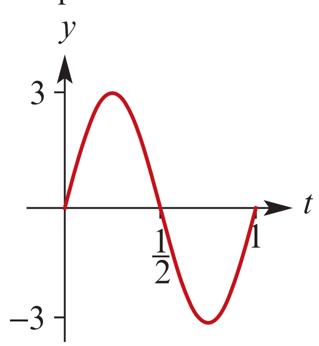
f Amplitude = 4

Period = 8π



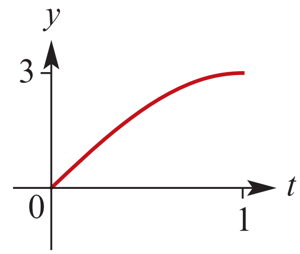
5 Period = $\frac{2\pi}{2\pi} = 1$

Amplitude = 3



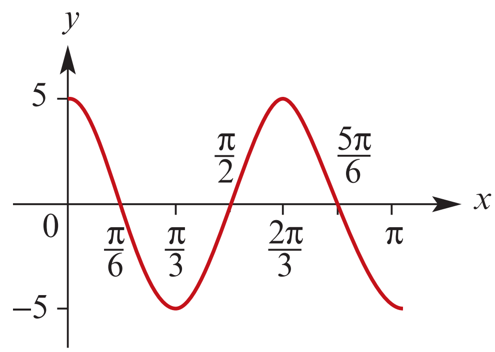
6 Period = = $\frac{2\pi}{\frac{\pi}{2}} = 4$

Amplitude = 3



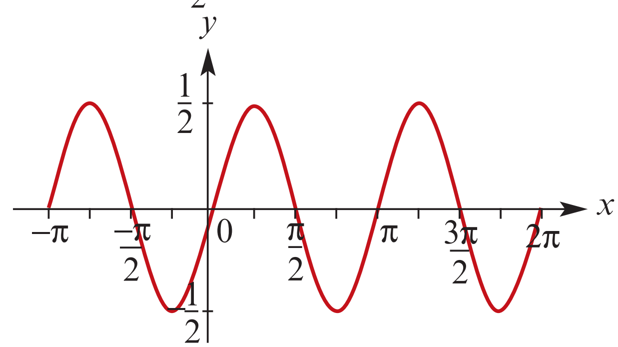
7 Period = = $\frac{2\pi}{3}$

Amplitude = 5



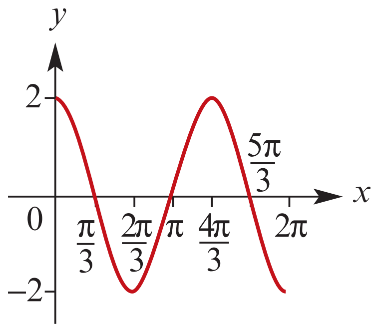
8 Period = = $\frac{2\pi}{2} = \pi$

Amplitude = $\frac{1}{2}$



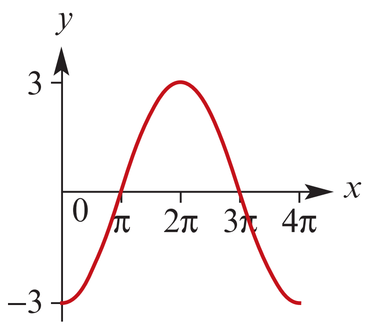
9 Period = = $\dfrac{2\pi}{\frac{3}{2}} = \dfrac{4\pi}{3}$

Amplitude = 2



10 Period = = $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$

Amplitude = 3



11 $y = \sin x$

Dilation of factor 2 from the x-axis

$\Rightarrow\ y = 2\sin x$

Dilation of factor 3 from the y-axis

$\Rightarrow\ y = 2\sin\left(\dfrac{x}{3}\right)$

12 $y = \cos x$

Dilation of factor $\dfrac{1}{2}$ from the x-axis

$\Rightarrow\ y = \dfrac{1}{2}\cos x$

Dilation of factor 3 from the y-axis

$\Rightarrow\ y = \dfrac{1}{2}\cos\left(\dfrac{x}{3}\right)$

13 $y = \sin x$

Dilation of factor $\dfrac{1}{2}$ from the x-axis

$\Rightarrow\ y = \dfrac{1}{2}\sin x$

Dilation of factor 2 from the y-axis

$\Rightarrow\ y = \dfrac{1}{2}\sin\left(\dfrac{x}{2}\right)$

Solutions to Exercise 6E

1 a $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$

b $\frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$

c $\frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}$

d $\frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$

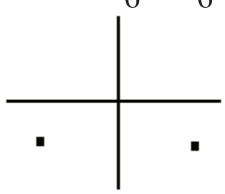
e $\frac{\pi}{2}, \frac{5\pi}{2}$

f $\pi, 3\pi$

2 a $\sin x = \frac{-1}{2}$

$$x = \sin^{-1}\left(\frac{1}{2}\right)$$

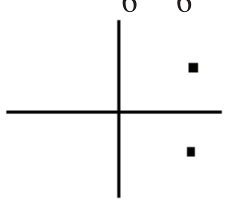
$$x = \frac{-5\pi}{6}, \frac{-\pi}{6}$$



b $\cos x = \frac{\sqrt{3}}{2}$

$$x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

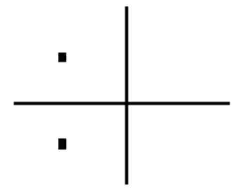
$$x = \frac{-\pi}{6}, \frac{\pi}{6}$$



c $\cos x = \frac{-\sqrt{3}}{2}$

$$x = \cos^{-1}\frac{-\sqrt{3}}{2}$$

$$x = \frac{-5\pi}{6}, \frac{5\pi}{6}$$



3 a $\sqrt{2}\sin x = 1$

$$\sin x = \frac{1}{\sqrt{2}}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$

b $\sqrt{2}\cos x = -1$

$$\cos x = -\frac{1}{\sqrt{2}}$$

$$x = \frac{3\pi}{4}, \frac{5\pi}{4}$$

c $2\cos x = -\sqrt{3}$

$$\cos x = -\frac{\sqrt{3}}{2}$$

$$x = \frac{5\pi}{6}, \frac{7\pi}{6}$$

d $2\sin x + 1 = 0$

$$\sin x = -\frac{1}{2}$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

e $\sqrt{2}\cos x = 1$

$$\cos x = \frac{1}{\sqrt{2}}$$

$$x = \frac{\pi}{4}, \frac{7\pi}{4}$$

f $4\cos x = -2$

$$\cos x = -\frac{1}{2}$$

$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

4 **a** $\sin x = 0.6$
calculator gives $x_1 \cong 0.6435$
second answer is $\pi - x_1 \cong 2.498$

b $\cos x = 0.8$
calculator gives $x_1 \cong 0.6435$
second answer is $2\pi - x_1 \cong 5.640$

c $\sin x = -0.45$
calculator gives $x_1 \cong 5.816$
second answer is $\pi - x_1 \cong 3.608$

d $\cos x = -0.2$
calculator gives $x_1 \cong 1.772$
second answer is $2\pi - x_1 \cong 4.511$

5 **a** $\sin\theta^\circ = 0.3$
calculator gives $\theta_1 = 17.46$
second answer is $180^\circ - \theta_1 = 162.54$

b $\cos\theta^\circ = 0.4$
calculator gives $\theta_1 = 66.42$
second answer is $360^\circ - \theta_1 = 293.58$

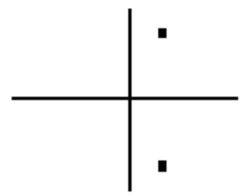
c $\sin\theta^\circ = -0.8$
calculator gives $\theta_1 = 306.87$
second answer is $180^\circ - \theta_1 = 233.13$

d $\cos\theta^\circ = -0.5$
$\theta_1 = 120$
second answer is $\theta_2 - 360^\circ - \theta_1 = 240$

6 **a** $\cos(\theta^\circ) = \frac{1}{2}$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

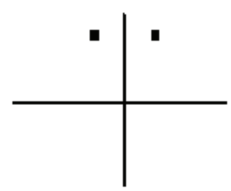
$$\theta = 60, 300$$



b $\sin(\theta^\circ) = \frac{\sqrt{3}}{2}$

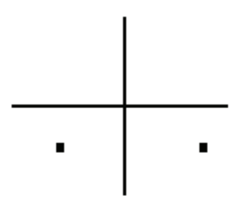
$$\theta^\circ = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\theta = 60, 120$$



c $\sin(\theta^\circ) = \frac{-1}{\sqrt{2}}$

$$\theta = 225, 315$$

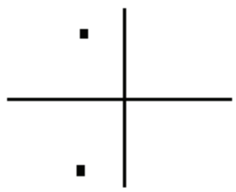


d $2\cos\theta^\circ + 1 = 0$

$$\cos\theta^\circ = \frac{-1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

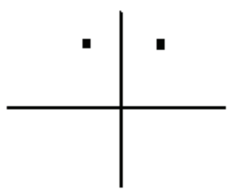
$$\theta = 120, 240$$



e $\sin(\theta^\circ) = \dfrac{\sqrt{3}}{2}$

$$\theta^\circ = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

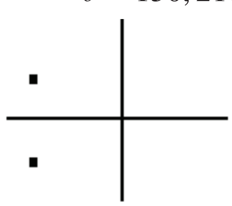
$$\theta = 60, 120$$



f $\cos(\theta^\circ) = \dfrac{-\sqrt{3}}{2}$

$$\theta^\circ = \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$$

$$\theta = 150, 210$$



7 a $\sin 2\theta = -\dfrac{1}{2}$

$$2\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$$

$$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

b $\cos 2\theta = \dfrac{\sqrt{3}}{2}$

$$2\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$$

c $\sin 2\theta = \dfrac{1}{2}$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

d $\sin 3\theta = -\dfrac{1}{\sqrt{2}}$

$$3\theta = \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}, \frac{21\pi}{4}, \frac{23\pi}{4}$$

$$\theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12}\left(= \frac{5\pi}{4}\right),$$

$$\frac{21\pi}{12}\left(= \frac{7\pi}{4}\right), \frac{23\pi}{12}$$

e $\cos 2\theta = -\dfrac{\sqrt{3}}{2}$

$$2\theta = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}$$

$$\theta = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$$

f $\sin 2\theta = -\dfrac{1}{\sqrt{2}}$

$$2\theta = \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}$$

$$\theta = \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$$

8 **a**

$$\cos 3x = -\frac{\sqrt{3}}{2}$$

$$3x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}, \frac{29\pi}{6}, \frac{31\pi}{6}$$

$$x = \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18}, \frac{19\pi}{18}, \frac{29\pi}{18}, \frac{31\pi}{18}$$

b $\sin 2x = \frac{1}{2}$

$$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

c $\cos 3x = \frac{1}{\sqrt{2}}$

$$3x = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}, \frac{23\pi}{4}$$

$$x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}$$

d $\sin 3x = \frac{1}{2}$

$$3x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$$

$$x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$$

e $\sin 2x = \frac{1}{\sqrt{2}}$

$$2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$$

$$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$$

f

$$\cos 3x = -\frac{\sqrt{3}}{2}$$

$$3x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}, \frac{29\pi}{6}, \frac{31\pi}{6}$$

$$x = \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18}, \frac{19\pi}{18}, \frac{29\pi}{18}, \frac{31\pi}{18}$$

g $x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{12}$

h $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

i $x = \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$

9 **a** 2.03444, 2.67795, 5.17604, 5.81954

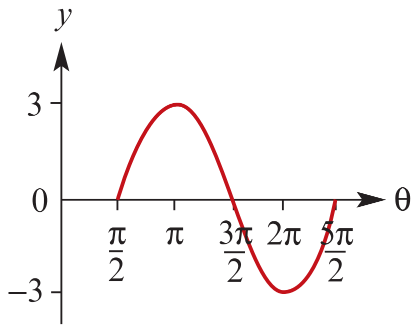
b 1.89255, 2.81984, 5.03414, 5.96143

c 0.57964, 2.56195, 3.72123, 5.70355

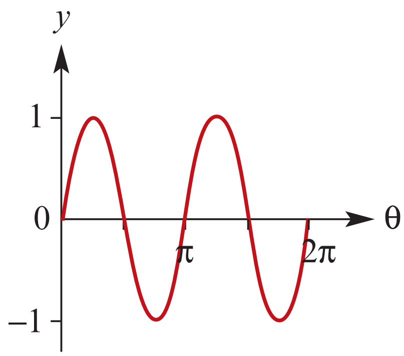
d 0.309098, 1.7853, 2.40349, 3.87969, 4.49789, 5.97409

Solutions to Exercise 6F

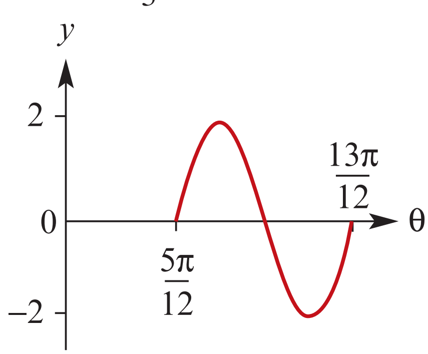
1 a Period = 2π; Amplitude = 3; $y = \pm 3$



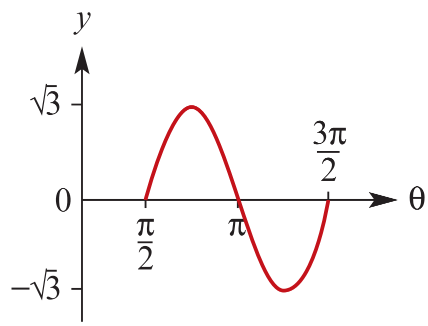
b Period = π; Amplitude = 1; $y = \pm 1$



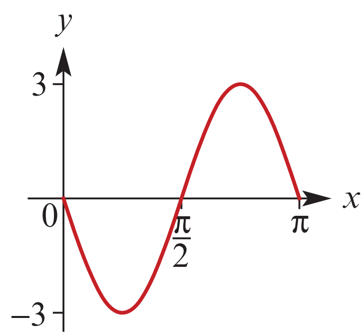
c Period = $\frac{2\pi}{3}$; Amplitude = 2; $y = \pm 2$



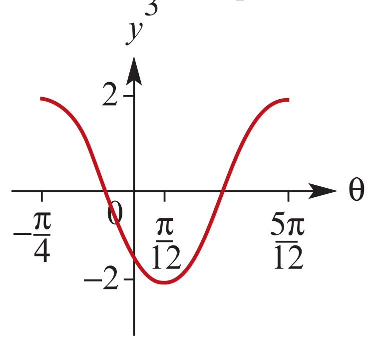
d Period = π; Amplitude = $\sqrt{3}$; $y = \pm\sqrt{3}$



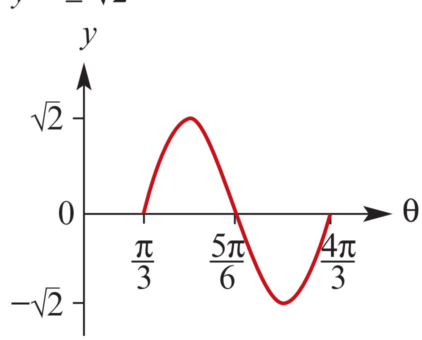
e Period = π; Amplitude = 3; $y = \pm 3$



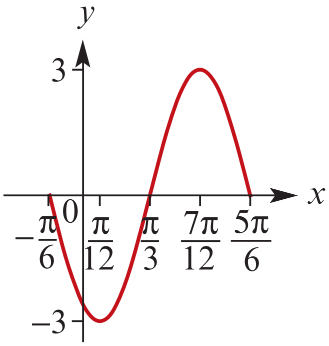
f Period = $\frac{2\pi}{3}$; Amplitude = 2; $y = \pm 2$



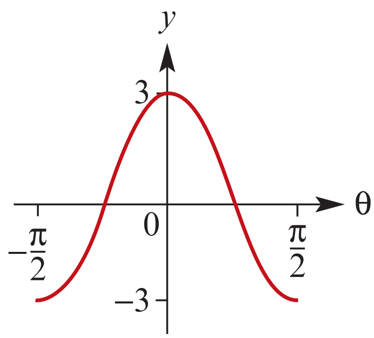
g Period = π; Amplitude = $\sqrt{2}$; $y = \pm\sqrt{2}$



h Period = π; Amplitude = 3; $y = \pm 3$

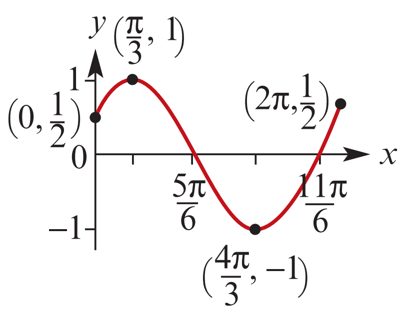


i Period = π; Amplitude = 3; $y = \pm 3$



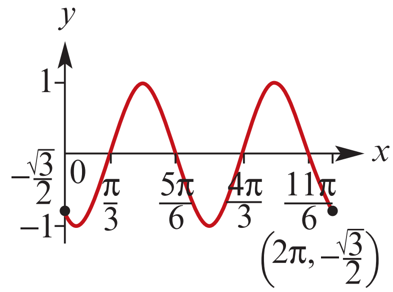
2 a $f(0) = \frac{1}{2}$, $f(2\pi) = \frac{1}{2}$

b



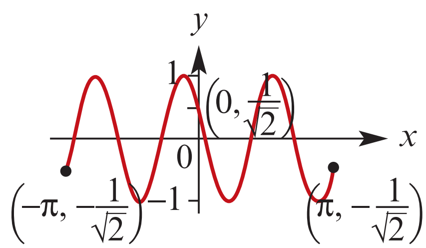
3 a $f(0) = -\frac{\sqrt{3}}{2}$, $f(2\pi) = -\frac{\sqrt{3}}{2}$

b



4 a $f(-\pi) = -\frac{1}{\sqrt{2}}$, $f(\pi) = -\frac{1}{\sqrt{2}}$

b



5 a $y = 3\sin\left(\frac{x}{2}\right)$

b $y = 3\sin(2x)$

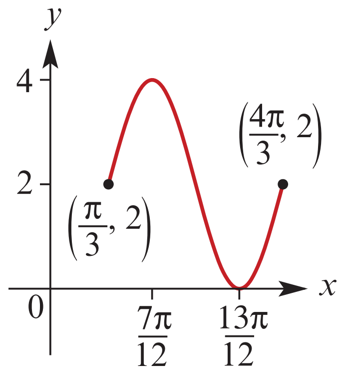
c $y = 2\sin\left(\frac{x}{3}\right)$

d $y = \sin 2\left(x - \frac{\pi}{3}\right)$

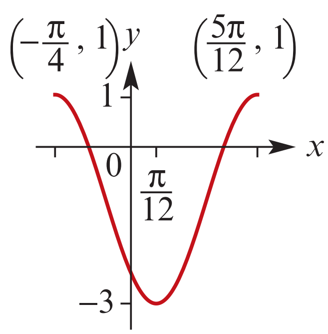
e $y = \sin \frac{1}{2}\left(x + \frac{\pi}{3}\right)$

Solutions to Exercise 6G

1 **a**



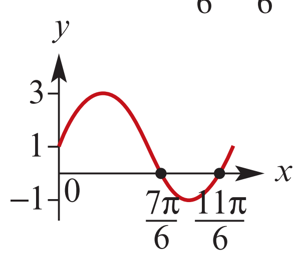
b



2 **a** $2\sin x + 1 = 0$

$$\sin x = -\frac{1}{2}$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

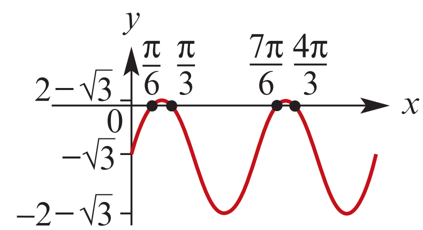


b $2\sin 2x - \sqrt{3} = 0$

$$\sin 2x = \frac{\sqrt{3}}{2}$$

$$2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$$

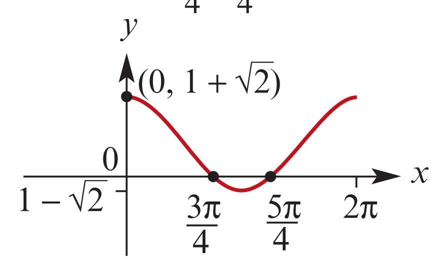
$$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$$



c $\sqrt{2}\cos x = -1$

$$\cos x = -\frac{1}{\sqrt{2}}$$

$$x = \frac{3\pi}{4}, \frac{5\pi}{4}$$

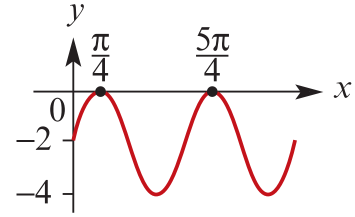


d $2\sin 2x - 2 = 0$

$$\sin 2x = 1$$

$$2x = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

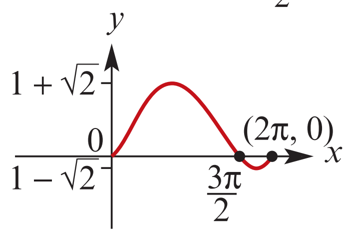


e $\sqrt{2}\sin(x - \frac{\pi}{4}) = -1$

$$\sin(x - \frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$$

$$x - \frac{\pi}{4} = \frac{5\pi}{4}, \frac{7\pi}{4}, -\frac{\pi}{4}$$

$$x = 0, \frac{3\pi}{2}, 2\pi$$



3 a ***y*-axis intercept**

$y = -2$

***x*-axis intercepts**

$$2\sin(3x) = 2$$

$$\sin(3x) = 1$$

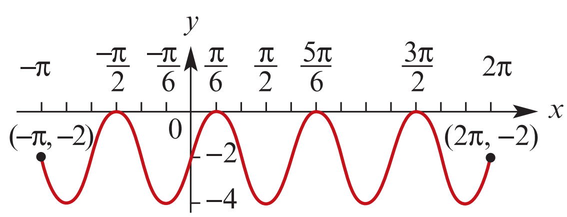
$$3x = -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$$

$$x = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

Endpoints

When $x = -\pi, y = -2$

When $x = 2\pi, y = -2$



b ***y*-axis intercept**

$y = -\sqrt{2}$

***x*-axis intercepts**

$$\cos(3(x - \frac{\pi}{4}) = 0$$

$$3(x - \frac{\pi}{4}) = -\frac{7\pi}{2}, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{6}, \frac{9\pi}{2}$$

$$(x - \frac{\pi}{4}) = -\frac{7\pi}{6}, -\frac{5\pi}{6}, -\frac{3\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}$$

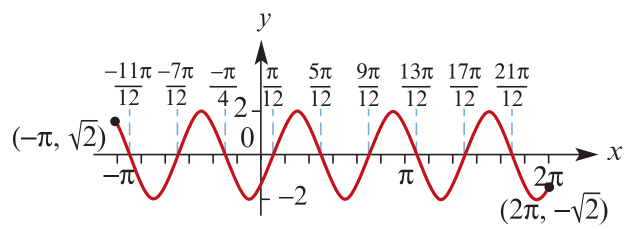
$$(x - \frac{3\pi}{12}) = -\frac{14\pi}{12}, -\frac{10\pi}{12}, -\frac{6\pi}{12}, -\frac{2\pi}{12}, \frac{2\pi}{12}, \frac{6\pi}{12}, \frac{10\pi}{12}, \frac{14\pi}{12}, \frac{18\pi}{12}$$

$$x = -\frac{11\pi}{12}, -\frac{7\pi}{12} - \frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12}$$

Endpoints

When $x = -\pi, y = \sqrt{2}$

When $x = 2\pi, y = -\sqrt{2}$



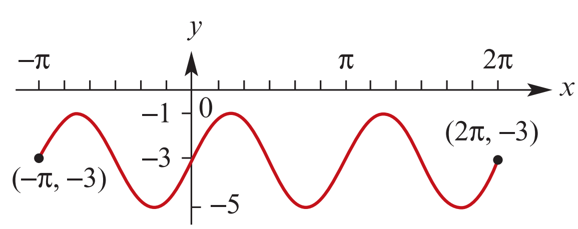
c ***y*-axis intercept**

$y = -3$

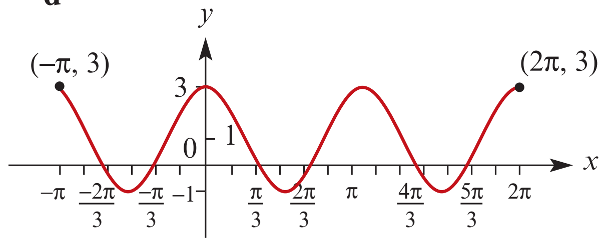
Endpoints

When $x = -\pi, y = -3$

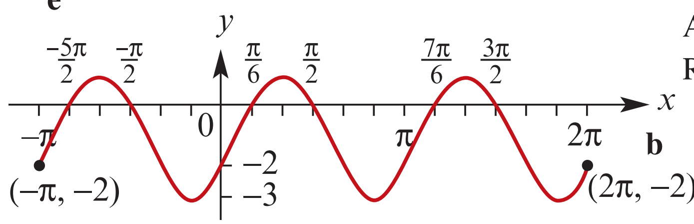
When $x = 2\pi, y = -3$



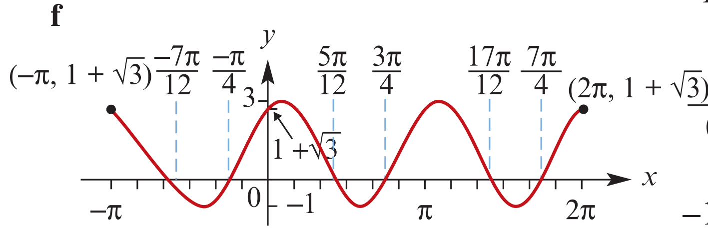
d



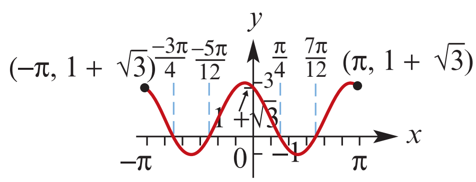
e



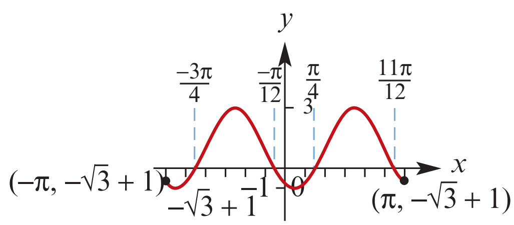
f



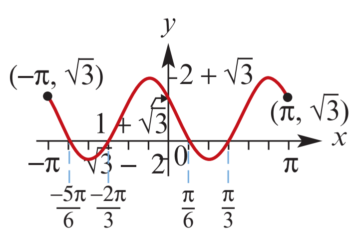
4 a



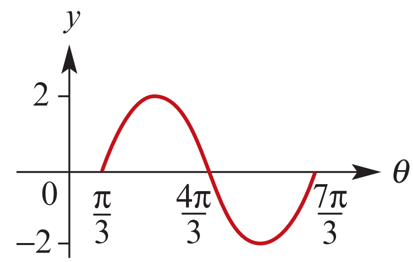
b



c



5 a

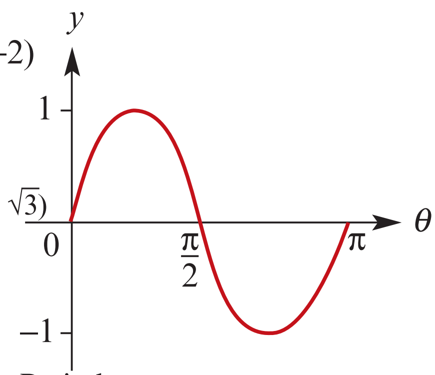


Period = 2π

Amplitude = 2

Range = $[-2, 2]$

b

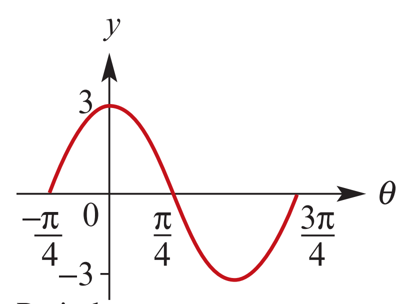


Period = π

Amplitude = 1

Range = $[-1, 1]$

c

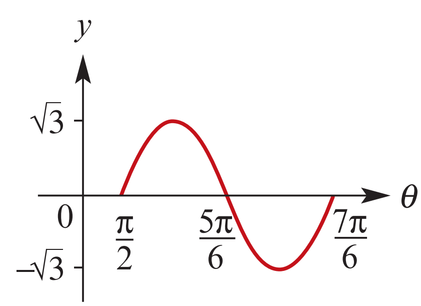


Period = π

Amplitude = 3

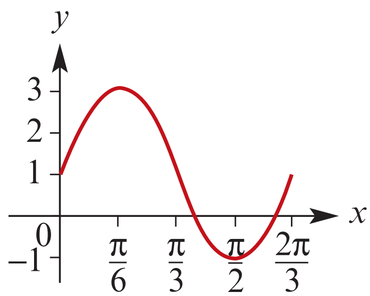
Range = $[-3, 3]$

d



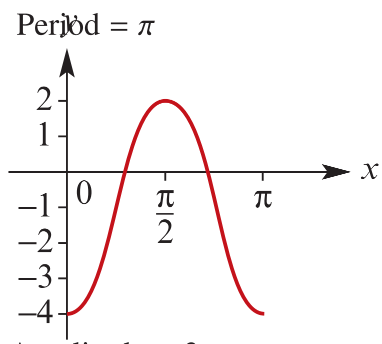
Period = $\dfrac{2\pi}{3}$
Amplitude = $\sqrt{3}$
Range = $[-\sqrt{3}, \sqrt{3}]$

e



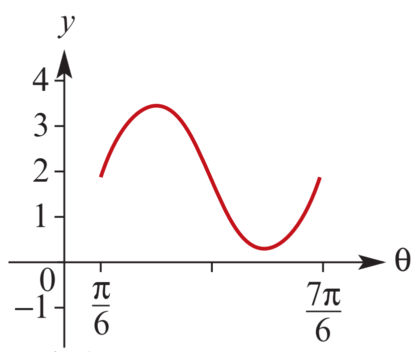
Period = $\dfrac{2\pi}{3}$
Amplitude = 2
Range = $[-1, 3]$

f Period = π



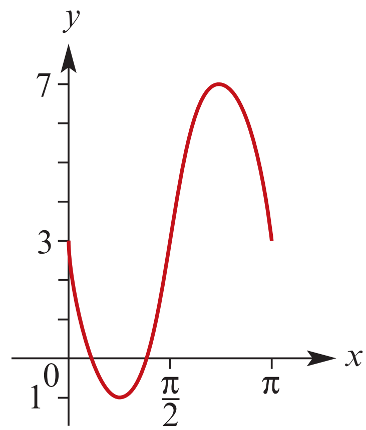
Amplitude = 3
Range = $[-4, 2]$

g



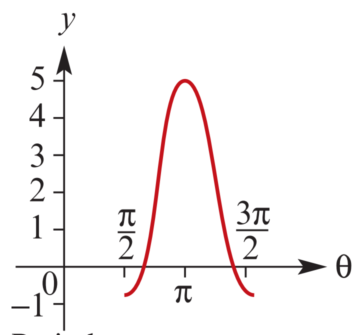
Period = π
Amplitude = $\sqrt{2}$
Range = $[-\sqrt{2} + 2, 2 + \sqrt{2}]$

h



Period = π
Amplitude = 4
Range = $[-1, 7]$

i



Period = π
Amplitude = 3
Range = $[-1, 5]$

6 **a** $y = \dfrac{1}{2}\cos\left(\dfrac{1}{3}\left(x - \dfrac{\pi}{4}\right)\right)$

b $y = 2\cos\left(x - \dfrac{\pi}{4}\right)$

c $y = -\dfrac{1}{3}\cos\left(x - \dfrac{\pi}{3}\right)$

7 **a**
- Dilation of factor 3 from the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Reflection in the x-axis

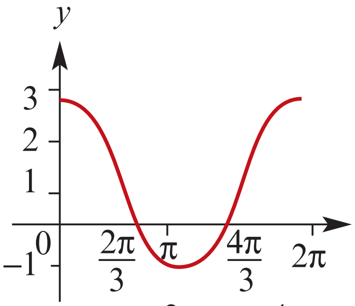
b
- Dilation of factor 3 from the x-axis

- Dilation of factor $\frac{1}{2}$ from the y-axis
- Reflection in the x-axis
- Translation $\frac{\pi}{3}$ units to the right

c
- Dilation of factor 3 from the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Translation $\frac{\pi}{3}$ units to the right and 2 units up

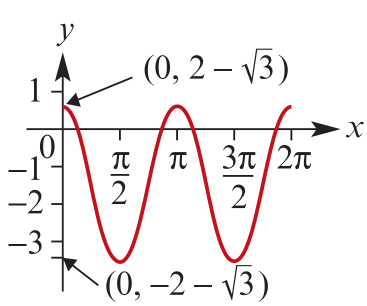
d
- Dilation of factor 2 from the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Reflection in the x-axis
- Translation $\frac{\pi}{3}$ units to the right and 5 units up

8 a



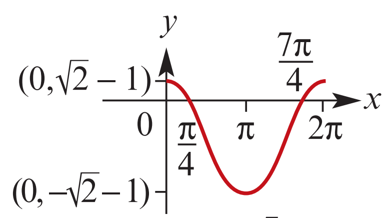
Intercepts: $\left(\frac{2\pi}{3},0\right),\left(\frac{4\pi}{3},0\right)$

b



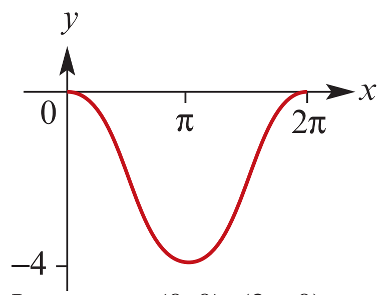
Intercepts: $\left(\frac{\pi}{12},0\right),\left(\frac{11\pi}{12},0\right)$, $\left(\frac{13\pi}{12},0\right),\left(\frac{23\pi}{12},0\right)$

c



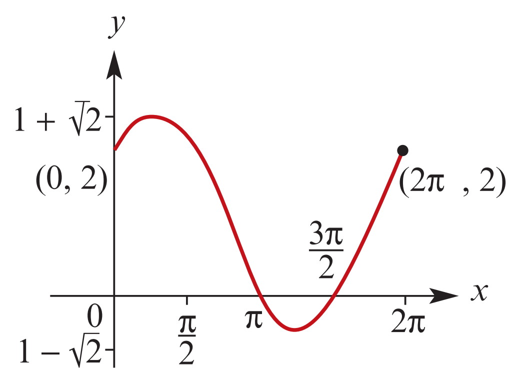
Intercepts: $\left(\frac{\pi}{4},0\right),\left(\frac{7\pi}{4},0\right)$t

d



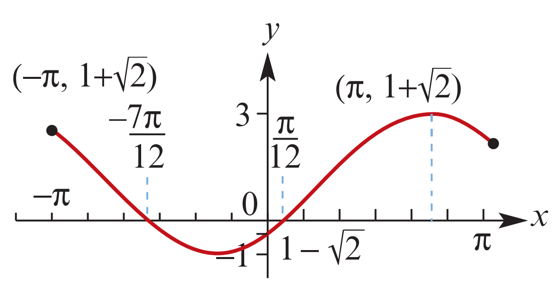
Intercepts: $(0,0)$, $(2\pi,0)$

e

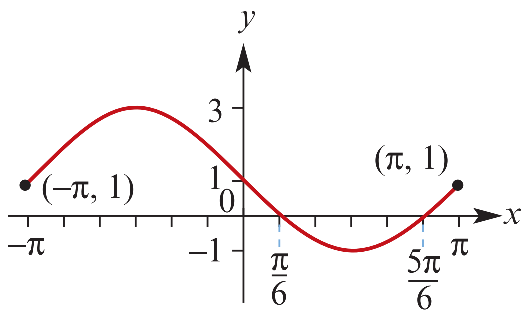


Intercepts: $(\pi,0),\left(\frac{3\pi}{2},0\right)$

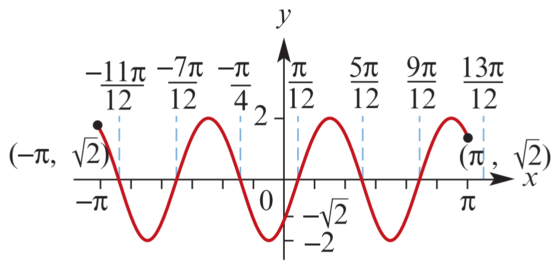
9 a



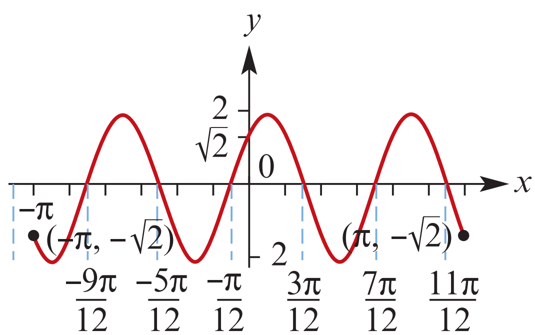
b


y
3
1
0
−1
(−π, 1)
(π, 1)
−π
π
x
π/6
5π/6

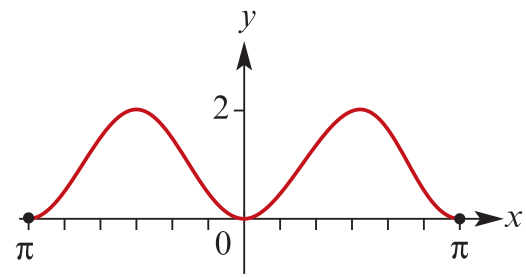
c


y
−11π/12
−7π/12
−π/4
π/12
5π/12
9π/12
13π/12
2
(−π, √2)
(π, √2)
−π
π
0
x
−√2
−2

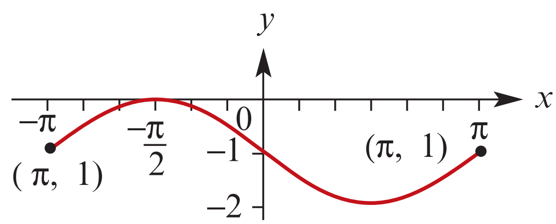
d


y
2
√2
0
x
−π
(−π, −√2)
(π, −√2)
−2
−9π/12
−5π/12
−π/12
3π/12
7π/12
11π/12

e

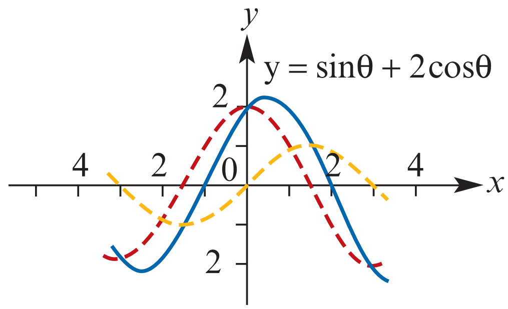

y
2
π
0
π
x

f

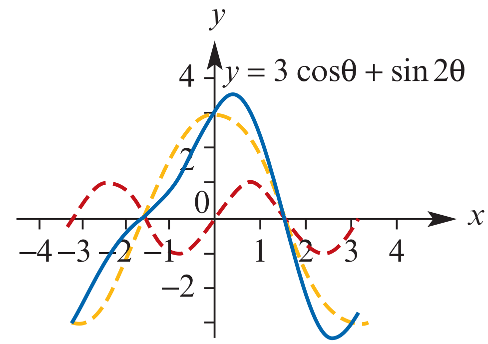

y
x
−π
0
π
(π, 1)
−π/2
−1
(π, 1)
−2

Solutions to Exercise 6H

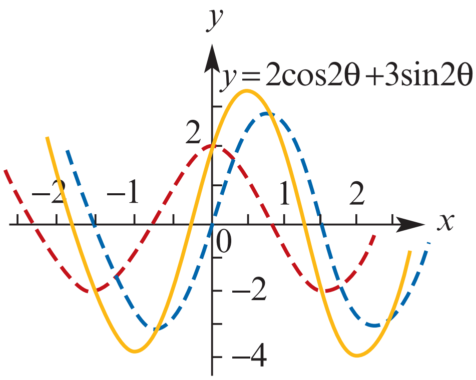
1 a



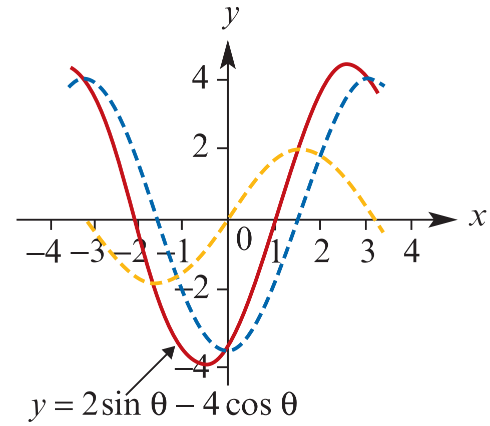
d



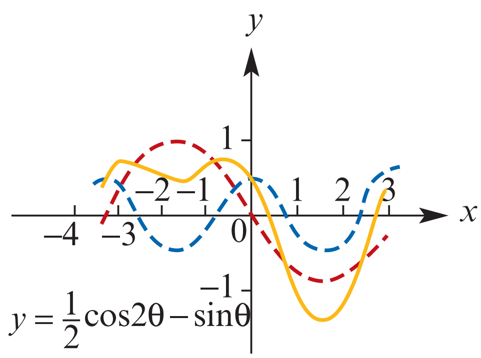
b



e



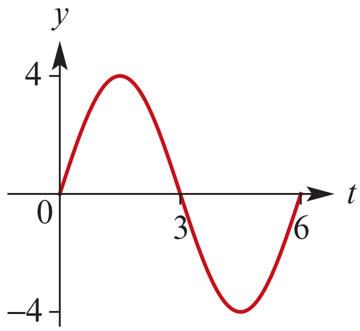
c



Solutions to Exercise 6I

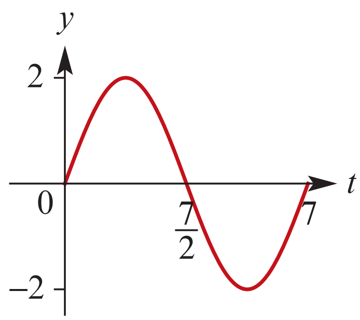
1 a $n = \frac{\pi}{3}$

$A = 4$



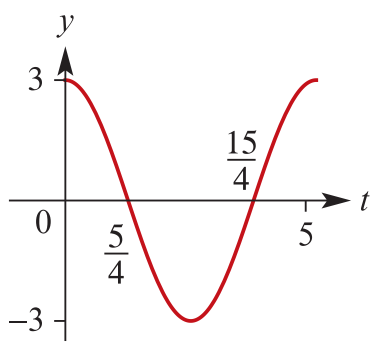
b $n = \frac{2\pi}{7}$

$A = 2$



c $n = \frac{2\pi}{5}$

$A = 3$



2 $A = 3,\ n = \frac{\pi}{4}$

3 $A = -4,\ n = \frac{\pi}{6}$

4 $A = 0.5,\ \varepsilon = \frac{-\pi}{3}$

5 $A = 3,\ n = 3,\ b = 5$

6

$$A = 4$$

$$\frac{2\pi}{n} = 8$$

$$n = \frac{\pi}{4}$$

$$t = 2,\ y = 0$$

$$0 = 4\sin\left(\frac{\pi}{4} \times 2 + \varepsilon\right)$$

$$\sin\left(\frac{\pi}{2} + \varepsilon\right) = 0$$

$$\varepsilon = -\frac{\pi}{2} + x\pi, x \in \mathbb{Z}$$

(*i.e.* ε can be an infinity of no.s, separated by π)

7

$$A = 2$$

$$\frac{2\pi}{n} = 6$$

$$n = \frac{\pi}{3}$$

$$t = 1,\ y = 1$$

$$1 = 2\sin\left(\frac{\pi}{3} + \varepsilon\right)$$

$$\sin\left(\frac{\pi}{3} + \varepsilon\right) = \frac{1}{2}$$

$$\frac{\pi}{3} + \varepsilon = \left(\frac{\pi}{6}, \frac{5\pi}{6}\right) + 2x\pi$$

$$\varepsilon = \left(\frac{-\pi}{6}, \frac{\pi}{2}\right) + 2x\pi,\ x \in \mathbb{Z}$$

8

$$A = \frac{6-(-2)}{2}$$

$$A = 4$$

$$d = 6 - A = 6 - 4$$

$$d = 2$$

$$\frac{2\pi}{n} = 8$$

$$n = \frac{\pi}{4}$$

$$t = 2,\ y = 2$$

$$2 = 4\sin\left(\frac{\pi}{4} \times 2 + \varepsilon\right) + 2$$

$$\sin\left(\frac{\pi}{2} + \varepsilon\right) = 0$$

$$\frac{\pi}{2} + \varepsilon = 0 + x\pi$$

$$\varepsilon = -\frac{\pi}{2} + x\pi, x \in \mathbb{Z}$$

9

$$A = \frac{4-0}{2} = 2$$

$$d = 4 - A = 2$$

$$\frac{2\pi}{n} = 6$$

$$n = \frac{\pi}{3}$$

$$t = 1,\ y = 3$$

$$3 = 2\sin\left(\frac{\pi}{3} \times 1 + \varepsilon\right) + 2$$

$$\sin\left(\frac{\pi}{3} + \varepsilon\right) = \frac{1}{2}$$

$$\frac{\pi}{3} + \varepsilon = \left(\frac{\pi}{6},\ \frac{7\pi}{6}\right) + 2x\pi$$

$$\varepsilon = \left(\frac{-\pi}{6},\ \frac{5\pi}{6}\right) + 2x\pi, x \in \mathbb{Z}$$

note: for $Q1$, $Q2$, $Q5$, $Q6$, $Q7$ & $Q8$ A could take the negative of the value given if ε is changed to $\varepsilon + \pi$

Solutions to Exercise 6J

1 a $T = \dfrac{\pi}{3}$

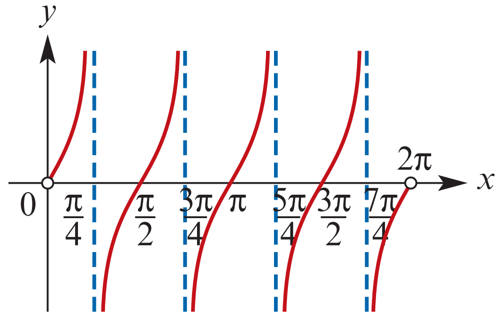
b $T = \dfrac{\pi}{\frac{1}{2}} = 2\pi$

c $T = \dfrac{\pi}{\frac{3}{2}} = \dfrac{2\pi}{3}$

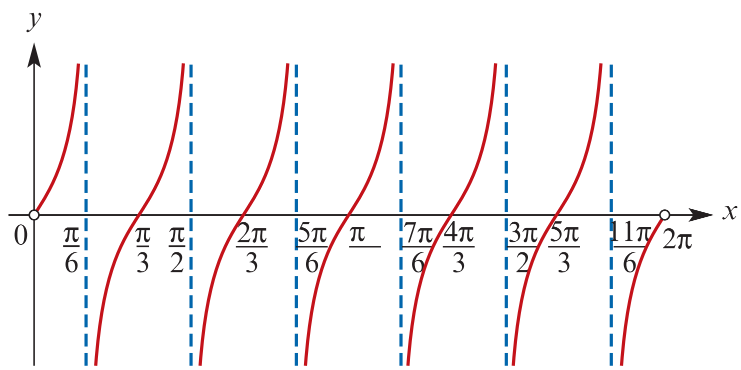
d $T = \dfrac{\pi}{\pi} = 1$

e $T = \dfrac{\pi}{\frac{\pi}{2}} = 2$

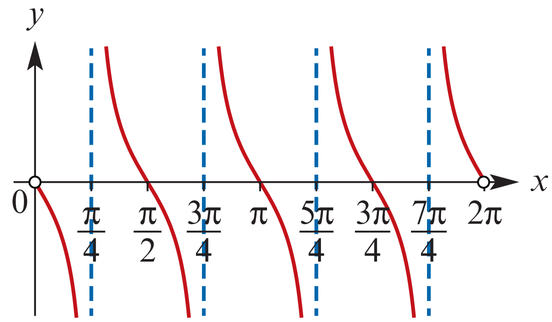
2 a



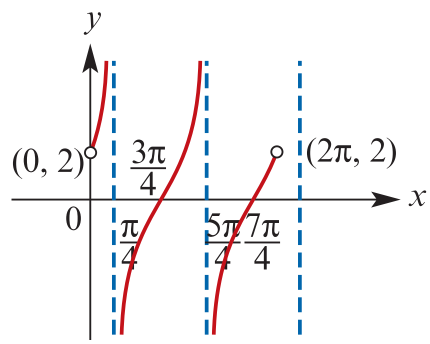
b



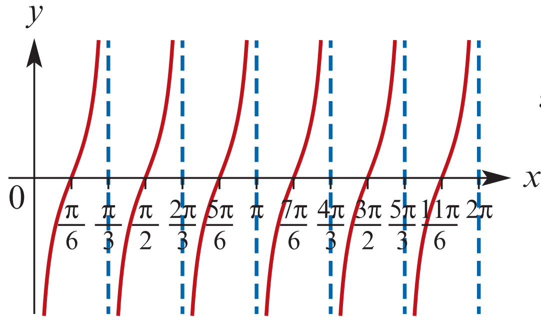
c



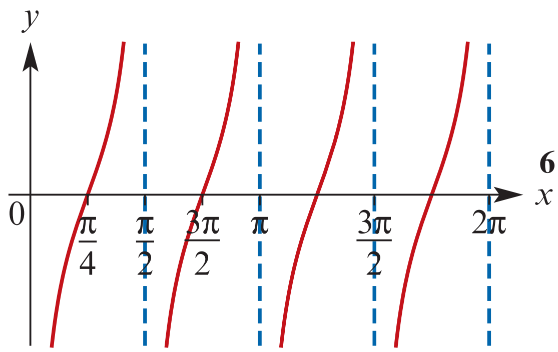
3 a



b



c



4 a $\tan(2x) = 1$

$$2x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$$

$$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$

b $\tan(2x) = -1$

$$2x = -\frac{\pi}{4}, -\frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4},$$

$$x = -\frac{\pi}{8}, -\frac{5\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8},$$

c $\tan(2x) = -\sqrt{3}$

$$2x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3},$$

$$x = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6},$$

d $-\frac{5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$

e $-\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}$

5 $\tan(2(x - \frac{\pi}{3})) = 1$

$$(2(x - \frac{\pi}{3})) = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$$

$$x - \frac{\pi}{3} = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}$$

$$x - \frac{8\pi}{24} = \frac{3\pi}{24}, \frac{15\pi}{24}, \frac{27\pi}{24}, \frac{39\pi}{44}$$

$$x = \frac{11\pi}{24}, \frac{23\pi}{24}, \frac{35\pi}{24}, \frac{47\pi}{24}$$

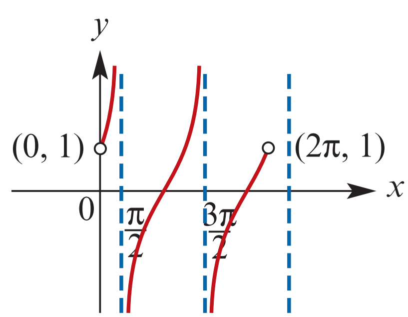
6 $\tan((x - \frac{\pi}{4})) = \sqrt{3}$

$$(x - \frac{\pi}{4}) = \frac{\pi}{3}, \frac{4\pi}{3}$$

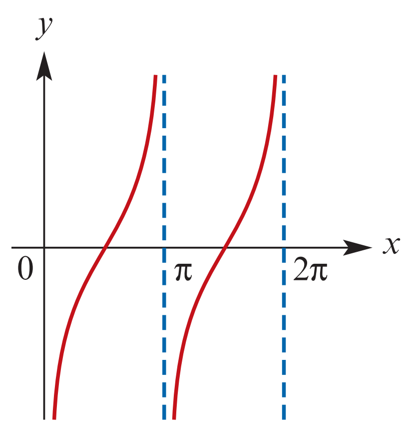
$$x - \frac{3\pi}{12} = \frac{4\pi}{12}, \frac{16\pi}{12}$$

$$x = \frac{7\pi}{12}, \frac{19\pi}{12}$$

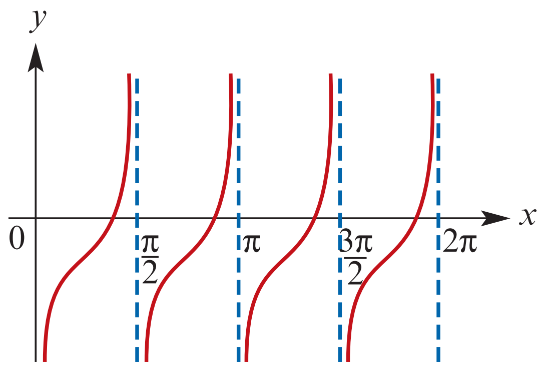
7 a



b



c



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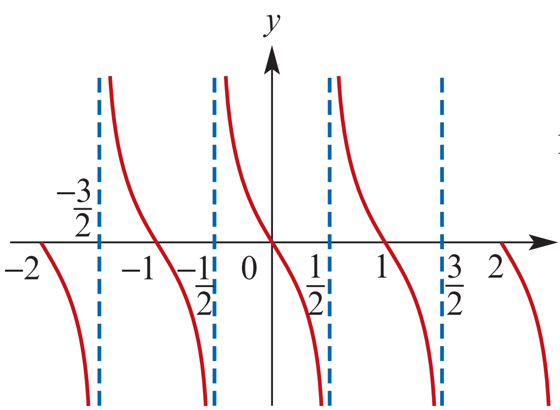
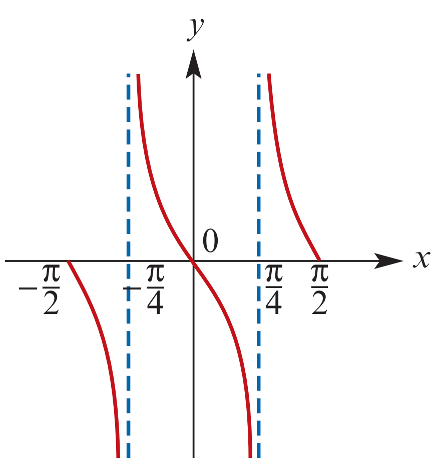
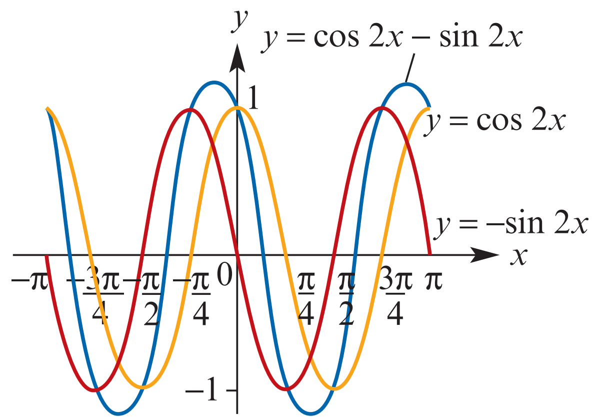


9



10 a c



b $\left(\frac{-5\pi}{8}, \frac{-1}{\sqrt{2}}\right), \left(\frac{-\pi}{8}, \frac{1}{\sqrt{2}}\right), \left(\frac{3\pi}{8}, \frac{-1}{\sqrt{2}}\right), \left(\frac{7\pi}{8}, \frac{1}{\sqrt{2}}\right)$

11 a $\sqrt{3}\sin x = \cos x$

$\sqrt{3}\tan x = 1$

$\tan x = \frac{1}{\sqrt{3}}$

$x = \frac{\pi}{6}, \frac{7\pi}{6}$

b $\sin(4x) = \cos(4x)$

$\tan(4x) = 1$

$4x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}, \frac{25\pi}{4}, \frac{29\pi}{4}$

$x = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \frac{17\pi}{16}, \frac{21\pi}{16}, \frac{25\pi}{16}, \frac{29\pi}{16}$

c $\sqrt{3}\sin(2x) = \cos(2x)$

$\tan(2x) = \frac{1}{\sqrt{3}}$

$2x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}$

$x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$

d $\tan(2x) = \dfrac{-1}{\sqrt{3}}$

$$2x = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$$

$$x = \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$$

e

$\sin(3x) = -\cos(3x)$

$\tan(3x) = -1$

$$3x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}, \frac{19\pi}{4}, \frac{23\pi}{4}$$

$$x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

f $\tan x = \dfrac{1}{2}$

using the CAS calculator $x = 0.4636$, 3.6052

g $\tan x = 2$

using the CAS calculator x = 1.1071, 4.2487

h $\tan(2x) = -1$

$$2x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$$

$$x = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$$

i $\sqrt{3}\sin(3x) = \cos(3x)$

$$\tan(3x) = \frac{1}{\sqrt{3}}$$

$$3x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}, \frac{25\pi}{6}, \frac{31\pi}{6}$$

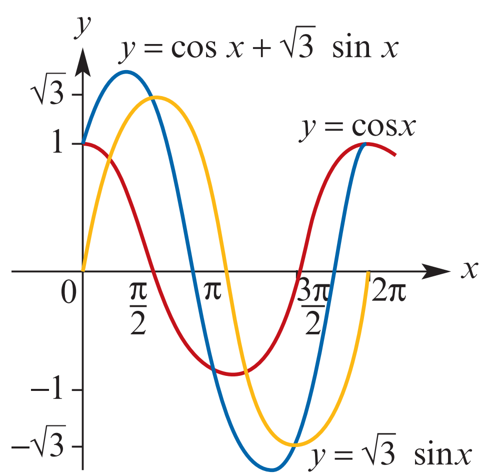
$$x = \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}, \frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}$$

j $\tan(3x) = \sqrt{3}$

$$3x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}, \frac{13\pi}{3}, \frac{16\pi}{3}$$

$$x = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$$

12a,c



b $\left(\dfrac{\pi}{6}, -\dfrac{\sqrt{3}}{2}\right), \left(\dfrac{7\pi}{6}, -\dfrac{\sqrt{3}}{2}\right)$

13 a $\tan\left(2x - \dfrac{\pi}{4}\right) = \sqrt{3}$

$$2x - \frac{\pi}{4} = \cdots \frac{-2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}, \cdots$$

$$2x = \cdots \frac{-5\pi}{12}, \frac{7\pi}{12}, \frac{19\pi}{12}, \frac{31\pi}{12}, \frac{43\pi}{12}, \cdots$$

$$x = \cdots \frac{-5\pi}{24}, \frac{7\pi}{24}, \frac{19\pi}{24}, \frac{31\pi}{24}, \frac{43\pi}{24}, \cdots$$

but $0 \le x \le 2\pi$

$$\therefore \quad x = \frac{7\pi}{24}, \frac{19\pi}{24}, \frac{31\pi}{24}, \frac{43\pi}{24}$$

b $\tan(2x) = \dfrac{-1}{\sqrt{3}}$

$$2x = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$$

$$x = \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$$

c $\tan\left(3x - \frac{\pi}{6}\right) = -1$

$$3x - \frac{\pi}{6} = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}, \frac{19\pi}{4}, \frac{23\pi}{4}$$

$$3x = \frac{11\pi}{12}, \frac{23\pi}{12}, \frac{35\pi}{12}, \frac{47\pi}{12}, \frac{59\pi}{12}, \frac{71\pi}{12}$$

$$x = \frac{11\pi}{36}, \frac{23\pi}{36}, \frac{35\pi}{36}, \frac{47\pi}{36}, \frac{59\pi}{36}, \frac{71\pi}{36}$$

14 asymptotes at $t = (2k+1)\frac{\pi}{6}$

period $= T = \frac{\pi}{3}$

$T = \frac{\pi}{n}$,

$\therefore \quad n = 3$

$t = \frac{\pi}{12},\ y = 5$

$5 = A\tan\left(\frac{\pi}{4}\right) = A \times 1$

$A = 5$

15 $T = \frac{\pi}{n}$

$T = 2$

$\therefore$ n $= \frac{\pi}{2}$

$t = \frac{1}{2},\ y = 6$

$6 = A\tan\left(\frac{\pi}{4}\right)$

$A = 6$

Solutions to Exercise 6K

1 a $\cos^{-1}(1) = 0$

i $2\pi \pm 0 = 2\pi$

ii $4\pi \pm 0 = 4\pi$

iii $-4\pi \pm 0 = -4\pi$

b $\cos^{-1}\left(\dfrac{-1}{2}\right) = \dfrac{2\pi}{3}$

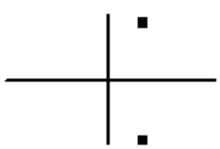
i $2\pi \pm \dfrac{2\pi}{3} = \dfrac{4\pi}{3}, \dfrac{8\pi}{3}$

ii $4\pi \pm \dfrac{2\pi}{3} = \dfrac{10\pi}{3}, \dfrac{14\pi}{3}$

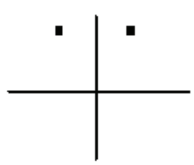
iii $-4\pi \pm \dfrac{2\pi}{3} = \dfrac{-14\pi}{3}, \dfrac{-10\pi}{3}$

2 a $\cos x = \dfrac{\sqrt{3}}{2}$

$$x = \left(\frac{\pi}{6}, \frac{-\pi}{6}\right) + 2n\pi \quad n \in \mathbb{Z}$$



b $2\sin 3x = \sqrt{3}$



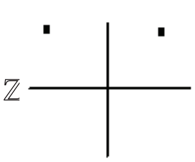
$$\sin 3x = \frac{\sqrt{3}}{2}$$

$$3x = \left(\frac{\pi}{3}, \frac{2\pi}{3}\right) + 2n\pi \quad n \in \mathbb{Z}$$

$$x = \left(\frac{\pi}{9}, \frac{2\pi}{9}\right) + \frac{2n\pi}{3} \quad n \in \mathbb{Z}$$

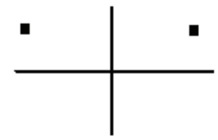
c $\tan x = \sqrt{3}$

$$x = \frac{\pi}{3} + n\pi, \quad n \in \mathbb{Z}$$



3 a $\sin x = \dfrac{1}{2}$

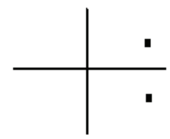
$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$



b $\cos 2x = \dfrac{\sqrt{3}}{2}$

$$2x = \frac{\pi}{6}, \frac{11\pi}{6}$$

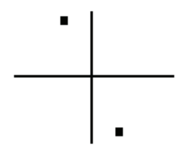
$$x = \frac{\pi}{12}, \frac{11\pi}{12}$$



c $\tan 2x = -\sqrt{3}$

$$2x = \frac{2\pi}{3}, \frac{5\pi}{3}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{6}$$



4 $x = n\pi + (-1)^n \sin^{-1}\left(\dfrac{1}{2}\right)$

$= n\pi + (-1)^n\left(\dfrac{\pi}{6}\right)$

$n = -2, \ x = -2\pi + \dfrac{\pi}{6} = -\dfrac{11\pi}{6}$

$n = -1, \ x = -\pi - \dfrac{\pi}{6} = -\dfrac{7\pi}{6}$

$n = 0, \ x = \dfrac{\pi}{6}$

$n = 1, \ x = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$

$\ldots$

$x = -\dfrac{11\pi}{6}, -\dfrac{7\pi}{6}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

5 $x = 2n\pi \pm (\pi = 2x\pi \pm) \cos^{-1}\left(\frac{1}{2}\right)$

$= 2n\pi \pm \frac{\pi}{3}$

$n = 0, 1 : \ x = \frac{-\pi}{3}, \ \frac{\pi}{3}, \ \frac{5\pi}{3}$

6 **a** $2\left(x + \frac{\pi}{3}\right) = \pm\frac{\pi}{3} + 2n\pi, \quad n \in \mathbb{Z}$

$x + \frac{\pi}{3} = n\pi \pm \frac{\pi}{6}$

$x = \left(\frac{-\pi}{2}, \ \frac{-\pi}{6}\right) + n\pi$

b $\tan\left(2\left(x + \frac{\pi}{4}\right)\right) = \sqrt{3}$

$2\left(x + \frac{\pi}{4}\right) = \frac{\pi}{3} + n\pi, n \in \mathbb{Z}$

$x + \frac{\pi}{4} = \frac{\pi}{6} + \frac{n\pi}{2}$

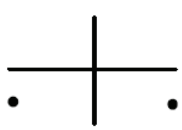
$x = \frac{-\pi}{12} + \frac{n\pi}{2}$

c $\sin\left(x + \frac{\pi}{3}\right) = \frac{-1}{2}$

$x + \frac{\pi}{3} = \frac{7\pi}{6}, \ \frac{11\pi}{6} + 2n\pi, n \in \mathbb{Z}$

$x = \left(\frac{5\pi}{6}, \ \frac{9\pi}{6}\right) + 2n\pi$

$= \left(\frac{5\pi}{6}, \ \frac{3\pi}{2}\right) + 2x\pi$



7 $\cos\left(2x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$2x + \frac{\pi}{4} = \pm\frac{\pi}{4} + 2n\pi n \in \mathbb{Z}$

$2x = \left(0, \ -\frac{\pi}{2}\right) + 2n\pi$

$x = \left(0, \ -\frac{\pi}{4}\right) + n\pi$

$x = -2\pi + \frac{3\pi}{4}, -\pi, -\pi + \frac{3\pi}{4},$

$0, \ \frac{3\pi}{4}, \ \pi, \ \pi + \frac{3\pi}{4}$

$x = \frac{-5\pi}{4}, -\pi, \ \frac{-\pi}{4}, 0, \ \frac{3\pi}{4}, \ \pi, \ \frac{7\pi}{4}$

8 $\tan\left(\frac{\pi}{6} - 3x\right) = \frac{1}{\sqrt{3}}$

$\frac{\pi}{6} - 3x = \frac{\pi}{6} + n\pi \quad n \in \mathbb{Z}$

$3x - \frac{\pi}{6} = n\pi - \frac{\pi}{6} \quad n \in \mathbb{Z}$

The−vebecomes part of n

$3x = n\pi$

$x = \frac{n\pi}{3} \quad n \in \mathbb{Z}$

$x = -\pi, \ \frac{-2\pi}{3}, \ \frac{-\pi}{3}, \ 0$

9

$\sin(4\pi x) = \frac{-\sqrt{3}}{2}$

$4\pi x = \left(\frac{4\pi}{3}, \ \frac{5\pi}{3}\right) + 2n\pi \quad n \in \mathbb{Z}$

$x = \left(\frac{1}{3}, \ \frac{5}{12}\right) + \frac{n}{2} \quad n \in \mathbb{Z}$

$x = -1 + \frac{1}{3}, -1 + \frac{5}{12}, \ \frac{-1}{2} + \frac{1}{3},$

$\frac{-1}{2} + \frac{5}{12}, \ \frac{1}{3}, \ \frac{5}{12}, \frac{1}{2} + \frac{1}{3}, \ \frac{1}{2} + \frac{5}{12}$

$x = \frac{-2}{3}, \frac{-7}{12}, \ \frac{-1}{6}, \frac{-1}{12},$

$\frac{1}{3}, \frac{5}{12}, \ \frac{5}{6}, \frac{11}{12}$

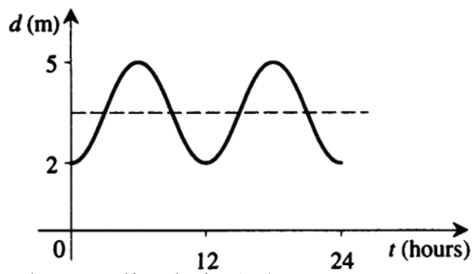
Solutions to Exercise 6L

1 a From the graph

i The range of the function is [2, 5] and the amplitude is 1.5.

ii The period is 12.

iii The function is of the form
$d = a\sin(nt + \varepsilon) + b$



The amplitude is 1.5.
Therefore $a = 1.5$
The period is 12.
Therefore $\dfrac{2p}{n} = 12$ and $n = \dfrac{\pi}{6}$
The centre of motion is at $d = 3\frac{1}{2}$.
Therefore $b = 3\frac{1}{2}$

$$\therefore \quad d = 1.5\sin\left(\frac{\pi t}{6} + \varepsilon\right) + 3.5$$

When $\quad t = 0,\ d = 2$

$$\therefore \quad 2 = 1.5\sin(\varepsilon) + 3.5$$

$$-1.5 = 1.5\sin(\varepsilon)$$

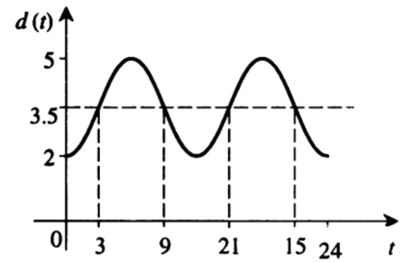
$\sin(\varepsilon) = -1$ and $\varepsilon = \dfrac{3\pi}{2}$

$$\therefore \quad d = 1.5\sin\left(\frac{\pi t}{6} + \frac{3\pi}{2}\right) + 3.5$$

But $\sin\left(\theta + \dfrac{3\pi}{2}\right) = \cos\theta$

$$\therefore \quad d = 3.5 - 1.5\cos\left(\frac{\pi t}{6}\right)$$

iv The length of the hour hand is 1.5 m. This is given by the amplitude.



b When is $d(t) \le 3.5$?
Consider $d(t) = 3.5$

$$3.5 = 3.5 - 1.5\cos\left(\frac{\pi t}{6}\right)$$

$$\therefore \quad \cos\left(\frac{\pi t}{6}\right) = 0$$

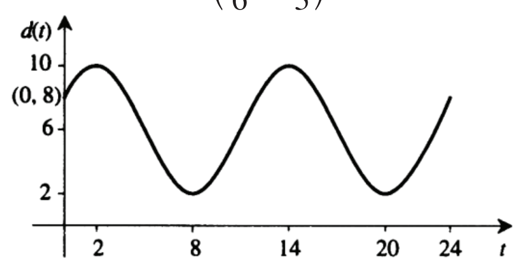
And $\dfrac{\pi t}{6} = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$ or $\dfrac{5\pi}{2}$ or $\dfrac{7\pi}{2}$ or $\cdots$

$\therefore \quad t = 3$ or 9 or 15 or 21 or $\cdots$

$\therefore \quad$ From the graph

$d(t) < 3.5$ for $t \in [0, 3) \cup (9, 15) \cup (21, 24]$

2 a $d(t) = 6 + 4\cos\left(\dfrac{\pi t}{6} - \dfrac{\pi}{3}\right)$



Centre: $d = 6$
Range: $[6 - 4,\ 6 + 4] = [2, 10]$
Period: $2\pi \div \dfrac{\pi}{6} = 12$

When $t = 0$, $d(0) = 6 + 4\cos\left(\dfrac{-\pi}{3}\right)$

$$= 6 + 4\frac{1}{2} = 8$$

When $t = 24$,

$$d(24) = 6 + 4\cos\left(4\pi - \frac{\pi}{3}\right)$$

$$= 6 + 4 \times \frac{1}{2} = 8$$

b Highest level is 10 m.
Consider $10 = 6 + 4\cos\left(\dfrac{\pi t}{6} - \dfrac{\pi}{3}\right)$

$1 = \cos\left(\frac{\pi t}{6} - \frac{\pi}{3}\right)$

$\therefore\ \frac{\pi t}{6} = \frac{\pi}{3}$ (No need to consider other solution as question asks for earliest time.)

$\therefore t = 2$

The water is first at its highest at 2:00 a.m.

c

When $d(t) = 2$

$$2 = 6 + 4\cos\left(\frac{\pi t}{6} - \frac{\pi}{3}\right)$$

$$-1 = \cos\left(\frac{\pi t}{6} - \frac{\pi}{3}\right)$$

$$\therefore\ \frac{\pi t}{6} = \frac{\pi}{3} = \pi \text{ or } 3\pi \text{ or } 5\pi \text{ or } \cdots$$

$$\therefore\ \frac{\pi t}{6} = \frac{4\pi}{3} \text{ or } \frac{10\pi}{3} \text{ or } \frac{16\pi}{3} \text{ or } \cdots$$

$$\therefore t = 8 \text{ or } 20 \text{ or } 32 \text{ or } \cdots$$

Only 8 and 20 are in the required domain.

$\therefore$ The water is 2 m up the wall at 8:00 a.m. and 8:00 p.m.

3 a The time between high tides is 12 hours, so the period = 12

$\frac{2\pi}{n} = 12 \Rightarrow n = \frac{\pi}{6}$

The average depth is 5 metres. Therefore $b = 5$.

The high tide is 8 m. Therefore amplitude = $8 - 5 = 3$ and $A = 3$

$$\therefore\ h(t) = 3\sin\left(\frac{\pi t}{6} + \varepsilon\right) + 5$$

When $t = 0$, $h = 8$ (t is the number of hours after 12:00 noon.)

$$\therefore\ 8 = 3\sin(\varepsilon) + 5$$

$$\therefore\ \sin(\epsilon) = 1 \text{ and } \varepsilon = \frac{\pi}{2}$$

$$\therefore\ h(t) = 3\sin\left(\frac{\pi t}{6} + \frac{\pi}{2}\right) + 5$$

$$= 3\cos\left(\frac{\pi t}{6}\right) + 5$$

b When $h = 6$

$$1 = 3\cos\left(\frac{\pi t}{6}\right)$$

$$\frac{1}{3} = \cos\left(\frac{\pi t}{6}\right)$$

$$\therefore\ \frac{\pi t}{6} = \cos^{-1}\left(\frac{1}{3}\right) \text{ or } 2\pi - \cos^{-1}\left(\frac{1}{3}\right) \text{ or } 2\pi + \cos^{-1}\left(\frac{1}{3}\right) \text{ or } 4\pi - \cos^{-1}\left(\frac{1}{3}\right)$$

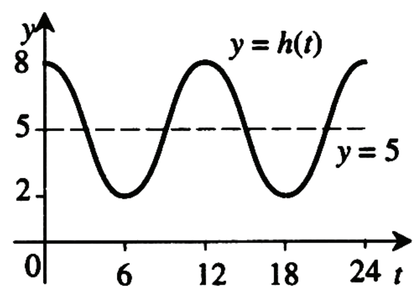
$$\therefore\ t = \frac{6}{\pi}\cos^{-1}\left(\frac{1}{3}\right) \text{ or } 12 - \frac{6}{\pi}\cos^{-1}\left(\frac{1}{3}\right) \text{ or } 12 + \frac{6}{\pi}\cos^{-1}\left(\frac{1}{3}\right) \text{ or } 24 - \frac{6}{\pi}\cos^{-1}\left(\frac{1}{3}\right)$$

$$\approx 2.351 \text{ or } 9.649 \text{ or } 14.351 \text{ or } 21.649$$

Depth of the water is 6 metres at the following times (times measured from 12 noon).

2:21 p.m. 9:39 p.m. 2:21 a.m. 9:39 a.m.

c



4 a Greatest distance occurs when $\sin 3t = 1$

$\therefore$ greatest distance = $3 + 2 = 5$m.

b Least distance occurs when $\sin 3t = -1$

$\therefore$ least distance = $3 - 2 = 1$m.

c When $x = 5$

$5 = 3 + 2\sin 3t$

$2 = 2\sin 3t$

$1 = \sin 3t$

$$3t = \frac{\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \frac{9\pi}{2} \text{ or } \cdots$$

$$\therefore \quad t = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } \frac{9\pi}{6} \text{ or } \cdots$$

For $0 \leq t \leq 5$ the times are: 0.524 sec, 2.618 sec, 4.712 sec

d When $x = 3$

$$3 = 3 + 2\sin 3t$$

$$0 = \sin 3t$$

$$3t = 0 \text{ or } \pi \text{ or } 2\pi \text{ or } 3\pi \text{ or } \cdots$$

$$t = 0 \text{ or } \frac{\pi}{3} \text{ or } \frac{2\pi}{3} \text{ or } \pi \text{ or } \cdots$$

For $0 \leq t \leq 3$ the times are: 0 sec, 1.047 sec, 2.094 sec

e The particle oscillates about the point $x = 3$ from $x = 1$ to $x = 5$.

5 $A = 21 - 3\cos\left(\frac{\pi t}{12}\right)$ or $0 \leq t \leq 24$ gives the temperature inside the house and $B = 22 - 5\cos\left(\frac{\pi t}{12}\right)$ for $0 \leq t \leq 24$ gives the temperature outside the house.

a When $t = 4$ (time measured from 4:00 a.m.)

$$A = 21 - 3\cos\frac{4\pi}{12}$$

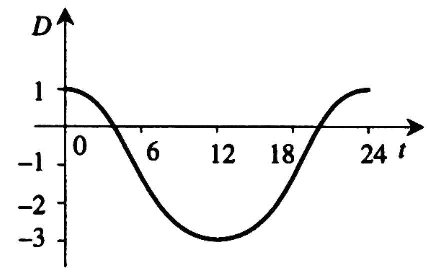
$$= 21 - 3\cos\frac{\pi}{3}$$

$$= 21 - 1.5$$

$$= 19.5$$

i.e. the temperature outside the house is 19.5°C at 8:00 a.m.

b $D = A - B = 2\cos\left(\frac{\pi t}{12}\right) - 1$



c

d The inside temperature is less than the outside temperature.

This occurs when

$A < B$

$\Leftrightarrow A - B < 0$

$\Leftrightarrow D < 0$

Consider $\quad D = 0$

$$0 = 2\cos\left(\frac{\pi t}{12}\right) - 1$$

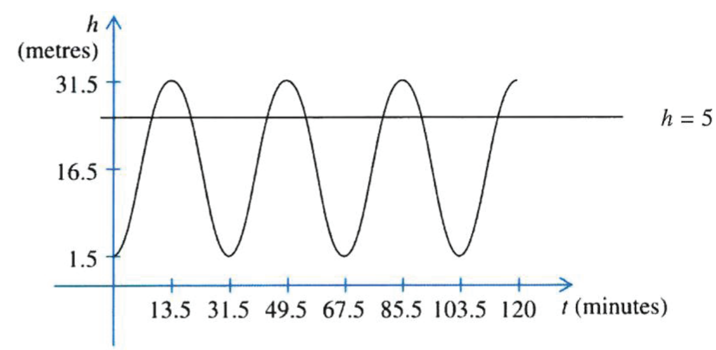
Implies $\quad \frac{1}{2} = \cos\left(\frac{\pi t}{12}\right)$

$$\therefore \quad \frac{\pi t}{12} = \frac{\pi}{3} \text{ or } \frac{5\pi}{3} \text{ or } \cdots$$

$$\therefore \quad t = 4 \text{ or } 20 \text{ or } \cdots$$

For $0 \leq t \leq 24$, $D < 0$ for $t \in (4, 20)$, i.e. $4 < t < 20$.

6 a



b When $t = 0,\ h = 15\sin(-45)^\circ + 16.5$
$= 5.89$ m (correct to two decimal places)

c Solving the equation $h(t) = 5$

$$\begin{aligned} 5 &= 15\sin(10t - 45)^\circ + 16.5 \\ -11.5 &= 15\sin(10t - 45)^\circ \\ -\frac{23}{30} &= \sin(10t - 45)^\circ \end{aligned}$$

The first positive solution is $t = 27.51$ seconds correct to two decimal places.

d There are 6 points of intersection with the graph of $h = 5$

e 20 times

f

$$\begin{aligned} t = 100,\ h(100) &= 15\sin(100 - 45) + 16.5 \\ &= 15\sin 955^\circ + 16.5 \\ &\approx 4.21 \text{ metres} \end{aligned}$$

g The phase shift will be different for Hamish; the range and the period will be the same. Consider $k(t) = 15\sin(10t + c)^\circ + 16.5$

When $t = 0,\ k(0) = 1.5$

$$\begin{aligned} 15\sin(+\,c)^\circ + 16.5 &= 1.5 \\ \sin c^\circ &= -1 \\ c^\circ &= 270^\circ \\ k(t) &= 15\sin(10t + 270)^\circ + 16.5 \\ t &= 100, \\ k(100) &= 15\sin(1270^\circ) + 16.5 \\ &\approx 13.9 \text{ metres} \end{aligned}$$

Solutions to technology-free questions

1 Note that $x \in [-\pi,\ 2\pi]$ throughout.

a $\sin x = \dfrac{1}{2}$

x is in the first or second quadrant.

$x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

b $2\cos x = -1$, so $\cos x = -\dfrac{1}{2}$

x is in the second or third quadrant.

$x = -\dfrac{2\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

c $2\cos x = \sqrt{3}$, so $\cos x = \dfrac{\sqrt{3}}{2}$

x is in the first or fourth quadrant.

$x = -\dfrac{\pi}{6}, \dfrac{\pi}{6}, \dfrac{11\pi}{6}$

d $\sqrt{2}\sin x + 1 = 0$, so $\sin x = -\dfrac{1}{\sqrt{2}}$

x is in the third and fourth quadrants.

$x = -\dfrac{3\pi}{4}, -\dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

e $4\sin x + 2 = 0$, so $\sin x = -\dfrac{1}{2}$

x is in the third or fourth quadrant.

$x = -\dfrac{5\pi}{6}, -\dfrac{\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

f $\sin 2x + 1 = 0$, so $\sin 2x = -1$

$2x = -\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{7\pi}{2}$

$x = -\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

g $\cos 2x = -\dfrac{1}{\sqrt{2}}$

$2x$ is in the second or third quadrant.

$2x = -\dfrac{5\pi}{4}, -\dfrac{3\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{11\pi}{4}, \dfrac{13\pi}{4}$

$x = -\dfrac{5\pi}{8}, -\dfrac{3\pi}{8}, \dfrac{3\pi}{8}, \dfrac{5\pi}{8}, \dfrac{11\pi}{8}, \dfrac{13\pi}{8}$

h $2\sin 3x - 1 = 0$, so $\sin 3x = \dfrac{1}{2}$

$3x$ is in the first or second quadrant.

$$3x = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$$

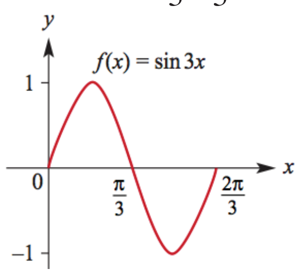
$$\frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$$

$$x = -\frac{11\pi}{18}, -\frac{7\pi}{18}, \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}$$

$$\frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$$

2 **a** $y = \sin 3x$

$y = 0 : x = 0, \frac{\pi}{3}, \frac{2\pi}{3}$ for one cycle



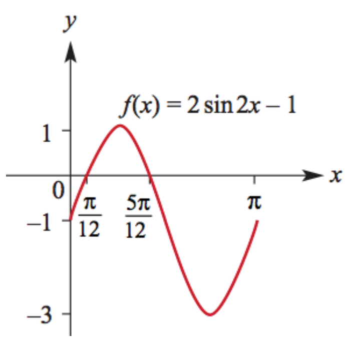
b $y = 2\sin 2x - 1$

$$y = 0 : \sin 2x = \frac{1}{2}$$

$2x = \frac{\pi}{6}, \frac{5\pi}{6}$ for one cycle

$$x = \frac{\pi}{12}, \frac{5\pi}{12}$$

$$x = 0 : y = -1$$



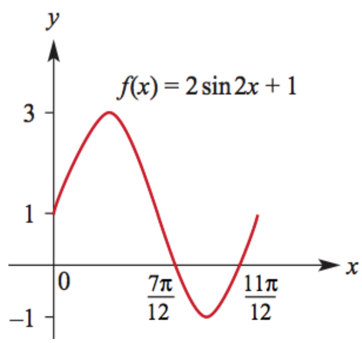
c $y = 2\sin 2x + 1$

$y = 0$: $\sin 2x = -\dfrac{1}{2}$

$2x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$ for one cycle

$x = \dfrac{7\pi}{12}, \dfrac{11\pi}{12}$

$x = 0$: $y = 1$

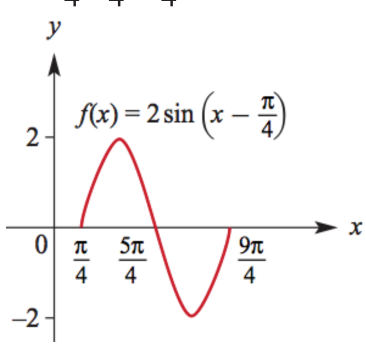


d $y = 2\sin\left(x - \dfrac{\pi}{4}\right)$ so translate the graph of $y = 2\sin x$ by $\dfrac{\pi}{4}$ to the right.

$y = 0$: $\sin\left(x - \dfrac{\pi}{4}\right) = 0$

$x - \dfrac{\pi}{4} = 0,\ \pi,\ 2\pi$ for one cycle

$x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}$

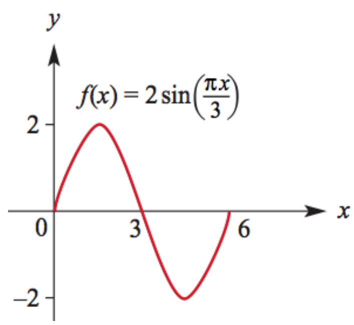


e $y = 2\sin\dfrac{\pi x}{3}$

$y = 0$: $\sin\dfrac{\pi x}{3} = 0$

$\dfrac{\pi x}{3} = 0, \pi, 2\pi$ for one cycle

$x = 0, 3, 6$



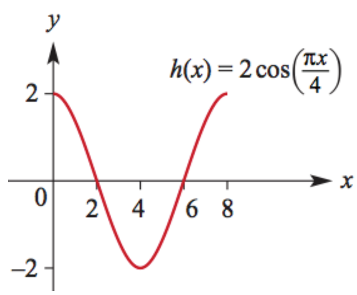
f $y = 2\cos\dfrac{\pi x}{4}$

$y = 0$: $\cos\dfrac{\pi x}{4} = 0$

$\dfrac{\pi x}{4} = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ for one cycle

$x = 2, 6$

$x = 0$: $y = 2$



3 Note that $x \in [0, 360]$ throughout.

a $\sin x° = 0.5$

$x°$ is in the first or second quadrant.

$x = 30, 150$

b $\cos(2x)° = 0$

$2x = 90, 270, 450, 630$

$x = 45, 135, 225, 315$

c $2\sin x° = -\sqrt{3}$, so $\sin x° = -\dfrac{\sqrt{3}}{2}$

$x°$ is in the third or fourth quadrant.

$x = 240, 300$

d $\sin(2x + 60)° = -\dfrac{\sqrt{3}}{2}$

$(2x + 60)°$ is in the third or fourth quadrant.

$$2x + 60 = 240, 300, 600, 660$$
$$2x = 180, 240, 540, 600$$
$$x = 90, 120, 270, 300$$

e $2\sin\left(\frac{1}{2}x\right)^\circ = \sqrt{3}$, so $\sin\left(\frac{1}{2}x\right)^\circ = \frac{\sqrt{3}}{2}$

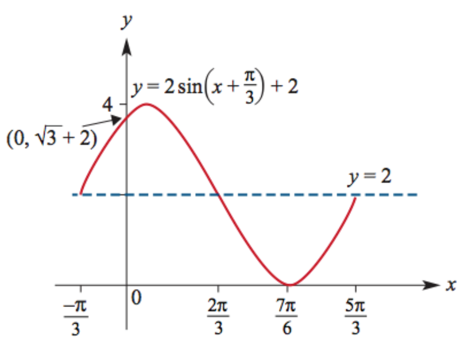
$\left(\frac{1}{2}x\right)^\circ$ is in the first or second quadrant.

$$\frac{1}{2}x = 60, 120$$
$$x = 120, 240$$

4 **a** $y = 2\sin\left(x + \frac{\pi}{3}\right) + 2$

The graph is that of $y = 2\sin x$ translated $\frac{\pi}{3}$ units to the left and 2 units up.

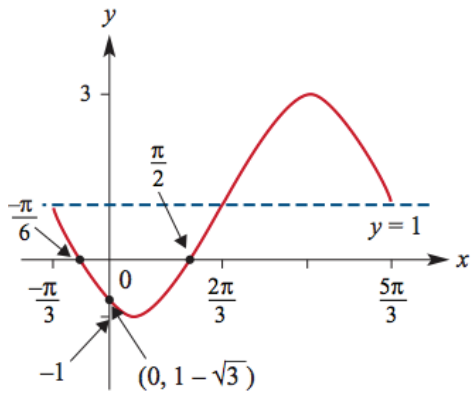
$$y = 0: \sin\left(x + \frac{\pi}{3}\right) = -1$$
$$x + \frac{\pi}{3} = \frac{3\pi}{2} \text{ for one cycle}$$
$$x = \frac{7\pi}{6}$$
$$x = 0: y = \sqrt{3} + 2$$



b $y = -2\sin\left(x + \frac{\pi}{3}\right) + 1$

The graph is that of $y = -2\sin x$ translated $\frac{\pi}{3}$ units to the left and 1 unit up.

$$y = 0: \sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$$
$$x + \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6} \text{ for one cycle}$$
$$x = -\frac{\pi}{6}, \frac{\pi}{2}$$
$$x = 0: y = 1 - \sqrt{3}$$



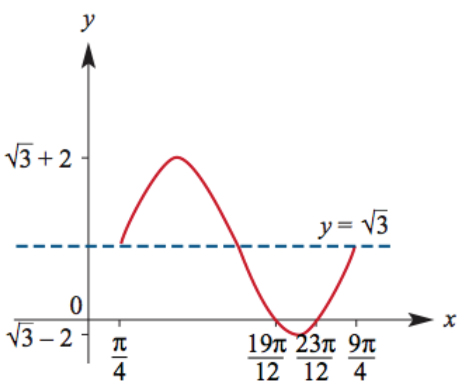
c $y = 2\sin\left(x - \frac{\pi}{4}\right) + \sqrt{3}$

The graph is that of $y = 2\sin x$ translated $\frac{\pi}{4}$ units to the right and $\sqrt{3}$ units up.

$$y = 0: \sin\left(x - \frac{\pi}{4}\right) = -\frac{\sqrt{3}}{2}$$

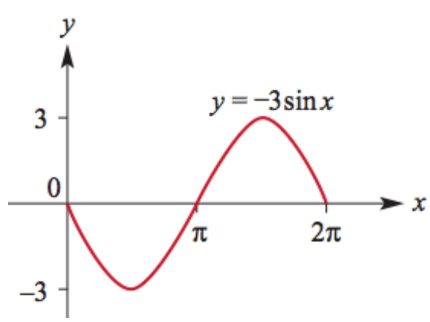
$$x - \frac{\pi}{4} = \frac{4\pi}{3}, \frac{5\pi}{3} \text{ for one cycle}$$

$$x = \frac{19\pi}{12}, \frac{23\pi}{12}$$



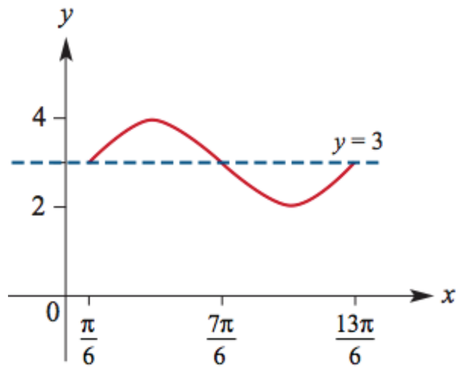
d $y = -3\sin x$

$y = 0: x = 0, \pi, 2\pi$ for one cycle



e $y = \sin\left(x - \dfrac{\pi}{6}\right) + 3$

The graph is that of $y = \sin x$ translated $\dfrac{\pi}{6}$ units to the right and 3 units up, so there are no x intercepts.



f $y = 2\sin\left(x - \dfrac{\pi}{2}\right) + 1$

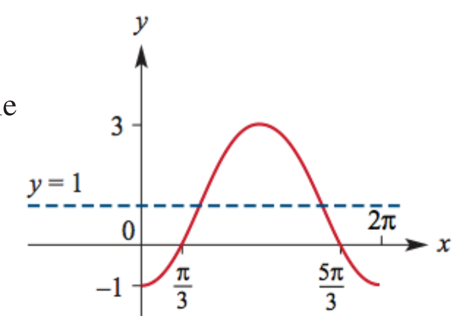
Now $\sin\left(x - \dfrac{\pi}{2}\right) = -\sin\left(\dfrac{\pi}{2} - x\right) = -\cos x$, so $y = -2\cos x + 1$ is an equivalent form.

The graph is that of $y = -2\cos x$ translated 1 unit up.

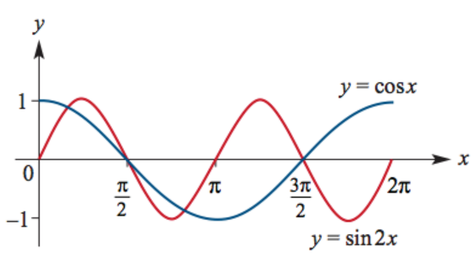
$y = 0$: $\cos x = \dfrac{1}{2}$

$x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$ for one cycle

$x = 0$: $y = -1$



5 The graphs are shown below.



a The line with equation $y = 0.6$ cuts the curve with equation $y = \sin 2x$ four times. The equation has 4 solutions.

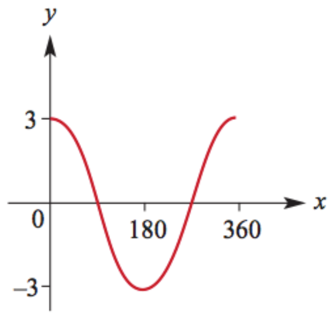
b The curve with equation $y = \sin 2x$ cuts the curve with equation $y = \cos x$ four times. The equation has 4 solutions.

c Rewrite the equation in the form:

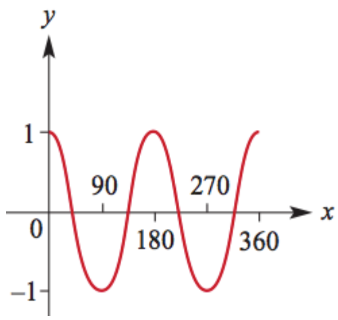
$\sin 2x - 1 = \cos x$

The curve with equation $y = \sin 2x - 1$ is that of $y = \sin 2x$ translated 1 unit down. Looking at the graphs above, it is clear that translating the sine graph 1 unit down means that two intersections with the cosine graph are lost and only two remain. The equation has 2 solutions.

6 **a** $y = 3\cos x°$



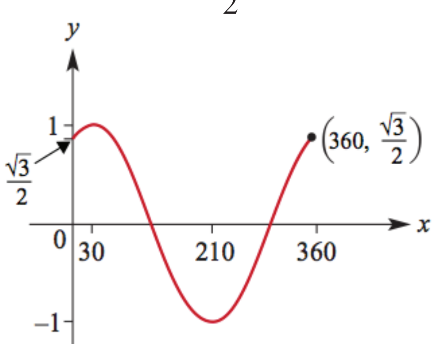
b $y = \cos 2x°$



c $y = \cos(x - 30)°$

The graph is that of $y = \cos x°$ translated 30° to the right.

$x = 0, 360: y = \dfrac{\sqrt{3}}{2}$



7 Note that $x \in [-\pi, \pi]$ throughout.

a $\tan x = \sqrt{3}$

x is in the first or third quadrant.

$X = -\frac{2\pi}{3}, \frac{\pi}{3}$

b $\tan x = -1$

x is in the second or fourth quadrant.

$x = -\frac{\pi}{4}, \frac{3\pi}{4}$

c $\tan 2x = -1$

$2x$ is in the second or fourth quadrant.

$$2x = -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$$

$$x = -\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}$$

d $\tan(2x) + \sqrt{3} = 0$, so $\tan 2x = -\sqrt{3}$

$2x$ is in the second or fourth quadrant.

$$2x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$$

$$x = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}$$

8 $\tan(x) = \sqrt{3}$

$$x = -\frac{2\pi}{3}, \frac{\pi}{3}$$

9 **a** $a \cos \frac{\pi}{6} = \sin \frac{\pi}{6}$

$$\frac{\sqrt{3}a}{2} = \frac{1}{2}$$

$$a = \frac{1}{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{3}$$

b $\tan x = \frac{\sqrt{3}}{3}$

$$x = \frac{\pi}{6}, -\frac{5\pi}{6}$$

10 **a** $\sin 2x = -1$

$$2x = -\frac{\pi}{2} + 2n\pi$$

$$x = -\frac{\pi}{4} + n\pi, n \in \mathbb{Z}$$

b $\cos 2x = 1$

$$3x = 2n\pi$$

$$x = -\frac{2n\pi}{3}, n \in \mathbb{Z}$$

c $\tan x = -1$

$$x = n\pi - \frac{\pi}{4}, n \in \mathbb{Z}$$

Solutions to multiple-choice questions

1 C $3\sin\left(\frac{1}{2}x - \pi\right) + 4$

Period $= \frac{2\pi}{n},\ n = \frac{1}{2}$

$= \frac{2\pi}{\frac{1}{2}}$

Period $= 4\pi$

2 A $f(x) = 5\cos\left(2x - \frac{\pi}{3}\right) - 7$

Range $= [-7+5,\ -7-5]$

Range $= [-2,\ -12]$

3 E $y = \sin x$

A dilation of factor $\frac{1}{2}$ from the y-axis:

$y = \sin(2x)$

A translation of $\frac{\pi}{4}$ in the positive direction of the x axis:

$y = \sin\left(2\left(x - \frac{\pi}{4}\right)\right)$

4 D $f : R \to R, f(x) = a\sin(bx) + c$

Period $= \frac{2\pi}{b}$

5 A $3\sin(x) - 1 = b$

It is only possible for the equation to have one positive real number solution at the turning point:

Max value $= 2$

6 C

7 C $f(x) = p\cos 5x + q,\ p > 0$

$f(x) \le 0$

$0 \ge p\cos 5x + q$

Maximum y value must be negative, this value occurs at $x = 0$

$\therefore\ 0 \ge p\cos 0 + q$

$\therefore\ p \le -q$

8 B One rotation = period

$\frac{2\pi}{6\pi} = \frac{1}{3}$

9 C

10 E $y = \cos x$

A dilation of factor 2 from the x-axis:

$y = 2\cos x$

A translation of $\frac{\pi}{4}$ in the positive direction of the x-axis:

$y = 2\cos\left(x - \frac{\pi}{4}\right)$

11 D

12 B Period of graph shown:

$\frac{2\pi}{n} = 8$

$n = \frac{\pi}{4}$

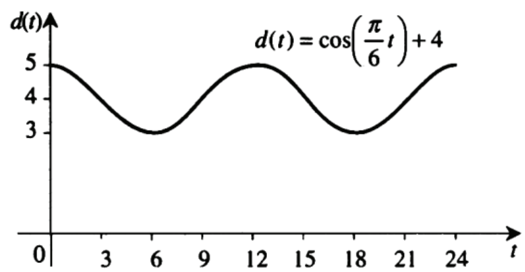
Graph is translated 3 units in the positive direction of the y-axis. As the graph is initially positive it must be a sine function.

$\therefore\ y = 3 + 3\sin\left(\frac{\pi x}{4}\right)$

Solutions to extended-response questions

1 The time between high tide and low tide is 6 hours.
Assume the function modelling the river is sinusoidal.
i.e. $d(t) = a\sin(nt + \varepsilon) + b$
where $d(t)$ is the depth at time t (measured from 0)
Period = 12 $\therefore \dfrac{2\pi}{n} = 12$ and $n = \dfrac{\pi}{6}$
Average depth = 4 m i.e. $d = 4$ is the centre $\therefore\ b = 4$
Highest value = 5 $\therefore$ amplitude = 5 − 4 = 1
Range = [3, 5] and $a = 1$

a $d(t) = \sin\left(\dfrac{\pi t}{6} + \varepsilon\right) + 4$



Also there is a high tide at 12:00.

$\therefore$ When $t = 12, d = 5$

$\therefore\quad 5 = \sin(2\pi + \varepsilon) + 4$

i.e. $\sin(\varepsilon) = 1$

and $\therefore\quad \varepsilon = \dfrac{\pi}{2}$

$\therefore\quad d(t) = \sin\left(\dfrac{\pi t}{6} + \dfrac{\pi}{2}\right) + 4$

But $\quad \sin\left(\theta + \dfrac{\pi}{2}\right) = \cos\theta$

$\therefore\quad d(t) = \cos\left(\dfrac{\pi t}{6}\right) + 4$

b For $d(t) \geq 4$ consider first $d = 4$

$4 = \cos\left(\dfrac{\pi t}{6}\right) + 4$

$\cos\left(\dfrac{\pi t}{6}\right) = 0$

$\therefore\quad \dfrac{\pi t}{6} = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$ or $\dfrac{5\pi}{2}$ or $\dfrac{7\pi}{2}$ or $\dfrac{9\pi}{2}$ or $\cdots$

$\therefore\quad t = 3$ or 9 or 15 or $\cdots$

$\therefore\quad d \geq 4$ for $t \in [0, 3] \cup [9, 15] \cup \ldots$

The boat may enter the harbour after 9:00 a.m. but it must leave by 3:00 p.m.

c For $d \geq 3.5$

First consider $d = 3.5$

$$3.5 = \cos\left(\frac{\pi t}{6}\right) + 4$$

$$\therefore \quad -0.5 = \cos\left(\frac{\pi t}{6}\right)$$

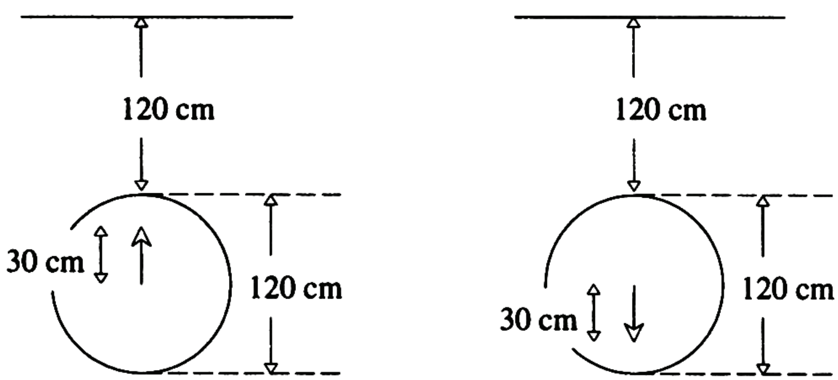
$$\therefore \quad \frac{\pi t}{6} = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3} \text{ or } \frac{8\pi}{3} \text{ or } \cdots$$

$$\therefore \quad t = 4 \text{ or } 8 \text{ or } 16 \text{ or } \cdots$$

$\therefore$ $d \geq 3.5$ for $t \in [0, 4] \cup [8, 16] \cup \ldots$ (See graph above)

A boat can enter the river after 8:00 a.m. but must leave before 4:00 p.m.

2 a



The minimum distance $= 120 + 30$ $= 150$ cm

The maximum distance $= 120 + 60 + 30$ $= 210$ cm

The mean distance $= 180$ cm

b $y = A\sin(nt + \varepsilon) + b$

From the above:

Mean distance is 180 cm $\therefore b = 180$

Range $= [150, 210]$

$\therefore$ amplitude $= 30, A = 30$

Period $= 12$ $\therefore \dfrac{2\pi}{n} = 12$

i.e. $n = \dfrac{\pi}{6}$

$$\therefore \quad y = 30\sin\left(\frac{\pi t}{6} + \varepsilon\right) + 180$$

When $t = 0$, distance is minimum

$$\therefore \quad y = 150$$

$$\therefore \quad 150 = 30\sin(\varepsilon) + 180$$

$$\therefore \quad \sin(\varepsilon) = -1$$

$$\varepsilon = -\frac{\pi}{2}$$

$$\therefore \quad y = 30\sin\left(\frac{\pi t}{6} - \frac{\pi}{2}\right) + 180$$

$$= 180 - 30\cos\left(\frac{\pi t}{6}\right) \text{ Since } \sin\left(\theta - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - \theta\right) = -\cos\theta.$$

c **i** When $t = 2$

$$y = 180 - 30\cos\left(\frac{\pi \times 2}{6}\right)$$

$$= 180 - 30\cos\left(\frac{\pi}{3}\right)$$

$$= 180 - 15$$

$$= 165$$

The distance from the ceiling to the tip of the hour hand is 165 cm at 2:00.

ii When $t = 23$

$$y = 180 - 30\cos\left(\frac{\pi \times 23}{6}\right)$$

$$= 180 - 30\cos\left(\frac{-\pi}{6}\right)$$

$$= 180 - 30 \times \frac{\sqrt{3}}{2}$$

$$= 180 - 15\sqrt{3}$$

$$\approx 154\,\text{cm}$$

The distance from the ceiling to the top of the hour hand is approx. 154 cm at 23:00.

d When $y = 200$

$$200 = 180 - 30\cos\left(\frac{\pi t}{6}\right)$$

$$\frac{-2}{3} = \cos\left(\frac{\pi t}{6}\right)$$

$$\therefore \quad \frac{\pi t}{6} = \pi - \cos^{-1}\left(\frac{2}{3}\right) \text{ or } \pi + \cos^{-1}\left(\frac{2}{3}\right) \text{ or } \cdots$$

$$\therefore \quad t = 6 - \frac{6}{\pi}\cos^{-1}\left(\frac{2}{3}\right) \text{ or } 6 + \frac{6}{\pi}\cos^{-1}\left(\frac{2}{3}\right)$$

$$\therefore \quad t \approx 4.39 \text{ or } 7.61$$

The tip of the hour hand is 200 cm below the ceiling at 7:36 and 4:24.

3 **a** Amplitude = 3 When $t = 0$, $y = -3$ and therefore $a = -3$

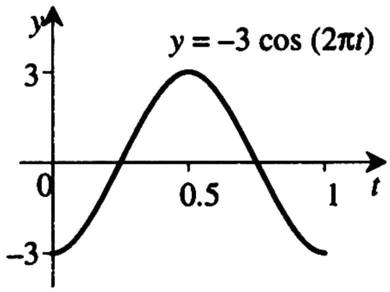
Period = 1 $\therefore \frac{2\pi}{n} = 1$

$$\therefore \quad n = 2\pi$$

$$\therefore \quad a = -3 \text{ and } n = 2\pi$$

$$y = -3\cos(2\pi t)$$

b



c **i** When $y = 1.5$

$$\begin{aligned}1.5 &= -3\cos(2\pi t)\\ -\frac{1}{2} &= \cos(2\pi t)\\ \frac{2\pi}{3} &= 2\pi t\\ \therefore\quad t &= \frac{1}{3}\end{aligned}$$

The centre of the weight is 1.5 cm above 0 after $\frac{1}{3}$ second.

ii When $y = -1.5$

$$\begin{aligned}-1.5 &= -3\cos(2\pi t)\\ \frac{1}{2} &= \cos(2\pi t)\\ \frac{\pi}{3} &= 2\pi t\\ \frac{1}{6} &= t\end{aligned}$$

The centre of the weight is 1.5 cm below 0 after $\frac{1}{6}$ second.

d When $y = -1$

$$\begin{aligned}-1 &= -3\cos(2\pi t)\\ \frac{1}{3} &= \cos(2\pi t)\\ \cos^{-1}\left(\frac{1}{3}\right) &= 2\pi t\\ \therefore t &= \frac{1}{2\pi}\cos^{-1}\left(\frac{1}{3}\right)\\ &\approx 0.196\end{aligned}$$

It reaches a point 1 cm below 0 after 0.196 seconds.

4 a $y = a\sin(nt + \varepsilon) + b$

The average inflow is $100\,000\ \text{m}^3/\text{day}$ $\therefore\ b = 100\,000$

Minimum flow is 80 000 and maximum 120 000.

$\therefore$ range = [80 000, 120 000]

Amplitude is 20 000 $\therefore\ a = 20\,000$

The period is 365 days $\therefore\ \dfrac{2\pi}{n} = 365$

and therefore $n = \dfrac{2\pi}{365}$

$\therefore\ y = 20\,000\sin\left(\dfrac{2\pi t}{365} + \epsilon\right) + 100\,000$

When $t = 121$, $y = 120\,000$

$\therefore\ 120\,000 = 20\,000\sin\left(\dfrac{2\pi \times 121}{365} + \varepsilon\right) + 100\,000$

$\therefore\ \sin\left(\dfrac{2\pi \times 121}{365} + \varepsilon\right) = 1$

$\dfrac{2\pi \times 121}{365} + \varepsilon = \dfrac{\pi}{2}$ or $\dfrac{5\pi}{2}$ or $\dfrac{9\pi}{2}$ or $\cdots$

$\therefore\ \varepsilon = \dfrac{\pi}{2} - \dfrac{2\pi \times 121}{365}$ or $\dfrac{5\pi}{2} - \dfrac{2\pi \times 121}{365}$ or $\dfrac{9\pi}{2} - \dfrac{2\pi \times 121}{365}$ or $\ldots$

≈ -0.512 or 5.77 or $\ldots$

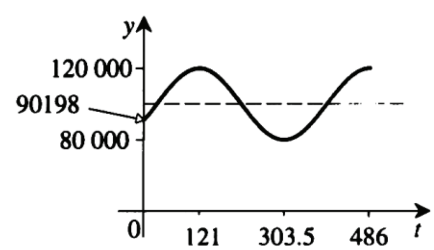
Choose $\varepsilon = 5.77$

$\therefore\ y = 20\,000\sin\left(\dfrac{2\pi t}{365} + 5.77\right) + 100\,000$

b When $t = 0$,

$y = 20\,000\sin(5.77) + 100\,000$

$\approx 90\,198.33$



c **i** When $y = 90\,000$

$90\,000 = 20\,000\sin\left(\dfrac{2\pi t}{365} + 5.77\right) + 100\,000$

$-\dfrac{1}{2} = \sin\left(\dfrac{2\pi t}{365} + 5.77\right)$

$\therefore\ \dfrac{2\pi t}{365} + 5.77 = \dfrac{7\pi}{6}$ or $\dfrac{11\pi}{6}$ or $\dfrac{19\pi}{6}$ or $\dfrac{23\pi}{6}$ or $\cdots$

$\therefore\ \dfrac{2\pi t}{365} = \dfrac{7\pi}{6} - 5.77$ or $\dfrac{11\pi}{6} - 5.77$ or $\dfrac{19\pi}{6} - 5.77$ or $\dfrac{23\pi}{6} - 5.77$ or $\cdots$

$\dfrac{2\pi t}{365} \approx -2.1058$ or -0.01141 or 4.1773 or 6.2717

(Negative values are not considered.)

$\therefore\ t \approx 242.7$ or $t \approx 364.3$

i.e. when $t = 242.7$ and $t = 364.3$ the inflow per day is $90\,000\ \text{m}^3/\text{day}$.

ii When $y = 110\,000$

$$110\,000 = 20\,000\sin\left(\frac{2\pi t}{365} + 5.77\right) + 100\,000$$

$$\frac{1}{2} = \sin\left(\frac{2\pi t}{365} + 5.77\right)$$

$$\frac{2\pi t}{365} + 5.77 = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6}$$

$$t = 60.2 \text{ or } t = 181.8$$

(Negative values not considered.)

i.e. when $t = 60.2$ and $t = 181.8$ the inflow is $110\,000\,\text{m}^3$/day.

d When $t = 152$

$$y = 20\,000\sin\left(\frac{2\pi \times 152}{365} + 5.77\right) + 100\,000$$

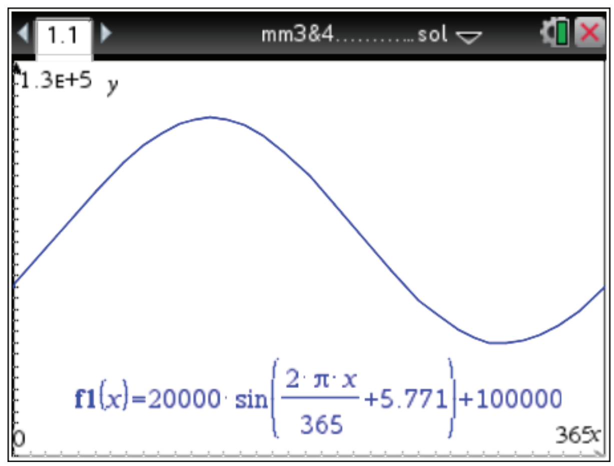
$$= 117219$$

The inflow rate is $117\,219\,\text{m}^3$/day on 1 June.

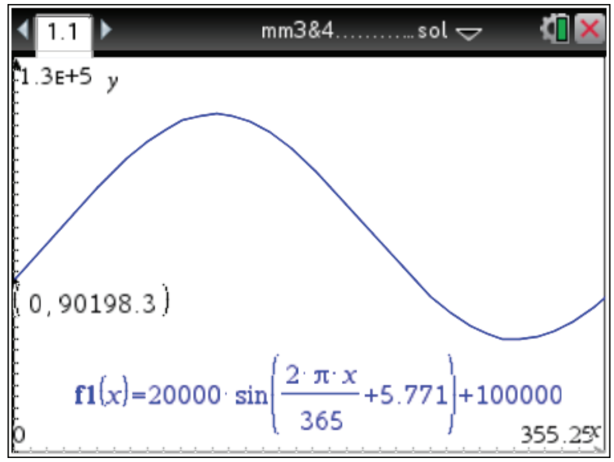
Graphic calculator techniques for question 7

In a **Graphs** page enter the rule (note that x must be used here instead of t) in the **function entry line.**

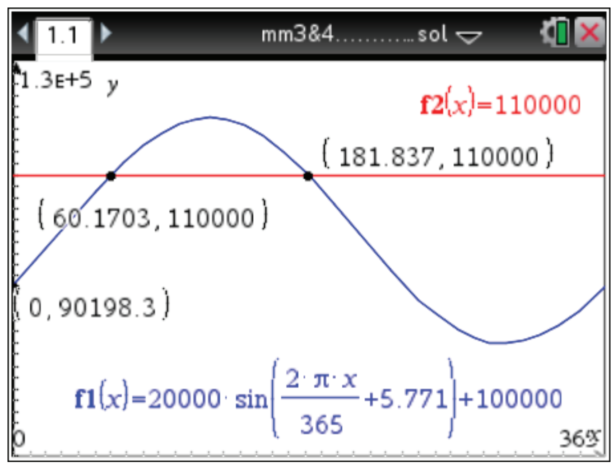
Because of the magnitude of the numbers in this problem it is useful to increase the number of display digits using b>**Settings** and change the **Display Digits** to **Auto** Set the **WINDOW** (b**>Window/Zoom>Window Settings)** at Xmin = 0, Xmax = 365; Ymin= 60000, Ymax = 130000. The graph appears as shown.



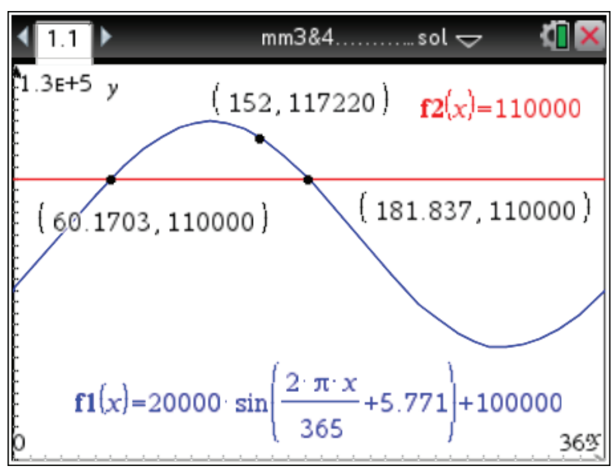
The value of Y when $x = 0$ can be found several ways.
Use b**>Geometry>Points & Lines>Point On** to place a point on the graph. Press d to exit the **Point On** tool. By double clicking on the x-coordinate you can edit this to 0.
Alternatively, use b**>Trace>Graph Trace** and type 0. (An "x =" box will appear as soon as you start typing a value).
Hence y = 90198.3 when x = 0.



In order to find the t values for which y = 110 000, press e or /+G) to show the function entry line and enter **f2(x)** = 110000 (use **Intersection** from the **Analyze Graph** menu to find each of the required values.
Hint: using b**>Geometry>Points & Lines>Intersection Point/s** will find all intersections at once.



The value when $x = 152$ can be found by double clicking on the x-coordinate of the point found earlier and changing to 152 or using the b**>Trace>Graph Trace** and editing the x-coordinate to 152.
Alternatively, insert (/+I)a **Calculator** page and type in f1(152)



5 $d = 12 + 12\cos\left(\frac{\pi}{6}\left(t + \frac{1}{3}\right)\right)$

a **i** When $t = 5.7$

$$d = 12 + 12\cos\frac{\pi}{6}\left(5.7 + \frac{1}{3}\right)$$
$$= 1.8276 \times 10^{-3} \approx 1.83 \times 10^{-3} \text{ hours}$$

ii When $t = 2.7$

$$d = 12 + 12\cos\frac{\pi}{6}\left(2.7 + \frac{1}{3}\right)$$
$$= 11.79 \text{ hours}$$

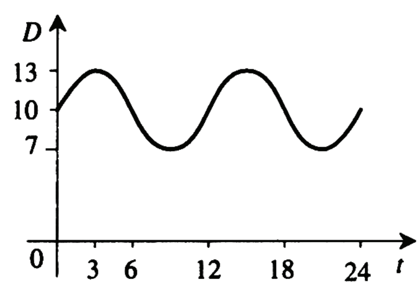
b When $d = 5$

$$5 = 12 + 12\cos\left(\frac{\pi}{6}\left(t + \frac{1}{3}\right)\right)$$
$$\frac{-7}{12} = \cos\left(\frac{\pi}{6}\left(t + \frac{1}{3}\right)\right)$$
$$\therefore \quad \frac{\pi}{6}\left(t + \frac{1}{3}\right) = 2.1936 \text{ or } 4.089$$
$$\therefore \quad t = 3.856 \text{ or } 7.477$$

There will be 5 hours of daylight on 25th April and 14th August.

6 **a** Period $= 2\pi \div \frac{\pi}{6}$

$$= 12$$



For $D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right)$

$$D(0) = 10$$
$$D(24) = 10 + 3\sin\left(\frac{\pi \times 24}{6}\right)$$
$$= 10$$

b For $D(t) \geq 8.5$, first consider

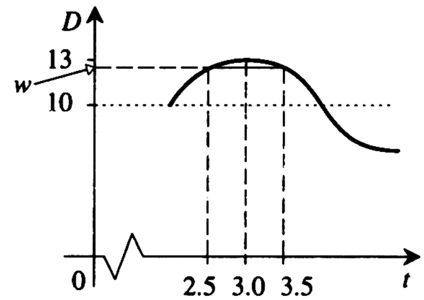
$$8.5 = 10 + 3\sin\left(\frac{\pi t}{6}\right)$$
$$-\frac{1}{2} = \sin\left(\frac{\pi t}{6}\right)$$
$$\therefore \quad \frac{\pi t}{6} = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{19\pi}{6} \text{ or } \frac{23\pi}{6} \text{ or } \ldots$$
$$\therefore \quad t = 7 \text{ or } 11 \text{ or } 19 \text{ or } 23 \text{ or } \ldots$$

From the graph is can be seen that $D(t) \geq 8.5$ for $t \in [0, 7] \cup [11, 19] \cup [23, 24]$

c The maximum depth is 13 m.
From the graph the required period of time is [2.5, 3.5]
The largest value of w occurs for $t = 2.5$

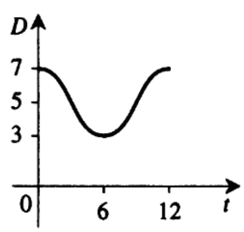
$$w = 10 + 3\sin\left(\frac{2.5\pi}{6}\right)$$
$$= 12.898$$

The largest value of w is 12.898.



7 a $D = p + q\cos(rt)°$
High tide is 7 m.
Low tide is 3 m.
Low tide occurs 6 hours after high tide.
High tide occurs when $\cos(rt)° = 1$ and low tide occurs when $\cos(rt)° = -1$.
$\therefore\ D = p + q\cos(rt)°$
gives $7 = p + q$ ①
and $3 = p - q$ ②
Adding ① and ② gives $2p = 10$
$p = 5$
Therefore from ① $q = 2$
Hence $D = 5 + 2\cos(rt)°$
The period is 12 $\therefore\ \dfrac{360}{r} = 12$
and $r = 30$
$\therefore\ D = 5 + 2\cos(30t)°$

b



c Low tide occurs when $t = 6$. The depth at low tide is 3 m.
$5 + 2\cos(30t)° = 4$

$$\cos(30t)° = -\frac{1}{2}$$
$$\therefore\ 30t = 120 \text{ or } 240 \text{ or } \ldots$$
$$\therefore\ t = 4 \text{ or } 8 \text{ or } \ldots$$

The ship may enter the harbour 2 hours after low tide.

8 **a** $a = b = 10,\ \theta = \dfrac{\pi}{3}$

i $A = \dfrac{1}{2} \times 100 \times \sin\dfrac{\pi}{3}$

$= 50 \times \dfrac{\sqrt{3}}{2}$

$= 25\sqrt{3}$ square units

ii $P = 10 + 10 + \sqrt{100 + 100 - 200\cos\dfrac{\pi}{3}}$

$= 20 + \sqrt{200 - 100}$

$= 30$

b $P = A$ implies

$20 + 10\sqrt{2 - 2\cos\theta} = 50\sin\theta$

$\Leftrightarrow 2 + \sqrt{2 - 2\cos\theta} = 5\sin\theta$

Plot the graphs of $y = 2 + \sqrt{2 - 2\cos\theta}$ and $y = 5\sin\theta$ to find the point of intersection.

Intersection occurs where $\theta = 0.53$ or $\theta = 2.27$

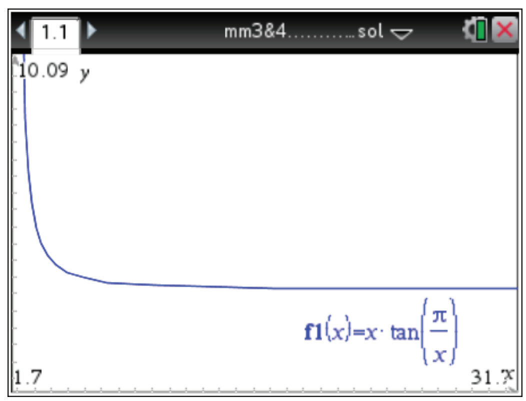
c If $a = b = 6$

$A = 18\sin\theta$ and

$P = 12 + \sqrt{72 - 72\cos\theta}$

$= 12 + 6\sqrt{2 - 2\cos\theta}$

Graph $y = P - A$ for $\theta \in (0, \pi)$ and note that the minimum > 0, so $P - A > 0 = P > A$.



d If $\theta = \dfrac{\pi}{2}$ and $a = 6$

$A = P$ implies

$$3b = 6 + b + \sqrt{36 + b^2}$$

i.e. $\quad 2b - 6 = \sqrt{36 + b^2}$

$\therefore \quad 4b^2 - 24b + 36 = 36 + b^2$

$\therefore \quad 3b^2 - 24b = 0$

$\therefore \quad 3b(b - 8) = 0$

$\therefore \quad b = 0$ or $b = 8$

$b = 0$ does not satisfy the original equation. Therefore $b = 8$

e If $a = 10$ and $b = 6$

$30\sin\theta = 16 + \sqrt{136 - 120\cos\theta}$

$15\sin\theta = 8 + \sqrt{34 - 30\cos\theta}$

$\theta = 0.927$ or $\theta = 1.837$ (from a cas calculator using the 'solve' command)

f If $a = b$ and $\theta = \dfrac{\pi}{3}$

$A = P$

implies

$$\frac{a^2}{2} \times \frac{\sqrt{3}}{2} = 2a + \sqrt{2a^2 - a^2}$$

$$\therefore \quad \frac{\sqrt{3a^2}}{4} = 2a + \sqrt{a^2}$$

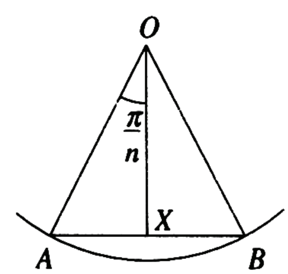
$$\frac{\sqrt{3a^2}}{4} = 3a$$

$$\therefore \quad \frac{\sqrt{3a^2}}{4} - 3a = 0$$

$$\therefore \quad a\left(\frac{\sqrt{3}}{4}a - 3\right) = 0$$

$\therefore\ a = \dfrac{12}{\sqrt{3}} = 4\sqrt{3}$ since $a > 0$.

9 a The n sided polygon consists of n isosceles triangles.



The angle for each triangle at the centre of the circle is $\dfrac{2\pi}{n}$

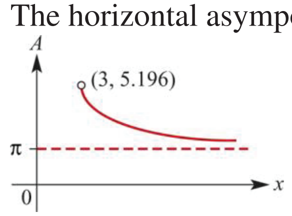
Length of $OX = 1$

Length of $AB = 2\tan\left(\dfrac{\pi}{n}\right)$

$\therefore$ Area of triangle $= \dfrac{1}{2} \times 2\tan\left(\dfrac{\pi}{n}\right) \times 1 = \tan\left(\dfrac{\pi}{n}\right)$

b Area of the polygon is $n\tan\left(\dfrac{\pi}{n}\right)$ (n triangles).

c The horizontal asympote is $y = \pi$



d i $n = 3$

Area of polygon $= 3\tan\left(\dfrac{\pi}{3}\right) = 3\sqrt{3}$ difference $= 3\sqrt{3} - \pi \approx 2.055$

ii $n = 4$

Area of polygon $= 4\tan\left(\dfrac{\pi}{4}\right) = 4$ difference $= 4 - \pi \approx 0.858$

iii $n = 12$

Area of polygon $= 12\tan\left(\dfrac{\pi}{12}\right) \approx 3.215$ difference ≈ 0.0738

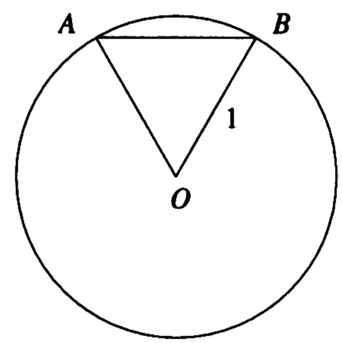
iv $n = 50$

Area of polygon $= 50\tan\left(\dfrac{\pi}{50}\right) =\approx 3.1457$difference ≈ 0.0041

e The circles are similar

$\therefore$ the area $= nr\tan\left(\dfrac{\pi}{n}\right)$

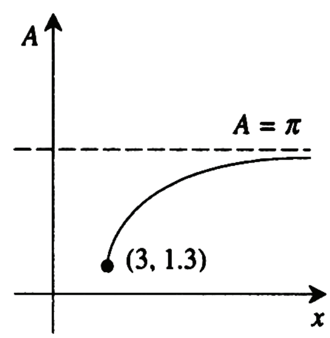
f **i** So the area is $n\sin\left(\dfrac{\pi}{n}\right)\cos\left(\dfrac{\pi}{n}\right)$



ii The polygon consists of n isosceles triangles.

The area of each triangle

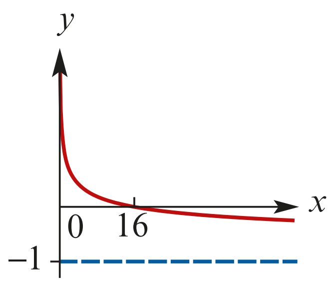
is $\dfrac{1}{2}\sin\left(\dfrac{2\pi}{n}\right) = \sin\left(\dfrac{\pi}{n}\right)\cos\left(\dfrac{\pi}{n}\right)$



Chapter 7 – Functions revisited

Solutions to Exercise 7A

1 a

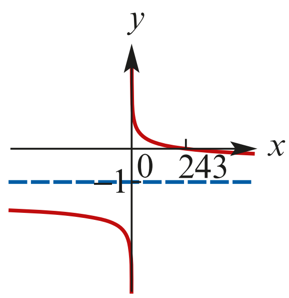


Domain $= \mathbb{R}^+$

Range $= (-1, \infty)$

Neither odd nor even

b



Domain $= \mathbb{R} \setminus \{0\}$

Range $= \mathbb{R} \setminus \{-1\}$

Neither odd nor even

2 a $32^{\frac{2}{5}} = (32^{\frac{1}{5}})^2 = 4$

b $(-32)^{\frac{2}{5}} = ((-32)^{\frac{1}{5}})^2 = 4$

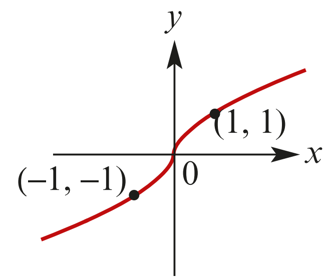
c $32^{\frac{3}{5}} = (32^{\frac{1}{5}})^3 = 8$

d $(-32)^{\frac{3}{5}} = ((-32)^{\frac{1}{5}})^3 = -8$

e $(-8)^{\frac{5}{3}} = ((-8)^{\frac{1}{3}})^5 = -32$

f $(-27)^{\frac{4}{3}} = ((-27)^{\frac{1}{3}})^4 = 81$

3 a

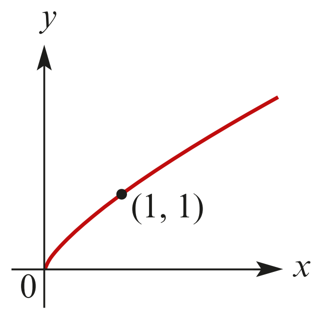


Domain $= \mathbb{R}$

Range $= \mathbb{R}$

Odd

b



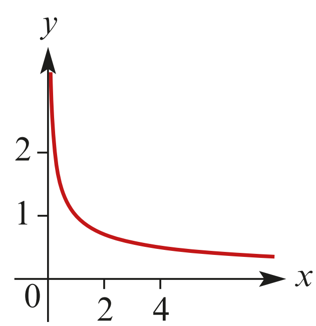
Domain $= \mathbb{R}^+ \cup \{0\}$

Range $= \mathbb{R}^+ \cup \{0\}$

Neither

4 a i Domain $= \mathbb{R}^+$; Range $= \mathbb{R}^+$;
Asymptotes: $x = 0$, $y = 0$

ii



b i Domain $= \mathbb{R}$; Range $= \mathbb{R}$

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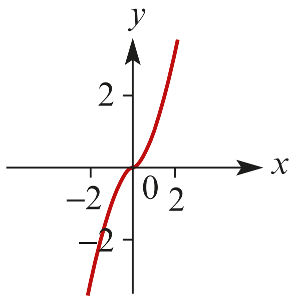
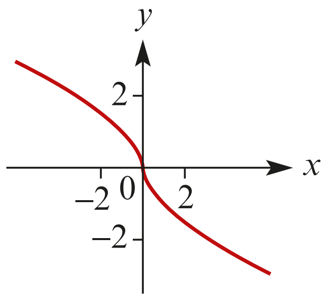
ii



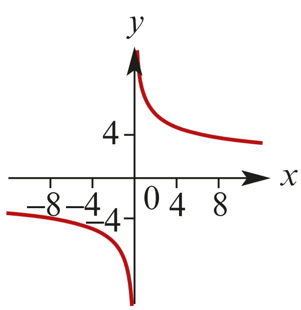
c **i** Domain = $\mathbb{R}$; Range = $\mathbb{R}$

ii



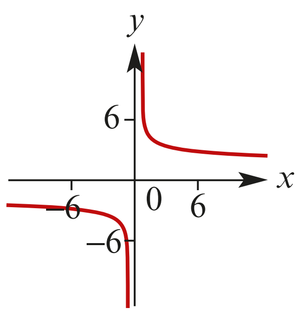
d **i** Domain = $\mathbb{R} \setminus \{0\}$;
Range = $\mathbb{R} \setminus \{0\}$;
Asymptotes: $x = 0, y = 0$

ii



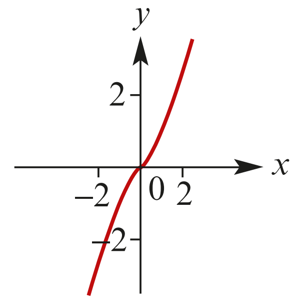
e **i** Domain = $\mathbb{R} \setminus \{0\}$;
Range = $\mathbb{R} \setminus \{0\}$;
Asymptotes: $x = 0, y = 0$

ii



f **i** Domain = $\mathbb{R}$; Range = $\mathbb{R}$

ii



5 We can assume $x \neq 0$ in the this question witout effecting the result.

a $x^{\frac{3}{2}} > x^2$

$1 > x^{2-\frac{3}{2}}$

$1 > x^{\frac{1}{2}}$

$x < 1$

$\therefore x \in (0, 1)$

b $x^{\frac{3}{2}} < x^{-2}$

$x^{2+\frac{3}{2}} < 1$

$x^{\frac{5}{2}} < 1$

$x < 1$

$\therefore x \in (0, 1)$

6 **a** Odd

b Even

c Odd

d Odd

e Even

f Odd

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Solutions to Exercise 7B

1 **a** $h(x) = f \circ g(x),\ f(x) = e^x,\ g(x) = x^3$

b $h(x) = f \circ g(x),\ f(x) = \sin x,$
$g(x) = 2x^2$

c $h(x) = f \circ g(x),\ f(x) = x^n,$
$g(x) = x^2 - 2x$

d $h(x) = f \circ g(x),\ f(x) = \cos x,$
$g(x) = x^2$

e $h(x) = f \circ g(x),\ f(x) = x^2,$
$g(x) = \cos x$

f $h(x) = f \circ g(x),\ f(x) = x^4,$
$g(x) = x^2 - 1$

g $h(x) = f \circ g(x),\ f(x) = x^2,$
$g(x) = \cos(2x)$

h $h(x) = f \circ g(x),\ f(x) = x^3 - 2x,$
$g(x) = x^2 - 2x$

2 **a** $f \circ f^{-1}(x) = x$

$$\Leftrightarrow 4e^{f^{-1}(x)} = x$$

$$\Leftrightarrow e^{f^{-1}(x)} = \frac{x}{4}$$

$$\Leftrightarrow f^{-1}(x) = \log_e\left(\frac{x}{4}\right)$$

Therefore,

$f^{-1}: (0, \infty) \to \mathbb{R},\ f^{-1}(x) = \dfrac{1}{3}\log_e\left(\dfrac{x}{4}\right)$

b $f \circ g^{-1}(x) = x$

$$\Leftrightarrow \frac{2}{\sqrt[3]{g^{-1}(x)}} = x$$

$$\Leftrightarrow 2 = x\sqrt[3]{g^{-1}(x)}$$

$$\Leftrightarrow g^{-1}(x) = \frac{8}{x^3}$$

Therefore,

$g^{-1}: \mathbb{R} \setminus \{0\} \to \mathbb{R},\ g^{-1}(x) = \dfrac{8}{x^3}$

c $f \circ g: \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f \circ g(x) = 4e^{\frac{6}{\sqrt[3]{x}}}$

d $g \circ f: \mathbb{R} \to \mathbb{R},\ g \circ f(x) = \dfrac{2}{\sqrt[3]{4e^{3x}}}$

e Let $h(x) = f \circ g(x)$

$h(x) = 4e^{\frac{6}{\sqrt[3]{x}}}$

$h \circ h^{-1}(x) = x$

$$\Leftrightarrow 4e^{\frac{6}{\sqrt[3]{h^{-1}(x)}}} = x$$

$$\Leftrightarrow e^{\frac{6}{\sqrt[3]{h^{-1}(x)}}} = \frac{x}{4}$$

$$\Leftrightarrow \frac{6}{\sqrt[3]{h^{-1}(x)}} = \log_e\left(\tfrac{x}{4}\right)$$

$$\Leftrightarrow h^{-1}(x) = \left(\frac{6}{\log_e\left(\frac{x}{4}\right)}\right)^3$$

Therefore,

$(f \circ g)^{-1}: \mathbb{R}^+ \to \mathbb{R},$

$(f \circ g)^{-1}(x) = \left(\dfrac{6}{\log_e\left(\frac{x}{4}\right)}\right)^3$

f $(g \circ f)^{-1}: \mathbb{R}^+ \to \mathbb{R},\ (g \circ f)^{-1}(x) = \dfrac{1}{3}\log_e\left(\dfrac{2}{x^3}\right)$

3 **a** $f^{-1}: \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f^{-1}(x) = x^{\frac{5}{2}}$
Both f and f^{-1} are strictly increasing

b $f^{-1}: \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f^{-1}(x) = -x^{\frac{5}{2}}$

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Both f and f^{-1} are strictly decreasing

c $f^{-1}: \mathbb{R}^+ \to \mathbb{R}$, $f^{-1}(x) = x^{\frac{2}{5}}$
Both f and f^{-1} are strictly increasing

4 a i $f \circ g(x) = 3\sin(2x^2)$,
$g \circ f(x) = g(3\sin 2x)^2 = 9\sin^2(2x)$

ii $\operatorname{ran}(f \circ g) = [-3, 3]$,
$\operatorname{dom}(f \circ g) = \mathbb{R}$,
$\operatorname{ran}(g \circ f) = [0, 9]$, $\operatorname{dom}(g \circ f) = \mathbb{R}$

b i $f \circ g(x) = -2\cos(2x^2)$,
$g \circ f(x) = g(-2\cos 2x) =$
$4\cos^2(2x)$

ii $\operatorname{ran}(f \circ g) = [-2, 2]$,
$\operatorname{dom}(f \circ g) = \mathbb{R}$,
$\operatorname{ran}(g \circ f) = [0, 4]$, $\operatorname{dom}(g \circ f) = \mathbb{R}$

c i $f \circ g(x) = e^{x^2}$, $g \circ f(x) = e^{2x}$

ii $\operatorname{ran}(f \circ g) = (1, \infty)$,
$\operatorname{dom}(f \circ g) = \mathbb{R}$,
$\operatorname{ran}(g \circ f) = (0, \infty)$,
$\operatorname{dom}(g \circ f) = \mathbb{R}$

d i $f \circ g(x) = e^{2x^2} - 1$,
$g \circ f(x) = (e^{2x} - 1)^2$

ii $\operatorname{ran}(f \circ g) = [0, \infty)$,
$\operatorname{dom}(f \circ g) = \mathbb{R}$,
$\operatorname{ran}(g \circ f) = [0, \infty)$,
$\operatorname{dom}(g \circ f) = \mathbb{R}$

e i $f \circ g(x) = -2e^{x^2} - 1$,
$g \circ f(x) = (2e^x + 1)^2$

ii $\operatorname{ran}(f \circ g) = (-\infty, -3]$,
$\operatorname{dom}(f \circ g) = \mathbb{R}$,
$\operatorname{ran}(g \circ f) = (1, \infty)$,
$\operatorname{dom}(g \circ f) = \mathbb{R}$

f i $f \circ g(x) = \log_e(2x^2)$,
$g \circ f(x) = (\log_e(2x))^2$

ii $\operatorname{ran}(f \circ g) = \mathbb{R}$,
$\operatorname{dom}(f \circ g) = \mathbb{R} \setminus \{0\}$,
$\operatorname{ran}(g \circ f) = [0, \infty)$,
$\operatorname{dom}(g \circ f) = \mathbb{R}^+$

g i $f \circ g(x) = \log_e(x^2 - 1)$,
$g \circ f(x) = (\log_e(x - 1))^2$

ii $\operatorname{ran}(f \circ g) = \mathbb{R}$,
$\operatorname{dom}(f \circ g) = \mathbb{R} \setminus [-1, 1]$,
$\operatorname{ran}(g \circ f) = [0, \infty)$,
$\operatorname{dom}(g \circ f) = (1, \infty)$

h i $f \circ g(x) = -\log_e(x^2)$,
$g \circ f(x) = (\log_e x)^2$

ii $\operatorname{ran}(f \circ g) = \mathbb{R}$,
$\operatorname{dom}(f \circ g) = \mathbb{R} \setminus \{0\}$,
$\operatorname{ran}(g \circ f) = [0, \infty)$,
$\operatorname{dom}(g \circ f) = \mathbb{R}^+$

5 a $g \circ f(x) = g\left(2x - \dfrac{\pi}{3}\right) = \sin\left(2x - \dfrac{\pi}{3}\right)$

b $(x, y) \to (2x' - \dfrac{\pi}{3}, y')$
$\therefore x' = \dfrac{x + \frac{\pi}{3}}{2} = \dfrac{1}{2}x + \dfrac{\pi}{6}$ and $y' = y$
Dilation of factor $\frac{1}{2}$ from the y-axis,
then translation $\frac{\pi}{6}$ units to the right

6 a $g \circ f: (\frac{1}{3}, \infty) \to \mathbb{R}$,
$$\begin{aligned} g \circ f(x) &= g(3x - 2) \\ &= \log_e(3x - 2 + 1) \\ &= \log_e(3x - 1) \end{aligned}$$

b Write $y' = \log_e(3x' - 1)$ and
$y = \log_e(x + 1)$
Then choose,
$y' = y$ and $3x' - 1 = x + 1$
That is $y' = y$ and $x' = \dfrac{x + 2}{3}$

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7 **a** $[g(x)]^2 - 7g(x) + 12 = 0$

$(g(x) - 3)(g(x) - 4) = 0$

$\therefore g(x) = 3 \text{ or } g(x) = 4$

b $[g(x)]^2 - 7xg(x) + 12x^2 = 0$

$(g(x) - 3x)(g(x) - 4x) = 0$

$\therefore g(x) = 3x \text{ or } g(x) = 4x$

8 $e^{g(x)} = 2x - 1$

$g(x) = \log_e(2x - 1)$

$\therefore g(x) = 3x \text{ or } g(x) = 4x$

9 $f(x) = e^{4x}, \qquad g(x) = 2\sqrt{x}$

a $g(f(x)) = 2\sqrt{e^{4x}} = 2(e^{4x})^{\frac{1}{2}}$

$= 2e^{2x}$

b $x = 2e^{2(g \circ f)^{-1}(x)}$

$\ln \dfrac{x}{2} = 2(g \circ f)^{-1}(x)$

$(g \circ f)^{-1}(x) = \dfrac{1}{2} \ln \dfrac{x}{2}$

c $x = 2\sqrt{g^{-1}(x)}$

$\dfrac{x}{2} = \sqrt{g^{-1}(x)}$

$g^{-1}(x) = \dfrac{x^2}{4}$

$(f \circ g^{-1})(x) = e^{4\left(\frac{x^2}{4}\right)}$

$= e^{x^2}$

10 $f(x) = e^{-2x}, g(x) = x^3 + 1$

a $x = e^{-2f^{-1}(x)}$

$\ln x = -2f^{-1}(x)$

$f^{-1}(x) = \dfrac{-1}{2} \ln x$

$x = (g^{-1}(x))^3 + 1$

$x - 1 = (g^{-1}(x))^3$

$g^{-1}(x) = (x - 1)^{\frac{1}{3}}$

b $f \circ g(x) = \mathrm{e}^{-2(x^3+1)}$

$= \mathrm{e}^{-2x^3-2}$

range $(f \circ g) = \mathbb{R}^+$

since range $(-2x^3 - 2) = \mathbb{R}$

and range $(e^x) = \mathbb{R}^+$

$g \circ f(x) = (e^{-2x})^3 + 1$

$= e^{-6x} + 1$

range $(g \circ f) = (1, \infty)$

11 **a** $f\colon (-1, \infty) \to \mathbb{R}, \quad f(x) = \dfrac{1}{x + 1}$

domain (f) $= (-1, \infty)$,

$\therefore$ range $(f) = \mathbb{R}^+$

$x = \dfrac{1}{f^{-1}(x) + 1}$

$f^{-1}(x) + 1 = \dfrac{1}{x}$

$f^{-1}(x) = \dfrac{1}{x} - 1$

range $(f^{-1}) = (-1, \infty)$,

domain $(f^{-1}) = \mathbb{R}^+$

$\therefore f^{-1}\colon \mathbb{R}^+ \to \mathbb{R}, \quad f^{-1}(x) = \dfrac{1}{x} - 1$

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b $f(x) = f^1(x) = x$

$$f(x) = x$$

$$\frac{1}{x+1} = x$$

$$(x+1)x = 1$$

$$x^2 + x - 1 = 0$$

$$x = \frac{-1 \pm \sqrt{1+4}}{2}$$

but $x > 0$, (domain(f^{-1}))

$$\therefore \quad x = \frac{-1 + \sqrt{5}}{2}$$

$$x = \frac{\sqrt{5} - 1}{2}$$

12 a $f(x) = \ln(x+1)$

$x = \ln(f^{-1}(x) + 1)$

$e^x = f^{-1}(x) + 1$

$f^{-1}(x) = e^x - 1$

$f^{-1}: R \to R, f^{-1}(x) = e^x - 1$

$g(x) = x^2 + 2x$, domain(g) $= (-1, \infty)$

range $(g) = (g(-1), \infty)$

$= (-1, \infty)$

$x = (g^{-1}(x))^2 + 2g^{-1}(x)$

$(g^{-1}(x))^2 + 2(g^{-1}(x)) - x = 0$

$(g^{-1}(x)^2 + 1)^2 - x - 1 = 0$

$g^{-1}(x) + 1 = \pm\sqrt{(x+1)}$

$g^{-1}(x) + 1 = \pm\sqrt{x+1}$

$g^{-1}(x) = -1 \pm \sqrt{x+1}$

but $g^{-1}(x) > -1$

$\therefore \quad g^{-1}(x) = -1 + \sqrt{x+1}$

$g^{-1}: (-1, \infty) \to R, g^{-1}(x) =$
$\sqrt{(x+1)} - 1$

b $f \circ g(x) = \ln(x^2 + 2x + 1)$

$$= \ln((x+1)^2)$$

$$= 2\ln(x+1)$$

$(Since\ domain\ g = (-1, \infty))$

13 $f \circ g(x) = \text{In}\left(\frac{1}{x}\right)$

$$= -\ln(x)$$

$$f(x) + f \circ g(x) = \ln x - \ln x = 0$$

14 $h(g(x)) = \sqrt{\frac{(5x^2+3)-3}{5}}$

$$= \sqrt{\frac{5x^2}{5}}$$

$$= \sqrt{x^2}$$

$$= |x|$$

15 a $f(g(x)) = (x^2 - 4 - 4)(x^2 - 4 - 6)$

$$= (x^2 - 8)(x^2 - 10)$$

$$f(g(x)) = x^4 - 18x^2 + 80$$

$$g(f(x)) = ((x-4)(x-6))^2 - 4$$

$$= (x^2 - 10x + 24)^2 - 4$$

$$= x^4 - 20x^3 + 48x^2 + 100x^2$$
$$- 480x + 576 - 4$$

$$g(f(x)) = x^4 - 20x^3 + 148x^2$$
$$- 480x + 572$$

b $g(f(x)) - f(g(x)) = 158$

$x^4 - 20x^3 + 148x^2 - 480x + 572$

$-x^4 + 18x^2 - 80 = 158$

$-20x^3 + 166x^2 - 480x + 334 = 0$

$10x^3 - 88x^2 + 240x - 167 = 0$

CAS calculator gives $x = 1$ as a solution

$\Rightarrow (x-1)(10x^2 - 73x + 167) = 0$

$x = 1, x = \frac{73 \pm \sqrt{5329 - 6680}}{20}$

$x = 1, \frac{73 \pm \sqrt{-1351}}{20}$

$\Downarrow$

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no real solutions

$\therefore \quad x = 1$

16 a $f(x) = 4 - x^2$

$f(f(x)) = 4 - (4 - x^2)^2$

$= 4 - (16 - 8x^2 + x^4)$

$= -12 + 8x^2 - x^4$

$f(f(x)) = 0$

$\Rightarrow x^4 - 8x^2 + 12 = 0$

$(x^2)^2 - 8x^2 + 12 = 0$

$(x^2 - 6)(x^2 - 2) = 0$

$x^2 = 2, 6$

$x = \pm\sqrt{2}, \pm\sqrt{6}$

17 $f(x) = e^x - e^{-x}$

a $LHS = e^{(-x)} - e^{-(-x)}$

$= e^{-x} - e^x$

$RHS = -e^x + e^{-x}$

$= e^{-x} - e^x$

$= LHS \quad QED$

b $RHS = e^{3x} - e^{-3x} - 3e^x + 3e^{-x}$

$= e^{3x} - 3e^x + 3e^{-x} - e^{-3x}$

$LHS = (e^x - e^{-x})^3$

$= e^{3x} - 3e^x + 3e^{-x} - e^{-3x}$

$= RHS \quad QED$

18 Consider,

$af^{-1}(x) + b = x$

$\therefore f^{-1}(x) = \frac{x}{a} - \frac{b}{a}$

If $f^{-1}(x) = 6x + 3$

$a = \frac{1}{6}$ and $-\frac{b}{a} = 3$

$\therefore a = \frac{1}{6}$ and $b = -\frac{1}{2}$

19 $\frac{f^{-1}(x) + 2}{f^{-1}(x) - 1} = x$

$f^{-1}(x) + 2 = x(f^{-1}(x) - 1)$

$f^{-1}(x)(1 - x) = -2 - x$

$f^{-1}(x) = \frac{x + 2}{x - 1}$

20 $\ln(g(x)) = ax + b$

$g(x) = e^{ax+b}$

$g(0) = 1$

$\therefore \quad 1 = e^b$

$\therefore \quad b = 0$

$g(x) = e^{ax}$

$g(1) = e^6$

$e^6 = e^a$

$a = 6$

$g(x) = e^{6x}$

21 Let $y = f^{-1}(x)$

a $\frac{e^y + e^{-y}}{2} = x$

$e^y + e^{-y} = 2x$

$e^{2y} + 1 = 2xe^y$

$e^2y - 2xe^y + 1 = 0$

$e^y = \frac{1}{2}(2x \pm \sqrt{4x^2 - 4})$

$y = \log_e(x \pm \sqrt{x^2 - 1})$

But Range of f^{-1}= Domain of $f = [0, \infty)$

and Domain of f^{-1}= Range of $f = [1, \infty) \therefore f^{-1}: [1, \infty) \to \mathbb{R}$,

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$f^{-1}(x) = \log_e(x + \sqrt{x^2 - 1})$

b $g^{-1}: \mathbb{R} \to \mathbb{R}, g^{-1}(x) = \log_e(x + \sqrt{x^2 + 1})$

c Yes

d Yes

22 **a** If $x > y$ then $f(x) > f(y)$ and If $x > y$ then $g(x) > g(y)$
Hence $x > y \Rightarrow f(x) > f(y) \Rightarrow g(f(x)) > g(f(y))$.

b If $x > y$ then $f(x) < f(y)$ and
If $x > y$ then $g(x) < g(y)$
Hence if $x > y \Rightarrow f(x) < f(y) \Rightarrow g(f(x)) > g(f(y))$.

c The composite function will be strictly decreasing.

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Solutions to Exercise 7C

1 $f(x) = e^{-2x}, g(x) = -2x$

a **i** $(f + g)(x) = e^{-2x} - 2x$

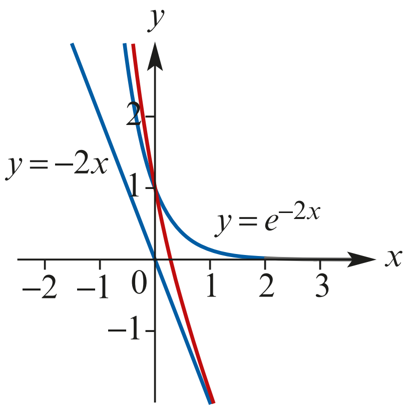
ii $(fg)(x) = -2xe^{-2x}$

b **i** $(f + g)\left(\frac{-1}{2}\right) = e^{-1} + 1$

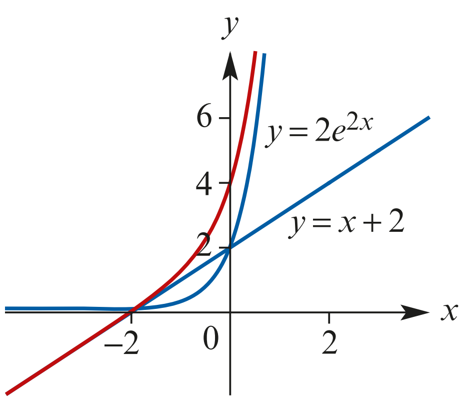
ii $(fg)\left(\frac{-1}{2}\right) = 1 \times e^{-1}$

$= e^{-1}$

2



3



4 $f(x) = \sin\left(\frac{\pi x}{2}\right), g(x) = -2x$

a **i** $(f + g)(x) = \sin\left(\frac{\pi x}{2}\right) - 2x$

ii $(fg)(x) = -2x\sin\left(\frac{\pi x}{2}\right)$

b **i** $(f + g)(1) = \sin\left(\frac{\pi}{2}\right) - 2$

$= 1 - 2$

$= -1$

ii $(fg)(1) = -2\sin\left(\frac{\pi}{2}\right)$

$= -2$

5 $f(x) = \cos\left(\frac{\pi x}{2}\right), g(x) = e^x$

a **i** $(f + g)(x) = \cos\left(\frac{\pi x}{2}\right) + e^x$

ii $(fg)(x) = e^x \cos\left(\frac{\pi x}{2}\right)$

b **i** $(f + g)(0) = \cos(0) + e^0$

$= 1 + 1$

$= 2$

ii $(fg)(0) = 1 \times 1$

$= 1$

6 Let $g(x) = \dfrac{f(x) + f(-x)}{2}$ and

$h(x) = \dfrac{f(x) - f(-x)}{2}$

We have, $g(-x) = g(x)$. That is $g(x)$ is even.

We have, $h(-x) = -h(x)$. That is $h(x)$ is odd.

$f(x) = h(x) + g(x)$

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Solutions to Exercise 7D

1 a $f(x-y) = 2(x-y)$
$= 2x - 2y$
$= f(x) - f(y)$

b $f(x-y) = (x-y-3)$
$\neq f(x) - f(y)$

2 $f(x-y) = k(x-y)$
$= kx - ky$
$= f(x) - f(y)$

3 $f(x+y) = 2(x+y) + 3$
$= 2x + 2y + 3$
$= 2x + 3 + 2y + 3 - 3$
$= f(x) + f(y) - 3$
$a = -3$

4 $f(x) + f(y) = \frac{3}{x} + \frac{3}{y}$
$= \frac{3(x+y)}{xy}$
$= (x+y)f(xy)$

5 $(g(x))^2 = g(x)$
$(g(x))^2 - g(x) = 0$
$g(x)(g(x) - 1) = 0$
$g(x) = 0, 1$

6 $\frac{1}{g(x)} = g(x)$
$(g(x))^2 = 1$
$g(x) = \pm 1$

7 $f(x) = x^3$
$f(x+y) = (x+y)^3$
$f(x) + f(y) = x^3 + y^3$
Let $x = 1, y = 1$
$f(x+y) = 8$
$f(x) + f(y) = 2$

8 $f(x) = \sin x$
$LHS = f(x+y) = \sin(x+y)$
$RHS = f(x) + f(y) = \sin x + \sin y$
let x $= \frac{\pi}{2}, y = \frac{\pi}{2}$
$LHS = \sin(\pi) = 0$
$RHS = \sin\frac{\pi}{2} + \sin\frac{\pi}{2} = 2 \neq LHS\, QED$
(any non zero numbers would work)

9 $f(x) = \frac{1}{x^2}$
$LHS = f(x) + f(y)$
$= \frac{1}{x^2} + \frac{1}{y^2}$
$= \frac{y^2}{x^2y^2} + \frac{x^2}{x^2y^2}$
$= (x^2 + y^2)\frac{1}{x^2y^2}$
$= (x^2 + y^2)\frac{1}{(xy)^2}$
$= (x^2 + y^2)f(xy)$
$= RHS\ QED$

10 $h(x) = x^2$

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a Let $x = 1, y = 1$

$LHS = (1+1)^2 = 4$

$RHS = (1)^2 + (1)^2 = 2 \neq LHS\ QED$

(any non zero numbers would work)

b
$$\begin{aligned} LHS &= (x+y)^2 \\ &= x^2 + 2xy + y^2 \\ RHS &= (x)^2 + (y)^2 \\ &= x^2 + y^2 \\ LHS &= RHS + 2xy \end{aligned}$$

$\therefore$ given LHS $= RHS$

$$2xy = 0$$

i.e., $x = 0 \text{ or } y = 0\ QED$

11 $g(x) = 2^{3x}$

$$\begin{aligned} LHS &= 2^{3(x+y)} \\ &= 2^{3x+3y} \\ &= 2^{3x} \times 2^{3y} \\ &= g(x) \times g(y) \\ &= RHS\ QED \end{aligned}$$

12 $f(x) = x^n$

$f(xy) = (xy)^n = x^n y^n$

(by the indices laws)

$= f(x)f(y)\ QED$

$f\left(\dfrac{x}{y}\right) = \left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$

(by the indices laws)

$= \dfrac{f(x)}{f(y)} \qquad QED$

13 $f(x) = ax, \quad a \in R\backslash\{0, 1\}$

$$\begin{aligned} f(xy) &= axy \\ &= f(x)f(y) = ax \times ay \\ &= a^2xy \end{aligned}$$

let $x = 1, \qquad y = 1$

(any non zero numbers would work)

$$\begin{aligned} f(xy) &= axy = a \\ f(x)f(y) &= a^2xy = a^2 \\ if\ f(x)f(y) &= f(xy) \\ a^2 &= a \\ a^2 - a &= 0 \\ a &= 0, 1 \end{aligned}$$

but $a \neq 0, 1$

$\therefore \quad f(x)f(y) \neq f(xy)$

for the case shown

14 $f : \mathbb{R}\backslash\{-1\} \to \mathbb{R}, f(x) = \dfrac{1}{x+1}$.

$f(f(x)) = f(\dfrac{1}{x+1}) = \dfrac{1}{\dfrac{1}{x+1} + 1} =$

$\dfrac{x+1}{x+2}$

$f(x+1) = \dfrac{1}{x+2}$

$$\begin{aligned} \therefore f(f(x)) + f(x+1) &= \frac{x+1}{x+2} + \frac{1}{x+2} \\ &= \frac{x+2}{x+2} \\ &= 1 \end{aligned}$$

15 Let $f(x) = x^2, g(x) = 2, h(x) = 3$

$f \circ (g+h)(x) = f(5) = 25$

$f \circ g(x) + f \circ h(x) = f(2) + f(3) = 13$

16
$$\begin{aligned} g + h \circ f(x) &= (g+h)f(x) \\ &= g(f(x) + h(f(x)) \\ &= g \circ f(x) + h \circ f(x) \end{aligned}$$

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17 For $x > 0$

$$\begin{aligned} f(g(x)) - f(x) &= f(xe^x) - \log_e x \\ &= \log_e(xe^x) - \log_e x \\ &= \log_e x + \log_e(e^x) - \log_e x \\ &= \log_e(e^x) \\ &= x \end{aligned}$$

$$\begin{aligned} \frac{g(f(x))}{f(x)} &= \frac{g(\log_e x)}{\log_e x} \\ &= \frac{\log_e x \times e^{\log_e x}}{\log_e x} \\ &= \frac{x \log_e x}{\log_e x} \\ &= x \end{aligned}$$

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Solutions to Exercise 7E

1 $f(x) = mx - 4, \quad m \in R\backslash\{0\}$

a $0 = mx - 4$

$mx = 4$

$x = \frac{4}{m}$

b $\frac{4}{m} \le 1$

$\therefore \quad 4 \le m, \quad m < 0$

c $x = mf^{-1}(x) - 4$

$x + 4 = mf^{-1}(x)$

$f^{-1}(x) = \frac{x+4}{m}, \quad \text{domain} = \mathbb{R}$

d $x = mx - 4$

$(m-1)x - 4 = 0$

$x = \frac{4}{m-1}$

check $\quad \mathrm{f}\left(\frac{4}{\mathrm{m}-1}\right) = \frac{4m}{m-1} - 4$

$= \frac{4m - 4m + 4}{m-1}$

$= \frac{4}{m-1}$

co-ordinates $\left(\frac{4}{m-1}, \frac{4}{m-1}\right)$,
$m \in \mathbb{R}\backslash\{0, 1\}$

e $y = ax + b$

$a = \frac{-1}{m} \quad$ (normalline)

$y = \frac{-x}{m} + b$

$(0, -4)$

$\Rightarrow \quad -4 = b$

$y = \frac{-x}{m} - 4$

2 $f(x) = -2x + c$

a $0 = -2x + c$

$-c = -2x$

$x = \frac{c}{2}$

b $\frac{c}{2} \le 1$
$c \le 2$

c $x = -2f^{-1}(x) + c$

$x - c = -2f^{-1}(x)$

$f^{-1}(x) = \frac{c-x}{2}, \quad \text{domain} = \mathbb{R}$

d $x = -2x + c$

$3x = c$

$x = \frac{c}{3}$

$y = x$

$co - ords = \left(\frac{c}{3}, \frac{c}{3}\right)$

e $y = ax + b$
$(0, c)$
$\Rightarrow \quad b = c$

$y = ax + c$
normal line
$\Rightarrow \quad a = \frac{-1}{-2}$

$a = \frac{1}{2}$

$y = \frac{x}{2} + c$

3 $y = x^2 - bx$

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a $x^2 - bx = 0$

$x(x - b) = 0$

$x = 0, b$

b $x^2 - bx + \frac{b^2}{4} - \frac{b^2}{4} = 0$

$\left(x - \frac{b}{2}\right)^2 - \frac{b^2}{4} = 0$

$co - ords: \left(\frac{b}{2}, \frac{-b^2}{4}\right)$

c **i** $-x = x^2 - bx$

$x^2 - (b-1)x = 0$

$x(x - (b-1)) = 0$

$x = 0, b - 1$

$y = -x = 0, 1 - b$

$co - ords: (0, 0), (b - 1, 1 - b)$

ii $b - 1 = 0,$

$b = 1$

iii $b - 1 \neq 0,$

$b \neq 1$

$b \in \mathbb{R}\backslash\{1\}$

4 $y = ax^2 + bx + c$

When $x = -1, y = 6$

When $x = 1, y = 4$

$6 = a - b + c \ldots (1)$

$4 = a + b + c \ldots (2)$

Equation (1) – Equation (2)

$2 = -2b$

$b = -1$

Substitute in (1)$6 = a + 1 + c$

$a = 5 - c$

$y = (5 - c)x^2 - x + c$

5 **a** $(1 + h)^2 = 8$

$1 + h = \pm 2\sqrt{2}$

$h = -1 \pm 2\sqrt{2}$

b $(a + 1)^2 = 8$

$a^2 = 8$

$a = \pm 2\sqrt{2}$

c $a = ax^2 + bx$

$y = a\left(x^2 + \frac{b}{a}x\right)$

$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a}$

$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a}$

vertex (1,8)

(**1**) $\frac{-b}{2a} = 1,$ (**2**) $-\frac{b^2}{4a} = 8$

$\frac{(\mathbf{2})}{(\mathbf{1})} \Rightarrow \frac{b}{2} = 8$

$b = 16$

Sub in (**1**) $\Rightarrow \frac{-16}{2a} = 1$

$\frac{-8}{a} = 1$

$a = -8$

check

$\Rightarrow (\mathbf{2}) \Rightarrow \frac{-(16)^2}{4^* - 8} = 8$

$\frac{256}{32} = 8$

$\frac{2}{2^5} = 2^3$ correct

$a = -8, b = 16$

6 $f(x) = \sqrt{2a - x}$

a $2a - x \geq 0$

$2a \geq x$, so domain $= (-\infty, 2a]$

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b $x = \sqrt{2a - x}$

$x^2 = 2a - x$

$x^2 + x - 2a = 0$

$x = \dfrac{-1 \pm \sqrt{1 + 8a}}{2}$

$but\ y > 0, \quad \therefore x > 0$

$\therefore \quad x = \dfrac{-1 \pm \sqrt{1 + 8a}}{2}$

$y = x,$

$\therefore \quad co - ords =$

$\left(\dfrac{-1 + \sqrt{1 + 8a}}{2}, \dfrac{-1 + \sqrt{1 + 8a}}{2}\right)$

c $\dfrac{-1 + \sqrt{1 + 8a}}{2} = 1$

$-1 + \sqrt{1 + 8a} = 2$

$\sqrt{1 + 8a} = 3$

$1 + 8a = 9$

$8a = 8$

$a = 1$

d $\dfrac{-1 + \sqrt{1 + 8a}}{2} = 2$

$\sqrt{1 + 8a} = 5$

$1 + 8a = 25$

$8a = 24$

$a = 3$

e $\dfrac{-1 + \sqrt{1 + 8a}}{2} = c$

$\sqrt{1 + 8a} = 2c + 1$

$1 + 8a = 4c^2 + 4c + 1$

$8a = 4c^2 + 4c$

$a = \dfrac{c^2 + c}{2}$

7 $f(x) = (x^2 - ax)^2$

a $0 = (x^2 - ax)^2$

$0 = x(x - a)$

$x = 0, a$

$co - ords = (0, 0), (a, 0)$

b $f(0) = (0 - 0)^2$

$= 0$

$co - ords = (0, 0)$

c $x \in [0, a]$

$f(x) = (ax - x^2)^2$

$= \left(-\left(x^2 - ax + \dfrac{a^2}{4}\right) + \dfrac{a^2}{4}\right)^2$

$= \left(-\left(x - \dfrac{a}{2}\right)^2 + \dfrac{a^2}{4}\right)^2$

maximum value is $\dfrac{a^4}{16}$

d $f(-1) = 16$

$16 = ((-1)^2 - a(-1))^2$

$\pm 4 = 1 + a$

$1 + a = \pm 4$

$a = -1 \pm 4$

$a = -5, 3$

8 **a** $-ae^{bx} + c = 0$

$e^{bx} = \dfrac{c}{a}$

$bx = \ln \dfrac{c}{a}$

$x = \dfrac{1}{b} \ln \dfrac{c}{a}$

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b $c\ln(x+a) = b$

$$\ln(x+a) = \frac{b}{c}$$

$$x + a = e^{\frac{b}{c}}$$

$$x = e^{\frac{b}{c}} - a$$

c $\ln(cx - a) = 0$

$$\ln(cx - a) = 1$$

$$cx = 1 + a$$

$$x = \frac{1+a}{c}$$

d $e^{ax+b} = c$

$$ax + b = \ln c$$

$$x = \frac{\ln c - b}{a}$$

9 $f(x) = c\ln(x - a)$

a $x - a = 0$

$$x = a$$

b $x - a = 1$

$$x = 1 + a$$

$$co - ords = (1 + a, 0)$$

c $1 = c\ln(x - a)$

$$x - a = e^{\frac{1}{c}}$$

$$x = a + e^{\frac{1}{c}}$$

$$co - ords = (a + e^{\frac{1}{e}}, 0)$$

d $1 = c\ln(2 - a)$

$$c = \frac{1}{\ln(2-a)}$$

10 $f(x) = e^{x-1} - b$

a $y = 0 - b$

$y = -b$

b $0 = e^{x-1} - b$

$x - 1 = \ln b$

$x = \ln b + 1$

$co - ords = (\ln b + 1, 0)$

c **i** $\ln b + 1 = 0$

$$\ln\ b = -1$$

$$b = e^{-1}$$

$$b = \frac{1}{e}.$$

ii $\ln b + 1 < 0$

$$\ln b < -1$$

$$b < e^{-1}$$

$$b < \frac{1}{e}$$

but $b > 0$ as given, else there is no intercept

$$\therefore\quad 0 < b < \frac{1}{e}$$

11 $y = ax^3 + bx^2 + cx + d$

$(-1, 6)$

$\Rightarrow$ ① $\quad 6 = -a + b - c + d$

$(1, -2)$

$\Rightarrow$ ② $\quad -2 = a + b + c + d$

② + ① $\quad 4 = 2b + 2d$

$$b = 2 - d$$

(2,4)

$\Rightarrow$ ③ $\quad 4 = 8a + 4b + 2c + d$

② − ① $\Rightarrow\quad 2a + 2c = -8$

$$a + c = -4$$

$$a = -4 - c$$

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Sub in ③ $\Rightarrow$

$$4 = 8(-4 - c) + 4(2 - d) + 2c + d$$
$$4 = -32 - 8c + 8 - 4d + 2c + d$$
$$4 = -24 - 6c - 3d$$
$$6c + 3d = -28$$
$$6c = -28 - 3d$$
$$c = \frac{-28 - 3d}{6}$$
$$a = -4 - \mathrm{c}$$
$$a = 4 - \mathrm{c}$$
$$a = \frac{-24 + 28 + 3d}{6}$$
$$a = \frac{3d + 4}{6}$$

12 $y = \left(\frac{c-8}{2}\right)x^2 + \left(\frac{20-3c}{2}\right)x + c$

$$b^2 - 4ac$$
$$= \left(\frac{20-3c}{2}\right)^2 - 4\left(\frac{c-8}{2}\right)c$$
$$= \frac{400 - 120c + 9c^2}{4} - 2c^2 + 16c$$
$$= 100 - 30c + \frac{9}{4}c^2 - 2c^2 + 4c$$
$$= \frac{1}{4}c^2 - 14c + 100$$

a $b^2 - 4ac = 0$

$$c^2 - 56c + 400 = 0$$
$$c = \frac{56 \pm \sqrt{1536}}{2}$$
$$c = 28 \pm 8\sqrt{6}$$

b $b^2 - 4ac > 0$

$c^2 - 56c + 400 > 0$

$c < 28 - 8\sqrt{6}$ or $c > 28 + 8\sqrt{6}$

but $c \neq 8$ (since if $c = 8$, the function becomes linear

$\therefore$ $c < 8$, or

$8 < c < 28 - 8\sqrt{6}$ or $c > 28 + 8\sqrt{6}$

13 a $y = ax^3 + bx^2 + cx + d$

$(-2, 8)$

$\Rightarrow$ ① $\quad 8 = -8a + 4b - 2c + d$

$(1,1)$

$\Rightarrow$ ② $\quad 1 = a + b + c + d$

$(3,4)$

$\Rightarrow$③ $\quad 4 = 27a + 9b + 3c + d$

③ − 3② ⇒ ④ $\quad 1 = 24a + 6b - 2d$

2③ + 3① ⇒ ⑤ $\quad 32 = 30a + 30b + 5d$

⑤ ⇒ $\quad \frac{32}{5} = 6a + 6b + d$

④ − ⑤ ⇒ $\quad 1 - \frac{32}{5} = 18a - 3d$

$$-\frac{9}{5} = 6a - d$$
$$a = \frac{5d - 9}{30}$$

Sub in ⑤ $\Rightarrow \frac{32}{5} = \frac{5d - 9}{5} + 6b + d$

$$32 = 5d - 9 + 30b + 5d$$
$$30b = 41 - 10d$$
$$b = \frac{41 - 10d}{30}$$

Sub in ② $\Rightarrow 1 = \frac{5d - 9}{30} + \frac{41 - 10d}{30} + c + d$

$$30d = 5d - 9 + 41 - 10d + 30c + 30d$$
$$-2 = 30c + 25d$$
$$c = \frac{-2 - 25d}{30}$$

14 a $AX + C = X'$

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$$A^{-1} = \begin{bmatrix} \frac{k}{\det(A)} & 0 \\ 0 & \frac{-4}{\det(A)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{-4} & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{-4} & 0 \\ 0 & \frac{1}{k} \end{bmatrix} \begin{bmatrix} x' - 3 \\ y' - 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x' - 3}{-4} \\ \frac{y' - 2}{k} \end{bmatrix}$$

b $$y = \frac{1}{x}$$

$$\frac{y' - 2}{k} = \frac{1}{\left(\frac{x' - 3}{-4}\right)} \quad \text{(from (a))}$$

$$y' - 2 = \frac{-4k}{x' - 3}$$

$$y' = \frac{-4k}{x' - 3} + 2$$

c $$0 = \frac{-4k}{0 - 3} + 2$$

$$\frac{4}{3}k = -2$$

$$k = \frac{-6}{4} = \frac{-3}{2}$$

15 a

$$AX + C = X'$$

$$X = A^{-1}(X' - C)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\left(\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} a \\ -2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \det(\mathrm{A}) = -4 \times 2 = -8$$

$$A^{-1} = \begin{bmatrix} \frac{2}{\det(\mathrm{A})} & 0 \\ 0 & \frac{-4}{\det(\mathrm{A})} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{-8} & 0 \\ 0 & \frac{-4}{-8} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{-4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{-4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x' - a \\ y' + 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x' - a}{-4} \\ \frac{y' + 2}{2} \end{bmatrix}$$

b

$$y = 2^x$$

$$\frac{y' + 2}{2} = 2^{\left(\frac{x' - a}{4}\right)}$$

$$y' + 2 = 2^{\left(\frac{x' - a}{4} + 1\right)}$$

$$y' = 2^{\left(\frac{x' + 4 - a}{4}\right)} - 2 = 2 \times 2^{\frac{(x' - a)}{4}} - 2$$

c $$0 = 2^{\left(\frac{0 + 4 - a}{4}\right)} - 2$$

$$2 = 2^{\left(\frac{4 - a}{4}\right)}$$

$$\frac{4 - a}{4} = 1$$

$$4 - a = 4$$

$$-a = 0$$

$$a = 0$$

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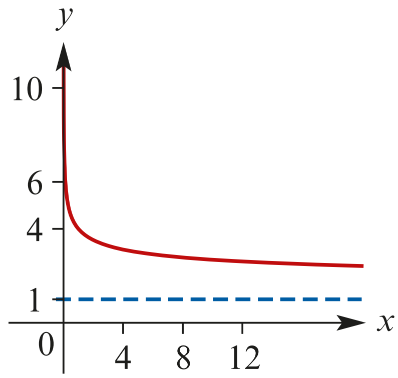
Solutions to Technology-free questions

1 a

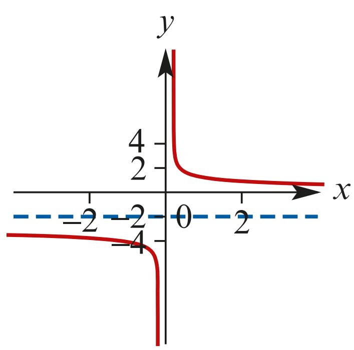


Domain = $\mathbb{R}^+$

Range = $(1, \infty)$

Neither

b



Domain = $\mathbb{R} \setminus \{0\}$

Range = $\mathbb{R} \setminus \{-2\}$

Neither

2 a $243^{\frac{2}{5}} = 3^2 = 9$

b $(-243)^{\frac{2}{5}} = (-3)^2 = 9$

c $243^{\frac{3}{5}} = 3^3 = 27$

d $(-243)^{\frac{2}{5}} = (-3)^2 = 9$

e $(-27)^{\frac{5}{3}} = (-3)^5 = -243$

f $(-125)^{\frac{4}{3}} = (-5)^4 = 625$

3 a i $f \circ g(x) = 3\cos(2x^2)$, $g \circ f(x) = 9\cos^2(2x)$

ii $\text{dom}(f \circ g) = \mathbb{R}$, $\text{ran}(f \circ g) = [-3, 3]$, $\text{dom}(g \circ f) = \mathbb{R}$, $\text{ran}(g \circ f) = [0, 9]$

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b **i** $f \circ g(x) = \log_e(3x^2)$, $g \circ f(x) = (\log_e(3x))^2$

ii $\text{dom}(f \circ g) = \mathbb{R} \setminus \{0\}$, $\text{ran}(f \circ g) = \mathbb{R}$, $\text{dom}(g \circ f) = \mathbb{R}^+$, $\text{ran}(g \circ f) = [0, \infty)$

c **i** $f \circ g(x) = \log_e(2 - x^2)$, $g \circ f(x) = (\log_e(2 - x))^2$

ii $\text{dom}(f \circ g) = (-\sqrt{2}, \sqrt{2})$, $\text{ran}(f \circ g) = (-\infty, \log_e 2)$, $\text{dom}(g \circ f) = (-\infty, 2)$, $\text{ran}(g \circ f) = [0, \infty)$

d **i** $f \circ g(x) = -\log_e(2x^2)$, $g \circ f(x) = (\log_e(2x))^2$

ii $\text{dom}(f \circ g) = \mathbb{R} \setminus \{0\}$, $\text{ran}(f \circ g) = \mathbb{R}$, $\text{dom}(g \circ f) = (0, \infty)$, $\text{ran}(g \circ f) = [0, \infty)$

4 **a** $h(x) = f \circ g(x)$, $g(x) = x^2$, $f(x) = \cos x$ (*Note:* answer not unique)

b $h(x) = f \circ g(x)$, $g(x) = x^2 - x$, $f(x) = x^n$ (*Note:* answer not unique)

c $h(x) = f \circ g(x)$, $g(x) = \sin x$, $f(x) = \log_e x$ (*Note:* answer not unique)

d $h(x) = f \circ g(x)$, $g(x) = \sin(2x)$, $f(x) = -2x^2$ (*Note:* answer not unique)

e $h(x) = f \circ g(x)$, $g(x) = x^2 - 3x$, $f(x) = x^4 - 2x^2$ (*Note:* answer not unique)

5 **a** **i** $(f + g)(x) = 2\cos\left(\frac{\pi x}{2}\right) + e^{-x}$

ii $(fg)(x) = 2e^{-x}\cos\left(\frac{\pi x}{2}\right)$

b **i** $(f + g)(0) = 3$

ii $(fg)(0) = 2$

6 $f : [a, \infty) \to \mathbb{R}, f(x) = -(3x - 2)^2 + 3$

Turning point at $(\frac{2}{3}, 3)$

a $\frac{2}{3}$

b $(-\infty, 3]$

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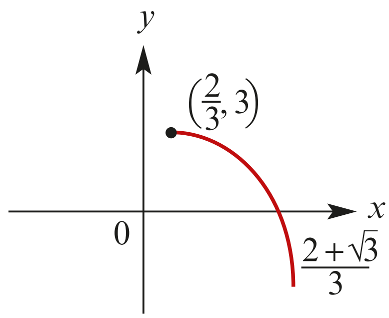
c



d $$f(f^{-1}(x)) = x$$

$$-3(f^{-1}(x) - 2)^2 + 3 = x$$

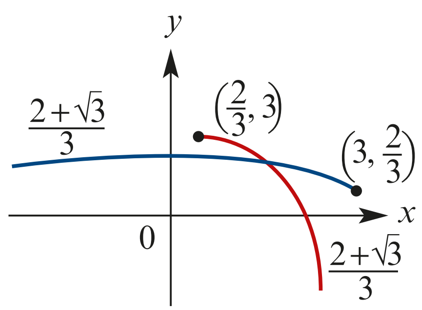
$$(f^{-1}(x) - 2)^2 = -\frac{x-3}{3}$$

$$f^{-1}(x) = 2 \pm \sqrt{-\frac{x-3}{3}}$$

Because of domain and range of f

$$f^{-1}(x) = \frac{2 + \sqrt{3-x}}{3}, \text{ ran} = [\tfrac{2}{3}, \infty), \text{ dom} = (-\infty, 3]$$

e



7 $f(x) = c\log_e(x - a)$

a $x = a$

b $c\log_e(x - a) = 0 \Rightarrow x - a = e^0 \Rightarrow x = a + 1$

Therefore cooordinates of the x-axis intercept are $(a + 1, 0)$

c $c\log_e(x - a) = c \Rightarrow \log_e(x - a) = 1 \Rightarrow x = a + e^1$

Therefore cooordinates of the point where the curve crosses the line $y = c$ is $(e + a, c)$

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d $$f(f^{-1}(x)) = x$$

$$c\log_e(f^{-1}(x) - a) = x$$

$$\log_e(f^{-1}(x) - a) = \frac{x}{c}$$

$$f^{-1}(x) = e^{\frac{x}{c}} + a$$

e (a, ∞)

f $f^{-1}(1) = 2 \Rightarrow f(2) = 1$
$f^{-1}(2) = 4 \Rightarrow f(4) = 2$

$$c\log_e(2 - a) = 1 \ldots (1)$$

$$c\log_e(4 - a) = 2 \ldots (2)$$

Equation (2) ÷ Equation (2)

$$\frac{\log_e(4 - a)}{\log_e(2 - a)} = 2$$

$$\log_e(4 - a) = 2\log_e(2 - a)$$

$$4 - a = (2 - a)^2$$

$$4 - a = 4 - 4a + a^2$$

$$a^2 - 3a = 0$$

$$a(a - 3) = 0$$

$$a = 0 \text{ or } a = 3$$

But $a = 3$ does not satisfy our equations
$\therefore c = \dfrac{1}{\log_e 2}, a = 0$

8 Consider,
$af^{-1}(x) + b = x$
$\therefore f^{-1}(x) = \dfrac{x}{a} - \dfrac{b}{a}$
If $f^{-1}(x) = 4x - 6$
$a = \dfrac{1}{4}$ and $-\dfrac{b}{a} = -6$
$\therefore a = \dfrac{1}{4}$ and $b = -\dfrac{3}{2}$

9 **a** $f^{-1}(x) = \left(\dfrac{x - 1}{3}\right)^3$

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b $f^{-1}(x) = \left(\dfrac{x+2}{4}\right)^3$

c $f^{-1}(x) = \dfrac{1}{3}\left((x-4)^{\frac{1}{3}} + 2\right)$

d $f^{-1}(x) = \left(\dfrac{3-x}{2}\right)^{\frac{1}{3}}$

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Solutions to multiple-choice questions

1 B

$h(x) = \dfrac{x^4 + 2}{x^2}$

Split $h(x)$ into two separate fractions:

$$h(x) = x^2 + \frac{2}{x^2}$$

$$\therefore \quad f(x) = x^2, g(x) = \frac{2}{x^2}$$

2 E The graph of

$f: R \to R, f(x) = \cos(x)$

Is not a one to one function.

3 E

A: $e^{x+y} = e^x \times e^y$, so A is true.

B: $\log_e xy = \log_e x + \log_e^y$, so B is true.

C: $\log_e x^y = y \log_e x$, so C is true.

D: $f^{-1}(1) = \log_e 1 = 0$, true for any x, y.

4 D

$$\begin{aligned} f(x) &= \cos x \\ g(x) &= 3x^2 \\ g(f(x)) &= 3(\cos x)^2 \\ g\left(f\left(\frac{\pi}{3}\right)\right) &= 3\left(\cos\frac{\pi}{3}\right)^2 \\ g\left(f\left(\frac{\pi}{3}\right)\right) &= 3\left(\frac{1}{4}\right) \\ g\left(f\left(\frac{\pi}{3}\right)\right) &= \frac{3}{4} \end{aligned}$$

5 E

$f: R \to R, f(x) = (x-2)^2$ is not an even function as it is a parabola that has been translated 2 units right, so it is not symmetrical about the y-axis.

6 E

$$y = 2ax + \cos 2x$$

When $x = \pi, y = 0$

$$\begin{aligned} 0 &= 2\pi a + \cos 2\pi \\ -1 &= 2\pi a \\ a &= -\frac{1}{2\pi} \end{aligned}$$

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7 B $$\begin{aligned} x > 5, g(x) &= \log_e(x-5) \\ 2[g(x)] &= g(f(x)) \\ 2[g(x)] &= \log_e(x-5)^2 \\ \log_e(x-5)^2 &= \log_e(f(x)-5) \\ (x-5)^2 &= f(x)-5 \\ x^2 - 10x + 25 &= f(x) - 5 \\ f(x) &= x^2 - 10x + 30 \end{aligned}$$

8 C

9 D

10 C Domain of $(f+g) = (-\infty, 3) \cap [2, \infty) = [2, 3)$

11 B $\dfrac{y'-2}{-4} = \sin\left(3x' - \dfrac{\pi}{3}\right)$

$\therefore$ choose $y' = -4y + 2$ and $x' = \dfrac{1}{3}x + \dfrac{\pi}{9}$

12 D $\dfrac{y+3}{2} = \sin\left(2x - \dfrac{\pi}{4}\right)$ to $y' = \sin x'$ $\therefore$ choose $y' = \dfrac{y+3}{2} = \dfrac{1}{2}y + \dfrac{3}{2}$ and $x' = 2x - \dfrac{\pi}{4}$

13 E

14 B

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Solutions to extended-response questions

1 **a** The range of $f\colon \mathbb{R}^+ \to \mathbb{R}, f(x) = e^{-x}$ is $(0,1)$

The range of $g\colon (-\infty, 0) \to R, g(x) = \dfrac{1}{x-1}$ is $\mathbb{R}^-$

b domain of f^{-1} = range of $f = (0, 1)$

domain of g^{-1} = range of $g = R^+$

To determine the rule for f^{-1} consider

$$x = e^{-y}$$

$$\log_e x = -y$$

$$-\log_e x = y$$

Therefore $f^{-1}(x) = -\log_e x$

To determine the rule for g^{-1} consider

$$x = \frac{1}{y-1}$$

Taking the reciprocal of both sides

$$y - 1 = \frac{1}{x}$$

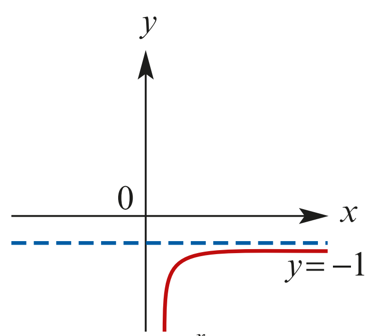
and $y = \dfrac{1}{x} + 1$

$$g^{-1}(x) = \frac{1}{x} + 1$$

c **i** $g \circ f(x)$ is defined as range of $f \subseteq$ domain of g

$$g \circ f(x) = g(f(x)) = g(e^{-x}) = \frac{1}{e^{-x}-1} = \frac{e^x}{1-e^x}$$

ii



$$g \circ f(x) = \frac{e^x}{1-e^x} = -1 + \frac{1}{1-e^x}$$

d **i** $g \circ f(x) = \dfrac{e^x}{1-e^x}$ with domain $= R^+$ For the inverse consider

$$\frac{e^y}{1-e^y} = x$$

Solve for y

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$x(1 - e^y) = e^y$

$x - xe^y = e^y$

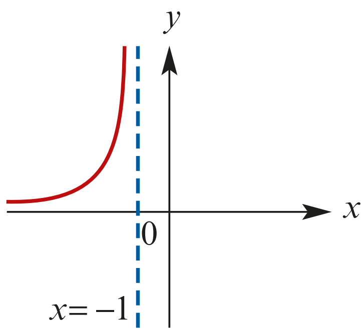
Therefore $\quad e^y(1 + x) = x$

and $\quad e^y = \dfrac{x}{1 + x}$

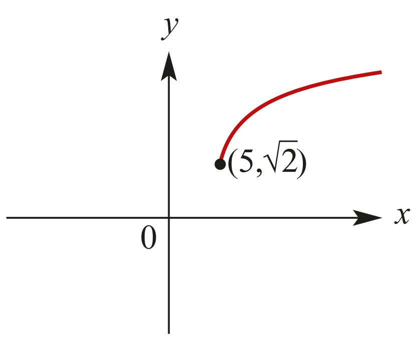
Therefore $\quad y = \log_e\left(\dfrac{x}{1 + x}\right)$

$(g \circ f)^{-1}(x) = \log_e\left(\dfrac{x}{1 + x}\right)$ and domain of $(g \circ f)^{-1}$ = range of $g \circ f = (-\infty, -1)$

ii



2 a i $f : [5, \infty) \to \mathbb{R}, f(x) = \sqrt{x - 3}$



ii range of = $[\sqrt{2}, \infty)$

iii For the inverse rule consider $x = \sqrt{y - 3}$

Square both sides and make y the subject.

$y = x^2 + 3$

and $f^{-1}(x) = x^2 + 3$. The domain of the inverse function is $[\sqrt{2}, \infty)$

b i h: $[4, \infty) \to \mathbb{R}$, $h(x) = \sqrt{x - p}$

The inverse function has domain $[1, \infty)$.

The function h is increasing and therefore $\sqrt{4 - p} = 1$

Therefore $p = 3$.

ii Proceeding as above the rule is $h^{-1}(x) = x^2 + 3$.

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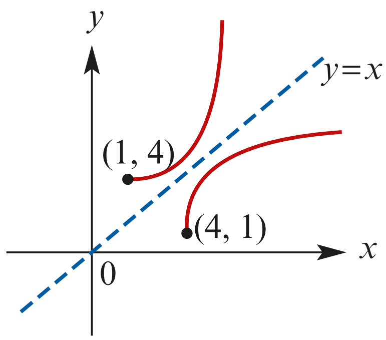
iii



3 $f: (0, \pi) \to \mathbb{R}, f(x) = \sin x$

$g: [1, \infty) \to \mathbb{R}, g(x) = \frac{1}{x}$

a range of $f = (0, 1)$

b range of $g = (0, 1]$

c $f \circ g$ is defined as the range of g $\subseteq$ domain of f as $1 < \pi$

$f \circ g(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \sin\left(\frac{1}{x}\right)$

d $g \circ f$ is not defined as the range of $f \not\subset$ domain of g

e Consider

$x = \frac{1}{y}$ and solve for y.

Therefore $g^{-1}(x) = \frac{1}{x}$

The domain of g^{-1} = range of g = $(0, 1]$

The range of g^{-1} = domain of $g = [1, \infty)$

f Range of $f = (0, 1) \subseteq (0, 1]$ = domain of g^{-1}.

Therefore g^{-1} of is defined and

$g^{-1} \circ f(x) = g^{-1}(f(x)) = g^{-1}(\sin x) = \frac{1}{\sin\ x}$

The domain of $g^{-1} \circ f$ = domain of $f = (0,\ \pi)$

The range of $g^{-1} \circ f = [1, \infty)$

4 **a** $a = 2$

b $c = 2 - k \log_e(2)$

c $k = \frac{10}{\log_e(\frac{d+2}{2})}$

d $k = 10$

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Chapter 8 – Revision of Chapters 1–7

Solutions to technology-free questions

1 **a** Domain = $\mathbb{R} \setminus \{0\}$; Range = $\mathbb{R} \setminus \{2\}$

b $3x - 2 \geq 0 \Rightarrow x \geq \dfrac{2}{3}$

The endpoint is $\left(\dfrac{2}{3}, 3\right)$

Domain = $\left[\dfrac{2}{3}, \infty\right)$; Range = $(-\infty, 3]$

c Domain = $\mathbb{R} \setminus \{2\}$; Range = $(3, \infty)$

d Domain = $\mathbb{R} \setminus \{2\}$; Range = $\mathbb{R} \setminus \{4\}$

e $x - 2 \geq 0 \Rightarrow x \geq 0$

The endpoint is $(2, -5)$

Domain = $[2, \infty)$; Range = $(-5, \infty]$

2 $\sqrt{f^{-1}(x) - 2} + 4 = x$

$\therefore \sqrt{f^{-1}(x) - 2} = x - 4$

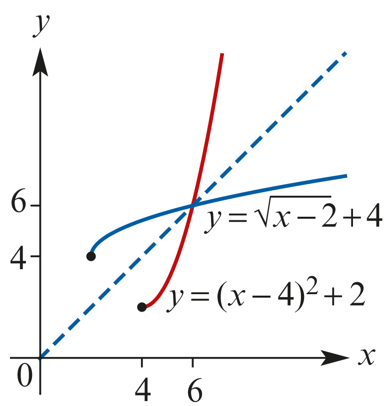
$\therefore f^{-1}(x) - 2 = (x - 4)^2$

$\therefore f^{-1}(x) = (x - 4)^2 + 2$

The domain of f^{-1} = range of f = $[4, \infty)$

Hence,

$f^{-1}: [4, \infty) \to \mathbb{R}, f^{-1}(x) = (x - 4)^2 + 2$



3 $\dfrac{f^{-1}(x) - 2}{f^{-1}(x) + 1} = x$

$\therefore f^{-1}(x) - 2 = x(f^{-1}(x) + 1)$

$\therefore f^{-1}(x) - xf^{-1}(x) = x + 2$

$\therefore f^{-1}(x)(1 - x) = x + 2$

$\therefore f^{-1}(x) = \dfrac{x + 2}{1 - x}$

The domain of f^{-1} = range of f = $\mathbb{R} \setminus \{1\}$

Hence,

$f^{-1}: \mathbb{R} \setminus \{1\} \to \mathbb{R}, f^{-1}(x) = \dfrac{x + 2}{1 - x}$

4 **a** $2e^{f^{-1}(x)} - 1 = x$

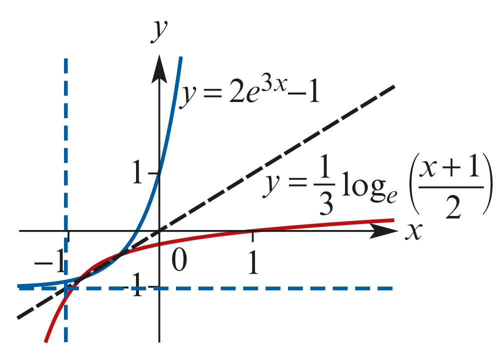
$\therefore e^{f^{-1}(x)} = \dfrac{x + 1}{2}$

$\therefore f^{-1}(x) = \log_e\left(\dfrac{x + 1}{2}\right)$

$f^{-1}(x) = \dfrac{1}{3}\log_e\left(\dfrac{x + 1}{2}\right)$,

dom f^{-1} = $(-1, \infty)$

b



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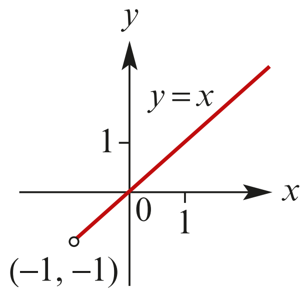
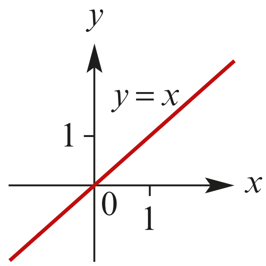
c



d



e $y = 2x$

5 $2\log_{10} 5 + 3\log_{10} 2 - \log_{10} 20$

$= \log_{10} 25 + \log_{10} 8 - \log_{10} 20$

$= \log_{10} \dfrac{200}{20}$

$= \log_{10} 10$

$= 1$

6
$$
\begin{aligned}
3\log_a x &= 3 + \log_a 12 \\
\log_a x^3 - \log_a 12 &= 3 \\
\log_a \left(\frac{x^3}{12}\right) &= 3 \\
\frac{x^3}{12} &= a^3 \\
x^3 &= 12a^3 \\
x &= \sqrt[3]{12}a
\end{aligned}
$$

7 $2^{-x} = 2^9$

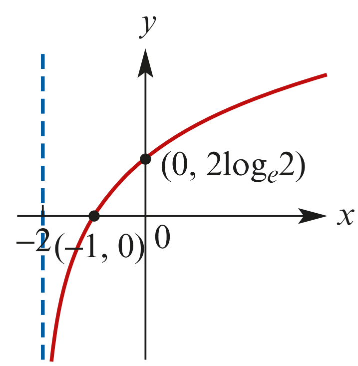
$x = -9$

8 $4e^{2x} = 9$
$$
\begin{aligned}
e^{2x} &= \frac{9}{4} \\
2x &= \log_e\left(\frac{9}{4}\right) \\
x &= \frac{1}{2}\log_e\left(\frac{9}{4}\right) = \log_e\left(\frac{3}{2}\right)
\end{aligned}
$$

9 **a** When $x = 0$, $f(0) = 2\log_e 2$

When $f(x) = 0$
$$
\begin{aligned}
2\log_e(x+2) &= 0 \\
\log_e(x+2)t &= 0 \\
x + 2 &= e^0 \\
x &= -1
\end{aligned}
$$
$\therefore a = -1$ and $b = 2\log_e 2$

b



10 $2^{4x} - 5 \times 2^{2x} + 4 = 0$

Let $a = 2^{2x}$
$$
\begin{aligned}
a^2 - 5a + 4 &= 0 \\
(a-4)(a-1) &= 0 \\
a = 4 \text{ or } a &= 1 \\
\therefore 2^{2x} = 4 \text{ or } 2^{2x} &= 1 \\
\therefore x = 0 \text{ or } x &= 1
\end{aligned}
$$

11 $\sin\left(\dfrac{3x}{2}\right) = \dfrac{1}{2}$

$\dfrac{3x}{2} = -\dfrac{7\pi}{6}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

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$x = -\frac{7\pi}{9}$ or $x = \frac{\pi}{9}$ or $x = \frac{5\pi}{9}$

12 **a** Range = $[2, 8]$; Period = 6

b $\cos\left(2x + \frac{\pi}{6}\right) = \frac{1}{2}$

$$2x + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$2x + \frac{\pi}{6} = \frac{2\pi}{6}, \frac{10\pi}{6}$$

$$2x = \frac{\pi}{6}, \frac{9\pi}{6}$$

$x = \frac{\pi}{12}$ or $x = \frac{3\pi}{4}$

13 Consider the gradients of the two lines
Gradient $\ell_1 = -m$
and Gradient $\ell_2 = -\frac{2}{m-1}$
If the gradients are equal

$$-m = -\frac{2}{m-1}$$

$$m = \frac{2}{m-1}$$

$$m^2 - m - 20$$

$$(m-2)(m+1) = 0$$

$$m = 2 \text{ or } m = -1$$

a Therefore a unique soluton when the lines are not parallel, $m \in \mathbb{R} \setminus \{-1, 2\}$

b If $m = 2$

$2x + y = 2$

$2x + y = -4$

The lines are parallel but do not coincide.
There is no solution.
$m = -1$ is checked in the next part.

c If $m = -1$

$-x + y = 2$

$2x - 2y = -4$

The lines coincide and there are infinitely many solutions.

14 $y = \frac{a}{x^2} + b$
When $x = 1, y = -1$
When $x = -2, y = \frac{1}{2}$

$$a + b = -1 \ldots (1)$$

$$\frac{a}{4} + b = \frac{1}{2} \ldots (2)$$

Equation (1) − Equation (2)

$$\frac{3a}{4} = -\frac{3}{2}$$

$\therefore a = -2$ and $b = 1$

15 $\Delta = m^2 - 8$

a $\Delta = 0 \Rightarrow m = \pm 2\sqrt{2}$

b $\Delta > 0 \Rightarrow m > 2\sqrt{2}$ or $m < -2\sqrt{2}$

c $\Delta < 0 \Rightarrow -2\sqrt{2} < m < 2\sqrt{2}$

16 **a** **i** $\frac{a+3}{2} = 0, \therefore a = -3$

ii $\sqrt{(a-3)^2 + (-2-1)^2} = \sqrt{13}$

$$a^2 - 6a + 9 + 9 = 13$$

$$a^2 - 6a + 5 = 0$$

$$(a-5)(a-1) = 0$$

$a = 5$ or $a = 1$

iii $\frac{3}{3-a} = \frac{1}{2}$

$$6 = 3 - a$$

$$a = -3$$

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b If $a = -2$ the gradient of the line is $\dfrac{3}{5}$
The equation of the line is
$y - 1 = \dfrac{3}{5}(x - 3)$
or $5y - 3x + 4 = 0$
The angle the line makes with the positive direction of the x-axis is $\tan^{-1}\left(\dfrac{3}{5}\right)$.

17 a Odd

b $f^{-1}(x) = \sqrt[3]{\dfrac{x}{2}}$

c i 2

ii -1

iii $f^{-1}(x) = f(x)$

$$\begin{aligned}\sqrt[3]{\frac{x}{2}} &= 2x^3\\ \frac{x}{2} &= 8x^9\\ x - 16x^9 &= 0\\ x(1 - 16x^8) &= 0\\ x = 0 \text{ or } x &= \left(\frac{1}{16}\right)^{\frac{1}{8}}\\ x = 0 \text{ or } x = 2^{-\frac{1}{2}} &\text{ or } x = -2^{-\frac{1}{2}}\end{aligned}$$

18 a 4

b $\sqrt{5}$

c $2 - 2a$

d $\sqrt{2a - 5}$

e $x = -8$

f $x = \dfrac{103}{2}$

g $x < 1$

19 a i $f \circ g(x) = 4x^2 + 8x - 3$

ii $g \circ f(x) = 16x^2 - 16x + 3$

iii $g \circ f^{-1}(x) = \dfrac{1}{16}(x^2 + 14x + 33)$

b Dilation of factor $\frac{1}{4}$ from the y-axis, then translation $\frac{3}{4}$ units to the right

c Translation 1 unit to the left and 1 unit down

20 $x = 2n\pi \pm \dfrac{\pi}{6}, n \in \mathbb{Z}$

21
$$\begin{aligned}Ae^k &= 4 \ldots (1)\\ Ae^{2k} &= 10 \ldots (2)\end{aligned}$$
Equation (2) ÷ Equation (1)
$$\begin{aligned}e^k &= \frac{5}{2}\\ k &= \log_e\left(\frac{5}{2}\right)\end{aligned}$$
$A = \dfrac{8}{5}$ and $k = \log_e\left(\dfrac{5}{2}\right)$

22 a
$$\begin{aligned}&2x^3 - 3x^2 - 11x + 6 \geq 0\\ \Leftrightarrow\ &(2x - 1)(x - 3)(x + 2) \geq 0\\ \Leftrightarrow\ &-2 \leq x \leq \frac{1}{2} \text{ or } x \geq 3\end{aligned}$$

b
$$\begin{aligned}&-x^3 + x^2 - 4x > 0\\ \Leftrightarrow\ &x(x^2 - x + 4) < 0\\ \Leftrightarrow\ &x < 0\end{aligned}$$

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Solutions to multiple-choice questions

1 D Domain $= [-1, 3)$ since -1 is included and 3 is excluded.

2 A For each value of $x > 0$, the rule $x = 2y^2$, $x \geq 0$ gives two value for y, so is not a function.

3 B Require $2 - x > 0$, i.e. $x < 2$. Implied domain $= (-\infty 2)$

4 E
$$f\left(-\frac{1}{a}\right) = \frac{-\frac{1}{a}}{-\frac{1}{a} - 1} = \frac{-1}{-1 - a} = \frac{1}{a + 1}$$

5 E
$$(f + g)\left(\frac{3\pi}{2}\right) = f\left(\frac{3\pi}{2}\right) + g\left(\frac{3\pi}{2}\right) = \sin(3\pi) + 2\sin\left(\frac{3\pi}{2}\right) = -2$$

6 C
$$f(g(3)) = f(18) = 56$$

7 A
$$\text{dom } f = [0, 6];\ \text{dom } g = (-\infty, 2]$$
$$\text{dom } (f + g) = \text{dom } f \cap \text{dom } g = [0, 2]$$

8 A For $x \leq 0$, the gradient is -2 and the y intercept is $(0, -2)$; the equation is $y = -2x - 2$ for $x \leq 0$. For $x > 0$, the gradient is 1 and the y intercept is $(0, -2)$; the equation is $y = x - 2$ for $x > 0$.

9 B
$$fg(x) = (2x^2 + 1)(3x + 2) = 6x^3 + 4x^2 + 3x + 2$$

10 C Require $4 - x^2 \geq 0$, i.e.
$(2 - x)(2 + x) \geq 0$
$-2 \leq x \leq 2$
Implied domain $= [-2, 2]$

11 C Reflect the graph of $y = f(x)$ in the line $y = x$.
Then the endpoint $(4, -2)$ reflects to $(-2, 4)$.
Only the third graph fits.

12 B

13 A For $x < 2$, the straight line has gradient | and the y-intercept is $(0, -3)$; the equations is $y = x - 3$ for $x < 2$. For $x \geq 2$, the curve has equation $y = (x - 2)^2$.

14 E $f(2) = 0$, $f(3) = 2$ so dom f^{-1} = ram $f = [0, 2]$. For f, $y = 2x - 4$. For f^{-1}, interchange x and y and solve for y.
$$x = 2y - 4$$
$$x + 4 = 2y$$
$$y = \frac{x + 4}{2}$$
Hence:
$$f^{-1} : [0, 2] \to R,\ f^{-1}(x) = \frac{x + 4}{2}$$

15 D Require a subset of R so that f is one-to-one. Either $x \geq 0$ or $x \leq 0$ would do. Only the fourth option fits.

16 C The graph of $y = h(x)$ has its vertex at $(1, 1)$, so for an inverse function to

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exist, $x \geq 1$. Hence $a = 1$.

17 E For f, $y = 3x - 2$
For f^{-1}, interchange x and y and solve for y.

$$x = 3y - 2$$
$$x + 2 = 3y$$
$$y = \frac{1}{3}(x + 2)$$

18 C
$$2x = \frac{3x}{2} - 4$$
$$2x - \frac{3x}{2} = -4$$
$$\frac{x}{2} = -4$$
$$x = -8$$

19 B The straight line has gradient 2 and y intercept $(0, -2)$.
Hence $y = 2x - 2$

20 D
$$\frac{2(x-1)}{3} - \frac{x+4}{2} = \frac{5}{6}$$
$$\frac{4(x-1) - 3(x+4)}{6} = \frac{5}{6}$$
$$4x - 4 - 3x - 12 = 5$$
$$x - 16 = 5$$
$$x = 21$$

21 C $m = \dfrac{0-3}{4-(-2)} = -\dfrac{3}{6} = -\dfrac{1}{2}$

Use $y - y_1 = m(x - x_1)$
$$y - 0 = -\frac{1}{2}(x - 4)$$
$$y = -\frac{1}{2}x + 2$$
$$2y = -x + 6$$
$$2y + x = 4$$

22

C
$$y = \frac{4}{5}x - 4 = \frac{4}{5}(x - 5)$$

$x -$ axis intcept is $(5, 0)$

$y -$ as is intercept is $(0, -4)$.

$$\text{area } OAB = \frac{1}{2}(OA)(OB)$$
$$= \frac{1}{2}(5)(4)$$
$$= 10 \text{ square units}$$

23 D
$$2x - 3y = 12 \quad ①$$
$$3x - 2y = 13 \quad ②$$
② − ① gives:
$x + y = 1$
(Note: in this case, you do not need to solve for x and y explicitly, although it is not wrong to do so.)

24 D
$$7x - 6y = 20 \quad ①$$
$$3x + 4y = 2 \quad ②$$

$3 \times$ ② $+ 2 \times (i)$ gives :
$$9x + 14x = 6 + 40$$
$$23x = 46$$
$$x = 2$$

25 E The graph is that of a hyperbola with asymptotes $x = 1$ and $y = -3$.
The equation is of the form
$$y = \frac{a}{x-1} - 3$$
$x = 0, y = -4$:
$$-4 = \frac{a}{-1} - 3$$
$$-4 = -a - 3$$
$$a = 1$$
$$y = \frac{1}{x-1} - 3$$
(Check: $x = \dfrac{4}{3}$, $y = 0$ as expected.)

26 A As $x \to \pm\infty$, $y = f(x) \to -2$

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$y = -2$ is an asymptote So the range is $R\backslash\{-2\}$

27 D Vertex at (2, 3) means $y = a(x-2)^2 + 3$.
Only the fourth option in which $a = 1$ fits.

28 D Require $f(-x) = f(x)$ for any value of x.
If $f(x) = -x^2$, then

$$f(-x) = -(-x)^2$$
$$= -x^2$$
$$= f(x)$$

So $f(x) = -x^2$ is an even function of x.
(A quick check reveals that none of the other functions is even.)

29 A The factor '$x + 2$' indicates a translation of 2 units to the left.
So $(x, y) \to (x - 2, y)$.
The factor '3' indicates a dilation of factor 3 from the x-axis.

30 E Require $x - 2 \geq 0$, i.e. $x \geq 2$.
Maximal domain = $[2, \infty)$

31 D The graph has endpoint (3, 1) so its equation must be of the form
$y = a\sqrt{x-3} + 1$
$x = 4,\ y = 0{:}0 = a\sqrt{1} + 1$

$$a = -1$$

$y = -\sqrt{x-3} + 1$

32 E $\dfrac{3}{(x-2)^2} > 0$ for any $x \neq 2$.
So $\dfrac{3}{(x-2)^2} + 4 > 4$ for any $x \neq 2$.
The range is $(4, \infty)$.

33 A $3(1)^2 + k(1) + 1 = k + 4$

$$= 0$$
$$k = -4$$

34 A $(x-5)(x+7) = 0$
$x^2 + 2x - 35 = 0$

35 E Let $P(x) = x^3 - 5x^2 + x + k$.
$P(x)$ is divisible by $x + 1$, so

$$P(-1) = 0.$$
$$(-1)^3 - 5(-1)^2 + (-1) + k = 0$$
$$-1 - 5 - 1 + k = 0$$
$$k = 7$$

36 C The graph could be a cubic with minimum turning point at $(-2, 0)$ and another x - intercept at $(2, 0)$.
Equation is $y = a(x+2)^2(x-2)$
$x = 0,\ y > 0{:} = 8a > 0$

$$a < 0$$

Only the third option fits.

37 D The graph could be a cubic with a Stationary points of inflexion as $(-1, 2)$.
Equation is $y = a(x+1)^3 + 2$.
Only the fourth option fits.
$y = -\dfrac{1}{2}(x+1)^3 + 2$, then when $x = 0$
(Check: If

$$y = -\frac{1}{2} + 2$$
$$= 1\frac{1}{2}$$

which is consistent with the graph.)

38 C $P(-1) = -1 + 2 + 5 - 6 = 0$
So $(x + 1)$ is a factor.
Option B expanded has a constant term of +6.
Option C expanded has constant

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term of −6.
Option D expanded has constant term of +6
Only option C fits.
(Alternatively, divide the cubic by $(x + 1)$ and factorise the resulting quadratic.)

39 E For $f.y = mx + 3$
For f^{-1}, interchange x and y and solve for y.

$$\begin{aligned} x &= m\,y + 3 \\ x - 3 &= m\,y \\ y &= \frac{1}{m}(x - 3) \\ &= \frac{1}{m}x - \frac{3}{m} \end{aligned}$$

Hence $a = \frac{1}{m}$, $b = -\frac{3}{m}$

40 B Remainder is given by $P(2)$

$$\begin{aligned} P(2) &= 2(2)^3 - 2(2)^2 + 3(2) + 1 \\ &= 16 - 8 + 6 + 1 \\ &= 15 \end{aligned}$$

41 B Let $P(x) = x^3 + 2x^2 + ax - 4$
Given $P(-1) = 1$

$$\begin{aligned} (-1)^3 + 2(-1)^2 + a(-1) - 4 &= 1 \\ -1 + 2 - a - 4 &= 1 \\ -a - 3 &= 1 \\ a &= -4 \end{aligned}$$

42 C The graph could be a quartic with minimum tuning point at $(-2, 0)$ and $(2, 0)$.

$$\begin{aligned} \text{Equation is } y &= a(x + 2)^2(x - 2)^2 \\ &= a\big((x + 2)(x - 2)\big)^2 \\ &= a(x^2 - 4)^2 \end{aligned}$$

When $x = 0$, $y = a \times (-4)^2 = 16a$
For the graph the y intercept is posture so $a > 0$
Only the third alternative fits.

43 D As $x \to \infty$, $f(x) \to 1$,
Since $e^{-x} > 0$ for all x, $f(x) > 1$ for all x.
Hence the range of f is $(1, \infty)$ and this is the domain of f^{-1}

44 B For f, $y = 2\log_e x + 1$
for f^{-1}, interchange x and y and solve for y.

$$\begin{aligned} x &= 2\log_e y + 1 \\ x - 1 &= \log_e y \\ \log_e y &= \frac{1}{2}(x - 1) \\ y &= e^{\frac{1}{2}(x-1)} \\ \text{So } f^{-1}(x) &= e^{\frac{1}{2}(x-1)} \end{aligned}$$

45 B $\log_e(-1 + 2) = \log_e 1 = 0$, so range of $g = (0, \infty)R^+$
$e^{-0} = 1$ and as $x \to \infty$, $e^{-x} \to 0$
Hence the range of the function with rule $y = f(g(x))$ is $(0, 1)$.

46 E For f, $y = e^x - 1$
For f^{-1}, interchange x and y and solve for y.

$$\begin{aligned} x &= e^y - 1 \\ x + 1 &= e^g \\ y &= \log_e(x + 1) \\ \text{So } f^{-1}(x) &= \log_e(x + 1). \end{aligned}$$

47 A $f(4) = \log_e(4 - 3) = \log_e 1 = 0$, so f has range $[0, \infty)$
and this is the domain of the inverse.

48 C For f, $y = e^{x-1}$

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For f^{-1}, interchange x and y and solve for y.

$$x = e^{y-1}$$
$$\log_e x = y - 1$$
$$y = 1 + \log_e x$$
So $f^{-1}(x) = 1 + \log_e x$

49 D For f, $y = \log_e \dfrac{x}{2}$
For f^{-1}, interchange x and y and solve for y.
$$x = \log_e \frac{y}{2}$$
$$e^x = \frac{y}{2}$$
$$y = 2e^x$$
So $f^{-1}(x) = 2e^x$.

50 B Require $3x - 2 > 0$, i.e.
$$3x > 2$$
$$x > \frac{2}{3}$$
So f is defined for $x \in \left(\dfrac{2}{3}, \infty\right)$.

51 C The Graph of f has asymptote $x = -2$.
Reflecting it in the line $y = x$ means its inverse has asymptote $y = -2$.
Only the third option fits.

52 C Method 1
$$\begin{aligned}\log_2 8x &= \log_2 8 + \log_2 x \\ &= \log_2 2^3 + \log_2 x \\ &= 3\log_2 2 + \log_2 x \\ &= \log_2 x + 3\end{aligned}$$
$$\begin{aligned}\log_2 2x &= \log_2 2 + \log_2 x \\ &= \log_2 x + 1\end{aligned}$$
So the equation becomes
$$\begin{aligned}\log_2 x + 3 + \log_2 x + 1 &= 6 \\ 2\log_2 x &= 2 \\ \log_2 x &= 1 \\ x &= 2\end{aligned}$$
Method 2
$$\begin{aligned}\log_2 8x - \log_2 2x &= 6 \\ \log_2(8x \times 2x) &= 6 \\ \log_2(16x^2) &= 6 \\ 16x^2 &= 2^6 \\ &= 64 \\ x^2 &= 4 \\ x &= \pm 2\end{aligned}$$
But $x > 0$, so $x = 2$.

53 A
$$\begin{aligned}\log_{10} x &= y(\log_{10} 3) + 1 \\ &= \log_{10} 3^y + \log_{10} 10 \\ &= \log_{10}(10(3^y)) \\ x &= 10(3^y)\end{aligned}$$

54 B Graph has gradient -2 and y intercept $(0, 2)$.
Equation is $\log_e N = -2t + 2$
$$N = e^{2-2t}$$

55 A As $x \to -\infty$, $y \to 1$, so the rule must involve e^x, and $e^{-x.}$
When $x = 0$, $y = 0$.
Only the first option fits both of these.

56 B $x = -2$ is a vertical asymptote and the domain is $(-2, \infty)$, so only the second and fourth options are possible.
The graph through $(0, 0)$.
B: $\log_e \dfrac{1}{2}(0 + 2) = \log_e 1 = 0$
E: $\dfrac{1}{2}\log_e(0 + 2) = \dfrac{1}{2}\log_e(2) \neq 0$

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So the second option fits.

57 D Period $= \frac{5\pi}{12} - \left(-\frac{\pi}{4}\right) = \frac{8\pi}{12} = \frac{2\pi}{3}$

Range $= [-4, 0]$ so amplitude $= 2$ and there is a vertical translation of 2 units down.

these rule out options A and B.

When $\theta = \frac{\pi}{4}$, $y = 0$.

In the third and fifth options, when $\theta = -\frac{\pi}{4}$, $y = -4$; in the fourth option:

$y = 2\cos 3\left(-\frac{\pi}{4} + \frac{\pi}{4}\right) - 2$

$2\cos 0 - 2$

$= 2 - 2$

$= 0$

So the fourth option fits.

58 D The minimum value of f is $2 - 3 = -1$

The maximum value of f is $2 + 3 = 5$.

The range of f is [-1, 5]

59 D $2\sin\theta + \sqrt{3} = 0$

$2\sin\theta = -\sqrt{3}$

$\sin\theta = -\frac{\sqrt{3}}{2}$

θ is in the third or fourth quadrants.

$\theta = \pi + \frac{\pi}{3}$ or $2\pi - \frac{\pi}{3}$

$\theta = \frac{4\pi}{3}$ or $\frac{5\pi}{3}$

60 A When $x = \frac{\pi}{2}$, $y = 0$. Only options A and C satisfy this.

When $\frac{\pi}{6} < x < \frac{7\pi}{6}$, $y > 0$.

This is true for option A but false for option C.

61 A Amplitude 3, period $= \frac{2\pi}{2} = \pi$

62 D The minimum value of f is -3.

The maximum value of f is 3.

The range of f is $[-3, 3]$

63 B $P(x) = 0 \Rightarrow x - 2a = 0$ or

$x + a = 0$ or $x^2 a = 0$.

So $x = 2a$ or $x = a$ or $x^2 = -a$.

But $\alpha > 0$ so $x^2 = -a$ has no solutions.

The equation has 2 decimal real solutions.

64 D The gradient of the given straight live is -2.

For perpendicular lines, $m_1 m_2 = -1$.

So $-2m_2 = -1$, going $m_2 = \frac{1}{2}$.

65 B Sine x and y are interchanged for the inverse, there must be an asymptote will equation $x = 6$ for the inverse function.

So the inverse has vertical asymptote with equation $x = 6$.

66 E Period $= \frac{2\pi}{6}$

67 C $f(18) = 32 = 2^5, f(34) = 64 = 2^6$.

$g(2^5) = \log_2 2^5 = 5$

$g(2^6) = \log_2 2^6 = 6$

The range of $g \circ f$ is $[5, 6]$

68 C Interchange x & y: $x = y^2 - 4y + 5$

Solve for y: $x = (y-2)^2 + 1$

$(y-2)^2 = x - 1$

$y = 2 \pm \sqrt{x-1}$

69 B

70 C

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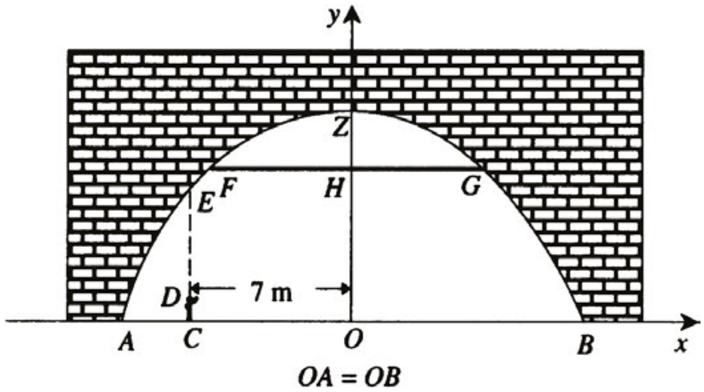
Solutions to extended-response questions



1 a The graph is of the form
$y = ax^2 + b$
The vertex is at $(0, 9)$.
Therefore $b = 9$
The width of the arch is 20m.
Therefore the x-axis intercepts are at $(10, 0)$ and $(-10, 0)$
When $x = 10$, $y = 0$.

Hence $b = 9$

and $\quad 0 = a \times 100 + 9$

$\therefore \quad a = \dfrac{-9}{100} = -0.09$

b The equation of the curve is $y = \dfrac{-9}{100}x^2 + 9$

When $x = -7$ $y = \dfrac{-9}{100} \times 49 + 9 = 4.59$

The man is 1.8 m high.

$\therefore$ E is $(4.56 - 1.8)$ m $= 2.79$ m above the man's head.

c OH is 6.3 m

$\therefore$ Consider $y = 6.3$

$$6.3 = \frac{-9}{100}x^2 + 9$$

$$\frac{-2.7}{-9} \times 100 = x^2$$

$\therefore \quad 30 = x^2$

$\therefore \quad x = \pm\sqrt{30}$

The length of the bar is $2\sqrt{30}$ m ≈ 10.95 m.

2 a Let $P(x) = 2x^3 + ax^2 - 72x - 18$

By the Remainder Theorem

$$P(-5) = 17$$

$$\text{i.e. } 2 \times (-5)^3 + a(-5)^2 - (72 \times -5) - 18 = 17$$

$$250 + 25a + 360 - 18 = 17$$

$$\therefore\ 25a = -75$$

$$a = -3$$

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b $2x^3 = x^2 + 5x + 2$

Let $P(x) = 2x^3 - x^2 - 5x - 2$

$P(-1) = -2 - 1 + 5 - 2 = 0$

$\therefore$ By the Factor Theorem $x + 1$ is a factor.

Dividing $P(x)$ by $x + 1$

$P(x) = (x + 1)(2x^2 - 3x - 2)$

For $P(x) = 0$

$x = -1$ or $2x^2 - 3x - 2 = 0$

and $2x^2 - 3x - 2 = 0$

implies $(2x + 1)(x - 2) = 0$

$\therefore\ x = -\frac{1}{2}$ or $x = 2$

i.e. $x = -\frac{1}{2}$, $x = 2$ and $x = -1$ are solutions to the equation $2x^3 = x^2 + 5x + 2$

c $x^2 - 5x + 7$ leaves the same remainder when divided by $x - b$ or $x - c$

By the Remainder Theorem

$$b^2 - 5b + 7 = c^2 - 5c + 7$$

$$\Leftrightarrow\quad b^2 - c^2 = 5(b - c)$$

$$\Leftrightarrow\quad (b - c)(b + c) = 5(b - c)$$

$$\Leftrightarrow\quad b + c = 5 \text{ as } b \neq c$$

$$\therefore\quad b = 5 - c$$

and if $\quad 4bc = 21$

$$4(5 - c)c = 21$$

$$20c - 4c^2 - 21 = 0$$

$$\therefore\quad 4c^2 - 20c + 21 = 0$$

$$(2c - 3)(2c - 7) = 0$$

which implies

$$c = \frac{3}{2} \text{ or } c = \frac{7}{2}$$

If $\quad c = \frac{3}{2},\ 4 \times b \times \frac{3}{2} = 21$

$\therefore\quad b = \frac{7}{2}$

If $\quad c = \frac{7}{2},\ b = \frac{3}{2}$

As $b > c$ the required values are $b = \frac{7}{2}$, $c = \frac{3}{2}$

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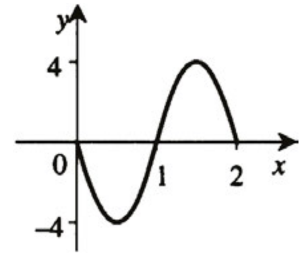
3 **a** $x = -4 \sin \pi t$

b **i** When $t = 0$, $x = -4 \sin 0 = 0$



ii When $t = \frac{1}{2}$, $x = -4 \sin \frac{\pi}{2} = -4$

iii When $t = 1$, $x = -4 \sin \pi = 0$

c When $x = 2$

$$2 = -4 \sin \pi t$$

$$\therefore \ -\frac{1}{2} = \sin(\pi t)$$

$$\text{i.e. } \frac{7\pi}{6} = \pi t$$

$$\therefore \ t = \frac{7}{6}$$

d Period $= \frac{2\pi}{n} = \frac{2\pi}{\pi} = 2$ seconds

4 **a** $y = -1.25 \cos(2\pi t) + 1.25$

i When $t = 0$

$$y = -1.25 \cos(0) + 1.25$$
$$= 0$$

ii When $t = \frac{1}{2}$

$$y = -1.25 \cos(\pi) + 1.25$$
$$= 1.25 + 1.25$$
$$= 2.5$$

iii When $t = 1$

$$y = -1.25 \cos 2\pi + 1.25$$
$$= 0$$

b Period $= \frac{2\pi}{n} = \frac{2\pi}{2\pi} = 1$

One revolution of the rope takes 1 second.

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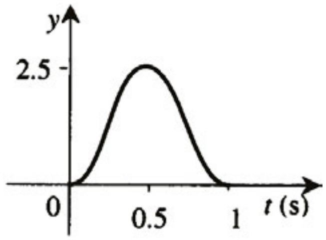
c



d $$2 = -1.25\cos(2\pi t) + 1.25$$

$$\frac{0.75}{-1.25} = \cos(2\pi t)$$

$$-0.6 = \cos(2\pi t)$$

$\therefore\ 2\pi t = \cos^{-1}(-0.6)$ (only first solution required)

$$t = \frac{1}{2\pi}\cos^{-1}(-0.6)$$

$$\approx 0.3524$$

It is 2 metres above the ground after 0.35 seconds.

5 $P(t) = 150 \times 10^6\, e^{kt}$

a From section 5.8, chapter 5 of *EMM Units 3 & 4*, $k = 0.0296$ (i.e. 2.96% as a decimal).

b $P(0) = 150 \times 10^6 e^0$

$\therefore$ Population on lst Jan 1950 is 150×10^6

c $P(50) = 150 \times 10^6 \times e^{0.0296 \times 50}$

$$= 150 \times 10^6 \times 4.3929$$

$$= 658941852.1$$

$$\approx 6.589418521 \times 10^8$$

Population is approximately 6.589×10^8 on January 1st 2000.

When $P = 300 \times 10^6$

d $300 \times 10^6 = 150 \times 10^6 e^{0.0296t}$

$\therefore\quad 2 = e^{0.0296t}$

Taking logarithms of both sides of the equation

$$\frac{1}{0.0296}\log_e 2 = t$$

$\therefore\ t \approx 23.417$ years

The population is 300×10^6 after 23.417 years.

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6 **a** $T = Ae^{-kt} + 15$, there $0 \le t \le 10$

When $t = 0, T = 95$

$$95 = A + 15$$

$$\therefore \quad A = 80$$

When $t = 2,\ T = 55$

$$55 = 80e^{-2k} + 15$$

$$\therefore \quad 0.5 = e^{-2k}$$

Taking logarithms both sides

$$-\frac{1}{2}\log_e 5 = k$$

$$\therefore \quad k = \log e(2^{\frac{1}{2}})$$

$k \approx 0.3466$

b At midnight $t = 0$

$$\therefore \quad T = 80e^{(-\log_e(2^{\frac{1}{2}}))\times 10}$$

$$= 80e^{\log_e 2^{-5}} + 15$$

$$= 80 \times \frac{1}{32} + 15$$

$$= 17.5$$

The temperature is 17.5°C at midnight.

c Graph is decreasing

When $T = 24°$

$$24 = 80e^{-\log_e(2^{\frac{1}{2}})t} + 15$$

$$\frac{9}{80} = e^{-\log_e(2^{\frac{1}{2}})t}$$

$$\log_e\left(\frac{90}{80}\right) = -\log_e(2^{\frac{1}{2}})t$$

$$\frac{\log_e\left(\frac{9}{80}\right)}{-\log_e(2^{\frac{1}{2}})} = t$$

$$\therefore \quad t = 6.304$$

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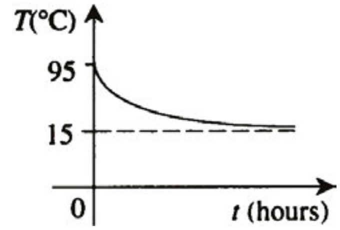
This is 6 hours 18 minutes and 14 seconds after 2:00 pm, i.e. 8:18:14 pm.
Jenny first recorded a temperature less than 24° at 9:00 p.m. (Note: temperature is recorded on the hour)



d $T = 80e^{-\log_e(2^{\frac{1}{2}})t} + 15$

7 **a** If $V = 25$ and $\alpha = 45°$

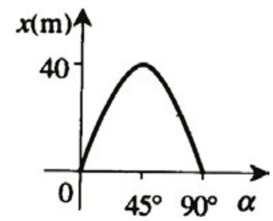
$$x = \frac{25^2 \sin 90}{10} = \frac{625}{10} = 62.5$$

The distance the ball is kicked is 62.5 m.

b For $V = 20$

$$x = \frac{400 \sin 2\alpha}{10} = 40 \sin 2\alpha$$

Period = 180°; amplitude = 40



c If $x = 30$ and $V = 20$

$$30 = \frac{400 \sin 2\alpha}{10}$$

$$\frac{3}{4} = \sin 2\alpha$$

$$\therefore \quad 2\alpha = \text{Sin}^{-1}\left(\frac{3}{4}\right) \text{ or } 180° - \sin^{-1}\left(\frac{3}{4}\right)$$

$$\alpha = \frac{1}{2}\text{Sin}^{-1}\left(\frac{3}{4}\right) \text{ or } 90° - \frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)$$

$$\approx 24.3° \text{ or } 65.7°$$

The angle projection is 24.3° or 65.7°

8 **a** Area $= 0.02\left(0.92^{\frac{1}{10}}\right)^x$

$$= 0.02(0.92)^{\frac{x}{10}}$$

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b When $x = \dfrac{5}{3}$

$$\text{Area} = (0.02)\left((0.92)^{\frac{5}{30}}\right)$$
$$= (0.02)\left((0.92)^{\frac{1}{6}}\right)$$
$$= 0.0197$$

Area is 0.0197 mm^2 when $x = \dfrac{5}{3}$

c load = strength × cross-sectional area

$$= (0.92)^{10-3x} \times (0.02) \times (0.92)^{\frac{x}{10}}$$
$$= (0.92)^{10-3x+\frac{x}{10}} \times 0.02$$
$$= (0.92)^{\frac{100-29x}{10}} \times 0.02 = 0.02(0.92)^{10-2.9x}$$

d If load $= 0.02 \times (0.92)^{2.5}$

$0.02 \times (0.92)^{2.5} = (0.92)^{\frac{100-29x}{10}} \times 0.02$

$(0.92)^{2.5} = (0.92)^{\frac{100-29x}{10}}$

$\therefore \dfrac{100-29x}{10} = 2.5$

i.e. $100 - 29x = 25$

$\therefore 75 = 29x$

and $x \approx 2.59$

Therefore the cable cannot exceed 2.59 m in length.

9 a The period of the function

$= \dfrac{2\pi}{b} = 2\pi \div = \dfrac{\pi}{6} = 12$

i Therefore length of OR is 12 units

ii Q is at the minimum value

$\therefore OQ = h - k$

R is at the maximum value

$\therefore OR = h + k$

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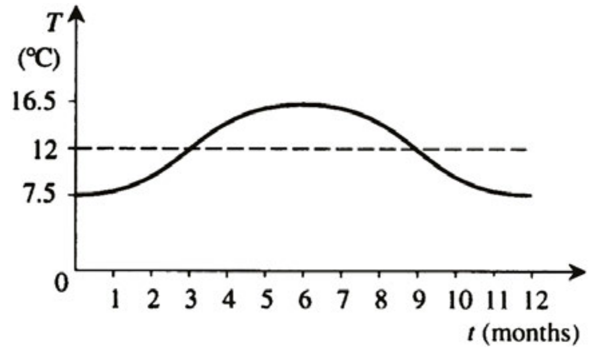
b



c From a $h + k = 16.5$ and $h - k = 7.5$

Consider as simultaneous equations

$h + k = 16.5$ ①

$h - k = 7.5$ ②

Add ① and ②

$2h = 24$

$h = 12$

and from $\langle 1 \rangle k = 4.5$

10 a *For Carriage A*

Stop 1 Illumination $= 0.83I$

Stop 2 Illumination $= (0.83)^2 I$

Stop n Illumination $= (0.83)^n I$

For Carriage B

Stop 1 Illumination $= 0.89 \times 0.66I$

Stop 2 Illumination $= (0.89)^2 \times 0.66I$

Stop n Illumination $= (0.89)^n \times 0.66I$

b *Illuminations equal implies*

$$(0.83)^n\ I = (0.89)^n \times 0.66I$$

$$\therefore\ \left(\frac{0.83}{0.89}\right)^n = 0.66$$

Taking logarithms of both sides

$$n \log_e\left(\frac{0.83}{0.89}\right) = \log_e(0.66)$$

$$n = \frac{\log_e(0.66)}{\log_e\left(\frac{0.83}{0.89}\right)}$$

$$\approx 5.95$$

The illumination is approximately equal after the sixth stop.

11 a i $y = 1 - a(x - 3)^2$

When $y = 0$

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$1 - a(x-3)^2 = 0$

$\therefore \quad 1 = a(x-3)^2$

$\therefore \quad (x-3)^2 = \dfrac{1}{a}$

$\therefore \quad x = 3 \pm \sqrt{\dfrac{1}{a}}$

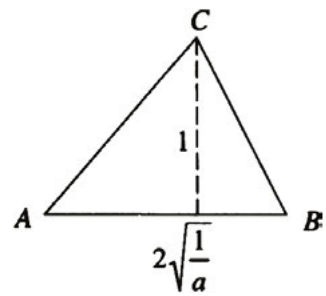
$\left(3 + \sqrt{\dfrac{1}{a}}, 0\right)$ and $\left(3 - \sqrt{\dfrac{1}{a}}, 0\right)$

ii AB has length $2\sqrt{\dfrac{1}{a}}$

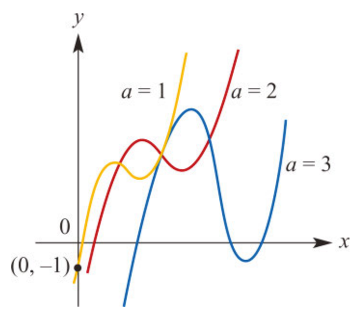
C has coordinates $(3, 1)$

Therefore the area $= \dfrac{1}{2} \times 2\sqrt{\dfrac{1}{a}} \times 1$

$= \dfrac{1}{\sqrt{a}}$ square units



b **i**



ii $-\dfrac{4}{27}a^3 + a = 0$

$\therefore \quad a\left(-\dfrac{4}{27}a^2 + 1\right) = 0$

$\therefore \quad a = 0$ or $a = \pm\sqrt{\dfrac{27}{4}}$

$= \pm\dfrac{3\sqrt{3}}{2}$

But $a > 0$. Therefore $a = \dfrac{3\sqrt{3}}{2}$

iii $-\dfrac{4}{27}a^3 + a < 0$

$\Leftrightarrow -\dfrac{4}{27}a^2 + 1 < 0$ (as $a > 0$)

$\Leftrightarrow a^2 > \dfrac{27}{4}$

$\Leftrightarrow a > \dfrac{3\sqrt{3}}{2} \qquad$ (as $a > 0$)

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iv $-\frac{4}{27}a^3 + a = -1$

$-4a^3 + 27a + 27 = 0$

Using a CAS calculator yields $a = 3$ is a solution.

Consider

$(a - 3)(-4a^2 - 12a - 9) = 0$

i.e. $(a - 3)(4a^2 + 12a + 9) = 0$

and $4a^2 + 12a + 9 > 0$ for all a

$\therefore\quad a = 3$ is the only solution.

v $-\frac{4}{27}a^3 + a = 1$

$-4a^3 + 27a - 27 = 0$

Using a graphics calculator yields $a = \frac{3}{2}$ is a solution.

$$\therefore \quad -4a^3 + 27a - 27 = (2a - 3)(-2a^2 - 3a + 9)$$

and $\quad -2a^2 - 3a + 9 = 0$

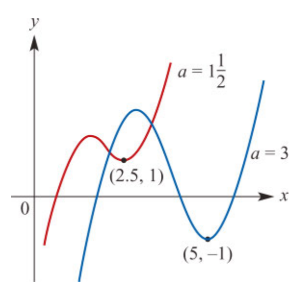
implies $\quad 2a^2 + 3a - 9 = 0$

$$\therefore \quad (2a - 3)(a + 3) = 0$$

$$\therefore \quad a = \frac{3}{2} \text{ or } a = -3$$

$$a = \frac{3}{2} \text{ is the only solution.}$$

vi



Graphic calculator techniques for 12b

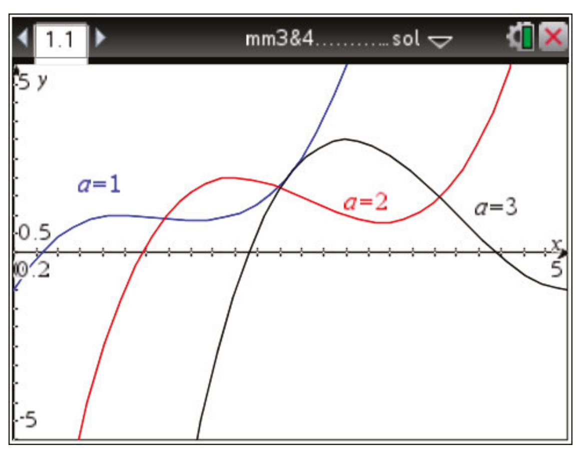
b i In a Graphs page enter

$f1(x) = (x - 1)^2(x - 2) + 1$,

$f2(x) = (x - 2)^2(x - 4) + 2$ an

d$f3(x) = (x - 3)^2(x - 6) + 3$

Set an appropriate window to show key points.



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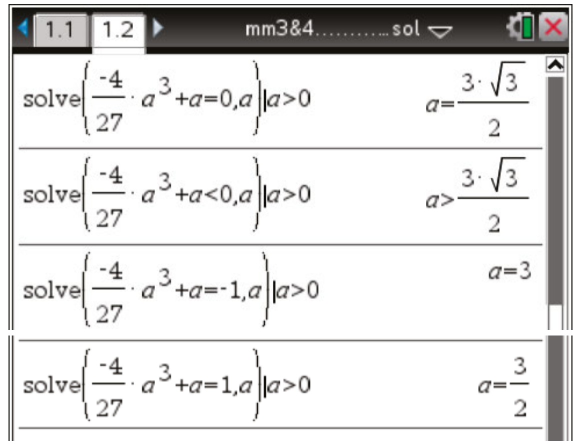
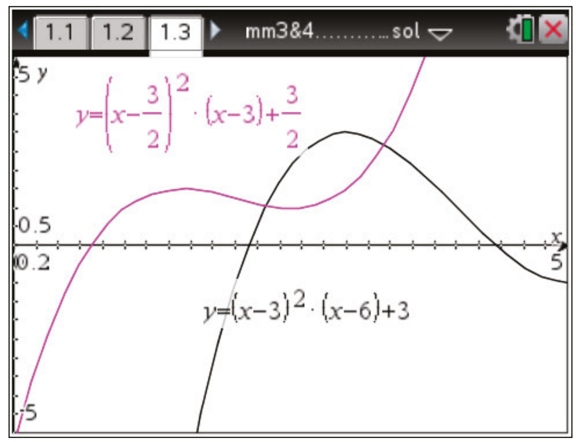
b ii – v In **a Calculator** page, use the **Solve** command. Note the domain restrictions.



b vi Plot in a **Graphs** page.



c **i** $\left(a, -\frac{4}{27}a^3 + a\right)$

ii $$PS = a - \left(-\frac{4}{27}a^3 + a\right)$$
$$= \frac{4}{27}a^3$$
$$SQ = \frac{5a}{3} - a$$
$$= \frac{2a}{3}$$

iii $$\text{Area} = \frac{1}{2} \times SQ \times PS$$
$$= \frac{1}{2} \times \frac{2a}{3} \times \frac{4}{27}a^3$$
$$= \frac{4a^4}{81}$$

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iv $\dfrac{4a^4}{81} = 4$

$\therefore\ a^4 = 81$

$\therefore\ a = 3$ (since $a > 0$)

v $\dfrac{4a^4}{81} = 1500$

$\therefore\ a^4 = \dfrac{81 \times 1500}{4}$

$\therefore\ a^4 = 81 \times 375$

$a = 3\sqrt[4]{375}$ (since $a > 0$)

12 a $D = at^2 + bt + c$

When $t = 0, D = 1.8$

Therefore $c = 1.8$

When $t = 1, D = 1.6$

Therefore

$1.6 = a + b + 1.8$ and rearranging gives,

$-0.2 = a + b$ ①

When $t = 3, D = 1.5$

Therefore

$1.5 = 9a + 3b + 1.8$ and rearranging gives,

$-0.3 = 9a + 3b$

Dividing both sides of the equation by 3 gives

$-0.1 = 3a + b$ ②

Subtract ① from ②

$0.1 = 2a$

Therefore $a = 0.05$. Substituting in (1) gives that $b - 0.25$

$D = 0.05t^2 - 0.25t + 1.8$

b When $t = 8, D = 0.05 \times 64 - 0.25 \times 8 + 1.8 = 3$

The deficit is 3 000 000 Ningteak dollars

Graphic calculator techniques for question

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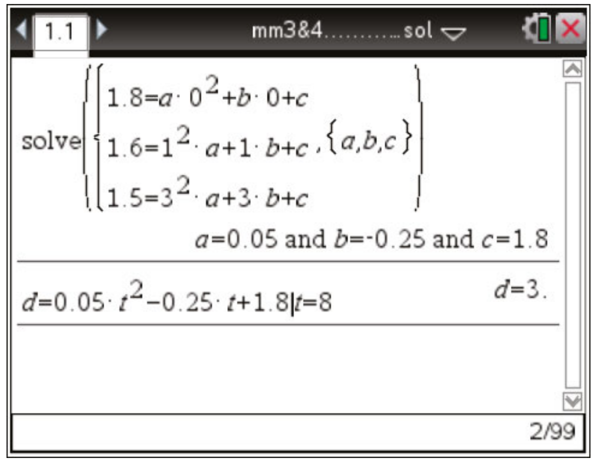
In a **Calculator** page use:
b>**Algebra**>**Solve System of Equations**>**Solve System of Equations** and enter as shown opposite.

Substitute $t = 8$ into equation.



13 $R = at^2 + bt + c$

When $t = 0, R = 7.5$

Therefore $c = 7.5$

When $t = 4, R = 9$

Therefore $9 = 16a + 4b + 7.5$

and $1.5 = 16a + 4b$

Diving both sides by 4 gives

$\frac{3}{8} = 4a + b$ ①

When $t = 6, R = 8$

$8 = 36a + 6b + 7.5$

$0.5 = 36a + 6b$

Divide both sides by 6

$\frac{1}{12} = 6a + b$ ②

Subtract ② from ①

$\frac{3}{8} - \frac{1}{12} = -2a$

Therefore $a = -\frac{7}{48}$ and substituting in ① gives $b = \frac{23}{24}$

Thus $R = -\frac{7}{48}t^2 + \frac{23}{24}t + \frac{15}{2}$

When $t = 8, R = \frac{35}{6}$. The rate is $\frac{35}{6}$mm/h at 12:00 noon.

The rainfall is greatest when $t = -\frac{-b}{2a} = -\frac{23}{24} \div -\frac{7}{24} = \frac{23}{7}$

The rainfall was heaviest at 7: 17 am.

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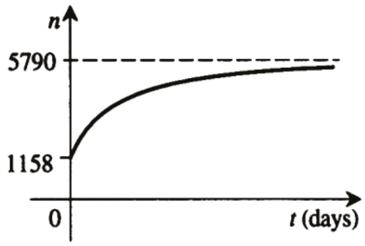
14 $n = \dfrac{c}{1 + ae^{-bt}} \quad t \geq 0$

$n = \dfrac{5790}{1 + 4e^{-0.03t}} \quad$ for $c = 5790, \ a = 4$ and $b = 0.03$

a **i** $n = 5790$ is the horizontal asymptote

ii when $t = 0, n = \dfrac{5790}{5} = 1158$

iii



iv

$$4000 = \frac{5790}{1 + 4e^{-0.03t}}$$

$$\therefore \quad 1 + 4e^{-0.03t} = \frac{579}{400}$$

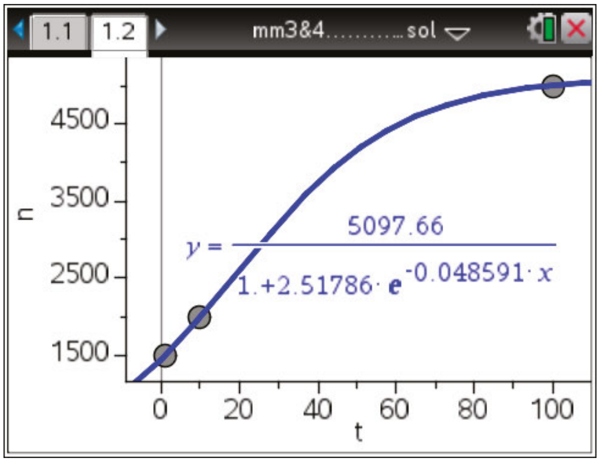
$$\therefore \quad 4e^{-0.03t} = \frac{179}{400}$$

$$\therefore \quad e^{-0.03t} = \frac{179}{1600}$$

$$\therefore \quad t = -\frac{100}{3}\log_e\left(\frac{179}{1600}\right)$$

$$= \frac{100}{3}\log_e\left(\frac{1600}{179}\right)$$

b **i** Enter the data in a **Lists & Spreadsheet** page.
Plot the data in a **Data & Statistics** page.
Determine the logistic regression using
b>**Analyze**>**Regression**>**Show Logistic** (**d**=0)
The result is as shown.



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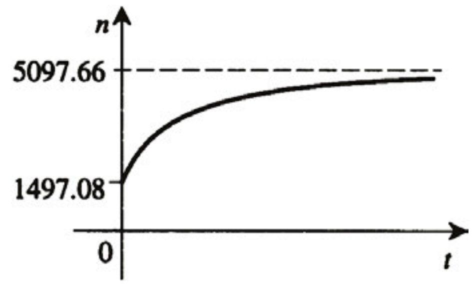
ii



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Chapter 9 – Differentiation of polynomials, power functions and rational functions

Solutions to Exercise 9A

1 $f(x) = -x^2 + 2x + 1$

$f(-1) = -(-1)^2 + 2(-1) + 1 = -2$

$f(4) = -(4)^2 + 2 \times 4 + 1 = -7$

$$\text{Average rate of change} = \frac{f(4) - f(-1)}{4 - (-1)} = \frac{-7 - (-2)}{5} = -1$$

2 $f(x) = 6 - x^3$

$f(-1) = 6 + 1 = 7$

$f(1) = 6 - 1 = 5$

$$\text{Average rate of change} = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{-2}{2} = -1$$

3 $f(x) = x^2 + 5x$

a Gradient

$$= \frac{(2 + h)^2 + 5(2 + h) - 14}{2 + h - 2}$$
$$= \frac{4 + 4h + h^2 + 10 + 5h - 14}{h}$$
$$= \frac{9h + h^2}{h}$$
$$= 9 + h$$

b $\lim\limits_{h \to 0} 9 + h = 9$

4 a $\lim\limits_{h \to 0} \dfrac{4x^2h^2 + xh + h}{h}$

$= \lim\limits_{h \to 0}(4x^2h + x + 1)$

$= x + 1$

b $\lim\limits_{h \to 0} \dfrac{2x^3h - 2xh^2 + h}{h}$

$= \lim\limits_{h \to 0}(2x^3 - 2xh + 1)$

$= 2x^3 + 1$

c $\lim\limits_{h \to 0}(40 - 50h)$

$= 40$

d $\lim\limits_{h \to 0} 5h$

$= 0$

e $\lim\limits_{h \to 0} 5$

$= 5$

f $\lim\limits_{h \to 0} \dfrac{30h^2x^2 + 20h^2x + h}{h}$

$= \lim\limits_{h \to 0}(30hx^2 + 20hx + 1)$

$= 1$

g $\lim\limits_{h \to 0} \dfrac{3h^2x^3 + 2hx + h}{h}$

$= \lim\limits_{h \to 0}(3hx^3 + 2x + 1)$

$= 2x + 1$

h $\lim\limits_{h \to 0} 3x$

$= 3x$

i $\lim\limits_{h \to 0} \dfrac{3x^3h - 5x^2h^2 + hx}{h}$

$= \lim\limits_{h \to 0}(3x^3 + 5x^2h + x)$

$= 3x^3 + x$

j $\lim\limits_{h \to 0}(6x - 7h)$

$= 6x$

5 $y = x^3 - x$

a $grad = \dfrac{rise}{run}$

$= \dfrac{(1+h)^3 - (1+h) - 0}{(1+h) - 1}$

$= \dfrac{1 + 3h + 3h^2 + h^3 - 1 - h}{h}$

$= \dfrac{h^3 + 3h^2 + 2h}{h}$

$= h^2 + 3h + 2$

b $grad = \lim\limits_{h\to 0} grad(PQ)$

$= \lim\limits_{h\to 0}(h^2 + 3h + 2)$

$= 2$

6 $f(x) = x^2 - 2$

$\dfrac{f(x+h) - f(x)}{h} = \dfrac{((x+h)^2 - 2) - (x^2 - 2)}{h}$

$= \dfrac{x^2 + 2xh + h^2 - x^2}{h}$

$= 2x + h$

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to 0}(2x + h)$

$f'(x) = 2x$

7

$y = x^2 + 2x + 5$

$grad(PQ) = \dfrac{rise}{run}$

$= \dfrac{((2+h)^2 + 2(a+h) + 5) - ((2)^2 + 2(2) + 5)}{(2+h) - (2)}$

$= \dfrac{4 + 4h + h^2 + 4 + 2h + 5 - 4 - 4 - 5}{h}$

$= \dfrac{h^2 + 6h}{h}$

$= h + 6$

$grad(P) = \lim\limits_{h\to 0}(grad(PQ))$

$= \lim\limits_{h\to 0}(h + 6)$

$= 6$

8 **a** $f(x) = 5x^2$

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{5(x+h)^2 - 5x^2)}{h}$

$= \lim\limits_{h\to 0} \dfrac{5x^2 + 10xh + 5h^2 - 5x^2}{h}$

$= \lim\limits_{h\to 0} 10x + 5h$

$= 10x$

b $f(x) = 3x + 2$

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{3(x+h) + 2 - 3x - 2}{h}$

$= \lim\limits_{h\to 0} \dfrac{3h}{h}$

$= \lim\limits_{h\to 0} 3$

$= 3$

c $f(x) = 5$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{5-5}{h}$$
$$= \lim_{h\to 0} 0$$
$$= 0$$

d

$$f(x) = 3x^2 + 4x + 3$$
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{(x+h)^2 + 4(x+h) + 3 - 3x^2 - 4x - 3}{h}$$
$$= \lim_{h\to 0}$$
$$\frac{3x^2 - 3x^2 + 6xh + 3h^2 + 4x - 4x + 4h + 3 - 3}{h}$$
$$= \lim_{h\to 0} \frac{6xh + 3h^2 + 4h}{h}$$
$$= \lim_{h\to 0}(6x + 3h + 4)$$
$$= 6x + 4$$

e

$$f(x) = 5x^3 - 5$$
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{5(x+h)^3 - 5 - 5(x)^3 + 5}{h}$$
$$= \lim_{h\to 0} \frac{5x^3 + 15x^2h + 15xh^2 + 5h^3 - 5x^3}{h}$$
$$= \lim_{h\to 0}(15x^2 + 15xh + 5h^2)$$
$$= 15x^2$$

f

$$f(x) = 5x^2 - 6x$$
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{5(x+h)^2 + 6(x+h) - 5x^2 + 6x}{h}$$
$$= \lim_{h\to 0} \frac{5x^2 + 10xh + 5h^2 - 6x - 6h - 5x^2 + 6x}{h}$$
$$= \lim_{h\to 0}(10x + 5h - 6)$$
$$= 10x - 6$$

Solutions to Exercise 9B

1 a $f(x) = x^5$

$f'(x) = 5x^4$

b $f(x) = 4x^7$

$f'(x) = 7 \times 4x^6$

$= 28x^6$

c $f(x) = 6x$

$f'(x) = 6$

d $f(x) = 5x^2 - 4x + 3$

$f'(x) = 2 \times 5x - 4$

$= 10x - 4$

e $f(x) = 4x^3 + 6x^2 + 2x - 4$

$f'(x) = 3 \times 4x^2 + 2 \times 6x + 2$

$= 12x^2 + 12x + 2$

f $f(x) = 5x^4 + 3x^3$

$f'(x) = 4 \times 5x^3 + 3 \times 3x^2$

$= 20x^3 + 9x^2$

g $f(x) = -2x^2 + 4x + 6$

$f'(x) = -4x + 4$

h $f(x) = 6x^3 - 2x^2 + 4x - 6$

$f'(x) = 18x^2 - 4x + 4$

2 a $f(x) = 2x^3 - 5x^2 + 1$

$f'(x) = 6x^2 - 10x$

$f'(1) = -4$

b $f(x) = -2x^3 - x^2 - 1$

$f'(x) = -6x^2 - 2x$

$f'(1) = -8$

c $f(x) = x^4 - 2x^3 + 1$

$f'(x) = 4x^3 - 6x^2$

$f'(1) = -2$

d $f(x) = x^5 - 3x^3 + 2$

$f'(x) = 5x^4 - 9x^2$

$f'(1) = -4$

3 a $f(x) = 2x^3 - 5x^2 + 2$

$f'(x) = 6x^2 - 10x$

$f'(1) = -4$

b $f(x) = -2x^3 - 3x^2 + 2$

$f'(x) = -6x^2 - 6x$

$f'(1) = -12$

4 a $\dfrac{dy}{dt} = 3t^2$

b $\dfrac{dx}{dt} = 3t^2 - 2t$

c $\dfrac{dz}{dx} = x^3 + 9x^2$

5 a $y = -2x$

$\dfrac{dy}{dx} = -2$

b $y = 7$

$\dfrac{dy}{dx} = 0$

c $y = 5x^3 - 3x^2 + 2x + 1$

$\frac{dy}{dx} = 15x^2 - 6x + 2$

d $y = \frac{2}{5}x^3 - \frac{8}{5}x + \frac{12}{5}$

$\frac{dy}{dx} = \frac{6}{5}x^2 - \frac{8}{5}$

e $y = (2x + 1)(x - 3)$

$= 2x^2 - 5x - 3$

$\frac{dy}{dx} = 4x - 5$

f $y = 3x(2x - 4)$

$= 6x^2 - 12x$

$\frac{dy}{dx} = 12x - 12$

g $y = \frac{10x^7 + 2x^2}{x^2}$

$= 10x^5 + 2$

$\frac{dy}{dx} = 50x^4$

h $y = \frac{9x^4 + 3x^2}{x}$

$= 9x^3 + 3x$

$\frac{dy}{dx} = 27x^2 + 3$

6 a $\frac{d}{dx}(2x^2 - 5x^3) = 4x - 15x^2$

b $\frac{d}{dz}(-2z^2 - 6z) = -4z - 6$

c $\frac{d}{dz}(6z^3 - 4z^2 + 3) = 18z^2 - 8z$

d $\frac{d}{dx}(-2x - 5x^3) = -2 - 15x^2$

e $\frac{d}{dz}(-2z^2 - 6z + 7) = -4z - 6$

f $\frac{d}{dz}(-z^3 - 4z^2 + 3) = -3z^2 - 8z$

7 a $y = 2x^2 - 4x + 1,\ \frac{dy}{dx} = -6$

$\frac{dy}{dx} = 4x - 4$

$-6 = 4x - 4$

$4x = -2$

$x = \frac{-1}{2}$

$y = \frac{1}{2} + 2 + 1 = \frac{7}{2}$

co-ords $= \left(\frac{-1}{2}, \frac{7}{2}\right)$

b $y = 4x^3,\ \frac{dy}{dx} = 48$

$\frac{dy}{dx} = 12x^2$

$48 = 12x^2$

$x^2 = 4$

$x = \pm 2$

$y = \pm 32$

co-ords $= (-2, -32)$ *and* $(2, 32)$

c $y = x(5 - x),\ \frac{dy}{dx} = 1$

$y = 5x - x^2$

$\frac{dy}{dx} = 5 - 2x$

$1 = 5 - 2x$

$-2x = -4$

$x = 2$

$y = 2(3) = 6$

co-ords $= (2, 6)$

d $y = x^3 - 3x^2,\ \dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = 3x^2 - 6x$$

$$0 = 3x^2 - 6x$$

$$0 = 3x(x - 2)$$

$$x = 0, 2$$

$$y = 0 - 0 = 0,\ y = 8 - 12 = -4$$

co-ords = (0, 0) *and* (2, −4)

8 **a** $\tan 45° = 1$

$\therefore$ gradient = 1

$$\frac{dy}{dx} = 4x - 3 \text{ Therefore}$$

When gradient $= 1$

$$4x - 3 = 1$$

$$\therefore x = 1$$

$$f(1) = 7$$

the tangent line at the point (1, 7) makes an angle of $\tan 45°$ with the positive direction of the x-axis.

b Gradient = 2

$$\frac{dy}{dx} = 4x - 3$$

When gradient $= 2$

$$4x - 3 = 2$$

$$\therefore x = \frac{5}{4}$$

$$f(\frac{5}{4}) = \frac{59}{8}$$

Therefore the tangent line at the point $\left(\dfrac{5}{4}, \dfrac{59}{8}\right)$ is parallel to the line $y = 2x + 8$

9 $\dfrac{dy}{dx} = 2x - 1$

a $2x - 1 = 1$

$$x = 1$$

b $2x - 1 = -1$

$$x = 0$$

c $2x - 1 = \sqrt{3}$

$$x = \frac{1}{2}\left(1 + \sqrt{3}\right)$$

$$= \frac{1 + \sqrt{3}}{2}$$

10 **a** $y = x^2 + 3x,\ (1, 4)$

Let θ be the angle between the tangent line and the x-axis.

$$\frac{dy}{dx} = 2x + 3$$

When $x = 1, \dfrac{dy}{dx} = 5$

$$\therefore \tan\theta = 5$$

$$\therefore \theta \approx 78.69°$$

b $y = -x^2 + 2x,\ (1, 1)$ Let θ be the angle between the tangent line and the x-axis. $\dfrac{dy}{dx} = -2x + 2$

When $x = 1, \dfrac{dy}{dx} = 0$

$$\therefore \tan\theta = 0$$

$$\therefore \theta = 0°$$

c $y = x^3 + x,\ (0, 0)$

Let θ be the angle between the tangent line and the x-axis.

$$\frac{dy}{dx} = 3x^2 + 1$$

When $x = 0, \dfrac{dy}{dx} = 1$

$$\therefore \tan\theta = 1$$

$$\therefore \theta = 45°$$

d $y = -x^3 - x,\ (0, 0)$

Let θ be the angle between the tangent line and the x-axis.

$$\frac{dy}{dx} = -3x^2 - 1$$

When $x = 0, \dfrac{dy}{dx} = -1$

$\therefore \tan\theta = -1$

$\therefore \theta = 135°$

e $y = x^4 - x^2,\ (1, 0)$

Let θ be the angle between the tangent line and the x-axis.

$$\frac{dy}{dx} = 4x^3 - 2x$$

When $x = 1, \dfrac{dy}{dx} = 2$

$\therefore \tan\theta = 2$

$\therefore \theta \approx 63.43°$

f $y = x^4 - x^2,\ (-1, 0)$

Let θ be the angle between the tangent line and the x-axis.

$$\frac{dy}{dx} = 4x^3 - 2x$$

When $x = -1, \dfrac{dy}{dx} = -2$

$\therefore \tan\theta = -2$

$\therefore \theta \approx 116.57°$

11 a
$$\begin{aligned} y &= (2x - 1)^2 \\ &= 4x^2 - 4x + 1 \\ \frac{dy}{dx} &= 8x - 4 \end{aligned}$$

b
$$\begin{aligned} y &= \frac{x^3 + 2x^2}{x} \\ &= x^2 + 2x \\ \frac{dy}{dx} &= 2x + 2 \end{aligned}$$

c
$$\begin{aligned} y &= 2x^3 - 6x^2 + 18x \\ \frac{dy}{dx} &= 6x^2 - 12x + 18 \\ &= 6(x^2 - 2x + 3) \end{aligned}$$

$b^2 - 4ac = 6(4 - 12) < 0$

$\therefore \dfrac{dy}{dx}$ *does not intersect the x-axis and since* $x = 0$ *gives* $\dfrac{dy}{dx} = 3, \dfrac{dy}{dx} > 0$ *for all* x (*as opposed to* $\dfrac{dy}{dx} < 0$ *for all* x)

d
$$\begin{aligned} y &= \frac{x^3}{3} - x^2 + x \\ \frac{dy}{dx} &= x^2 - 2x + 1 \\ &= (x - 1)^2 \end{aligned}$$

$\therefore \dfrac{dy}{dx} \geq 0,$

since any number squared is non-negative

12 a
$$\begin{aligned} y &= x^2 + 2x + 1,\ x = 3 \\ y &= 3^2 + 2(3) + 1 \\ &= 9 + 6 + 1 \\ y &= 16 \\ \frac{dy}{dx} &= 2x + 2 \\ \frac{dy}{dx} &= 2(3) + 2 \\ &= 8 \end{aligned}$$

b
$$\begin{aligned} y &= x^2 - x - 1,\ x = 0 \\ y &= -1 \\ \frac{dy}{dx} &= 2x - 1 \\ \frac{dy}{dx} &= -1 \end{aligned}$$

c $y = 2x^2 - 4x,\ x = -1$

$y = 2(-1)^2 - 4(-1)$

$= 2 + 4$

$y = 6$

$\dfrac{dy}{dx} = 4x - 4$

$\dfrac{dy}{dx} = 4(-1) - 4$

$= -8$

d $y = (2x + 1)(3x - 1)(x + 2),\ x = 4$

$y = 6x^3 + 13x^2 + x - 2$

$= 6(4)^3 + 13(4)^2 + (4) - 2$

$y = 6 \times 64 + 13 \times 16 + 4 - 2$

$y = 384 + 208 + 2$

$y = 594$

$y = 6x^3 + 13x^2 + x - 2$

$\dfrac{dy}{dx} = 18x^2 + 26x + 1$

$x = 4, \quad \dfrac{dy}{dx} = 18 \times 16 + 26 \times 4 + 1$

$= 393$

e $y = (2x + 5)(3 - 5x)(x + 1), x = +1$

$y = -10x^3 - 25x^2 + 6x^2 - 10x + 6x$

$- 25x + 15x + 15$

$y = -10x^3 - 29x^2 - 4x + 15$

$x = +1,\ y = -10 - 29 - 4 + 15$

$y = -28$

$\dfrac{dy}{dx} = -30x^2 - 58x - 4$

$x = +1,\ \dfrac{dy}{dx} = -30 - 58 - 4$

$= -92$

f $y = (2x - 5)^2, x = 2\frac{1}{2}$

$x = 2\frac{1}{2},\ y = (5 - 5)^2$

$y = 0$

$y = 4x^2 - 20x + 25$

$\dfrac{dy}{dx} = 8x - 20$

$x = 2\frac{1}{2},\ \dfrac{dy}{dx} = 4 \times 5 - 20$

$= 0$

13 $f(x) = 3(x - 1)^2$

a $0 = 3(x - 1)^2$

$x = 1$

b $f'(x) = 3(x^2 - 2x + 1)$

$f'(x) = 3(2x - 2)$

$= 6(x - 1)$

$0 = 6(x - 1)$

$x = 1$

c $0 < 6(x - 1)$

$x - 1 > 0$

$x > 1$; i.e.$(1, \infty)$

d $0 > 6(x - 1)$

$x - 1 < 0$

$x < 1$; i.e.$(-\infty, 1)$

e $10 = 6(x - 1)$

$x - 1 = \dfrac{5}{3}$

$x = \dfrac{8}{3}$

f $27 = 3(x-1)^2$

$9 = (x-1)^2$

$x - 1 = \pm 3$

$x = -2, 4$

14 a $x < -1,\ x > 1$
i.e. $x \in R\backslash[-1, 1]$

b $-1 < x < 1$
i.e. $x \in (-1, 1)$

c $x = -1, 1$

15 a $-1 < x < 0.5,\ x > 2$
i.e. $x \in \left(-1, \frac{1}{2}\right) \cup (2, \infty)$

b $x < -1,\ \frac{1}{2} < x < 2$
i.e. $x \in (-\infty, -1) \cup \left(\frac{1}{2}, 2\right)$

c $x = -1,\ \frac{1}{2}, 2$

16 a $x > -1,\ x \neq 2$
i.e. $x \in \left(\frac{-1}{4}, 2\right) \cup (2, \infty)$

b $x < \frac{-1}{4}$
i.e. $x \in \left(-\infty, \frac{-1}{4}\right)$

c $x = \frac{-1}{4}, 2$

17 $y = x^2 - 4x - 8$

a $\frac{dy}{dx} = 2x - 4$

$\frac{dy}{dx} = 0,$

$0 = 2x - 4$

$2x = 4$

$x = 2$

$y = 4 - 8 - 8 = -12$
co-ords $= (2, -12)$

b $\frac{dy}{dx} = 2$

$2 = 2x - 4$

$2x = 6$

$x = 3$

$y = -11$
co-ords $= (3, -11)$

c $3x + 2y = 8$

$\Rightarrow y = \frac{8}{2} - \frac{3}{2}x$

$= 4 - \frac{3}{2}x$

$\frac{dy}{dx} = \frac{-3}{2}$

$\frac{-3}{2} = 2x - 4$

$x = \frac{5}{4}$

$y = -\frac{183}{16}$

co-ords $= \left(\frac{5}{4}, -\frac{183}{16}\right)$

18 a $f'(x) = x^2 > 0$ for all $x \neq 0$.
Therefore strictly increasing for $\mathbb{R}\backslash\{0\}$.
Also $f(0) = 0$ and $f(b) > 0$ for all $b > 0$ and $f(b) < 0$ for all $b < 0$.
Therefore strictly increasing for all

$x \in \mathbb{R}$.

b $f'(x) = -x^2 < 0$ for all $x \neq 0$.
Therefore strictly decreasing for $\mathbb{R}\backslash\{0\}$.
Also $f(0) = 0$ and $f(b) < 0$ for all $b > 0$ and $f(b) > 0$ for all $b < 0$.
Therefore strictly decreasing for all $x \in \mathbb{R}$.

19 a Let x_1 and $x_2 \in [0,\infty)$ such that $x_1 > x_2$ and $x_1 \geq 0$ and $x_2 \geq 0$.

Then

$$x_1 > x_2$$
$$\Leftrightarrow x_1 - x_2 > 0$$
$$\Leftrightarrow (x_1 - x_2)(x_1 + x_2) >$$
$$0 \Leftrightarrow {x_1}^2 - {x_2}^2 > 0$$
$$\Leftrightarrow {x_1}^2 > {x_2}^2$$
$$\Leftrightarrow f(x_1) > f(x_2)$$

$\therefore f(x)$ is an increasing function on $[0, \infty)$

b Let x_1 and $x_2 \in (-\infty,0]$ such $x_1 > x_2$ and $x_1 \leq 0$ and $x_2 \leq 0$.

Then

$$x_1 > x_2$$
$$\Leftrightarrow x_1 - x_2 > 0$$
$$\Leftrightarrow (x_1 - x_2)(x_1 + x_2) < 0$$
$$\Leftrightarrow {x_2}^2 - {x_2}^2 < 0$$
$$\Leftrightarrow {x_1}^2 < {x_2}^2$$
$$\Leftrightarrow f(x_1) < f(x_2)$$

$\therefore f(x)$ is a decreasing function on $(-\infty,0]$

20 $f'(x) = 2x - 1$
$2x - 1 > 0 \Leftrightarrow x > \frac{1}{2}$
$\therefore$ strictly increasing for $x > \frac{1}{2}$ We also know that $f(x) > f(\frac{1}{2})$ for all $x \in \mathbb{R}\backslash\left\{\frac{1}{2}\right\}$
$\therefore$ strictly increasing for $\left[\frac{1}{2}, \infty\right)$
If $x < \frac{1}{2}$ then $f(x) > \frac{1}{2}$
Hence $\left[\frac{1}{2}, \infty\right)$ is the largest interval for which f is strictly increasing.

21 Note that answers in text are given for strictly increasing

a $(\infty, -1]$

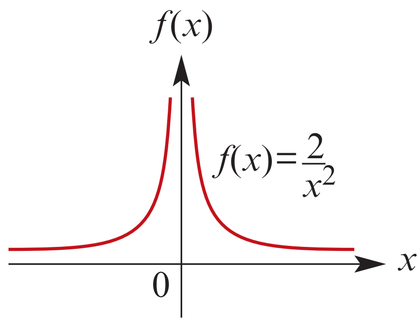
b $[2, \infty)$

c $[-\infty, 0]$

d $[\frac{3}{2}, \infty)$

Solutions to Exercise 9C

1 a



b $$\begin{aligned}grad(PQ) &= \frac{rise}{run}\\ &= \frac{f(1+h)-f(1)}{(1+h)-1}\\ &= \frac{\frac{2}{(1+h)^2}\ \frac{2}{1^2}}{h}\\ &= \frac{1}{h}\left(\frac{2}{(1+h)^2} - 2\frac{(1+h)^2}{(1+h)^2}\right)\\ &= \frac{1}{h}\left(\frac{2-2(1+2h+h^2)}{1+2h+h^2}\right)\\ &= \frac{1}{h}\left(\frac{-4h-2h^2}{1+2h+h^2}\right)\\ grad(PQ) &= \frac{-4-2h}{1+2h+h^2}\end{aligned}$$

c $$\begin{aligned}grad(P) &= \lim_{h\to 0} grad(PQ)\\ &= \lim_{h\to 0}\frac{-4-2h}{1+2h+h^2}\\ &= \frac{-4}{1}\\ grad(P) &= -4\end{aligned}$$

2 a $$\begin{aligned}&\frac{f(x+h)-f(x)}{h}\\ &= \left(\frac{1}{x+h-3} - \frac{1}{x-3}\right)\times\frac{1}{h}\\ &= \left(\frac{x-3-(x+h-3}{(x+h-3)(x-3)}\right)\times\frac{1}{h}\\ &= \left(\frac{-h}{(x+h-3)(x-3)}\right)\times\frac{1}{h}\\ &= \left(\frac{-1}{(x+h-3)(x-3)}\right)\end{aligned}$$

Hence

$$\begin{aligned}&\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{-1}{(x+h-3)(x-3)}\\ &= -\frac{1}{(x-3)^2}\end{aligned}$$

b $$\begin{aligned}&\frac{f(x+h)-f(x)}{h}\\ &= \left(\frac{1}{x+h+2} - \frac{1}{x+2}\right)\times\frac{1}{h}\\ &= \left(\frac{x+2-(x+h+2}{(x+h+2)(x+2)}\right)\times\frac{1}{h}\\ &= \left(\frac{-h}{(x+h+2)(x+2)}\right)\times\frac{1}{h}\\ &= \left(\frac{-1}{(x+h+2)(x+2)}\right)\end{aligned}$$

Hence

$$\begin{aligned}&\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{-1}{(x+h+2)(x+2)}\\ &= -\frac{-1}{(x+2)^2}\end{aligned}$$

3

$$\frac{f(x+h)-f(x)}{h}$$
$$= \frac{(x+h)^{-4}-x^{-4}}{h}$$
$$= \left(\frac{1}{(x+h)^4}-\frac{1}{x^4}\right)\times\frac{1}{h}$$
$$= \left(\frac{x^4-(x+h)^4}{x^4(x+h)^4}\right)\times\frac{1}{h}$$
$$= \left(\frac{x^4-(x^4+4x^3h+6x^2h^2+4xh^3+h^4)}{x^4(x+h)^4}\right)\times\frac{1}{h}$$
$$= \left(\frac{-(4x^3h+6x^2h^2+4xh^3+h^4)}{x^4(x+h)^4}\right)\times\frac{1}{h}$$
$$= \frac{-(4x^3+6x^2h+4xh^2+h^3)}{x^4(x+h)^4}$$

Hence

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\left(\frac{-(4x^3+6x^2h+4xh^2+h^3)}{x^4(x+h)^4}\right)$$
$$= -\frac{4x^3}{x^8}$$
$$= -\frac{4}{x^5}$$

4 a $y = 3x^{-2}+5x^{-1}+6$

$$\frac{dy}{dx} = -6x^{-3}-5x^{-2}$$

b $y = 5x^{-3}+6x^2$

$$\frac{dy}{dx} = -15x^{-4}+12x$$

c $f(x) = -5x^{-3}+4x^{-2}+1$

$f'(x) = 15x^{-4}-8x^{-3}$

d $f(x) = 6x^{-3}+3x^{-2}$

$f'(x) = -18x^{-4}-6x^{-3}$

e $y = 4+2x^{-1}$

$$\frac{dy}{dx} = -2x^{-2}$$

5 a $y = 2-4z^{-1}$

$$\frac{dy}{dz} = 4z^{-2}$$

b $y = 6z^{-3}+z^{-2}$

$$\frac{dy}{dz} = -18z^{-4}-2z^{-3}$$

c $y = 16-z^{-3}$

$$\frac{dy}{dz} = 3z^{-4}$$

d $f(z) = 4z^{-1}+z-z^2$

$f'(z) = -4z^{-2}+1-2z$

e $f(z) = 6z^{-2}-2z^{-3}$

$f'(z) = -12z^{-3}+6z^{-4}$

f $f(x) = 6x^{-1}-3x^2$

$f'(x) = -6x^{-2}-6x$

6 a $y = x^{-2}+x^3$

$$\frac{dy}{dx} = -2x^{-3}+3x^2$$

$x = 2$,

$$\frac{dy}{dx} = \frac{-2}{8}+3\times 4$$
$$= \frac{-1}{4}+12$$
$$\frac{dy}{dx} = \frac{47}{4} = 11\frac{3}{4}$$

b $y = x^{-2} - x^{-1}$

$$\frac{dy}{dx} = -2x^{-3} + x^{-2}$$

$x = 4,$

$$\frac{dy}{dx} = \frac{-2}{64} + \frac{1}{16}$$

$$= \frac{-1}{32} + \frac{1}{16}$$

$$\frac{dy}{dx} = \frac{1}{32}$$

c $y = x^{-2} - x^{-1}$

$$\frac{dy}{dx} = -2x^{-3} + x^{-2}$$

$x = 1,$

$$\frac{dy}{dx} = -2 + 1$$

$$\frac{dy}{dx} = -1$$

d $y = 1 + x^3 - x^{-2}$

$$\frac{dy}{dx} = 3x^2 + 2x^{-3}$$

$x = 1,$

$$\frac{dy}{dx} = 3 + 2$$

$$\frac{dy}{dx} = 5$$

7 $f'(x) = 10x^{-4} > 0$ for all $x \neq 0$

8 $y = \dfrac{x^2 - 1}{x} = x - \dfrac{1}{x} = x - x^{-1}$

$$\frac{dy}{dx} = 1 + x^{-2}$$

$$\frac{dy}{dx} = 5$$

$$5 = 1 + \frac{1}{x^2}$$

$$\frac{1}{x^2} = 4$$

$$x^2 = \frac{1}{4}$$

$$x = \pm\frac{1}{2}$$

9 $y = ax^2 + bx^{-1}$

$x = 2,\ y = -2$

$$1 - 2 = 4a + \frac{b}{2}$$

$$\frac{dy}{dx} = 2ax - bx^{-2}$$

$$x = 2,\ \frac{dy}{dx} = -5$$

$$2 - 5 = 4a - \frac{b}{4}$$

$$1 - 2 \Rightarrow 3 = \frac{3b}{4}$$

$b = 4$

Sub in 1 $\Rightarrow -2 = 4a + 2$

$4a = -4$

$a = -1$

$$y = -x^2 + \frac{4}{x}$$

10 $y = 2x^{-1} - 4x^{-2}$

$y = 0,$

$0 = 2x^{-1} - 4x^{-2}$

$0 = 2x - 4$

$x = 2$

$\frac{dy}{dx} = -2x^{-2} + 8x^{-3}$

$x = 2,$

$\frac{dy}{dx} = \frac{-2}{4} + \frac{8}{8}$

$= \frac{-1}{2} + 1$

$= \frac{1}{2}$

11 $y = \frac{9}{x} + bx^2$

$= ax^{-1} + bx^2$

$x = 3,\ y = 6$

$1\ 6 = \frac{9}{3} + 9b$

$\frac{dy}{dx} = ax^{-2} + 2bx$

$x = 3,\ \frac{dy}{dx} = 7$

$2\ 7 = \frac{-a}{9} + 6b$

$31 \Rightarrow 18 = a + 27b \quad b = 1$

$92 \Rightarrow 63 = -a + 54b$

$31 + 92 \Rightarrow 81 = 81b$

Sub in 2 $\Rightarrow 7 = \frac{-a}{9} + 6$

$\frac{-a}{9} = 1$

$a = -9$

$y = \frac{-9}{x} + x^2$

12 $y = \frac{5}{3}x + kx^2 - \frac{8}{9}x^3$

$\frac{dy}{dx} = \frac{5}{3} + 2kx - \frac{8}{3}x^2$

at $x = \frac{-1}{2}$

$\frac{dy}{dx} = \frac{5}{3} - k - \frac{2}{3}$

$\frac{dy}{dx} = 1 - k$

at $x = 1,$

$\frac{dy}{dx} = \frac{5}{3} + 2k - \frac{8}{3}$

$\frac{dy}{dx} = 2k - 1$

$2k - 1 = \frac{-1}{1-k}$(*perpendicular*)

$(2k-1)(k-1) = 1$

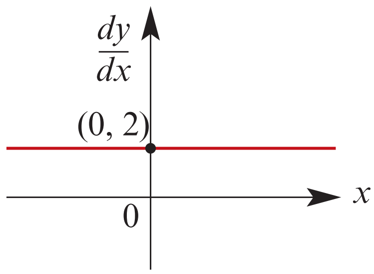
$2k^2 - 3k + 1 = 1$

$2k^2 - 3k = 0$

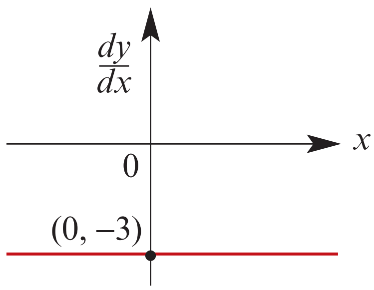
$k(2k-3) = 0 \Rightarrow k = 0, \frac{3}{2}$

Solutions to Exercise 9D

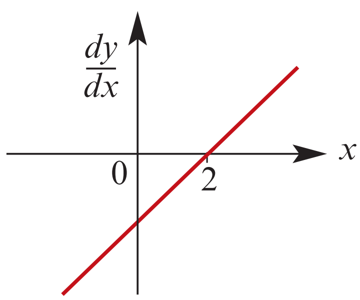
1 **a**



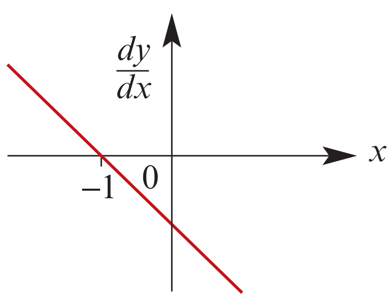
b



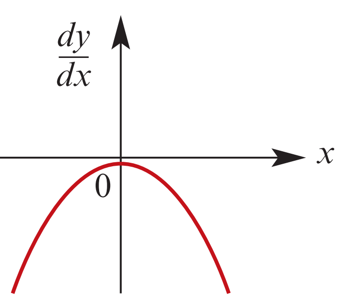
c



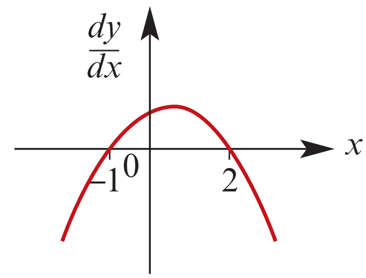
d



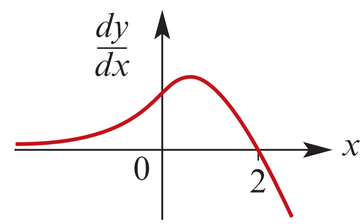
e



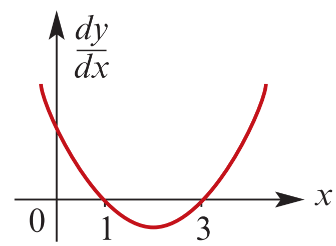
f



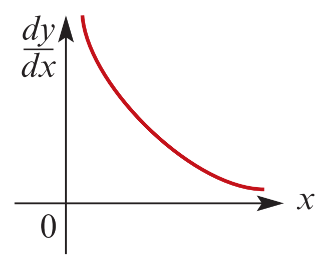
g



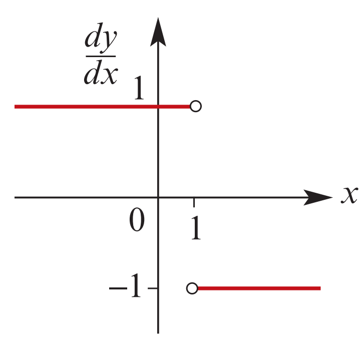
h



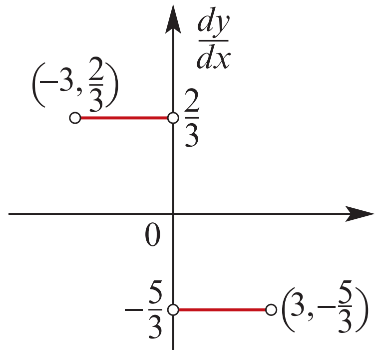
i



2 **a**

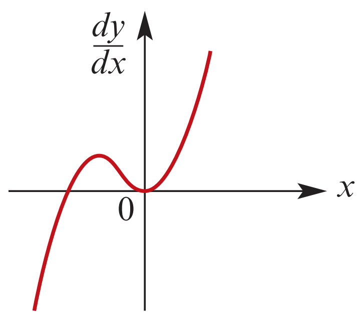


b



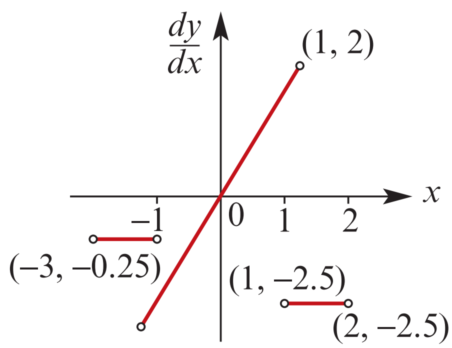
dy/dx
(−3, 2/3)
2/3
0
x
−5/3
(3, −5/3)

c



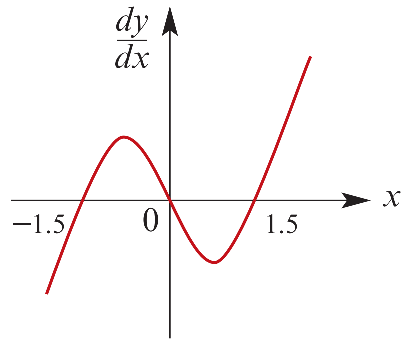
dy/dx
0
x

d



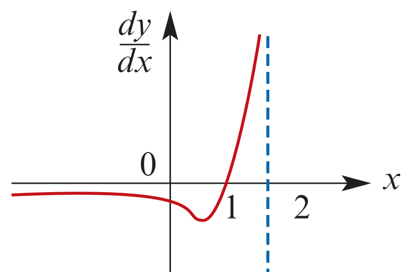
dy/dx
(1, 2)
x
−1
0
1
2
(−3, −0.25)
(1, −2.5)
(2, −2.5)

e



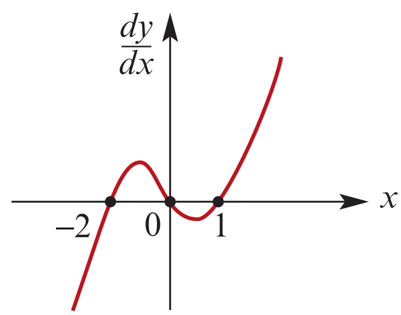
dy/dx
x
−1.5
0
1.5

f



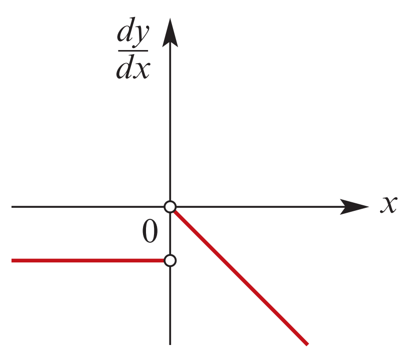
dy/dx
0
x
1
2

g



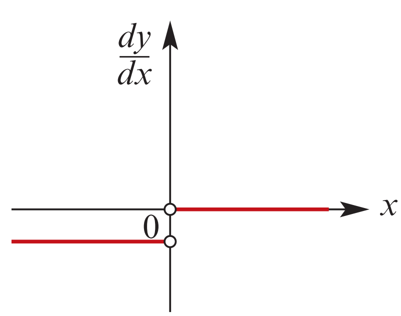
dy/dx
x
−2
0
1

h



dy/dx
x
0

i



dy/dx
x
0

3 a D

b F

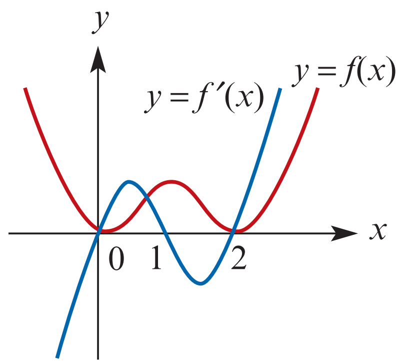
c B

d C

e A

f E

4 a



b **i** 0

ii 0

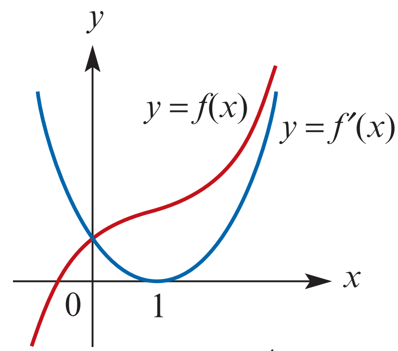
iii 0

iv 96

c **i** 1

ii 0.423

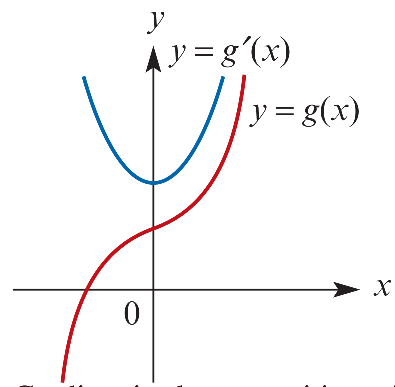
5



Gradient is 0 at $\left(1, \frac{4}{3}\right)$

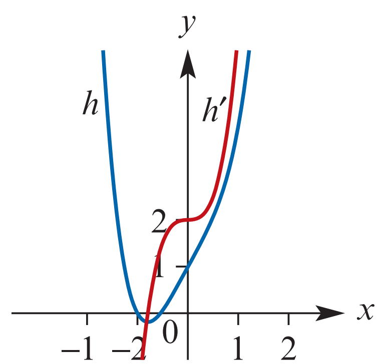
Gradient is positive for $R \backslash \{1\}$

6



Gradient is always positive,minimum gradient where $x = 0$

7 a



b **i** $x = -1.495$ *or* $x = 0.798$

ii $x = 0.630$

Solutions to Exercise 9E

1 a $y = (x^2+1)^4$

Let $u = x^2+1,\ y = u^4$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= 4u^3 \times 2x\\ &= 4(x^2+1)^3 \times 2x\\ &= 8x(x^2+1)^3\end{aligned}$$

b $y = (2x^2-3)^5$

Let $u = 2x^2-3,\ y = u^5$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= 4x \times 5u^4\\ &= 4x \times 5(2x^2-3)^4\\ &= 20x(2x^2-3)^4\end{aligned}$$

c $y = (6x+1)^4$

Let $u = 6x+1,\ y = u^4$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= 6 \times 4u^3\\ &= 24(6x+1)^3\end{aligned}$$

d $y = (ax+b)^n$

Let $u = ax+b,\ y = u^n$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= a \times nu^{(n-1)}\\ &= an(ax+b)^{n-1}\end{aligned}$$

e $y = (ax^2+b)^n$

Let $u = ax^2+b,\ y = u^n$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= 2ax \times nu^{(n-1)}\\ &= 2anx(ax^2+b)^{n-1}\end{aligned}$$

f $y = (1-x^2)^{-3}$

Let $u = 1-x^2, y = u^{-3}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= -2x - 3u^{-4}\\ &= 6x(1-x^2)^{-4}\end{aligned}$$

g $y = (x^2-x^{-2})^{-3}$

Let $u = x^2-x^{-2}, y = u^{-3}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= (2x+2x^{-3}) \times -3u^{-4}\\ &= -6(x+x^{-3})(x^2-x^{-2})^{-4}\end{aligned}$$

h $y = (1-x)^{-1}$

Let $u = 1-x,\ y = u^{-1}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= -1 \times -u^{-2}\\ &= (1-x)^{-2}\end{aligned}$$

2 a

$$y = (x^2 + 2x + 1)^3$$
$$y = ((x + 1)^2)^3$$
$$y = (x + 1)^6$$

Let $u = x + 1,\ y = u^6$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 1 \times 6u^5$$
$$= 6(x + 1)^5$$

b

$$y = (x^3 + 2x^2 + x)^4$$

Let $u = x^3 + 2x^2 + x,\ y = u^4$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= (3x^2 + 4x + 1) \times 4u^3$$
$$= 4(3x + 1)(x + 1)(x^3 + 2x^2 + x)^3$$
$$= 4(3x + 1)(x + 1)(x(x + 1)^2)^3$$
$$= 4x^3(3x + 1)(x + 1)(x + 1)^6$$
$$= 4x^3(3x + 1)(x + 1)^7$$

c

$$y = (6x^3 + 2x^{-1})^4$$

Let $u = 6x^3 + 2x^{-1},\ y = u^4$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= (18x^2 - 2x^{-2}) \times 4u^3$$
$$= 8(9x^2 - x^{-2})(6x^3 + 2x^{-1})^3$$

d

$$y = (x^2 + 2x + 1)^{-2}$$
$$= ((x + 1)^2)^{-2}$$
$$= (x + 1)^{-4}$$

Let $u = x + 1,\ y = u^{-4}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 1 \times -4u^{-5}$$
$$= -4(x + 1)^{-5}$$

3 Let $y = \dfrac{16}{3x^3 + x} = 16(3x^3 + x)^{-1}$

Let $u = 3x^3 + x$

Then $y = 16u^{-1}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= -16u^{-2} \times (9x^2 + 1)$$
$$= \frac{-16(9x^2 + 1)}{(3x^3 + x)^2}$$

When $x = 1$

$$\frac{dy}{dx} = -10$$

4 Let $y = \dfrac{1}{x^2 + 1} = (x^2 + 1)^{-1}$

Let $u = x^2 + 1$

Then $y = u^{-1}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= -u^{-2} \times (2x)$$
$$= \frac{-2x}{(x^2 + 1)^2}$$

When $x = 1$

$$\frac{dy}{dx} = -\frac{1}{2}$$

When $x = -1$

$$\frac{dy}{dx} = \frac{1}{2}$$

5 $F(x) = f(g(x))$

$$F'(x) = g'(x)f'(g(x))$$
$$= 2x\sqrt{3g(x) + 4}$$
$$= 2x\sqrt{3x^2 + 1}$$

6 a

$$\begin{aligned} \textit{Let } h(x) &= [f(x)]^n \\ \textit{Let } g(x) &= x^n \\ \textit{then } h(x) &= g(f(x)) \\ h'(x) &= g'(f(x)) \times f'(x) \\ &= n(f(x))^{n-1} \times f'(x) \end{aligned}$$

b

$$\begin{aligned} \textit{Let } h(x) &= (f(x))^{-1} \\ \textit{Let } g(x) &= x^{-1} \\ \textit{then } h(x) &= g(f(x)) \\ h'(x) &= g'(f(x)) \times f'(x) \\ &= -(f(x))^{-2} \times f'(x) \end{aligned}$$

Solutions to Exercise 9F

1

$$\frac{f(x+h)-f(x)}{h}$$

$$= (2\sqrt{x+h} - 2\sqrt{x}) \times \frac{1}{h}$$

$$= (2\sqrt{x+h} - 2\sqrt{x}) \times \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} \times \frac{1}{h}$$

$$= \frac{2(x+h-x)}{\sqrt{x+h} + \sqrt{x}} \times \frac{1}{h}$$

$$= \frac{2}{\sqrt{x+h} + \sqrt{x}}$$

Hence

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{x+h} + \sqrt{x}}$$

$$= -\frac{1}{\sqrt{x}}$$

2 a $\dfrac{d\left(x^{\frac{1}{5}}\right)}{dx} = \dfrac{1}{5}x^{-\frac{4}{5}}$

b $\dfrac{d\left(x^{\frac{5}{2}}\right)}{dx} = \dfrac{5}{2}x^{\frac{3}{2}}$

c $\dfrac{d\left(x^{\frac{5}{2}} - x^{\frac{3}{2}}\right)}{dx} = \dfrac{5}{2}x^{\frac{3}{2}} - \dfrac{3}{2}x^{\frac{1}{2}}$

d $\dfrac{d\left(3x^{\frac{1}{2}} - 4x^{\frac{5}{3}}\right)}{dx} = \dfrac{3}{2}x^{-\frac{1}{2}} - 4 \times \dfrac{5}{3}x^{\frac{2}{3}} = \dfrac{3}{2}x^{-\frac{1}{2}} - \dfrac{20}{3}x^{\frac{2}{3}}$

e $\dfrac{d\left(x^{-\frac{6}{7}}\right)}{dx} = \dfrac{-6}{7}x^{-\frac{13}{7}}$

f $\dfrac{d\left(x^{\frac{1}{4}} + 4x^{\frac{1}{2}}\right)}{dx} = \dfrac{-1}{4}x^{-\frac{5}{4}} + 2x^{-\frac{1}{2}}$

3 a

$$f(x) = x^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$f'(27) = \frac{1}{3} \times \frac{1}{(27^{\frac{1}{3}})^2}$$

$$= \frac{1}{3} \times \frac{1}{3^2} = \frac{1}{27}$$

b

$$f(x) = x^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$f'(-8) = \frac{1}{3} \times (-8)^{-\frac{2}{3}}$$

$$= \frac{1}{3}(-2)^{-2}$$

$$= \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$$

c

$$f(x) = x^{\frac{2}{3}}$$

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$$

$$f'(27) = \frac{2}{3} \times (27)^{-\frac{1}{3}}$$

$$= \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$$

d

$$f(x) = x^{\frac{5}{4}}$$

$$f'(x) = \frac{5}{4}x^{\frac{1}{4}}$$

$$f'(16) = \frac{5}{4} \times (16)^{\frac{1}{4}}$$

$$= \frac{5}{4} \times 2 = \frac{5}{2}$$

4 a $\dfrac{d}{dx}(\sqrt{2x+1})$

$= 2 \times \dfrac{1}{2\sqrt{2x+1}}$

$= \dfrac{1}{\sqrt{2x+1}}$

b $\dfrac{d}{ax}(\sqrt{4-3x})$

$= -3 \times \dfrac{1}{2\sqrt{4-3x}}$

$= \dfrac{-3}{2\sqrt{4-3x}}$

c $\dfrac{d}{dx}(\sqrt{x^2+2})$

$= 2x \times \dfrac{1}{2\sqrt{x^2+2}}$

$= \dfrac{x}{\sqrt{x^2+2}}$

d $\dfrac{d}{dx}(4-3x)^{\frac{1}{3}}$

$= -3 \times \dfrac{1}{3(4-3x)^{\frac{2}{3}}}$

$= -(4-3x)^{-2/3}$

e $\dfrac{d}{dx}\left(\dfrac{x^2+2}{\sqrt{x}}\right)$

$= \dfrac{d}{dx}\left(x^{\frac{3}{2}} + 2x^{-\frac{1}{2}}\right)$

$= \dfrac{3}{2}x^{\frac{1}{2}} - x^{-\frac{3}{2}}$

f $\dfrac{d}{dx}(3\sqrt{x}(x^2+2x))$

$= \dfrac{d}{dx}(3x^{\frac{5}{2}} + 6x^{\frac{3}{2}})$

$= \dfrac{15}{2}x^{\frac{3}{2}} + \dfrac{18}{2}x^{\frac{1}{2}}$

$= \dfrac{15}{2}x^{\frac{3}{2}} + 9\sqrt{x}$

5 a *Let* $u = x^2 \pm a^2$

$LHS = \dfrac{d}{dx}(\sqrt{x^2 \pm a^2}) = \dfrac{d}{dx}(u^{\frac{1}{2}})$

$= \dfrac{d}{dx}(u^{\frac{1}{2}}) \times \dfrac{du}{dx}$

$= \dfrac{1}{2\sqrt{u}} \times 2x$

$= \dfrac{x}{\sqrt{x^2 \pm a^2}} = RHS \quad QED$

b *Let* $u = a^2 - x^2$

$LHS = \dfrac{d}{dx}(\sqrt{a^2 - x^2}) = \dfrac{d}{dx}(\sqrt{u})$

$= \dfrac{d}{ax}(\sqrt{u}) \times \dfrac{du}{dx}$

$= \dfrac{1}{2\sqrt{u}} \times -2x$

$= \dfrac{x}{\sqrt{a^2 - x^2}}$

6 $y = (x + \sqrt{x^2 + 1})^2$

Let $u = x + \sqrt{x^2 + 1},\ y = u^2$

$$LHS = \frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$= \left(\frac{d(x)}{dx} + \frac{d(\sqrt{x^2 + 1})}{dx}\right) \times \frac{dy}{du}$$

Let $w = x^2 + 1$

$$\frac{dy}{dx} = \left(\frac{d(x)}{dx} + \frac{d(\sqrt{w})}{dw} \times \frac{dw}{dx}\right) \times \frac{dy}{du}$$

$$= \left(1 + \frac{1}{2\sqrt{w}} \times 2x\right) \times 2\text{u}$$

$$\frac{dy}{dx} = \left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) \times 2(x + \sqrt{x^2 + 1})$$

$$= \left(\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}\right) \times 2(x + \sqrt{x^2 + 1})$$

$$= \frac{2\left(x + \sqrt{x^2 + 1}\right)^2}{\sqrt{x^2 + 1}}$$

$$= \frac{2y}{\sqrt{x^2 + 1}} = RHS\ \ QED$$

7 **a** Let $u = x^2 + 2$

$$\frac{d(\sqrt{x^2 + 2})}{dx} = \frac{d(\sqrt{u})}{du} \times \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \times 2x$$

$$= \frac{x}{\sqrt{x^2 + 2}}$$

b Let $u = x^3 - 5x$

$$\frac{d\left((x^3 - 5x)^{\frac{1}{3}}\right)}{dx} = \frac{d\left(u^{\frac{1}{3}}\right)}{du} \times \frac{du}{dx}$$

$$= \frac{1}{3}u^{-\frac{2}{3}} \times (3x^2 - 5)$$

$$= \frac{1}{3}(3x^2 - 5)(x^3 - 5x)^{-\frac{2}{3}}$$

$$= \frac{3x^2 - 5}{3\sqrt[3]{(x^3 - 5x)^2}}$$

c Let $u = x^2 + 2x$

$$\frac{d\left((x^2 + 2x)^{\frac{1}{5}}\right)}{dx} = \frac{d\left(u^{\frac{1}{5}}\right)}{du} \times \frac{du}{dx}$$

$$= \frac{1}{5}u^{-\frac{4}{5}} \times 2x + 2$$

$$= \frac{2x + 2}{5(x^2 + 2x)^{\frac{4}{5}}}$$

Solutions to Exercise 9G

1 a $f(x) = e^{5x}$

$f'x = 5e^{5x}$

b $f(x) = 7e^{-3x}$

$f'(x) = -21e^{-3x}$

c $f(x) = 3e^{-4x} + e^{x} - x^2$

$f'(x) = -12e^{-4x} + e^{x} - 2x$

d $f(x) = e^{x} - 1 + e^{-x}$

$f'(x) = e^{x} - e^{-x}$

e $f(x) = \dfrac{4e^{2x} - 2e^{x} + 1}{2e^{2x}}$

$= -e^{-x} + \dfrac{1}{2}e^{-2x}$

$f'(x) = e^{-x} - e^{-2x}$

$= e^{-2x}(e^{x} - 1)$

f $f(x) = e^{2x} + e^{4} + e^{-2x}$

$f'(x) = 2e^{2x} - 2e^{-2x}$

2 a $-6x^2e^{-2x^3}$

b $2xe^{x^2} + 3$

c $(2x - 4)e^{x^2-4x} + 3$

d $(2x - 2)e^{x^2-2x+3} - 1$

e $-\dfrac{1}{x^2}e^{\frac{1}{x}}$

f $\dfrac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}$

3 Let $y = e^{\frac{x}{2}} + 4x$

Then $\dfrac{dy}{dx} = \dfrac{1}{2}e^{\frac{x}{2}} + 4$

a When$x = 0, y = \dfrac{9}{2}$

b When$x = 1, \dfrac{1}{2}e^{\frac{1}{2}} + 4$

4 Let $y = e^{x^2+3x} + 2x$

Then $\dfrac{dy}{dx} = (2x + 3)e^{x^2+3x} + 2$

a When$x = 0, y = 5$

b When$x = 1, 5e^{4} + 2$

5 a $2f'(x)e^{2f(x)}$

b $2e^{2x}f'(e^{2x})$

6 a $y = (e^{2x} - 1)^4$

Let $u = e^{2x} - 1, y = u^4$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= (2e^{2x}) \times 4u^3$

$= 8e^{2x}(e^{2x} - 1)^3$

b $y = e^{\sqrt{x}}$

Let $u = \sqrt{x}, y = e^{u}$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= e^{u} \times \dfrac{1}{2\sqrt{x}}$

$= e^{\sqrt{x}} \times \dfrac{1}{2\sqrt{x}}$

$= \dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}$

c $y = (e^x - 1)^{\frac{1}{2}}$

Let $u = e^x - 1, y = u^{\frac{1}{2}}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= \frac{1}{2}u^{-\frac{1}{2}} \times e^x\\ &= \frac{1}{2}e^x(e^x - 1)^{-\frac{1}{2}}\end{aligned}$$

d $y = e^{x^{\frac{2}{3}}}$

Let$u = x^{\frac{2}{3}}, y = e^u$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= e^u \times \frac{2}{3}x^{-\frac{1}{3}}\\ &= \frac{2}{3}x^{-\frac{1}{3}}e^{x^{\frac{2}{3}}}\end{aligned}$$

e $(2x - 3)e^{(x-1)(x-2)}$

f e^{e^x+x}

Solutions to Exercise 9H

1 **a** $\dfrac{dy}{dx} = \dfrac{2}{x}$

b $\dfrac{dy}{dx} = \dfrac{4}{2x} = \dfrac{2}{x}$

c $\dfrac{dy}{dx} = 2x + \dfrac{3}{x}$

d $\dfrac{dy}{dx} = \dfrac{3}{x} - \dfrac{1}{x^2} = \dfrac{3x - 1}{x^2}$

e $\dfrac{dy}{dx} = \dfrac{3}{x} + 1 = \dfrac{3 + x}{x}$

f $\dfrac{dy}{dx} = \dfrac{1}{x + 1}$

g $\dfrac{dy}{dx} = \dfrac{2}{2x + 4} = \dfrac{1}{x + 2}$

h $\dfrac{dy}{dx} = \dfrac{3}{3x - 1}$

i $\dfrac{dy}{dx} = \dfrac{6}{6x - 1}$

2 **a** $\dfrac{dy}{dx} = \dfrac{3}{x}$

b $\dfrac{dy}{dx} = \dfrac{3(\log_e x)^2}{x}$

c $\dfrac{dy}{dx} = \dfrac{2x + 1}{x^2 + x - 1}$

d $\dfrac{dy}{dx} = \dfrac{3x^2 + 2x}{x^3 + x^2}$

e $\dfrac{dy}{dx} = \dfrac{4}{2x + 3}$

f $\dfrac{dy}{dx} = \dfrac{4}{2x - 3}$

3 **a** $f(x) = \log_e(x^2 + 1)$

$$f'(x) = 2x \times \frac{1}{x^2 + 1}$$

$$= \frac{2x}{x^2 + 1}$$

b $f(x) = \log_e(e^x)$

$$f'(x) = e^x \times \frac{1}{e^x}$$

$$= 1$$

4 **a** $y = \log_e x$

$x = e,$

$y = \ln e = 1$

$$\frac{dy}{dx} = \frac{1}{x}$$

$x = e,$

$$\frac{dy}{dx} = \frac{1}{e} = e^{-1}$$

b $y = \ln(x^2 + 1)$

$x = e,$

$y = \ln(e^2 + 1)$

$$\frac{dy}{dx} = 2x \times \frac{1}{x^2 + 1}$$

$$= \frac{2x}{x^2 + 1}$$

$x = e,$

$$\frac{dy}{dx} = \frac{2e}{e^2 + 1}$$

c $y = \ln(-x)$

$x = -e,$

$y = \log_e e = 1$

$$\frac{dy}{dx} = \frac{-1}{-x} = \frac{1}{x}$$

$x = -e,$

$$\frac{dy}{dx} = \frac{-1}{e} = -e^{-1}$$

d $y = x + \log_e x$

$x = 1,$

$y = 1 + \log_e 1 = 1$

$$\frac{dy}{dx} = 1 + \frac{1}{x}$$

$x = 1,$

$$\frac{dy}{dx} = 1 + 1 = 2$$

e $y = \log_e(x^2 - 2x + 2)$

$x = 1,$

$y = \log_e 1 = 0$

$$\frac{dy}{dx} = (2x - 2)\frac{1}{x^2 - 2x + 2}$$

$x = 1,$

$$\frac{dy}{dx} = 0$$

f $y = \log_e(2x - 1)$

$x = \frac{3}{2},$

$y = \log_e 2$

$$\frac{dy}{dx} = \frac{2}{2x - 1}$$

$x = \frac{3}{2},$

$$\frac{dy}{dx} = \frac{2}{2} = 1$$

5 $f(x) = \ln(\sqrt{x^2 + 1})$

$$f'(x) = 2x \times \frac{1}{2\sqrt{x^2 + 1}} \times \frac{1}{\sqrt{x^2 + 1}}$$

$$= \frac{x}{x^2 + 1}$$

alternatively,

$$f(x) = \ln\left((x^2 + 1)^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}\ln(x^2 + 1)$$

$$f'(x) = \frac{1}{2} \times 2x \times \frac{1}{x^2 + 1}$$

$$= \frac{x}{x^2 + 1}$$

$$f'(1) = \frac{1}{1 + 1} = \frac{1}{2}$$

6 $\frac{d}{dx}(\ln(x^2 + x + 1))$

$$= (2x + 1) \times \frac{1}{x^2 + x + 1}$$

$$= \frac{2x + 1}{x^2 + x + 1}$$

7 $f(x) = \ln(x^2 + 1)$

$$f'(x) = 2x \times \frac{1}{x^2 + 1}$$

$$= \frac{2x}{x^2 + 1}$$

$$f'(3) = \frac{6}{9 + 1} = \frac{3}{5}$$

8 $\frac{d}{dx}(\ln(f(x))) = f'(x) \times \frac{1}{f(x)}$

$$= \frac{f'(x)}{f(x)}$$

$$
\begin{aligned}
x &= 0, \\
\frac{d}{dx}(\ln(f(x))) &= \frac{f'(0)}{f(0)} \\
&= \frac{4}{2} \\
&= 2
\end{aligned}
$$

Solutions to Exercise 9I

1 **a** $5\cos 5x$

b $-5\sin 5x$

c $5\sec^2 5x$

d $\cos x \times 2\sin x = \sin 2x$

e $3\sec^2(3x+1)$

f $-2x\sin(x^2+1)$

g $2\sin\left(x-\frac{\pi}{4}\right)\cos\left(x-\frac{\pi}{4}\right)$

h $-2\cos\left(x-\frac{\pi}{3}\right)\sin\left(x-\frac{\pi}{3}\right)$

i $6\sin^2\left(2x+\frac{\pi}{6}\right)\cos\left(2x+\frac{\pi}{6}\right)$

j $6\cos\left(2x+\frac{\pi}{4}\right)\sin^2\left(2x+\frac{\pi}{4}\right)$

2 **a** $y = \sin 2x$

$\frac{dy}{dx} = 2\cos 2x$

$x = \frac{\pi}{8}$,

$y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\frac{dy}{dx} = 2\cos\frac{\pi}{4} = \frac{2}{\sqrt{2}} = \sqrt{2}$

b $y = \sin 3x$

$\frac{dy}{dx} = 3\cos 3x$

$x = \frac{\pi}{6}$,

$y = \sin\frac{\pi}{2} = 1$

$\frac{dy}{dx} = 3\cos\frac{\pi}{2} = 0$

c $y = 1 + \sin 3x$

$\frac{dy}{dx} = 3\cos 3x$

$x = \frac{\pi}{6}$,

$y = 1 + \sin\frac{\pi}{2} = 1 + 1 = 2$

$\frac{dy}{dx} = 3\cos\frac{\pi}{2} = 0$

d $y = \cos^2 2x$

$$\begin{aligned}\frac{dy}{dx} &= -2\sin 2x \times 2\cos 2x \\ &= -4\sin 2x\cos 2x \\ &= -2\sin 4x\end{aligned}$$

$x = \frac{\pi}{4}$,

$y = \cos^2\frac{\pi}{2} = 0$

$\frac{dy}{dx} = -2\sin\pi = 0$

e $y = \sin^2 2x$

$$\begin{aligned}\frac{dy}{dx} &= -2\cos 2x \times 2\sin 2x \\ &= 4\cos 2x\sin 2x \\ &= 2\sin 4x\end{aligned}$$

$x = \frac{\pi}{4}$,

$y = \sin^2\frac{\pi}{2} = 1$

$\frac{dy}{dx} = 2\sin\pi = 0$

f $y = \tan 2x$

$\frac{dy}{dx} = 2\sec^2 2x$

$x = \frac{\pi}{8},$

$y = \tan\frac{\pi}{4} = 1$

$\frac{dy}{dx} = 2\sec^2\frac{\pi}{4}$

$= 2 \times (\sqrt{2})^2$

$= 4$

3 a $f(x) = 5\cos x - 2\sin 3x$

$f'(x) = -5\sin x - 6\cos 3x$

b $f(x) = \cos x + \sin x$

$f'(x) = -\sin x + \cos x$

$= \cos x - \sin x$

c $f(x) = \sin x + \tan x$

$f'(x) = \cos x + \sec^2 x$

d $f(x) = \tan^2 x$

$f'(x) = \sec^2 x \times 2\tan x$

$= 2\tan x \sec^2 x$

4 a $y = 2\cos\left(\frac{\pi x}{180}\right)$

$\frac{dy}{dx} = \frac{-2\pi}{180}\sin\left(\frac{\pi x}{180}\right)$

$= \frac{-\pi}{90}\sin(x^\circ)$

b $y = 3\sin\left(\frac{\pi x}{180}\right)$

$\frac{dy}{dx} = \frac{3\pi}{180}\cos\left(\frac{\pi x}{180}\right)$

$= \frac{\pi}{60}\cos(x^\circ)$

c $y = \tan\left(\frac{3\pi x}{180}\right)$

$y = \tan\left(\frac{\pi x}{60}\right)$

$\frac{dy}{dx} = \frac{\pi}{60}\sec^2\left(\frac{\pi x}{60}\right)$

$= \frac{\pi}{60}\sec^2(3x^\circ)$

5 a $y = -\ln(\cos x)$

$\frac{dy}{dx} = -\sin x \times -1 \times \frac{1}{\cos x}$

$= \frac{\sin x}{\cos x}$

$= \tan x$

b $y = -\log_e(\tan x)$

$\frac{dy}{dx} = -\sec^2 x \times \frac{1}{\tan x}$

$= -\frac{1}{\cos x \sin x}$

6 a $2\cos x e^{2\sin x}$

b $-2\sin(2x)e^{\cos 2x}$

Solutions to Exercise 9J

1 a

$$y = (2x^2 + 6)(2x^3 + 1)$$

$$\frac{dy}{dx} = (2x^2 + 6)\frac{d}{dx}(2x^3 + 1) + (2x^3 + 1)\frac{d}{dx}(2x^2 + 6)$$

$$= (2x^2 + 6)(6x^2) + (2x^3 + 1)(4x)$$

$$= 12x^4 + 36x^2 + 8x^4 + 4x$$

$$= 20x^4 + 36x^2 + 4x$$

b

$$y = 3x^{\frac{1}{2}}(2x + 1)$$

$$\frac{dy}{dx} = 3x^{\frac{1}{2}}\frac{d}{dx}(2x + 1) + 3(2x + 1)\frac{d}{dx}x^{\frac{1}{2}}$$

$$= 3x^{\frac{1}{2}} \times 2 + 3(2x + 1) \times \frac{1}{2x^{\frac{1}{2}}}$$

$$= 6x^{\frac{1}{2}} + 3x^{\frac{1}{2}} + \frac{3}{2}x^{-\frac{1}{2}}$$

$$= 9x^{\frac{1}{2}} + \frac{3}{2}x^{-\frac{1}{2}}$$

c

$$y = 3x(2x - 1)^3$$

$$\frac{dy}{dx} = 3x\frac{d}{dx}((2x - 1)^3) + 3(2x - 1)^3\frac{d}{dx}(x)$$

Let $u = 2x - 1$

$$\frac{dy}{dx} = 3x\frac{du}{dx} \times \frac{d(u)^3}{dx} + 3(2x - 1)^3$$

$$= 3x(2 \times 3u^2) + 3(2x - 1)^3$$

$$= 18x(2x - 1)^2 + 3(2x - 1)^3$$

$$= 3(2x - 1)^2(6x + (2x - 1))$$

$$= 3(2x - 1)^2(8x - 1)$$

d

$$y = 4x^2(2x^2 + 1)^2$$

$$\frac{dy}{dx} = 4x^2\frac{d}{dx}(2x^2 + 1)^2 + 4(2x^2 + 1)^2\frac{dy}{dx}(x^2)$$

$$= 4x^2\left((2x^2 + 1)\frac{d}{dx}(2x^2 + 1) + (2x^2 + 1)\frac{d}{dx}(2x^2 + 1)\right) + 4(2x^2 + 1)^2 \times 2x$$

$$= 8x^2(2x^2 + 1)(4x) + 8x(2x^2 + 1)^2$$

$$= 32x^3(2x^2 + 1) + 8x(2x^2 + 1)^2$$

$$= 8x(2x^2 + 1)(4x^2 + 2x^2 + 1)$$

$$= 8x(2x^2 + 1)(6x^2 + 1)$$

e

$$y = (3x + 1)^{\frac{3}{2}}(2x + 4)$$

$$\frac{dy}{dx} = (3x + 1)^{\frac{3}{2}}\frac{d}{dx}(2x + 4) + (2x + 4)\frac{d}{dx}(3x + 1)^{\frac{3}{2}}$$

Let $u = 3x + 1$

$$\frac{dy}{dx} = (3x + 1)^{\frac{3}{2}}(2) + (2x + 4)\left(\frac{d\left(u^{\frac{3}{2}}\right)}{dx} \times \frac{du}{dx}\right)$$

$$= 2(3x + 1)^{\frac{3}{2}} + (2x + 4)\left(\frac{3}{2}u^{\frac{1}{2}} \times 3\right)$$

$$= 2(3x + 1)^{\frac{3}{2}} + \frac{9}{2}(2x + 4)(3x + 1)^{\frac{1}{2}}$$

$$= 2(3x + 1)^{\frac{3}{2}} + 9(x + 2)(3x + 1)^{\frac{1}{2}}$$

$$= (2(3x + 1) + 9(x + 2))(3x + 1)^{\frac{1}{2}}$$

$$= (6x + 2 + 9x + 18)(3x + 1)^{\frac{1}{2}}$$

$$= (15x + 20)(3x + 1)^{\frac{1}{2}}$$

$$= 5(3x + 4)(3x + 1)^{\frac{1}{2}}$$

f

$$y = (x^2+1)(2x-4)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = (x^2+1)\frac{d}{dx}(2x-4)^{\frac{1}{2}} + (2x-4)^{\frac{1}{2}}\frac{d}{dx}(x^2+1)$$

Let $u = 2x - 4$

$$\frac{dy}{dx} = (x^2+1)\left(\frac{d\left(u^{\frac{1}{2}}\right)}{dx} \times \frac{du}{dx}\right) + (2x-4)^{\frac{1}{2}} + 2x$$

$$= (x^2+1)\left(\frac{1}{2u^{\frac{1}{2}}} \times 2\right) + 2x(2x-4)^{\frac{1}{2}}$$

$$= (x^2+1)(2x-4)^{-\frac{1}{2}} + 2x(2x-4)^{\frac{1}{2}}$$

$$= \frac{(x^2+1)+2x(2x-4)}{\sqrt{2x-4}}$$

$$= \frac{(x^2+1)+4x^2-8x}{\sqrt{2x-4}}$$

$$= \frac{5x^2-8x+1}{\sqrt{2x-4}}$$

g

$$y = x^3(3x^2+2x+1)^{-1}$$

$$\frac{dy}{dx} = x^3\frac{d}{dx}(3x^2+2x+1)^{-1} + (3x^2+2x+1)^{-1}\frac{d}{dx}x^3$$

Let $u = 3x^2 + 2x + 1$

$$\frac{dy}{dx} = x^3\left(\frac{d(u^{-1})}{dx} \times \frac{du}{dx}\right) + (3x^2+2x+1)^{-1} + 3x^2$$

$$= x^3(-u^{-2} \times (6x+2)) + 3x^2(3x^2+2x+1)^{-1}$$

$$= -x^3(6x+2)(3x^2+2x+1)^{-2} + 3x^2(3x^2+2x+1)^{-1}$$

$$= \frac{-x^3(6x+2)+3x^2(3x^2+2x+1)}{(3x^2+2x+1)^2}$$

$$= \frac{-6x^4-2x^3+9x^4+6x^3+3x^2}{(3x^2+2x+1)^2}$$

$$= \frac{3x^4+4x^3+3x^2}{(3x^2+2x+1)^2}$$

$$= \frac{x^2(3x^2+4x+3)}{(3x^2+2x+1)^2}$$

h

$$y = x^4(2x^2-1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = x^4\frac{d}{dx}(2x^2-1)^{\frac{1}{2}} + (2x^2-1)^{\frac{1}{2}}\frac{d}{dx}x^4$$

Let $u = 2x^2 - 1$

$$\frac{dy}{dx} = x^4\left(\frac{d\left(u^{\frac{1}{2}}\right)}{dx} \times \frac{du}{dx}\right) + (2x^2-1)^{\frac{1}{2}} + 4x^3$$

$$= x^4\left(\frac{1}{2}u^{-\frac{1}{2}} \times 4x\right) + 4x^3(2x^2-1)^{\frac{1}{2}}$$

$$= 2x^5(2x^2-1)^{-\frac{1}{2}} + 4x^3(2x^2-1)^{\frac{1}{2}}$$

$$= (2x^5+4x^3(2x^2-1))(2x^2-1)^{-\frac{1}{2}}$$

$$= (2x^5+8x^5-4x^3)(2x^2-1)^{-\frac{1}{2}}$$

$$= (10x^5-4x^3)(2x^2-1)^{-\frac{1}{2}}$$

$$= 2x^3(5x^2-2)(2x^2-1)^{-\frac{1}{2}}$$

i

$$y = x^2(x^2 + 2x)^{\frac{1}{3}}$$

$$\frac{dy}{dx} = x^2\frac{d}{dx}(x^2 + 2x)^{\frac{1}{3}} + (x^2 + 2x)^{\frac{1}{3}}\frac{d}{dx}(x^2)$$

$$Let\ u = x^2 + 2x$$

$$\frac{dy}{dx} = x^2\left(\frac{d\left(u^{\frac{1}{3}}\right)}{dx} \times \frac{du}{dx}\right) + (x^2 + 2x)^{\frac{1}{3}} \times 2x$$

$$= x^2\left(\frac{1}{3}u^{-\frac{2}{3}} \times (2x + 2)\right) + 2x(x^2 + 2x)^{\frac{1}{3}}$$

$$= \frac{2}{3}x^2(x + 1)(x^2 + 2x)^{-\frac{2}{3}} + 2x(x^2 + 2x)^{\frac{1}{3}}$$

$$= \left(\frac{2}{3}x^3 + \frac{2}{3}x^2 + 2x(x^2 + 2x)\right)(x^2 - 2x)^{-\frac{2}{3}}$$

$$= \left(\frac{2}{3}x^3 + \frac{2}{3}x^2 + 2x^3 + 4x^2\right)(x^2 - 2x)^{-\frac{2}{3}}$$

$$= x^2\left(\frac{8}{3}x + \frac{14}{3}\right)(x^2 + 2x)^{-\frac{2}{3}}$$

$$= \frac{2}{3}x^2(4x + 7)(x^2 + 2x)^{-\frac{2}{3}}$$

j $\dfrac{4(5x^2 - 4)^2(5x^2 + 2)}{x^3}$

k $\dfrac{3(x^6 - 16)}{x^4}$

l $\dfrac{2x^3(9x^2 - 8)}{5(x(x^2 - 1))^{4/5}}$

2 **a**

$$f(x) = e^x(x^2 + 1)$$

$$f'(x) = e^x + 2x + (x^2 + 1) \times e^x$$

$$= e^x(x^2 + 2x + 1)$$

$$= ex(x + 1)^2$$

b

$$f(x) = e^{2x}(x^3 + 3x + 1)$$

$$f'(x) = e^{2x}(3x^2 + 3) + (x^3 + 3x + 1) \times 2e^{2x}$$

$$= e^{2x}(3x^2 + 3 + 2x^3 + 6x + 2)$$

$$= e^{2x}(2x^3 + 3x^2 + 6x + 5)$$

c

$$
\begin{aligned}
f(x) &= e^{4x+1}(x+1)^2 \\
f'(x) &= e^{4x+1} \times 2(x+1) + (x+1)^2 \times 4e^{4x+1} \\
&= e^{4x+1}(4(x+1)^2 + 2(x+1)) \\
&= e^{4x+1}(4x^2 + 8x + 4 + 2x + 2) \\
&= e^{4x+1}(4x^2 + 10x + 6) \\
&= e^{4x+1}(2x+2)(2x+3)
\end{aligned}
$$

d

$$
\begin{aligned}
f(x) &= e^{-4x}(x+1)^{\frac{1}{2}} \\
f'(x) &= e^{-4x} \times \frac{1}{2}(x+1)^{\frac{-1}{2}} + (x+1)^{\frac{1}{2}} \times -4e^{-4x} \\
&= e^{-4x}\left(\frac{1}{2}(x+1)^{\frac{-1}{2}} - 4(x+1)^{\frac{-1}{2}}\right) \\
&= e^{-4x}(x+1)^{\frac{-1}{2}}\left(\frac{1}{2} - 4(x+1)\right) \\
&= e^{-4x}(x+1)^{\frac{-1}{2}}\left(-4x - \frac{7}{2}\right) \\
&= \frac{-8x-7}{2e^{4x}\sqrt{x+1}}
\end{aligned}
$$

3 a $f'(x) = \ln x \times 1 + x + \frac{1}{x}$

$= \ln x + 1$

b $f'(x) = \ln x \times 4x + 2x^2 \times \frac{1}{x}$

$= 2x(1 + 2\ln x)$

c $f'(x) = e^x \times \frac{1}{x} + \ln x \times e^x$

$= e^x\left(\frac{1}{x} + \ln x\right)$

d $f'(x) = \ln(-x) \times 1 + x + \frac{-1}{-x}$

$= \ln(-x) + 1$

4 a $f'(x) = 4x^3e^{-2x} - 2x^4e^{-2x}$

$= 2x^3e^{-2x}(2 - x)$

b $f'(x) = 2e^{2x+3}$

c Let $y = (e^{2x} + x)^{\frac{3}{2}}$

Let $u = e^{2x} + x$

Then $y = u^{\frac{3}{2}}$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
&= \frac{3}{2}u^{\frac{1}{2}} \times (2e^{2x} + 1) \\
&= \frac{3}{2}(e^{2x} + x)^{\frac{1}{2}} \times (2e^{2x} + 1)
\end{aligned}
$$

d Let $y = \frac{1}{x}e^x$

$$
\begin{aligned}
\frac{dy}{dx} &= -\frac{1}{x^2} \times e^x + \frac{1}{x} \times e^x \\
&= \frac{e^x(x-1)}{x^2}
\end{aligned}
$$

e Let $y = e^{\frac{1}{2}x^2}$

$\frac{dy}{dx} = xe^{\frac{1}{2}x^2}$

f Let $y = (x^2 + 2x + 2)e^{-x}$

$$
\begin{aligned}
\frac{dy}{dx} &= (2x+2)e^{-x} - (x^2 + 2x + 2)e^{-x} \\
&= e^{-x}(2x + 2 - x^2 - 2x - 2) \\
&= -x^2e^{-x}
\end{aligned}
$$

5 a $\frac{d}{dx}\left(e^x f(x)\right) = e^x f(x) + e^x f'(x)$

$= e^x(f(x) + f'(x))$

b $\frac{d}{dx}\left(\frac{e^x}{f(x)}\right) = \frac{e^x f(x) - e^x f'(x)}{(f(x))^2}$

c $\dfrac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$

d

$\dfrac{d}{dx}\left(e^x(f(x))^2\right) = e^x(f(x))^2 + 2e^x f(x)f'(x)$

j $\dfrac{d}{dx}(e^x \tan x)$

$= e^x \tan x + e^x \sec^2 x$

$= e^x(\tan x + \sec^2 x)$

6 a $\dfrac{d}{dx}(x^3 \cos x) = 3x^2 + (-\sin x)x^3$

$= x^2(3\cos x - x\sin x)$

b $2x\cos x - (1 + x^2)\sin x$

c $\dfrac{d}{dx}(e^{-x}\sin x)$

$= e^{-x}\sin x + e^{-x}\cos x$

$= e^{-x}(\cos x - \sin x)$

d $6\cos x - 6x\sin x$

e $3\cos(3x)\cos(4x) - 4\sin(4x)\sin(3x)$

f $2\sin(2x) + 2\tan(2x)\sec(2x)$

g $12\sin x + 12x\cos x$

h $\dfrac{d}{dx}(x^2 e^{\sin x})$

$= 2xe^{\sin x} + x^2\cos xe^{\sin x}$

$= xe^{\sin x}(2 + x\cos x)$

i $\dfrac{d}{dx}(x^2\cos^2 x)$

$= 2x\cos^2 x - 2\sin x\cos x \times x^2$

$= 2x\cos^2 x - x^2\sin 2x$

7 a $f(x) = e^x \sin x$

$f'(x) = e^x\sin x + e^x\cos x$

$= e^x(\sin x + \cos x)$

$f'(\pi) = e^{\pi}(\sin\pi + \cos\pi)$

$= -e^{\pi}$

b $f(x) = \cos^2 2x$

$f'(x) = -2\sin 2x \times 2\cos 2x$

$= -4(\sin 2x\cos 2x)$

$= -2\sin 4x$

$f'(\pi) = -2\sin 4\pi$

$= -2\sin 0$

$= 0$

8 Let $y = \dfrac{d}{dx}(f(x)\log_e x)$

$= f'(x)\log_e x + \dfrac{1}{x}f(x)$

When $x = 1$

$y = 4\log_e 1 + 2 = 2$

Solutions to Exercise 9K

1 a

$$y = \frac{x}{x+4}$$

$$\frac{dy}{dx} = \frac{(x+4)\frac{dx}{dx} - x\frac{d(x+4)}{dx}}{(x+4)^2}$$

$$= \frac{(x+4) - x}{(x+4)^2}$$

$$= \frac{4}{(x+4)^2}$$

b

$$y = \frac{x^2-1}{x^2+1}$$

$$\frac{dy}{dx} = \frac{(x^2+1)\frac{d(x^2-1)}{dx}(x^2-1)\frac{d(x^2+1)}{dx}}{(x^2+1)^2}$$

$$= \frac{(x^2+1)\times 2x + (1-x^2)\times 2x}{(x^2+1)^2}$$

$$= \frac{4x}{(x^2+1)^2}$$

c

$$y = \frac{x^{\frac{1}{2}}}{1+x}$$

$$\frac{dy}{dx} = \frac{(1+x)\frac{d\left(x^{\frac{1}{2}}\right)}{dx}x^{\frac{1}{2}}\frac{d(1+x)}{dx}}{(1+x)^2}$$

$$= \frac{(1+x)\frac{1}{2}x^{-\frac{1}{2}} - x^{\frac{1}{2}}}{(1+x)^2}$$

$$= \frac{\frac{1}{2}(1+x) - x}{x^{\frac{1}{2}}(1+x)^2}$$

$$= \frac{\frac{1}{2} - \frac{1}{2}x}{x^{\frac{1}{2}}(1+x)^2}$$

$$= \frac{x^{-\frac{1}{2}} - x^{\frac{1}{2}}}{2(1+x)^2}$$

d

$$y = \frac{(x+2)^3}{x^2+1}$$

$$\frac{dy}{dx} = \frac{(x^2+1)\frac{d(x+2)^3}{dx} - (x+2)^3\frac{d(x^2+1)}{dx}}{(x^2+1)^2}$$

$$= \frac{(x^2+1)\times 3(x+2)^2 - (x+2)^3\times 2x}{(x^2+1)^2}$$

$$= \frac{(3(x^2+1) - 2x(x+2))(x+2)^2}{(x^2+1)^2}$$

$$= \frac{(3x^2+3-2x^2-4x)(x+2)^2}{(x^2+1)^2}$$

$$= \frac{(x^2-4x+3)(x+2)^2}{(x^2+1)^2}$$

$$= \frac{(x-3)(x-1)(x+2)^2}{(x^2+1)^2}$$

e

$$y = \frac{x-1}{x^2+2}$$

$$\frac{dy}{dx} = \frac{(x^2+2)\frac{d(x-1)}{dx} - (x-1)\frac{d(x^2+2)}{dx}}{(x^2+2)^2}$$

$$= \frac{(x^2+2) - (x-1)\times 2x}{(x^2+2)^2}$$

$$= \frac{x^2+2-2x^2+2x}{(x^2+2)^2}$$

$$= \frac{-x^2+2x+2}{(x^2+2)^2}$$

f

$$y = \frac{x^2+1}{x^2-1}$$

$$\frac{dy}{dx} = \frac{(x^2-1)\frac{d(x^2+1)}{dx}(x^2+1)\frac{d(x^2-1)}{dx}}{(x^2-1)^2}$$

$$= \frac{(x^2-1)\times 2x - (x^2+1)\times 2x}{(x^2-1)^2}$$

$$= \frac{-4x}{(x^2-1)^2}$$

g

$$y = \frac{3x^2 + 2x + 1}{x^2 + x + 1}$$

$$\frac{dy}{dx} = \frac{(x^2 + x + 1)\frac{d}{dx}(3x^2 + 2x + 1) - (3x^2 + 2x + 1)\frac{d}{dx}(x^2 + x + 1)}{(x^2 + x + 1)^2}$$

$$= \frac{(x^2 + x + 1)(6x + 2) - (3x^2 + 2x + 1) \times (2x + 1)}{(x^2 + x + 1)^2}$$

$$= \frac{6x^3 + 8x^2 + 8x + 2 - 6x^3 - 7x^2 - 4x - 1}{(x^2 + x + 1)^2}$$

$$= \frac{x^2 + 4x + 1}{(x^2 + x + 1)^2}$$

h

$$y = \frac{2x + 1}{2x^3 + 2x}$$

$$\frac{dy}{dx} = \frac{(2x^3 + 2x)\frac{d}{dx}(2x + 1) - (2x + 1)\frac{d}{dx}(2x^3 + 2x)}{(2x^3 + 2x)^2}$$

$$= \frac{(2x^3 + 2x) \times 2 - (2x + 1)(6x^2 + 2)}{(2x^3 + 2x)^2}$$

$$= \frac{4x^3 + 4x - 12x^3 - 6x^2 - 4x - 2}{(2x^3 + 2x)^2}$$

$$= \frac{-8x^3 - 6x^2 - 2}{(2x^3 + 2x)^2}$$

$$= \frac{-(4x^3 + 3x^2 + 1)}{2(x^3 + x)^2}$$

2 a

$$y = (2x + 1)^4 x^2$$

$$x = 1,\ y = (2(1) + 1)^4 \times (1)^2$$

$$y = 3^4$$

$$y = 81$$

$$\frac{dy}{dx} = x^2 \frac{d((2x + 1)^4)}{dx} + (2x + 1)^4 \frac{d(x^2)}{dx}$$

$$= x^2 \times 2 \times 4(2x + 1)^3 + (2x + 1)^4 \times 2x$$

$$= 8x^2(2x + 1)^3 + 2x(2x + 1)^4$$

$$= (8x^2 + 4x^2 + 2x)(2x + 1)^3$$

$$= (8x^2 + 2x(2x + 1))(2x + 1)^3$$

$$= 2x(6x + 1)(2x + 1)^3$$

$$x = 1,$$

$$\frac{dy}{dx} = 2(1)(6(1) + 1)(2(1) + 1)^3$$

$$= 2(7)(3)^3$$

$$= 14 \times 27$$

$$\frac{dy}{dx} = 378$$

b $y = x^2(x+1)^{\frac{1}{2}}$

$x = 0,$

$y = (0)^2(0+1)^{\frac{1}{2}}$

$y = 0$

$$\frac{dy}{dx} = x^2\frac{d\left((x+1)^{\frac{1}{2}}\right)}{dx} + (x+1)^{\frac{1}{2}}\frac{d(x^2)}{dx}$$

$$= x^2 \times \frac{1}{2\sqrt{x+1}} + \sqrt{x+1} \times 2x$$

$$= \frac{x^2 + 2x(x+1) \times 2x}{2\sqrt{x+1}}$$

$$= \frac{x^2 + 4x^2 + 4x}{2\sqrt{x+1}}$$

$$= \frac{5x^2 + 4x}{2\sqrt{x+1}}$$

$x = 0,$

$$\frac{dy}{dx} = \frac{5(0)^2 + 4(0)}{2\sqrt{0+1}}$$

$$\frac{dy}{dx} = 0$$

c $y = x^2(2x+1)^{\frac{1}{2}}$

$x = 0,$

$y = 0$

$$\frac{dy}{dx} = x^2\frac{d(2x+1)^{\frac{1}{2}}}{dx} + (2x+1)\frac{dx^2}{dx}$$

$$= x^2\frac{d(2x+1)^{\frac{1}{2}}}{dx} + 2x(2x+1)$$

$x = 0,$

$$\frac{dy}{dx} = 0 + 0$$

$$\frac{dy}{dx} = 0$$

d $y = \frac{x}{x^2+1}$

$x = 1,$

$y = \frac{1}{1+1}$

$y = \frac{1}{2}$

$$\frac{dy}{dx} = \frac{(x^2+1)\frac{dx}{dx} - x\frac{d(x^2+1)}{dx}}{(x^2+1)^2}$$

$$= \frac{x^2 + 1 - x \times 2x}{(x^2+1)^2}$$

$$= \frac{1 - x^2}{(x^2+1)^2}$$

$x = 1,$

$$\frac{dy}{dx} = \frac{1-1}{(1+1)^2}$$

$$\frac{dy}{dx} = 0$$

e

$y = \frac{2x+1}{x^2+1}$

$x = 1,$

$y = \frac{2+1}{1+1}$

$y = \frac{3}{2}$

$$\frac{dy}{dx} = \frac{(x^2+1)\frac{d(2x+1)}{dx} - (2x+1)\frac{d(x^2+1)}{dx}}{(x^2+1)^2}$$

$$= \frac{(x^2+1) \times 2 - (2x+1) \times 2x}{(x^2+1)^2}$$

$$= \frac{2x^2 + 2 - 4x^2 - 2x}{(x^2+1)^2}$$

$$= \frac{2(-x^2 - x + 1)}{(x^2+1)^2}$$

$x = 1,$ $\frac{dy}{dx} = \frac{2(-1-1+1)}{(2)^2} = -\frac{1}{2}$

3 a

$$f(x) = (x+1)(x^2+1)^{\frac{1}{2}}$$

$$f'(x) = (x+1)\frac{d(x^2+1)^{\frac{1}{2}}}{dx}$$

$$+ (x^2+1)^{\frac{1}{2}}\frac{d(x+1)}{dx}$$

$$= (x+1)\left(\frac{d(x^2+1)^{\frac{1}{2}}}{d(x^2+1)} \times \frac{d(x^2+1)}{dx}\right)$$

$$+ (x^2+1)^{\frac{1}{2}}$$

$$= (x+1)\left(\frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x\right) + (x^2+1)^{\frac{1}{2}}$$

$$= (x^2+1)^{-\frac{1}{2}}(x^2+x+x^2+1)$$

$$= (x^2+1)^{-\frac{1}{2}}(2x^2+x+1)$$

b

$$f(x) = (x^2+1)(x^3+1)^{\frac{1}{2}}$$

$$f'(x) = (x^2+1)\frac{d(x^3+1)^{\frac{1}{2}}}{dx}$$

$$+ (x^3+1)^{\frac{1}{2}}\frac{d(x^2+1)}{dx}$$

$$= (x^2+1)\left(\frac{d(x^3+1)^{\frac{1}{2}}}{d(x^3+1)} \times \frac{d(x^3+1)}{dx}\right)$$

$$+ (x^3+1)^{\frac{1}{2}} \times 2x$$

$$= (x^2+1)\left(\frac{1}{2}(x^3+1)^{-\frac{1}{2}} \times 3x^2\right) + 2x(x^3+1)^{\frac{1}{2}}$$

$$= \frac{3}{2}x^2(x^2+1)(x^3+1)^{-\frac{1}{2}} + 2x(x^3+1)^{\frac{1}{2}}$$

$$= (x^3+1)^{-\frac{1}{2}}\left(\frac{3}{2}x^4 + \frac{3}{2}x^2 + 2x^4 + 2x\right)$$

$$= (x^3+1)^{-\frac{1}{2}}\left(7x^4 + \frac{3}{2}x^2 + 2x\right)$$

$$= x\left(\frac{7}{2}x^3 + \frac{3}{2}x + 2\right)(x^3+1)^{-\frac{1}{2}}$$

c

$$f(x) = \frac{2x+1}{x+3}$$

$$f^1(x) = \frac{(x+3)\frac{d(2x+1)}{dx} - (2x+1)\frac{d(x+3)}{dx}}{(x+3)^2}$$

$$= \frac{(x+3) \times 2 - (2x+1)}{(x+3)^2}$$

$$= \frac{2x+6-2x-1}{(x+3)^2}$$

$$= \frac{5}{(x+3)^2}$$

4 a $$f'(x) = \frac{e^x(e^{3x}+3) - 3e^{3x}e^x}{(e^{3x}+3)^2}$$

$$= \frac{3e^x - 2e^{4x}}{(e^{3x}+3)^2}$$

b $$f'(x) = \frac{-\sin x(x+1) - \cos x}{(x+1)^2}$$

$$= -\frac{\sin x(x+1) + \cos x}{(x+1)^2}$$

c $$f'(x) = \frac{\frac{1}{x} \times (x+1) - \log_e x}{(x+1)^2}$$

$$= \frac{(x+1) - x\log_e x}{x(x+1)^2}$$

5 a $$f'(x) = \frac{\frac{1}{x} \times x - \log_e x}{(x^2}$$

$$= \frac{1 - \log_e x}{x^2}$$

b $$f'(x) = \frac{\frac{1}{x}(x^2+1) - 2x\log_e x}{(x^2+1)^2}$$

$$= \frac{x^2 + 1 - 2x^2\log_e x}{x(x^2+1)^2}$$

6 a

$$f(x) = \frac{e^{3x}}{e^{3x}+3}$$

$$f'(x) = \frac{(e^{3x}+3)\frac{d}{dx}e^{3x} - e^{3x}\frac{d}{dx}(e^{3x}+3)}{(e^{3x}+3)^2}$$

$$= \frac{3e^{3x}(e^{3x}+3) - 3e^{3x}(e^{3x})}{(e^{3x}+3)^2}$$

$$= \frac{9e^{3x}}{(e^{3x}+3)^2}$$

b $$f(x) = \frac{e^x+1}{e^x-1}$$

$$f'(x) = \frac{(e^x-1)\frac{d}{dx}(e^x+1) - (e^X+1)\frac{d}{dx}(e^x-1)}{(e^x-1)^2}$$

$$= \frac{(e^x-1)e^x + (-e^x-1)(e^x)}{(e^x-1)^2}$$

$$= \frac{-2e^x}{(e^x-1)^2}$$

c $$f(x) = \frac{e^{2x}+2}{e^{2x}-2}$$

$$f'(x) = \frac{(e^x-2)\frac{d}{dx}(e^{2x}+2) - (e^{2x}+2)\frac{d}{dx}(e^{2x}-2)}{(e^{2x}-2)^2}$$

$$= \frac{(e^{2x}-2)2e^{2x} - (e^{2x}+2)2e^{2x}}{(e^{2x}-2)^2}$$

$$= \frac{-8e^{2x}}{(e^{2x}-2)^2}$$

7 a $$f(x) = \frac{2x}{\cos x}$$

$$f'(x) = \frac{\cos x \times 2 - 2x(-\sin x)}{\cos^2 x}$$

$$= \frac{2\cos x + 2x\sin x}{\cos^2 x}$$

$$f'(\pi) = \frac{2\cos(\pi) + 2\pi\sin(\pi)}{(\cos(\pi))^2}$$

$$= \frac{-2}{1}$$

$$= -2$$

b

$$f(x) = \frac{3x^2+1}{\cos x}$$

$$f'(x) = \frac{\cos x(6x) - (3x^2+1)(-\sin x)}{\cos^2 x}$$

$$= \frac{6x\cos x + (3x^2+1)\sin x}{\cos^2 x}$$

$$f'(\pi) = \frac{6\pi\cos(\pi) + (3\pi^2+1)\sin(\pi)}{(\cos(\pi))^2}$$

$$= \frac{-6\pi}{1}$$

$$= -6\pi$$

c $$f(x) = \frac{e^x}{\cos\ x}$$

$$f'(x) = \frac{\cos xe^x + \sin xe^x}{\cos^2 x}$$

$$f'(\pi) = \frac{(\cos\pi + \sin\pi)e^\pi}{\cos^2\pi}$$

$$= -e^\pi$$

d $$f(x) = \frac{\sin x}{x}$$

$$f'(x) = \frac{x\cos x - \sin x}{x^2}$$

$$f'(\pi) = \frac{\pi\cos\pi - \sin\pi}{\pi^2}$$

$$= \frac{-\pi}{\pi^2}$$

$$= \frac{-1}{\pi}$$

Solutions to Exercise 9L

1 **a** $\lim_{x\to 2}(17) = 17$

b $\lim_{x\to 6}(x-3) = 6-3 = 3$

c $\lim_{x\to \frac{1}{2}}(2x-5) = 1-5 = -4$

d $\lim_{t\to -3}\left(\frac{(t+2)}{(t-5)}\right) = \frac{-3+2}{-3-5} = \frac{-1}{-8} = \frac{1}{8}$

e $\lim_{t\to 2} = \left(\frac{t^2+2t+1}{t+1}\right)$

$= \lim_{t\to 2}\left(\frac{(t+1)^2}{t+1}\right) = \lim_{t\to 2}(t+1) = 3$

f $\lim_{x\to 0}\left(\frac{(x+2)^2-4}{x}\right)$

$= \lim_{x\to 0}\left(\frac{x^2+4x+4-4}{x}\right)$

$= \lim_{x\to 0}\left(\frac{x^2+4x}{x}\right) = \lim_{x\to 0}(x+4) = 4$

g $\lim_{t\to 1} = \left(\frac{t^2-1}{t-1}\right)$

$= \lim_{t\to 1}\left(\frac{(t+1)(t-1)}{\mathrm{t}-1}\right)$

$= \lim_{t\to 1}(t+1) = 2$

h $\lim_{x\to 9}(\sqrt{x}+3) = \sqrt{9}+3$

$= \sqrt{12} = 2\sqrt{3}$

i $\lim_{x\to 0}\left(\frac{x^2-2x}{x}\right)$

$= \lim_{x\to 0}(x-2) = -2$

j $\lim_{x\to 2} = \left(\frac{x^3-8}{x-2}\right)$

$= \lim_{x\to 2}\left(\frac{(x-2)(x^2+2x+4)}{(x-2)}\right)$

$= \lim_{x\to 2}(x^2+2x+4) = 12$

k $\lim_{x\to 2} = \left(\frac{3x^2-x-10}{x^2+5x-14}\right)$

$= \lim_{x\to 2}\left(\frac{(3x+5)(x-2)}{(x+7)(x-2)}\right)$

$= \lim_{x\to 2}\frac{3x+5}{x+7} = \frac{11}{9}$

l $\lim_{x\to 1} = \left(\frac{x^2-3x+2}{x^2-6x+5}\right)$

$= \lim_{x\to 1}\left(\frac{(x-1)(x-2)}{(x-1)(x-4)}\right)$

$= \lim_{x\to 1}\frac{x-2}{x-5} = \frac{1}{4}$

2 **a** $x = 3$, *since* $f(3) \neq \lim_{x\to 3}(f(x))$, $x = 4$, *since* $\lim_{x\to 4^+}(f(x)) \neq \lim_{x\to 4^-}(f(x))$

b $x = 7$, *since* $\lim_{x\to 7^+}(f(x)) \neq \lim_{x\to 7^-}(f(x))$

3 **a** *value to test*: $x = 0$

$\lim_{x\to 0^-}(f(x)) = \lim_{x\to 0}(f(-2x+2)) = 2$

$\lim_{x\to 0^+}(f(x)) = \lim_{x\to 0}(3x) = 0 \neq$

$\lim_{x\to 0^-}(f(x))$

$\therefore$ *there is a discontinuity at* $x = 0$

b *value to test*: $x = 1$

$\lim_{x\to 1^-}(f(x)) = \lim_{x\to 1}(f(-2x+1)) =$

$-2(1)+1 = -1$

$\lim_{x\to 1^+}(f(x)) = \lim_{x\to 1}(x^2+2) = 1^2+2$

$= 3 \neq \lim_{x\to 1^-}(f(x))$

$\therefore$ *there is a discontinuity at* $x = 1$

c *value to test*: $x = -1,\ 0$

$\lim\limits_{x \to 1^-} (f(x)) = \lim\limits_{x \to -1} (-x) = -(-1) = 1$

$\lim\limits_{x \to -1^+} (f(x)) = \lim\limits_{x \to -1} (x^2) = (-1)^2 = 1$

$= \lim\limits_{x \to -1^-} (f(x))$

$f(-1) = -(-1) = 1 = \lim\limits_{x \to -1} (f(x))$

$\therefore$ *$f(x)$ is continuous at $x = -1$*

$\lim\limits_{x \to 0} -(f(x)) = \lim\limits_{x \to 0}(x^2) = (0)^2 = 0$

$\lim\limits_{x \to 0^+} (f(x)) = \lim\limits_{x \to 0}(-3x + 1) = -3(0) + 1$

$= 1 \neq \lim\limits_{x \to 0^-} (f(x))$

$\therefore$ *there is one discontinuity at $x = 0$*

4 a *value to test*: $x = 1, 7$

$\lim\limits_{x \to 1^-} (f(x)) = \lim\limits_{x \to -1} (2) = 2$

$\lim\limits_{x \to -1^+} (f(x))$

$= \lim\limits_{x \to -1} ((x - 4)^2 - 9)$

$= (1 - 4)^2 - 9$

$= 0 \neq \lim\limits_{x \to -1^-} (f(x))$

$\therefore$ *there is a discontinuity at $x = 1$*

$\lim\limits_{x \to 7^-} (f(x)) = (\lim\limits_{x \to 7}(x - 4)^2 - 9)$

$= (7 - 4)^2 - 9 = 0$

$\lim\limits_{x \to 7^-} (f(x)) = \lim\limits_{x \to 7}(x - 7) = 7 - 7 =$

$\lim\limits_{x \to 7^-} (f(x))$

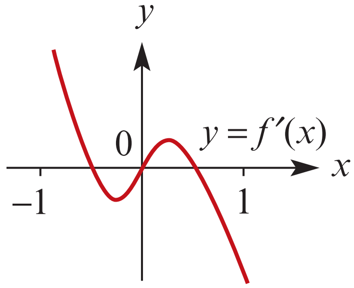
$f(7) = 7 - 7 = 0 = \lim\limits_{x \to 7} f(x)$

$f(x)$ is continuous at $x = 7$

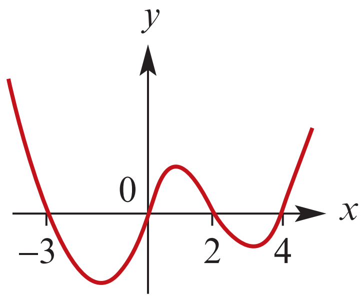
$\therefore$ *$f(x)$ is continuous for all $x \in R \backslash \{1\}$*

Solutions to Exercise 9M

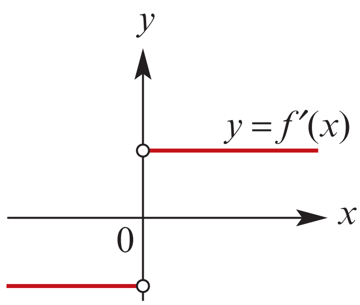
1 **a**



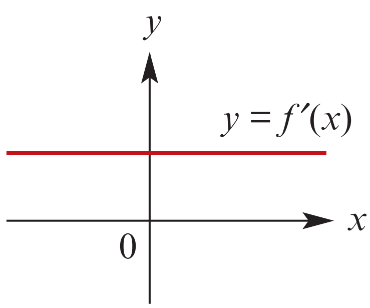
b



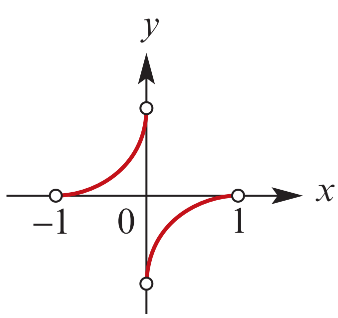
c



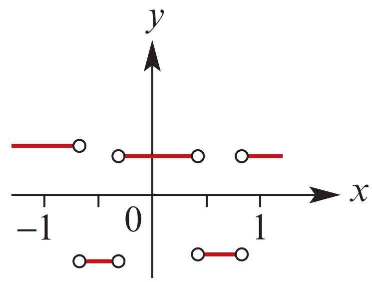
d



e



f



2 $x > 0,\ f'(x) = \dfrac{d}{dx}(-x^2 + 3)$

$= -2x + 3$

$x < 0,\ f'(x) = \dfrac{d}{dx}(3x + 1)$

$= 3$

test $x = 0$

$\lim_{x\to 0^-}(f(x)) = \lim_{x\to 0}(3x + 1) = 1$

$\lim_{x\to 0^+}(f(x)) = \lim_{x\to 0}(-x^2 + 3x + 1) = 1 = \lim_{x\to 0^-}(f(x))$

$f(0) = -(0)^2 + 3(0) + 1 = 1 = \lim_{x\to 0}(f(x))$

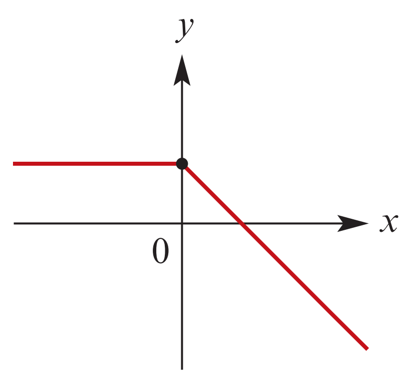
$\therefore$ $f(x)$ is continous at $x = 0$

$\lim_{x\to 0^-}(f'(x)) = \lim_{x\to 0}(3) = 3$

$\lim_{x\to 0^+}(f'(x)) = \lim_{x\to 0}(-2x + 3) = 3 = \lim_{x\to 0^-}(f'(x))$

$f(x)$ is differentiable at $x = 0$

$$f'(x) = \begin{cases} -2x + 3 & if\ x \geq 0 \\ 3 & if\ x < 0 \end{cases}$$



3

$x > 1,\ f'(x) = \dfrac{d}{dx}(x^2 + 2x + 1)$

$= 2x + 2$

$x < 1,\ f'(x) = \dfrac{d}{dx}(-2x + 3)$

$= -2$

test $x = 1$

$\lim\limits_{x\to1^-}(f(x)) = \lim\limits_{x\to1}(-2x + 3) = -2 + 3 = 1$

$\lim\limits_{x\to1^+}(f(x)) = \lim\limits_{x\to1}(x^2 + 2x + 1) = 1 + 2 + 1$

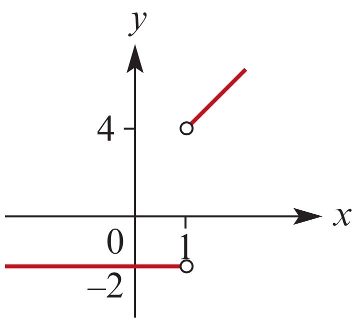
$= 4 \neq \lim\limits_{x\to1^-}(f(x))$

$\therefore$ $f(x)$is discontinous &

$\therefore$ not differentiable at $x = 1$

$\therefore$ $f'(x)$ is defined for $x \in R\backslash\{1\}$

$$f'(x) = \begin{cases} 2x + 2 & if\ x > 1 \\ -2 & if\ x < 0 \end{cases}$$



4 $x > -1,\ f'(x) = \dfrac{d}{dx}(-x^2 - 2x + 1)$

$= -2x - 2$

$x < -1,\ f'(x) = \dfrac{d}{dx}(-2x + 3)$

$= -2$

test $x = -1$

$\lim\limits_{x\to-1^-}(f(x)) = \lim\limits_{x\to-1}(-2x + 3) = 2 + 3 = 5$

$\lim\limits_{x\to-1^+}(f(x)) = \lim\limits_{x\to-1}(-x^2 - 2x + 1) = -1 + 2 + 1$

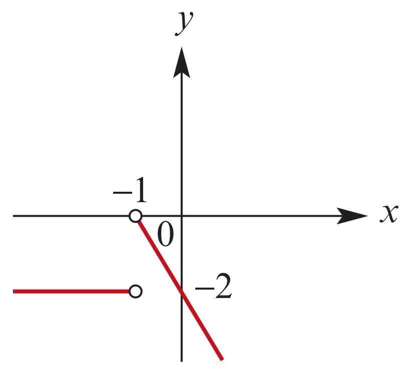
$= 2 \neq \lim\limits_{x\to-1^-}(f(x))$

$\therefore$ $f(x)$ is not continuous &

$\therefore$ not differentiable at$x = -1$

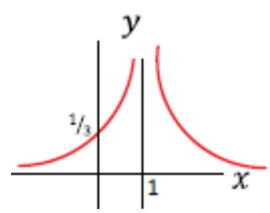
$\therefore$ $f'(x)$ is defined for$x \in R\backslash\{-1\}$

$$f'(x) = \begin{cases} -2x - 2 & if\ x > -1 \\ -2 & if\ x < -1 \end{cases}.$$



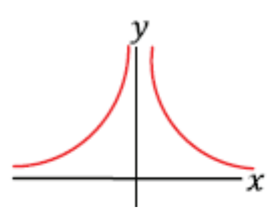
5 **a** $f'(x) = \dfrac{1}{3}(x - 1)^{-\frac{2}{3}}$

$f'(x)$ is defined for $x \in \mathbb{R}\backslash\{1\}$ (since $x = 1$ gives $f'(x) = \dfrac{1}{0}$)



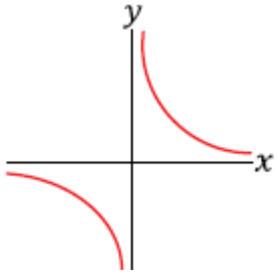
b $f'(x) = \dfrac{1}{5}x^{\frac{-4}{5}}$

$f'(x)$ is defined for $x \in \mathbb{R}\backslash\{0\}$ (*since* $x = 0$ gives $f'(x) = \dfrac{1}{0}$)



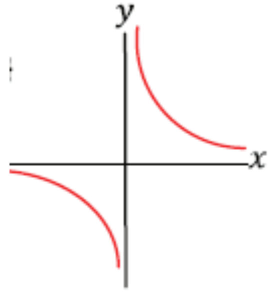
c $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$

$f'(x)$ is defined for $x \in \mathbb{R}\backslash\{0\}$



d $f'(x) = \frac{2}{5}(x+2)^{-\frac{3}{5}}$

$f'(x)$ is defined for $x \in \mathbb{R}\backslash\{-2\}$



Solutions to Technology-free questions

1 a Average rate of change $= \dfrac{26-10}{2}$

$= 8$

b $\dfrac{dy}{dx} = 2x$

When $x = -4$, $\dfrac{dy}{dx} = -8$

2 a
$$y = x + \sqrt{1-x^2} = x + (1-x^2)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = 1 + \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times (-2x)$$
$$= 1 - \frac{x}{\sqrt{1-x^2}}$$

b
$$y = \frac{4x+1}{x^2+3}$$
$$\frac{dy}{dx} = \frac{(x^2+3)(4) - (4x+1)(2x)}{(x^2+3)^2}$$
$$= \frac{12 - 2x - 4x^2}{(x^2+3)^2}$$

c
$$y = \sqrt{1+3x} = (1+3x)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2}(1+3x)^{-\frac{1}{2}} \times 3$$
$$= \frac{3}{2\sqrt{1+3x}}$$

d
$$y = \frac{2+\sqrt{x}}{x} = 2x^{-1} + x^{-\frac{1}{2}}$$
$$\frac{dy}{dx} = -2x^{-2} - \frac{1}{2}x^{-\frac{3}{2}}$$
$$= -\frac{2}{x^2} - \frac{1}{2x^{\frac{3}{2}}}$$

e
$$y = (x-9)\sqrt{x-3} = (x-9)(x-3)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = (1)(x-3)^{\frac{1}{2}} + (x-9) \times \frac{1}{2}(x-3)^{-\frac{1}{2}}$$
$$= \frac{2(x-3) + (x-9)}{2\sqrt{x-3}}$$
$$= \frac{3x-15}{2\sqrt{x-3}}$$

f
$$y = x\sqrt{1+x^2} = x(1+x^2)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = (1)(1+x^2)^{\frac{1}{2}} + x \times \frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x)$$
$$= \frac{1+2x^2}{\sqrt{1+x^2}}$$

g
$$y = \frac{x^2-1}{x^2+1}$$
$$\frac{dy}{dx} = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}$$
$$= \frac{4x}{(x^2+1)^2}$$

h
$$y = \frac{x}{x^2+1}$$
$$\frac{dy}{dx} = \frac{(x^2+1)(1) - (x)(2x)}{(x^2+1)^2}$$
$$= \frac{1-x^2}{(x^2+1)^2}$$

i
$$y = (2+5x^2)^{\frac{1}{3}}$$
$$\frac{dy}{dx} = \frac{1}{3}(2+5x^2)^{-\frac{2}{3}} \times 10x$$
$$= \frac{10x}{3}(2+5x^2)^{-\frac{2}{3}}$$

j $y = \dfrac{2x+1}{x^2+2}$

$$\frac{dy}{dx} = \frac{(x^2+2)(2)-(2x+1)(2x)}{(x^2+2)^2}$$

$$= \frac{4-2x-2x^2}{(x^2+2)^2}$$

k $y = (3x^2+2)^{\frac{2}{3}}$

$$\frac{dy}{dx} = \frac{2}{3}(3x+2)^{-\frac{1}{3}} \times 6x$$

$$= 4x(3x^2+2)^{-\frac{1}{3}}$$

3 a $y = 3x^2 - 4$

$$\frac{dy}{dx} = 6x$$

$$= -6(\text{ at } x = -1)$$

b $y = \dfrac{x-1}{x^2+1}$

$$\frac{dy}{dx} = \frac{(x^2+1)(1)-(x-1)(2x)}{(x^2+1)^2}$$

$$= \frac{1+2x-x^2}{(x^2+1)^2}$$

$$= 1(\text{at } x = 0)$$

c $y = (x-2)^5$

$$\frac{dy}{dx} = 5(x-2)^4$$

$$= 5 \text{ (at } x = 1)$$

d $y = (2x+2)^{\frac{2}{3}}$

$$\frac{dy}{dx} = \frac{1}{3}(2x+2)^{-\frac{1}{3}} \times 2$$

$$= \frac{2}{3}(2x+2)^{-\frac{2}{3}}$$

$$= \frac{2}{3}(8)^{-\frac{2}{3}} \text{ (at } x = 3)$$

$$= \frac{2}{3}(2^3)^{-\frac{2}{3}}$$

$$= \frac{2}{3} \times 2^{-2}$$

$$= \frac{1}{6}$$

4 a $y = \log_e(x+2)$

$$\frac{dy}{dx} = \frac{1}{x+2}$$

b $y = \sin(3x+2)$

$$\frac{dy}{dx} = 3\cos(3x+2)$$

c $y = \cos\left(\dfrac{x}{2}\right)$

$$\frac{dy}{dx} = -\frac{1}{2}\sin\left(\frac{x}{2}\right)$$

d $y = e^{x^2-2x}$

$$\frac{dy}{dx} = (2x-2)e^{x^2-2x}$$

e $y = \log_e(3-x)$

$$\frac{dy}{dx} = -\frac{1}{3-x} = \frac{1}{x-3}$$

f $y = \sin(2\pi x)$

$$\frac{dy}{dx} = 2\pi\cos(2\pi x)$$

g $y = \sin^2(3x + 1)$

$$\frac{dy}{dx} = 2\sin(3x + 1) \times 3\cos(3x + 1)$$
$$= 6\sin(3x + 1)\cos(3x + 1)$$
$$= 3\sin(6x + 2)$$

as $\sin(2a) = 2\sin(a)\cos(a)$

h $y = \sqrt{\log_e x} = (\log_e x)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(\log_e x)^{\frac{1}{2}} \times \frac{1}{x} = \frac{1}{2x\sqrt{\log_e x}}$$

i

$$y = \frac{2\log_e 2x}{x} = 2x^{-1}\log_e 2x$$
$$\frac{dy}{dx} = -2x^{-2}\log_e 2x + 2x^{-1} \times \frac{2}{2x}$$
$$= -\frac{2\log_e 2x}{x^2} + \frac{2}{x^2} = \frac{2 - 2\log_e 2x}{x^2}$$

j $y = x^2\sin(2\pi x)$

$$\frac{dy}{dx} = 2x\sin(2\pi x) + 2\pi x^2\cos(2\pi x)$$

5 a $y = e^x\sin 2x$

$$\frac{dy}{dx} = e^x\sin 2x + 2e^x\cos 2x$$

b $y = 2x^2\log_e x$

$$\frac{dy}{dx} = 4x\log_e x + 2x^2 \times \frac{1}{x}$$
$$= 4x\log_e x + 2x$$

c $y = \dfrac{\log_e x}{x^3} = x^{-3}\log_e x$

$$\frac{dy}{dx} = -3x^{-4}\log_e x + x^{-3} \times \frac{1}{x}$$
$$= \frac{1 - 3\log_e x}{x^4}$$

d

$$y = \sin 2x\cos 3x$$
$$\frac{dy}{dx} = (2\cos 2x)\cos 3x + \sin 2x(-3\sin 3x)$$
$$= 2\cos 2x\cos 3x - 3\sin 2x\sin 3x$$

e $y = \dfrac{\sin 2x}{\cos 2x} = \tan 2x$

$$\frac{dy}{dx} = 2\sec^2 2x$$

(Alternatively, use the quotient rule.)

f $y = \cos^3(3x + 2)$

$$\frac{dy}{dx} = 3\cos^2(3x + 2) \times -3\sin(3x + 2)$$
$$= -9\cos^2(3x + 2)\sin(3x + 2)$$

g

$$y = x^2\sin^2(3x)$$
$$\frac{dy}{dx} = 2x\sin^2(3x)$$
$$+ x^2(2\sin(3x) \times 3\cos(3x))$$
$$= 2x\sin^2(3x) + 6x^2\sin(3x)\cos(3x)$$
$$= 2x\sin^2(3x) + 3x^2\sin(6x)$$

as $\sin(2a) = 2\sin(a)\cos(a)$

6 a $y = e^{2x} + 1$

$$\frac{dy}{dx} = 2e^{2x}$$
$$= 2e^2 \text{ (at } x = 1)$$

b $y = e^{x^2+1}$

$$\frac{dy}{dx} = 2xe^{x^2+1}$$
$$= 0 \text{ (at } x = 0)$$

c $y = 5e^{3x} + x^2$

$$\frac{dy}{dx} = 15e^{3x} + 2x$$
$$= 15e^3 + 2 \text{ (at } x = 1)$$

d $y = 5 - e^{-x}$

$\dfrac{dy}{dx} = e^{-x}$

$= 1 \text{ (at } x = 0)$

7 a $y = e^{ax}$

$\dfrac{dy}{dx} = ae^{ax}$

b $y = e^{ax+b}$

$\dfrac{dy}{dx} = ae^{ax+b}$

c $y = e^{a-bx}$

$\dfrac{dy}{dx} = -be^{a-bx}$

d $y = be^{ax} - ae^{bx}$

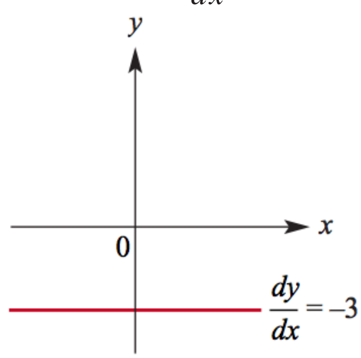
$\dfrac{dy}{dx} = abe^{ax} - abe^{bx}$

$= ab(e^{ax} - e^{bx})$

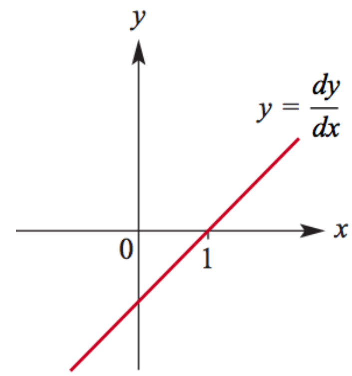
e $y = \dfrac{e^{ax}}{e^{bx}} = e^{ax-bx} = e^{(a-b)x}$

$\dfrac{dy}{dx} = (a-b)e^{(a-b)x}$

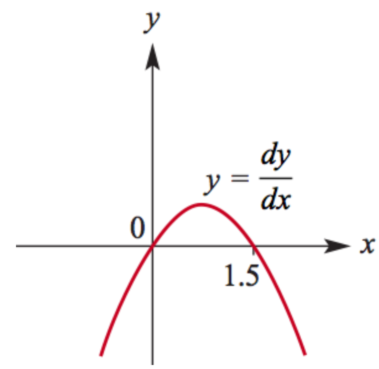
8 a $y = 3 - 3x$ so $\dfrac{dy}{dx} = -3$



b Graph looks parabolic, so derivative graph will be linear. Also, there is a turning point where $x = 1$, so the derivative function will be zero at $x = 1$.



c Graph looks cubic, so derivative graph will be quadratic. Also, there are turning points where $x = 0$ and $x = 1.5$, so derivative function will be zero at $x = 0, 1.5$. Finally, the gradient goes from negative to positive to negative, so the gradient graph will be an inverted parabola.



9 $y = \left(4x + \dfrac{9}{x}\right)^2$

$\dfrac{dy}{dx} = 2\left(4x + \dfrac{9}{x}\right)\left(4 - \dfrac{9}{x^2}\right)$

$= \dfrac{2(4x^2 + 9)(4x^2 - 9)}{x^3}$

Then $\dfrac{dy}{dx} = 0$ provided $4x^2 - 9 = 0$ (since $4x^2 + 9 > 0$ for all values of x).

Hence $x = \pm\dfrac{3}{2}$.

10 a $y = \dfrac{2x-3}{x^2+4}$

$$\frac{dy}{dx} = \frac{(x^2+4)(2)-(2x-3)(2x)}{(x^2+4)^2}$$

$$= \frac{2x^2+8-4x^2+6x}{(x^2+4)^2}$$

$$= \frac{8+6x-2x^2}{(x^2+4)^2}$$

b Note that $x^2+4>0$ for all values of x. So only check the numerators.

$y>0$ provided $2x-3>0$, i.e. $x>\dfrac{3}{2}$.

$\dfrac{dy}{dx}=0$ provided $8+6x-2x^2>0$, which is equivalent to $4+3x-x^2>0$.

$4+3x-x^2=(4-x)(1+x)>0$ provided $-1<x<4$ (since the corresponding quadratic graph is an inverted parabola with x- axis intercepts of -1 and 4).

So y and $\dfrac{dy}{dx}$ are both positive provided $x\in\left(\dfrac{3}{2},\ \infty\right)\cap(-1,4)$, i.e. $\left(\dfrac{3}{2},4\right)$.

11 a $y = xf(x)$

$$\frac{dy}{dx} = (x)(f'(x))+(1)(f(x))$$

$$= xf'(x)+f(x)$$

b $y = \dfrac{1}{f(x)}$

$$\frac{dy}{dx} = \frac{-f'(x)}{[f(x)]^2}$$

c $y = \dfrac{x}{f(x)}$

$$\frac{dy}{dx} = \frac{f(x)+xf'(x)}{[f(x)]^2}$$

d $y = \dfrac{x^2}{[f(x)]^2}$

$$\frac{dy}{dx} = \frac{[f(x)]^2(2x)-(x^2)(2f(x)f'(x))}{[f(x)]^2}$$

$$= \frac{[f(x)](2xf(x)-2x^2f'(x))}{[f(x)]^4}$$

$$= \frac{2xf(x)-2x^2f'(x)}{[f(x)]^3}$$

12 a $f\circ g(x) = 2\cos^3 x - 1$

b $g\circ f(x) = \cos(2x^3-1)$

c $g'\circ f(x) = -\sin(2x^3-1)$

d $(g\circ f)'(x) = -(6x^2)\sin(2x^3-1)$

e $\dfrac{3}{2}$

f $-\dfrac{3\sqrt{3}}{4}$

Solutions to multiple-choice questions

1 A

$$\text{Average rate of change} = \frac{e + 1 - (1)}{1} = e$$

2 B $f(x) = \dfrac{4x^4 - 12x^2}{3x}$

$f(x) = \dfrac{4x^3}{3} - 4x$

$\therefore\ f'(x) = 4x^2 - 4$

3 C $f : R\backslash\{7\} \to R, f(x) = 5 + \dfrac{5}{(7 - x)^2}$

$f(x) = 5 + 5(7 - x)^{-2}$

$f'(x) = 10(7 - x)^{-3}$

$f'(x) = \dfrac{10}{(7 - x)^3}$

$f'(x) > 0$

$\therefore\ (7 - x)^3 > 0$

$x < 7$

4 A $y = f(g(x))$

$g(x) = 2x^4$

$\therefore\ y = f(2x^4)$

$\dfrac{dy}{dx} = 8x^3 f'(2x^4)$

5 A $f(x) = x^{\frac{1}{3}}$

$f'(x) = \dfrac{1}{3x^{\frac{2}{3}}}$

As $3x^{\frac{2}{3}} \neq 0$, the gradient is undefined at this point.

6 B $y = \dfrac{k}{2(x^3 + 1)}$

$y' = \dfrac{-3kx^2}{2(x^3 + 1)^2}$

$\therefore\ 1 = \dfrac{-3kx^2}{2(x^3 + 1)^2}$

$1 = \dfrac{-3k}{8}$

$k = \dfrac{-8}{3}$

7 C The gradient is positive when:
$x < -3$ or $x > 2$

8 D $f(x) = 4x(2 - 3x)$

$f(x) = 8x - 12x^2$

$f'(x) = 8 - 24x$

$f'(x) < 0$

$8 - 24x < 0$

$x > \dfrac{8}{24}$

$x > \dfrac{1}{3}$

9 E $y = (3 - 2f(x))^{\frac{1}{2}}$

$\dfrac{dy}{dx} = \dfrac{1}{2} \times -2f'(x) \times (3 - 2f(x)^{\frac{-1}{2}})$

$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{3 - 2f(x)}}$

10 A $y = (x+3)(x-2)$

$y = x^2 + x - 6$

$\frac{dy}{dx} = 2x + 1$

When $\frac{dy}{dx} = -7$

$-8 = 2x$

$x = -4$

When $x = -4$

$y = (-4)^2 - 4 - 6$

$y = 6$

Coordinates $= (-4, 6)$

11 B $y = ax^2 - bx$

$\frac{dy}{dx} = 2ax - b$

When $\frac{dy}{dx} = 0, x = 2$

$0 = 4a - b$

$4a = b$

Sub into: $y = ax^2 - bx$

$y = ax^2 - 4ax$

$y = ax(x-4)$

Using null factor theorem:

$x = 0,\ x = 4$

12 C Derivative of $e^{-2ax}\cos(ax)$

Using Product rule

$u = e^{-2ax}$

$u' = -2ae^{-2ax}$

$v = \cos(ax)$

$v' = -a\sin(ax)$

$u'v + v'u$

$\therefore -2ae^{-2ax}\cos(ax) - ae^{-2ax}\sin(ax)$

13 B $f(x) = \frac{\cos(x)}{x-a}$

Using quotient rule

$u = \cos(x)$

$u' = -\sin(x)$

$v = x - a$

$v' = 1$

$\frac{u'v - v'u}{v^2}$

$f'(x) = \frac{-\sin(x)(x-a) - \cos(x)}{(x-a)^2}$

$f'(x) = -\frac{\sin(x)}{(x-a)} - \frac{\cos(x)}{(x-a)^2}$

Solutions to extended-response questions

1 $f(1) = 6,\ g(1) = -1,\ g(6) = 7$ and $f(-1) = 8$
$f'(1) = 6,\ g'(1) = -2, f'(-1) = 2$ and $g'(6) = -1$

a **i** $f \circ g)'(1) = g'(1)f'(g(1)) = -2 \times f'(-1) = -2 \times 2 = -4$

ii $(g \circ f)'(1) = f'(1)g'(f(1)) = 6 \times g'(6) = 6 \times (-1) = -6$

iii $(fg)'(1) = f'(1)g(1) + g'(1)f(1) = 6 \times (-1) + (-2) \times 6 = -18$

iv $(gf)'(1) = f'(1)g(1) + g'(1)f(1) = 6 \times (-1) + (-2) \times 6 = -18$

v $\left(\dfrac{f}{g}\right)'(1) = \dfrac{f'(1)g(1) - g'(1)f(1)}{[g(1)]^2} = \dfrac{6 \times -1 - (-2 \times 6)}{[-1]^2} = 6$

vi $\left(\dfrac{g}{f}\right)'(1) = \dfrac{g'(1)f(1) - f'(1)g(1)}{[f(1)]^2} = \dfrac{-2 \times 6 - (6 \times -1))}{[6]^2} = -\dfrac{1}{6}$

b For $f(x) = ax^3 + bx^2 + cx + d.$, $f(1) = 6$ and $f(-1) = 8$
Therefore

$a + b + c + d = 6 \qquad 1$

$-a + b - c + d = 8 \qquad 2$

Also $f'(x) = 3ax^2 + 2bx + c$, $f'(1) = 6$ and $f'(-1) = 2$
Therefore

$3a + 2b + c = 6 \qquad 3$

$3a - 2b + c = 2 \qquad 4$

Subtract 4 from 3 to give $4b = 4$ and $b = 1$
Add 1 and 2
$2b + 2d = 14$ and as $b = 1,\ d = 6$
From 1

$a + c = -1 \qquad 5$

and from 4

$3a + c = 4 \qquad 6$

Subtract 5 from 6
$2a = 5$ and therefore $a = \dfrac{5}{2}$ and $c = -\dfrac{7}{2}$

2 $f'(x) = 0$ for $x = 1$ and $x = 5$
$f'(x) > 0$ for $x > 5$ and $x < 1$
$f'(x) < 0$ for $1 < x < 5$
$f(1) = 6$ and $f(5) = 1$

a The graph of $y = f(x+2)$ is obtained from the graph of $y = f(x)$ by a translation of 2 units in the negative direction of the x-axis.

i Therefore $\dfrac{dy}{dx} = 0$ for $x = -1$ and $x = 3$

ii $\dfrac{dy}{dx} > 0$ for $x > 3$ and $x < -1$

b The graph of $y = f(x-2)$ is obtained from the graph of $y = f(x)$ by a translation of 2 units in the positive direction of the x-axis.

i Therefore $\dfrac{dy}{dx} = 0$ for $x = 3$ and $x = 7$.

ii The coordinates at which the gradient is zero are $(3, 6)$ and $(7, 1)$

c The graph of $y = f(2x)$ is obtained from the graph of $y = f(x)$ by a dilation of factor $\dfrac{1}{2}$ from the y-axis.

i Therefore $\dfrac{dy}{dx} = 0$ for $x = \dfrac{1}{2}$ and $\dfrac{5}{2}$

ii The coordinates at which the gradient is zero are $\left(\dfrac{1}{2}, 6\right)$ and $\left(\dfrac{5}{2}, 1\right)$

d The graph of $y = f\left(\dfrac{x}{2}\right)$ is obtained from the graph of $y = f(x)$ by a dilation of factor 2 from the y-axis.

i Therefore $\dfrac{dy}{dx} = 0$ for $x = 2$ and $x = 10$

ii The coordinates at which the gradient is zero are $(2, 6)$ and $(10, 1)$

e The graph of $y = 3f\left(\dfrac{x}{2}\right)$ is obtained from the graph of $y = f(x)$ by a dilation of factor 2 from the y-axis and factor 3 from the x-axis.

i Therefore $\dfrac{dy}{dx} = 0$ for $x = 2$ and $x = 10$

ii The coordinates at which the gradient is zero are $(2, 18)$ and $(10, 3)$

3 $f(x) = (x-\alpha)^n(x-\beta)^m$ where m and n are positive integers with $m > n$ and $\beta > \alpha$

a $f(x) = 0$ implies $x = \alpha$ or $x = \beta$

b Using the product rule

$$f'(x) = n(x-\alpha)^{n-1}(x-\beta)^m + m(x-\alpha)^n(x-\beta)^{m-1}$$
$$= (x-\alpha)^{n-1}(x-\beta)^{m-1}[n(x-\beta) + m(x-\alpha)]$$
$$= (x-\alpha)^{n-1}(x-\beta)^{m-1}[x(n+m) - (n\beta + m\alpha)]$$

c $f'(x) = 0$ implies $x = \alpha$ or $x = \beta$ or $x = \dfrac{n\beta + m\alpha}{n+m}$

d **i** If m and n are odd then $m-1$ and $n-1$ are even.
Therefore $(x-\alpha)^{n-1}(x-\beta)^{m-1} \geq 0$ for all x
and $f'(x) > 0$ for $x > \dfrac{n\beta + m\alpha}{n+m}$ and $x \neq \beta$

ii If m is odd then $m-1$ is even and $(x-\beta)^{m-1} \geq 0$ for all x
Therefore $f'(x) > 0$ if and only if $(x-\alpha)^{n-1}[x(n+m) - (n\beta + m\alpha)] > 0$
If n is even then $(x-\alpha)^{n-1} > 0$ if and only if $x - \alpha > 0$.
Together gives
$(x-\alpha)^{n-1}[x(n+m) - (n\beta + m\alpha)] > 0$ is equivalent to both factors positive or both factors negative.

If both are positive:
$x > \alpha$ and $x > \dfrac{n\beta + m\alpha}{n+m}$
and as $\beta > \alpha$, $\dfrac{n\beta + m\alpha}{n+m} > \alpha+$ and thus $x > \dfrac{n\beta + m\alpha}{n+m}$

If both are negative
$x < \alpha$ and $x < \dfrac{n\beta + m\alpha}{n+m}$ and hence $x < \alpha$

4 $f(x) = \dfrac{x^n}{1+x^n}$ where n is an even integer.

a $1 - \dfrac{1}{x^n + 1} = \dfrac{x^n + 1 - 1}{x^n + 1} = \dfrac{x^n}{1 + x^n}$

b $f(x) = \dfrac{nx^{n-1}}{(x^n + 1)^2}$

c $0 < \dfrac{1}{x^n + 1} \leq 1$ as n is even. Therefore $-1 \leq -\dfrac{1}{x^n + 1} < 0$ and $0 \leq 1 - \dfrac{1}{x^n + 1} < 1$

d $f'(x) = 0$ implies $\dfrac{n^{xn-1}}{(x^n + 1)^2} = 0$ implies $x = 0$

e $f'(x) > 0$ for $\dfrac{nx^{n-1}}{(x^n + 1)^2} > 0$ which implies $x > 0$

Chapter 10 – Applications of differentiation

Solutions to Exercise 10A

1 $y = x^2 - 1$

$\dfrac{dy}{dx} = 2x$

$x = 2,$

$\dfrac{dy}{dx} = 4$

tangent: $y = 4x + c$

$x = 2, y = 3$

$3 = 8 + c$

$c = -5$

$y = 4x - 5$

2 $y = x^2 + 3x - 1$

$x = 0, y = -1$

$\dfrac{dy}{dx} = 2x + 3$

$x = 0,$

$\dfrac{dy}{dx} = 3$

normal:

$grad = -\dfrac{-1}{4}$

$y = \dfrac{-1}{3}x + c$

$x = 0, y = -1$

$-1 = c$

$y = \dfrac{-x}{3} - 1$

3 $y = x^2 - 5x + 6$

$= (x - 3)(x - 2)$

$y = 0, x = 2, 3$

$\dfrac{dy}{dx} = 2x - 5$

When $x = 2$,

$\dfrac{dy}{dx} = 4 - 5$

$= -1$

Gradient of normal = 1

$y = x + c$

$= x + c$

$x = 2, y = 0$

$0 = 2 + c$

$c = -2$

$y = x - 2$

When $x = 3$,

$\dfrac{dy}{dx} = 6 - 5$

$= 1$

Gradient of normal = -1

$y = -x + c$

$= -x + c$

$x = 3, y = 0$

$0 = -3 + c$

$c = 3$

$y = 3 - x$

4 $y = (2x + 1)^9$

$\dfrac{dy}{dx} = 2 \times 9(2x + 1)^8$

$= 18(2x + 1)^8$

$x = 0,$

$\dfrac{dy}{dx} = 18(1)^8$

$= 18$

tangent:

$y = 18x + c$

$x = 0, y = 1$

$1 = c$

$y = 18x + 1$

normal:

$y = \dfrac{-1}{18}x + c$

$x = 0, y = 1$

$1 = c$

$y = \dfrac{-1}{18}x + 1$

5 $y = x^2 - 5$

$\dfrac{dy}{dx} = 2x$

$\dfrac{dy}{dx} = 3$

$3 = 2x$

$x = \dfrac{3}{2}$

$x = \dfrac{3}{2},$

$y = \left(\dfrac{3}{2}\right)^2 - 5$

$= \dfrac{9}{4} - 5$

$y = \dfrac{-11}{4}$

co-ords $= \left(\dfrac{3}{2}, \dfrac{-11}{4}\right)$

$y = 3x + c$

$x = \dfrac{3}{2}, y = \dfrac{-11}{4}$

$\dfrac{-11}{4} = 3 \times \dfrac{3}{2} + c$

$c = \dfrac{-11}{4} - \dfrac{9}{2}$

$= \dfrac{-11}{4} - \dfrac{18}{4}$

$c = \dfrac{-29}{4}$

6 **a** $y = x^2 - 2$

$x = 1, y = 1 - 2 = -1$

$\dfrac{dy}{dx} = 2x$

$x = 1, \dfrac{dy}{dx} = 2$

i tangent:

$$y = 2x + c$$
$$x = 1, y = -1$$
$$-1 = 2 + c$$
$$c = -3$$
$$y = 2x - 3$$

ii normal:

$$y = \frac{-1}{2}x + c$$
$$x = 1, y = -1$$
$$-1 = \frac{-1}{2} + c$$
$$c = \frac{-1}{2}$$
$$y = \frac{-1}{2}x - \frac{1}{2}$$

b $y = x^2 - 3x - 1$

$$x = 0, y = -1$$
$$\frac{dy}{dx} = 2x - 3$$
$$x = 0, \frac{dy}{dx} = -3$$

i tangent:

$$y = -3x + c$$
$$x = 0, y = -1$$
$$-1 = c$$
$$y = -3x - 1$$

ii normal:

$$y = \frac{-1}{-3}x + c$$
$$= \frac{1}{3}x + c$$
$$x = 0, y = -1$$
$$-1 = c$$
$$y = \frac{1}{3}x - 1$$

c $y = \frac{1}{x}$

$$x = -1, y = -1$$
$$\frac{dy}{dx} = \frac{-1}{x^2}$$
$$x = -1, \frac{dy}{dx} = -1$$

i tangent:

$$y = -x + c$$
$$x = -1, y = -1$$
$$-1 = 1 + c$$
$$c = -2$$
$$y = -x - 2$$

ii normal:

$$y = \frac{-1}{-1}x + c$$
$$= x + c$$
$$x = -1, y = -1$$
$$-1 = -1 + c$$
$$c = 0$$
$$y = x$$

d $y = (x-2)(x^2+1)$

$= x^3 - 2x^2 + x - 2$

$x = -1,$

$y = -1 - 2 - 1 - 2$

$= -6$

$\frac{dy}{dx} = 3x^2 - 4x + 1$

$x = -1,$

$\frac{dy}{dx} = 3 + 4 + 1$

$= 8$

i tangent:

$y = 8x + c$

$x = -1, y = -6$

$-6 = -8 + c$

$c = 2$

$y = 8x + 2$

ii normal:

$y = \frac{-1}{8}x + c$

$x = -1$

$y = -6$

$-6 = \frac{1}{8} + c$

$c = -6\frac{1}{8} = -\frac{-49}{8}$

$y = \frac{-1}{8}x - \frac{49}{8}$

e $y = \sqrt{3x+1}$

$x = 0, \quad y = \sqrt{1} = 1$

$\frac{dy}{dx} = 3 \times \frac{1}{2\sqrt{3x+1}}$

$= \frac{3}{2\sqrt{3x+1}}$

$x = 0,$

$\frac{dy}{dx} = \frac{3}{2\sqrt{1}}$

$\frac{dy}{dx} = \frac{3}{2}$

i tangent:

$y = \frac{3}{2}x + c$

$x = 0, y = 1$

$1 = c$

$y = \frac{3}{2}x + 1$

ii normal:

$y = \frac{-2}{3}x + c$

$x = 0, y = 1$

$1 = c$

$y = \frac{-2}{3}x + 1$

f $y = \sqrt{x}$

$x = 1, y = 1$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$x = 1,$

$\frac{dy}{dx} = \frac{1}{2}$

i tangent:

$$y = \frac{1}{2}x + c$$

$$x = 1, y = 1$$

$$1 = \frac{1}{2} + c$$

$$c = \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2}$$

ii normal:

$$y = -2x + c$$

$$x = 1, y = 1$$

$$1 = -2 + c$$

$$c = 3$$

$$y = -2x + 3$$

g $y = x^{\frac{2}{3}} + 1$

$$x = 1, y = 2$$

$$\frac{dy}{dx} = \frac{2}{3}x^{\frac{-1}{3}}$$

$$x = 1,$$

$$\frac{dy}{dx} = \frac{2}{3}$$

i tangent:

$$y = \frac{2}{3}x + c$$

$$x = 1, y = 2$$

$$2 = \frac{2}{3} + c$$

$$c = \frac{4}{3}$$

$$y = \frac{2x + 4}{3}$$

ii normal:

$$y = \frac{-3}{2}x + c$$

$$x = 1, y = 2$$

$$2 = \frac{-3}{2} + c$$

$$c = \frac{7}{2}$$

$$y = \frac{7 - 3x}{2}$$

h $y = x^3 - 8x$

$$x = 2,$$

$$y = 8 - 16$$

$$= -8$$

$$\frac{dy}{dx} = 3x^2 - 8$$

$$x = 2,$$

$$\frac{dy}{dx} = 12 - 8$$

$$= 4$$

i tangent:

$$y = 4x + c$$

$$x = 2, y = -8$$

$$-8 = 8 + c$$

$$c = -16$$

$$y = 4x - 16$$

ii normal:

$$y = \frac{-1}{4}x + c$$

$$x = 2, y = -8$$

$$-8 = \frac{-1}{2} + c$$

$$c = -7\frac{1}{2}$$

$$= \frac{-15}{2}$$

$$y = \frac{-x}{4} - \frac{15}{2}$$

i $y = x^3 - 3x^2 + 2$

$x = 2,$

$y = 8 - 3 \times 4 + 2$

$y = -2$

$$\frac{dy}{dx} = 3x^2 - 6x$$

$x = 2,$

$$\frac{dy}{dx} = 3 \times 4 - 6 \times 2$$

$$= 0$$

i tangent:

$y = c$

$x = 2, y = -2$

$c = -2,$

$y = -2$

ii normal:

$x = 2$

j $y = 2x^3 + x^2 - 4x + 1$

$x = 1,$

$y = 2 + 1 - 4 + 1$

$y = 0$

$$\frac{dy}{dx} = 6x^2 + 2x - 4$$

$x = 1,$

$$\frac{dy}{dx} = 6 + 2 - 4$$

$$= 4$$

i tangent:

$y = 4x + c$

$x = 1, y = 0$

$0 = 4 + c$

$c = -4$

$y = 4x - 4$

ii normal:

$$y = \frac{-1}{4}x + c$$

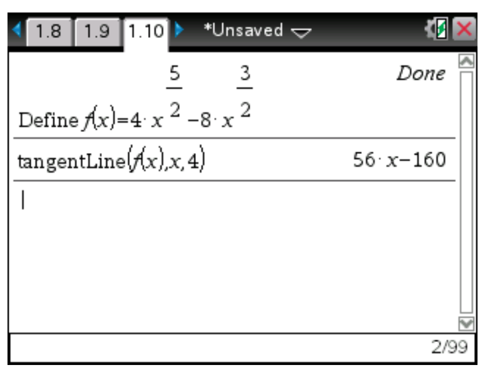
$$x = 1, y = 0$$

$$0 = \frac{-1}{4} + c$$

$$c = \frac{1}{4}$$

$$y = \frac{-1}{4}x + \frac{1}{4}$$

7 $y = 56x - 160$



8 **a** $y = \dfrac{x^2 - 1}{x^2 + 1}$

$x = 0, y = \dfrac{-1}{1} = -1$

$y = \dfrac{x^2 + 1 - 2}{x^2 + 1}$

$y = 1 - \dfrac{2}{x^2 + 1}$

$y = 1 - 2(x^2 + 1)^{-1}$

$\dfrac{dy}{dx} = 2x \times -1 \times -2(x^2 + 1)^{-2}$

$= \dfrac{4x}{(x^2 + 1)^2}$

$x = 0, \dfrac{dy}{dx} = 0$

tangent:

$y = 0 \times x + c$

$x = 0, y = -1$

$-1 = c$

$y = -1$

b $y = \sqrt{3x^2 + 1}$

$x = 1,$

$y = \sqrt{3 + 1}$

$= 2$

$\dfrac{dy}{dx} = 6x \times \dfrac{1}{2\sqrt{3x^2 + 1}}$

$= \dfrac{3x}{\sqrt{3x^2 + 1}}$

$x = 1,$

$\dfrac{dy}{dx} = \dfrac{3}{\sqrt{3 + 1}}$

$= \dfrac{3}{2}$

tangent:

$y = \dfrac{3}{2}x + c$

$x = 1, y = 2$

$2 = \dfrac{3}{2} + c$

$c = \dfrac{1}{2}$

$y = \dfrac{3}{2}x + \dfrac{1}{2}$

c $y = \dfrac{1}{2x - 1}$

$x = 0,$

$y = \dfrac{1}{-1}$

$y = -1$

$\dfrac{dy}{dx} = 2 \times -1 \times \dfrac{1}{(2x - 1)^2}$

$= \dfrac{-2}{(2x - 1)^2}$

$x = 0,$

$\dfrac{dy}{dx} = \dfrac{-2}{(-1)^2}$

$= -2$

tangent:

$y = -2x + c$

$x = 0, y = -1$

$c = -1$

$y = -2x - 1$

d $y = \frac{1}{(2x-1)^2}$

$x = 1,$

$y = \frac{1}{(1)^2}$

$y = 1$

$\frac{dy}{dx} = -2 \times 2 \times \frac{1}{(2x-1)^3}$

$= \frac{-4}{(2x-1)^3}$

$x = 1,$

$\frac{dy}{dx} = \frac{-4}{(1)^3}$

$= -4$

tangent:

$y = -4x + c$

$x = 1, y = 1$

$1 = -4 + c$

$c = 5$

$y = -4x + 5$

9 a $y = \sin 2x$

$\frac{dy}{dx} = 2\cos 2x$

$x = 0,$

$y = \sin 0 = 0$

$\frac{dy}{dx} = 2\cos 0 = 2$

$y = 2x + c$

$x = 0, \ y = 0$

$c = 0$

$y = 2x$

b $y = \cos 2x$

$\frac{dy}{dx} = -2\sin 2x$

$x = \frac{\pi}{2},$

$y = \cos\pi = -1$

$\frac{dy}{dx} = -\sin\pi = 0$

$y = -1$

c $y = \tan x$

$\frac{dy}{dx} = \sec^2 x$

$x = \frac{\pi}{4},$

$y = \tan\frac{\pi}{4} = 1$

$\frac{dy}{dx} = \sec^2\frac{\pi}{4} = \sqrt{2}^2 = 2$

$y = 2x + \mathrm{c}$

$x = \frac{\pi}{4}, \ y = 1$

$1 = \frac{\pi}{2} + c$

$c = 1 - \frac{\pi}{2}$

$y = 2x + 1 - \frac{\pi}{2}$

d $y = \tan 2x$

$\frac{dy}{dx} = 2\sec^2 2x$

$x = 0,$

$y = \tan 0 = 0$

$\frac{dy}{dx} = 2\sec^2 0 = 2$

$y = 2x + c$

$x = 0, \ y = 0$

$0 = c$

$y = 2x$

e $y = \sin x + x\sin 2x$

$\frac{dy}{dx} = \cos x + \sin 2x + 2x\cos 2x$

$x = 0,$

$y = \sin 0 + 0\sin 0 = 0$

$\frac{dy}{dx} = \cos 0 + 2\sin 0 + 0 = 1$

$y = x + c$

$x = 0, \quad y = 0$

$c = 0$

$y = x$

f $y = x - \tan x$

$\frac{dy}{dx} = 1 - \sec^2 x$

$x = \frac{\pi}{4},$

$y = \frac{\pi}{4} - \tan\frac{\pi}{4} = \frac{\pi}{4} - 1$

$\frac{dy}{dx} = 1 - \sec^2\frac{\pi}{4} = 1 - 2 = -1$

$y = -x + c$

$x = \frac{\pi}{4}, \ y = \frac{\pi}{4} - 1$

$\frac{\pi}{4} - 1 = \frac{-\pi}{4} + c$

$c = \frac{\pi}{2} - 1$

$y = -x + \frac{\pi}{2} - 1$

10 a $f(x) = e^x + e^{-x}$

$f'(x) = e^x - e^{-x}$

$f'(0) = 1 - 1 = 0$

$y = c$

$f(0) = 1 + 1 = 2$

$2 = c$

$y = 2$

b $f(x) = \frac{e^x - e^{-x}}{2}$

$f'(x) = \frac{e^x + e^{-x}}{2}$

$f'(0) = \frac{1+1}{2} = 1$

$y = x + c$

$f(0) = \frac{1-1}{2} = 0$

$0 = c$

$y = x$

c $f(x) = x^2e^{2x}$

$f'(x) = 2xe^{2x} + 2x^2e^{2x}$

$= 2xe^{2x}(x^2 + x)$

$f'(1) = 2e^2(1+1)$

$= 4e^2$

$y = 4e^2x + c$

$f(1) = 1 \times e^2 = e^2$

$e^2 = 4e^2 + c$

$c = -3e^2$

$y = 4e^2x - 3e^2$

d $f(x) = e^{\sqrt{x}}$

$f'(x) = \frac{1}{2\sqrt{x}}e^{\sqrt{x}}$

$f'(1) = \frac{e}{2}$

$y = \frac{e}{2}x + c$

$f(1) = e^1 = e$

$e = \frac{e}{2} + c$

$c = \frac{e}{2}$

$y = \frac{e}{2}(x + 1)$

e $f(x) = xe^{x^2}$

$$f'(x) = e^{x^2} + 2x^2e^{x^2}$$
$$= e^{x^2}(2x^2 + 1)$$
$$f'(1) = e^1(2 + 1)$$
$$= 3e$$
$$y = 3ex + c$$
$$f'(1) = 1 \times e^1 = e$$
$$e = 3e + c$$
$$c = -2e$$
$$y = 3ex - 2e$$

f $f(x) = x^2e^{-x}$

$$f'(x) = 2xe^{-x} - x^2e^{-x}$$
$$= e^{-x}(2x - x^2)$$
$$f'(2) = e^{-2}(4 - 4) = 0$$
$$y = c$$
$$f(2) = 2^2e^{-2} = \frac{4}{e^2}$$
$$\frac{4}{e^2} = c$$
$$y = \frac{4}{e^2}$$

11 a $f(x) = \ln x$

$$f'(x) = \frac{1}{x}$$
$$f'(1) = 1$$
$$y = x + c$$
$$(1, 0) \Rightarrow 0 = 1 + c$$
$$c = -1$$
$$y = x - 1$$

For the normal the gradient is -1
The equation of the normal is
$y = x + 1$

b $f(x) = \ln(2x)$

$$f'(x) = \frac{2}{2x} = \frac{1}{x}$$
$$f'\left(\frac{1}{2}\right) = 2$$
$$y = 2x + c$$
$$\left(\frac{1}{2}, 0\right) \Rightarrow 0 = 1 + c$$
$$c = -1$$
$$y = 2x - 1$$

c $f(x) = \ln(kx)$

$$f'(x) = \frac{k}{kx} = \frac{1}{x}$$
$$f\left(\frac{1}{k}\right) = k$$
$$y = kx + c$$
$$\left(\frac{1}{k}, 0\right) \Rightarrow 0 = 1 + c$$
$$c = -1$$
$$y = kx - 1$$

12 a $y = x^{\frac{1}{5}}$

$$\frac{dy}{dx} = \frac{1}{5}x^{-\frac{4}{5}}$$

When $x = 0, y = 0, \dfrac{dy}{dx}$ not defined.

Therefore equation of tangent

$$x = 0$$

b $y = x^{\frac{3}{5}}$

$$\frac{dy}{dx} = \frac{3}{5}x^{-\frac{2}{5}}$$

When $x = 0, y = 0, \dfrac{dy}{dx}$ not defined.

Therefore equation of tangent

$$x = 0$$

c $y = (x-4)^{\frac{1}{3}}$

$y = 0 \Rightarrow x = 4$

$\frac{dy}{dx} = \frac{1}{3}(x-4)^{-\frac{2}{3}}$

$x = 4,$

$\frac{dy}{dx}$ is undefined

$\therefore$ tangent is $x = 4$

d $y = (x+5)^{\frac{2}{3}}$

$y = 0 \Rightarrow x = -5$

$\frac{dy}{dx} = \frac{2}{3}(x+5)^{-\frac{1}{3}}$

$x = -5,$

$\frac{dy}{dx}$ is undefined

$\therefore$ tangent is $x = -5$

e $y = (2x+1)^{\frac{1}{3}}$

$y = 0 \Rightarrow x = -\frac{1}{2}$

$\frac{dy}{dx} = \frac{2}{3}(2x+1)^{-\frac{2}{3}}$

$x = -\frac{-1}{2},$

$\frac{dy}{dx}$ is undefined

$\therefore$ tangent is $x = -\frac{1}{2}$

f $y = (x+5)^{\frac{4}{5}}$

$y = 0 \Rightarrow x = -5$

$\frac{dy}{dx} = \frac{4}{5}(x+5)^{-\frac{1}{5}}$

$x = -5,$

$\frac{dy}{dx}$ is undefined

$\therefore$ tangent is $x = -5$

13 $y = \tan 2x$

$\frac{dy}{dx} = 2\sec^2 2x$

$x = \frac{\pi}{8},$

$y = \tan\frac{\pi}{4} = 1$

$\frac{dy}{dx} = 2\sec^2\frac{\pi}{4} = 4$

$y = 4x + c$

$x = \frac{\pi}{8},\ y = 1$

$1 = \frac{\pi}{2} + c$

$c = 1 - \frac{\pi}{2}$

$y = 4x + 1 - \frac{\pi}{2}$

$x = 0,$

$y = 1 - \frac{\pi}{2}$

$A = \left(0, 1 - \frac{\pi}{2}\right)$

$OA = \frac{\pi}{2} - 1$

14 $y = 2e^x$

$\frac{dy}{dx} = 2e^x$

$\therefore \frac{dy}{dx} = 2e^a$ when $x = a$

Gradient of the line segment joining $(a, 2e^a)$ and the origin is $\frac{2e^a}{a}$

$\therefore \frac{2e^a}{a} = 2e^a$

$\therefore a = 1$

15 $y = \log_e x$

$\dfrac{dy}{dx} = \dfrac{1}{x}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{a}$ when $x = a$

Gradient of the line segment joining $(a, \log_e a)$ and the origin is $\dfrac{\log_e a}{a}$

$\therefore \dfrac{\log_e a}{a} = \dfrac{1}{a}$

$\therefore \log_e a = 1 \therefore a = e$

16 $y = x^2 + 2x$

$\dfrac{dy}{dx} = 2x + 2$

$\therefore \dfrac{dy}{dx} = 2a + 2$ when $x = a$

Gradient of the line segment joining $(a, a^2 + 2a)$ and the origin is $\dfrac{a^2 + 2a}{a}$

$\therefore \dfrac{a^2 + 2a}{a} = 2a + 2$

$\therefore a^2 + 2a = 2a^2 + 2a$

$\therefore a = 0$

17 $y = x^3 + x$

$\dfrac{dy}{dx} = 3x^2 + 1$

$\therefore \dfrac{dy}{dx} = 3a^2 + 1$ when $x = a$

Gradient of the line segment joining $(a, a^3 + a)$ and the point $(1, 1)$ is

$\dfrac{a^3 + a - 1}{a - 1}$

$\therefore 3a^2 + 1 = \dfrac{a^3 + a - 1}{a - 1}$

$\therefore (3a^2 + 1)(a - 1) = a^3 + a - 1$

$\therefore 3a^3 + a - 3a^2 - 1 = a^3 + a - 1$

$\therefore 2a^3 - 3a^2 = 0$

$\therefore a^2(2a - 3) = 0$

$\therefore a = 0$ or $a = \dfrac{3}{2}$

Solutions to Exercise 10B

1 a

$$\text{Average rate of change} = \frac{f(3)-f(2)}{3-2} = \frac{45-24}{1} = 21$$

b Average rate of change

$$= \frac{f(2+h)-f(2)}{2+h-2}$$
$$= \frac{3(2+h)^2+6(2+h)-24}{h}$$
$$= \frac{3(4+4h+h^2)+6(2+h)-24}{h}$$
$$= \frac{18h+3h^2)}{h}$$
$$= 18+3h$$

c $f'(x) = 6x+6$

$f'(2) = 18$

2 a $\frac{dV}{dt}$

b $\frac{dS}{dr}$

c $\frac{dV}{dx}$

d $\frac{dA}{dt}$

e $\frac{dV}{dh}$

3 a

$$I = \frac{4}{(t+1)^2}$$
$$\frac{dI}{dt} = \frac{-8}{(t+1)^3}$$
$$t = 10,$$
$$\frac{dI}{dt} = \frac{-8}{11^3} = \frac{-8}{1331}$$

i.e. I wanes by ≈ 0.006 units/day

4 $V(t) = 1000(90-t)^3$

a $V'(t) = -3000(90-t)^2$

it empties at $3000(90-t)^2$ m^3/day

b

$$V(t) = 0,$$
$$1000(90-t)^3 = 0$$
$$t = 90 \text{ days}$$

c

$$V(0) = 1000(90)^3 = 729\,000\,000 \text{ m}^3$$

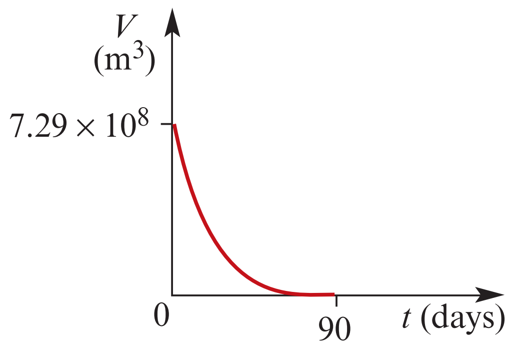
d

$$V'(t) = -300\,000$$
$$-300\,000 = -3000(90-t)^2$$
$$(90-t)^2 = 100$$
$$90-t = \pm 10$$
$$t = 90 \pm 10$$

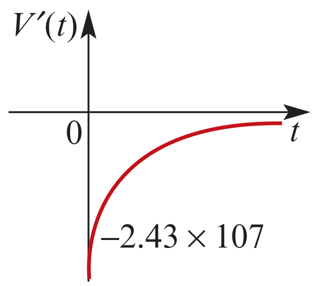
since $t \in [0, 90]$

$$t = 80\text{th day}$$

e



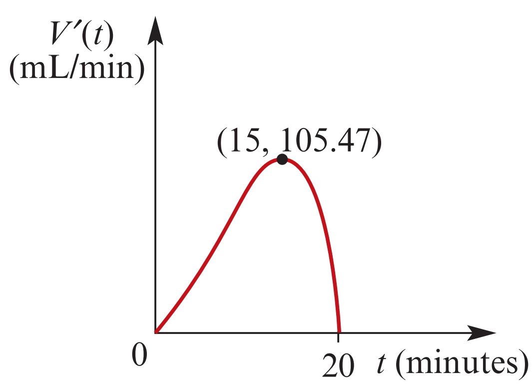
f



5 $V(t) = \frac{1}{160}\left(5t^4 - \frac{t^5}{5}\right), 0 \le t \le 20$

a $V'(t) = \frac{1}{160}(20t^3 - t^4)$ ml/min

b



c

$$\frac{d}{dt} = \frac{1}{160}(60t^2 - 4t^3)$$
$$= \frac{1}{40}(15t^2 - t^3)$$
$$\frac{d}{dt} = 0,$$
$$15t^2 - t^3 = 0$$
$$t^2(15 - t) = 0$$
$$t = 0, 15$$

max flow occurs at
$t = 15$
(using graph to determine max. or min. status)

6 **a** $t \approx 100, 250, 500$
(read off graph-turning points)

b draw tangent at $t = 200$,
use $\frac{rise}{run}$ to find gradient
$\frac{dV}{dt} \cong 430\,000\ \text{m}^3/\text{day}$
(be careful *re*: vertical scale)

c $t = 100, V \cong 4 \times 10^7$
$t = 250, V \cong 8 \times 10^7$

$$\frac{rise}{run} \approx \frac{4 \times 10^7}{150} \approx 270\,000\ \text{m}^3/\text{day}$$

d $100 < t < 250$ or $t > 500$

7 a

$$P = P_0e^{-kt}$$

When $t = 0, P = 30$

$\therefore\quad 30 = P_0e^0$

$\therefore\quad P_0 = 30$

When $t = 8, P = 10$

$\therefore\quad 10 = 30e^{-8k}$

$\therefore\quad \frac{1}{3} = e^{-8k}$

$\therefore\quad \log_e\left(\frac{1}{3}\right) = -8k$

and $\quad k = -\frac{1}{8}\log_e\left(\frac{1}{3}\right)$

$\quad = \frac{1}{8}\log_e(3) \approx 0.1373$

b When $P = 8, 8 = 30\, e^{\left(-\frac{1}{8}\log_e(3)\right)t}$

$\therefore\quad \frac{4}{15} = e^{\left(\log_e(3)^{\frac{1}{8}}\right)t}$

$\therefore\quad \frac{4}{15} = 3^{-\frac{t}{8}}$

$\therefore\quad \frac{15}{4} = 3^{\frac{t}{8}}$

$$\therefore\quad t = \frac{8\log_e\left(\frac{15}{4}\right)}{\log_e(3)} \approx 9.625$$

The pressure would be 8 units after approximately 9.625 hours.

c i $\quad \frac{dP}{dt} = -30ke^{-kt}$

where $k = \frac{1}{8}\log_e 3$

When $t = 0, \frac{dP}{dt} = -30ke^0$

$\quad = -30k$

$\quad = -\frac{30}{8}\log_e 3$

$\quad = -\frac{15}{4}\log_e 3$

The rate of loss is $\frac{15}{4}\log_e 3 \approx 4.120$ units per hour when $t = 0$.

ii When $t = 8, \frac{dP}{dt} = -30ke^{-8k}$

$\quad = -\frac{15}{4} \times \log_e 3 \times e^{-\log_e 3}$

$\quad = -\frac{15}{4} \times \log_e 3 \times \frac{1}{3}$

$\quad = -\frac{5}{4}\log_e 3$

This rate of loss is $\frac{5}{4}\log_e 3 \approx 1.373$ units per hour when $t = 8$.

8 a $\quad \frac{dT}{dt} = -\frac{45}{2}e^{-0.3t}$

Also, $e^{-0.3t} = \frac{1}{75}(T - 15)$

$\therefore\quad \frac{dT}{dt} = -\frac{45}{150}(T - 15)$

$\quad = -0.3(T - 15)$

b i When $T = 90, t = 0$

$\therefore\quad \frac{dT}{dt} = -\frac{45}{2}$

ii When $T = 60$

$\frac{dT}{dt} = -0.3(60 - 15) = -13.5$

iii When $T = 30$

$\frac{dT}{dt} = -0.3(30 - 15) = -4.5$

9

$$y = 3x + 2\cos x$$

$$\frac{dy}{dx} = 3 - 2\sin x$$

$$-1 \leq \sin 2x \leq 1$$

$$-2 \leq -2\sin 2x \leq 2$$

$$-1 \leq 3 - 2\sin 2x \leq 5 \therefore \frac{dy}{dx} > 0 \quad QED$$

10 $V(t) = 3 + 2\sin\frac{t}{4}$

a $V(10) = 3 + 2\sin\left(\frac{5}{2}\right)$

≈ 4.197

b $V'(t) = \frac{1}{2}\cos\frac{t}{4}$

$V'(10) = \frac{1}{2}\cos\left(\frac{5}{2}\right)$

≈ -0.4

11 $x = 2t^3 - 9t^2 + 12t$

a $V = \frac{dx}{dt} = 6t^2 - 18t + 12$

b $V = 0$

$\Rightarrow t^2 - 3t + 2 = 0$

$(t - 2)(t - 1) = 0$

$t = 1, 2$

$t = 1,$

$x = 2 - 9 + 12$

$t = 1, x = 5\,\text{cm}$

$t = 2, x = 16 - 36 + 24 = 4\,\text{cm}$

c $a = \frac{dV}{dt} = 12t - 18$

$t = 1, a = -6\,\text{cm/s}^2$

$t = 2, a = 6\,\text{cm/s}^2$

d $a = 0$

$12t = 18$

$t = \frac{3}{2}$

$V = 6 \times \frac{9}{4} - 18 \times \frac{3}{2} + 12$

$= \frac{27}{2} - 27 + 12$

$= \frac{24}{2} - \frac{27}{2}$

$V = \frac{-3}{2}\,\text{cm/s}$

12 $x = 8 + 2t - t^2$

a $t = 0,$

$x = 8\,\text{cm}$

b $V = 2 - 2t$

$t = 0,$

$v = 2\,\text{cm/s}$

c $V = 0$

$t = 1s$

$x = 8 + 2 - 1 = 9\,\text{cm}$

d $a = -2\,\text{cm/s}^2$

13 $x = \sqrt{2t^2 + 2}$

a $V = 4t \times \dfrac{1}{\sqrt{2t^2 + 2}}$

$= \dfrac{\sqrt{2}t}{\sqrt{2t^2 + 2}}$

$V = \dfrac{\sqrt{2}t}{\sqrt{t^2 + 1}}$

b

$$a = \frac{\sqrt{t^2+1}\frac{d}{dt}(\sqrt{2}t) - \sqrt{2}t\frac{d}{dt}(\sqrt{t^2+1})}{t^2+1}$$

$$a = \frac{\sqrt{2t^2+2} - \sqrt{2}t \times \frac{2t}{2\sqrt{t^2+1}}}{t^2+1}$$

$$= \frac{\sqrt{2}(t^2+1) - \sqrt{2}t^2}{(t^2+1)^{\frac{3}{2}}}$$

$$= \frac{\sqrt{2}}{(t^2+1)^{\frac{3}{2}}}$$

c $t = 1$

$V = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$ cm/s

$a = \dfrac{\sqrt{2}}{(2)^{\frac{3}{2}}} = \dfrac{1}{2}$ cm/s^2

14 $x = 0.4e^t$

$v = 0.4e^t$

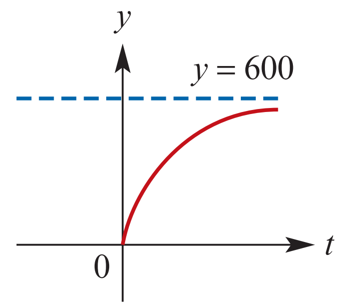
$t = 0,\ v = 0.4$ m/s

$t = 1,\ v = 0.4e \approx 1.087$ m/s

$t = 2,\ v = 0.4e^2 \approx 2.956$ m/s

15 $y = 600(1 - e^{-0.5t})$

a



b $\dfrac{dy}{dx} = 600(0.5e^{-0.5t})$

$= 300e^{-0.5t}$

$t = 10,$

$\dfrac{dy}{dx} = 300e^{-5} \approx 2.02$

16 a $y = e^{-2x}$

$-2x = \ln y$

$x = \dfrac{-1}{2}\ln y$

$\dfrac{dy}{dx} = \dfrac{-1}{2y}$

$\dfrac{dy}{dx} = -2y$

b $y = Ae^{kx}$

$\dfrac{dy}{dx} = Ake^{kx}$

$= k(Ae^{kx})$

$= ky$

17 $m = 2e^{-0.2t}$

a $t = 12,$

$m = 2e^{-2.4}$

$\approx 0.18\ kg$

b $t = 0,$

$m = 2$

$m - 1,$

$1 = 2e^{-0.2t}$

$-0.2t = \ln\frac{1}{2}$

$0.2t = \ln 2$

$t = 5\ln 2 \approx 3.47$ *hours*

c **i** $e^{-0.2t} = \frac{1}{4}$

$0.2t = \ln 4$

$t = 10\ln 2 \approx 6.93$ *hours*

ii $e^{-0.2t} = \frac{1}{8}$

$0.2t = \ln 8$

$t = 15\ln 2 \approx 10.4$ *hours*

d
$$\frac{dm}{dt} = -\frac{2}{5}e^{\frac{-t}{5}}$$
$$= -\frac{1}{5}(2e^{\frac{-t}{5}})$$
$$= -\frac{1}{5}\text{ m/hr}$$

Rate of decay $= \frac{1}{5}$ m

Solutions to Exercise 10C

1 a

$$\begin{aligned} f(x) &= x^3 - 12x \\ f'(x) &= 3x^2 - 12 \\ f'(x) &= 0 \\ 3x^2 - 12 &= 0 \\ x^2 - 4 &= 0 \\ x &= \pm 2 \\ f(\pm 2) &= \pm 8 \mp 24 \\ &= \mp 16 \\ \text{co-ords} &= (-2, 16), (2, -16) \end{aligned}$$

b

$$\begin{aligned} g(x) &= 2x^2 - 4x \\ g'(x) &= 4x - 4 \\ g'(x) &= 0, \\ 4x - 4 &= 0 \\ x &= 1, \\ g(1) &= 2 - 4 \\ &= -2 \\ \text{co-ords} &= (1, -2) \end{aligned}$$

c

$$\begin{aligned} h(x) &= 5x^4 - 4x^5 \\ h'(x) &= 20x^3 - 20x^4 \\ h'(x) &= 0, \\ 20x^3 - 20x^4 &= 0 \\ x^3(1 - x) &= 0 \\ x &= 0, 1 \\ h(0) &= 0, \\ h(1) &= 1 \\ \text{co-ords} &= (0, 0), (1, 1) \end{aligned}$$

d

$$\begin{aligned} f(t) &= 8t + 5t^2 - t^3, t > 0 \\ f'(t) &= 8 + 10t - 3t^2 \\ f'(t) &= 0, \\ 3t^2 + 10t + 8 &= 0 \\ t &= \frac{10 \pm \sqrt{100 + 16}}{6} \\ &= \frac{10 \pm 14}{6} \\ &= \frac{-2}{3}, 4 \\ t > 0, \therefore\ t &= 4 \\ f(4) &= 32 + 80 - 64 = 48 \\ \text{co-ords} &= (4, 48) \end{aligned}$$

e

$$\begin{aligned} g(z) &= 8z^2 - 3z^4 \\ g'(z) &= 16z - 12z^3 \\ g'(z) &= 0, \\ 16z - 12z^3 &= 0 \\ (3z^2 - 4)z &= 0 \\ z &= 0, \frac{\pm 2}{\sqrt{3}} \\ g(0) &= 0, \\ g\left(\frac{\pm 2}{\sqrt{3}}\right) &= 8 \times \frac{4}{3} - 3 \times \frac{16}{9} \\ &= \frac{32}{3} - \frac{16}{3} \\ &= \frac{16}{3} \\ \text{co-ords} &= \left(\frac{\pm 2}{\sqrt{3}}, \frac{16}{3}\right), (0, 0) \end{aligned}$$

f $f(x) = 5 - 2x + 3x^2$

$f'(x) = -2 + 6x$

$f'(x) = 0,$

$6x - 2$

$x = \frac{1}{3}$

$f\left(\frac{1}{3}\right) = 5 - \frac{2}{3} + \frac{1}{3}$

$= 4\frac{2}{3} = \frac{14}{3}$

co-ords $= \left(\frac{1}{3}, \frac{14}{3}\right)$

g $h(x) = x^3 - 4x^2 - 3x + 20,$

$x > 0$

$h'(x) = 3x^2 - 8x - 3$

$h'(x) = 0$

$3x^2 - 8x - 3 = 0$

$x = \frac{8 \pm \sqrt{64 + 36}}{6}$

$= \frac{-2}{6}, \frac{18}{6}$

$x > 0, \therefore x = \frac{18}{6} = 3$

$h(3) = 27 - 36 - 9 + 20$

$= 2$

co-ords $= (3, 2)$

h

$f(x) = 3x^4 - 16x^3 + 24x^2 - 10$

$f'(x) = 12x^3 - 48x^2 + 48x$

$f'(x) = 0,$

$x(x^2 - 4x + 4) = 0$

$x(x - 2)^2 = 0$

$x = 0, 2$

$f(0) = -10$

$f(2) = 3 \times 16 - 16 \times 8 + 24 \times 4 - 10$

$= 48 - 128 + 96 - 10$

$= -80 + 86$

$= 6$

co-ords $= (0, -10), (2, 6)$

2 a $f(x) = e^{2x} - 2x$

$f'(x) = 2e^{2x} - 2$

$f'(x) = 0 \Rightarrow e^{2x} = 1$

$\Rightarrow x = 0$

Coordinates of stationary point: (0,1)

b $f(x) = x \log_e(3x)$

$f'(x) = \log_e(3x) + 1$

$f'(x) = 0 \Rightarrow \log_e(3x) = -1$

$\Rightarrow x = \frac{1}{3e}$

Coordinates of stationary point: $(\frac{1}{3e}, -\frac{1}{3e})$

c $f(x) = \cos(2x), x \in [-\pi, \pi]$

$f'(x) = -2\sin(2x)$

$f'(x) = 0 \Rightarrow \sin(2x) = 0$

$\Rightarrow 2x = -2\pi, -\pi, 0, \pi, 2\pi$

$\Rightarrow x = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$

Coordinates of stationary point :

$(-\pi, 1), (-\frac{\pi}{2}, -1), (0, 1), (\frac{\pi}{2}, -1), (\pi, 1)$

d $f(x) = xe^x$

$f'(x) = e^x + xe^x = e^x(1 + x)$

$f'(x) = 0 \Rightarrow x = -1$

Coordinates of stationary point: $(-1, -\frac{1}{e})$

e

$$f(x) = x^2e^x$$

$$f'(x) = x^2e^x + 2xe^x = xe^x(2 + x)$$

$f'(x) = 0 \Rightarrow x = -2, 0$

Coordinates of stationary point:

$(-2, \frac{4}{e^2}), (0, 0)$

f

$$f(x) = 2x\log_e(x)$$

$$f'(x) = 2\log_e(3x) + 2$$

$f'(x) = 0 \Rightarrow \log_e(x) = -1$

$$\Rightarrow x = \frac{1}{e}$$

Coordinates of stationary point:$(\frac{1}{e}, -\frac{2}{e})$

3 **a** $f(x) = x^2 - ax + 9$

$f'(x) = 2x - a$

$f'(3) = 0,$

$6 - a = 0$

$a = 6$

b

$$h(x) = x^3 - bx^2 - 9x + 7$$

$$h'(x) = 3x^2 - 2bx - 9$$

$$h'(-1) = 0,$$

$$3 + 2b - 9 = 0$$

$$2b = 6$$

$$b = 3$$

4

$$y = x^3 + bx^2 + cx + d$$

$$\frac{dy}{dx} = 3x^2 + 2bx + c$$

When $x = 0, y = 3$

$\therefore d = 3$

When $x = 1, y = 3$

$\therefore 1 + b + c + 3 = 3$

$\therefore b + c = -1 \ldots (1)$

When $x = 1, \frac{dy}{dx} = 0$

$\therefore 2b + c = -3 \ldots (2)$

Subtract (1) from (2)

$b = -2$

$\therefore c = 1$

5 $y = ax^2 + bx + c$

$x = 1, y = -3$

(1) $-3 = a + b + c$

$$\frac{dy}{dx} = 2ax + b$$

$x = 2, \frac{dy}{dx} = 4$

(2) $4 = 4a + b$

$x = 1, \frac{dy}{dx} = 0$

(3) $0 = 2a + b$

(2) − (3) $\Rightarrow$ $4 = 2a$

$a = 2$

sub in (3) $\Rightarrow$ $b + 4 = 0$

$b = -4$

sub in (1) $\Rightarrow$ $-3 = 2 - 4 + c$

$c = -1$

$y = 2x^2 - 4x - 1$

6
$$y = ax^3 + bx^2 + cx + d$$
$$x = 0, y = 7\tfrac{1}{2}$$
$$d = \frac{15}{2}$$
$$x = 3, y = -6$$
$$-6 = 27a + 9b + 3c + \frac{15}{2}$$
$$-\frac{27}{2} = 27a + 9b + 3c$$
$$-\frac{9}{2} = 9a + 3b + c \ldots (1)$$
$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$
$$x = 0, \frac{dy}{dx} = -3$$
$$-3 = c$$
$$\text{sub in (1)} \Rightarrow -\frac{9}{2} = 9a + 3b - 3$$
$$-\frac{3}{2} = 9a + 3b \ldots (2)$$
$$x = 3, \frac{dy}{dx} = 0$$
$$0 = 27a + 6b - 3$$
$$9a + 2b = 1 \ldots (3)$$
$$(2) - (3) \Rightarrow b = \frac{-5}{2}$$
$$\text{sub in (3)} \Rightarrow 9a - 5 = 1$$
$$9a = 6$$
$$a = \frac{2}{3}$$
$$y = \frac{2x^2}{3} - \frac{5x^2}{2} - 3x + \frac{15}{2}$$

7 $y = ax + b(2x - 1)^{-1}$

a $x = 2, y = 7$
$$1 \quad 7 = 2a + \frac{b}{3}$$
$$\frac{dy}{dx} = a - \frac{2b}{(2x-1)^2}$$
$$x = 2, \frac{dy}{dx} = 0$$
$$0 = a - \frac{2b}{9}$$
$$2 \quad a = \frac{2b}{9}$$
$$\text{sub in 1} \Rightarrow 7 = \frac{4b}{9} + \frac{b}{3}$$
$$7 = \frac{7b}{9}$$
$$b = 9$$
$$\text{sub in 2} \Rightarrow a = 2$$

b
$$y = 2x + \frac{9}{(2x-1)}$$
$$\frac{dy}{dx} = 2 - \frac{18}{(2x-1)^2}$$
$$\frac{dy}{dx} = 0$$
$$\frac{18}{(2x-1)^2} = 2$$
$$(2x-1)^2 = 9$$
$$2x - 1 = \pm 3$$
$$2x = 1 \pm 3$$
$$x = -1, 2$$
$$x = -1,$$
$$y = -2 + \frac{9}{-3}$$
$$= -5$$
$$\text{co-ords} = (-1, -5)$$

8

$$y = (2x-1)^n(x+2)$$

$$\frac{dy}{dx} = \frac{d}{dx}(2x-1)^n \times (x+2)(2x-1)^n$$

$$\times \frac{d}{dx}(x+2)$$

$$= 2n(2x-1)^{n-1} \times (x+2)$$

$$+ (2x-1)^n$$

$$= (2x-1)^{n-1}(2n(x+2) + (2x-1))$$

$$= (2x-1)^{n-1}(2nx + 4n + 2x - 1)$$

$$= (2x-1)^{n-1}((2n+2)x + (4n-1))$$

$$\frac{dy}{dx} = 0,$$

$$0 = (2x-1)^{n-1}((2n+2)x + (4n-1))$$

$$2x - 1 = 0 \textit{ or } (2n+2)x + (4n-1) = 0$$

$$x = \frac{1}{2} \textit{ or } x = \frac{1-4n}{2n+2}$$

9 $y = (x^2-1)^n$

$$\frac{dy}{dx} = 2x \times n(x^2-1)^{n-1}$$

$$= 2nx(x^2-1)^{n-1}$$

$$= 2nx((x+1)(x-1))^{n-1}$$

$$\frac{dy}{dx} = 0,$$

$$2nx((x+1)(x-1))^{n-1} = 0$$

$$x = 0, -1, 1$$

10

$$y = \frac{x}{x^2+1}$$

$$\frac{dy}{dx} = \frac{(x^2+1)\frac{d}{dx}(x) - x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

$$\frac{dy}{dx} = 0,$$

$$1 - x^2 = 0$$

$$x = \pm 1$$

$$y = \frac{\pm 1}{2}$$

$$\text{co-ords} = \left(1, \frac{1}{2}\right), \left(-1, \frac{-1}{2}\right)$$

Solutions to Exercise 10D

1 a $0 = 4x^2$

$x = 0$

	0	
+	0	+
/	—	/
	inflexion	

b $0 = (x-2)(x+5)$

$x = -5, 2$

	−5		2	
+	0	−	0	+
/	—	\	__	/
	max.		min.	

c $0 = (x+1)(2x-1)$

$x = -1, \dfrac{1}{2}$

	−1		$\frac{1}{2}$	
+	0	−	0	+
/	—	\	__	/
	max.		min.	

d $0 = -x^2 + x + 12$

$0 = -(x^2 - x - 12)$

$0 = -(x-4)(x+3)$

$x = -3, 4$

	−3		4	
−	0	+	0	−
\	__	/	—	\
	min.		max.	

e $0 = x^2 - x - 12$

$0 = -(x-4)(x+3)$

$x = -3, 4$

	−3		4	
+	0	−	0	+
/	—	\	__	/
	max.		min.	

f $0 = 5x^4 - 27x^3$

$0 = x^3(5x - 27)$

$x = 0, \dfrac{27}{5}$

	0		$\frac{27}{5}$	
+	0	−	0	+
/	—	\	__	/
	max.		min.	

g $0 = (x-1)(x-3)$

$x = 1, 3$

	1		3	
+	0	−	0	+
/	—	\	__	/
	max.		min.	

h $0 = -(x-1)(x-3)$

$x = 1, 3$

	1		3	
−	0	+	0	−
\	__	/	—	\
	min.		max.	

2 a $y = x^3 - 12x$

$\frac{dy}{dx} = 3x^2 - 12$

$\frac{dy}{dx} = 0$

$x^2 - 4 = 0$

$x = -2, +2$

$x = -2.5,$

$\frac{dy}{dx} > 0$

$x = -1.5,$

$\frac{dy}{dx} < 0$

$\therefore\ x = -2$ is a max

$x = 1.5,$

$\frac{dy}{dx} < 0$

$x = 2.5,$

$\frac{dy}{dx} > 0$

$\therefore\ x = 2$ is a min

b $y = 3x^2 - x^3$

$\frac{dy}{dx} = 6x - 3x^2$

$\frac{dy}{dx} = 0,$

$2x - x^2 = 0$

$x(x - 2) = 0$

$x = 0, 2$

$x = -0.5,$

$\frac{dy}{dx} < 0$

$x = 0,$

$\frac{dy}{dx} = 0$

$x = 1,$

$\frac{dy}{dx} > 0,$

$x = 2,$

$\frac{dy}{dx} = 0$

$x = 2.5,$

$\frac{dy}{dx} < 0$

$\therefore\ x = 0$ is a min.

$\therefore\ x = 2$ is a max.

c $y = x^3 - 5x^2 + 3x$

$\frac{dy}{dx} = 3x^2 - 10x + 3$

$\frac{dy}{dx} = 0,$

$(3x - 1)(x - 3) = 0$

$x = \frac{1}{3}, 3$

$x = 0,$

$\frac{dy}{dx} > 0$

$x = \frac{1}{3},$

$\frac{dy}{dx} = 0$

$x = 1,$

$\frac{dy}{dx} < 0$ \

$x = 3,$

$\frac{dy}{dx} = 0$ —

$x = 4,$

$\frac{dy}{dx} > 0$ /

$\therefore\ x = \frac{1}{3}$ is a max.

$x = 3$ is a min.

d

$$y = 3 - x^3$$

$$\frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = 0,$$

$$x = 0$$

$x = -1,$

$\frac{dy}{dx} < 0$ \

$x = 0,$

$\frac{dy}{dx} = 0$ —

$x = 1,$

$\frac{dy}{dx} < 0$ \

$x = 0$ is a stationary point of infection

e

$$y = 3x^4 + 16x^3 + 22x^2 + 3$$

$$\frac{dy}{dx} = 12x^3 + 48x^2 + 48x$$

$$\frac{dy}{dx} = 0,$$

$$x(x^2 + 4x + 4) = 0$$

$$x(x + 2)^2 = 0$$

$$x = -2, 0$$

$x = -3,$

$\frac{dy}{dx} = -27 \times 12$

$+ 48 \times 9 - 48 \times 3 < 0$ \

$x = -2,$

$\frac{dy}{dx} = 0$ —

$x = -1,$

$\frac{dy}{dx} = -12 + 48 - 48 < 0$ \

$x = 0,$

$\frac{dy}{dx} = 0$ —

$x = 1,$

$\frac{dy}{dx} = 12 + 48 + 48 > 0$ /

$\therefore\ x = -2$ is a stationary point of infection

$x = 0$ is a min.

f $y = x^3 - x$

$$\frac{dy}{dx} = 3x^2 - 1$$

$$\frac{dy}{dx} = 0,$$

$$x^2 = \frac{1}{3}$$

$$x = \frac{\pm 1}{\sqrt{3}}$$

$x = -1$,

$$\frac{dy}{dx} > 0$$

$x = \frac{-1}{\sqrt{3}}$,

$$\frac{dy}{dx} = 0$$

$x = 0$,

$$\frac{dy}{dx} < 0$$

$x = \frac{+1}{\sqrt{3}}$,

$$\frac{dy}{dx} = 0$$

$x = 1$,

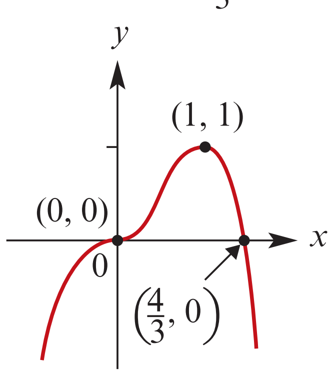
$$\frac{dy}{dx} > 0$$

$\therefore\ x = \frac{-1}{\sqrt{3}}$ is a max

$x = \frac{1}{\sqrt{3}}$ is a min

3 a i

$$y = 0$$
$$4x^3 - 3x^4 = 0$$
$$x^3(4 - 3x) = 0$$
$$x = 0, \frac{4}{3}$$



ii

$$\frac{dy}{dx} = 12x^2 - 12x^3$$
$$\frac{dy}{dx} = 0,$$
$$12x^2 - 12x^3 = 0$$
$$x^2(1 - x) = 0$$
$$x = 0, 1$$
$$x = 0, y = 0$$

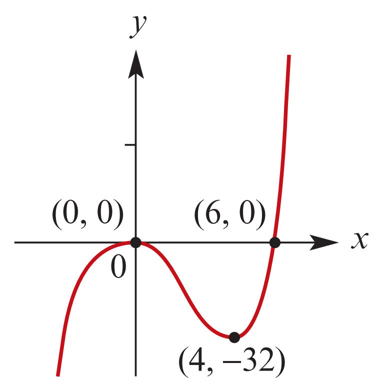
(0, 0) is a stationary point of inflection

$x = 1, y = 4 - 3 = 1$

(1, 1) is a maximum turning point

b i

$$y = x^3 - 6x^2$$
$$y = 0$$
$$x^2(x - 6) = 0$$
$$x = 0, 6$$



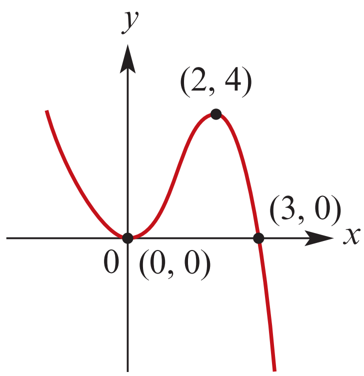
ii $\frac{dy}{dx} = 3x^2 - 12x$

$$\frac{dy}{dx} = 0$$

$$3x(x-4)=0$$
$$x=0,4$$
$$x=0, y=0$$
(0, 0) is a maximum turning point
$$x=4,$$
$$y=64-96$$
$$=-32$$
(4, −32) is a minimum turning point

c i

$$y=3x^2-x^3$$
$$y=0$$
$$x^2(3-x)=0$$
$$x=0,3$$

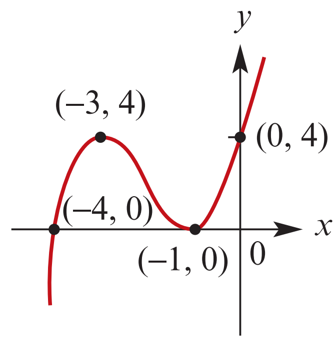


ii

$$\frac{dy}{dx}=6x-3x^2$$
$$\frac{dy}{dx}=0$$
$$3x(2-x)=0$$
$$x=0,2$$
$$x=0, y=0$$
(0,0) is a minimum turning point
$$x=2,$$
$$y=3\times4-8$$
$$=4$$
(2, 4) is a maximum turning point

d i

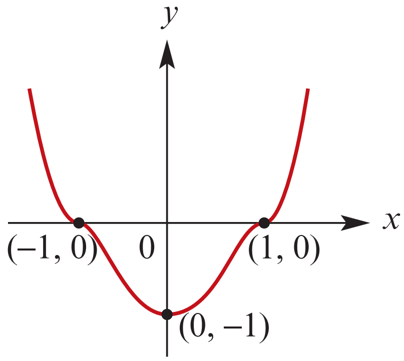
$$y=x^3+6x^2+9x+4$$
$$y=0$$
$$x^3+6x^2+9x+4=0$$
$$(x+4)(x+1)^2=0$$
$$x-\text{ints}: x=-4,-1$$
$$y-\text{int}: y=4$$



ii

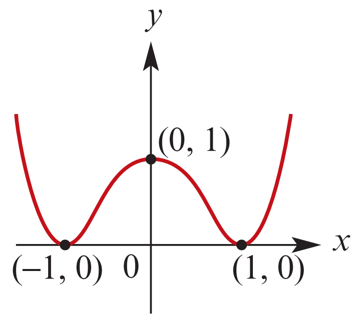
$$\frac{dy}{dx}=3x^2+12x+9$$
$$\frac{dy}{dx}=0$$
$$3x^2+12x+9=0$$
$$x^2+4x+3=0$$
$$(x+3)(x+1)=0$$
$$x=-3,-1$$
$$x=-3,$$
$$y=-27+6\times9-9\times3+4$$
$$y=-27+54-27+4$$
$$y=4$$
$(-3,4)$
is a maximum turning point
$$x=-1,$$
$$y=0$$
$(-1,0)$
is a minimum turning point

e **i**
$$y = (x^2 - 1)^5$$
$$y = 0$$
$$x^2 - 1 = 0$$
$$x - \text{ints} : x = \pm 1$$
$$x = 0, y = -1$$
$$y - \text{int} : y = -1$$



ii
$$\frac{dy}{dx} = 2x \times 5(x^2 - 1)^4$$
$$= 10x(x^2 - 1)^4$$
$$\frac{dy}{dx} = 0$$
$$10x(x^2 - 1)^4 = 0$$
$$x = 0, \pm 1$$
$$x = 0,$$
$$y = (-1)^5 = -1$$
$(0, -1)$
is a minimum turning point
$x = \pm 1,$
$y = 0$
$(\pm 1, 0)$
are stationary point of inflection

f **i** $y = (x^2 - 1)^4$
$x^2 = 1$
x-ints: $x = \pm 1$
$x = 0, y = 1$
y-int: $y = 1$



ii $\frac{dy}{dx} = 8x(x^2 - 1)^3$
$\frac{dy}{dx} = 0,$
$x(x^2 - 1)^3 = 0$
$x = 0, \pm 1$
$(-1, 0), (1, 0)$
are minimum turning points
$(0, 1)$ are maximum turning points

4 **a**
$$y = 2x^3 + 3x^2 - 12x + 7$$
$$\frac{dy}{dx} = 6x^2 + 6x - 12$$
$$\frac{dy}{dx} = 0,$$
$$0 = 6x^2 + 6x - 12$$
$$x^2 + x - 2 = 0$$
$$(x - 1)(x + 2) = 0$$

$x = -2, 1$ both are turning points

$x = -3,$

$\frac{dy}{dx} = 6(9 - 3 - 2)$

> 0

$x = 0,$

$\frac{dy}{dx} = -12 < 0$

$\therefore\ x = -2$ is a max

$x = -2,$

$y = -2 \times 8 + 3 \times 4 + 12 \times 2 + 7$

$y = -16 + 12 + 24 + 7$

$= 27$

$(-2, 27)$ is a max

$x = 2$

$\frac{dy}{dx} = 6(4 + 2 - 2)$

>0

$\therefore\ x = 1$ is a min

$x = 1,$

$y = 2 + 3 - 12 + 7$

$x = 1, y = 0$

(1,0) is a *min*

b see above, (1,0)
is a point on the curve

c $y = 0$
$2x^3 + 3x^2 - 12x + 7 = 0$ (from **a**),
we know (1,0) is a turning point.

$\therefore (x - 1)^2$ is a factor,

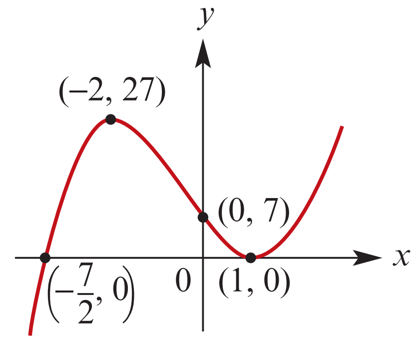
$\therefore\ y = (x - 1)^2(2x + 7)$

$(x - 1)^2\,(2x + 7) = 0$

$x = ints : x = 1, \frac{-7}{2}$

$x = 0, y = 7, y - \text{int} : y = 7$

d



5 a $P(x) = x^3 + ax^2 + b$

$P'(x) = 3x^2 + 2ax$

$= x(3x + 2a)$

$P'(0) = 0$

$\therefore\ x = 0$

is a stationary point for all values of a and b

b

$$P'(-2) = 0$$
$$3(-2)^2 + 2a(-2) = 0$$
$$12 - 4a = 0$$
$$a = 3$$
$$P(-2) = 6$$
$$(-2)^3 + 3(-2)^2 + b = 6$$
$$-8 + 12 + b = 6$$
$$4 + b = 6$$
$$b = 2$$
$$P(x) = x^3 + 3x^2 + 6$$
$$P'(x) = 3x^2 + 6x$$
$$P'(-1) = 3 - 6$$
$$<0 \quad \diagdown$$
$$P'(1) = 3 + 6$$
$$>0 \quad /$$
$$x = 0 \text{ is a min.}$$

Local minimum at $(0, 2)$

$$P'(-3) = 27 - 18 > 0 \quad /$$
$$x = -2 \text{ is a max.}$$

Local maximum at $(-2, 6)$

6 a

$$f(x) = (2x - 1)^5(2x - 4)^4$$
$$f(0) = (-1)^5(-4)^4$$
$$y\text{-intercept} = -256$$
$$\text{co-ords}(0, -256)$$
$$f(x) = 0,$$
$$0 = (2x - 1)^5(2x - 4)^4$$
$$x = \frac{1}{2}, 2$$
$$x\text{-intercepts} = \frac{1}{2}, 2$$
$$\text{co-ords } \left(\frac{1}{2}, 0\right), (2, 0)$$

b

$$f'(x) = (2x - 1)^5$$
$$\left(\frac{d}{dx}((2x - 4)^4) + (2x - 4)^4 \frac{d}{dx}((2x - 1)^5)\right)$$
$$= (2x - 1)^5 \times 2 \times 4(2x - 4)^3$$
$$+ (2x - 4)^4 \times 2 \times 5(2x - 1)^4$$
$$= (2x - 4)^3(2x - 1)^4((2x - 1) \times 8$$
$$+ (2x - 4) \times 10)$$
$$= (2x - 4)^3(2x - 1)^4(16x - 8 + 20x - 40)$$
$$= (2x - 4)^3(2x - 1)^4(36x - 48)$$
$$= 12(3x - 4)(2x - 4)^3(2x - 1)^4$$
$$f'(x) = 0,$$
$$12(3x - 4)(2x - 4)^3(2x - 1)^4 = 0$$

$x = \frac{4}{3}$ (turning point)

or $x = 2$ (turning point)

or $x = \frac{1}{2}$ (stationary point of inflection)

$f'(1) = 12(3-4)(2-4)^3(2-1)^4$

> 0

$f'(1.5) = 12(4.5-4)(3-4)^3(3-1)^4$

< 0

$f'(3) = 12(9-4)(6-4)^3(6-1)^4$

> 0

$x = \frac{4}{3}$ is a max.

$x = 2$ is a min.

$f\left(\frac{4}{3}\right) = \left(\frac{8}{3}-1\right)^5\left(\frac{8}{3}-4\right)^4\left(\frac{5}{3}\right)^5\left(\frac{-4}{3}\right)^4$

≈ 40.6

$\left(\frac{4}{3}, 40.6\right)$ is a max.

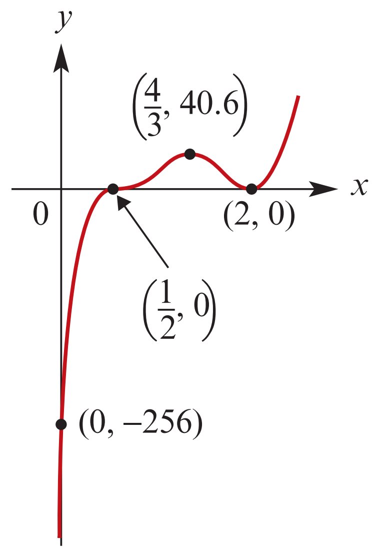
$f(2) = (4-1)^5(4-4)^4$

$= 0$

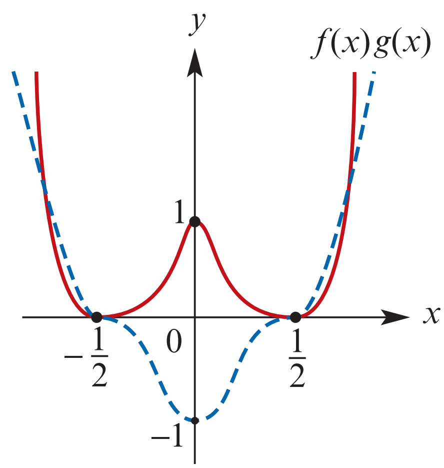
$(2, 0)$ is a min.

$f\left(\frac{1}{2}\right) = 0$

$\left(\frac{1}{2}, 0\right)$ is a stationary point of inflection



7 a



b **i** $(4x^2-1)^6 > (4x^2-1)^5$

$4x^2 - 1 > 1 \; if \; (4x^2-1)^5 > 0$

$4x^2 > 2 \; if \; (4x^2-1)^5 > 0$

$x^2 > \frac{1}{2} \; if \; (4x^2-1)^5 > 0$

$|x| > \frac{1}{\sqrt{2}} \; if \; (4x^2-1)^5 > 0$

$x^2 < \frac{1}{4} \; if \; (4x^2-1)^5 < 0$

$|x| < \frac{1}{2} \; if \; (4x^2-1)^5 < 0$

$\therefore \; |x| > \frac{1}{\sqrt{2}}$ or $|x| < \frac{1}{2}$

ii

$f'(x) = 8x \times 6(4x^2-1)^5$

$g'(x) = 8x \times 6(4x^2-1)^4$

$f'(x) > g'(x)$

$8x \times 6(4x^2-1)^5 > 8x \times 6(4x^2-1)^4$

$6x(4x^2-1) > 5x$

$if \; x > 0,$

$6(4x^2-1) > 5$

$(4x^2-1) > \frac{5}{6}$

$4x^2 > \frac{11}{6}$

$x^2 > \frac{11}{24}$

$x > \sqrt{\frac{11}{24}}$

$x > \frac{\sqrt{66}}{12}$

$if \; x < 0,$

$6(4x^2-1) < 5$

$(4x^2-1) < \frac{5}{6}$

$4x^2 < \frac{11}{6}$

$x^2 < \frac{11}{24}$

$x^2 < \frac{66}{144}$

$x > \frac{-\sqrt{66}}{12}, x < 0$

$x \neq \pm\frac{1}{2}$

$\therefore \; \frac{-\sqrt{66}}{12} < x < \frac{-1}{2},$

or $\frac{-1}{2} < x < 0,$

or $x > \frac{\sqrt{66}}{12}$

8 **a**

$y = x^3 + x^2 - 8x - 12$

$x = 0, y = -12$

y-intercept $= (0, -12)$

$y = 0$

$x^3 + x^2 - 8x - 12 = 0$

try $x = 3$ (a factor of -12)

$27 + 9 - 24 - 12$

$= 0$

$\therefore (x-3)$ is a factor

$(x-3)(x^2+4x+4) = 0$

$(x-3)(x+2)^2 = 0$

$x = 3, -2$

x-intercepts $= (-2, 0), (3, 0)$

$$\frac{dy}{dx} = 3x^2 + 2x - 8$$
$$= (3x - 4)(x + 2)$$
$$\frac{dy}{dx} = 0,$$
$$(3x - 4)(x + 2) = 0$$
$$x = -2, \frac{4}{3}$$
$$x = -2, y = 0$$
$$x = \frac{4}{3}$$
$$y = \left(\frac{4}{3}\right)^3 + \left(\frac{4}{3}\right)^2 - 8\left(\frac{4}{3}\right) - 12$$
$$= \frac{64}{27} + \frac{16}{9} - \frac{32}{3} - 12$$
$$= \frac{64 + 48 - 288 - 324}{27}$$
$$= \frac{-500}{27}$$
$$= -18\frac{14}{27}$$

stationary points are $(-2, 0)$ max and $\left(\frac{4}{3}, \frac{-500}{27}\right)$ min

b $y = 4x - 18x^2 + 48x - 290$

$$= 2(2x^3 - 9x^2 + 24x - 145)$$
$$x = 0, y = -290$$

y-intercept $= (0, -290)$

$$y = 0$$
$$2x^3 - 9x^2 + 24x - 145 = 0$$

using CAS calculator

$$x = 5$$
$$y = 2(x - 5)(2x^2 + x + 29)$$
$$2x^2 + x + 29 = 0,$$
$$x = \frac{-1 \pm \sqrt{1 - 232}}{4}$$

no real solutions

$$y = 0, x = 5$$

x-intercept $= (5, 0)$

$$\frac{dy}{dx} = 12x^2 - 36x + 48$$
$$= 12(x^2 - 3x + 4)$$
$$\frac{dy}{dx} = 0,$$
$$x^2 - 3x + 4 = 0$$
$$x = \frac{3 \pm \sqrt{9 - 16}}{2}$$

no real solutions

$\therefore$ y has no stationary points

9 a
$$f(x) = 3x^4 + 4x^3$$
$$f(x) = 12x^3 + 12x^2$$
$$f'(x) = 0,$$
$$12x^2(x + 1) = 0$$
$$x = -1, 0$$
$$f'(0) = 0$$

$(0, 0)$, stationary point of inflection

$$f(-1) = 3 - 4$$
$$= -1$$

$(-1, -1)$ *min.*, since
$f(x)$ is shaped and $(0, 0)$
is a stationary point of inflection

b $$f(x) = x^4 + 2x^3 - 1$$
$$f'(x) = 4x^3 + 6x^2$$
$$f'(x) = 0,$$
$$2x^2(2x + 3) = 0$$

$x = 0$ (stationary point of inflection)

$x = \frac{-3}{2}$ (turning point)

$f(0) = -1$

$(0, -1)$

is a stationary point of inflection

$$f'(-2) = 4 \times -8 + 6 \times 4$$

$$< 0$$

$$f'(-1) = -4 + 6$$

$$> 0$$

$$f'\left(-\frac{3}{2}\right) = \frac{81}{16} + \frac{-27}{4} - 1$$
$$= \frac{-43}{16}$$

$\left(\frac{-3}{2}, \frac{-43}{16}\right) = (-1.5, -2.6875)$ is a min.

c $$f(x) = 3x^3 - 3x^2 + 12x + 9$$
$$= 3(x^3 - x^2 + 4x + 3)$$
$$f'(x) = 3(3x^2 - 2x + 4)$$
$$f'(x) = 0,$$
$$3x^2 - 2x + 4 = 0$$
$$x = \frac{2 \pm \sqrt{4 - 48}}{6}$$

no real solutions

$\therefore f(x)$ has no stationary points

10 $f(x) = \frac{1}{8}(x - 1)^3(8 - 3x) + 1$

a $$f(0) = \frac{1}{8}(-1)^3(8) + 1$$
$$= 1 - 1$$
$$= 0 \ QED$$
$$f(3) = \frac{1}{8}(2)^3(-1) + 1$$
$$= -1 + 1$$
$$= 0 \quad QED$$

b $$f'(x) = \frac{1}{8}(x - 1)^3 \frac{d}{dx}(8 - 3x) + \frac{1}{8}(8 - 3x)\frac{d}{dx}(x - 1)^3$$
$$= \frac{-3}{8}(x - 1)^3 \frac{3}{8}(8 - 3x)(x - 1)^2$$
$$= \frac{3}{8}(x - 1)^2((8 - 3x) - (x - 1))$$
$$= \frac{3}{8}(x - 1)^2(9 - 4x) \quad QED$$

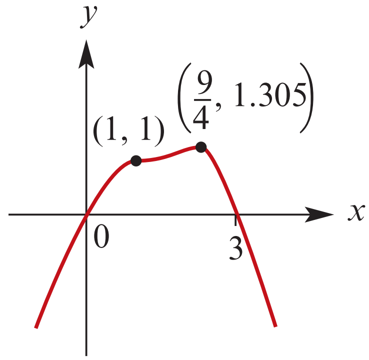
want x such that $f'(x) \geq 0$

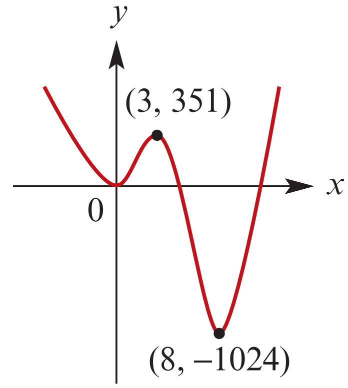
$$\frac{3}{8}(x - 1)^2(9 - 4x) \geq 0$$

since $(x - 1)^2 \geq 0$,

$$9 - 4x \geq 0$$
$$x \leq \frac{9}{4}$$

c





11 $y = 3x^4 - 44x^3 + 144x^2$

$\frac{dy}{dx} = 12x^3 - 132x^2 + 288x$

$= 12x(x^2 - 11x + 24)$

$= 12x(x - 8)(x - 3)$

$\frac{dy}{dx} = 0,$

$x = 0, 3, 8$

$x = 0,$

$y = 0$

(0, 0) is a minimum turning point

$x = 3,$

$y = 3^5 - 44 \times 27 + 144 \times 9$

$= 243 - 1188 + 1296$

$= 351$

(3,351) is a maximum turning point

$x = 8,$

$y = 3 \times 8^4 - 44 \times 8^3 + 144 \times 64$

$= 12288 - 22528 + 9216$

$= -1024$

(8, −1024) is a minimum turning point

12 a

$x = -1$ (stationary point of inflection)

$x = 1$ (*min*)

$x = 5$ (*max*)

b $x = 0$ (*max*)

$x = 2$ (*min*)

c $x = -4$ (*min*)

$x = 0$ (*max*)

d $x = -3$ (*min*)

$x = 2$ (stationary point of inflection)

13 a $y = x^4 - 16x^2$

$\frac{dy}{dx} = 4x^3 - 32x$

$= 4x(x^2 - 8)$

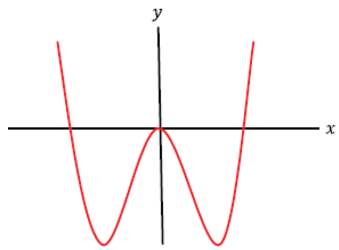
$\frac{dy}{dx} = 0,$

$x = 0, \pm 2\sqrt{2}$

$x = 0, x = \pm 2\sqrt{2},$

$y = 0\ y = -64$

Since the x-intercepts are ±4, 0 we can sketch the graph.



hence (0,0) is a maximum
$(\pm 2\sqrt{2}, -64)$ are minimums

b $y = x^{2m} - 16x^{2m-2}$

$$\frac{dy}{dx} = (x^{2m} - 16x^{2m-2})$$
$$= 2mx^{2m-1} - 16(2m-2)x^{2m-3}$$
$$= 2x^{2m-3}(mx^2 - 16(m-1))$$
$$\frac{dy}{dx} = 0$$
$$x^{2m-3}(mx^2 - 16(m-1)) = 0$$
$$x = 0, mx^2 - 16(m-1) = 0$$
$$x^2 = \frac{16(m-1)}{m}$$
$$x = \pm\sqrt{\frac{16(m-1)}{m}}$$
$$= \pm 4\sqrt{\frac{(m-1)}{m}}$$

If $x = 0$ then $y = 0$.

If$x = 4\sqrt{\frac{(m-1)}{m}}$

$y =$

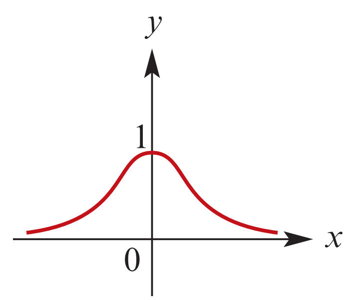
$$\left(4\sqrt{\frac{(m-1)}{m}}\right)^{2m} - 16\left(4\sqrt{\frac{(m-1)}{m}}\right)^{2m-2}$$
$$= 16m\left(\frac{(m-1)}{m}\right)^{m} - 16 \times 16^{m-1}\left(\frac{(m-1)}{m}\right)^{m-1}$$
$$= 16m\left(\frac{m-1}{m}\right)^{m} - \left(\frac{(m-1)}{m}\right)^{m-1}$$
$$= 16^m\left(\frac{(m-1)}{m}\right)^{m-1}\left(1 - \frac{(m-1)}{m}\right)$$
$$= 16^m\left(\frac{(m-1)}{m}\right)^{m-1}\frac{1}{m}$$

stationary points are :

$(0, 0)min$

$$\left(\pm 4\sqrt{\frac{(m-1)}{m}}, \frac{16^m(m-1)^{m-1}}{m^m}\right)max$$

14



15

$$f(x) = x^2e^x \qquad in\ set$$
$$f'(x) = 2xe^x + x^2e^x$$
$$= e^x(x^2 + 2x)$$
$$f'(x) < 0,$$
$$e^x(x^2 + 2x) < 0$$
$$x^2 + 2x < 0$$
$$x(x + 2) < 0$$
$$x < 0 \ \& \ x > -2$$
$$\therefore \ -2 < x < 0$$
$$notation, \{x: \ -2 < x < 0\}$$

in set notation, $\{x: -2 < x < 0\}$

16

$$f(x) = 100e^{-x^2+2x-5}$$

$$f'(x) = 100(-2x+2)e^{-x^2+2x-5}$$

$$f'(x) > 0,$$

$$100(-2x+2)e^{-x^2+2x-5} > 0$$

$$-2x+2 > 0$$

$$x < 1$$

hence maximum $f(x)$ *occurs at* $x = 1$.

$f'(1) = 100e^{-1+2-5}$

$= 100e^{-4}$

$\cong 1.83$

17 $f(x) = e^x - 1 - x$

a $f'(x) = e^x - 1$

$f'(x) = 0,$

$e^x = 1$

$x = 0$

$f(0) = e^0 - 1 - 0$

min $f(x) = 0$

b *min* $f(x) = 0$

$\therefore\ f(x) \geq 0$

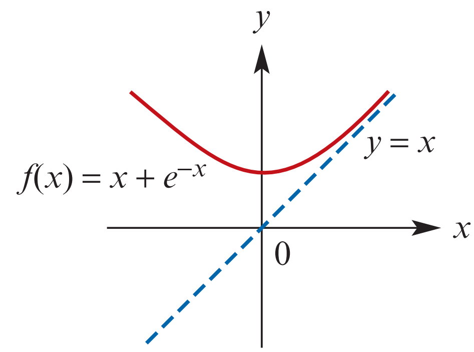
$e^x - 1 - x \geq 0$

$e^x \geq 1 + x$ *QED*

18 **a** $(0, 1)$ min

b $y = x$

c



19 $y = e^x(px^2 + qx + r)$

$\frac{dy}{dx} = e^x(px^2 + qx + r) + e^x(2px + q)$

$= e^x(px^2(q + 2p)x + (r + q))$

$x = 0, y = 9$

$9 = e^0(0 + 0 + r)$

$9 = r$

$y = e^x(px^2 + qx + 9)$

$\frac{dy}{dx} = e^x(px^2(q + 2p)x + (q + 9))$

$x = 1, y = 0$

$0 = e^1(p + q + 2p + q + 9)$

$13p + 2q + 9 = 0$

$x = 3, y = 0$

$0 = e^3(9p + 3(q + 2p) + (q + 9))$

$9p + 3q + 6p + q + 9 = 0$

$215p + 4q + 9 = 0$

$2 - 21 \Rightarrow 9p - 9 = 0$

$p = 1$

sub in $1 \Rightarrow 3 + 2q + 9 = 0$

$2q = -12$

$q = -6$

$y = e^x(x^2 - 6x + 9)$

20 **a** $y = e^{4x^2-8x}$

$$\frac{dy}{dx} = (8x-8)e^{4x^2-8x}$$

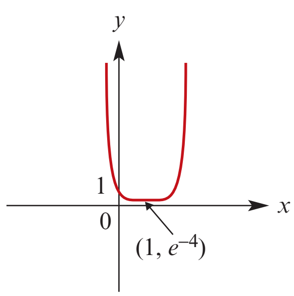
b
$$\frac{dy}{dx} = 0$$
$$8(x-1)e^{4x^2-8x} = 0$$
$$x - 1 = 0$$
$$x = 1$$
$$x < 1, f'(x) < 0$$
$$x > 1, f'(x) > 0$$
$$\left(\text{since } e^{4x^2-8x} > 0\right)$$
$\therefore x = 1$ *is a min.*

$x = 1,$

$y = e^{4-8} = e^{-4}$

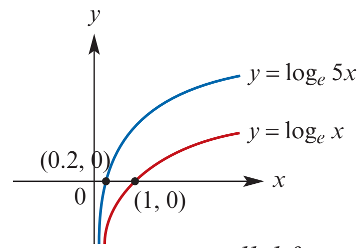
$\therefore (1, e^{-4})$ *is a min.*

c



d
$$x = 2,$$
$$y = e^{16-16} = 1$$
$$\frac{dy}{dx} = 8(x-1)e^{16-16}$$
$$= 8$$
$$x = 2, y = 1$$
$$1 = -\frac{1}{4} + c$$
$$c = \frac{5}{4}$$
$$y = -\frac{1}{8}x + \frac{5}{4}$$

21



tangents are parallel for any given value of x

22 $f(x) = x^2 \ln x$

a $f'(x) = 2x\ln x + \dfrac{x^2}{x}$

$= x(2\ln x + 1)$

b $f(x) = 0,$

$x^2 \ln x = 0$

$x^2 = 0, \quad \ln x = 0$

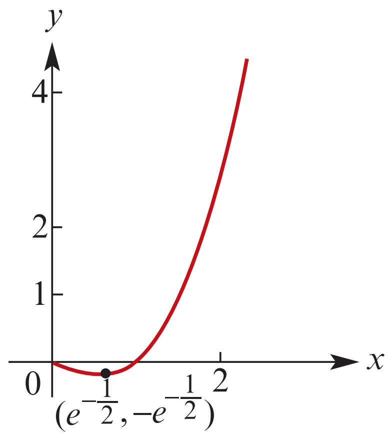
$x = 0, \quad x = 1$

$x = 0, 1$ *but* $x > 0, \quad \therefore x = 1$

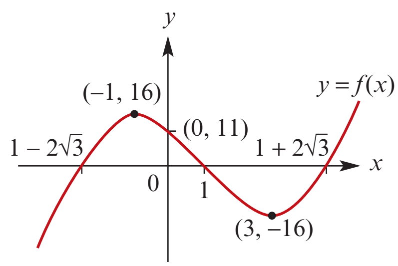
c
$$f'(x) = 0$$
$$x(2\ln x + 1) = 0$$
$$x = 0,\ 2\ln x + 1 = 0$$
$$\ln x = \frac{-1}{2}$$
$$x = \frac{1}{\sqrt{e}}$$
$x = 0, \dfrac{1}{\sqrt{e}}$ *but* $x > 0,$

$$\therefore x = \frac{1}{\sqrt{e}} = e^{-\frac{1}{2}}$$

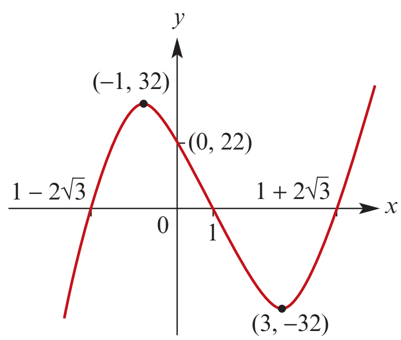
d $x = e^{-\frac{1}{2}}, y = \left(e^{-\frac{1}{2}}\right)^2 \ln\left(e^{-\frac{1}{2}}\right) = -\frac{1}{2}e^{-1}$



23 **a**

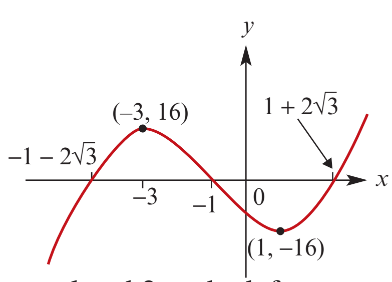


b



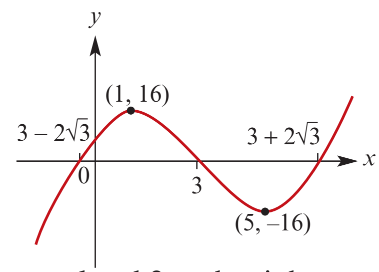
graph dilated by factor 2 from x-axis

c



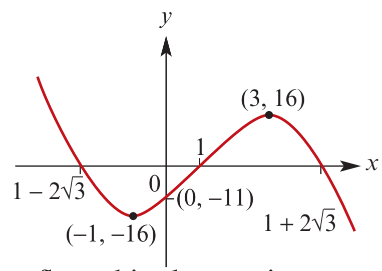
translated 2 to the left

d



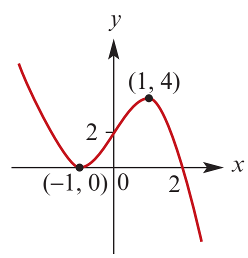
translated 2 to the right

e

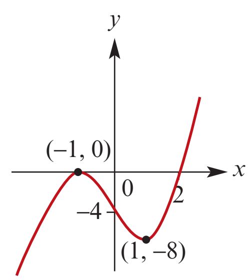


reflected in the x-axis

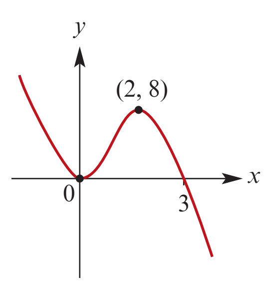
24 **a**



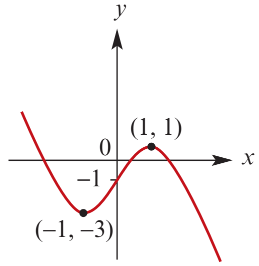
b



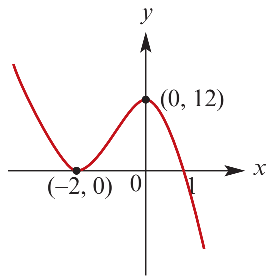
c



d



e



25 **a** $A' = (a + l, 0)$

$B' = (b + l, 0)$

b $P' = (h + l, kp)$

26 **a**

$f(x) = 2\cos x - 2\cos^2 x + 1$

$f'(x) = -2\sin x + 4\sin x\cos x$

$f'(x) = 0 \Rightarrow 2\sin x(2\cos x - 1) = 0$

$\Rightarrow \sin x = 0 \text{ or } \cos x = \dfrac{1}{2}$

$\Rightarrow x = 0, \pi, 2\pi \text{ or}$

$x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

$(0, 1)$, $(\pi, -3)$, $(2\pi, 1)$ *are min.*

$\left(\dfrac{\pi}{3}, \dfrac{3}{2}\right), \left(\dfrac{5\pi}{3}, \dfrac{3}{2}\right)$ *are max.*

b

$f(x) = 2\cos x + 2\sin x cox$

$f'(x) = -2\sin x + 2\cos x\cos x - 2\sin x\sin x$

$= -2\sin x + 2(1 - 2\sin^2 x)$

$= 2(-2\sin^2 x - \sin x + 1)$

$f'(x) = 0,$

$\sin x = \dfrac{1 \pm \sqrt{1+8}}{-4}$

$\sin x = \dfrac{-1 \pm 3}{4}$

$\sin x = \dfrac{1}{2}, -1$

$x = \dfrac{\pi}{6}, \dfrac{3\pi}{2}, \dfrac{5\pi}{6}$

$f\left(\dfrac{\pi}{6}\right) = 2\cos\left(\dfrac{\pi}{6}\right) + \sin\left(\dfrac{\pi}{3}\right)$

$= \sqrt{3} + \dfrac{\sqrt{3}}{2}$

$= \dfrac{3\sqrt{3}}{2}$

$f\left(\dfrac{3\pi}{2}\right) = 2\cos\left(\dfrac{3\pi}{2}\right) + \sin 3\pi$

$= 0$

$f\left(\dfrac{5\pi}{6}\right) = 2\cos\left(\dfrac{5\pi}{6}\right) + \sin(\dfrac{10\pi}{6})$

$= -\sqrt{3} - \dfrac{\sqrt{3}}{2}$

$= \dfrac{-3\sqrt{3}}{2}$

$\left(\dfrac{\pi}{6}, \dfrac{3\sqrt{3}}{2}\right)$ *max.*

$\left(\dfrac{3\pi}{2}, 0\right)$ *stationary point of inflection*

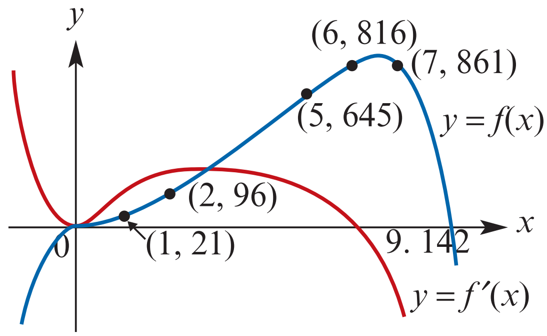
$\left(\dfrac{5\pi}{6}, \dfrac{-3\sqrt{3}}{2}\right)$ *min.*

c Max $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$; Min $x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

d Max $x = \dfrac{\pi}{3}$; Infl $x = \pi$; Min $x = \dfrac{5\pi}{3}$

27 **a** and

b



Using a CAS calculator:
$y = -x^4 + 8x^3 + 10x^2 + 4x$
loc max at (6.761,867.07)
no stationary point of inflexion, since at $x = 0, \frac{dy}{dx} = 4$.

c −960

d Use the 'solve' command of a CAS calculator, giving:
$x = 4.317$ or $x = 8.404$

Solutions to Exercise 10E

1 $f : [-3, 3] \to \mathbb{R}, f(x) = 2 - 8x^2$
Local maximum at $(0, 2)$
$f(-3) = 2 - 8(-3)^2 = 2 - 72 = -70$
$f(3) = 2 - 2(3)^2 = 2 - 72 = -70$
Therefore absolute maximum of f is 2 and absolute minimum is -70

2
$$f : [-3, 2] \to R,$$
$$f(x) = x^3 + 2x + 3$$
$$f'(x) = 3x^2 + 2$$
$f'(x)$ has no real solution

$\therefore$ $f(x)$ has no stationary points

$\therefore$ $f(-3)$ is absolute minimum

$f(2)$ is absolute maximum

$$f(-3) = -27 - 6 + 3$$
$$abs.\ min. = -30$$
$$f(2) = 8 + 4 + 3$$
$$abs.\ max. = 15$$

3 $f : \left[-\frac{3}{2}, \frac{5}{2}\right] \to R,$

$$\begin{aligned} f(x) &= 2x^3 - 6x^2 \\ f'(x) &= 6x^2 - 12x \\ f'(x) &= 0, \\ x(x-2) &= 0 \\ x &= 0, 2 \\ f(0) &= 0 \\ f(2) &= 16 - 24 \\ &= -8 \\ f\left(\frac{-3}{2}\right) &= \frac{-27}{4} - \frac{27}{2} \\ &= \frac{-81}{4} = -20.25 \\ f(\frac{5}{2}) &= \frac{125}{4} - \frac{75}{2} \\ &= \frac{-25}{4} \end{aligned}$$

absolute $min = \dfrac{-81}{4}$

absolute $max = 0$

4 $f:[-2,6]\to R,\ f(x)=2x^4-8x^2$

$$f'(x)=8x^3-16x$$

$$f'(x)=0,$$

$$8x(x^2-2)=0$$

$$x=\pm\sqrt{2},0$$

$$f(\pm\sqrt{2})=8-16$$

$$=-8$$

$$f(0)=0$$

$$f(-2)=32-32=0$$

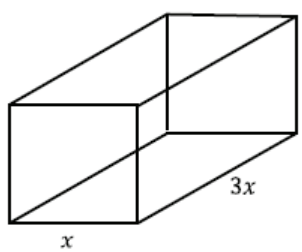
$$f(6)=2\times6^4-8\times6^2$$

$$=2592-288$$

$$=2304$$

absolute *min* $=-8$
absolute *max* $=2304$

5



$$4x+4(3x)+4y=20$$

$$4x+y=5$$

$$y=5-4x$$

a $V=x(3x)y$

$$=3x^2(5-4x)$$

$$V=15x^2-12x^3$$

QED

b $\dfrac{dV}{dx}=30x-36x^2$

c $\dfrac{dV}{dx}=0,$

$$30x-36x^2=0$$

$$x(5-6x)=0$$

$$x=0,\frac{5}{6}$$

since $x=0$ gives
$V=0$, it is not the *max*,
$\therefore\ x=\dfrac{5}{6}$ is the max
co-ords $=\left(\dfrac{5}{6},\dfrac{125}{36}\right)$

d

there are no turning points, so test the end points,

$x=0,$

$V=0$

$x=0.8,$

$$V=15\times\frac{16}{25}-12\times\frac{64}{125}$$

$$=\frac{1200-768}{125}$$

absolute *max* : $V=\dfrac{432}{125}=3.456\text{ cm}^3$
when $x=0.8$

e turning point at $x=\dfrac{5}{6}$,

test the endpoints, $\dfrac{5}{6}$

$x=0,$

$V=0,$

$x=1,$

$$V=15-12$$

$$=3$$

$x=\dfrac{5}{6},$

$$V=15\times\frac{16}{25}-12\times\frac{125}{216}$$

$$=\frac{375-250}{36}$$

absolute *max* : $V = \frac{125}{36} = 3.472\,\text{cm}^3$

when $x = \frac{5}{6}$

6 $x + y = 30, z = xy$

a $x \in [2, 5]$,

$y = 30 - x$

$y \in [25, 28]$, i.e. $25 \le y \le 28$

b $z = x(30 - x)$

$z = 30x - x^2$

$\frac{dz}{dx} = 30 - 2x$

$\frac{dz}{dx} = 0, x = 15$

this is outside the domain $x \in [2, 5]$

$\therefore$ values to test are

$x = 2, 5$

$x = 2$,

$z = 60 - 4 = 56$,

$x = 5$,

$z = 150 - 25 = 125$

absolute minimum = 56

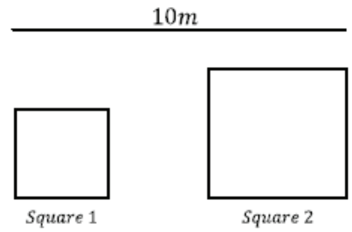
absolute maximum = 125

7 **a** $\frac{1}{(x-4)^2} - \frac{1}{(x-1)^2}$

b $\left(\frac{5}{2}, \frac{4}{3}\right)$

c Absolute max $= \frac{4}{3}$; Absolute min $= \frac{3}{2}$

8



a square 1 has perimeter x, i.e. side $\left(\frac{x}{4}\right)$

$A_1 = \frac{x^2}{16}$

square 2 has perimeter $(10 - 4)$, *i.e.* side $\left(\frac{10-x}{4}\right)$

$A_2 = \frac{(10-x)^2}{16}$

$A = A_1 + A_2$

$= \frac{x^2 + (10-x)^2}{16}$

$= \frac{x^2 + 100 - 20x + x^2}{16}$

$= \frac{2x^2 - 20x + 100}{16}$

$= \frac{1}{8}(x^2 - 10x + 50)$ *QED*

b $\frac{dA}{dx} = \frac{1}{8}(2x - 10)$

$= \frac{1}{4}(x - 5)$

c $\frac{dA}{dx} = 0$,

$x = 5$

d $A = \frac{1}{8}(x^2 - 10x + 50)$

$x \in [0, 1]$ $A(0) = \frac{25}{4}$ and $A(1) = \frac{41}{8}$

The maximum is $\frac{25}{4}\text{m}^2$ but only one square is formed

9 $g : [2.1, 8] \to R,\ g(x) = x + \dfrac{1}{x-2}$

$$g'(x) = 1 - \frac{1}{(x-2)^2}$$

$$g'(x) = 0,$$

$$\frac{1}{(x-2)^2} = 1$$

$$(x-2)^2 = 1$$

$$x - 2 = \pm 1$$

$$x = 2 \pm 1$$

$$x \in [2.1, 8]$$

$$x = 3,$$

values to test :

$f(2.1), f(3), f(8)$

$$f(2.1) = 2.1 + \frac{1}{0.1}$$

$$= 12.1$$

$$f(3) = 3 + \frac{1}{1}$$

$$= 4$$

$$f(8) = 8 + \frac{1}{6}$$

$$= 8\frac{1}{6}$$

absolute minimum = 4

absolute maximum = 12.1

10 $f : [0, 3] \to R,\ f(x) = \dfrac{1}{x+1} + \dfrac{1}{4-x}$

$$f(x) = \frac{1}{x+1} - \frac{1}{x-4}$$

a $f'(x) = \dfrac{-1}{(x+1)^2} - \dfrac{-1}{(x-4)^2}$

$$= \frac{1}{(x-4)^2} - \frac{1}{(x+1)^2}$$

b $f'(x) = 0,$

$$\frac{1}{(x-4)^2} = \frac{1}{(x+1)^2}$$

$$(x-4)^2 = (x+1)^2$$

$$(x-4) = \pm(x+1)$$

$$if\ x - 4 = x + 1$$

$$-4 = -1$$

does not work

$$\therefore\ x - 4 = -(x+1) = -x - 1$$

$$2x - 3 = 0$$

$$x = \frac{3}{2}$$

$$f\left(\frac{3}{2}\right) = \frac{1}{\left(\frac{5}{2}\right)} + \frac{1}{\left(\frac{5}{2}\right)}$$

$$= \frac{4}{5}$$

$$\text{co-ords} = \left(\frac{3}{2}, \frac{4}{5}\right)$$

c values to test:

$f(0), f(3), f\left(\dfrac{3}{2}\right)$

$$f(\frac{3}{2}) = \frac{4}{5}$$

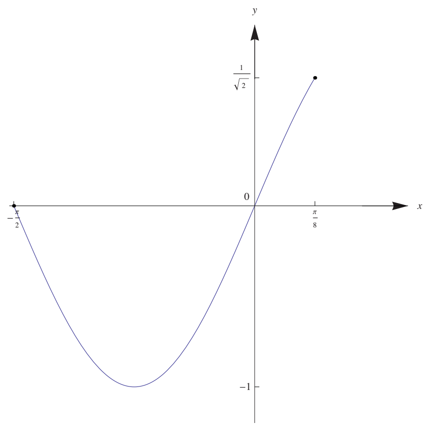
$$f(0) = \frac{1}{1} + \frac{1}{4} = \frac{5}{4}$$

$$f(3) = \frac{1}{4} + \frac{1}{1} = \frac{5}{4}$$

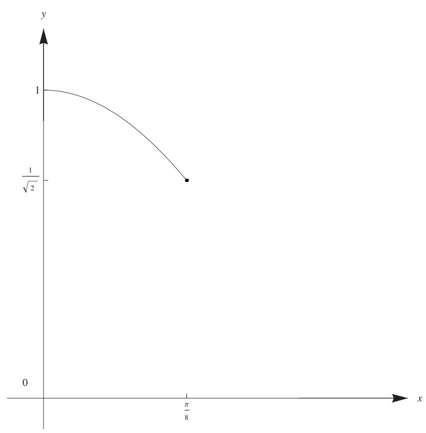
absolute minimum = $\dfrac{4}{5}$

absolute maximum = $\dfrac{5}{4}$

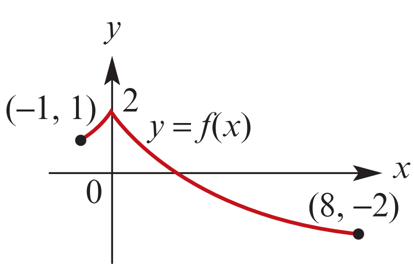
11 Absolute max $= \dfrac{\sqrt{2}}{2}$; Absolute min $= -1$



12 Absolute max $= 1$; Absolute min $= \dfrac{\sqrt{2}}{2}$



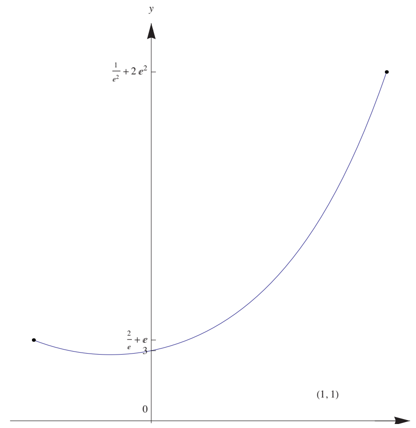
13



absolute maximum = 2
absolute minimum = −2

14 Absolute max $= \dfrac{1}{e^2} + 2e^2$;

Absolute min $= 2\sqrt{2}$



15 $f(x) = 2e^{(x-1)^2}$

$f(-2) = 2e^9$ and $f(2) = 2e$

$f'(x) = 4(x-1)e^{(x-1)^2}$

$f'(x) = 0$ implies $x = 1$

$f(1) = 2$

Absolute max $= 2e^9$; Absolute min $= 2$

16 Absolute max $= -\log_e 10$;

Absolute min $= -\dfrac{10}{e}$

Solutions to Exercise 10F

1 Let x m be the width of the rectangle

Let y m be the length of the rectangle

$2x + 2y = 100 \Rightarrow x + y = 50$

Area, $A = xy = x(50 - x) = 50x - x^2$

$\frac{dA}{dx} = 50 - 2x$

$\frac{dA}{dx} = 0 \Rightarrow x = 25$

$\therefore$ maximum area $= 25 \times 25 = 625$ m^2.

2 $x = y = 4; x, y > 0;$

$x^3 + y^2$ is a min.

let $z = x^3 + y^2$

$y = 4 - x$

$z = x^3 + (4 - x)^2$

$= x^3 + 16 - 8x + x^2$

$= x^3 + x^2 - 8x + 16$

$\frac{dz}{dx} = 3x^2 + 2x - 8$

$\frac{dz}{dx} = 0,$

$3x^2 + 2x - 8 = 0$

$x = \frac{-2 \pm \sqrt{4 + 96}}{6}$

$x = \frac{-2 \pm 10}{6}$

$x = -2, \frac{4}{3}$

but $x > 0$,

$\therefore \ x = \frac{4}{3}$

$y = 4 - x = \frac{8}{3}$

3 $x + y = 100 \quad P = xy$

$y = 100 - x$

$P = x(100 - x)$

$= 100x - x^2$

$\frac{dp}{dx} = 100 - 2x$

$\frac{dp}{dx} = 0,$

$x = 50$

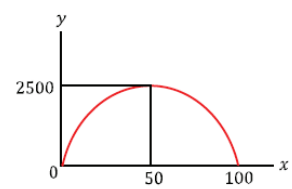
this gives max P

$x = 50, y = 100 - 50 = 50 = x$

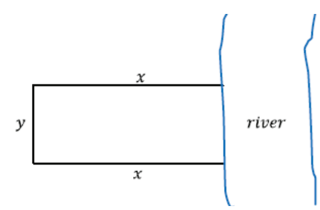
QED

$P = 50^2$

$P = 2500$



4



$y + 2x = 4$

$y = 4 - 2x$

$A = xy$

$= x(4 - 2x)$

$= 4x - 2x^2$

$\frac{dA}{dx} = 4 - 4x$

$\frac{dA}{dx} = 0,$

$x = 1$ km

the farmer should make one side 2 km long and the other two sides 1 km long, using 2 km of river.

5 $p, q > 0$

$p^3 q = 9$

$\Rightarrow q = \dfrac{9}{p^3}$

$z = 16p + 3q$

$z = 16p + \dfrac{27}{p^3}$

$\dfrac{dz}{dx} = 16 - \dfrac{81}{p^4}$

$\dfrac{dz}{dx} = 0,$

$\dfrac{81}{p^4} = 16$

$\dfrac{3}{p} = 2 \textit{ since } p > 0$

$p = \dfrac{3}{2}$

$q = \dfrac{9}{p^3} = \dfrac{9}{\left(\frac{27}{8}\right)} = \dfrac{8}{3}$

6 $SA = 150$, base has side $x(x >, not \geq 0)$

a $\text{SA} = 2x^2 + 4xh$

$150 = 2x^2 + 4xh$

$h = \dfrac{75 - x^2}{2x}\ QED$

b $V = x^2 h$

$= x^2\left(\dfrac{75 - x^2}{2x}\right)$

$V = \dfrac{75x - x^3}{2}$

c $\dfrac{dv}{dx} = \dfrac{75}{2} - \dfrac{3}{2}x^2$

$\dfrac{dv}{dx} = 0,$

$x^2 = \dfrac{75}{2} \times \dfrac{2}{3}$

$= 25$

$x = 5\,\text{cm}$

$V = \dfrac{75 \times 5 - 5^2}{2} = \dfrac{375 - 125}{2} = 125\,\text{cm}^3$

7 $P = 100n - 0.4n^2 - 160$

a **i** $\dfrac{dP}{dn} = 100 - 0.8n$

$\dfrac{dP}{dn} = 0$ implies $100 = 0.8n$

$\therefore n = 125$

A maximum occurs when $n = 125$ as P is a quadratic with negative coeffient of n^2.

ii When $n = 125$, $P = 100 \times 125 - 0.4 \times 125^2 - 160 = 6090$

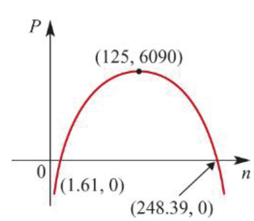
Maximum daily profit is \$ 6090.

b When $P = 0$,

$n = -\dfrac{100 \pm \sqrt{100^2 - .4 \times 04 \times 160}}{-0.8}$

$\therefore\ n \approx 1.6, 248.4$

In this problem a continuous model for a discrete situation has been used.



c $P > 0$ implies $2 \leq n \leq 248$ (Note: n can only take integer values)

d Let \$$P$ be the profit per article

$$\therefore\ p = \frac{P}{n}\left(= \frac{\text{total profit}}{\text{no. of articles}}\right)$$
$$= \frac{100n - 0.4n^2 - 160}{n}$$
$$= 100 - 0.4n - \frac{160}{n}$$

In order to find the maximum profit per article consider the derivative of p with resect to n.

$$\frac{dp}{dn} = -0.4 + \frac{160}{n^2}$$
$$\frac{dp}{dn} = 0 \text{ implies } 0.4 = \frac{160}{n^2}$$
$$\therefore\ n^2 = \frac{160}{0.4}$$

i.e. $n^2 = 400$

$\therefore\ n = 20$

The gradient chart indicates maximum:

	< 20	20	> 20
sign $f'(x)$	+ve	0	−ve
shape	/	−	\

i.e. selling 20 articles maximises the profit per article.

8

$$S(x) = -x^3 + 3x^2 + 360x + 5000,$$
$$x \in [6, 20]$$
$$S'(x) = -3x^2 + 6x + 360$$
$$S'(x) = 0,$$
$$x^2 - 2x - 120 = 0$$
$$(x + 10)(x - 12) = 0$$
$$x = -10, 12$$
$$\text{but } x \in [6, 20]$$
$$x = 12$$

values to test :

$S(6), S(12), S(20)$

$$S(6) = -216 + 108 + 2160 + 5000$$
$$= 2052 + 5000 = 7052$$
$$S(12) = -1728 + 432 + 4320 + 5000$$
$$= 3024 + 5000 = 8024$$
$$S(20) = -8000 + 1200 + 7200 + 5000$$
$$= 400 + 5000 = 5400$$

absolute maximum = 12°C

$S(12) = 8024$ salmon

9 $M(x) = \frac{-1}{30}(x^3 - 14x^2 + 32x - 50),$
$0 \le x \le 10$

$$M'(x) = \frac{-1}{30}(3x^2 - 28x + 32)$$
$$M'(x) = 0,$$
$$3x^2 - 28x + 32 = 0$$
$$x = \frac{28 \pm \sqrt{784 - 384}}{6}$$
$$x = \frac{28 \pm 20}{6}$$
$$x = \frac{4}{3}, 8$$

values to test:

$$x = 0, x = \frac{4}{3}, x = 8, x \to \infty$$

$$M(0) = \frac{50}{30} = \frac{5}{3}$$

$$M\left(\frac{4}{3}\right) = \frac{1}{30}\left(50 - 30 \times \frac{4}{3} + 14 \times \frac{16}{9} - \frac{64}{27}\right)$$

$$= \frac{1}{30}\left(\frac{1}{27}(1350 - 1152 + 672 - 64)\right)$$

$$= \frac{806}{810}$$

$$M\left(\frac{4}{3}\right) = \frac{403}{405}$$

$$M(8) = \frac{1}{30}(50 - 32 \times 8 + 14$$

$$\times 64 - 512)$$

$$= \frac{178}{30}$$

$$= \frac{89}{15}$$

.

Maximum M occurs at $x = 8$ mm

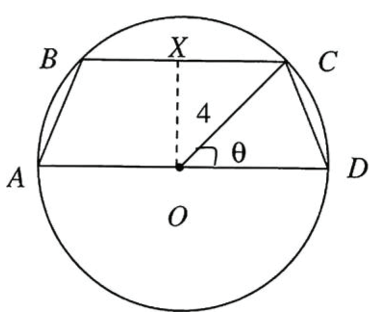
Minimum when $x = \frac{4}{3}$

10 a Let X be the midpoint of BC.

Angle $XCO = \theta$

Therefore $XC = 4\cos\theta$

$BC = 8\cos\theta$



b

$$\text{Area} = \frac{1}{2} \times (8 + 8\cos\theta) \times 4\sin\theta$$

$$= 16\sin\theta(1 + \cos\theta)$$

$$A = 16\sin\theta(1 + \cos\theta)$$

$$\frac{dA}{d\theta} = 16[\cos\theta(1 + \cos\theta) - \sin^2\theta)]$$

$$= 16[\cos^2\theta - \sin^2\theta + \cos\theta]$$

$$= 16[\cos^2\theta - (1 - \cos^2\theta) + \cos\theta]$$

$$= 16[2\cos^2\theta + \cos\theta - 1]$$

$\frac{dA}{d\theta} = 0$ implies

$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

For the figure to exist $\cos\theta = \frac{1}{2}$

which implies $\theta = \frac{\pi}{3}$

Therefore maximum area

$$= 16\sin\frac{\pi}{3}\left(1 + \cos\frac{\pi}{3}\right)$$

$$= 16 \times \frac{\sqrt{3}}{2} \times \frac{3}{2} = 12\sqrt{3} \text{ square units}$$

11 $$\textit{distance} = \sqrt{(x-3)^2 + y^2}$$

$$= \sqrt{x^2 - 6x + 9 + x^4}$$

want minimum distance

$$\frac{d}{dx}(x^4 + x^2 - 6x + 9) = 0$$

$$4x^3 + 2x - 6 = 0$$

$$2x^3 + x - 3 = 0$$

try $x = 1$

$2 + 1 - 3 = 0$ ✓

$$(x-1)(2x^2+2x+3)=0$$

$$x-1=0, 2x^2+2x+3=0$$

$$x=1 \quad x=\frac{-2\pm\sqrt{4-24}}{4}$$

no solution

$x=1$ is the only solution

$y=1$

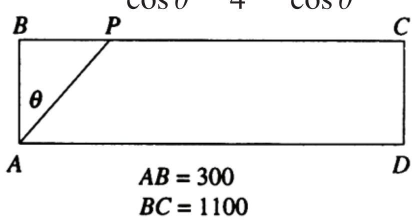
$\therefore$ (1, 1) is the point on $y=x^2$ closest to (3, 0)

12 a $\dfrac{AB}{AP}=\cos\theta$

$\therefore AP=\dfrac{300}{\cos\theta}$

$\therefore$ time taken to run from

A to $P=\dfrac{300}{\cos\theta}\times\dfrac{1}{4}=\dfrac{75}{\cos\theta}$



b

$$PC=BC-BA\tan\theta$$
$$=1100-300\tan\theta$$

$\therefore$ the time taken to run from P to C

$$=\frac{1100-300\tan\theta}{5}$$
$$=220-60\tan\theta$$

c Let T denote the total time then $T=$ time to run from A to $P+$ time taken to run from P to C

$$=\frac{75}{\cos\theta}+220-60\tan\theta$$
$$=\frac{75}{\cos\theta}+220-60\frac{\sin\theta}{\cos\theta}$$
$$=\frac{75-60\sin\theta}{\cos\theta}+220$$

d The quotient rule gives

$$\frac{dT}{d\theta}=\frac{\cos\theta(-60\cos\theta)+\sin\theta(75-60\sin\theta)}{\cos^2\theta}$$

(Note: $\dfrac{d}{d\theta}(220)=0$)

$$=\frac{-60\cos^2\theta+75\sin\theta-60\sin^2\theta}{\cos^2\theta}$$
$$=\frac{-60[\cos^2\theta+\sin^2\theta]+75\sin\theta}{\cos^2\theta}$$
$$=\frac{75\sin\theta-60}{\cos^2\theta}$$

e $\dfrac{dT}{d\theta}=0$

implies $\dfrac{75\sin\theta-60}{\cos^2\theta}=0$

$$\therefore \sin\theta=\frac{60}{75}=\frac{4}{5}$$
$$\therefore \theta=\sin^{-1}\left(\tfrac{4}{5}\right)$$

(Only the acute angle solution needs to be considered).

$\theta\approx 53.13°$

In order to confirm a minimum consider the following

When $\theta=60°$,

$$\frac{dT}{d\theta}=\frac{75\sin 60°-60}{\cos^2\theta}>0$$

When $\theta=50°$,

$$\frac{dT}{d\theta}=\frac{75\sin 50°-60}{\cos^2\theta}<0$$

$\therefore$ a minimum occurs when $\theta=\sin^{-1}\left(\dfrac{4}{5}\right)$

f

When $\sin\theta=\dfrac{4}{5}$, $T=\dfrac{75-60\times\frac{4}{5}+220}{\frac{3}{5}}$

$$=45+220$$
$$=265$$

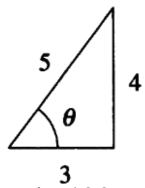
$\therefore$ minimum time taken is 265 seconds.

If $\sin\theta = \dfrac{4}{5}, \tan\theta = \dfrac{4}{5}$

$\therefore BP = BA\tan\theta$

$= 300 \times \dfrac{4}{5}$

$= 400$



P is 400 metres from B for a minimum time.

13

$N(t) = 50te^{-0.1t}$

$N'(t) = 50e^{-0.1t} - 5te^{-0.1t}$

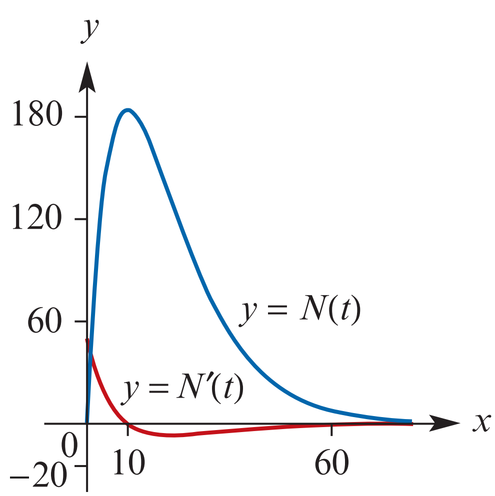
$= 5e^{-0.1t}(10 - t)$

$N'(t) = 0 \Rightarrow t = 10$

Therefore maximum population when $t = 10$

$N(10) = 500e^{-1}$

14 a



b Maximum rate of increase $= N'(0) = 50$

$N''(t) = -5e^{-0.1x} - 0.5e^{-0.1x}(10 - x)$

$N''(t) = 0 \Rightarrow t = 20$

$N'(20) = -\dfrac{50}{e^2}$

Maximum rate of decrease $= N'(20) = \dfrac{50}{e^2}$

15 $V(t) = \dfrac{3}{4}\left(10t^2 - \dfrac{t^3}{3}\right)\ 0 \le t \le 20$

a **i** $V(0) = 0$ The volume of water is 0 mL when $t = 0$

ii $V(20) = \dfrac{3}{4}\left(10 \times 20^2 - \dfrac{20^3}{3}\right)$

$= \dfrac{3 \times 400}{4}\left(10 - \dfrac{20}{3}\right)$

$= 3 \times 100\left(\dfrac{30 - 20}{3}\right)$

$= 1000$

The volume of water is 1000 mL when $t = 20$

b $V'(tf) = \dfrac{3}{4}\left(20t - \dfrac{3t^3}{3}\right)$

$= \dfrac{3}{4}(20t - t^2)$

c

Domain of $V(t) = [0, 20]$

$V(0) = 0$ and $V(20) = 1000$

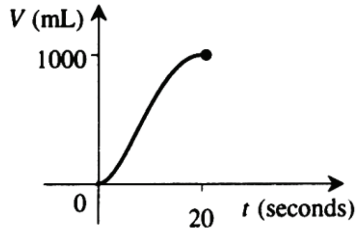
$V'(t) = 0$ implies $20t - t^2 = 0$

$\therefore\ t(20 - t) = 0$

$\therefore\ t = 0$ or $t = 20$

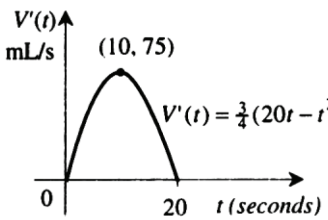
A gradient chart

t	<	0	<<	20	>
sign of $V'(t)$	$-ve$	0	$+ve$	0	$-ve$
shape	\	–	/	–	\



reveals a local minimum at (0,0) and a local minimum at (20,1000)

d The graph of $V'(t)$ against t is a parabola with t intercepts 0 and 20.



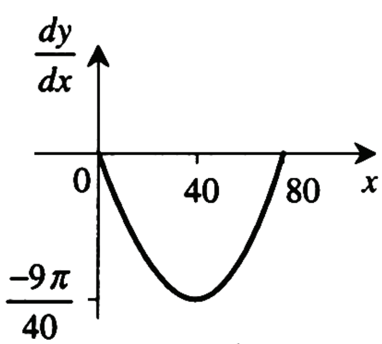
The maximum occurs when $t = 10$ and

$$V'(10) = \frac{3}{4}(200 - 100)$$
$$= 75$$

e The flow is greatest after 10 seconds and the flow is 75 mL/s.

16 a $$\frac{dy}{dx} = -\frac{18\pi}{80}\sin\left(\frac{\pi x}{80}\right) \quad x \in [0, 80]$$
$$= -\frac{9\pi}{40}\sin\left(\frac{\pi x}{80}\right)$$

b



For the graph of $\frac{dy}{dx}$ against x

amplitude $= \frac{18\pi}{80} = \frac{9\pi}{40}$

period $= 2\pi \div \frac{\pi}{80} = 2\pi \times \frac{80}{\pi} = 160$

When $x = 0$, $\frac{dy}{dx} = 0$

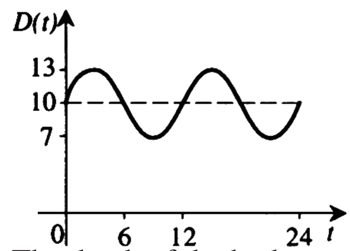
When $x = 80$, $\frac{dy}{dx} = \frac{18\pi}{80}\sin\left(\frac{\pi \times 80}{80}\right)$

$$= \frac{18\pi}{80}\sin\pi = 0$$

c Maximum gradient magnitude occurs where $\sin\left(\frac{\pi x}{80}\right) = \pm 1$

This occurs when $x = 40$ for $0 \le x \le 80$

17 a



The depth of the harbour at time t is given by

$D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right) 0 \le t \le 24$

amplitude = 3

period $= 2\pi \div \frac{\pi}{6} = 2\pi \times \frac{6}{\pi} = 12$

centre $D = 10$

range $= [10 - 3, 10 + 3] = [7, 13]$

b $D(t) \geq 8.5$

$\Leftrightarrow 10 + 3\sin\left(\frac{\pi t}{6}\right) \geq 8.5$

which is equivalent to

$$3\sin\left(\frac{\pi t}{6}\right) \geq -1.5$$

$$\sin\left(\frac{\pi t}{6}\right) \geq -\frac{1}{2}$$

Consider $\quad \sin\left(\frac{\pi t}{6}\right) = -\frac{1}{2}$

then $\quad \frac{\pi t}{6} = \frac{7\pi}{6}$

or $\frac{11\pi}{6}$ or $\frac{19\pi}{6}$ or $\frac{23\pi}{6}$ or $\cdots$

$t = 7$ or 11 or 19 or 23 or $\ldots$

From the graph and considering the domain $[0, 24]$

$\{t : D(t) \geq 8.5\} = [0, 7] \cup [11, 19] \cup [23, 24]$

c The rate of change of depth is given by the derivative function

$D'(t) = \frac{3\pi}{6}\cos\left(\frac{\pi t}{6}\right) = \frac{\pi}{2}\cos\left(\frac{\pi t}{6}\right)$

i $D'(t) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) = 0$

The rate at which the depth is changing when $t = 3$ is 0 metres/hour.

ii $D'(6) = \frac{\pi}{2}\cos\left(\frac{6\pi}{6}\right) = \frac{\pi}{2}\cos(\pi) = -\frac{\pi}{2}$

The rate at which the depth is changing when $t = 3$ is $-\frac{\pi}{2}$ metres/hours.

(This means that the depth is decreasing at a rate of $\frac{\pi}{2}$ metres/hour).

iii $D'(12) = \frac{\pi}{2}\cos(2\pi) = \frac{\pi}{2}$

The depth is increasing at a rate of $\frac{\pi}{2}$ metres/hours

d The function which describes the rate is

$D'(t) = \frac{\pi}{2}\cos\left(\frac{\pi t}{6}\right)$

i $D'(t)$ has a maximum when

$\cos\left(\frac{\pi t}{6}\right) = 1$

$\therefore \frac{\pi t}{6} = 0$ or 2π or 4π or $\cdots$

$t = 0$ or 12 or 24 or $\cdots$

For the required domain the depth is increasing most rapidly when $t = 0$ or $t = 12$ or $t = 24$

ii The depth is decreasing most rapidly when

$\cos\left(\frac{\pi t}{6}\right) = -1$

$\therefore$ when $\frac{\pi t}{6} = \pi$ or 3π or 5π or $\cdots$

$\therefore t = 6$ or 18 or 30 or$\cdots$

For the required domain the depth is decreasing most rapidly when $t = 6$ or 18

Solutions to Exercise 10G

1 $f(x) = (x-1)^2(x-b)\,,\, b > 1$

a $f(x) = (x^2 - 2x + 1)(x - b)$

$= x^3 - (2+b)x^2 + (1+2b)x - b$

$f'(x) = 3x^2 - 2(2+b)x + 1 + 2b$

$= (x-1)(3x - 2b - 1)$

b $f'(x) = 0,$

$(x-1)(3x - (1+2b)) = 0$

$x = 1, x = \dfrac{1+2b}{3}$

$f(1) = 1 - 2 - b + 1 + 2b - b = 0$

$f\left(\dfrac{1+2b}{3}\right) = \left(\dfrac{2b-2}{3}\right)^2\left(\dfrac{2b+1-3b}{3}\right)$

$= \dfrac{4}{9}(b-1)^2\left(\dfrac{1-b}{3}\right)$

$= \dfrac{-4}{27}(b-1)^3$

co-ords $= (1,0)$ & $\left(\dfrac{1+2b}{3}, \dfrac{-4(b-1)^3}{27}\right)$

c $\dfrac{2b+1}{3} > 1$ as $b > 1$ so the other stationary point is a local minimum; hence the point (1,0) is always a local maximum.

d $\dfrac{1+2b}{3} = 4$

$1 + 2b = 12$

$b = \dfrac{11}{2}$

2 $y = x^4 - 4x^2$

a $\dfrac{dy}{dx} = 4x^3 - 8x$

$\dfrac{dy}{dx} = 0,$

$4x(x^2 - 2) = 0$

$x = 0, \pm\sqrt{2}$

$x = 0, y = 0$

$(0, 0)$

$x = \pm\sqrt{2}$

$y = 4 - 4(2)$

$= -4$

$(\pm\sqrt{2}, -4)$

b $(x, y) \to (x + a, y + b)$

$(0, 0) \to (a, b)$

$(\pm\sqrt{2}, -4) \to (a \pm \sqrt{2}, b - 4)$

3 a

$f(x) = ax^3 + bx^2 + cx$

$f(1) = 10 \Rightarrow a + b + c = 10 \ldots (1)$

$f'(x) = 3ax^2 + 2bx + c$

$f'(1) = 0 \Rightarrow 3a + 2b + c = 0 \ldots (2)$

Multiply (1) by 2 and subtract from (2)

$a - c = -20 \Rightarrow a = c - 20$

Substitute for a in (1)

$c - 20 + b + c = 10$

$\therefore b = 30 - 2c$

b

$$f'(3) = 0 \Rightarrow 27a + 6b + c = 0$$
$$\therefore 27(c-20) + 6(30-2c) + c = 0$$
$$27c - 540 + 180 - 12c + c = 0$$
$$16c - 360 = 0$$
$$c = \frac{360}{16} = \frac{45}{2}$$

4 $f : [0, \infty] \to R, f(x) = x^2 - ax^3$
$a > 0$

a $f'(x) = 2x - 3ax^2$
$= x(2 - 3ax)$
$f'(x) < 0,$
$x(2-3ax) < 0$
since $x \geq 0, 2 - 3ax < 0$
$\Rightarrow x > \dfrac{2}{3a}$
$f(x)$ is decreasing when $x > \dfrac{2}{3a}$
$f'(x) > 0,$
$x(2-3ax) > 0$
since $x \geq 0, 2 - 3ax > 0$
$\Rightarrow x < \dfrac{2}{3a}$
$0 < x < \dfrac{2}{3a}$
$f(x)$ is increasing when $0 < x < \dfrac{2}{3a}$

b $f'\left(\dfrac{1}{a}\right) = \dfrac{1}{a}\left(2 - 3\dfrac{a}{a}\right)$
$= \dfrac{-1}{a}$
$y = \dfrac{-x}{a} + c$
$x = \dfrac{1}{a}, y = 0$
$0 = \dfrac{-1}{a^2} + c$
$c = \dfrac{1}{a^2}$
$y = \dfrac{-x}{a} + \dfrac{1}{a^2}$

c $y = ax + c$
$x = \dfrac{1}{a}, y = 0$
$0 = 1 + c$
$c = -1$
$y = ax - 1$

d $max.$ at $x = \dfrac{2}{3a}$
$f\left(\dfrac{2}{3a}\right) = \dfrac{4}{9a^2} - \dfrac{a \times 8}{27a^3} \quad = \dfrac{1}{a^2}\left(\dfrac{4}{9} - \dfrac{8}{27}\right)$
$= \dfrac{4}{27a^2}$
range $= \left(-\infty, \dfrac{4}{27a^2}\right]$

5 a i $y = (x-3)^2$
$\dfrac{dy}{dx} = 2(x-3)$
$= 2x - 6$
$x = a,$
$\dfrac{dy}{dx} = 2a - 6$

ii $m = 2a - 6$

b $x = a$

$$y = (a-3)^2$$
$$= a^2 - 6a + 9$$
$$\Rightarrow (a, a^2 - 6a + 9)$$

c $$y = mx + c$$
$$= (2a-6)x + c$$
$$x = a, y = (a-3)^2$$
$$(a-3)^2 = 2a(a-3) + c$$
$$c = (a-3)(a-3-2a)$$
$$c = (a-3)(-a-3)$$
$$y = (a-3)(2x-a-3)$$
$$= 2(a-3)x - a^2 + 9$$

d $$y = 0,$$
$$2x - a - 3 = 0$$
$$2x = a + 3$$
$$x = \frac{a+3}{2}$$

6 a $$f(x) = x^4$$
$$\Rightarrow f(x+h) = (x+h)^4$$
$$f(1+h) = 16$$
$$(1+h)^4 = 16$$
$$1 + h = \pm 2$$
$$h = -1 \pm 2$$
$$h = -3, 1$$

b $$f(x) = x^3$$
$$\Rightarrow f(ax) = (ax)^3$$
$$f(a) = 8$$
$$a^3 = 8$$
$$a = 2$$

c

$$y = ax^4 - bx^3$$
$$\frac{dy}{dx} = 4ax^3 - 3bx^2$$
$$x = 1,$$
$$\frac{dy}{dx} = 0$$
$$4a = 3b \ldots (1)$$
$$x = 1, y = 16$$
$$16 = a - b$$
$$a = 16 + b \ldots (2)$$
$$\text{sub in (1)} \quad \Rightarrow 4(16+b) = 3b$$
$$64 + 4b = 3b$$
$$b = -64$$
$$\text{sub in (2)} \quad \Rightarrow a = -48$$

7 a $$f(x) = (x-a)^2(x-1)$$
$$= (x^2 - 2ax + a^2)(x-1)$$
$$= x^3 - (2a+1)x^2 + (a^2+2a)x - a^2$$
$$f'(x) = 3x^2 - (4a+2)x + (a^2+2a)$$
$$f'(x) = 0,$$
$$3x^2 - (4a+2)x + (a^2+2a) = 0$$
$$x = \frac{4a + 2 \pm \sqrt{4(2a+1)^2 - 4(3a^2+6a)}}{6}$$
$$= \frac{2a + 1 \pm \sqrt{4a^2 + 4a + 1 - 3a^2 - 6a}}{3}$$
$$= \frac{2a + 1 \pm \sqrt{a^2 - 2a + 1}}{3}$$
$$= \frac{(2a+1) \pm (a-1)}{3}$$
$$x = a, \frac{a+2}{3}$$
$$f(a) = 0,$$

$$f\left(\frac{a+2}{3}\right) = \left(\frac{a+2}{3} - \frac{3a}{a}\right)^2\left(\frac{a+2}{3} - \frac{3}{3}\right)$$
$$= \left(\frac{2-2a}{3}\right)^2\left(\frac{a-1}{3}\right)$$
$$= \frac{4}{9}(1-a)^2\frac{-1}{3}(1-a)$$
$$= \frac{-4(1-a)^3}{27}$$

co-ords $= (a, 0), \left(\frac{a+2}{3}, \frac{-4(1-a)^3}{27}\right)$

b Since $a > 1$, $\frac{-4(1-a)^3}{27} > 0$,
Hence $(a, 0)$ is a local minimum and $\left(\frac{a+2}{3}, \frac{-4(1-a)^3}{27}\right)$ is a local maximum.

c **i** $f'(1) = 3 - 4a - 2 + a^2 + 2a$
$$= a^2 - 2a + 1$$
$$= (a-1)^2$$
$$y = (a-1)^2 x + c$$
$f'(1) = 0$,
$$0 = (a-1)^2 + c$$
$$c = -(a-1)^2$$
$$y = (a-1)^2(x-1)$$

ii $f'(a) = 0$
$$y = c$$
$$f(a) = 0 = c$$
$$y = 0$$

iii $f'\left(\frac{a+1}{2}\right)$
$$= \frac{3(a+1)^2}{4} - (2a+1)(a+1) + (a^2+2a)$$
$$= \frac{3a^2 + 6a + 3 - 8a^2 - 12a - 4 + 4a^2 + 8a}{4}$$
$$= \frac{-a^2 + 2a - 1}{4}$$
$$= \frac{-(a-1)^2}{4}$$
$$f\left(\frac{a+1}{2}\right) = \left(\frac{a+1}{2} - a\right)^2\left(\frac{a+1}{2} - 1\right)$$
$$= \frac{1}{4}(1-a)^2\frac{1}{2}(a-1)$$
$$= \frac{1}{8}(a-1)^3$$
$$y = \frac{-(a-1)^2}{4}x + c$$
$$x = \frac{a+1}{2}, y = \frac{1}{8}(a-1)^3$$

$$\frac{1}{8}(a-1)^3 = \frac{-1}{4}(a-1)^2\left(\frac{a+1}{2}\right) + c$$

$$c = \frac{1}{4}(a-1)^2\left(\frac{a-1}{2} + \frac{a+1}{2}\right)$$

$$= \frac{1}{4}(a-1)^2$$

$$y = \frac{1}{4}(a-1)^2(-x+a)$$

$$= \frac{-1}{4}(a-1)^2(x-a)$$

8 **a** $f'(x) = (x-1)^2\frac{d}{dx}(x-b)^2 + (x-b)^2\frac{d}{dx}(x-1)^2$

$$= 2(x-1)(x-b)((x-1)+(x-b))$$

$$= 2(x-1)(x-b)(2x-b-1)$$

b $f'(x) = 0,$

$$x = 1, b, \frac{b+1}{2}$$

$$f(1) = 0, f(b) = 0,$$

$$f\left(\frac{b+1}{2}\right) = \left(\frac{b-1}{2}\right)^2\left(\frac{1-b}{2}\right)^2$$

$$= \left(\frac{b-1}{2}\right)^4$$

$$\text{co-ords} = (1,0)\,(b,0)\left(\frac{b+1}{2}, \frac{(b-1)^4}{16}\right)$$

c $\frac{b+1}{2} = 2$

$$b = 1 = 4$$

$$b = 3$$

9 $a = \frac{1}{486}$, $b = 0$, $c = \frac{-1}{161}$, $d = \frac{1459}{243}$

10 $f(x) = ax^4 + bx^3 + cx^2 + dx$

a $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$f(1) = 1$

$1 = a + b + c + d \ldots (1)$

$f'(1) = 0$

$0 = 4a + 3b + 2c + d \ldots (2)$

$f'(-1) = 4$

$4 = a - b + c - d \ldots (3)$

$(1) - (3) \Rightarrow \quad -3 = 2b + 2d$

$b = \frac{-3}{2} - d$

$(2) - (1) - (3) \Rightarrow \; -5 = 2a + 3b + d$

$-5 = 2a - \frac{9}{2} - 3d + d$

$-\frac{1}{2} = 2a - 2d$

$a = d - \frac{1}{4}$

sub in (1) $\Rightarrow \quad 1 = \left(d - \frac{1}{4}\right) + \left(\frac{-3}{2} - d\right)c + d$

$1 = -\frac{7}{4} + c + d$

$c = \frac{11}{4} - d$

b $f'(4) = 0$

$0 = 4a(64) + 3b(16) + 2c(4) + d$

$0 = 256a + 48b + 8c + d$

$0 = 256\left(\frac{4d-1}{4}\right) + 48\left(\frac{-3-2d}{2}\right)$

$+ 8\left(\frac{11-4d}{4}\right) + d$

$0 = 256d - 64 - 72 - 48d + 22 - 8d + d$

$0 = 201d - 114$

$d = \frac{114}{201}$

$d = \frac{38}{67}$

Solutions to Technology-free questions

1 **a** $y = x^3 - 8x^2 + 15x$

$\frac{dy}{dx} = 3x^2 - 16x + 15$

$= -1 \text{ (at } x = 4)$

For the tangent:

$y + 4 = -1(x - 4)$

$y = -x$

b Tangent meets curve again when

$x^3 - 8x^2 + 15x = -x$

$x^3 - 8x^2 + 16x = 0$

$x(x^2 - 8x + 16) = 0$

$x(x - 4)^2 = 0$

Thus $x = 0$ and then $y = 0$ ($x = 4$ corresponds to the given point).

The tangent meets the curve again at the point $(0, 0)$.

2 At $x = a, y = 3a^2$

$y = 3x^2$

$\frac{dy}{dx} = 6x$

$= 6a \text{ (at } x = a)$

For the tangent:

$y - 3a^2 = 6a(x - a)$

$y = 6ax - 3a^2$

$x = 0 : y = -3a^2$, so the tangent meets the y axis where $y = -3a^2$.

3 $y = x^3 - 7x^2 + 14x - 8$

$\frac{dy}{dx} = 3x^2 - 14x + 14$

$= 3 \text{ (at } x = 1)$

For the tangent, $x = 1$ gives $y = 0$, so:

$y - 0 = 3(x - 1)$

$y = 3x - 3$

A parallel tangent has gradient 3, so:

$\frac{dy}{dx} = 3$

$3x^2 - 14x + 14 = 3$

$3x^2 - 14x + 11 = 0$

$(3x - 11)(x - 1) = 0$

$x = 1, \frac{11}{3}$

The x coordinate of a second point with the same gradient is $x = \frac{11}{3}$.

4 **a** Average rate is given by

$\frac{A(3) - A(2)}{3 - 2} = \frac{9\pi - 4\pi}{1} = 5\pi$

b $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$= 6\pi \text{ (at } r = 3)$

Instantaneous rate is 6π

5 **a** $f(x) = 4x^3 - 3x^4$

$f'(x) = 12x^2 - 12x^3$

$= 12x^2(1 - x)$

$= 0 \text{ if } x = 0, 1$

$x = 0, y = 0; x = 1, y = 1$

The stationary points have coordinates $(0, 0)$ and $(1, 1)$.

$x < 0, f'(x) > 0; f'(0) = 0;$

$0 < x < 1, f'(x) > 0$; so $(0, 0)$ is a stationary point of inflexion.

$0 < x < 1, f'(x) > 0; f'(1) = 0;$

$x > 1, f'(x) < 0$; so $(1, 1)$ is a maximum.

b $g(x) = x^3 - 3x - 2$

$g'(x) = 3x^2 - 3$

$= 3(x + 1)(x - 1)$

$= 0$ if $x = -1, 1$

$x = -1, y = 0; x = 1, y = -4$

The stationary points have coordinates $(-1, 0)$ and $(1, -4)$.

$x < -1, f'(x) > 0; f'(-1) = 0;$

$-1 < x < 1, f'(x) < 0;$ so $(-1, 0)$ is a maximum.

$-1 < x < 1, f'(x) < 0; f'(1) = 0;$

$x > 1, f'(x) > 0;$ so $(1, -4)$ is a minimum.

c $h(x) = x^3 - 9x + 1$

$g'(x) = 3x^2 - 9$

$= 3(x^2 - 3)$

$= 3(x + \sqrt{3})(x - \sqrt{3})$

$= 0$ if $x = -\sqrt{3}, \sqrt{3}$

$x = -\sqrt{3}, y = 6\sqrt{3} + 1$

$x = \sqrt{3}, y = -6\sqrt{3} + 1$

The stationary points have coordinates $(-\sqrt{3}, 6\sqrt{3} + 1)$ and $(\sqrt{3}, -6\sqrt{3} + 1)$.

$x < -\sqrt{3}, f'(x) > 0; f'(-\sqrt{3}) = 0;$

$-\sqrt{3} < x < \sqrt{3}, f'(x) < 0;$

so there is a maximum at

$(-\sqrt{3}, 6\sqrt{3} + 1)$.

$-\sqrt{3} < x < \sqrt{3}, f'(x) < 0;$

$f'(\sqrt{3}) = 0;$

$x > \sqrt{3}, f'(x) > 0;$

so there is a minimum at

$(\sqrt{3}, -6\sqrt{3} + 1)$.

6 $y = x^3 - 6x^2 + 9x = x(x - 3)^2$

$\dfrac{dy}{dx} = 3x^2 - 12x + 9$

$= 3(x^2 - 4x + 3)$

$= 3(x - 1)(x - 3)$

$= 0$ if $x = 1, 3$

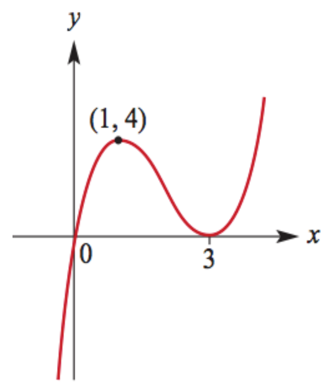
$x = 1, y = 4; x = 3, y = 0$

The stationary points have coordinates $(1, 4)$ and $(3, 0)$. Also it is evident from the factorised form that $(3, 0)$ is a stationary point of inflexion. Then $(1, 4)$ must be a maximum.

For the intercepts:

$y = 0, x = 0, 3$

The graph is shown below.



7 $\dfrac{dy}{dx} = (x - 1)^2(x - 2)$

$= 0$ if $x = 1, 2$

There are stationary points where $x = 1$ and $x = 2$.

$x < 1, \dfrac{dy}{dx} < 0; \dfrac{dy}{dx} = 0$ at $x = 1;$

$1 < x < 2, \dfrac{dy}{dx} < 0;$

so there is a stationary point of inflexion at $x = 1$.

$1 < x < 2, \dfrac{dy}{dx} < 0; \dfrac{dy}{dx} = 0$ at $x = 2;$

$x > 2, \dfrac{dy}{dx} > 0;$ so there is a minimum at $x = 2$.

8 $y = x^3 - 3x^2 - 9x + 11$

$\dfrac{dy}{dx} = 3x^2 - 6x - 9$

$= -9$ (at $x = 2$)

Also when $x = 2, y = -11$

For the tangent:

$y + 11 = -9(x - 2)$

$y = -9x + 7$

9 $f(x) = 3 + 6x^2 - 2x^3$

$f'(x) = 12x - 6x^2$

$= 6x(2 - x)$

$= 0$ if $x = 0, 2$

The graph of the gradient function is an inverted parabola with x intercepts 0 and 2. So the gradient function is positive if $0 < x < 2$, i.e. the graph of $y = f(x)$ has a positive gradient in the interval $(0, 2)$.

10

$y = x^3 \Rightarrow \dfrac{dy}{dx} = 3x^2$

$y = x^3 + x^2 + x - 2 \Rightarrow \dfrac{dy}{dx} = 3x^2 + 2x + 1$

So the graphs have the same gradient when $3x^2 = 3x^2 + 2x + 1$, i.e. when $2x + 1 = 0$, or $x = -\dfrac{1}{2}$.

11 $f(x) = (x - 1)^{\frac{4}{5}}$

a The function is differentiable for $R\backslash\{1\}$. $f'(x) = \dfrac{4}{5}(x - 1)^{-\frac{1}{5}}$

b $f'(0) = \dfrac{4}{5}, f'(2) = -\dfrac{4}{5}$

For the tangent at (2, 1):

$y - 1 = \dfrac{4}{5}(x - 2)$

$y = \dfrac{4}{5}x - \dfrac{3}{5}$

For the tangent at (0, 1):

$y - 1 = -\dfrac{4}{5}(x - 0)$

$y = -\dfrac{4}{5}x + 1$

c $\dfrac{4}{5}x - \dfrac{3}{5} = -\dfrac{4}{5}x + 1$

$\dfrac{8}{5}x = \dfrac{8}{5}$

$x = 1$

When $x = 1, y = \dfrac{1}{5}$, so $\left(1, \dfrac{1}{5}\right)$ is the point of intersection of the tangents.

12 For a sphere of radius r and volume V, $V = \dfrac{4}{3}\pi r^3$.

a $\dfrac{dV}{dr} = 4\pi r^2$

$= 64\pi$ if $r = 4$

The rate of increase of volume with respect to the change in radius is 64π cm^3/cm when the radius is 4 cm.

b $\dfrac{dV}{dt} = \dfrac{dV}{dr} \times \dfrac{dr}{dt}$

$= 4\pi r^2 \times 1$

$= 4\pi r^2$

$= 64\pi$ if $r = 4$

The rate of increase of volume with respect to time is 64π cm^3/s when the radius is 4 cm.

(An alterative is to use the initial conditions to express r in terms of t, i.e. $r = 1 + t$, and then V in terms of t; then differentiate to get the result directly.)

13 $x = 0.25e^t$

$$v = \frac{dx}{dt} = 0.25e^t$$

At $t = 0, 1, 2, 4$, the velocity v in m/s is respectively $0.25, 0.25e, 0.25e^2, 0.25e^4$.

14 $\theta = \frac{1}{4}e^{100t}$

a $\frac{d\theta}{dt} = 25e^{100t}$ °C/s

b $\frac{d\theta}{dt} = 25e^5 \left(\text{at } t = \frac{1}{20}\right)$

So the rate of increase is $25e^5$ °C/s.

15

$$\begin{aligned} y &= e^x \\ \frac{dy}{dx} &= e^x \\ &= e \text{ (at } x = 1) \quad y - e = e(x-1) \\ y - e &= ex - e \\ y &= ex \end{aligned}$$

16 $D = 50e^{kt}$

a

$$\begin{aligned} \frac{dD}{dt} &= 50ke^{kt} \\ &= k \times 50e^{kt} = kD \end{aligned}$$

Thus $\frac{dD}{dt} = cD$, where $c = k$.

b $\frac{dD}{dt} = kD = 0.2 \times 100 = 20$ cm/year

17

$$\begin{aligned} y &= e^{3x} + e^{-3x} \\ \frac{dy}{dx} &= 3e^{3x} - 3e^{-3x} \\ &= 0 \text{ if} \\ 3e^{3x} &= 3e^{-3x} \\ e^{6x} &= 1 \\ x &= 0 \end{aligned}$$

When $x = 0, y = 2$.
Since $y \to \infty$ when $x \to \pm\infty$, it is evident that $y = 2$ is a minimum.

18 a

$$\begin{aligned} y &= \log_e x \\ \frac{dy}{dx} &= \frac{1}{x} \\ &= \frac{1}{e} \text{ (at } x = e) \\ y - 1 &= \frac{1}{e}(x - e) \\ y - 1 &= \frac{1}{e}x - 1 \\ y &= \frac{1}{e}x \end{aligned}$$

b

$$\begin{aligned} y &= 2\sin\left(\frac{x}{2}\right) \\ \frac{dy}{dx} &= \cos\left(\frac{x}{2}\right) \\ &= \frac{1}{\sqrt{2}} \left(\text{at } x = \frac{\pi}{2}\right) \\ y - \sqrt{2} &= \frac{1}{\sqrt{2}}(x - \frac{\pi}{2}) \\ y &= \frac{1}{\sqrt{2}}x - \frac{\pi}{2\sqrt{2}} + \sqrt{2} \end{aligned}$$

c $y = \cos x$

$$\frac{dy}{dx} = -\sin x$$

$$= 1 \left(\text{at } x = \frac{3\pi}{2}\right)$$

$$y = (1)\left(x - \frac{3\pi}{2}\right)$$

$$y = x - \frac{3\pi}{2}$$

d $y = \log_e(x^2)$

$$\frac{dy}{dx} = \frac{2}{x}$$

$$= -\frac{2}{\sqrt{e}} \text{ (at } x = -\sqrt{e}\text{)}$$

$$y - 1 = -\frac{2}{\sqrt{e}}(x + \sqrt{e})$$

$$y = -\frac{2}{\sqrt{e}}x - 1$$

Solutions to multiple-choice questions

1 A $y = x^2 - x - 5$

$$\frac{dy}{dx} = 2x - 1$$

Gradient of tangent equation=4

$$\therefore \frac{dy}{dx} = 4$$

$$\therefore 4 = 2x - 1$$

$$x = \frac{5}{2}$$

$$y = \left(\frac{5}{2}\right)^2 - \frac{5}{2} - 5$$

$$y = -\frac{5}{4}$$

Sub x and y values into $y = 4x + c$

$$-\frac{5}{4} = 10 + c$$

$$c = -\frac{45}{4}$$

2 E Tangent of $y = x^4$ at $x = 1$

$$\frac{dy}{dx} = 4x^3$$

$$\frac{dy}{dx} = 4$$

Equation of tangent:

$$y = 4x + c$$

At $x = 1, y = 1$

$$1 = 4 + c$$

$$c = -3$$

$$\therefore y = 4x - 3$$

3 E Since the gradient changes from negative to positive at point a, this is a local minimum.
Since the gradient remains the same at, before and after point b, this is a stationary point of inflection.

4 E The graph of the second function is obtained from the graph of the first function by this sequence of transformations:
(1) a reflection in the x-axis
(2) a dilation of factor 2 from the x-axis
(3) a dilation of factor 2 from y-axis
(4) a translation of k units vertically up
The point $(0, 0)$ transforms to $(0, k)$ and is now a maximum due to the reflection (the dilation leave no effect).
The maximum point $a, f(a)$ of the original graph transforms as follows:
(1) $(a, -f(a)$; local minimum
(2) $(a, -2f(a)$; local minimum
(3) $(2a, -2f(a)$; local minimum
(4) $(2a, -2f(a) + k)$ local minimum

5 B $f(x) = x^3 - x^2 - 1$

$$f'(x) = 3x^2 - 2x$$

Stationary points occur when

$$f'(x) = 0$$

$$3x^2 - 2x = 0$$

$$x(3x - 2) = 0$$

Using the null factor theorem:

$$x = 0 \text{ and } x = \frac{2}{3}$$

6 C As it is a local minimum the gradient of the tangent is 0. Therefore it is a horizontal line which goes through the point (2, 4)

$$y = mx + c$$

$$m = 0$$

$$\therefore\ y = 4$$

7 B $V = -10x(2x^2 - 6)$

$V = -20x^3 + 60x$

$\frac{dV}{dx} = -60x^2 + 60$

$0 = -60x^2 + 60$

$60x^2 = 60$

$x = \pm 1$

When $x = -1, V = -80$

$V \neq -80$

$\therefore$ Maximum volume occurs when $x = 1$

8 B $y = x^2$

$\frac{dy}{dx} = 2x$

At $x = a$, gradient of tangent $= 2a$

Normal at $x = a : -\frac{1}{2a}$

$\therefore y = -\frac{1}{2a}x + c$

At $x = a, y = a^2$

$\therefore a^2 = -\frac{1}{2a}a + c$

$\therefore c = a^2 + \frac{1}{2}$

Equation of normal:

$y = -\frac{1}{2a}x + \frac{1}{2} + a^2$

9 D $f : R \to R, f(x) = e^x - ex$

$f'(x) = e^x - e$

$f'(x) = 0$

$e^x - e = 0$

$e^x = e$

$x = 1$

Turning point occurs at $x = 1$

Sub into $f(x)$ to find y coordinate:

$y = e - e$

$y = 0$

$\therefore (1, 0)$

10 E $y = e^{ax}$

Tangent at point $\left(\frac{1}{a}, e\right)$

$\frac{dy}{dy} = ae^{ax}$

At $x = \frac{1}{a}$

$\frac{dy}{dx} = ae^{\frac{a}{a}}$

$\frac{dy}{dx} = ae$

Equation of tangent: $y = aex + c$

Sub in point $\left(\frac{1}{a}, e\right)$

$e = ae\frac{1}{a} + c$

$e = e + c$

$c = 0$

$\therefore$ equation of tangent:

$y = aex$

11 A $N = 4000e^{0.2t}$

$\frac{dN}{dt} = 800e^{0.2t}$

When $t = 3$

$\frac{dN}{dt} = 800e^{0.6} \approx 1458$

12 E $y = x^2 \cos(5x)$

Tangent at $x = \pi$

Using product rule to find derivative:

$\frac{dy}{dx} = 2x\cos(5x) - 5x^2 \sin(5x)$

Gradient of tangent:

$\frac{dy}{dx} = 2\pi\cos(5\pi) - 5\pi^2 \sin(5\pi)$

$\frac{dy}{dx} = -2\pi$

13 B $y = e^{-x} - 1$

Point where equation crosses the y-axis:

$x = 0, y = 0$ coordinate: $(0, 0)$

$$\frac{dy}{dx} = -e^{-x}$$

Gradient of tangent at $x = 0$:

$$\frac{dy}{dx} = -1$$

Equation of tangent:

$y = -x + c$

Sub in point $(0, 0)$:

$c = 0$

$\therefore$ Equation of tangent:

$y = -x$

14 C

$$f(x) = e^{ax} - \frac{ax}{e}$$

$$f'(x) = ae^{ax} - \frac{a}{e}$$

$$= 0 \text{ if}$$

$$ae^{ax} = \frac{a}{e}$$

$$e^{ax} = \frac{1}{e} = e^{-1}$$

$$ax = -1$$

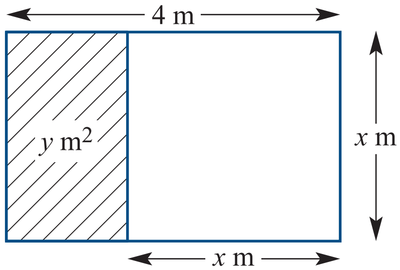
$$x = -\frac{1}{a}$$

$$f\left(-\frac{1}{a}\right) = e^{-1} - \frac{-1}{e} = \frac{2}{e}$$

The coordinates of the turning point are $\left(-\frac{1}{a}, \frac{2}{e}\right)$.

Solutions to extended-response questions

1

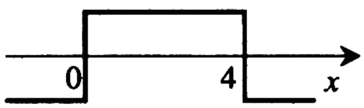


a Shaded area $= 4x - x^2$

i.e. $y = 4x - x^2$

b As $y > 0, 4x - x^2 > 0$

i.e $x(4 - x) > 0$



$\therefore\ y > 0$ for $0 < x < 4$

The possible values of x are $0 < x < 4$.

c $\dfrac{dy}{dx} = 4 - 2x$

$\dfrac{dy}{dx} = 0$ implies $x = 2$

Note: $y = 4x - x^2$ is a quadratic with negative coefficient of x^2.

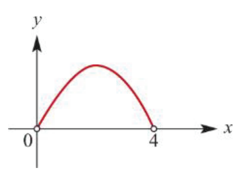
When $x = 2, y = 8 - 4 = 4$.

The maximum value of y is 4 and this occurs when $x = 2$.

d $y = 4x - x^2$ is a quadratic with negative coefficient of x^2

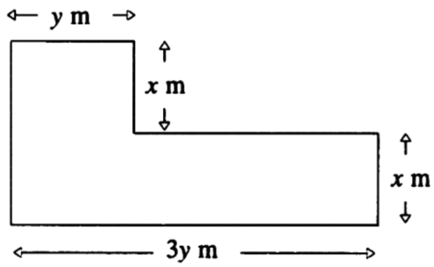
or gradient to the left of $x = 2$ is positive and to the right negative.

e



f From the graph the possible values for y are $0 < y < 4$.

2



a $A = xy + 3xy = 4xy$

b Perimeter $= 48$

$$\therefore\ 48 = 6y + 4x$$

$$y = \frac{48 - 4x}{6}$$

$$= 8 - \frac{2}{3}x$$

c $A = 4xy$

$$= 4x\left(8 - \frac{2}{3}x\right)$$

$$= 32x - \frac{8x^2}{3}$$

d $\frac{dA}{dx} = 32 - \frac{16x}{3}$

$\frac{dA}{dx} = 0$ implies $x = \frac{96}{16} = 6$. Maximum as quadratic with negative coefficient of x^2

When $x = 6, y = 8 - \frac{2}{3} \times 6 = 4$

e When $x = 6$

$$A = 32 \times 6 - \frac{8}{3} \times 36$$

$$= 96$$

The maximum area is 96 m^2.

3 **a** Cost is $(12 + 0.008x)$ dollars per kilometre plus \$14.40 per hour for the driver, where x is the speed of the truck in km/h

i Cost per kilometre for tmck travelling at 40 km/h

$$= (12 + 0.008 \times 40) + 14.40 \times \frac{1}{40}$$

$$= 12.68$$

i.e. the cost per kilometre is \$12.68.

ii Cost per kilometre for truck travelling at 64 km/h

$$= (12 + 0.008 \times 64) + \frac{1}{64} \times 14.40$$

$$= 12.737$$

i.e. the cost per kilometre is \$12.74.

b Let C be the cost per kilometre.

$$C = (12 + 0.008x) + \frac{14.40}{x}$$

$$= 12 + 0.008x + \frac{14.40}{x}$$

c To sketch the graph we first differentiate to determine tuming points.

For $C = 12 + 0.008x + \dfrac{14.40}{x}$

$$\frac{dc}{dx} = 0.008 - \frac{14.40}{x^2}$$

and stationary points occur for $\dfrac{dC}{dx} = 0$.

This implies

$$0.008x^2 = 14.40$$

$$x^2 = 1800$$

$$x = 30$$

$$\approx 42.426$$

A sign chart is used to determine the nature of the stationary point.

	$< 30\sqrt{2}$	$30\sqrt{2}$	$> 30\sqrt{2}$
sign $f'(x)$	-ve	0	+ve
shape	\	–	/

$\therefore$ A minimum occurs where $x = 30\sqrt{2}$.

When $x = 30\sqrt{2}$

$$C = 12 + 0.008 \times 30\sqrt{2} + \frac{14.40}{30\sqrt{2}}$$

$$= 12 + 0.24\sqrt{2} + \frac{0.48}{\sqrt{2}}$$

$$= 12 + 0.24\sqrt{2} + 0.24\sqrt{2}$$

$$= 12 + 0.48\sqrt{2}$$

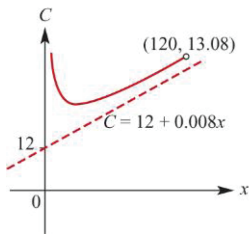
$$\approx 12.679$$

$\therefore$ minimum at $(30\sqrt{2}, 12 + 0.48\sqrt{2})$

When $x = 120$

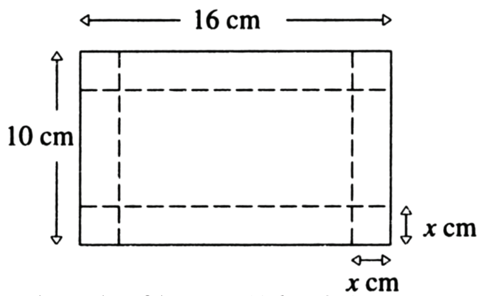
$$C = 12 + 0.008 \times 120 + \frac{14.40}{120}$$
$$= 12 + 0.96 + 0.12$$
$$= 13.08$$

It is also observed that as $x \to 0, C \to \infty$ and that as x large, the graph gets close to that of $c = 12 + 0.08x$.



d From the above, the truck should be driven at $30\sqrt{2} \approx 42.43$ km/hr.

4 a



$\therefore$ length of box $= (16 - 2x)$ cm

width of box $= (10 - 2x)$ cm

height of box $= x$ cm

$$\text{Volume} = \text{length} \times \text{width} \times \text{height}$$
$$= (16 - 2x)(10 - 2x)x$$
$$= 4(8 - x)(5 - x)x$$
$$= 4(40 - 13x + x^2)x$$
$$= 4(x^3 - 13x^2 + 40x)\text{ cm}^3$$

b All dimensions are positive.

$10 - 2x > 0$ and $16 - 2x > 0$ and $x > 0$

$\therefore \quad x < 5$ and $x < 8$ and $x > 0$

$\therefore \quad 0 < x < 5$

c Let $V = 4(x^3 - 13x^2 + 40x)$

$\frac{dV}{dx} = 4(3x^2 - 26 + 40)$

$\frac{dV}{dx} = 0$ implies $4(3x^2 - 26x + 40) = 0$

$\therefore\ 3x^2 - 26x + 40 = 0$

$\therefore\ (3x - 20)(x - 2) = 0$

$\therefore\ x = \frac{20}{3}$ or $x = 2$

but $0 < x < 5 \therefore\ x = 2$

d A gradient chart reveals there is a maximum when $x = 2$:

	< 2	2	> 2
sign $\frac{dV}{dx}$	+ve	0	−ve
shape	/	−	\

When$x = 2$,

$16 - 2x = 12$

$10 - 2x = 6$

$\therefore$ The dimensions of the box for maximum volume are:
2 cm, 6 cm, 12 cm

e Maximum when $x = 2$

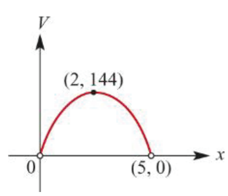
$\therefore\ V_{\max} = 4(5 - 2)(8 - 2)2$

$= 4 \times 3 \times 6 \times 2$

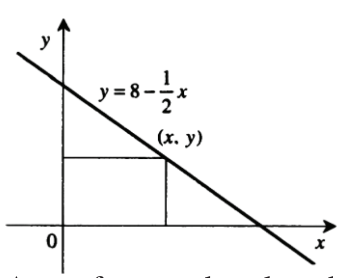
$= 144$

The maximum volume is 144 cm^3

f



5



Area of rectangle = length × width
Let A denote the area.

Let x denote the width.
Let y denote the length.

$$A = xy$$
$$= x\left(8 - \frac{x}{2}\right)$$
$$= 8x - \frac{x^2}{2}$$

Consider the derivative of A with respect to x.

$$\frac{dA}{dx} = 8 - x$$

$\frac{dA}{dx} = 0$ implies $x = 8$

As A is a quadratic function with negative coefficient of x^2, a maximum occurs where $x = 8$.

When $x = 8$, $A = 8 \times 8 - \frac{8^2}{2} = 32$

$\therefore$ Maximum area = 32 square units.

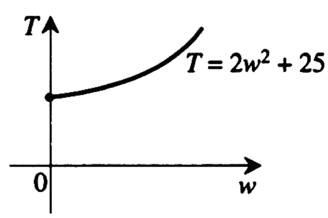
6 a $T = k + 2w^2$

When $w = 5, T = 75$

$\therefore\ 75 = k + 50$

i.e. $k = 25$

So: $T = 2w^2 + 25$

b 

c Average time in seconds per kg = $\frac{T}{w} = \frac{25}{w} + 2w$

d **i** Let A be the average time.

$$A = \frac{25}{w} + 2w$$

Minimum occurs when $\frac{dA}{dw} = 0$.

$$\frac{dA}{dw} = -25w^{-2} + 2 = 0$$

which implies $w^2 = \frac{25}{2}$

i.e. $w = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$

A gradient chart confirms minimum:

w	$< \frac{5}{\sqrt{2}}$	$\frac{5}{\sqrt{2}}$	$> \frac{5}{\sqrt{2}}$
sign of $\frac{dA}{dw}$	−ve	0	+ve
shape	/	−	\

$\therefore \frac{5\sqrt{2}}{2}$kg $\approx$ 3.54 kg yields the minimum average machinery time.

ii When $w = \frac{5\sqrt{2}}{2}$

$$A = \frac{2 \times 5\sqrt{2}}{2} + \frac{25}{\left(\frac{5\sqrt{2}}{2}\right)}$$

$$= 5\sqrt{2} + 5\sqrt{2}$$

$$= 10\sqrt{2}$$

$\therefore$ minimum average machine time is $10\sqrt{2} \approx 14.14$ seconds.

7 **a** Area of bottom $= x^2 + x^2 = 2x^2$

Area of top $= x^2$

Area of sides $= xh + xh + xh + xh = 4xh$

$\therefore$ total area $= 4xh + 3x^2$

i.e. $C = 4xh + 3x^2$

b Volume $V = x^2h$

For Volume $= 12\,\text{m}^3$

$$12 = x^2h$$

i.e. $h = \dfrac{12}{x^2}$

and $C = 4x\left(\dfrac{12}{x^2}\right) + 3x^2$

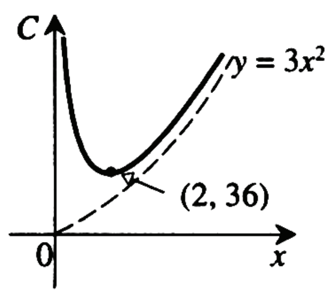
$$= \frac{48}{x} + 3x^2$$

c It is preferable to complete **d** before sketching the graph.

d **i**
$$\frac{dC}{dx} = -\frac{48}{x^2} + 6x$$
$$\frac{dC}{dx} = 0 \text{ implies } -\frac{48}{x^2} + 6x = 0$$

which implies $6x = \dfrac{48}{x^2}$

$\therefore\ x^3 = 8$ and $x = 2$



The gradient chart is as shown:

x	<2	2	>2
sign $\dfrac{dC}{dx}$	$-$ve	0	$+$ve
shape	\	$-$	/

$\therefore$ a minimum when $x = 2$ When $x = 2$, the dimensions are $2\,\text{m}, 2\,\text{m}, 3\,\text{m}\left(h = \dfrac{12}{2^2}\right)$

ii When $x = 2$

$$C = 12 + \frac{48}{2} = 12 + 24 = 36$$

$\therefore$ The minimum area is $36\,\text{m}^2$

8 Let the base have dimension x m by x m and h m be the height of the tank.

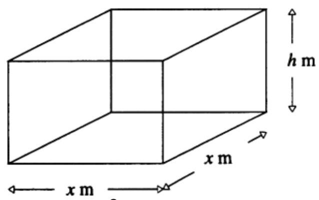
The volume of a cuboid = length × width × height

$$= x \times x \times h$$

$$= x^2h$$

For this tank volume = 500 m^3

$$\therefore \; x^2h = 500 \quad 1$$



Let A m^2 be the area of sheet metal required.

$A = x^2 + 4xh$

(Note: The tank is open.)

From equation (1)

$$h = \frac{500}{x^2}$$

$$\therefore \; A = x^2 + 4x\left(\frac{500}{x^2}\right)$$

$$= x^2 + \frac{2000}{x}$$

Differentiating to find a minimum:

$$\frac{dA}{dx} = 2x - \frac{2000}{x^2}$$

$$\therefore \quad \frac{dA}{dx} = 0 \text{ implies } x^3 = 1000$$

$$\therefore \quad x = 10$$

The gradient chart shows a minimum occurs when $x = 10$.

x	<	10	>
sign of $\frac{dA}{dx}$	−ve	0	+ve
shape	\	−	/

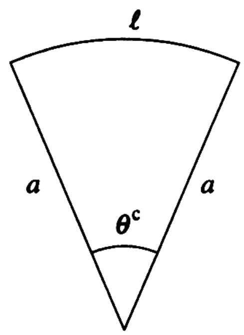
When $x = 10, h = 5$

Therefore the dimensions necessary for a minimum surface area are 10 m × 10 m × 5 m

9 **a** The area of a sector $A = \frac{1}{2}r^2\,\theta$

In this case $r = a$

$$\therefore \; A = \frac{1}{2}a^2\theta$$



b The length of the wire $= a + a + \ell$

$$= 2a + \ell$$

where $\ell = a\theta$

Therefore as the wire is 1 m = 100 cm in length

$$100 = 2a + a\theta$$

$$\therefore\ 100 = a(\theta + 2)$$

$$\text{i.e. } a = \frac{100}{\theta + 2}$$

$$\therefore\ A = \frac{1}{2}\left(\frac{100}{\theta + 2}\right)^2 \theta$$

c Differentiating to find maximum

$$A = \frac{10^4}{2}\left(\frac{1}{\theta + 2}\right)^2 \theta$$

Using the product rule

$$\frac{dA}{d\theta} = 5000\left[\frac{1}{(\theta + 2)^2} - \frac{2\theta}{(\theta + 2)^3}\right]$$

$$\frac{dA}{d\theta} = 0 \text{ implies } \frac{1}{(\theta + 2)^2} = \frac{2\theta}{(\theta + 2)^3}$$

$$\therefore (\theta + 2)^3 - 2\theta(\theta + 2)^2 = 0$$

$$\therefore (\theta + 2)^2[\theta + 2 - 2\theta] = 0$$

$$\therefore \theta = 2 \text{ or } \theta = -2$$

but $\theta > 0 \therefore\ \theta = 2$

The gradient chart will show a maximum i.e. A is maximum when $\theta = 2$

d When $\theta = 2, A = \frac{1}{2}\left(\frac{100}{2 + 2}\right)^2 \times 2$

$$= \frac{1}{2} \times 25^2 \times 2$$

$$= 625$$

The maximum area is 625 cm^2

10 a $L = 2r + r\theta$

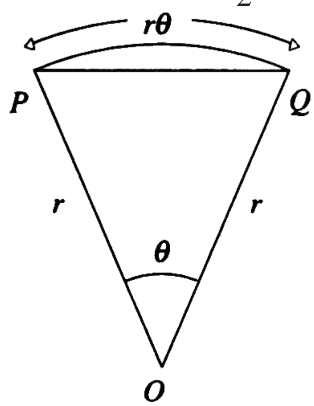
$\therefore\ \theta = \frac{1}{r}(L - 2r)$ 1

Area of sector $= \frac{1}{2}r^2\theta$

$= \frac{1}{2}r^2\left(\frac{1}{r}(L - 2r)\right)$

$= \frac{1}{2}r(L - 2r)$

$= \frac{1}{2}rL - r^2$



b **i** The area of the sector $A = \frac{1}{2}rL - r^2$

$\therefore\ \frac{dA}{dr} = \frac{1}{2}L - 2r$

and $\frac{dA}{dr} = 0$ implies $\frac{1}{2}L - 2r = 0$, so $r = \frac{L}{4}$.

ii Substituting $r = \frac{L}{4}$ in 1 gives

$\theta = \frac{1}{\frac{L}{4}}\left(L - 2 \times \frac{L}{4}\right)$

$= \frac{4}{L}\left(L - \frac{L}{2}\right)$

$= \frac{4}{L} \times \frac{L}{2} = 2$

iii A stationary point occurs when $r = \frac{L}{4}$.

If $r < \frac{L}{4}, \frac{dA}{dr} > 0$

$r < \frac{L}{4}, \frac{dA}{dr} < 0$ (gradients considered locally)

So the stationary point is a maximum.

c Area of sector $= \frac{1}{2}r^2\theta$

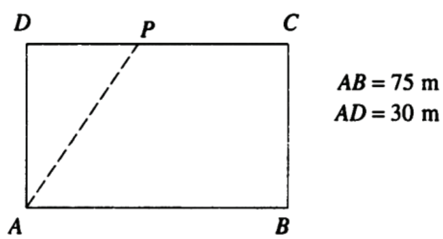
When $\theta = 2$, $r = \frac{L}{4}$

$$\therefore \text{Area of sector} = \frac{1}{2} \times \frac{L^2}{16} \times 2 = \frac{L^2}{16}$$

$$\text{Area of triangle} = \frac{1}{2}r^2 \sin 2 = \frac{1}{2} \times \frac{L^2}{16} \sin 2 = \frac{L^2 \sin 2}{32}$$

$$\frac{\text{Area of triangle}}{\text{Area of sector}} = \frac{L^2 \sin 2}{32} \div \frac{L^2}{16} = \frac{L^2 \sin 2}{32} \times \frac{16}{L^2} = \frac{\sin 2}{2} = 0.4546\cdots$$

$\therefore$ Area of triangle $\approx 45.5\%$ area of sector

11 

a Let T be the total time in seconds and $DP = x$ (m) (Note: the position of P varies)

T = time to swim to AP + time to run PC + time to get out

$$\text{time to swim } AP = \frac{AP}{\text{speed of swimming}} = \frac{\sqrt{900 + x^2}}{1} \text{ (Pythagoras' Theorem)}$$

$$\text{time to run } PC = \frac{75 - x}{1\frac{2}{3}} = \frac{3}{5}(75 - x)$$

time to get out = 2

$$T = \sqrt{900 + x^2} + \frac{3}{5}(75 - x) + 2$$

b $\dfrac{d}{dx}\left(\sqrt{900 + x^2}\right) = 2x \times \dfrac{1}{2}(900 + x^2)^{-\frac{1}{2}}$ (Chain rule)

$\therefore \dfrac{dT}{dx} = x(x^2 + 900)^{-\frac{1}{2}} - \dfrac{3}{5}$

c **i** Minimum occurs when $\dfrac{dT}{dx} = 0$

$\dfrac{dT}{dx} = 0$ implies $x(x^2 + 900)^{-\frac{1}{2}} = \dfrac{3}{5}$

$$\therefore \frac{x}{x(x^2 + 900)^{\frac{1}{2}}} = \frac{3}{5}$$

and $5x = 9(x^2 + 900)^{\frac{1}{2}}$

Squaring both sides yields

$25x^2 = 9x^2 + 8100$

$\therefore 6x^2 = 8100$

$\therefore x^2 = \dfrac{8100}{16}$

and $x = \dfrac{90}{4}$ (Note: $x \geq 0$ and so positive root is chosen)

$= 22\dfrac{1}{2}$

A gradient chart reveals a local minimum when $x = 22\dfrac{1}{2}$

ii The minimum time occurs when $x = \dfrac{90}{4}$

$$\text{When } x = \frac{90}{4}, T = \sqrt{900 + \left(\frac{90}{4}\right)^2} + \frac{3}{5}\left(75 - \frac{90}{4}\right) + 2$$

$$= \sqrt{\frac{22500}{16}} + 3\left(15 - \frac{18}{4}\right) + 2$$

$$= \frac{150}{4} + 3 \times \frac{42}{4} + 2$$

$$= 71$$

The minimum time is 71 seconds.

d If the boy runs from A to D and then from D to C

$$\text{time} = \frac{30}{1\frac{2}{3}} + \frac{75}{1\frac{2}{3}}$$

$$= \frac{30}{\frac{5}{3}} + \frac{75}{\frac{5}{3}}$$

$$= \frac{3}{5} \times \frac{105}{1} = 63$$

It takes 63 seconds to run from A to D and then from D to C.

12 **a** For $y = e^x$, $\dfrac{dy}{dx} = e^x$

When $x = 1$, $y = e$ and $\dfrac{dy}{dx} = e$

Therefore the equation of the tangent is given by

$$y - e = e(x - 1)$$

i.e. $\quad y - e = ex - e$

$\therefore$ $y = ex$ is the equation of the tangent.

b For $y = e^{2x}$, $\dfrac{dy}{dx} = 2e^{2x}$

when $x = \dfrac{1}{2}$, $y = e + \dfrac{dy}{dx} = 2e$

The equation of the tangent at $\left(\dfrac{1}{2}, e\right)$ is given by

$$y - e = 2e\left(x - \frac{1}{2}\right)$$

i.e. $\quad y - e = 2ex - e$

$\therefore \qquad y = 2ex$ is the equation of the tangent at $\left(\dfrac{1}{2}, e\right)$

c For $y = e^{kx}$, $\dfrac{dy}{dx} = ke^{kx}$

when $x = \dfrac{1}{k}$, $y = e$ and $\dfrac{dy}{dx} = ke$

The equation of the tangent is

$$y - e = ke\left(x - \frac{1}{k}\right)$$

i.e. $\qquad y = kex$

d Consider the equation of the tangent at the point (a, e^{ka})

which passes through the origin for the curve with equation $y = e^{kx}$

$\dfrac{dy}{dx} = ke^{kx}$ and at (a, e^{ka}) , $\dfrac{dy}{dx} = ke^{ka}$

$\therefore$ The equation of the tangent is

$y - 0 = ke^{ka}(x - 0)$

i.e. $y = ke^{ka}x$

Also the gradient of the tangent can be determined as the gradient of a straight line joining the point (a, e^{ka}) and (0,0)

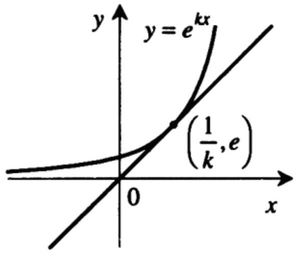
$$\text{Gradient} = \frac{e^{ka} - 0}{a - 0} = \frac{e^{ka}}{a}$$

$$\therefore \qquad ke^{ka} = \frac{e^{ka}}{a}$$

$$\therefore \qquad a = \frac{1}{k}$$

and $\qquad e^{ka} = e$

$\therefore$ Equation of tangent is $y = kex$

e 

Solving $e^{kx} = x$ is equivalent to solving the pair of equations
$y = e^{kx}$
$y = x$
simultaneously.

i There is a single solution to the equation $e^{kx} = x$ if $y = x$ is a tangent to the curve $y = e^{kx}$.

From (d) this occurs only if $ke = 1$ i.e. if $k = \frac{1}{e}$ for $k > 0$.

There is always a unique real root for $k \leq 0$ (check the graph of $y = e^{kx}$ for $k \leq 0$)

ii For no real roots, there are no solutions to the pair of equations

$y = e^{kx}$

and $y = x$

For $k = \frac{1}{e}$, $y = x$ is a tangent.

For $k > \frac{1}{e}$, the curve $y = e^{kx}$ does not meet the line$y = x$.

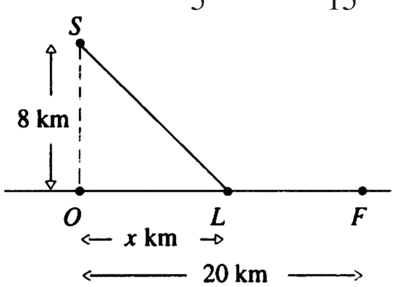
13 a Distance $SL = \sqrt{64 + x^2}$ (Pythagoras' Theorem)

Time taken = time taken for SL+ time taken for LF

$$= \frac{SL}{\text{rowing speed}} + \frac{LF}{\text{running speed}}$$

$$= \frac{\sqrt{64 + x^2}}{5} + \frac{20 - x}{15}$$

$$\therefore T(x) = \frac{\sqrt{64 + x^2}}{5} + \frac{20 - x}{15}$$



b Differentiating to find minimum

$$T'(x) = \frac{x}{5}(64 + x^2)^{-\frac{1}{2}} - \frac{1}{15}$$

$$T'(x) = 0$$

implies $\dfrac{x}{5(64 + x^2)^{\frac{1}{2}}} = \dfrac{1}{15}$

$\therefore\ 15x = 5(64 + x^2)^{\frac{1}{2}}$

Squaring both sides yields

$$225x^2 = 25(64 + x^2)$$

i.e. $200x^2 = 25 \times 64$

and $x = \dfrac{5 \times 8}{10\sqrt{2}}$ (Note: positive root is chosen as $x \geq 0$)

$$= \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

A gradient chart reveals a minimum.

When $x = 2\sqrt{2}$

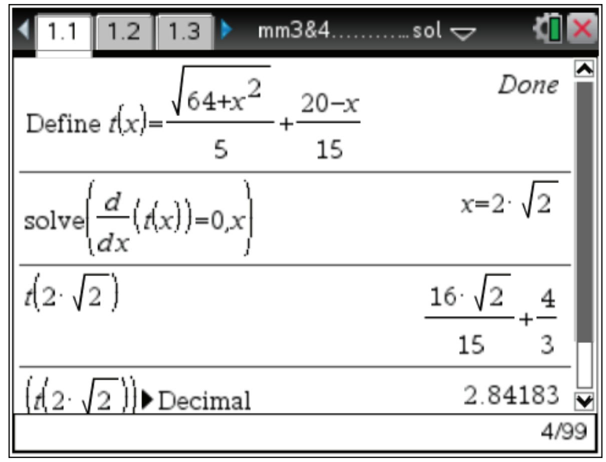
$$\begin{aligned} T &= \frac{\sqrt{64 + 8}}{5} + \frac{20 - 2\sqrt{2}}{15} \\ &= \frac{6\sqrt{2}}{5} + \frac{4}{3} - \frac{2\sqrt{2}}{15} \\ &= \frac{18\sqrt{2} - 2\sqrt{2} + 20}{15} \\ &= \frac{16\sqrt{2} + 20}{15} \\ &\approx 2.84 \end{aligned}$$

The minimum time is $\dfrac{16\sqrt{2} + 20}{15}$ hours ≈ 2.84 hours ≈ 2 hours 50 minutes 31 seconds

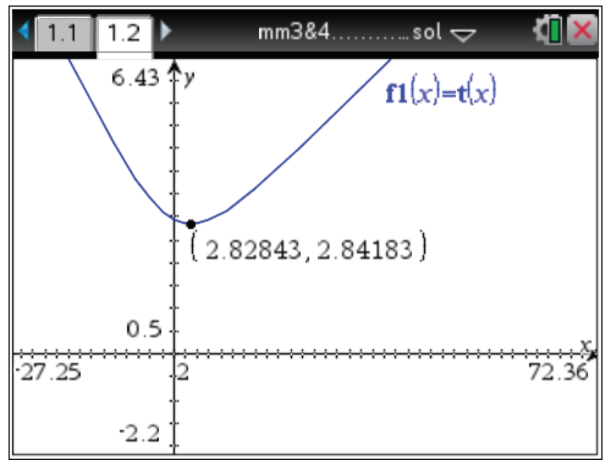
Graphic calculator techniques for question 13

In a **Calculator** page, define the function.
To find the x-value where the minimum occurs, solve the derivative equalling zero.

Hence minimum occurs at $\left(2\sqrt{2}, \dfrac{16\sqrt{2}}{15} + \dfrac{4}{3}\right)$



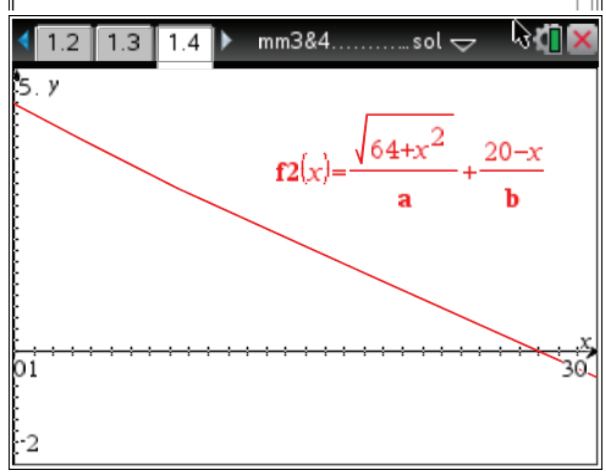
In a **Graphs** page enter the function t(x).
Find the minimum
using b>**Analyze Graph**>**Minimum**.



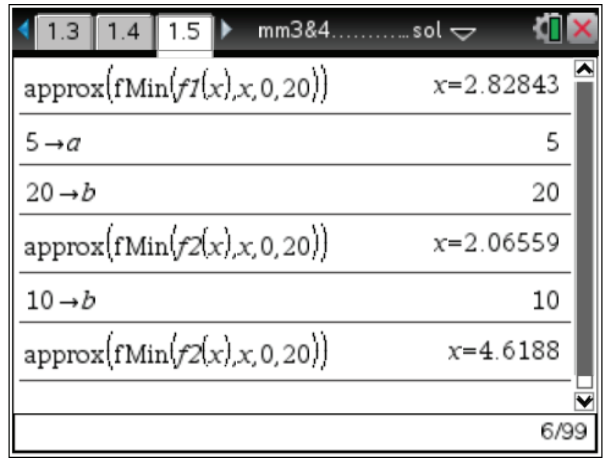
A further investigation can be made by considering

$$T(x) = \frac{\sqrt{64 + x^2}}{A} + \frac{20 - x}{B}$$

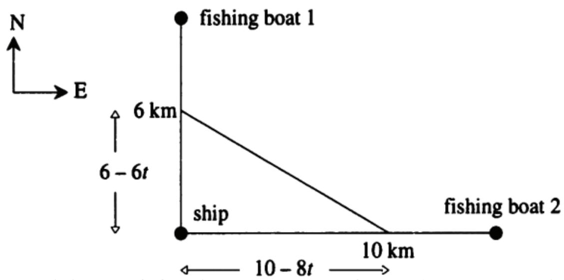
where A is the rowing speed and B is the running speed.
In the problem, store A = 20 and B = 5 in a **Calculator** page
Enter the formula in the **Function Entry Line** as shown by the graph label.
The result is as expected. It is best to row straight to F.



Investigate the minimum value, holding A = 5 and varying B. The use of **fMin(**, b>**Calculus**>**Function Minimum**, is a good way of doing this. The screen shows a possible investigation. The first value of **fMin(** was obtained with A = 5, B = 15. i.e. f1(x). The following values are obtained by altering the B value (A is constant) in function f2(x).
Note: **approx(** is used to give decimal answers for easier comparison. Other options to give decimal answers can also be used.



14



Position of fishing boat 1 after t hours = $(6 - 6t)$ km North
Position of fishing boat 2 after t hours $(10 - 8t)$ km East
Distance apart after t hours

$= \sqrt{(6 - 6t)^2 + (10 - 8t)^2}$ (Pythagoras' Theorem)

Let D km be the distance apart after t hours

$$D = \sqrt{(6 - 6t)^2 + (10 - 8t)^2}$$

and $D^2 = (6 - 6t)^2 + (10 - 8t)^2$

The minimum value of D will occur for the same value of t as the minimum of D^2.

$$\frac{d(D^2)}{dt} = -12(6 - 6t) - 16(10 - 8t)$$

$$= -72 + 72t - 160 + 128t$$

$$= 200t - 232$$

$$\frac{d(D^2)}{dt} = 0 \text{ implies}$$

$$t = \frac{232}{200} = 1.16$$

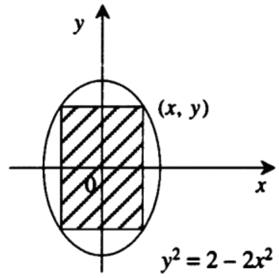
This is a local minimum as D^2 vs t is a parabola with positive coefficient of t^2.
The boats are closest 1.16 hours after noon, i.e. after 1 hour 9 minutes and 36 seconds.
When $t = 1.16$

$$D = \sqrt{(6 - 6 \times 1.16)^2 + (10 - 8 \times 116)^2}$$

$$= 1.2$$

The least distance between the two fishing boats is 1.2 km.

15 **a**



Area of rectangle, $A = 2x \times 2y$

$$= 4x\sqrt{2 - 2x^2}$$

$$= 4x(2 - 2x^2)^{\frac{1}{2}}$$

b $2 - 2x^2 > 0$ for the relation to be defined

$\therefore\ 1 > x^2$

and $-1 < x < 1$

But $0 < x < 1$ (as x is the half–width of the beam)

$\therefore$ allowable values are $x \in (0, 1)$

c Using the product rule and chain rule

$$A = 4x(2 - 2x^2)^{\frac{1}{2}}$$

$$\frac{dA}{dx} = 4(2 - 2x^2)^{\frac{1}{2}} - 2x(2 - 2x^2)^{\frac{1}{2}} \times 4x$$

$$= 4(2 - x^2)^{\frac{1}{2}} - \frac{8x^2}{(2 - 2x^2)^{\frac{1}{2}}}$$

$$= \frac{4(2 - x^2) - 8x^2}{(2 - 2x^2)^{\frac{1}{2}}}$$

$$= \frac{8 - 8x^2 - 8x^2}{(2 - 2x^2)^{\frac{1}{2}}} = \frac{8 - 16x^2}{(2 - 2x^2)^{\frac{1}{2}}}$$

Maximum will occur when $\frac{dA}{dx} = 0$

When $\frac{dA}{dx} = 0$

$$8 = 16x^2$$

$$x^2 = \frac{1}{2}$$

$$\therefore\ x = \pm\frac{1}{\sqrt{2}}$$

but $x \in (0, 1)$ $\therefore\ x = \frac{1}{\sqrt{2}}$

A gradient chart shows local maximum when $x = \frac{1}{\sqrt{2}}$. When $x = \frac{1}{\sqrt{2}}$, $y = \pm 1$

d When $x = \dfrac{1}{\sqrt{2}}$

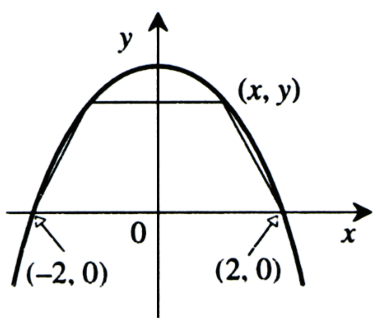
$$A = \frac{4}{\sqrt{2}} \times \left(2 - 2 \times \left(\frac{1}{\sqrt{2}}\right)^2\right)^{\frac{1}{2}}$$

$$= \frac{4}{\sqrt{2}} \times (2 - 1)^{\frac{1}{2}}$$

$$= 2\sqrt{2}$$

$\therefore$ The maximum cross-sectional area of the beam is $2\sqrt{2}$ square units.

16



a Area of a trapezoid $= \dfrac{h}{2}(a + b)$

where h is the height of the trapezoid and a and b are the lengths of the opposite parallel sides.

$\therefore$ Area of the trapezoid $= \dfrac{y}{2}(4 + 2x)$

But $y = 4 - x^2$

$$\therefore \text{Area, } A = \frac{(4 - x^2)}{2}(4 + 2x)$$

$$= \frac{1}{2}(4 - x^2)(2x + 4)$$

b Using the product rule

$$\frac{dA}{dx} = \frac{1}{2}[-2x(2x + 4) + 2(4 - x^2)]$$

$$= \frac{1}{2}[-4x^2 - 8x + 8 - 2x^2]$$

$$= \frac{1}{2}[-6x^2 - 8x + 8] = -3x^2 - 4x + 4$$

$\frac{dA}{dx} = 0$ implies $3x^2 + 4x - 4 = 0$

$\therefore\ (3x - 2)(x + 2) = 0$

$\therefore\ x = \frac{2}{3}$ or $x = -2$

$\frac{dA}{dx} = -(3x - 2)(x + 2)$

When $x > \frac{2}{3}, \frac{dA}{dx} < 0$ (locally)

When $x < \frac{2}{3}, \frac{dA}{dx} > 0$

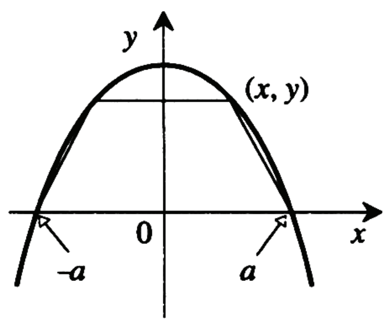
$\therefore$ local maximum when $x = \frac{2}{3}$

$\therefore$ The trapezoid has its greatest area when $x = \frac{2}{3}$

c **i** $A = \frac{1}{2} \times y(2a + 2x)$

$= y(a + x)$

$= (a^2 - x^2)(a + x)$



ii Using the product rule

$\frac{dA}{dx} = a^2 - x^2 + (-2x)(a + x)$

$= a^2 - x^2 - 2xa - 2x^2$

$= a^2 - 2xa - 3x^2$

$= (a + x)(a - 3x)$

iii $\frac{dA}{dx} = 0$ implies $x = \frac{a}{3}$ or $x = -a$

when $x > \frac{a}{3}, \frac{dA}{dx} < 0$ (locally)

When $x < \frac{a}{3}, \frac{dA}{dx} > 0$

$\therefore$ maximum when $x = \frac{a}{3}$

17 $N(t) = 24te^{-0.2t}$

$$N'(t) = 24e^{\frac{-t}{5}} - \frac{24}{5}te^{\frac{-t}{5}}$$

$$= e^{\frac{-t}{5}}\left(24 - \frac{24t}{5}\right)$$

$N'(t) = 0,$

$$24 - \frac{24t}{5} = 0$$

$$\frac{t}{5} = 1$$

$$t = 5$$

$$N(5) = 120e^{-1}$$

$$= \frac{120}{e} = 44 \text{ bacteria}$$

(*round because it is a discrete quantity not a continuous one*)

18 a $y = -t^3 + bt^2 + ct$

When $t = 1, y = 10$

When $t = 2, y = 24$

$\therefore\ 10 = -1 + b + c$

and $24 = -8 + 4b + 2c$

$\therefore\ 11 = b + c$ 1

and $32 = 4b + 2c$ 2

Subtract 2×1 from 2

$10 = 2b$

$\therefore\ b = 5$ and from 1, $c = 6$

$\therefore\ y = -t^3 + 5t^2 + 6t$

b $y = -t^3 + 5t^2 + 6t$

i y is the rate of increase.

$\therefore$ to determine when area covered by the plant is a maximum consider $y = 0$

i.e. $-t^3 + 5t^2 + 6t = 0$

$-t(t^2 - 5t - 6) = 0$

$\therefore\ t = 0$ or $(t - 6)(t + 1) = 0$

$\therefore\ t = 6$ or $t = -1$ or t (Note: $t \geq 0$ and when $t = 0, y = 0$)

$y = -t(t - 6)(t + 1)$

When $t > 6, y < 0$ (locally)

When $t < 6, y > 0$

$\therefore$ local maximum then $t = 6$

The area is a maximum 6 weeks after planting.

ii The rate of increase $y = -t^3 + 5t^2 + 6t$

To determine maximum rate consider

$$\frac{dy}{dt} = -3t^2 + 10t + 6$$

$$\frac{dy}{dt} = 0 \text{ implies } -3t^2 + 10t + 6 = 0$$

The quadratic formula gives

$$t = \frac{-10 \pm \sqrt{100 + 72}}{-6}$$

$$= \frac{-10 \pm \sqrt{172}}{-6}$$

$$= \frac{-10 \pm 2\sqrt{43}}{-6}$$

$$= \frac{10 \mp 2\sqrt{43}}{6} = \frac{-5 \mp \sqrt{43}}{3} \approx 3.852 \text{ or } -0.519$$

$t \geq 0$ in this example and a gradient chart reveals that a maximum rate of increase occurs when $t = 3.852$.

i.e. The rate of increase is a maximum after 3.852 weeks.

c This question requires antidifferentiation at year 11 MM 1 & 2 standard.

$y = -t^3 + 5t^2 + 6t$

$$\therefore\ \text{Area} = -\frac{t^4}{4} + \frac{5t^3}{3} + \frac{6t^2}{2} + c$$

When $t = 0$, area = 100 cm^2 $\therefore\ c = 100$

$$\therefore\ \text{Area} = \frac{-t^4}{4} + \frac{5t^3}{3} + 3t^2 + 100$$

When $t = 4$

$$\text{Area} = \frac{-4^4}{4} + \frac{5 \times 4^3}{3} + 3 \times 16 + 100$$

$$= -4^3 + \frac{5}{3} \times 4^3 + 3 \times 16 + 100$$

$$= \frac{2 \times 4^3}{+} 3 \times 16 + 100$$

$$= \frac{2 \times 64}{3} + 48 + 100$$

$$= 42\frac{2}{3} + 48 + 100$$

$$= 190\frac{2}{3}$$

The plant will cover $190\frac{2}{3}$ cm^2 after 4 weeks

d After 6 weeks the rate becomes negative which implies the plant begins to recede.

Area $= 244$ cm^2 after 6 weeks

$7 = 218$ cm^2 after 7 weeks

$= 121$ cm^2 after 8 weeks

The area becomes "negative" between 8 and 9 weeks. The model is not valid after this. Once the area begins to decrease the model is questionable.

19 $f(x) = x^3 - 3x^2 + 6x - 10$

a $f'(x) = 3x^2 - 6x + 6$

$f'(x) = 3$ implies $3x^2 - 6x + 6 = 3$

$$\therefore\ x^2 - 2x + 2 = 1$$

$$\therefore\ x^2 = 2x + 1 = 0$$

$$\therefore\ (x - 1)^2 = 0$$

$$x = 1$$

and $\quad f(1) = 1 - 3 + 6 - 10 = -6$

The coordinates of the point where $f'(x) = 3$ are $(1, -6)$

b $f(x) = 3x^2 - 6x + 6$

$$= 3[x^2 - 2x + 2]$$

$$= 3[x^2 - 2x + 1 + 1]$$

$$= 3[(x - 1)^2 + 1]$$

$$= 3(x - 1)^2 + 3$$

c $(x - 1)^2 > 0$ for all $x \in R\backslash\{1\}$

$\therefore\ f(x) > 3$ for all $x \in R\backslash\{1\}$

20 a $y = ax^3 + bx^2 + cx + d$ passes through the x-axis at $(1, 0)$

$\therefore\ 0 = a + b + c + d \ldots\ldots (1)$

Gradient $= 0$ when $x = 1$ and $x = \dfrac{1}{3}$

gradient function, $\dfrac{dy}{dx} = 3ax^2 + 2abx + c$

$\therefore\ 0 = 3a + 2b + c \ldots\ldots (2)$

and $0 = \dfrac{a}{3} + \dfrac{2}{3}b + c \ldots\ldots (3)$

Finally it passes through the point $\left(\dfrac{1}{3}, \dfrac{4}{27}\right)$

$\therefore\ \dfrac{4}{27} = \dfrac{a}{27} + \dfrac{b}{9} + \dfrac{c}{3} + d \ldots\ldots (4)$

Subtract 3 and 2

$0 = \dfrac{8a}{3} + \dfrac{4b}{3}$

$\therefore\ 0 = 2a + b$

i.e. $b = -2a \ldots (5)$

Substitute in (1) for b

$0 = a - 2a + c + d$

i.e. $0 = -a + c + d \ldots (6)$

Substitute in (4) for b

$\dfrac{4}{27} = \dfrac{a}{27} - \dfrac{2a}{9} + \dfrac{c}{3} + d$

$\dfrac{4}{27} = -\dfrac{5a}{27} + \dfrac{c}{3} + d \ldots (7)$

Subtract (7) from (6)

$$-\frac{4}{27} = -\frac{22a}{27} + \frac{2c}{3}$$

i.e $-4 = -22a + 18c$

and $-2 = -11a + 9c$

$$\therefore\ c = \frac{-2 + 11a}{9}$$

Substitute in (2) for b and c

$$0 = 3a - 4a + 2 - \frac{2 + 11a}{9}$$

$$0 = \frac{9a + -2 + 11a}{9}$$

$\therefore\ a = 1$

and $b = -2a = -2$

and $c = -\frac{2 + 11}{9} = 1$

From (1)

$$0 = a + b + c + d$$

$$0 = 1 - 2 + 1 + d$$

$\therefore\ d = 0$

i.e. $a = 1, b = -2, c = 1, d = 0$

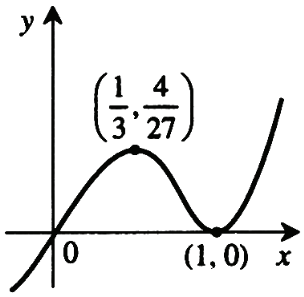
b $\dfrac{dy}{dx} = 3ax^2 + 2bx + c$

$= 3x^2 - 4x + 1$

$= (3x - 1)(x - 1)$

$\dfrac{dy}{dx} < 0$ for $\dfrac{1}{3} < x < 1$

c



$y = x^3 - 2x^2 + x$

$= x(x - 1)^2$

$\dfrac{dy}{dx} = 0$ when $x = \dfrac{1}{3}$

and $x = 1$

when $x = \dfrac{1}{3}, y = \dfrac{4}{27}$

and $\dfrac{dy}{dx} < 0$ for $x \in \left(\dfrac{1}{3}, 1\right)$

21 $V = \dfrac{\pi}{3}((y + 630)^3 - 630^3)$

a When $y = 40$

$V = \dfrac{\pi}{3}((40 + 630)^3 - 630^3)$

$= \dfrac{\pi}{3}(670^3 - 630^3)$

$= \dfrac{\pi}{3}(50\,716\,000)$

$\approx 53109\,671.0$

Volume of water in reservoir = 53 109 671.0 m^3

b $\dfrac{dV}{dy} = \dfrac{\pi}{3}(3(y + 630)^2)$

$= \pi(y + 630)^2$

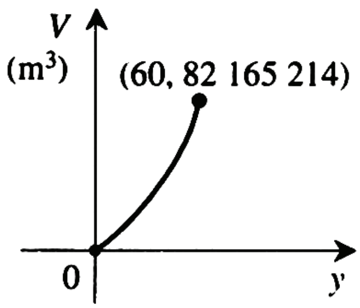
c $\dfrac{dV}{dy} > 0$ for all $v \in R$, gives that the function is increasing and the gradient increases as y increases.

d $V = \dfrac{\pi}{3}((y + 630)^3 - 630^3)$

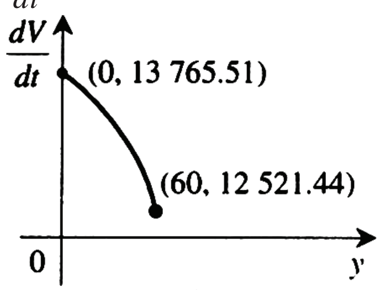
$= \dfrac{\pi}{3}((690)^3 + (630)^3)$ when $y = 60$

$= \dfrac{\pi}{3}(78\ 462\ 000)$

$= 82\ 165\ 214\,\text{m}^3$



e $\dfrac{dV}{dt} = 20\ 000 - 0.005\pi(y + 630)^2$



The graph of $\dfrac{dV}{dt}$ against y is a parabola.

It is the graph of $z = -x^2$ transformed by a dilation of 0.005π from the x-axis followed by a translation of 630 units "to the left" and 20 000 units "up".

The domain is $0 \le y \le 60$.

When $y = 60$, $\dfrac{dV}{dt} = 12521.44$

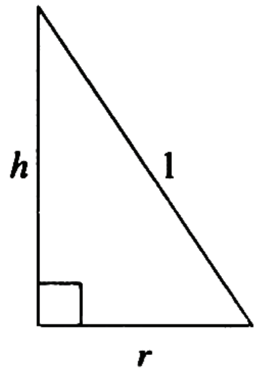
When $y = 0$, $\dfrac{dV}{dt} = 13765.51$

22 a i The circumference of the base of the cone is equal to the length of the sector formed. Hence $2\pi r = (2\pi - \theta)$ (radius of circle is one)

$\therefore\ 2\pi r = 2\pi - \theta$

$r = \dfrac{2\pi - \theta}{2\pi}$

ii



$$h^2 + r^2 = 1$$

$$\therefore\ h^2 = 1 - r^2$$

From (i), $r = \dfrac{2\pi - \theta}{2\pi}$

$$\therefore\ h^2 = 1 - \left(\frac{2\pi - \theta}{2\pi}\right)^2$$

and $h = \sqrt{1 - \left(\dfrac{2\pi - \theta}{2\pi}\right)^2}$

iii $V = \dfrac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi\left(\frac{2\pi - \theta}{2\pi}\right)^2 \sqrt{1 - \left(\frac{2\pi - \theta}{2\pi}\right)^2}$$

b When $\theta = \dfrac{\pi}{2}$

$$V = \frac{\pi}{3}\left(\frac{2\pi - \frac{\pi}{4}}{2\pi}\right)^2 \sqrt{\frac{4\pi^2 - (4\pi^2 - 4\pi\theta + \theta^2)}{4\pi^2}}$$

$$= \frac{\pi}{3}\left(\frac{7}{8}\right)^2 \sqrt{\frac{4\pi \times \frac{\pi}{4} - (\frac{\pi}{4})^2}{4\pi^2}}$$

$$= \frac{49\pi}{192}\sqrt{\frac{1 - \frac{1}{16}}{4}}$$

$$= \frac{49\pi}{384}\sqrt{\frac{15}{16}}$$

$$= \frac{49\pi}{1536}\sqrt{15}$$

c $0.3 = \frac{\pi}{3}\left(\frac{2\pi - \theta}{2\pi}\right)^2 \sqrt{\frac{4\pi\theta - \theta^2}{4\pi^2}}$ Solving using a CAS calculator $\theta = 0.3281$

$= \frac{(2\pi - \theta)^2}{24\pi^2}\sqrt{4\pi\theta - \theta^2}$

d (Note: $0 < \theta < \pi$)

i maximum occurs at $\theta \approx 1.153$

ii maximum volume is $V \approx 0.403 \text{ cm}^2$

e $V = \frac{(2\pi - \theta)^2}{24\pi^2}(4\pi\theta - \theta^2)^{\frac{1}{2}}$

$\frac{dV}{d\theta} = \frac{1}{24\pi^2}\left[-2(2\pi - \theta)(4\pi\theta - \theta^2)^{\frac{1}{2}} + \frac{1}{2}(2\pi - \theta)^2(4\pi - 2\theta)(4\pi\theta - \pi^2)^{-\frac{1}{2}}\right]$

$= \frac{(2\pi - \theta)}{24\pi^2}\left[\frac{-2(4\pi\theta - \theta^2) + (2\pi - \theta)^2}{(4\pi\theta - \theta^2)^{\frac{1}{2}}}\right]$

$\frac{dV}{d\theta} = 0$ implies $\theta = 2\pi$

or $-8\pi\theta + 2\theta^2 + 4\pi^2 - 4\pi\theta + \theta^2 = 0$

i.e $3\theta^2 - 12\pi\theta + 4\pi^2 = 0$

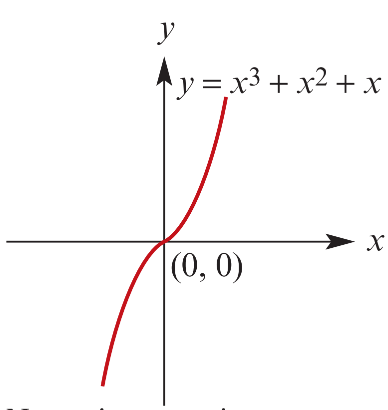
$\therefore\ \theta = \frac{12\pi \pm \sqrt{144\pi^2 - 48\pi^2}}{6}$

$\theta = \frac{12\pi \pm 4\sqrt{6\pi^2}}{6}$

$\therefore\ \theta = \frac{6\pi - 2\pi\sqrt{6}}{3} \approx 1.153$

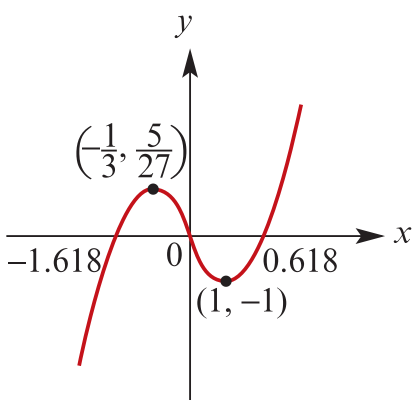
Maximum volume is 0.403 cm^2

23 a i

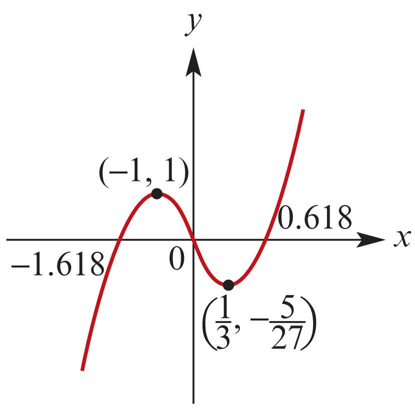


No stationary points

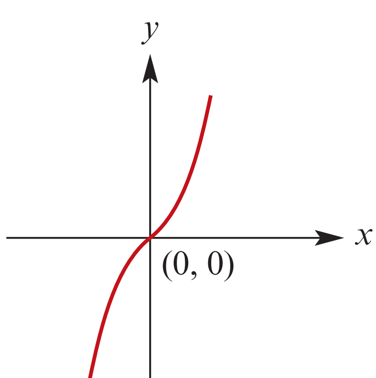
ii



iii



iv



No stationary point

b **i** $f'(x) = 3x^2 + 2ax + b$

ii $f'(x) = 0$ implies $3x^2 + 2ax + b = 0$

$$\therefore\ x = \frac{-2a \pm \sqrt{4a^2 - 4 \times 3 \times b}}{6}$$

$$= \frac{-2a \pm \sqrt{4a^2 - 12b}}{6}$$

$$= \frac{-a \pm \sqrt{a^2 - 3b}}{3}$$

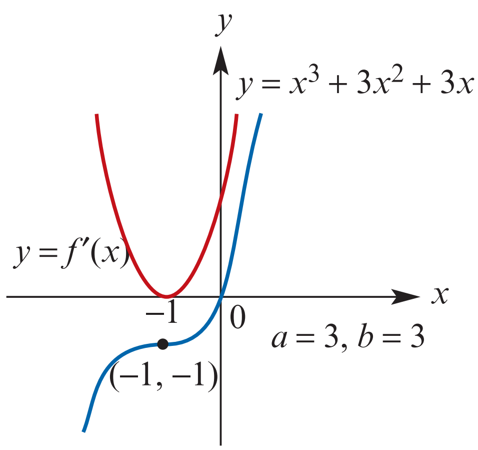
c **i** If $a^2 - 3b = 0$, the cubic has one stationary point given by $x = \dfrac{-9}{3} = -3$.

ii If $b = 3, a^2 = 9$ and $a = \pm 3$
$\therefore x = -1$ or $x = 1$
If $b = 3, a = 3$ and $x = -1$
$y = (-1)^3 + 3(-1)^2 + 3(-1)$
$= -1 + 3 - 3$
$= -1$
Coordinates $(-1, -1)$

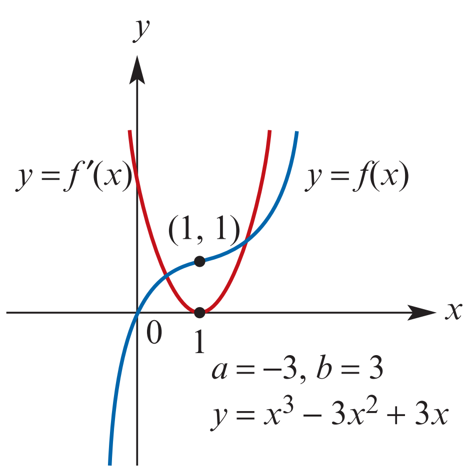
If $b = 3, a = -3$ and $x = 1$
$y = (1)^3 - 3(1)^2 + 3(1)$
$= 1 - 3 + 3$
$= 1$
Coordinates $(1, 1)$

Each is a stationary point of inflexion

iii



iv



d No stationary points exist if $a^2 < 3b$

24 Let $y = \dfrac{\log_e x}{x}$

$$\frac{dy}{dx} = \frac{x \times \frac{1}{x} - \log_e x}{x^2}$$

$$= \frac{1 - \log_e x}{x^2}$$

$\dfrac{dy}{dx} = 0$ implies $\dfrac{1 - \log_e x}{x^2} = 0$

$\therefore x = e$

$\dfrac{dy}{dx} < 0$ for $1 - \log_e x < 0 \Leftrightarrow \log_e x > 1 \Leftrightarrow x > e$

$\dfrac{dy}{dx} < 0$ for $1 - \log_e x > 0 \Leftrightarrow \log_e x < 1 \Leftrightarrow x < e$

$\therefore$ a maximum for $x = e$

When $x = e, y = \dfrac{\log_e}{e} = \dfrac{1}{e}$

i.e. The ratio of the logarithm of a number to the number is a maximum when $x = e$.

25 a i $f(x) = 6x^4 - x^3 + ax^2 - 6x + 8$

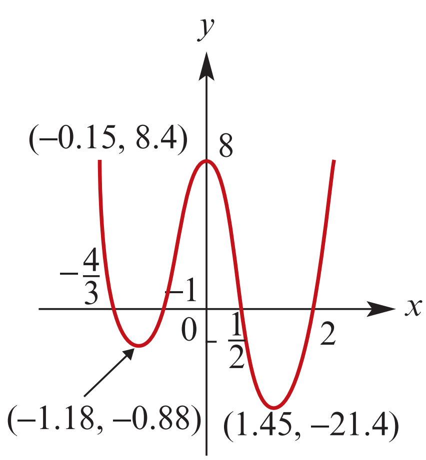
If $x + 1$ is a factor $f(-1) = 0$

i.e. $f(-1) = 6 + 1 + a + 6 + 8 = 0$

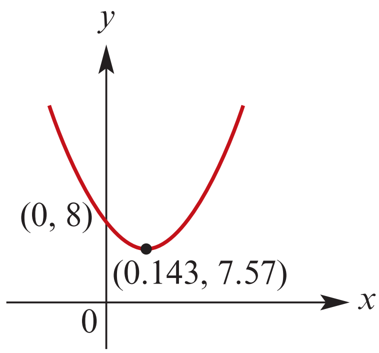
$\therefore\ a + 21 = 0$

$a = -21$

ii



b **i**



ii minimum = 7.57 when $x = 0.143$

iii $g'(x) = 24x^3 - 3x^2 + 42x - 6$

iv
$$\begin{aligned} g'(x) &= 0 \\ 24x^3 - 3x^2 + 42x - 6 &= 0 \\ x &= 0.1427 \end{aligned}$$

v $g'(\mathrm{o}) = -6; g'(10) = 24\ 114$

vi $\frac{d}{dx}(g(x))$ can be written as $g''(x)$, meaning the derivative of derivative.
$g''(x) = 72x^2 - 6x + 42$

vii $g''(x) = 0$ implies

$12x^2 - x + 7 = 0$

But $\Delta = 1 - 4 \times 12 \times 7 < 0$

$\therefore$ no stationary points

Hence the graph of

$y = g'(x)$ has positive gradient for all x. There is only one solution of $g'(x) = 0$.

26 **a** $f(x) = (x-a)^2(x-b)^2 a > 0\ b > 0$

$$\begin{aligned} f'(x) &= 2(x-a)(x-b)^2 + 2(x-b)(x-a)^2 \\ &= 2(x-a)(x-b)(x-b+x-a) \\ &= 2(x-a)(x-b)(2x-(b+a)) \end{aligned}$$

b **i** $f'(x) = 0$ implies $x = a$ or $x = b$ or $x = \dfrac{b+a}{2}$

ii $x = a$ or $x = b$

c Stationary points
$(a, 0)\ (b, 0)$

When $x = \dfrac{a+b}{2}$

$$f\left(\frac{a+b}{2}\right) = \left(\frac{a+b}{2} - a\right)^2 \left(\frac{a+b}{2} - b\right)^2$$

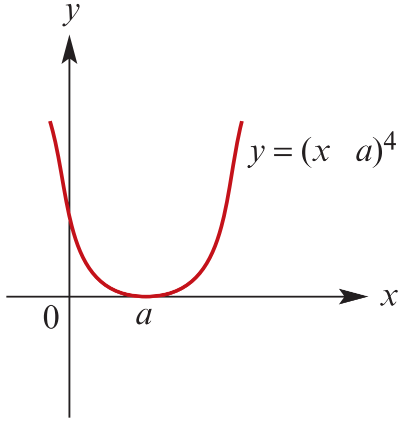
$$= \left(\frac{a+b-2a}{2}\right)^2 \left(\frac{a+b-2b}{2}\right)^2$$

$$= \left(\frac{b-a}{4}\right)^2 \left(\frac{a-b}{4}\right)^2$$

$$= \frac{(a-b)^4}{16}$$

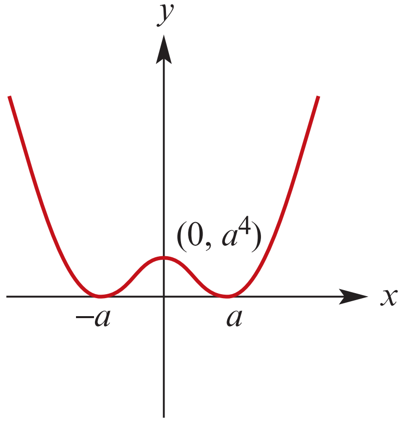
$\therefore$ coordinates $\left(\dfrac{(a+b)}{2}, \dfrac{(a-b)^4}{16}\right)$

d **i**



ii If $a = -b$ coordinates are $(a, 0)$ $(-a, 0)$ $(0, a^4)$

iii



27 **a** $f(x) = (x-a)^3(x-b)$

$$f'(x) = 3(x-a)^2(x-b) + (x-a)^3$$

$$= (x-a)^2[3(x-b) + (x-a)]$$

$$= (x-a)^2[4x - (3b+a)]$$

b **i** $f'(x) = 0$ implies $x = a$ or $x = \dfrac{3b + a}{4}$

ii $f(x) = 0$ implies $x = a$ or $x = b$

c $(a, 0)$ is a stationary point of inflection as $f'(a + h)$ and $f'(a - h)$ have the same sign where h is a small number.

If $x = \dfrac{3b + a}{4}$

$$\begin{aligned} f(x) &= \left(\frac{3b + a}{4} - a\right)^3 \left(\frac{3b + a}{4} - b\right) \\ &= \left(\frac{3b + a - 4a}{4}\right)^3 \left(\frac{3b + a - 4b}{4}\right) \\ &= \left(\frac{3b - 3a}{4}\right)^3 \left(\frac{a - b}{4}\right) \\ &= -\frac{27}{256}(b - a)^4 \end{aligned}$$

If $x > \dfrac{3b + a}{4}$ then $f'(x) > 0$

If $x < \dfrac{3b + a}{4}$ then $f'(x) < 0$

$\therefore$ local minimum at $\left(\dfrac{3b + a}{4}, -\dfrac{27}{256}(b - a)^4\right)$

d Calculator

e If $a = -b$

stationary points are $(a, 0)$ and $\left(-\dfrac{a}{2}, \dfrac{27a^4}{16}\right)$

f **i** If a local minimum at $x = 0$, $\dfrac{3b + a}{4} = 0$, i.e. $a = -3b$ or $b = -\dfrac{a}{3}$.

g If there is a turning point for $x = \dfrac{a + b}{2}$

then $\dfrac{a + b}{2} = \dfrac{3b + a}{4}$

$$\begin{aligned} \therefore\ 2a + 2b &= 3b + a \\ \therefore\ 0 &= b - a \\ \therefore\ b &= a \end{aligned}$$

If $b = a$ $\quad f(x) = (x - a)^4$

28 $f\colon (0, 6] \to R$, $f(x) = x \log_e x + 1$

a $f'(x) = \log_e x + x \times \dfrac{1}{x} = \log_e x + 1$ (product rule)

b $f'(x) = 0$ implies $\log_e x = -1$

$\therefore x = e^{-1}$

When $x > e^{-1}$, $\quad \log_e x + 1 > 0$

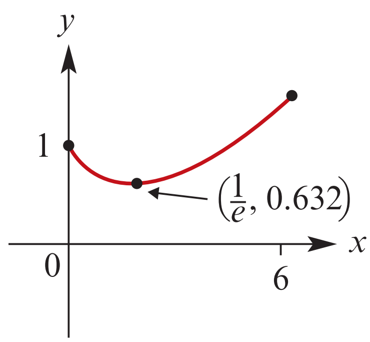
When $x < e^{-1}$, $\quad \log_e x + 1 < 0$

$\therefore$ a minimum when $x = e^{-1} \approx 0.37$,

i.e. during the fourth month of its life.

$$\begin{aligned}\text{When } x = e^{-1},\ f(x) &= \frac{1}{e}\log_e e^{-1} + 1\\ &= -\frac{1}{e} + 1\\ &\approx 0.632\end{aligned}$$

c



d The mouse's ability to memorise is a maximum after 6 years.

29 a i $y^2 + r^2 = 100$

$\therefore\ y = \sqrt{100 - r^2}$

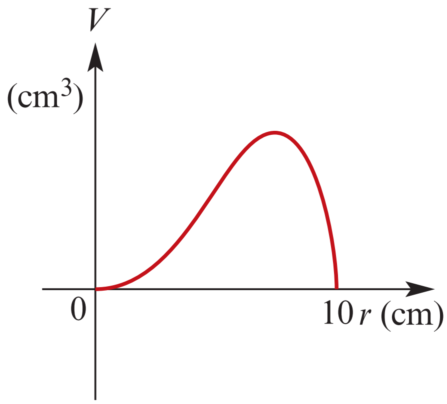
$\therefore$ height $= 2y = 2\sqrt{100 - r^2}$

ii $V = \pi r^2 h$

$= \pi r^2 \left(2\sqrt{100 - r^2}\right)$

$= 2\pi r^2 \sqrt{100 - r^2}$

b **i**



ii Maximum volume is $2418.4\,\text{cm}^3$

This occurs when $r = 8.165$ and $h = 11.55$

iii Use the 'solve' command of a CAS calculator.

$r = 6.456$ or $r = 9.297$

c **i**

$$V = 2\pi r^2(100 - r^2)^{\frac{1}{2}}$$

$$\begin{aligned}\frac{dV}{dr} &= 4\pi r(100 - r^2)^{\frac{1}{2}} - \frac{2r}{2}(100 - r^2)^{-\frac{1}{2}} \times 2\pi r^2\\
&= 2\pi r\left[2(100 - r^2)^{\frac{1}{2}} - \frac{r^2}{(100 - r^2)^{\frac{1}{2}}}\right]\\
&= 2\pi r\frac{[2(100 - r^2) - r^2]}{(100 - r^2)^{\frac{1}{2}}}\\
&= 2\pi r\frac{[200 - 3r^2]}{(100 - r^2)^{\frac{1}{2}}}\end{aligned}$$

ii If $\dfrac{dV}{dr} = 0, 200 - 3r^2 = 0$

$\therefore\ 3r^2 = 200$

$$\therefore\ r^2 = \frac{200}{3}$$

$$r = \sqrt{\frac{200}{3}} = \sqrt{\frac{600}{3}}$$

$$= \frac{10\sqrt{6}}{3}$$

$\therefore$ maximum volume is given by

$$V\max = 2\pi \times \frac{200}{3}\left(100 - \frac{200}{3}\right)^{\frac{1}{2}}$$

$$= \frac{400\pi}{3}\left(\frac{100}{3}\right)^{\frac{1}{2}} = \frac{4000\pi}{3} \times \frac{\sqrt{3}}{3}$$

$$= \frac{4000\sqrt{3}\pi}{9}$$

d **i** Calculator

ii $\frac{dV}{dr} > 0$ for $r \in \left(0, \frac{20\sqrt{6}}{6}\right)$

iii $\frac{dV}{dr}$ is increasing for $r \in (0, 5.21)$

30 **a** Surface area $= \pi r^2 + 2\pi rh + 2\pi r^2$

$\therefore 3\pi r^2 + 2\pi rh = 100\pi$

$\therefore 3r^2 + 2rh = 100$

$\therefore h = \frac{100 - 3r^2}{2r}$

b $V = \pi r^2 h + \frac{2}{3}\pi r^3$

$$= \pi r^2\left(\frac{100 - 3r^2}{2r}\right) + \frac{2}{3}\pi r^3$$

$$= \pi r\left(\frac{100 - 3r^2}{2}\right) + \frac{2}{3}\pi r^3$$

$$= \frac{\pi r}{6}(300 - 9r^2 + 4r^2)$$

$$= \frac{\pi r}{6}(300 - 5r^2)$$

c defined for $r > 0$ and $300 - 5r^2 > 0$

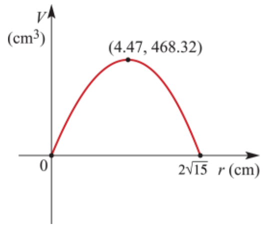
i.e. $r^2 < 60$

$r < 2\sqrt{15}$

d $V = \frac{\pi}{6}(300r - 5r^3)$

$\frac{dV}{dr} = \frac{\pi}{6}(300 - 15r^2)$

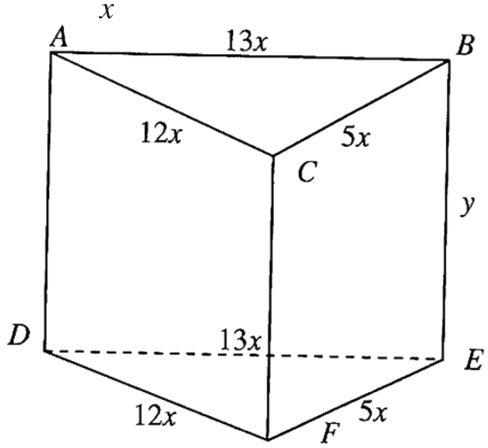
e



31 **a** **i** $30x^2y = 3000$

$$y = \frac{100}{x^2}$$

ii $$\begin{aligned} S &= 12xy + 5xy + 13xy + 60x^2 \\ &= 30xy + 60x^2 \\ &= 30x\frac{100}{x^2} + 60x^2 \\ &= \frac{3000}{x} + 60x^2 \end{aligned}$$



b **i** $\dfrac{dS}{dx} = -\dfrac{3000}{x^2} + 120x$

ii $\dfrac{dS}{dx} = 0$ implies $3000 = 120x^3$

Therefore $x^3 = 250$

and hence $x = 5^{\frac{2}{3}}$

When $x = 5^{\frac{2}{3}}$, $S \approx 1539\,\text{cm}^2$

c $$\frac{dS}{dt} = \frac{dS}{dx}\frac{dx}{dt} = \left(-\frac{3000}{x2} + 120x\right) \times 0.5$$

When $x = 10$, $\dfrac{dS}{dt} = \left(-\dfrac{3000}{10^2} + 1200\right) \times 0.5$

$= 585\,\text{cm}^2/\text{s}$

32 a $f(x) = \dfrac{100\,000}{1 + 100e^{-3x}}$

Using the Chain rule

$$f'(x) = -\frac{100\,000}{(1 + 100e^{-0.3x})^2} \times -30e^{-0.3x}$$

$$= \frac{3000\,000e^{-0.3x}}{(1 + 100e^{-0.3x})^2}$$

b **i** When $x = 0$, $f'(0) = \dfrac{3\,000\,000}{(1 + 100)^2} = 294.08$

The rate of growth is 294 kangaroos per year when $x = 0$

ii When $x = 4$, $f'(4) = \dfrac{3\,000\,000e^{-1.2}}{(1 + 100e^{-1.2})^2} = 933.0498$

The rate of growth is 933 kangaroos per year when $x = 4$

33 a f is defined for

$6 - 0.2x > 0$

$\Leftrightarrow 6 > 0.2x$

$\Leftrightarrow \dfrac{6}{0.2} > x$

$\Leftrightarrow 30 > x$

$\therefore a = 30$

b $f(0) = 8\log_e 6$

When $f(x) = 0$

$8\log_e(6 - 0.2x) = 0$

which implies

$6 - 0.2x = 1$

$5 = \dfrac{1}{5}x$

$25 = x$

$\therefore$ $(25, 0)$ and $(0, 8\log_e 6)$ are the coordinates of the axes intercepts

c $f(x) = 8\log_e(6 - 0.2x)$

$$f'(x) = \frac{-8}{5(6 - 0.2x)}$$

when $x = 20$

$$f'(20) = \frac{-8}{5(6 - 4)}$$

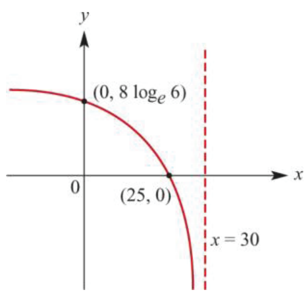
$$= \frac{-4}{5} = -0.8$$

d Consider

$$x = 8\log_e(6 - 0.2y)$$
$$e^{\frac{x}{8}} = 6 - \frac{y}{5}$$
$$\therefore\ y = 5\left(6 - e^{\frac{x}{8}}\right)$$
$$\therefore\ f^{-1}(x) = 5\left(6 - e^{\frac{x}{8}}\right)$$

e The domain of f^{-1} is R

f



34 a Calculator

b $g'(x) = \cos(x)e^{\sin x}$

$g'(x) = 0$ implies $\cos x = 0$ as $e^{\sin x} \neq 0$

$\therefore$ the stationary points occur at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$

The coordinates of the stationary points are

$\left(\frac{\pi}{2}, e\right)$ and $\left(\frac{3\pi}{2}, \frac{1}{e}\right)$

c range $= \left[\frac{1}{e}, e\right]$

d period $= 2\pi$ as $g(x + 2\pi) = g(x)$

35 a

$$y = e^x$$
$$\frac{dy}{dx} = e^x$$

When $x = 0$, $\frac{dy}{dx} = 1$

Therefore equation of tangent is $y = x + 1$

b Identical transformations applied to the curve and the tangent retain the relationship, i.e. the image of the tangent is tangent to the image of the curve.

c Consider the curve with equation $y = a\,f(bx)$

Then the gradient at $\left(\frac{x_1}{b}\right)$ is given by $\frac{dy}{dx} = ab\,f'(bx) = ab\,f'(x_1)$

But the gradient of $y = f(x)$ at x_1 is $f'(x_1) = m$

$\therefore$ gradient of $y = a\,f(bx)$ is abm

$\therefore$ equation of the tangent at $\left(\frac{x_1}{b}, y_1 a\right)$

is $y - y_1 a = abm\left(x - \frac{x_1}{b}\right)$

$$\therefore y = bam\left(x - \frac{x_1}{b}\right) + y_1 a$$
$$= bamx - amx_1 + y_1 a$$

But $y_1 = mx_1 + c$ and

$$\therefore y = a(bmx - mx_1 + y_1)$$
$$y = a(bmx - mx_1 + mx_1 + c)$$
$$\therefore y = a(bmx + c)$$

36 **a** **i** When $t = 0$, $x = \dfrac{60}{5e^0 - 3} = \dfrac{60}{2} = 30$

When $t = 0$, there are 30 g not dissolved.

ii When $t = 5$, $x = \dfrac{60}{5e^{5\lambda} - 3}$ where $\lambda = \dfrac{1}{2}\log_e \dfrac{6}{5}$

$$= \frac{60}{5e^{\frac{5}{2}\log\left(\frac{6}{5}\right)} - 3}$$
$$= \frac{60}{5 \times \left(\frac{6}{5}\right)^{\frac{5}{2}} - 3}$$
$$\approx 12.2769$$

When $t = 5$ there are 12.28 g not dissolved.

b $\dfrac{dx}{dt} = 5\lambda e^{\lambda t} \times -\dfrac{60}{(5e^{\lambda t} - 3)^2}$ (Chain rule)

$$= -\frac{300\lambda e^{\lambda t}}{(5e^{\lambda t} - 3)^2}$$

c **i** $$x = \frac{60}{5e^{\lambda t} - 3}$$

$$\therefore \quad 5xe^{\lambda t} - 3x = 60$$

$$\therefore \quad e^{\lambda t} = \frac{3x + 60}{5x}$$

$$\therefore \quad \frac{dx}{dt} = -\frac{300\lambda e^{\lambda t}}{\left(5e^{\lambda t} - 3\right)^2}$$

$$= -300\lambda\left(\frac{3x + 60}{5x}\right) \div \left(\frac{5(3x + 60)}{5x} - 3\right)^2$$

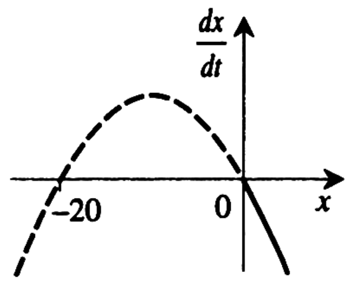
$$= -60\lambda\left(\frac{3x + 60}{x}\right) \div \left(\frac{3x + 60 - 3x}{x}\right)^2$$

$$= -60\lambda\left(\frac{3x + 60}{x}\right) \times \frac{x^2}{3600}$$

$$= -\lambda(x + 20) \times \frac{x}{20}$$

$$= -\frac{\lambda x^2}{20} - \lambda x$$

ii



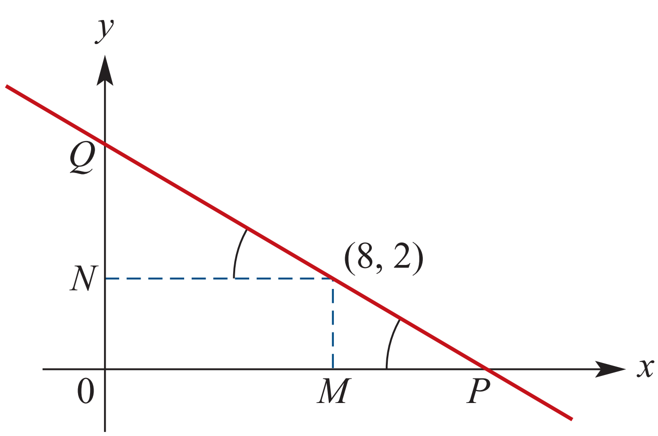
$$\frac{dx}{dt} = -\lambda\left(\frac{x^2}{20} + x\right)$$

$$= -\lambda x\left(\frac{x}{20} + 1\right)$$

x-axis intercepts $x = 0$ and $x = -20$ domain $= [0, \infty)$

iii Rate of dissolving increases where x is the amount of material not dissolved.

37



a Let $y = \dfrac{1}{\tan\theta}$
Let $u = \tan\theta$

Then $y = \dfrac{1}{u}$ and $\dfrac{dy}{d\theta} = \dfrac{dy}{du} \times \dfrac{du}{d\theta}$

$$= -\frac{1}{u^2} \times \sec^2\theta$$

$$= -\frac{1}{\tan^2\theta} \times \sec^2\theta$$

$$= -\frac{\cos^2\theta}{\sin^2\theta} \times \sec^2\theta$$

$$= -\frac{1}{\sin^2\theta}$$

$$= -\text{cosec}^2\theta$$

b $\dfrac{2}{MP} = \tan\theta$

$\therefore MP = \dfrac{2}{\tan\theta}$

c $\dfrac{NQ}{8} = \tan\theta$

$\therefore NQ = 8\tan\theta$

d $OP + OQ = OM + MP + ON + NQ$

$$= 8 + \frac{2}{\tan\theta} + 2 + 8\tan\theta$$

$$= 10 + 8\tan\theta + \frac{2}{\tan\theta}$$

e Let $x = OP + OQ$

i.e. $x = 10 + 8\tan\theta + \dfrac{2}{\tan\theta}$

$\dfrac{dx}{d\theta} = -2\,\text{cosec}^2\theta + 8\sec^2\theta$

f minimum occurs when $\dfrac{dx}{d\theta} = 0$

$$-2[\text{cosec}]^2\theta + 8\sec^2\theta = 0$$

$$\therefore\ -\frac{2}{\sin^2\theta} + \frac{8}{\cos^2\theta} = 0$$

$$\therefore\ \frac{2}{\sin^2\theta} = \frac{8}{\cos^2\theta}$$

$$\frac{\sin^2\theta}{\cos^2\theta} = \frac{2}{8} = \frac{1}{4}$$

$$\therefore \tan^2\theta = \frac{1}{4}$$

$$\text{and } \tan\theta = \pm\frac{1}{2}$$

$$\text{We know } 0 < \theta < \frac{\pi}{2}$$

$$\therefore \tan\theta = \frac{1}{2}$$

$$\therefore \theta > 26.6^\circ$$

If $\theta > 26.6^\circ, \dfrac{dx}{d\theta} < 0$ (locally)

If $\theta < 26.6^\circ, \dfrac{dx}{d\theta} > 0$

$\therefore$ minimum when $\theta = 26.6^\circ$

If $\tan\theta = \dfrac{1}{2}$

$$\begin{aligned} x &= \frac{2}{\tan\theta} + 8\tan\theta + 10 \\ &= \frac{2}{\frac{1}{2}} + 8 \times \frac{1}{2} + 10 \\ &= 4 + 4 + 10 = 18 \end{aligned}$$

The minimum value of x is 18 units.

38 Let $f : R \to R, f(x) = e^x - e^{-x}$

a $f'(x) = e^x + e^{-x}$

b $f(x) = 0$ implies $e^x - e^{-x} = 0$

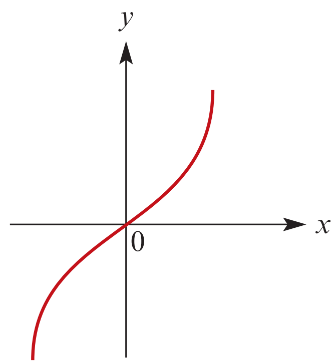
i.e. $e^x - \dfrac{1}{e^x} = 0$

$\therefore \quad e^{2x} - 1 = 0$

$\therefore \quad 2x = 0$

which implies $x = 0$

c $f'(x) = e^x + e^{-x}$ and both e^x and e^{-x} are positive, so $f'(x) > 0$ for all x.

d 

The result that $f'(x) > 0$ for all x is used

39 **a** $(\log_e x)^2 = 2\log_e x$
is equivalent to $(\log_e x)^2 - 2\log_e x = 0$
i.e. $\log_e x[\log_e x - 2] = 0$
which implies $\log_e x = 0$ or $\log_e x = 2$
$\therefore x = 1$ or $x = e^2$

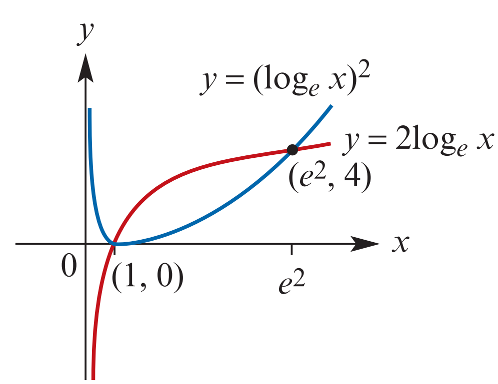
b

For $y = 2\log_e x$ $\quad \dfrac{dy}{dx} = \dfrac{2}{x}$	For $y = (\log_e x)^2$ Let $u = \log_{2^e} x$ Then $y = u^2$ and $\dfrac{dy}{dx} = \dfrac{dy}{du}.\dfrac{du}{dx}$ $= 2u.\dfrac{1}{x}$ $= \dfrac{2\log_e x}{x}$

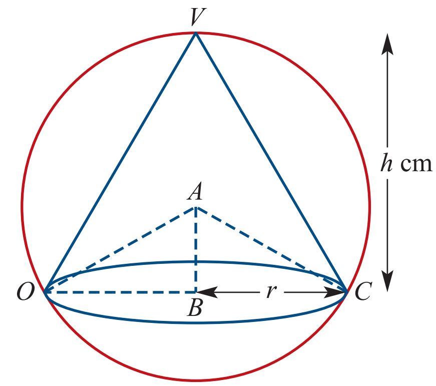
The gradient of $y = 2\log_e x$ at $x = 1$ is 2
The gradient of $y = (\log_e x)^2$ at $x = 1$ is 0
This information is now used to sketch the graphs.

c Note: $y = (\log_e x)^2 \geq 0$ for all x and
$(\log_e x)^2 \to \infty$ as $x \to 0$

d $\therefore \left\{x\colon 2\log_e x > (\log_e x)^2\right\} = \left\{x\colon 1 < x < e^2\right\}$



40 A cross-section of the solids is as shown.



a $h = VA + AE$

$= a + a\cos\theta$

where a is the radius of the sphere

b $r = a\sin\ \theta$

c $V = \dfrac{1}{3}\pi r^2 h$

$= \dfrac{1}{3}\pi(a^2\sin^2\theta)(a + a\cos\theta)$

$= \dfrac{1}{3}\pi a^3\sin^2\theta(1 + \cos\theta)$

d Using the product rule

$$\frac{dV}{d\theta} = \frac{1}{3}\pi a^3[\sin^2\theta \times -\sin\theta + 2\sin\theta\cos\theta(1 + \cos\theta)]$$

$$= \frac{1}{3}\pi a^3[-\sin^3\theta + 2\sin\theta\cos\theta(1 + \cos\theta)]$$

$$\frac{dV}{d\theta} = 0 \text{ implies } \frac{1}{3}\pi a^3[-\sin^3\theta + 2\sin\theta\cos\theta(1 + \cos\theta)] = 0$$

$\therefore$ $\sin^3\theta = 2\sin\theta\cos\theta(1 + \cos\theta)$

For $\sin\theta \neq 0$

$\sin^2\theta = 2\cos\theta + 2\cos^2\theta$

Using $\sin^2\theta = 1 - \cos^2\theta$

$1 - \cos^2\theta = 2\cos\theta + 2\cos^2\theta$

which implies $3\cos^2\theta + 2\cos\theta - 1 = 0$

This is a quadratic equation in $\cos\theta$. It factorises to give the following:

$(3\cos\theta - 1)(\cos\theta + 1) = 0$

$$\therefore \cos\theta = \frac{1}{3} \text{ or } \cos\theta = -1$$

A gradient chart confirms a maximum volume occurs when

$\cos\theta = \dfrac{1}{3}\left(\text{when } \theta = \cos^{-1}\left(\dfrac{1}{3}\right)\right)$, i.e. $\theta \approx 70.53°$

e $$V = \frac{1}{3}\pi a^3 \sin^2 4(1 + \cos\theta)$$

Using $\sin^2\theta = 1 - \cos^2\theta$

$$V = \frac{1}{3}\pi a^3(1 - \cos^2\theta)(1 + \cos\theta)$$

When $\cos\theta = \frac{1}{3}$

$$V = \frac{1}{3}\pi a^3\left(1 - \frac{1}{9}\right)\left(1 + \frac{1}{3}\right)$$

$$= \frac{1}{3}\pi a^3 \times \frac{8}{9} \times \frac{4}{3}$$

$$= \frac{32\pi a^3}{81}$$

The maximum volume is $\frac{32\pi a^3}{81}$ cm^3

41 a $y = \dfrac{Ae^{bt}}{1 + Ae^{bt}}$

Dividing through by $1 + Ae^{bt}$ gives

$y = 1 - \dfrac{1}{1 + Ae^{bt}}$

and as $Ae^{bt} > 0$ for all t, $\dfrac{1}{1 + Ae^{bt}} < 1$

Hence $0 < y < 1$

b By using the quotient rule

$$\frac{dy}{dt} = \frac{(1 + Ae^{bt})bAe^{bt} - bA^2e^{2bt}}{(1 + Ae^{bt})^2}$$

$$= \frac{bAe^{bt}}{(1 + Ae^{bt})^2}$$

c As $$y = \frac{Ae^{bt}}{1 + Ae^{bt}}$$

$$y(1 + Ae^{bt}) = Ae^{bt}$$

$$\therefore \quad y + yAe^{bt} = Ae^{bt}$$

and $$y = Ae^{bt}(1 - y)$$

$$\therefore \quad Ae^{bt} = \frac{y}{1 - y}$$

d **i** From the result of **b**

$$\frac{dy}{dt} = \frac{bAe^{bt}}{(1 + Ae^{bt})^2}$$

Substituting $Ae^{bt} = \dfrac{y}{1 - y}$

$$\begin{aligned}\frac{dy}{dx} &= b\left(\frac{y}{1-y}\right) \div \left(1 + \frac{y}{1-y}\right)^2 \\ &= b\left(\frac{y}{1-y}\right) \times (1-y)^2 \\ &= by(1-y)\end{aligned}$$

ii $\dfrac{dy}{dx} = by(1 - y)$

is a quadratic expression in y with negative coefficient of y^2 (b is a positive constant)

$\therefore$ a maximum occurs when $y = 0.5$.

e From **c** $Ae^{bt} = \dfrac{y}{1 - y}$

$\therefore$ when $A = 0.01$, $b = 0.7$ and $y = 0.5$

$$0.01e^{0.7t} = 1$$

$$\therefore \quad e^{0.7t} = 100$$

$$\therefore \quad 0.7t = \log_e 100$$

$$\therefore \quad t = \frac{10}{7}\log_e 100 \approx 6.578$$

$\therefore$ The bacteria are increasing at the fastest rate when $t = 7$ (to the nearest hour).

42 Let $f(x) = \dfrac{e^x}{x}$

a $f(x) = e^x.x^{-1}$

The product rule gives

$$\begin{aligned}f'(x) &= e^x x^{-1} - e^x x^{-2} \\ &= \frac{e^x}{x} - \frac{e^x}{x^2} \\ &= \frac{xe^x - e^x}{x^2}\end{aligned}$$

b If

$$f'(x) = 0$$

$$\frac{xe^x - e^x}{x^2} = 0$$

which implies $e^x(x - 1) = 0$

and as $e^x \neq 0$ for all $x \in R^+$, $x = 1$

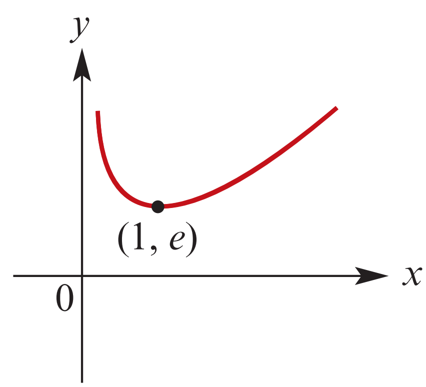
c There is a stationary point when $x = 1$ and $f(1) = e$.

Therefore there is a stationary point at $(1, e)$.

If $x > 1$ $f'(x) = \dfrac{e^x(x-1)}{x^2} > 0$ (Note: domain of f' is R^+)

If $x < 1$ $f'(x) = \dfrac{e^x(x-1)}{x^2} < 0$

$\therefore$ there is a minimum at$(1, e)$



d **i** $\dfrac{f'(x)}{f(x)} = \dfrac{xe^x - e^x}{x^2} \times \dfrac{x}{e^x}$

$= \dfrac{x-1}{x}$

ii $\lim_{x\to\infty} \dfrac{f'(x)}{f(x)} = \lim\limits_{x\to\infty}\left(1 - \dfrac{1}{x}\right) = 1$

$\therefore$ $f'(x) \approx f(x)$ when x is very large

e $n = \dfrac{ae^{kt}}{t}$

When $t = 65, n = \dfrac{ae^{65k}}{65}$

When $t = 30, n = \dfrac{ae^{30k}}{30}$

The population of birds is the same for the years 1930 and 1965.

$\therefore \quad \dfrac{ae^{65k}}{65} = \dfrac{ae^{30k}}{30}$

$\therefore \quad e^{35k} = \dfrac{65}{30} = \dfrac{13}{6}$

$\therefore \quad 35k = \log_e\left(\dfrac{13}{6}\right)$

$\therefore \quad k = \dfrac{1}{35}\log_e\left(\dfrac{13}{6}\right) \approx 0.0221$

f Minimum occurs when $\dfrac{dn}{dt} = 0$

Using the quotient rule

$\dfrac{dn}{dt} = \dfrac{a(kte^{kt} - e^{kt})}{t^2}$

$= \dfrac{ae^{kt}(kt-1)}{t^2}$

$\dfrac{dn}{dt} = 0$ implies $t = \dfrac{1}{k}$

This is a local minimum

and $t = \dfrac{1}{\frac{1}{35}\log_e\left(\frac{13}{6}\right)} = \dfrac{35}{\log_e\left(\frac{13}{6}\right)} \approx 45.27$

The minimum population occurred in 1945.

43 **a** When $t = 0, N = 1000$

When $t = 5, N = 10\,000$

As $N = Ae^{kt}$

$1000 = Ae^0$

which implies $A = 1000$

Also $10\,000 = 1000e^{5k}$

which implies

$e^{5k} = 10$

$\therefore k = \frac{1}{5}\log_e 10$

i.e. $A = 1000$ and $k = \frac{1}{5}\log_e 10 \approx 0.46$

b $\dfrac{dN}{dt} = kAe^{kt}$, where $A = 1000$ and $k = \frac{1}{5}\log_e 10$

c $\dfrac{dN}{dt} = kN$ as $N = Ae^{kt}$

d **i** When $t = 4$

$$\frac{dN}{dt} = \frac{1}{5}\log_e 10 \times 1000e^{\frac{4}{5}\log_e 10}$$
$$= 200\log_e 10 \times 10^{\frac{4}{5}}$$

i.e. the rate of growth when $t = 4$ is 2905.7 bacteria/hour.

ii When $t = 50$

$$\frac{dN}{dt} = 200(\log_e 10)e^{\frac{50}{5}\log_e 10}$$
$$= 200\log_e 10 \times 10^{10}$$
$$= 2\log_e 10 \times 10^{12}$$
$$\approx 4.61 \times 10^{12}$$

i.e. the rate of growth when $t = 50$ is 4.61×10^{12} bacteria/hour.

44 **a** For the populations to be equal

$2 \times 10^4 e^{0.03t} = 10^4 e^{0.05t}$

which implies

$e^{0.02t} = 2$

$\therefore t = \dfrac{1}{0.02}\log_e 2 \approx 34.657$

The populations will be equal after 34.66 years.

b $\dfrac{dA}{dt} = 600e^{0.03t}$ and $\dfrac{dB}{dt} = 500e^{0.05t}$

Rates are equal implies

$$\frac{dA}{dt} = \frac{dB}{dt}$$

$$\therefore 600e^{0.03t} = 500e^{0.05t}$$

$$\frac{6}{5} = e^{0.02t}$$

which implies

$$t = \frac{1}{0.02}\log_e\left(\frac{6}{5}\right)$$

$$= 50\log_e\left(\frac{6}{5}\right) \approx 9.116$$

Rates are equal after 9.12 years.

45 $h = 0.5 + 0.2\sin(3\pi t) t \geq 0$

where h metres is the height of the particular time t seconds.

a The greatest height above the floor is reached at values of t such that $\sin(3\pi t) = 1$

i.e. the greatest height is 0.7 metres

and this occurs when

$$3\pi t = \frac{\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \cdots$$

$$t = \frac{1}{6} \text{ or } \frac{5}{6} \text{ or } \cdots$$

The question required the first time and this is given by $t = \dfrac{1}{6}$

b Period of oscillation is $2\pi \div 3\pi = \dfrac{2}{3}$ seconds.

c $\dfrac{dh}{dt} = 0.6\pi\cos(3\pi t)$

i When $t = \dfrac{1}{3}$, $\dfrac{dh}{dt} = 0.6\pi\cos(\pi) = -0.6\pi$

The speed is $0.6\,\pi$ metres/second moving downwards.

ii When $t = \dfrac{2}{3}$, $\dfrac{dh}{dt} = 0.6\pi\cos(2\pi) = 0.6\pi$

The speed is 0.6π metres/second moving upwards.

iii When $t = \dfrac{1}{6}$, $\dfrac{dh}{dt} = 0.6\pi\cos\left(\dfrac{\pi}{2}\right) = 0$

The speed is 0 metres/second.

46 $T(t) = p + q\cos(\pi rt)$ where p, q and r are constants

a From the graph:

i The period is 12 $\therefore \quad \dfrac{2\pi}{\pi r} = 12$

which implies $r = \dfrac{1}{6}$

ii The amplitude is $\dfrac{(20-4)}{2} = 8$ which implies $q = 8$

The centre is $T = \dfrac{20+4}{2} = 12$ which implies $p = 12$

b $T'(t) = -\pi rq\sin(\pi rt)$

$$\therefore T'(3) = -\frac{4\pi}{3}\sin\left(\frac{3\pi}{6}\right)$$
$$= -\frac{4\pi}{3}\sin\left(\frac{\pi}{2}\right) = -\frac{4\pi}{3}$$

The hours of night are decreasing at a rate of $\dfrac{4\pi}{3}$ hours/month when $t = 3$

$$T'(9) = -\frac{4\pi}{3}\sin\left(\frac{3\pi}{2}\right) = \frac{4\pi}{3}$$

The hours of night are increasing at a rate of $\dfrac{4\pi}{3}$ hours/month when $t = 9$

c Average rate of change from $t = 0$ to $t = 6$

$$= \frac{T(6) - T(0)}{6 - 0}$$

when $T(t) = 12 + 8\cos\left(\dfrac{\pi t}{6}\right)$

$$\therefore \text{ Average rate of change } = \frac{12 + 8\cos(\pi) - (12 + 8\cos(0))}{6}$$
$$= \frac{12 - 8 - 12 - 8}{6}$$
$$= -\frac{16}{6} = -\frac{8}{3}$$

i.e. the average rate of change for time interval $[0, 6]$ is $-\dfrac{8}{3}$ hours/month.

d $T'(t) = -\dfrac{4\pi}{3}\sin\left(\dfrac{\pi t}{6}\right)$

Rate of change of hours is maximum (in the sense of maximum increasing rate)

when $\sin\left(\dfrac{\pi t}{6}\right) = -1$

This occurs when

$$\frac{\pi t}{6} = \frac{3\pi}{2} \text{ or } \frac{7\pi}{2} \text{ or } \ldots$$

i.e. $t = 9$ or 21 or $\ldots$

The rate of change of hours of night is a miximum after 9 months.

47 **a** Area A = length × width

$= x \times 2\cos(3x)$

$= 2x\ \cos(3x)$

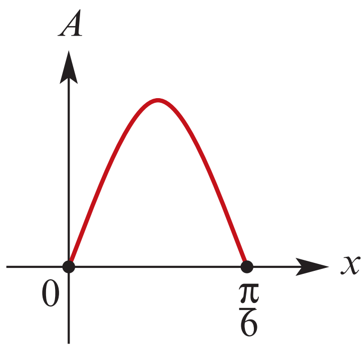
b **i** $\dfrac{dA}{dx} = 2\cos(3x) - 6x\sin(3x)$

ii When $x = 0$, $\dfrac{dA}{dx} = 2$

When $x = \dfrac{\pi}{6}$, $\dfrac{dA}{dx} = -6 \times \dfrac{\pi}{6}\sin\dfrac{\pi}{2}$

$= -\pi$

c **i**



ii Either use the 'Intersect' feature of a CAS calculator of the graph screen or use the 'solve' command at the calculator screen to solve the equation $2x\cos 3x = 0.2$.
$x = 0.105$ or $x = 0.449$

iii maximum area is 0.374
Use the 'max' feature of a CAS calculator of the graph screen or use the 'flex' command at the calculator screen with the instruction $0 < x < \dfrac{y}{6}$. when $x = 0.287$

d **i** $\dfrac{dA}{dx} = 2\cos(3x) - 6x\sin(3x)$

$\dfrac{dA}{dx} = 0$ implies $\tan(3x) = \dfrac{1}{3x}$

ii The co-ordinates of the points of intersection are (0.287, 1.16), founded as with **c ii**.

48 **a** **i** $N'(t) = -1 + \dfrac{1}{10}e^{\frac{t}{20}}$

ii $N'(t) = 0$ implies $10 = e^{\frac{t}{20}}$

$\therefore \quad t = 20\log_e 10 \approx 46.05$

When $t = 20\log_e 10$

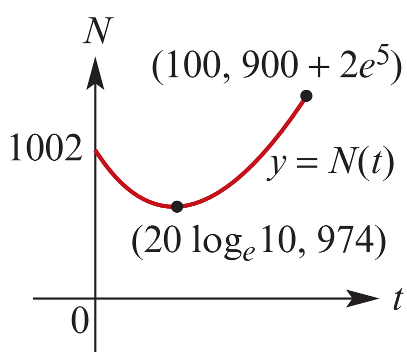
$$
\begin{aligned}
N(t) &= 1000 - 20\log_e 10 + 2e^{\log_e 10} \\
&= 1000 - 20\log_e 10 + 20 \\
&= 1020 - 20\log_e 10 \\
&\approx 973.95
\end{aligned}
$$

Minimum population is 974

iii $N(0) = 1000 + 2 = 1002$

iv
$$
\begin{aligned}
N(100) &= 1000 - 100 + 2e^5 \\
&= 900 + 2e^5 \\
&\approx 1196.826
\end{aligned}
$$

v



b $N_2(t) = 1000 - t^{\frac{1}{2}} + 2e^{\frac{t^{\frac{1}{2}}}{20}}$

i
$$
\begin{aligned}
N_2(0) &= 1000 - 0 + 2 \\
&= 1002
\end{aligned}
$$

ii
$$
\begin{aligned}
N_2(100) &= 1000 - 10 + 2e^{\frac{1}{2}} \\
&= 990 + 2e^{\frac{1}{2}}
\end{aligned}
$$

iii
$$
\begin{aligned}
N_2'(t) &= -\frac{1}{2}t^{-\frac{1}{2}} + \frac{1}{2}t^{-\frac{1}{2}} \times \frac{1}{20} \times 2e^{\frac{t^{\frac{1}{2}}}{20}} \\
&= \frac{1}{2}t^{\frac{1}{2}}\left(-1 + \frac{1}{10}e^{\frac{t^{\frac{1}{2}}}{20}}\right)
\end{aligned}
$$

$N_2'(t) = 0$ implies $e^{\frac{t^{\frac{1}{2}}}{20}} = 10$

$\therefore \quad \dfrac{t^{\frac{1}{2}}}{20} = \log_e 10$

$\therefore \quad t^{\frac{1}{2}} = 20\log_e 10$

$\therefore \quad t = (20\log_e 10)^2$

When $\quad t = (20\log_e 10)^2$

$$N_2(t) = 1000 - 20\log_e 10 + 2e^{\log_s 10}$$

$$= 1000 - 20\log_e 10 + 20$$

Minimum population is 974

c $N_3(t) = 1000 - t^{\frac{3}{2}} + 2e^{\frac{t}{20}}$

i Using a CAS calculator with the 'min' feature at the graph screen, the minimum population is
297 when $t = 100.24$

d **i** $N_3'(t) = -\dfrac{3}{2}t^{\frac{1}{2}} + \dfrac{1}{10}e^{\frac{t}{20}}$

ii $N_3'(t) = 0$

$\dfrac{3}{2}t^{\frac{1}{2}} = \dfrac{1}{10}e^{\frac{t}{20}}$

$15t^{\frac{1}{2}} = e^{\frac{t}{20}}$

$t = 20\log_e(15\sqrt{t})$

49 a $y = (2x^2 - 5x)e^{ax}$

$(3, 10):\ 10 = 3e^{3a}$

$e^{3a} = \dfrac{10}{3}$

$a = \dfrac{1}{3}\log e\left(\dfrac{10}{3}\right)$

b i $y = 0$: $2x^2 - 5x = 0$ (since $e^{ax} > 0$)

$x(2x - 5) = 0$

$x = 0, \dfrac{5}{2}$

ii $\dfrac{dy}{dx} = (4x - 5)e^{ax} + (2x^2 - 5x) \times ae^{ax}$

$= (2ax^2(4 - 5a)x - 5)e^{ax}$

$= 0$ if

$2ax^2 + (4 - 5a)x - 5 = 0$

$$x = \frac{-4 + 5a \pm \sqrt{16 - 40a + 25a^2 + 40a}}{4a}$$

$$= \frac{-4 + 5a \pm \sqrt{25a^2 + 16}}{4a}$$

Chapter 11 – Integration

Solutions to Exercise 11A

1 a $A \approx \frac{1}{2} \times \frac{1}{2}(2.5)^2 + \frac{1}{2} \times \frac{1}{2}(3)^2$

$= \frac{1}{4} \times \frac{25}{4} + \frac{1}{4} \times 9$

$= \frac{25}{16} + \frac{9}{4}$

$= \frac{25}{16} + \frac{36}{16}$

$A \approx \frac{61}{16} \approx 3.81$ square units

b $A \approx \frac{\pi}{4} \times \cos 0 + \frac{\pi}{4} \times \cos\frac{\pi}{4}$

$= \frac{\pi}{4} + \frac{\pi}{4} \times \frac{1}{\sqrt{2}}$

$A \approx \frac{\left(1 + \sqrt{2}\pi\right)\pi}{4\sqrt{2}} \approx 1.34$ square units

c $A \approx 1 \times \frac{1}{2}(2)^3 + 1 \times \frac{1}{2}(3)^3$

$= 4 + \frac{27}{2}$

$A \approx \frac{35}{2} \approx 17.5$ square units

2 a $A \approx 1 \times (f(1) + f(2) + f(3) + f(4))$

$= 5 + 3.5 + 2.5 + 2.2$

$A \approx 13.2$ square units

b $A \approx 1 \times (f(2) + f(3) + f(4) + f(4))$

$= 3.5 + 2.5 + 2.2 + 2$

$A \approx 10.2$ square units

3 a $y = x(4 - x)$

using left end estimate

$A \approx 1.0 \times ((0)(4 - 0) + (1)(4 - 1)$

$+ (2)(4 - 2) + (3)(4 - 3))$

$= 0 + 3 + 4 + 3$

$A \approx 10$ square units

using right end estimate

$A \approx 1.0 \times ((1)(4 - 1) + (2)(4 - 2)$

$+ (3)(4 - 3) + (4)(4 - 4)$

$= 3 + 4 + 3 + 0$

$A \approx 10$ square units

b using the CAS calculator

10.64 square units

4 a $y = \frac{1}{1 + x^2}$

$A \approx 0.25\left(\frac{1}{1 + (0.25)^2} + \frac{1}{1 + (0.5)^2} + \frac{1}{1 + (0.75)^2} + \frac{1}{1 + (1)^2}\right)$

$= \frac{1}{4}\left(\frac{1}{\left(\frac{17}{16}\right)} + \frac{1}{\left(\frac{5}{4}\right)} + \frac{1}{\left(\frac{25}{16}\right)} + \frac{1}{2}\right)$

$= \frac{1}{4}\left(\frac{16}{17} + \frac{4}{5} + \frac{16}{25} + \frac{1}{2}\right)$

$= \frac{14}{17} + \frac{1}{5} + \frac{4}{25} + \frac{1}{8}$

$A \approx \frac{2449}{3400} \approx 0.72$ square units

b $\frac{\pi}{4} = A \approx 0.72$ square units

$\pi \approx 2.88$

this approximation could be improved by adding more sub – intervals to the area calculation

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5 a $A \approx 1.0 \times (3 + 3.5 + 3.7 + 3.8 + 3.9$
$+ 3.9 + 4.0 + 4.0 + 3.7 + 3.3)$
$A \approx 36.8$ square units

b $A \approx 1.0 \times (3.5 + 3.7 + 3.8 + 3.9 + 3.9$
$+ 4.0 + 4.0 + 3.7 + 3.3 + 2.9)$
$A \approx 36.7$ square units

6 a

$y = 2^x$

$A \approx 0.5\left(2^{0.5} + 2^1 + 2^{1.5} + 2^2 + 2^{2.5} + 2^3\right)$

$= \frac{1}{\sqrt{2}} + 1 + \sqrt{2} + 2 + 2\sqrt{2} + 4$

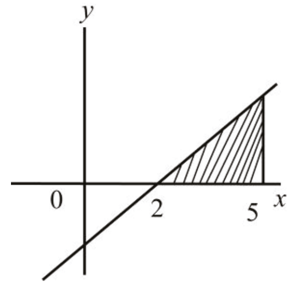
$A \approx 7 + \frac{7\sqrt{2}}{2} \approx 11.9$ square units

7 a $A \approx 1.0 \times (4 + 5.6 + 7 + 8 + 8.2$
$+ 8.1 + 7.6)$
$A \approx 48.5$ square units
note: since the values are read off the graph, the value of A is approximate

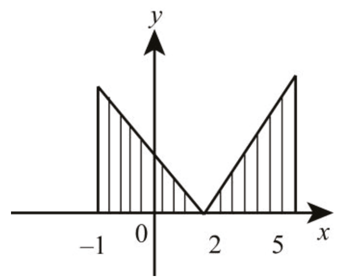
b this area represents the distance travelled between t = 2 and t = 9

8 a The definite integral represents the trangular region shown.
The trangle has base $5 - 2 = 3$ units.
When $x = 5$, $y = 5 - 2 = 3$, so the triangle has height 3 units Area
$= \frac{1}{2} \times 3 \times 3$
$= \frac{9}{2}$ square units

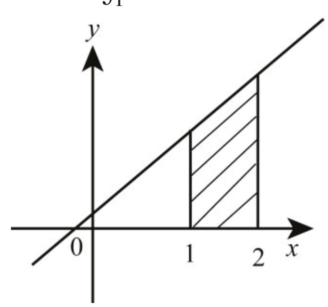
Hence $\int_2^5 (x - 2)\, dx = \frac{9}{2}$.



b The definits integral represents two equal traingular regions show.
Formal, area $= 2 \times \frac{9}{2} = 9$ square units
Hence $\int_{-1}^5 |x - 2|\, dx = 9$.



c The definits integral represents the trapezium region shown.
The distance between the parallel sides is 1 unit.
When $x = 1, y = 2 + 1 = 3$ units.
When $x = 2, y = 4 + 1 = 5$ units.
Area $= \frac{1}{2} \times 1 \times (3 + 5)$
$= 4$ square units
Hence $\int_1^2 (2x + 1)\, dx = 4$



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Solutions to Exercise 11B

1 a $\int \frac{1}{2}x^3\,dx$

$= \frac{1}{2} \times \frac{1}{4}x^4 + c$

$= \frac{1}{8}x^4 + c$

b $\int 5x^3 - 2x\,dx$

$= 5 \times \frac{x^4}{4} - 2 \times \frac{x^2}{2} + c$

$= \frac{5}{4}x^4 - x^2 + c$

c $\int \frac{4}{5}x^3 - 3x^2\,dx$

$= \frac{4}{5} \times \frac{x^4}{4} - 3\frac{x^3}{3} \times + c$

$= \frac{1}{5}x^4 - x^3 + c$

d $\int 6z - 3z^2 - z + 2\,dz$

$= \int 3z^2 + 5z + 2\,dz$

$= -3\frac{z^3}{3} + 5\frac{z^2}{2} + 2z + c$

$= -z^3 + \frac{5}{2}z^2 + 2z + c$

2 a $\frac{dy}{dx} = x^{-3}$

$y = -\frac{1}{2}x^{-2} + c = -\frac{1}{2x^2} + c$

b $\frac{dy}{dx} = 4\sqrt[3]{x} = 4x^{\frac{1}{3}}$

$y = 4 \times \frac{3}{4}x^{\frac{4}{3}} + c = 3x^{\frac{4}{3}} + c$

c $\frac{dy}{dx} = x^{\frac{1}{4}} + x^{-\frac{3}{5}}$

$y = \frac{4}{5}x^{\frac{5}{4}} + \frac{5}{2}x^{\frac{2}{5}} + c$

3 a $\int 3x^{-2}\,dx$

$= 3\frac{x^{-1}}{-1} + c$

$= \frac{-3}{x} + c$

b $\int 2x^{-4} + 6x\,dx$

$= 2\frac{x^{-3}}{-3} + 6\frac{x^2}{2} + c$

$= \frac{-2}{3}x^{-3} + 3x^2 + c$

c $\int 2x^{-2} + 6x^{-3}\,dx$

$= -2x^{-1} - 3x^{-2} + c$

d $\int 3x^{\frac{1}{3}} - 5x^{\frac{5}{4}}\,dx$

$= 3\frac{x^{\frac{4}{3}}}{\frac{4}{3}} - 5\frac{x^{\frac{9}{4}}}{\frac{9}{4}} + c$

$= \frac{9}{4}x^{\frac{4}{3}} - \frac{20}{9}x^{\frac{9}{4}} + c$

e $\int 3x^{\frac{3}{4}} - 7x^{\frac{1}{2}}\,dx$

$= \frac{12}{7}x^{\frac{7}{4}} - \frac{14}{3}x^{\frac{3}{2}} + c$

f $\int 4x^{\frac{3}{5}} + 12x^{\frac{5}{3}}\,dx$

$= \frac{5}{2}x^{\frac{8}{5}} + \frac{9}{2}x^{\frac{8}{3}} + c$

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4 a $\frac{dy}{dx} = 2x - 3$

$y = \frac{2x^2}{2} - 3x + c$

$= x^2 - 3x + c$

$x = 1,\ y = 1$

$1 = 1 - 3 + c$

$c = 3$

$y = x^2 - 3x + 3$

b $\frac{dy}{dx} = x^3$

$y = \frac{x^4}{4} + c$

$x = 0,\ y = 6$

$6 = c$

$y = \frac{x^4}{4} + 6$

c $y = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^2}{2} + c$

$= \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 + c$

$x = 4,\ y = 6$

$6 = \frac{2}{3} \times 8 + \frac{1}{2} \times 16 + c$

$6 = \frac{16}{3} + 8 + c$

$c = -2 - \frac{16}{3} = \frac{-22}{3}$

$y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 - \frac{22}{3}$

5 a $\int \sqrt{x}(2 + x)\,dx = \int 2x^{\frac{1}{2}} + x^{\frac{3}{2}}\,dx$

$= \frac{4}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + c$

b $\int \frac{3z^4 + 2z}{z^3}\,dz = \int 3z + 2z^{-2}\,dz$

$= \frac{3z^2}{2} - 2z^{-1} + c$

$= \frac{3z^2}{2} - \frac{2}{z} + c$

$= \frac{3z^3 - 4}{2z} + c$

c $\int \frac{5x^3 + 2x^2}{x},dx = \int 5x^2 + 2x\,dx$

$= \frac{5x^3}{3} + x^2 + c$

$= \frac{5x^3 + 3x^2}{3} + c$

d $\int \sqrt{x}(2x + x^2)\,dx = \int 2x^{\frac{3}{2}} + x^{\frac{5}{2}}\,dx$

$= \frac{4}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}} + c$

e $\int x^2(2 + 3x^2)\,dx = \int 2x^2 + 3x^4\,dx$

$= \frac{2x^3}{3} + \frac{3x^5}{5} + c$

f

$\int \sqrt[3]{x}(x + x^4)\,dx = \int x^{\frac{4}{3}} + x^{\frac{13}{3}}\,dx$

$= \frac{3}{7}x^{\frac{7}{3}} + \frac{3}{16}x^{\frac{16}{3}} + c$

6 $f'(x) = 3x^2 - x^{-2}$

$f(x) = 3\frac{x^3}{3} - \frac{x^{-1}}{-1} + c$

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$$= x^3 + \frac{1}{x} + c$$

$$f(2) = 0$$

$$0 = 8 + \frac{1}{2} + c$$

$$c = \frac{-17}{2}$$

$$f(x) = x^3 + \frac{1}{x} - \frac{17}{2}$$

7 $\frac{ds}{dt} = 3t - \frac{8}{t^2} = 3t - 8t^{-2}$

$$s = 3\frac{t^2}{2} - \frac{8t^{-1}}{-1} + c$$

$$= \frac{3}{2}t^2 + \frac{8}{t} + c$$

$$t = 1,\ s = \frac{3}{2}$$

$$\frac{3}{2} = \frac{3}{2} + 8 + c$$

$$c = -8$$

$$s = \frac{3}{2}t^2 + \frac{8}{t} - 8$$

8 **a** $f'(x) = 16x + k$

$$f'(2) = 0\text{s}$$

$$0 = 32 + k$$

$$k = -32$$

$$f'(x) = 16x - 32$$

b $f(x) = 16\frac{x^2}{2} - 32x + c$

$$= 8x^2 - 32x + c$$

$$f(2) = 1$$

$$1 = 32 - 64 + c$$

$$c = 33$$

$$f(x) = 8x^2 - 32x + 33$$

$$f(7) = 8 \times 49 - 32 \times 7 + 33$$

$$= 201$$

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Solutions to Exercise 11C

1 a $\int (2x-1)^2\,dx$

$= \dfrac{1}{2\times 3}(2x-1)^3 + c$

$= \dfrac{1}{6}(2x-1)^3 + c$

b $\int (2-t)^3\,dt$

$= \dfrac{1}{-1\times 4}(2-t)^4 + c$

$= \dfrac{-1}{4}(2-t)^4 + c$

c $\int (5x-2)^3\,dx$

$= \dfrac{1}{5\times 4}(5x-2)^4 + c$

$= \dfrac{1}{20}(5x-2)^4 + c$

d $\int (4x-6)^{-2}\,dx$

$= -\dfrac{1}{4}(4x-6)^{-1} + c$

$= -\dfrac{1}{4(4x-6)} + c$

$= \dfrac{1}{24-16x} + c$

e $\int (6-4x)^{-3}\,dx$

$= \dfrac{1}{8}(6-4x)^{-2} + c$

$= \dfrac{1}{8(6-4x)^2} + c$

f $\int (4x+3)^{-3}\,dx$

$= -\dfrac{1}{8}(4x+3)^{-2} + c$

$= -\dfrac{1}{8(4x+3)^2} + c$

g $\int (3x+6)^{\frac{1}{2}}\,dx$

$= \dfrac{1}{3\times\frac{3}{2}}(3x+6)^{\frac{3}{2}} + c$

$= \dfrac{2}{9}(3x+6)^{\frac{3}{2}} + c$

h $\int (3x+6)^{\frac{-1}{2}}\,dx$

$= \dfrac{1}{3\times\frac{1}{2}}(3x+6)^{\frac{1}{2}} + c$

$= \dfrac{2}{3}(3x+6)^{\frac{1}{2}} + c$

i $\int (2x-4)^{\frac{7}{2}}\,dx$

$= \dfrac{1}{2\times\frac{9}{2}}(2x-4)^{\frac{9}{2}} + c$

$= \dfrac{1}{9}(2x-4)^{\frac{9}{2}} + c$

j $\int (3x+11)^{\frac{4}{2}}\,dx$

$= \dfrac{1}{3\times\frac{7}{3}}(3x+11)^{\frac{7}{2}} + c$

$= \dfrac{1}{7}(3x+11)^{\frac{7}{2}} + c$

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k $\int (2-3x)^{\frac{1}{2}}\,dx$

$= \dfrac{1}{-3 \times \frac{3}{2}}(2-3x)^{\frac{3}{2}} + c$

$= \dfrac{-2}{9}(2-3x)^{\frac{3}{2}} + c$

l $\int (5-2x)^4\,dx$

$= \dfrac{1}{-2 \times 5}(5-2x)^5 + c$

$= \dfrac{-1}{10}(5-2x)^5 + c$

2 a $\int \dfrac{1}{2}x^{-1}\,dx$

$= \dfrac{1}{2}\log_e x + c$

b $\int \dfrac{1}{3x+2}\,dx$

$= \dfrac{1}{3}\log_e(3x+2) + c$

c $\int \dfrac{4}{1+4x}\,dx$

$= \log_e(1+4x) + c$

d $\int \dfrac{5}{3x-2}\,dx$

$= \dfrac{5}{3}\log_e(3x-2) + c$

e $\int \dfrac{3}{1-4x}\,dx$

$= -\dfrac{3}{4}\log_e(1-4x) + c$

f $\int \dfrac{3}{2-\frac{x}{2}}\,dx$

$= -6\log_e(\dfrac{4-x}{2}) + c$

$= -6\log_e(x-4) + c_2$

3 a $\int \dfrac{5}{x}\,dx$

$= 5\log_e|x| + c$

b $\int \dfrac{3}{x-4}\,dx$

$= 3\log_e|x-4| + c$

c $\int \dfrac{10}{2x+1}\,dx$

$= \dfrac{10}{2}\log_e|2x+1| + c$

$= 5\log_e|2x+1| + c$

d $\int \dfrac{6}{5-2x}\,dx$

$= \dfrac{6}{-2}\log_e|5-2x| + c$

$= -3\log_e|2x-5| + c$

e $\int 6(1-2x)^{-1}\,dx$

$= -3\log_e|1-2x| + c$

f $\int (4-3x)^{-1}\,dx$

$= \dfrac{1}{-3}\ln|4-3x| + c$

$= \dfrac{-1}{3}\ln|3x-4| + c$

4 a $3x + \log_e|x| + c$

b $x + \log_e|x| + c$

c $-\dfrac{1}{x+1} + c$

d $2x + \dfrac{x^2}{2} + \log_e|x| + c$

e $-\dfrac{3}{2(x-1)^2} + c$

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f $-2x + \log_e |x| + c$

5 a $\frac{dy}{dx} = \frac{1}{2x}\ x > 0$

$y = \frac{1}{2}\log_e x + c$

$x = e^2,\ y = 2$

$2 = \frac{1}{2}\log_e e^2 + c$

$2 = \frac{1}{2} \times 2 + c$

$c = 1$

$y = \frac{1}{2}\log_e x + 1\ ,\ x > 0$

b $\frac{dy}{dx} = \frac{2}{5 - 2x}$

$y = \frac{2}{-2}\log_e |5 - 2x| + c$

$y = -\log_e |2x - 5| + c$

$x = 2, y = 10$

$10 = -\log_e 1 + c$

$c = 10$

$y = -\log_e 2x - 5 + 10\ ,\ x < \frac{5}{2}$

6 $f'(x) = \frac{10}{x - 5}$

$f(x) = 10\log_e |x - 5| + c$

$f(5 + e) = 10$

$10 = 10\log_e e + c$

$c = 0$

$f(x) = 10\log_e x - 5\ ,\ x > 5$

7 a $\int \frac{x}{x+1}\,dx$

$= \int 1 - \frac{1}{x+1}\,dx$

$= x - \log_e |x + 1| + c$

b $\int \frac{1-2x}{x+1}\,dx$

$= \int -2 + \frac{3}{x+1}\,dx$

$= -2x + 3\log_e |x + 1| + c$

c $\int \frac{2x+1}{x+1}\,dx$

$= \int 2 - \frac{1}{x+1}\,dx$

$= 2x - \log_e |x + 1| + c$

8 $\frac{dy}{dx} = \frac{3}{x - 2}$

$y = 3\log_e |x - 2| + c$

$x = 0,\ y = 10$

$10 = 3\log_e |-2| + c$

$c = 10 - 3\log_e 2$

$y = 3\log_e |x - 2| + 10 - 3\log_e 2$

$y = 3\log_e\left(\frac{|x-2|}{2}\right) + 10$

You can complete it without using the absolute value function.

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$$\frac{dy}{dx} = \frac{3}{x-2}$$
$$= -\frac{3}{2-x}$$
$$y = 3\log_e(2-x) + c$$
$$x = 0,\ y = 10$$
$$10 = 3\log_e 2 + c$$
$$c = 10 - 3\log_e 2$$
$$y = 3\log_e(2-x) + 10 - 3\log_e 2$$
$$y = 3\log_e\left(\frac{2-x}{2}\right) + 10$$

9 $\dfrac{dy}{dx} = \dfrac{5}{2-4x}$

$$y = \frac{5}{-4}\log_e|2-4x| + c$$
$$y = \frac{-5}{4}\log_e|4x-2| + c$$
$$x = -2,\ y = 10$$
$$10 = \frac{-5}{4}\log_e|-8-4| + c$$
$$10 = \frac{-5}{4}\log_e 10 + c$$
$$c = 10 + \frac{5}{4}\log_e 10$$
$$y = \frac{5}{4}\log_e 10 - \frac{5}{4}\log_e|4x-2| + 10$$
$$y = \frac{5}{4}\log_e\left|\frac{10}{4x-2}\right| + 10$$
$$y = \frac{5}{4}\log_e\left|\frac{5}{2x-1}\right| + 10$$

To satisy the conditions you can write the rule as $y = \dfrac{5}{4}\log_e\dfrac{5}{1-2x} + 10$

10 $\dfrac{dy}{dx} = \dfrac{5}{2-4x}$

$$y = -\frac{5}{4}\log_e|2x-1| + c$$
$$x = 1,\ y = 10$$
$$10 = -\frac{5}{4}\log_e|1| + c$$
$$c = 10$$
$$y = -\frac{5}{4}\log_e|2x-1| + 10$$
$$y = \frac{5}{4}\log_e\left|\frac{1}{2x-1}\right| + 10$$

To satisy the conditions you can write the rule as $y = \dfrac{5}{4}\log_e\dfrac{1}{2x-1} + 10$

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Solutions to Exercise 11D

1 **a** $\frac{1}{6}e^{6x} + c$

b $\frac{1}{2}e^{2x} + \frac{3}{2}x^2 + c$

c $-\frac{1}{3}e^{-3x} + x^2 + c$

d $-\frac{1}{2}e^{-2x} + \frac{1}{2}e^{2x} + c$

2 **a** $\int e^{2x} - e^{\frac{x}{2}}\,dx$

$= \frac{1}{2}e^{2x} - 2e^{\frac{x}{2}} + c$

b $\int e^{x} + e^{-x}\,dx$

$= e^{x} - e^{-x} + c$

c $\int 2e^{3x} - e^{-x}\,dx$

$= \frac{2}{3}e^{3x} + e^{-x} + c$

d $\int 5e^{\frac{x}{3}} - 3e^{\frac{x}{5}}\,dx$

$= 15e^{\frac{x}{3}} - 15e^{\frac{x}{5}} + c$

e $\frac{9}{2}e^{\frac{2x}{3}} - \frac{15}{7}e^{\frac{7x}{5}} + c$

f $\frac{15}{4}e^{\frac{4x}{3}} - \frac{9}{2}e^{\frac{2x}{3}} + c$

3 **a** $\frac{dy}{dx} = e^{2x} - x$

$y = \frac{1}{2}e^{2x} - \frac{x^2}{2} + c$

$x = 0, y = 5$

$5 = \frac{1}{2} - 0 + c$

$c = \frac{9}{2}$

$y = \frac{1}{2}\left(e^{2x} - x^2 + 9\right)$

b $\frac{dy}{dx} = 3e^{-x} - e^{x}$

$y = -3e^{-x} - e^{x} + c$

$x = 0, y = 4$

$4 = -3 - 1 + c$

$c = 8$

$y = -3e^{-x} - e^{x} + 8$

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4 $\dfrac{dy}{dx} = ae^{-x} + 1$

$x = 0, \dfrac{dy}{dx} = 3$

$3 = a + 1$

$a = 2$

$\dfrac{dy}{dx} = 2e^{-x} + 1$

$y = -2e^{-x} + x + c$

$x = 0, y = 5$

$5 = -2 + 0 + c$

$c = 7$

$y = -2e^{-x} + x + 7$

$x = 2,$

$y = -2e^{-2} - 2 + 7$

$y = 9 - \dfrac{2}{e^2}$

5 $\dfrac{dy}{dx} = e^{kx}$

a $x = 1,$

$\dfrac{dy}{dx} = e^k$

Tangent

$y = e^k x + c$

$x = 0, y = 0$

$0 = c$

$y = e^k x$

$x = 1, y = e^2$

$e^2 = e^k$

$k = 2$

$\dfrac{dy}{dx} = e^{2x}$

b $y = \dfrac{1}{2}e^{2x} + c$

$x = 1, y = e^2$

$e^2 = \dfrac{1}{2}e^2 + c$

$c = \dfrac{1}{2}e^2$

$y = \dfrac{1}{2}e^{2x} + \dfrac{1}{2}e^2$

6 $\dfrac{dy}{dx} = -e^{kx}$

a $x = 1,$

$\dfrac{dy}{dx} = -e^k$

Tangent

$y = -e^k x + c$

$x = 0, y = 0$

$0 = c$

$y = -e^k x$

$x = 1, y = -e^3$

$-e^3 = -e^k$

$k = 3$

$\dfrac{dy}{dx} = -e^{3x}$

b $y = -\dfrac{1}{3}e^{3x} + c$

$x = 1, y = -e^3$

$-e^3 = -\dfrac{1}{3}e^3 + c$

$c = -\dfrac{2}{3}e^3$

$y = -\dfrac{1}{3}e^{3x} - \dfrac{2}{3}e^3$

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Solutions to Exercise 11E

1 a $\int_1^2 x^2\,dx$

$= \left[\frac{x^3}{3}\right]_1^2$

$= \frac{2^3}{3} - \frac{1^3}{3}$

$= \frac{7}{3}$

b $\int_{-1}^3 x^3\,dx$

$= \left[\frac{x^4}{4}\right]_{-1}^2$

$= \frac{81}{4} - \frac{1}{4}$

$= 20$

c $\int_0^1 x^3 - x dx$

$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1$

$= \left(\frac{1}{4} - \frac{1}{2}\right) - (0 - 0)$

$= \frac{-1}{4}$

d $\int_{-1}^2 (x+1)^2\,dx$

$= \left[\frac{1}{3}(x+1)^3\right]_{-1}^2$

$= \frac{1}{3}(3)^3 - \frac{1}{3}(0)^3$

$= 9$

e $\int_1^2 \frac{1}{x^2}\,dx$

$= \left[\frac{-1}{x}\right]_1^2$

$= \frac{-1}{2} - \frac{-1}{1}$

$= \frac{1}{2}$

f $\int_1^4 x^{\frac{1}{2}} + 2x^2\,dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{3}x^3\right]_1^4$

$= \frac{2}{3}\left(4^{\frac{3}{2}} + 4^3\right) - \frac{2}{3}\left(1^{\frac{3}{2}} + 1^3\right)$

$= \frac{2}{3}(8 + 64) - \frac{4}{3}$

$= \frac{140}{3}$

g $\int_0^2 x^3 + 2x^2 + x + 2\,dx$

$= \left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + 2x\right]_0^2$

$= \frac{16}{4} + \frac{2}{3} \times 8 + \frac{4}{2} + 2 \times 2 - 0$

$= 4 + \frac{16}{3} + 2 + 4$

$= \frac{46}{3} = 15\frac{1}{3}$

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h $\displaystyle\int_1^4 2x^{\frac{3}{2}} + 5x^3\,dx$

$= \left[\frac{4}{5}x^{\frac{5}{2}} + \frac{5}{4}x^4\right]_1^4$

$= \left(\frac{4}{5}\times 32 + \frac{5}{4}\times 256\right) - \left(\frac{4}{5}+\frac{5}{4}\right)$

$= \frac{128}{5} + 320 - \frac{4}{5} - \frac{5}{4}$

$= \frac{6871}{20} = 343\frac{11}{20}$

2 **a** $\displaystyle\int_0^1 (2x+1)^3\,dx$

$= \left[\frac{1}{8}(2x+1)^4\right]_0^1$

$= \frac{1}{8}(3)^4 - \frac{1}{8}(1)^4$

$= \frac{81}{8} - \frac{1}{8}$

$= 10$

b $\displaystyle\int_0^2 (4x+1)^{\frac{-1}{2}}\,dx$

$= \left[\frac{2}{4}(4x+1)^{\frac{1}{2}}\right]_0^2$

$= \frac{1}{2}(9)^{\frac{1}{2}} - \frac{1}{2}(1)^{\frac{1}{2}}$

$= \frac{3}{2} - \frac{1}{2}$

$= 1$

c $\displaystyle\int_1^2 (1-2x)^2\,dx$

$= \left[\frac{1}{-6}(1-2x)^3\right]_1^2$

$= \frac{-1}{6}(-3)^3 + \frac{1}{6}(-1)^3$

$= \frac{27}{6} - \frac{1}{6}$

$= \frac{13}{3}$

d $\displaystyle\int_0^1 (3-2x)^{-2}\,dx$

$= \left[\frac{1}{2}(3-2x)^{-1}\right]_0^1$

$= \frac{1}{2}(1)^{-1} - \frac{1}{2}(3)^{-1}$

$= \frac{1}{2} - \frac{1}{6}$

$= \frac{1}{3}$

e $\displaystyle\int_0^2 (3+2x)^{-3}\,dx$

$= \left[\frac{1}{-4}(3+2x)^{-2}\right]_0^2$

$= \frac{-1}{4}(7)^{-2} + \frac{1}{4}(3)^{-2}$

$= \frac{1}{36} - \frac{1}{196}$

$= \frac{10}{441}$

f $\displaystyle\int_{-1}^1 (4x+1)^3\,dx$

$= \left[\frac{1}{16}(4x+1)^4\right]_{-1}^1$

$= \frac{1}{16}(5)^4 - \frac{1}{16}(-3)^4$

$= \frac{625}{16} - \frac{81}{16}$

$= \frac{544}{16}$

$= 34$

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g $\displaystyle\int_0^1 (2-x)^{\frac{1}{2}}\,dx$

$= \left[\frac{2}{-3}(2-x)^{\frac{3}{2}}\right]_0^1$

$= \frac{-2}{3}(1)^{\frac{3}{2}} + \frac{2}{3}(2)^{\frac{3}{2}}$

$= \frac{2}{3}(2^{\frac{3}{2}} - 1)$

$\simeq 1.22$

h $\displaystyle\int_3^4 (2x-4)^{\frac{-1}{2}}\,dx$

$= \left[\frac{2}{2}(2x-4)^{\frac{1}{2}}\right]_3^4$

$= \sqrt{4} - \sqrt{2}$

$= 2 - \sqrt{2}$

i $\displaystyle\int_0^1 (3+2x)^{-2}\,dx$

$= \left[\frac{1}{-2}(3+2x)^{-1}\right]_0^1$

$= \frac{-1}{2}(5)^{-1} + \frac{1}{2}(3)^{-1}$

$= \frac{1}{6} - \frac{1}{10}$

$= \frac{1}{15}$

3 a $\displaystyle\int_0^1 e^{2x}\,dx$

$= \left[\frac{1}{2}e^{2x}\right]_0^1$

$= \frac{1}{2}e^2 - \frac{1}{2}$

b $\displaystyle\int_0^1 e^{-2x} + 1\,dx$

$= \left[\frac{-1}{2}e^{-2x} + x\right]_0^1$

$= \left(\frac{-1}{2}e^{-2} + 1\right) - \left(\frac{-1}{2}\right)$

$= \frac{3}{2} - \frac{1}{2e^2}$

c $\displaystyle\int_0^1 2e^{\frac{x}{3}} + 2\,dx$

$= \left[6e^{\frac{x}{3}} + 2x\right]_0^1$

$= (6e^{\frac{1}{3}} + 2) - (6)$

$= 6e^{\frac{1}{3}} - 4$

d $\displaystyle\int_{-2}^2 \frac{e^x + e^{-x}}{2}\,dx$

$= \left[\frac{e^x - e^{-x}}{2}\right]_{-2}^2$

$= \frac{e^2 - e^{-2}}{2} - \frac{e^{-2} - e^2}{2}$

$= e^2 - e^{-2}$

4 a $\displaystyle\int_0^4 h(x)\,dx = 5$

$\displaystyle\int_0^4 2h(x)\,dx$

$\displaystyle= 2\int_0^4 h(x)\,dx = 10$

b $\displaystyle\int_0^4 (h(x) + 3)\,dx$

$\displaystyle= \int_0^4 h(x)\,dx + \int_0^4 3\,dx$

$= 5 + 12$

$= 17$

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c $\int_4^0 h(x)\,dx$

$= -\int_0^4 h(x)\,dx$

$= -5$

d $\int_0^4 (h(x) + 1)\,dx$

$= \int_0^4 h(x)\,dx + \int_0^4 dx$

$= 5 + 4$

$= 9$

e $\int_0^4 (h(x) - x)\,dx$

$= \int_0^4 h(x)\,dx - \int_0^4 x dx$

$= 5 - \left[\frac{x^2}{2}\right]_0^4$

$= 5 - 8$

$= -3$

5 a $\int_0^4 \frac{1}{x-6}\,dx$

$= -\int_0^4 \frac{1}{6-x}\,dx$

$= \left[\log_e(6-x)\right]_0^4$

$= (\log_e(2) - \log_e(6))$

$= \log_e\left(\frac{1}{3}\right)$

b $\int_2^4 \frac{1}{2x-3}\,dx$

$= \left[\frac{1}{2}\log_e(2x-3)\right]_2^4$

$= \frac{1}{2}\log_e 5 - \frac{1}{2}\log_e 1$

$= \frac{1}{2}\log_e 5$

c $\int_5^6 \frac{3}{2x+7}\,dx$

$= \left[\frac{3}{2}\log_e(2x+7)\right]_5^6$

$= \frac{3}{2}\log_e 19 - \frac{3}{2}\log_e 17$

$= \frac{3}{2}\log_e\left(\frac{19}{17}\right)$

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Solutions to Exercise 11F

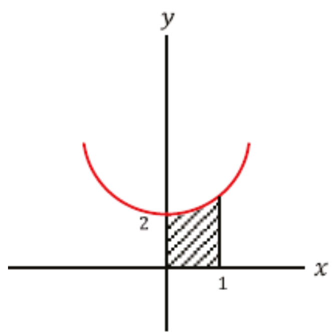
1 **a** $A = \int_0^1 y\,dx$

$= \int_0^1 3x^2 + 2\,dx$

$= \left[x^3 + 2x\right]_0^1$

$= (1 + 2) - (0 + 0)$

$= 3$



b $A = \int_2^4 y\,dx$

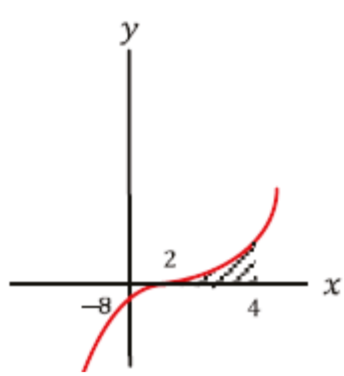
$= \int_2^4 x^3 - 8\,dx$

$= \left[\frac{x^4}{4} - 8x\right]_2^4$

$= (64 - 32) - (4 - 16)$

$= 32 + 12$

$= 44$



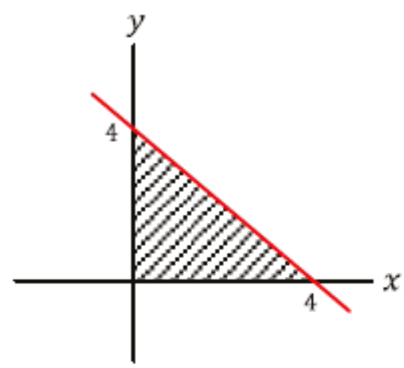
c **i** $A = \int_0^4 y\,dx$

$= \int_0^4 4 - x\,dx$

$= \left[4x - \frac{x^2}{2}\right]_0^4$

$= (16 - 8) - 0$

$= 8$



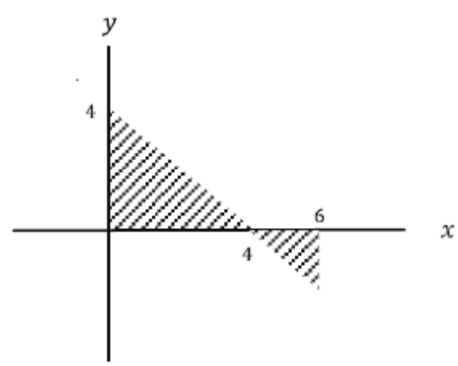
ii $A = \int_0^4 y\,dx - \int_4^6 y\,dx$

$= 8 - \left[4x - \frac{x^2}{2}\right]_4^6$ (*from* (*i*))

$= 8 - ((24 - 18) - (16 - 8))$

$= 8 - (6 - 8)$

$= 10$



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2 a $A = x^2 - 2x$

$= x(x-2)$

$y = 0,\ x = 0, 2$

$$A = -\int_0^2 y\,dx$$

$$= -\int_0^2 x^2 - 2x\,dx$$

$$= -\left[\frac{x^3}{3} - x^2\right]_0^2$$

$$= -\left(\frac{8}{3} - 4\right)$$

$$= \frac{4}{3}$$

b $y = (4-x)(3-x)$

$y = 0,\ x = 3, 4$

$$A = -\int_3^4 y\,dx$$

$$A = -\int_3^4 (x-4)(x-3)\,dx$$

$$A = -\int_3^4 -x^2 - 7x + 12\,dx$$

$$= -\left[\frac{x^3}{3} - \frac{7x^2}{2} + 12x\right]_3^4$$

$$= \left(\left(\frac{64}{3} - \frac{7\times 16}{2} + 48\right) - \left(\frac{27}{3} - \frac{7\times 9}{2} + 36\right)\right)$$

$$= -\left(\left(\frac{40}{3}\right) - \left(\frac{27}{2}\right)\right)$$

$$= \frac{1}{6}$$

c $y = (x+2)(x+7)$

$y = 0,\ x = -2, 7$

$$A = \int_{-2}^{7} y\,dx$$

$$= \int_{-2}^{7} -x^2 + 5x + 14\,dx$$

$$= \left[\frac{-x^3}{3} + \frac{5x^2}{2} + 14x\right]_{-2}^{7}$$

$$= \left(\frac{-343}{3} + \frac{5\times 49}{2} + 98\right) - \left(\frac{8}{3} + \frac{20}{2} - 28\right)$$

$$= \left(\frac{637}{6}\right) + \left(\frac{46}{3}\right)$$

$$= 121.5$$

d $y = x^2 - 5x + 6$

$= (x-2)(x-3)$

$y = 0,\ x = 2, 3$

$$A = -\int_2^3 y\,dx$$

$$= -\int_2^3 x^2 - 5x + 6\,dx$$

$$= -\left[\frac{x^3}{3} - \frac{5x^2}{2} + 6x\right]_2^3$$

$$= -\left(\left(\frac{27}{3} - \frac{45}{2} + 18\right) - \left(\frac{8}{3} - \frac{20}{2} + 12\right)\right)$$

$$= \left(\frac{14}{3}\right) - \left(\frac{9}{2}\right)$$

$$= \frac{1}{6}$$

e $y = 3 - x^2$

$= (\sqrt{3} + x)(\sqrt{3} + x)$

$y = 0, x = \pm\sqrt{3}$

$$A = \int_{-\sqrt{3}}^{\sqrt{3}} y\,dx$$

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$$= \int_{-\sqrt{3}}^{\sqrt{3}} 3 - x^2 \, dx$$

$$= \left[3x - \frac{x^3}{3}\right]_{-\sqrt{3}}^{\sqrt{3}}$$

$$= \left(3\sqrt{3} - \frac{3\sqrt{3}}{3}\right) - \left(-3\sqrt{3} + \frac{3\sqrt{3}}{3}\right)$$

$$= 2\sqrt{3} + 2\sqrt{3}$$

$$= 4\sqrt{3}$$

f $y = x^3 - 6x^2$

$$= x^2(x - 6)$$

$y = 0, \; x = 0, 6$

$$A = -\int_0^6 y dx$$

$$= -\int_0^6 x^3 - 6x^2 \, dx$$

$$= \int_0^6 6x^2 - x^3 \, dx$$

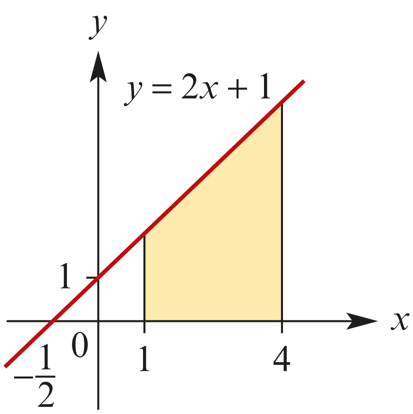
$$= \left[2x^3 - \frac{x^4}{4}\right]_0^6$$

$$= 2 \times (216) - \frac{1296}{4}$$

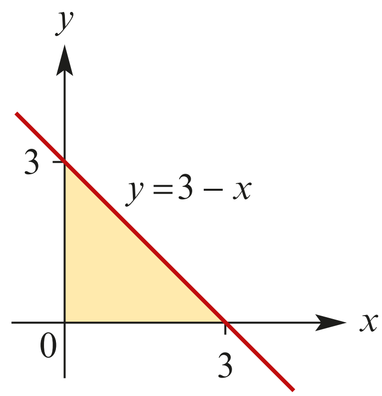
$$= 432 - 324$$

$$= 108$$

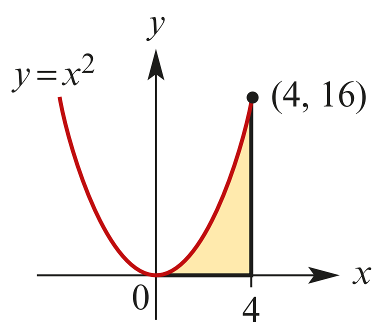
3 a



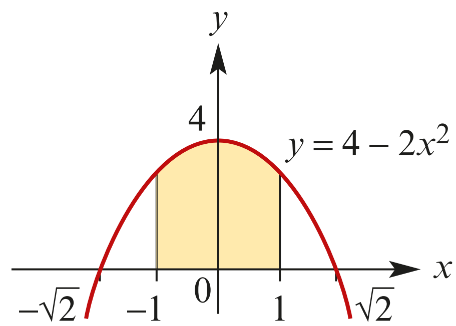
b



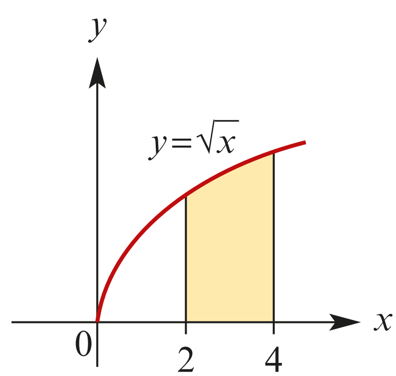
c



d



e



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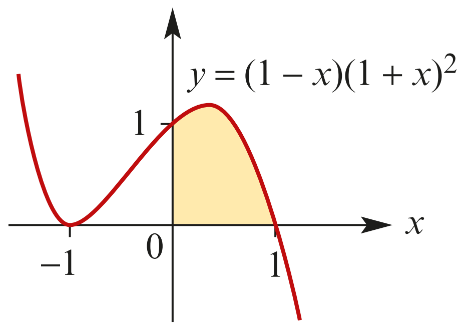
f



4 $y = 3x + 2x^{-2}$

$$y = 0,$$

$$3x = \frac{-2}{x^2}$$

$$3x^3 = -2$$

$$x^3 = \frac{-2}{3}$$

$$x = \left(\frac{-2}{3}\right)^{\frac{1}{3}}$$

which is not in the region under consideration

$$\therefore A = \int_2^5 3x + 2x^{-2}\,dx$$

$$= \left[\frac{3}{2}x^2 - 2x^{-1}\right]_2^5$$

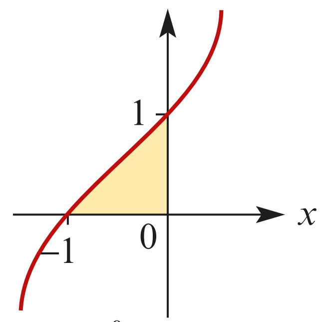
$$= \left(\frac{3}{2}\times 25 - \frac{2}{5}\right) - \left(\frac{3}{2}\times 4 - \frac{2}{2}\right)$$

$$= \frac{75}{2} - \frac{2}{5} - 6 + 1$$

$$= \frac{375 - 4 - 50}{10}$$

$$= \frac{321}{10} \text{ square units}$$

5



$$A = \int_{-1}^{0} f(x)\,dx \quad (\textit{from graph})$$

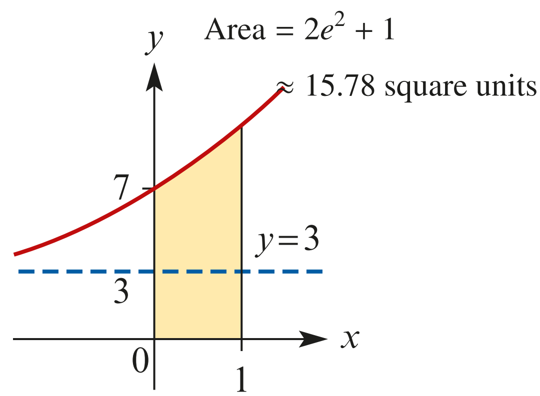
$$= \int_{-1}^{0} 1 + x^3\,dx$$

$$= \left[x + \frac{x^4}{4}\right]_{-1}^{0}$$

$$= 0 - \left(-1 + \frac{1}{4}\right)$$

$$= \frac{3}{4} \text{ square units}$$

6



$$A = \int_0^1 f(x)\,dx \quad (\textit{from graph})$$

$$= \int_0^1 4e^{2x} + 3\,dx$$

$$= \left[2e^{2x} + 3x\right]_0^1$$

$$= (2e^2 + 3) - (2 + 0)$$

$$= 2e^2 + 1 \approx 15.78 \text{ square units}$$

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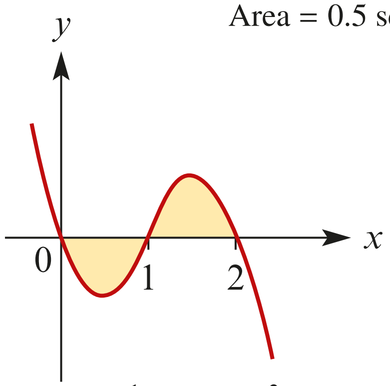
7 Area = 0.5 square units



$$A = -\int_0^1 y\,dx + \int_1^2 y\,dx \quad (\textit{from graph})$$

$$= -\int_0^1 x(2-x)(x-1)\,dx$$

$$+ \int_1^2 x(2-x)(x-1)\,dx$$

$$= -\int_0^1 -x^3 + 3x^2 - 2x\,dx$$

$$+ \int_1^2 -x^3 + 3x^2 - 2x\,dx$$

$$= \left[\frac{-x^4}{4} + \frac{3x^3}{3} - \frac{2x^2}{2}\right]_0^1$$

$$+ \left[\frac{-x^4}{4} + \frac{3x^3}{3} - \frac{2x^2}{2}\right]_1^2$$

$$= \left[\frac{-x^4}{4} + x^3 - x^2\right]_0^1 + \left[\frac{-x^4}{4} + x^3 - x^2\right]_1^2$$

$$= 0 - \left(\frac{-1}{4} + 1 - 1\right) + \left(\frac{-16}{4} + 8 - 4\right)$$

$$- \left(\frac{-1}{4} + 1 - 1\right)$$

$$= \frac{1}{2} \text{ square units}$$

8 a $$\int_{-1}^{4} x(3-x)\,dx$$

$$= \int_{-1}^{4} 3x - x^2\,dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_{-1}^{4}$$

$$= \left(\frac{3*16}{2} - \frac{64}{3}\right) - \left(\frac{3}{2} + \frac{1}{3}\right)$$

$$= 24 - \frac{64}{3} - \frac{3}{2} - \frac{1}{3}$$

$$= \frac{45}{2} - \frac{65}{3}$$

$$= \frac{5}{6} \text{ square units}$$

b assuming the graph shown is $y = x(3-x)$,

$$A = -\int_{-1}^{0} x(3-x)\,dx + \int_0^3 x(3-x)\,dx$$

$$- \int_3^4 x(3-x)\,dx$$

$$= \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3 - \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_{-1}^0 - \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_3^4$$

from (a)

$$= \left(\frac{3\times 9}{2} - \frac{27}{3}\right) - (0) - 0 + \left(\frac{3}{2} + \frac{1}{3}\right)$$

$$- \left(\frac{3}{2}\times 16 - \frac{64}{3}\right) + \left(\frac{3\times 9}{2} - \frac{27}{3}\right)$$

$$= \frac{49}{6}$$

9 a $y^2 = 9(1-x)$

A: $x = 0$,

$y^2 = 9(1-0)$

$= 9$

$y = \pm 3$

but A is above the x-axis

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$$so\ A = (0, 3)$$

$$B{:}y = 0$$

$$0 = 9(1 - x)$$

$$1 - x = 0$$

$$x = 1$$

$$B = (1, 0)$$

b $A = \int_0^3 \left(1 - \frac{y^2}{9}\right) dy$

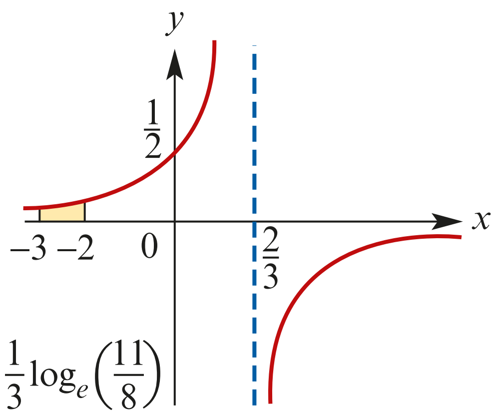
$(0\ to\ A,\ where\ 0\ is\ the\ origin)$

$$= \left[y - \frac{y^3}{27}\right]_0^3$$

$$= \left(3 - \frac{27}{27}\right) - 0$$

$= 2$ square units

10



$A = -\int_{-3}^{-2} y\,dx \quad (from\ graph)$

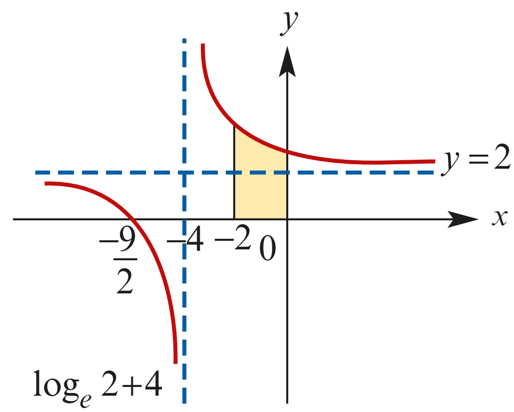
$$= -\int_{-3}^{-2} \frac{1}{2 - 3x}\,dx$$

$$= \left[\frac{-1}{3}\log_e |2 - 3x|\right]_{-3}^{-2}$$

$$= \frac{-1}{3}\log_e |8| + \frac{1}{3}\log_e |11|$$

$$= \frac{1}{3}\log_e\left(\frac{11}{8}\right)$$

11



$A = -\int_{-2}^{0} y\,dx \quad (from\ graph)$

$$= \int_{-2}^{0} 2 + \frac{1}{x + 4}\,dx$$

$$= \left[2x + \log_e |x + 4|\right]_{-2}^{0}$$

$$= (0 + \log_e 4) - (-4 + \log_e 2)$$

$$= 2\log_e 2 + 4 - \log_e 2$$

$$= \log_e 2 + 4$$

12 a $\text{RHS} = e^{x(\ln a)}$

$$= e^{(\ln a^x)}$$

$$= a^x$$

$$= LHS \quad QED$$

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b $\dfrac{d}{dx}(a^x)$

$= \dfrac{d}{dx}(e^{x(\ln a)})$

$= \ln a e^{x \ln a}$

$= a^x \ln a$

$\displaystyle\int a^x \, dx$

$= \displaystyle\int e^{x \ln a} \, dx$

$= \dfrac{1}{\ln a} e^{x \ln a} + c$

$= \dfrac{a^x}{\ln a} + c$

c $\displaystyle\int_0^b a^x \, dx$

$= \left[\dfrac{a^x}{\ln a}\right]_0^b \quad \textit{from} \quad (b)$

$= \dfrac{a^b}{\ln a} - \dfrac{1}{\ln a}$

$= \dfrac{1}{\ln a}(a^b - 1) \quad QED$

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Solutions to Exercise 11G

1 a $\int \cos 3x\, dx$

$= \frac{1}{3} \sin 3x$

b $\int \sin \frac{1}{2}x\, dx$

$= -2\cos \frac{1}{2}x$

c $\int 3\cos 3x\ x\, dx$

$= \sin 3x$

d $\int 2\sin \frac{1}{2}x\, dx$

$= -4\cos \frac{1}{2}x$

e $\int \sin\left(2x - \frac{\pi}{3}\right) dx$

$= \frac{-1}{2}\cos\left(2x - \frac{\pi}{3}\right)$

f $\int \cos 3x + \sin 2x\, dx$

$= \frac{1}{3}\sin 3x - \frac{1}{2}\cos 2x$

g $\int \cos 4x - \sin 4x\, dx$

$= \frac{1}{4}\sin 4x + \frac{1}{4}\cos 4x$

h $\int \frac{-1}{2}\sin 2x + \cos 3x\, dx$

$= \frac{1}{4}\cos 2x + \frac{1}{3}\sin 3x$

i $\int \frac{-1}{2}\cos\left(2x + \frac{\pi}{3}\right) dx$

$= \frac{-1}{4}\sin\left(2x + \frac{\pi}{3}\right)$

j $\int \sin \pi x\, dx$

$= \frac{-1}{\pi}\cos \pi x$

2 a $\int_0^{\frac{\pi}{4}} \sin x\, dx$

$= [-\cos x]_0^{\frac{\pi}{4}}$

$= \frac{-1}{\sqrt{2}} + 1$

$= 1 - \frac{1}{\sqrt{2}}$

b $\int_0^{\frac{\pi}{4}} \cos 2x\, dx$

$= \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}(1) - 0$

$= \frac{1}{2}$

c $\int_{\frac{-\pi}{2}}^{\frac{\pi}{4}} \cos\theta\, d\theta$

$= [\sin\theta]_{\frac{-\pi}{2}}^{\frac{\pi}{4}}$

$= \frac{1}{\sqrt{2}} - -1$

$= 1 + \frac{1}{\sqrt{2}}$

d $\int_0^{\frac{\pi}{2}} \sin\theta + \cos\theta\, d\theta$

$= [-\cos\theta + \sin\theta]_0^{\frac{\pi}{2}}$

$= (0 + 1) - (-1 + 0)$

$= 2$

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e $\int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta$

$= \left[-\frac{1}{2}\cos 2\theta\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}(-1) + \frac{1}{2}(1)$

$= 1$

f $\int_0^{\frac{\pi}{3}} \cos 3\theta + \sin 3\theta \, d\theta$

$= \left[\frac{1}{3}\sin 3\theta - \frac{1}{3}\cos 3\theta\right]_0^{\frac{\pi}{3}}$

$= \left(0 - \frac{1}{3}(-1)\right) - \left(0 - \frac{1}{3}(1)\right)$

$= \frac{2}{3}$

g $\int_0^{\frac{\pi}{3}} \cos 3\theta + \sin\left(\theta - \frac{\pi}{3}\right) d\theta$

$= \left[\frac{1}{3}\sin 3\theta - \cos\left(\theta - \frac{\pi}{3}\right)\right]_0^{\frac{\pi}{3}}$

$= (0 - 1) - \left(0 - \frac{1}{2}\right)$

$= \frac{-1}{2}$

h $\int_0^{\pi} \sin\frac{x}{4} + \cos\frac{x}{4} \, dx$

$= \left[-4cos\frac{x}{4} + 4\sin\frac{x}{4}\right]_0^{\pi}$

$= \left(-4\left(\frac{1}{\sqrt{2}}\right) + 4\left(\frac{1}{\sqrt{2}}\right)\right) - (-4(1) + 0)$

$= 4$

i $\int_0^{\frac{\pi}{4}} \sin\left(2x - \frac{\pi}{3}\right) dx$

$= \left[\frac{-1}{2}\cos\left(2x - \frac{\pi}{3}\right)\right]_0^{\frac{\pi}{4}}$

$= \frac{-1}{2}\cos\left(\frac{\pi}{6}\right) + \frac{1}{2}\cos\left(\frac{-\pi}{3}\right)$

$= \frac{-\sqrt{3}}{4} + \frac{1}{4}$

$= \frac{1 - \sqrt{3}}{4}$

j

$\int_0^{\pi} \cos 2x - \sin\frac{x}{2} \, dx$

$= \left[\frac{1}{2}\sin\ 2x + 2\cos\frac{x}{2}\right]_0^{\pi}$

$= \left(\frac{1}{2}\sin\ 2\pi + 2\cos\frac{\pi}{2}\right) - \left(\frac{1}{2}\sin 0 + 2\cos 0\right)$

$= -2$

3 $A = \int_0^{\frac{\pi}{2}} y \, dx$

$= \int_0^{\frac{\pi}{2}} \sin\frac{1}{2}x \, dx$

$= \left[-2\cos\frac{1}{2}x\right]_0^{\frac{\pi}{2}}$

$= -2\cos\frac{\pi}{4} + 2\cos 0$

$= 2 - \sqrt{2}$

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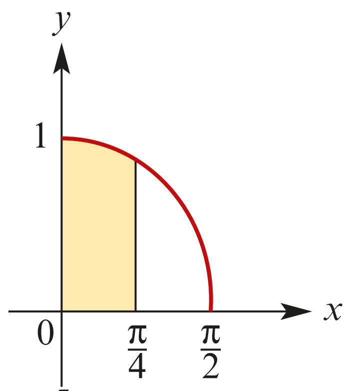
4 a



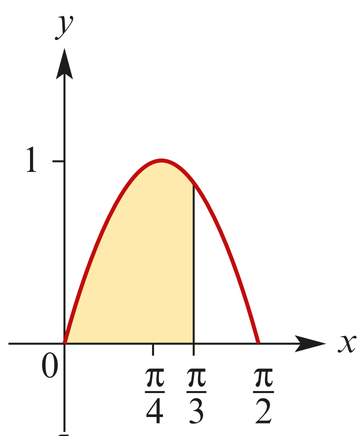
$$\int_0^{\frac{\pi}{4}} \cos x\,dx$$

$$= [\sin x]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}} - 0$$

$$= \frac{1}{\sqrt{2}}$$

b



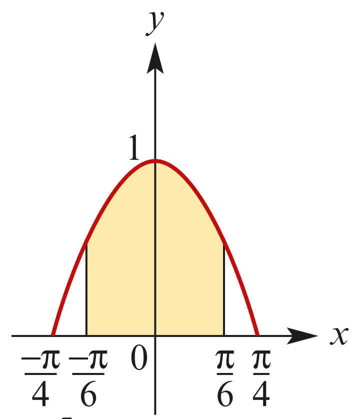
$$\int_0^{\frac{\pi}{3}} \sin 2x\,dx$$

$$= \left[\frac{-1}{2}\cos 2x\right]_0^{\frac{\pi}{3}}$$

$$= \frac{-1}{2}\left(\frac{-1}{2}\right) + \frac{-1}{2}(1)$$

$$= \frac{3}{4}$$

c



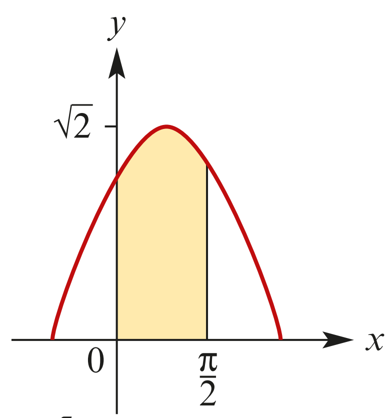
$$\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \cos 2x\,dx$$

$$= \left[\frac{1}{2}\sin 2x\right]_{\frac{-\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(\frac{-\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{2}$$

d



$$\int_0^{\frac{\pi}{2}} \cos\theta + \sin\ \theta\,d\theta$$

$$= [\sin\theta - \cos\theta]_0^{\frac{\pi}{2}}$$

$$= (1 - 0) - (0 - 1)$$

$$= 2$$

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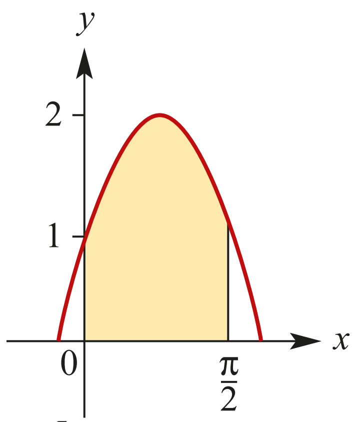
e



$$\int_0^{\frac{\pi}{2}} \sin 2\theta + 1 \, d\theta$$

$$= \left[\frac{-1}{2}\cos 2\theta + \theta\right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{-1}{2}(-1) + \frac{\pi}{2}\right) - \left(\frac{-1}{2}(-1) + 0\right)$$

$$= 1 + \frac{\pi}{2}$$

f $$\int_{\frac{-x}{4}}^{\frac{x}{4}} \cos 2\theta \, d\theta$$

$$= \left[\frac{1}{2}\sin 2\theta\right]_{\frac{-\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{2}(1) - \frac{1}{2}(-1)$$

$$= 1$$

5 **a** $$\int_0^{\frac{\pi}{2}} \sin\left(2x + \frac{\pi}{4}\right) dx$$

$$= \left[\frac{-1}{2}\cos\left(2x + \frac{\pi}{4}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{-1}{2}\cos\left(\frac{5\pi}{4}\right) + \frac{1}{2}\cos\left(\frac{\pi}{4}\right)$$

$$= \frac{-1}{2}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}$$

b $$\int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{6}\right) dx$$

$$= \left[\frac{1}{3}\sin\left(3x + \frac{\pi}{6}\right)\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{3}\sin\left(\frac{7\pi}{6}\right) - \frac{1}{3}\sin\left(\frac{\pi}{6}\right)$$

$$= \frac{1}{3}\left(\frac{-1}{2}\right) - \frac{1}{3}\left(\frac{1}{2}\right)$$

$$= \frac{-1}{3}$$

c $$\int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{3}\right) dx$$

$$= \left[\frac{1}{3}\sin\left(3x + \frac{\pi}{3}\right)\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{3}\sin\left(\frac{4\pi}{3}\right) - \frac{1}{3}\sin\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{3}\left(\frac{-\sqrt{3}}{2}\right) - \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{-1}{\sqrt{3}}$$

d $$\int_0^{\frac{\pi}{4}} \cos(3\pi - x) \, dx$$

$$= \left[-\sin(3\pi - x)\right]_0^{\frac{\pi}{4}}$$

$$= [-\sin(x - 3\pi)]_0^{\frac{\pi}{4}}$$

$$= \sin\left(-2\pi - \frac{3\pi}{4}\right) - \sin(-2\pi - \pi)$$

$$= \sin\left(-\frac{3\pi}{4}\right) - \sin(-\pi)$$

$$= \frac{-1}{\sqrt{2}} - 0$$

$$= \frac{-1}{\sqrt{2}}$$

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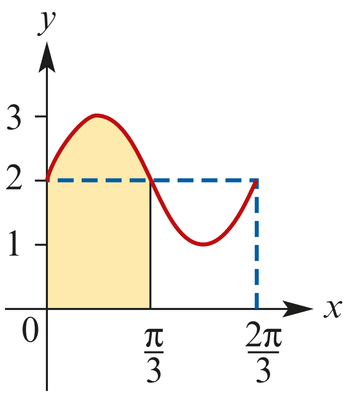
6



$$
\begin{aligned}
A &= \int_0^{\frac{\pi}{3}} 2 + \sin 3x \, dx \\
&= \left[2x - \frac{1}{3}\cos 3x\right]_0^{\frac{\pi}{3}} \\
&= \left(\frac{2x}{3} - \frac{1}{3}(-1)\right) - \left(0 - \frac{1}{3}(1)\right) \\
&= \frac{2}{3}(\pi + 1)
\end{aligned}
$$

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Solutions to Exercise 11H

1 **a** $\displaystyle\int_1^4 \sqrt{x}\,dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4$

$= \frac{16}{3} - \frac{2}{3}$

$= \frac{14}{3}$

b $\displaystyle\int_{-1}^1 (1+x)^2\,dx$

$= \left[\frac{1}{3}(1+x)^3\right]_{-1}^1$

$= \frac{1}{3}(2)^3 - \frac{1}{3}(0)^3$

$= \frac{8}{3}$

c $\displaystyle\int_0^8 x^{\frac{1}{3}}\,dx$

$= \left[\frac{3}{4}x^{\frac{4}{3}}\right]_0^8$

$= \frac{3}{4}\times 16 - 0$

$= 12$

d

$\displaystyle\int_0^{\frac{\pi}{3}} \cos 2x - \sin\frac{1}{2}x\,dx$

$= \left[\frac{1}{2}\sin 2x + 2\cos\frac{1}{2}x\right]_0^{\frac{\pi}{3}}$

$= \frac{1}{2}\sin\frac{2\pi}{3} + 2\cos\frac{\pi}{6} - \frac{1}{2}\sin 0 + 2\cos 0$

$= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right) - 0 - 2(1)$

$= \frac{5\sqrt{3}}{4} - 2$

e $\displaystyle\int_1^2 e^{2x} + \frac{4}{x}\,dx$

$= \left[\frac{1}{2}e^{2x} + 4\log_e|x|\right]_1^2$

$= \frac{1}{2}e^4 + 4\log_e 2 - \frac{1}{2}e^2 - 4\log_e 1$

$= \frac{1}{2}e^4 - \frac{1}{2}e^2 + 4\log_e 2$

f

$\displaystyle\int_0^{\frac{\pi}{2}} \sin 2x + \cos 3x\,dx$

$= \left[\frac{-1}{2}\cos 2x + \frac{1}{3}\sin 3x\right]_0^{\frac{\pi}{2}}$

$= \left(\frac{-1}{2}(-1) + \frac{1}{3}(-1)\right) - \left(\frac{-1}{2}(1) + \frac{1}{3}(0)\right)$

$= \frac{1}{2} - \frac{1}{3} + \frac{1}{2}$

$= \frac{2}{3}$

g

$\displaystyle\int_0^{\pi} \sin\frac{x}{4} + \cos\frac{x}{4}\,dx$

$= \left[-4\cos\frac{x}{4} + 4\sin\frac{x}{4}\right]_0^{\pi}$

$= \left(-4\left(\frac{1}{\sqrt{2}}\right) + 4\left(\frac{1}{\sqrt{2}}\right)\right) - (-4(1) + 4(0))$

$= 4$

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h $\displaystyle\int_0^{\frac{\pi}{2}} 5x + \sin 2x\,dx$

$$= \left[\frac{5x^2}{2} - \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{5\left(\frac{\pi}{2}\right)^2}{2} - \frac{1}{2}(-1)\right) - \left(0 - \frac{1}{2}(1)\right)$$

$$= \frac{5\pi^2}{8} + \frac{1}{2} + \frac{1}{2}$$

$$= 1 + \frac{5\pi^2}{8}$$

i

$$\int_1^4 \left(2 + \frac{1}{x}\right)^2 dx$$

$$= \int_1^4 4 + \frac{4}{x} + \frac{1}{x^2}\,dx$$

$$= \left[4x + 4\log_e|x| - \frac{1}{x}\right]_1^4$$

$$= \left(16 + 4\log_e 4 - \frac{1}{4}\right) - (4 + 4\log_e 1 - 1)$$

$$= 16 + 8\log_e 2 - \frac{1}{4} - 4 + 1$$

$$= 12\frac{3}{4} + 8\log_e 2$$

j $\displaystyle\int_0^1 x^2 - x^3\,dx$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4}$$

$$= \frac{1}{12}$$

2 $\displaystyle A = \int_0^{\frac{\pi}{3}} \sin x\,dx$

$$= \left[-\cos x\right]_0^{\frac{\pi}{3}}$$

$$= -\cos\frac{\pi}{3} + \cos 0$$

$$= \frac{-1}{2} + 1$$

$= \frac{1}{2}$ square units

3 a $\displaystyle\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$

$$= \frac{\cos x\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

Therefore, $\displaystyle\int \frac{1}{\cos^2 x}\,dx$

$$= \frac{\sin x}{\cos x} + c$$

$$= \tan x + c$$

b

$$\frac{d}{dx}\left(\frac{\cos 2x}{\sin 2x}\right)$$

$$= \frac{2\sin 2x\frac{d}{dx}(\cos 2x) - 2\cos 2x\frac{d}{dx}(\sin 2x)}{\sin^2(2x)}$$

$$= \frac{2(-\sin^2 2x - \cos^2 2x)}{\sin^2 2x}$$

$$= -\frac{2}{\sin^2 2x}$$

Therefore, $\displaystyle\int \frac{1}{\sin^2 2x}\,dx$

$$= -\frac{\cos 2x}{2\sin 2x} + c$$

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c $\dfrac{d}{dx}(\log_e(3x^2+7))$

$= 6x \times \dfrac{1}{3x^2+7}$

$= \dfrac{6x}{3x^2+7}$

$\displaystyle\int \frac{x}{3x^2+7}\,dx$

$= \dfrac{1}{6}\displaystyle\int \frac{6x}{3x^2+7}\,dx$

$= \dfrac{1}{6}\log_e [3x^2+7]_0^2$

$= \dfrac{1}{6}\log_e\left(\dfrac{19}{7}\right)$

d $\dfrac{d}{dx}(x\sin x))$

$= x\cos x + \sin x$

Therefore,

$\int x\cos x + \sin x\,dx = x\sin x + c$

$\int_0^{\frac{\pi}{4}} x\cos x + \sin x\,dx = \Big[x\sin x\Big]_0^{\frac{\pi}{4}}$

$\int_0^{\frac{\pi}{4}} x\cos x\,dx = \Big[\cos x + x\sin x\Big]_0^{\frac{\pi}{4}}$

$= \dfrac{\pi\sqrt{2}}{8} + \dfrac{\sqrt{2}}{2} - 1$

c

$\dfrac{d}{dx}\left(x + \sqrt{1+x^2}\right)$

$= 1 + \dfrac{2x}{2\sqrt{1+x^2}}$

$= 1 + \dfrac{x}{\sqrt{1+x^2}}$

$\dfrac{d}{dx}\left(\log_e(x + \sqrt{1+x^2})\right)$

$= \left(1 + \dfrac{x}{\sqrt{1+x^2}}\right) \times \dfrac{1}{x+\sqrt{1+x^2}}$

$= \left(\dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right) \times \dfrac{1}{x+\sqrt{1+x^2}}$

$= \dfrac{1}{\sqrt{1+x^2}}$

$\displaystyle\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$

$= \Big[\log_e(x+\sqrt{1+x^2})\Big]_0^1$

$= \log_e(1+\sqrt{1+1}) - \log_e(0+\sqrt{1+0})$

$= \log_e(1+\sqrt{2}) - \log_e 1$

$= \log_e(1+\sqrt{2})$

4 **a** $1 + \log_e(2x),\ -x + \log_e(2x)$

b $x + 2x\log_e(2x),\ \dfrac{1}{2}x^2\log_e(2x) - \dfrac{x^2}{4}$

5 $\dfrac{d}{dx}\left(e^{\sqrt{x}}\right)$

$= \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

Therefore,

$\int_1^2 \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx = \Big[e^{\sqrt{x}}\Big]_1^2 = 2e^{\sqrt{2}} - 2e$

6 $6\sin^2(2x)\cos(2x),\ \dfrac{1}{6}$

7 *using the CAS calculator's 'integral' command:*

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a 139.68

b 18.50

c −0.66

d −23.76

e 2.06

f 0.43

8 a $LHS = \dfrac{2x+3}{x-1}$

$= \dfrac{2x-2+2+3}{x-1}$

$= \dfrac{2(x-1)}{x-1} + \dfrac{5}{x-1}$

$= 2 + \dfrac{5}{x-1}$

$= RHS\ QED$

b $\displaystyle\int_2^4 \frac{2x+3}{x-1}\,dx$

$= \displaystyle\int_2^4 2 + \frac{5}{x-1}\,dx$

$= \Big[2x + 5\log_e|x-1|\Big]_2^4$

$= (8 + 5\log_e 3) - (4 + 5\log_e 1)$

$= 4 + 5\log_e 3$

9 a $LHS = \dfrac{5x-4}{x-2}$

$= \dfrac{5x-4-6+6}{x-2}$

$= \dfrac{5(x-2)}{x-2} + \dfrac{6}{x-2}$

$= 5 + \dfrac{6}{x-2}$

$= RHS\ QED$

b $\displaystyle\int_3^4 \frac{5x-4}{x-2}\,dx$

$= \displaystyle\int_3^4 5 + \frac{6}{x-2}\,dx$

$= \Big[5x + 6\log_e|x-2|\Big]_3^4$

$= (20 + 6\log_e 2) - (15 + 6\log_e 1)$

$= 5 + 6\log_e 2$

10 a $y = \left(1 - \frac{1}{2}x\right)^8$

$\dfrac{dy}{dx} = \dfrac{-1}{2} \times 8\left(1 - \frac{1}{2}x\right)^7$

$= -4\left(1 - \frac{1}{2}x\right)^7$

$\displaystyle\int \left(1 - \frac{1}{2}x\right)^7 dx$

$= \dfrac{-1}{4}\displaystyle\int -4\left(1 - \frac{1}{2}x\right)^7 dx$

$= \dfrac{-1}{4}\left(1 - \frac{1}{2}x\right)^8 + c$

b $y = \log_e|\cos x|$

$\dfrac{dy}{dx} = -\sin x \times \dfrac{1}{\cos x}$

$= -\tan x$

$\displaystyle\int_0^{\frac{\pi}{3}} \tan x\,dx$

$= \displaystyle\int_0^{\frac{\pi}{3}} -\tan x\,dx$

$= -\Big[\log_e|\cos x|\Big]_0^{\frac{\pi}{3}}$

$= -\log_e|\cos\frac{\pi}{3}| + \log_e|\cos 0|$

$= -\log_e\frac{1}{2} + \log_e 1$

$= \log_e 2$

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11 $f'(x) = \sin\left(\frac{1}{2}x\right)$

$f(x) = -2\cos\left(\frac{1}{2}x\right) + c$

$f\left(\frac{4\pi}{3}\right) = 2,$

$2 = -2\cos\left(\frac{2\pi}{3}\right) + c$

$2 = -2\left(\frac{-1}{2}\right) + c$

$2 = 1 + c$

$c = 1$

$f(x) = -2\cos\left(\frac{1}{2}x\right) + 1$

12 **a** $f'(x) = \cos 2x$

$f(x) = \frac{1}{2}\sin 2x + c$

$f(\pi) = 1,$

$1 = \frac{1}{2}\sin 2\pi + c$

$c = 1$

$f(x) = \frac{1}{2}\sin 2x + 1$

b $f'(x) = \frac{3}{x}$

$f(x) = 3\log_e |x| + c$

$f(1) = 6,$

$6 = 3\log_e 1 + c$

$c = 6$

$f(x) = 3\log_e |x| + 6$

c $f'(x) = e^{\frac{x}{2}}$

$f(x) = 2e^{\frac{x}{2}} + c$

$f(0) = 1,$

$1 = 2 + c$

$c = -1$

$f(x) = 2e^{\frac{x}{2}} - 1$

13 $\frac{d}{dx}(x\sin 3x) = \sin 3x + 3x\cos 3x$

$\int_0^{\frac{\pi}{6}} x\cos 3x\,dx$

$= \frac{1}{3}\int_0^{\frac{\pi}{6}} 3x\cos 3x + \sin 3x - \sin 3x\,dx$

$= \frac{1}{3}\int_0^{\frac{\pi}{6}} 3x\cos 3x + \sin 3x - \frac{1}{3}\int_0^{\frac{\pi}{6}} \sin 3x\,dx$

$= \frac{1}{3}\Big[x\sin 3x\Big]_0^{\frac{\pi}{6}} - \frac{1}{3}\left[\frac{-1}{3}\cos 3x\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{3}\left(\frac{\pi}{6}\sin\frac{\pi}{2} - 0\right) - \frac{1}{3}\left(\frac{-1}{3}\cos\frac{\pi}{2} + \frac{1}{3}\cos 0\right)$

$= \frac{\pi}{18} - \frac{1}{9}$

14 $y = a + b\sin\left(\frac{\pi x}{2}\right)$

$(0, 1)$

$\Rightarrow 1 = a + b\sin 0$

$1 = a$

$(3, 3)$

$\Rightarrow 3 = 1 + b\sin\left(\frac{3\pi}{2}\right)$

$3 = 1 - b$

$b = -2$

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$y = 1 - 2\sin\left(\frac{\pi x}{2}\right)$

$x = 0, y = 1$

$x = 1, y = 1 - 2\sin\frac{\pi}{2}$

$\quad = -1$

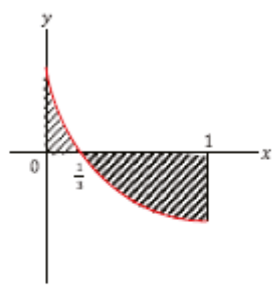
$\therefore$ *there is an x − intercept*

$$0 = 1 - 2\sin\left(\frac{\pi}{2}x\right)$$

$$\sin\frac{\pi}{2}x = \frac{1}{2}$$

$$\frac{\pi}{2}x = \frac{\pi}{6}$$

$$x = \frac{1}{3}$$



$$\therefore A = \int_0^{\frac{1}{3}} y dx - \int_{\frac{1}{3}}^{1} y\,dx$$

$$= \int_0^{\frac{1}{3}} 1 - 2\sin\frac{\pi x}{2}\,dx - \int_{\frac{1}{3}}^{1} 1 - 2\sin\frac{\pi x}{2}\,dx$$

$$= \left[x + \frac{4}{\pi}\cos\frac{\pi x}{2}\right]_0^{\frac{1}{3}} - \left[x + \frac{4}{\pi}\cos\frac{\pi x}{2}\right]_{\frac{1}{3}}^{1}$$

$$= \left(\frac{1}{3} + \frac{4}{\pi}\cos\frac{\pi}{6}\right) - \left(0 + \frac{4}{\pi}\cos 0\right) - \left(1 + \frac{4}{\pi}\cos\frac{\pi}{2}\right) + \left(\frac{1}{3} + \frac{4}{\pi}\cos\frac{\pi}{6}\right)$$

$$= \frac{-1}{3} + \frac{4\sqrt{3}}{\pi} - \frac{4}{\pi} \approx 0.5987 \text{ square units}$$

15 *using the CAS calculator's 'integral' command:*

a 1.450 square units

b 1.716 square units

16 *using the CAS calculator's 'integral' command:*
0.1345

17 $f'(x) = x + \sin 2x$

$$f(x) = \frac{x^2}{2} - \frac{1}{2}\cos 2x + c$$

$$f(0) = 1,$$

$$1 = 0 - \frac{1}{2}\cos 0 + c$$

$$c = \frac{3}{2}$$

$$f(x) = \frac{x^2 - \cos 2x + 3}{2}$$

18 **a** $\int f(x)\,dx$

$$= \int g'(x)\,dx$$

$$= g(x) + c$$

$$= (x^2 + 1)^3 + c$$

b $\int h(x)\,dx$

$$= \int k'(x)\,dx$$

$$= k(x) + c$$

$$= \sin x^2 + c$$

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c $\int f(x) + h(x)\,dx$

$= \int g'(x) + k'(x)\,dx$

$= g(x) + k(x) + c$

$= (x^2 + 1)^3 + \sin x^2 + c$

d $\int -f(x)\,dx$

$= -\int g'(x)\,dx$

$= -g(x) + c$

$= -(x^2 + 1)^3 + c$

e $\int f(x) - 4\,dx$

$= \int g'(x)\,dx - \int 4\,dx$

$= g(x) - 4x + c$

$= (x^2 + 1)^3 - 4x + c$

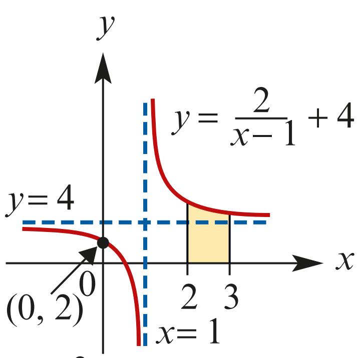
f $\int 3h(x)\,dx$

$= 3\int k'(x)\,dx$

$= 3k(x) + c$

$= 3\sin x^2 + c$

19



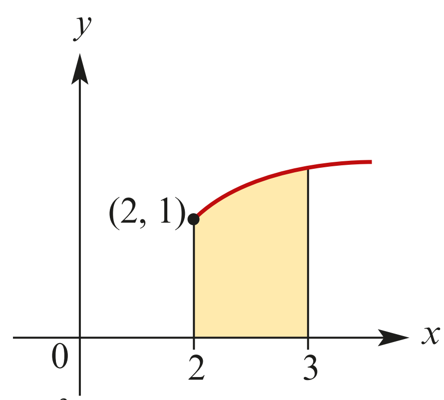
$\int_2^3 \frac{2}{x-1} + 4\,dx$

$= [2\log_e|x-1| + 4x]_2^3$

$= (2\log_e 2 + 12) - (2\log_e 1 + 8)$

$= 4 + 2\log_e 2$

20



$\int_2^3 \sqrt{2x-4} + 1\,dx$

$= \left[\frac{2}{3\times 2}(2x-4)^{\frac{3}{2}} + x\right]_2^3$

$= \left(\frac{1}{3}(2)^{\frac{3}{2}} + 3\right) - \left(\frac{1}{3}(0)^{\frac{3}{2}} + 2\right)$

$= 1 + \frac{1}{3}(2)^{\frac{3}{2}}$

21 a $\int_3^4 \sqrt{x-2}\,dx$

$= \left[\frac{2}{3}(x-2)^{\frac{3}{2}}\right]_3^4$

$= \frac{2}{3}(2)^{\frac{3}{2}} - \frac{2}{3}(1)^{\frac{3}{2}}$

$= \frac{2}{3}(2\sqrt{2} - 1)$

b $\int_0^2 \sqrt{2-x}\,dx$

$= \left[\frac{-2}{3}(2-x)^{\frac{3}{2}}\right]_0^2$

$= \frac{-2}{3}(0)^{\frac{3}{2}} + \frac{2}{3}(2)^{\frac{3}{2}}$

$= \frac{2^{\frac{5}{2}}}{3}$

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c $\int_0^1 \frac{1}{3x+1}\,dx$

$= \left[\frac{1}{3}\log_e|3x+1|\right]_0^1$

$= \frac{1}{3}\log_e 4 + \frac{1}{3}\log_e 1$

$= \frac{2}{3}\log_e 2$

d $\int_1^2 \frac{1}{2x-1} + 3\,dx$

$= \left[\frac{1}{2}\log_e|2x-1| + 3x\right]_1^2$

$= \left(\frac{1}{2}\log_e|3| + 6\right) - \left(\frac{1}{2}\log_e 1 + 3\right)$

$= \frac{1}{2}\log_e 3 + 3$

e $\int_{2.5}^3 \sqrt{2x-5} - 6\,dx$

$= \left[\frac{2}{3} \times \frac{1}{2} \times (2x-5)^{\frac{3}{2}} - 6x\right]_{2.5}^3$

$= \left(\frac{1}{3}(1)^{\frac{3}{2}} - 18\right) - \left(\frac{1}{3}(0)^{\frac{3}{2}} - 15\right)$

$= \frac{1}{3} - 3$

$= \frac{-8}{3}$

f $\int_3^4 \frac{1}{\sqrt{x-2}}\,dx$

$= \left[2(x-2)^{\frac{1}{2}}\right]_3^4$

$= 2(2)^{\frac{1}{2}} - 2(1)^{\frac{1}{2}}$

$= 2\sqrt{2} - 2$

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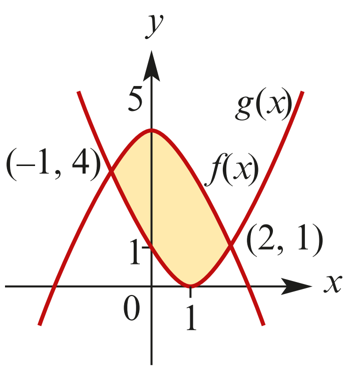
Solutions to Exercise 11I

1
$$y_1 = 12 - x - x^2, y_2 = x + 4$$
$$12 - x - x^2 = x + 4$$
$$x^2 + 2x - 8 = 0$$
$$(x + 4)(x - 2) = 0$$
$$x = 2, -4$$
to test which graph is higher in this interval:
$$x = 0, y_1 = 12, y_2 = 4$$
$$A = \int_{-4}^{2} y_1 - y_2 \, dx$$
$$= \int_{-4}^{2} (12 - x - x^2) - (x + 4) \, dx$$
$$= \int_{-4}^{2} 8 - 2x - x^2 \, dx$$
$$= \left[8x - x^2 - \frac{x^3}{3}\right]_{-4}^{2}$$
$$= \left(16 - 4 - \frac{8}{3}\right) - \left(-32 - 16 + \frac{64}{3}\right)$$
$$= 36 \text{ units}^2$$

2



$$f(x) = 5 - x^2, g(x) = (x - 1)^2$$
$$5 - x^2 = (x - 1)^2$$
$$5 - x^2 = x^2 - 2x + 1$$
$$2x^2 - 2x - 4 = 0$$
$$x^2 - x - 2 = 0$$
$$(x - 2)(x - 1) = 0$$
$$x = -1, 2$$
to test which graph is higher in this interval:
$$f(0) = 5, \; g(0) = (-1)^2 = 1$$
$$\therefore A = \int_{-1}^{2} f(x) - g(x) \, dx$$
$$= \int_{-1}^{2} 5 - x^2 - (x^2 - 2x + 1) \, dx$$
$$= \int_{-1}^{2} 4 + 2x - 2x^2 \, dx$$
$$= \left[4x + x^2 - \frac{2x^3}{3}\right]_{-1}^{2}$$
$$= \left(8 + 4 - \frac{16}{3}\right) - \left(-4 + 1 + \frac{2}{3}\right)$$
$$= 16 - 6 - 1$$
$$= 9 \text{ units}^2$$

3 **a** $y_1 = x + 3, y_2 = 12 + x - x^2$
$$x + 3 = 12 + x - x^2$$
$$x^2 - 9 = 0$$
$$(x + 3)(x - 3) = 0$$
$$x = -3, 3$$
to test which graph is higher in this interval:

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$x = 0, y_1 = 3, y_2 = 12$

$$A = \int_{-3}^{3} y_2 - y_1\,dx$$
$$= \int_{-3}^{3} (12 + x - x^2) - (x + 3)\,dx$$
$$= \int_{-3}^{3} 9 - x^2\,dx$$
$$= \left[9x - \frac{x^3}{3}\right]_{-3}^{3}$$
$$= \left(27 - \frac{27}{3}\right) - \left(-27 + \frac{27}{3}\right)$$
$$= 54 - \frac{54}{3}$$
$$= 54 - 18$$
$$= 36\,\text{units}^2$$

b $y_1 = 3x + 5, y_2 = x^2 + 1$

$3x + 5 = x^2 + 1$

$x^2 - 3x - 4 = 0$

$(x - 4)(x + 1) = 0$

$x = -1, 4$

to test which graph is higher in this interval:

$x = 0, y_1 = 5, y_2 = 1$

$$A = \int_{-1}^{4} y_1 - y_2\,dx$$
$$= \int_{-1}^{4} (3x + 5) - (1 + x^2)\,dx$$
$$= \int_{-1}^{4} 4 + 3x - x^2\,dx$$
$$= \left[4x + \frac{3}{2}x^2 - \frac{x^3}{3}\right]_{-1}^{4}$$
$$= \left(16 - 24 - \frac{64}{3}\right) - \left(-4 + \frac{3}{2} + \frac{1}{3}\right)$$
$$= \frac{125}{6}\,\text{units}^2$$

c $y_1 = 3 - x^2, y_2 = 2x^2$

$3 - x^2 = 2x^2$

$x^2 = 1$

$x = \pm 1$

to test which graph is higher in this interval:

$x = 0, y_1 = 3, y_2 = 0$

$$A = \int_{-1}^{1} y_1 - y_2\,dx$$
$$= \int_{-1}^{1} (3 - x^2) - (2x^2)\,dx$$
$$= \int_{-1}^{1} 3 - 3x^2\,dx$$
$$= [3x - x^3]_{-1}^{1}$$
$$= (3 - 1) - (-3 + 1)$$
$$= 4\,\text{units}^2$$

d $y_1 = x^2, y_2 = 3x$

$x^2 = 3x$

$x^2 - 3x = 0$

$x(x - 3) = 0$

$x = 0, 3$

to test which graph is higher in this interval:

$x = 1, y_1 = 1, y_2 = 3$

$$A = \int_{0}^{3} y_2 - y_1\,dx$$
$$= \int_{0}^{3} 3x - x^2\,dx$$
$$= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_{0}^{3}$$
$$= \left(\frac{27}{2} - 9\right) - (0 - 0)$$
$$= \frac{9}{2}\,\text{units}^2$$

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e $y_{1^2} = x,\ x - y_2 = 2$

$y_1 = \pm\sqrt{x},\ y_2 = x - 2$

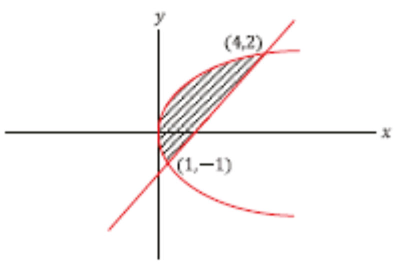
$\pm\ \sqrt{x} = x - 2$

$x = x^2 - 4x + 4$

$x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$x = 1, 4$



$$A = \int_0^1 \sqrt{x} - \sqrt{x}\,dx + \int_1^4 \sqrt{x} - (x-2)\,dx$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2} + 2x\right]_1^4$$

$$= \frac{4}{3} + \left(\frac{16}{3} - 8 + 8\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right)$$

$$= \frac{20}{3} - \frac{2}{3} - \frac{3}{2}$$

$$= 6 - \frac{3}{2}$$

$$= \frac{9}{2}\text{ units}^2$$

4 a $P = \int_{-1}^0 e - e^{-x}dx + \int_0^1 e - e^x\,dx$

$$= 2\int_0^1 e - e^x\,dx$$

$$= 2[ex - e^x]_0^1$$

$$= 2(e-e) - 2(0-1)$$

$$= 2\text{ units}^2$$

b $Q = \int_0^1 e^x - e^{-x}\,dx$

$$= \left[e^x + e^{-x}\right]_0^1$$

$$= \left(e + \frac{1}{e}\right) - (1+1)$$

$$= e + \frac{1}{e} - 2 \approx 1.086\text{ units}^2$$

5

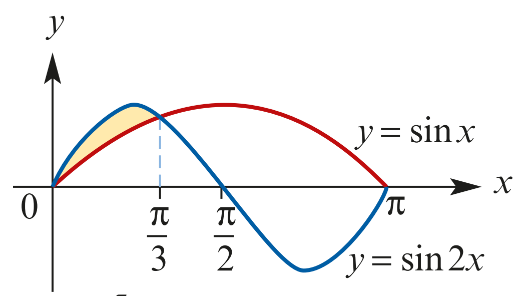
$$A = \int_0^{\frac{7\pi}{6}} (\sin x) - \left(\frac{-1}{2}\right)dx$$

$$= \left[-\cos x + \frac{1}{2}x\right]_0^{\frac{7\pi}{6}}$$

$$= \left(-\left(\frac{-\sqrt{3}}{2}\right) + \frac{1}{2}\times\frac{7\pi}{6}\right) - \left(-(1) + \frac{1}{2}(0)\right)$$

$$= \frac{\sqrt{3}}{2} + \frac{7\pi}{12} + 1 \approx 3.699\text{ units}^2$$

6



$$A = \int_0^{\frac{\pi}{3}} \sin 2x - \sin x\,dx \text{ (\textit{from the graph})}$$

$$= \left[\frac{-1}{2}\cos 2x + \cos x\right]_0^{\frac{\pi}{3}}$$

$$= \left(\frac{-1}{2}\cos\frac{2\pi}{3} + \cos\frac{\pi}{3}\right)$$

$$-\left(\frac{-1}{2}\cos 0 + \cos 0\right)$$

$$= \frac{1}{4} + \frac{1}{2} - \frac{1}{2}$$

$$= \frac{1}{4}\text{ units}^2$$

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7

$$A = \int_0^{\frac{\pi}{6}} \cos x - \sin 2x \, dx$$

$$+ \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin 2x - \cos x \, dx$$

$$= \left[\sin x + \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{6}}$$

$$+ \left[\frac{-1}{2}\cos 2x - \cos x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(\sin\frac{\pi}{6} + \frac{1}{2}\cos\frac{\pi}{3}\right) - \left(\sin 0 + \frac{1}{2}\cos 0\right)$$

$$+ \left(\frac{-1}{2}\cos\pi - \sin\frac{\pi}{2}\right) - \left(\frac{-1}{2}\cos\frac{\pi}{3} - \sin\frac{\pi}{6}\right)$$

$$= \frac{1}{2} + \frac{1}{4} - \frac{1}{2} + \frac{1}{2} - 1 + \frac{1}{2} + \frac{1}{4}$$

$$= \frac{1}{2} \text{ units}^2$$

8

$$e^x = 2 + 3e^{-x}$$

$$e^{2x} - 2e^x - 3 = 0$$

$$(e^x - 3)(e^x + 1) = 0$$

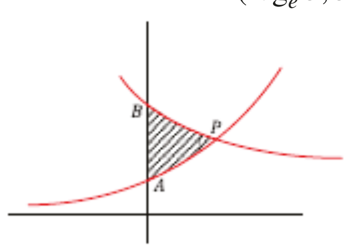
$$e^x = -1, 3$$

Since $e^x > 0$, $e^x = 3$

$$x = \log_e 3$$

$$y = e^x = 3$$

$$P = (\log_e 3, 3)$$



$$Area = \int_0^{\log_e 3} 2 + 3e^{-x} - e^x \, dx$$

$$= \left[2x - 3e^{-x} - e^x\right]_0^{\log_e 3}$$

$$= (2\log_e 3 - 3e^{\log_e \frac{1}{3}} - e^{\log_e 3}) - (0 - 3 - 1)$$

$$= 4 + 2\log_e 3 - 1 - 3$$

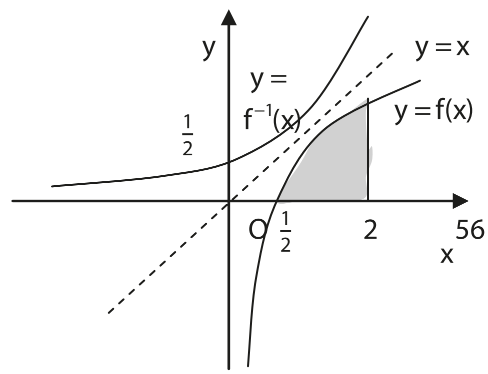
$$= 2\log_e 3$$

$$\approx 2.197 \text{ units}^2$$

9 **a** $f : R^+ \to R, f(x) = \log_e(2x)$

Consider $x = \log_e(2y)$. Solving for y gives $y = \frac{1}{2}e^x$

$f^{-1}(x) = \frac{1}{2}e^x$ and the domain of $f^{-1} = R$



b $b \int_0^{\log_e 4} f^{-1}(x) \, dx$

$$= \int_0^{\log_e 4} \frac{1}{2}e^x \, dx$$

$$= \left[\frac{1}{2}e^x\right]_0^{\log_e 4} = \frac{3}{2}$$

c By symmetry $\int_{\frac{1}{2}}^{2} f(x) \, dx$

$$= 4\log_e(2) - \frac{3}{2}$$

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Solutions to Exercise 11J

1 a $\text{av} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

$\text{av} = \dfrac{1}{2-0}\displaystyle\int_0^2 x(2-x)\,dx$

$= \dfrac{1}{2}\displaystyle\int_0^2 (2x - x^2)\,dx$

$= \dfrac{1}{2}\left[x^2 - \dfrac{1}{3}x^3\right]_0^2$

$= \dfrac{1}{2}\left(4 - \dfrac{8}{3}\right) = \dfrac{2}{3}$

b $\text{av} = \dfrac{1}{\pi - 0}\displaystyle\int_0^\pi \sin(x)\,dx$

$= \dfrac{1}{\pi}\left[-\cos(x)\right]_0^\pi$

$= \dfrac{1}{\pi}((-(-1)) - (-1)) = \dfrac{2}{\pi}$

c $\text{av} = \dfrac{1}{\frac{\pi}{2} - 0}\displaystyle\int_0^{\frac{\pi}{2}} \sin(x)\,dx$

$= \dfrac{2}{\pi}\left[-\cos(x)\right]_0^{\frac{\pi}{2}}$

$= \dfrac{2}{\pi}((-0) - (-1)) = \dfrac{2}{\pi}$

d $\text{av} = \dfrac{1}{\frac{2\pi}{n} - 0}\displaystyle\int_0^{\frac{2\pi}{n}} \sin(nx)\,dx$

$= \dfrac{n}{2\pi}\left[-\dfrac{1}{n}cos(nx)\right]_0^{\frac{2x}{n}}$

$\dfrac{1}{2\pi}((-1) - (-1)) = 0$

$= \dfrac{1}{2\pi}((-1) - (-1)) = 0$

e $\text{av} = \dfrac{1}{2-(-2)}\displaystyle\int_{-2}^2 (e^x + e^{-x})\,dx$

$= \dfrac{1}{4}\left[e^x - e^{-x}\right]_{-2}^2$

$= \dfrac{1}{4}((e^2 - e^{-2}) - (e^{-2} - e^2))$

$= \dfrac{1}{2}(e^2 - e^{-2})$

2 $\text{av temp} = \dfrac{1}{10-0}\displaystyle\int_0^{10} 50e^{-\frac{t}{2}}\,dt$

$= \dfrac{1}{10}\left[\dfrac{50}{\left(-\frac{t}{2}\right)}e^{-\frac{t}{2}}\right]_0^{10}$

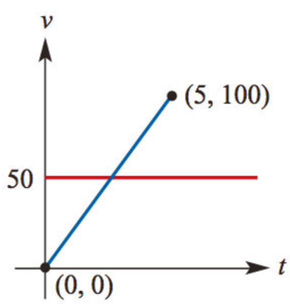
$= -10(e^{-5} - e^{-0})$

$= 10(1 - e^{-5}) \approx 9.93°\text{C}$

3 a $\text{av speed} = \dfrac{1}{5-0}\displaystyle\int_0^5 20t\,dt$

$= \dfrac{1}{5}\left[10t^2\right]_0^5$

$= 50\text{m/s}$



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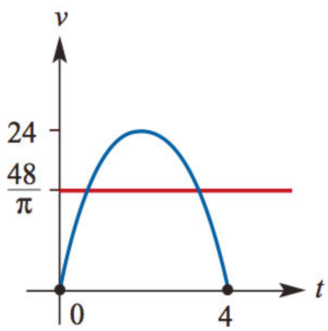
b av speed $= \dfrac{1}{4-0}\displaystyle\int_0^4 24\sin\left(\frac{1}{4}\pi t\right)dt$

$= \dfrac{1}{4}\left[-\dfrac{24\times 4}{\pi}\cos(\dfrac{1}{4}\pi t)\right]_0^4$

$= -\dfrac{24}{\pi}(-1-1)$

$= \dfrac{48}{\pi}$ m/s

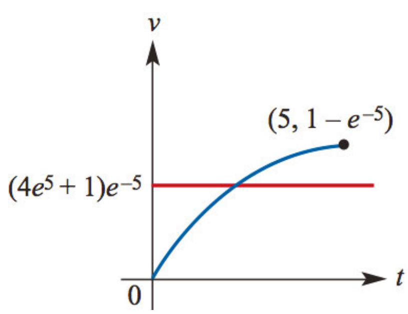


c av speed $= \dfrac{1}{5-0}\displaystyle\int_0^5 5(1-e^{-t})\,dt$

$= \dfrac{1}{5}\left[5(t+e^{-t})\right]_0^5$

$= \dfrac{1}{5}(5(5+e^{-5})-5e^{-0})$

$= 4 + e^{-5}$ m/s



4 av velocity $= \dfrac{1}{3-0}\displaystyle\int_0^3 9.8t\,dt$

$= \dfrac{1}{3}\left[4.9t^2\right]_0^3$

$= 14.7$ m/s

5 mean value $= \dfrac{1}{a-0}\displaystyle\int_0^a x(a-x)\,dx$

$= \dfrac{1}{a}\displaystyle\int_0^a (ax-x^2)\,dx$

$= \dfrac{1}{a}\left[\dfrac{1}{2}ax^2-\dfrac{1}{3}x^3\right]_0^a$

$= \dfrac{1}{a}\left(\dfrac{1}{2}a^3-\dfrac{1}{3}a^3\right)$

$= \dfrac{a^2}{6}$

6 a $pv^{0.9} = 300 \Rightarrow p = 300v^{-0.9}$

av pressure $= \dfrac{1}{1-\frac{1}{2}}\displaystyle\int_{\frac{1}{2}}^1 300v^{-0.9}dv$

$2\left[\dfrac{300}{0.1}v^{0.1}\right]_{\frac{1}{2}}^1$

$= 6000\left(1-\left(\dfrac{1}{2}\right)^{0.1}\right)$

$= 3000(2-2^{0.9})$

≈ 401.8 N/m^2

b $v = 3t+1$, so:

$t=0,\ v=1,\ t=1,\ v=4$

av pressure

$= \dfrac{1}{4-1}\displaystyle\int_1^4 300v^{-0.9}dv$

$= \dfrac{1}{3}\left[\dfrac{300}{0.1}v^{0.1}\right]_1^4$

$= 1000((4)^{0.1}-1)$

≈ 148.7 N/m^2

7 $v = \dfrac{dx}{dt}$ and $v = 2t-3$

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a $x = \int (2t - 3)\,dt$

$= t^2 - 3t + c$

$t = 0,\ x = 0$ so $c = 0$

$x = t^2 - 3t$

b $t = 3,\ x = 0$, so the body is at O.

c av velocity $= \dfrac{x(3) - x(0)}{3 - 0} = 0$
(Note: this is exactly the same as av velocity $= \dfrac{1}{3-0}\int_0^3 (2t - 3)\,dt$; but the integration is already done in part **a**.)

d $v = 0$ when $2t - 3 = 0$, i.e. when $t = 1.5$.
When $t = 1.5,\ x = -2.25$, so the body goes from $x = 0$ to $x = -2.25$ in the first 1.5 s and then returns to $x = 0$ in the next 1.5 s. It travels 4.5 m in the first 3 s.

e av speed $= \dfrac{4.5}{3} = 1.5$ m/s

8 **a** $v = 2t^2 - 8t + 6,\ v = \dfrac{dx}{dt}, a = \dfrac{dv}{dt}$

$x = \int (2t^2 - 8t + 6)\,dt$

$= \dfrac{2}{3}t^3 - 4t^2 + 6t + c$

$t = 0,\ x = 4$

so $c = 4$

$x = \dfrac{2}{3}t^3 - 4t^2 + 6t + 4$

$a = \dfrac{d}{dt}(2t^2 - 8t + 6)$

$= 4t - 8$

b $v = 0$ when $2t^2 - 8t + 6 = 0$, i.e.

$t^2 - 4t + 3 = 0$

$(t - 1)(t - 3) = 0$

$t = 1, 3$

$x(1) = \dfrac{20}{3}$ m, $x(3) = 4$ m

c $a(1) = -4$ m/s^2, $a(3) = 4$ m/s^2

9 $a = \dfrac{dv}{dt}$ so $\dfrac{dv}{dt} = 8$

$v = 8t + c$

$v = \dfrac{dx}{dt}$ so $\dfrac{dx}{dt} = 8t + c$

$x = 4t^2 + ct + d$

$t = 1,\ x = 0$ so $4 + c + d = 0$, i.e.
$c + d = -4$ ①
$t = 3,\ x = 30$ so $36 + 3c + d = 30$, i.e.
$3c + d = -6$ ②
② -① gives $2c = -2$, so $c = -1$
Substitution into (1) gives $d = -3$, so
$x = 4t^2 - t - 3$
$t = 0,\ x = -3$
Initial displacement is-3 m.

10 $a = \dfrac{dv}{dt}$ so $\dfrac{dv}{dt} = 2t - 3$

$v = t^2 - 3t + c$

$t = 0,\ v = 3$ so $c = 3$

$v = t^2 - 3t + 3$

$v = \dfrac{dx}{dt}$ so $\dfrac{dx}{dt} = t^2 - 3t + 3$

$x = \dfrac{1}{3}t^3 - \dfrac{3}{2}t^2 + 3t + d$

$t = 0,\ x = 2$ so $d = 2$

$x = \dfrac{1}{3}t^3 - \dfrac{3}{2}t^2 + 3t + 2$

$v(10) = 73$ m/s, $x(10) = \dfrac{646}{3}$ m

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11 $a = \dfrac{dv}{dt}$ so $\dfrac{dv}{dt} = -10$

a $v = -10t + c; t = 0,\ v = 25$, so
$c = 25\ v = -10t + 25$

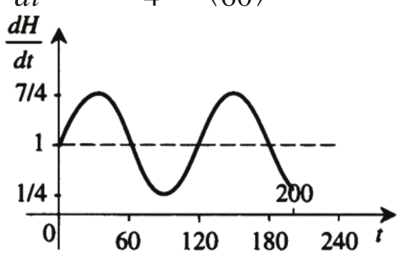
b $x = -5t^2 + 25\,t + d; t = 0,\ x = 0$, so
$d = 0\ x = -5t^2 + 25t$

c max height when $v = 0$, i.e. $t = 2.5$ s

d $x(2.5) = \dfrac{125}{4}$ m

e $x = 0$ when$-5\,t^2 + 25t = 0$, giving
$t = 0$ or $t = 5$; so returns after 5 s.

12 **a** $\dfrac{dH}{dt} = 1 + \dfrac{3}{4}\sin\left(\dfrac{\pi t}{60}\right)$ for $t \in [0,200]$



b $\dfrac{dH}{dt} > 1.375 \Leftrightarrow 1 + 4\dfrac{3}{4}\sin\left(\dfrac{\pi t}{60}\right) > \dfrac{11}{8}$

$\Leftrightarrow \dfrac{3}{4}\sin\left(\dfrac{\pi t}{60}\right) > \dfrac{3}{8}$

$\Leftrightarrow \sin\left(\dfrac{\pi t}{60}\right) > \dfrac{1}{2}$

Consider the equation $\sin\left(\dfrac{\pi t}{60}\right) = \dfrac{1}{2}$
This is equivalent to
$\dfrac{\pi t}{60} = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$ or $\dfrac{13\pi}{6}$ or $\dfrac{17\pi}{6}$ or ...
i.e. $t = 10$ or 50 or 130 or 170 or $\cdots$
For the required domain and by observation from graph $\dfrac{dH}{dt} > 1.375$
for $t \in (10, 50) \cup (130, 170)$

c The rate of heat loss is greatest when
$\sin\left(\dfrac{\pi t}{60}\right) = 1$
This occurs when
$\dfrac{\pi t}{60} = \dfrac{\pi}{2}$ or $\dfrac{5\pi}{2}$ or $\dfrac{9\pi}{2}$ or ...
$\Rightarrow t = 30$ or 150 or 270 or$\cdots$
$\therefore$ rate of heat loss is greatest when
$t = 30$ or 150 for t in $[0, 200]$

d **i** The total heat loss for $t \in$
$[0, 120] = \int_0^{120} 1 + \dfrac{3}{4}\sin\left(\dfrac{\pi t}{60}\right) dt$

$$= \left[t - \frac{3}{4} \times \frac{60}{\pi}\cos\left(\frac{\pi t}{60}\right)\right]_0^{120}$$

$$= 120 - \frac{45}{\pi}\cos 2\pi$$

$$-\left(0 - \frac{45}{\pi}\cos 0\right)$$

$$= 120 - \frac{45}{\pi} + \frac{45}{\pi}$$

$$= 120$$

$\therefore$ 120 kilojoules lost over the 120 days.

ii Total heat lost for $t \in$
$[0, 200] = \int_0^{200)} 1 + \dfrac{3}{4}\sin\left(\dfrac{\pi t}{60}\right) dt$

$$= \left[t - \frac{45}{\pi}\cos\left(\frac{\pi t}{60}\right)\right]_0^{200}$$

$$= 200 - \frac{45}{\pi}\cos\left(\frac{200\pi}{60}\right) - \left(0 - \frac{45}{\pi}\right)$$

$$= 200 - \frac{45}{\pi}\cos\left(\frac{10\pi}{3}\right) + \frac{45}{\pi}$$

$$= 200 - \frac{45}{\pi} \times -\frac{1}{2} + \frac{45}{\pi}$$

$$= 200 + \frac{45}{2\pi} + \frac{45}{\pi}$$

$$= 200 + \frac{135}{2\pi}$$

≈ 221.48 kilojules

13 $\dfrac{dV}{dt} = 1000 - 30t^2 + 2t^3\ 0 \le t \le 15$

a When $t = 0$, $\dfrac{dV}{dt} = 1000$
The rate of flow is 1000 million

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litres/hour= 10^9 litres/hour.
When $t = 2$,
$= 1000 - 30 \times 4 + 2 \times 2^3 =$
$1000 - 120 + 16$
$= 896$ million litres/hour $= 8.96 \times 10^8$ litres/hour

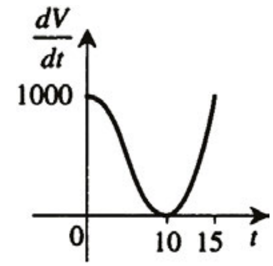
b **i** To find stationary points, let
$R = \dfrac{dV}{dt} = 1000 - 30t^2 + 2t^3$
Stationary points occur when
$\dfrac{dR}{dt} = 0$
$\dfrac{dR}{dt} = -60t + 6t^2$
$= -6t(10 - t)$
$\dfrac{dR}{dt} = 0$ implies $t = 0$ or $t = 10$
A gradient chart for $\dfrac{dR}{dt}$ is as shown:

t	< 0	0	<<	10	> 10
sing of $\frac{dR}{dt}$	+ ve	0	-ve	0	+ ve
shape	/	−	\	−	/

$\therefore$ a local maximum at (0,1000) and a local minimum when $t = 10$.
when $t = 10$
$R = \dfrac{dV}{dt} = 1000 - 30 \times 10^2 + 2 \times 10^3$
$= 1000 - 3000 + 2000$
$= 0$
$\therefore$ local minimum at $(10,\ 0)$
When $t = 15$, $\dfrac{dV}{dt} =$
$1000 - 30 \times 15^2 + 2 \times 15^3$
$= 1000 - 30 \times 225 + 2 \times 3375$
$= 1000$
$\therefore$ The maximum flow occurs when $t = 0$ and $t = 15$

ii The maximum flow is 1000 millon litres/hour.

c

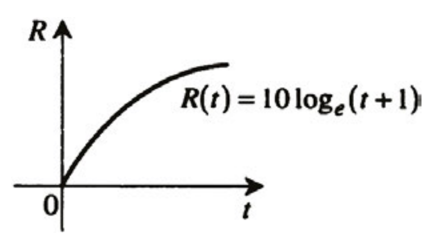


d **i** Area beneath the graph between $t = 0$ and $t = 10$
$= \displaystyle\int_0^{10} 1000 - 30t^2 + 2t^3\, dt$
$= \left[1000t - \dfrac{30 \times t^3}{3} + \dfrac{2t^4}{4}\right]_0^{10}$
$= 5000$

ii 5000 million litres flowed out in the first 10 hours.

14 **a** $R{:}[0,\ \infty) \to R,\ R(t) = 10\log_e(t + 1)$
When $t = 5,\ R(5) = 10\log_e(6) \approx 17.918$
When $t = 5$, the rate of growth is ≈ 17.918 penguins per year.
When $t = 10,\ R(10) = 10\log_e(10) \approx 23.978$
When $t = 10$, the rate of growth is 23.978 penguins per year.
When $t = 100,\ R(100) = 10\log_e(100) \approx 46.151$
When $t = 100$, the rate of growth is 46.151 penguins per year.

b



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c For the inverse function consider

$$t = 10\log_e(y + 1)$$

$$\therefore \frac{t}{10} = \log_e(y + 1)$$

$$\therefore e^{\frac{t}{10}} = y + 1$$

$$\therefore y = e^{\frac{t}{10}} - 1$$

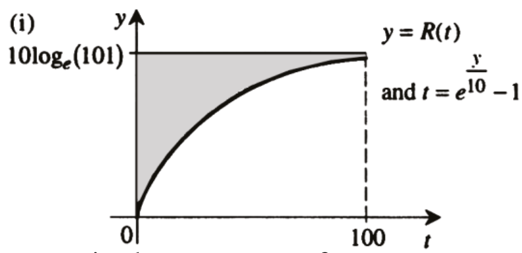
and the inverse function is

$R^{-1}(t) = e^{\frac{t}{10}} - 1$

The domain of R^{-1} = range of

$R = R^{+} \cup \{0\}$

d i



$\therefore$ required area = area of rectangle − area

$$= 100 \times 10\log_e(101) - \int_0^{10\log_e(101)} (e^{\frac{y}{10}} - 1)\,dy$$

$$= 1000\log_e(101) - \left[10e^{\frac{y}{10}} - y\right]_0^{10\log_e 101}$$

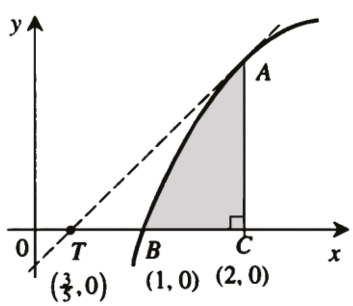
$$= 1000\log_e(101) - \left[10e^{\log_e 101} - 10\log_e(101) - (10e^0 - 0)\right]$$

$$= 1000\log e(101) - [1010 - 10\log_e(101) - 10]$$

$$= 1000\log_e(101) - 1000$$

$$\approx 3661.27$$

ii The penguin population has increased by 3661 penguins over 100 years.



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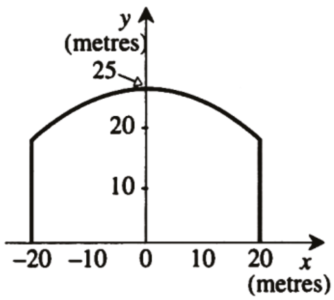
15



Area of cross section

$= \int_{-20}^{20}(25 - 0.02x^2)\,dx$

The symmetry of f gives that the area of cross section

$= 2\int_{0}^{20}(25 - 0.02x^2)\,dx$

$= 2\left[25x - \dfrac{x^3}{150}\right]_0^{20}$

$= 2\left(25 \times 20 - \dfrac{20^3}{150}\right)$

$= 2\left(500 - \dfrac{8000}{150}\right)$

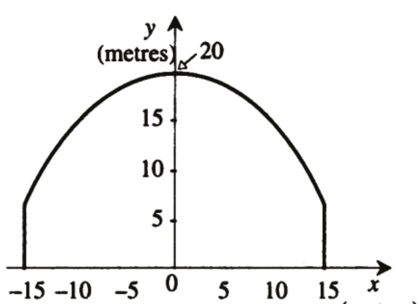
$= 893\dfrac{1}{3}$

The volume of the hall = area of cross section × length

$= 893\dfrac{1}{3} \times 80$

$= 71\#466\dfrac{2}{3}\text{ m}^3$

16 **a**



Area of cross section

$= \int_{-15}^{15}(20 - 0.06x^2)\,dx$

The symmetry of f gives that the area of cross section

$= 2\int_{0}^{15}(20 - 0.06x^2)\,dx$

$= 2\left[20x - \dfrac{x^3}{50}\right]_0^{15}$

$= 2\left(20 \times 15 - \dfrac{15^3}{30}\right)$

$= 465$

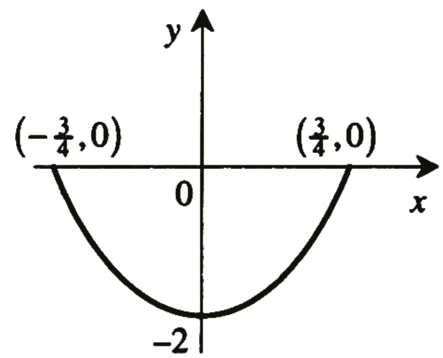
The area of the cross section is 465 m^2

b The volume of the hangar = area of cross section × length

$= 465 \times 100$

$= 46\,500\text{ m}^3$

17



The parabola is of the form $y = ax^2 + b$

When $x = 0,\ y = -2$

When $y = 0,\ x = \pm\dfrac{3}{4}$

$0 = a(\dfrac{3}{4})^2 - 2$

and $\dfrac{2 \times 16}{9} = a$

i.e. $a = \dfrac{32}{9}$

∴ The equation of the parabola is

$y = \dfrac{32}{9}x^2 - 2$

The total volume of the trough = area of cross section × length To determine the area of cross section

consider $2\int_0^{\frac{3}{4}} \dfrac{32x^2}{9} - 2\,dx$

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$$= 2\left[\frac{32x^3}{27} - 2x\right]^{\frac{3}{4}}$$

$$= 2\left(\frac{32}{27} \times \frac{27}{64} - 2 \times \frac{3}{4}\right)$$

$$= 2\left(\frac{1}{2} - 1\frac{1}{2}\right)$$

$$= -2$$

$\therefore$ The cross sectional area is 2 m^2

The total volume $= 2 \times l = 2l$ m^3, where ℓ is the length of the trough.

When the trough is half full the volume is l m^3 and the cross sectional area is 1 m^3 The shaded area = 1 m^2

$$\text{Shaded area} = 2\int_0^a \left(\frac{32}{9}a^2 - 2\right) - \left(\frac{32}{9}a^2 - 2\right) dx$$

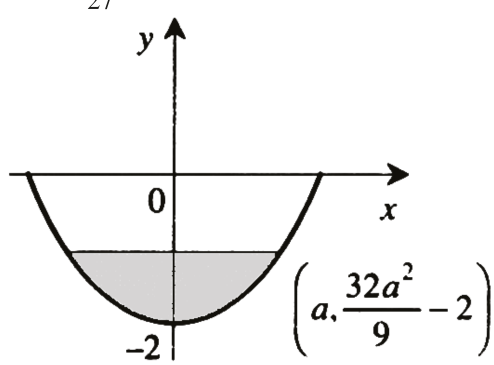
$$= 2\left[\frac{32}{9}a^2x - 2x - \frac{32}{27}x^3 + 2x\right]_0^a$$

$$= 2\left[\frac{32a^3}{9} - \frac{32a^3}{27}\right]$$

$$= 2 \times \left[\frac{64a^3}{27}\right]$$

$$= \frac{128a^3}{27}$$

$$\therefore \frac{128\,a^3}{27} = 1$$



which implies $a^3 = \frac{27}{128}$

$$\therefore a^3 = \frac{3^3}{2^7}$$

$$\therefore a = \frac{3}{2^{\frac{7}{3}}}$$

When $x = \frac{3}{2^{\frac{7}{3}}}$ $\quad y = \frac{32}{9} \times \left(\frac{3}{2^{\frac{7}{3}}}\right)^2 - 2$

$$= \frac{32}{9} \times -\frac{9}{2^{\frac{14}{3}}} - 2$$

$$= \frac{2^5}{2^{\frac{14}{3}}} - 2$$

$$= 2^{\frac{1}{3}} - 2$$

$\therefore$ the depth $= 2 - (2 - 2^{\frac{1}{3}})$

$= 2^{\frac{1}{3}}$ metres ≈ 1.26 metres

The depth of the water is 1.26 metres when it is half full.

18 **a** $y = 3 - 3\cos\left(\frac{x}{3}\right)$ for $x \in [-3\pi, 3\pi]$. The maximum value of the function is 6 and hence the maximum height of 6 metres

b The area $= 2\int_0^{3\pi}\left(3 - 3\cos\left(\frac{x}{3}\right)\right)dx$

$= 2[3x - 9\sin(\frac{x}{3})]_0^{3\pi} = 18\pi$

The area is 18π m^2

c **i** $\frac{dy}{dx} = \sin\left(\frac{x}{3}\right)$. When

$x = a$, $y = 3 - 3\cos\left(\frac{a}{3}\right)$ and $\frac{dy}{dx} = \sin\left(\frac{a}{3}\right)$.

Therefore the equation of the normal is

$y - \left(3 - 3\cos\left(\frac{a}{3}\right)\right)$

$= -\frac{1}{\sin\left(\frac{a}{3}\right)}(x - a)$

ii If it passes through (9,0),

$0 - 3 + 3\cos\left(\frac{a}{3}\right) = -\frac{1}{\sin\left(\frac{a}{3}\right)}(9 - a)$

Solving numerically gives $a = 5.409$

19 **a** $\frac{dV}{dt} = 3\left[\cos\left(\frac{\pi t}{2}\right) + \sin\left(\frac{\pi t}{8}\right) + 2\right]$

i When $t = 0$, $\frac{dV}{dt} = 3[1 + 0 + 2] = 9$

ii When $t = 2$, $\frac{dV}{dt}$

$= 3\left[-1 + \frac{1}{\sqrt{2}} + 2\right] = 3\frac{(\sqrt{2} + 2)}{2}$

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iii When $t = 4$, $\frac{dV}{dt} = 3[1 + 1 + 2]$
$= 12$

b From the graph maximum value is 12 and the minimum value is 0.834

c The volume through the pipe in the first 8 minutes

$$= 3\int_0^8 \left[\cos\left(\frac{\pi t}{2}\right) + \sin\left(\frac{\pi t}{8}\right) + 2\right] dt$$

$$= 3\left[\frac{2}{\pi}\sin\left(\frac{\pi t}{2}\right) - \frac{8}{\pi}\cos\left(\frac{\pi t}{8}\right) + 2t\right]_0^8$$

$$= 3\left(\frac{8}{\pi} + 16 - \left(-\frac{8}{\pi}\right)\right) = 48\left(1 + \frac{1}{\pi}\right)$$

$$= \frac{48(\pi + 1)}{\pi} \text{ litres}$$

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Solutions to Technology-free questions

1 a $\int_2^3 x^3\,dx = \left[\frac{1}{4}x^4\right]_2^3 = \frac{65}{4}$

b Since $\sin x$ is an odd function, the integral is 0. (Alternatively work through the integral.)

c $\int_a^{4a} (a^{\frac{1}{2}} - x^{\frac{1}{2}})\,dx$

$= \left[a^{\frac{1}{2}}x - \frac{2}{3}x^{\frac{3}{2}}\right]_a^{4a}$

$= \left(4a^{\frac{3}{2}} - \frac{2}{3} \times 8a^{\frac{3}{2}}\right) - \left(a^{\frac{3}{2}} - \frac{2}{3}a^{\frac{3}{2}}\right)$

$= -\frac{5a^{\frac{3}{2}}}{3}$

d $\int_1^4 \frac{3}{\sqrt{x}} - 5\sqrt{x} - x^{-\frac{3}{2}}\,dx$

$= \int_1^4 3x^{-\frac{1}{2}} - 5x^{\frac{1}{2}} - x^{-\frac{3}{2}}\,dx$

$= \left[6x^{\frac{1}{2}} - \frac{10}{3}x^{\frac{3}{2}} + 2x^{\frac{-1}{2}}\right]_1^4$

$= \left(12 - \frac{80}{3} + 1\right) - \left(6 - \frac{10}{3} + 2\right)$

$= -\frac{55}{3}$

e $\int_0^{\frac{\pi}{4}} \cos 2\,\theta d\theta = \left[\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$

$= \left(\frac{1}{2}\sin\frac{\pi}{2}\right) - 0 = \frac{1}{2}$

f $\int_1^e \frac{1}{x}\,dx = [\log_e x]_1^e$

$= \log_e e - \log_e 1 = 1$

g $\int_0^{\frac{\pi}{2}} \sin 2\left(\theta + \frac{\pi}{4}\right)d\theta$

$= \left[-\frac{1}{2}\cos 2\left(\theta + \frac{\pi}{4}\right)\right]_0^{\frac{\pi}{2}}$

$= \left(-\frac{1}{2}\cos\frac{3\pi}{2}\right) - \left(-\frac{1}{2}\cos\frac{\pi}{2}\right)$

$= 0$

h $\int_0^{\pi} \sin 4\theta\,d\theta = \left[-\frac{1}{4}\cos 4\theta\right]_0^{\pi}$

$= \left(-\frac{1}{4}\cos 4\pi\right) - \left(-\frac{1}{4}\cos 0\right)$

$= 0$

2 $\int_{-1}^{2} x + 2f(x)\,dx$

$= \int_{-1}^{2} x dx + 2\int_{-1}^{2} f(x)\,dx$

$= \left[\frac{x^2}{2}\right]_{-1}^{2} + 2 \times 5$

$= \frac{23}{2}$

3 $\int_1^5 f(x)\,dx = \int_0^5 f(x)\,dx - \int_0^1 f(x)\,dx$

$= 1 - (-2) = 3$

4 $\int_3^{-2} f(x)\,dx = -\int_{-2}^{3} f(x)\,dx$

$= -\int_{-2}^{1} f(x)\,dx - \int_1^3 f(x)\,dx$

$= -2 - (-6) = 4$

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5 $\int_0^2 (x+1)^7\,dx = \left[\frac{1}{8}(x+1)^8\right]_0^2 = 820$

6 $\int_0^1 (3x+1)^3\,dx = \left[\frac{1}{3\times 4}(3x+1)^4\right]_0^1$

$= \frac{85}{4}$

7 If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_0^9 f(x)\,dx = [F(x)]_0^9$$

$$= F(9) - F(0) = 5$$

Also by the chain rule

$$\frac{d}{dx}(F(3x)) = 3f(3x), \text{so:}$$

$$\int_0^3 f(3x)\,dx = \frac{1}{3}[F(3x)]_0^3$$

$$= \frac{1}{3}(F(9) - F(0))$$

$$= \frac{5}{3}$$

8 If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_1^4 f(x)\,dx = [F(x)]_1^4$$

$$= F(4) - F(1) = 5$$

Also by the chain rule

$$\frac{d}{dx}(F(3x+1)) = 3f(3x+1), \text{so:}$$

$$\int_0^1 f(3x+1)\,dx = \frac{1}{3}[F(3x+1)]_0^1$$

$$= \frac{1}{3}(F(4) - F(1))$$

$$= \frac{5}{3}$$

9 The area of the shaded region from $x = a$ to $x = b$ is given by $\int_a^b f(x) - g(x)\,dx$.
The area of the shaded region from $x = b$ to $x = c$ is given by $\int_b^c g(x) - f(x)\,dx$.
The area of the shaded region from $x = c$ to $x = d$ is given by $\int_c^d f(x) - g(x)\,dx$.
The area of the shaded region is
$\int_a^b f(x) - g(x)\,dx + \int_b^c g(x) - f(x)\,dx$
$+ \int_c^d f(x) - g(x)\,dx$

10 a The curves intersect where $2x + x^2 = 15$, i.e. $x^2 + 2x - 15 = 0$.
Hence $(x-3)(x+5) = 0$, so P has coordinates $(3, 9)$.
Q has coordinates $(7.5,\ 0)$.

b The area of the shaded region is

$$\int_0^3 x^2\,dx + \int_3^{7.5} 15 - 2x\,dx$$

$$= \left[\frac{1}{3}x^3\right]_0^3 + [15x - x^2]_3^{7.5}$$

$$= 9 + (112.5 - 56.25) - (45 - 9)$$

$$= 29.25$$

11 a area $A = \int_1^2 10x^{-2}\,dx$

$$= [-10x^{-1}]_1^2$$

$$= -5 - (-10) = 5$$

b $\int_2^p 10x^{-2}\,dx = \int_p^5 10x^{-2}\,dx$

$$[-10x^{-1}]_2^p = [-10x^{-1}]_p^5$$

$$5 - \frac{10}{p} = -2 + \frac{10}{p}$$

$$\frac{20}{p} = 7$$

$$p = \frac{20}{7}$$

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12 The area of the shaded region is

$$\int_2^4 16x^{-2} - 0.5x + 1\,dx$$
$$+ \int_4^5 0.5x - 1 - 16x^{-2}\,dx$$
$$= [-16x^{-1} - 0.25x^2 + x]_2^4$$
$$+ [0.25x^2 - x + 16x^{-1}]_4^5$$
$$= (-4 - 4 + 4) - (-8 - 1 + 2)$$
$$+ (6.25 - 5 + 3.2) - (4 - 4 + 4)$$
$$= 3.45$$

13 **a** When $x = 0, 6y - y^2 = 0$, so $y = 0, 6$. Thus A has coordinates $(0, 6)$. For point B, solve $y = 6y - y^2$, i.e. $y^2 - 5y = 0$, so $y = 0$ or $y = 5$. As $y = x$, then B has coordinates (5, 5).

b area

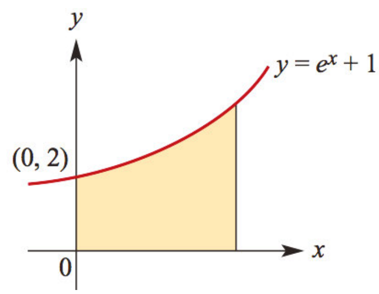
$$P = \int_0^5 y\,dy + \int_5^6 6y - y^2\,dy$$
$$= \left[\frac{1}{2}y^2\right]_0^5 + \left[3y^2 - \frac{1}{3}y^3\right]_5^6$$
$$= \frac{25}{2} + (108 - 72) - \left(75 - \frac{125}{3}\right)$$
$$= \frac{91}{6} = 15\frac{1}{6}$$

c Area bounded by the parabola and the y axis is given by

$$\int_0^6 6y - y^2\,dy = \left[3y^2 - \frac{1}{3}y^3\right]_0^6$$
$$= 36$$

So area $Q = 36 -$ area $P = 20\frac{5}{6} = \frac{125}{6}$.

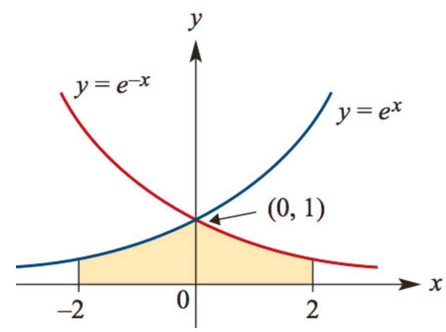
14 **a** y intercept is $(0, 2)$



b $\displaystyle\int_0^2 e^x + 1\,dx = [e^x + x]_0^2$

$$= (e^2 + 2) - (1 + 0)$$
$$= e^2 + 1$$

15 **a** The graphs intersect at $(0, 1)$.



b $\displaystyle\int_0^2 e^{-x}\,dx + \int_{-2}^0 e^x\,dx = 2\int_{-2}^0 e^x\,dx$

$$= 2[e^x]_{-2}^0$$
$$= 2 - 2e^{-2}$$

16 **a** $\displaystyle\int_0^1 e^x\,dx = [e^x]_0^1$

$$= e - 1$$

b area $= 2(e - 1)$

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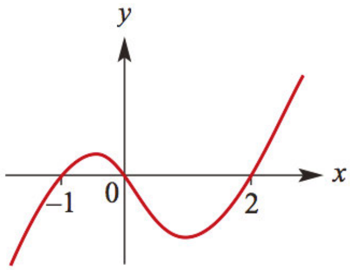
17 area $= \int_0^1 2e^{2x} + 3\,dx$

$= [e^{2x} + 3x]_0^1$

$= (e^2 + 3 + -(1 + 0)$

$= e^2 + 2$

18 The intercepts are $(-1, 0)$, $(0, 0)$, $(2, 0)$.



area $= \int_{-1}^0 x(x-2)(x+1)\,dx$

$- \int_0^2 x(x-2)(x+1)\,dx$

$= \int_{-1}^0 x^3 - x^2 - 2x\,dx$

$- \int_0^2 x^3 - x^2 - 2x\,dx$

$= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2\right]_{-1}^0$

$- \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2\right]_0^2$

$= -\left(\frac{1}{4} + \frac{1}{3} - 1\right) - \left(4 - \frac{8}{3} - 4\right)$

$= \frac{37}{12} = 3\frac{1}{12}$

19 a $\int_0^2 e^{-x} + x\,dx = \left[-e^{-x} + \frac{1}{2}x^2\right]_0^2$

$= (-e^{-2} + 2) - (-1)$

$= 3 - e^{-2}$

b $\int_{-2}^{-1} x + \frac{1}{x-1}\,dx$

$= \left[\frac{1}{2}x^2 + \log_e|x-1|\right]_{-2}^{-1}$

$= \left(\frac{1}{2} + \log_e|-2|\right) - (2 + \log_e|-3|)$

$= \log_e\frac{2}{3} - \frac{3}{2}$

c $\int_0^{\frac{\pi}{2}} \sin x + x\,dx = \left[-\cos x + \frac{1}{2}x^2\right]_0^{\frac{\pi}{2}}$

$= \left(0 + \frac{\pi^2}{8}\right) - (-1)$

$= \frac{\pi^2}{8} + 1$

d $\int_{-4}^{-5} e^x + \frac{1}{2-2x}\,dx$

$= \int_{-4}^{-5} e^x + \frac{1}{2} \times \frac{1}{1-x}\,dx$

$= \left[e^x - \frac{1}{2}\log_e|1-x|\right]_{-4}^{-5}$

$= \left(e^{-5} - \frac{1}{2}\log_e|6|\right) - \left(e^{-4} - \frac{1}{2}\log_e|5|\right)$

$= e^{-5} - e^{-4} + \frac{1}{2}\log_e\frac{5}{6}$

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Solutions to multiple-choice questions

1 C $\int_0^2 3f(x) + 2\,dx$

$= \int_0^2 3f(x)\,dx + \int_0^2 2\,dx$

$= 3\int_0^2 f(x)\,dx + [2x]_0^2$

$= 3\int_0^2 f(x)\,dx + 4$

2 C $\sqrt{(ax-b)^3} = (ax-b)^{\frac{3}{2}}$

$\int (ax-b)^{\frac{3}{2}}\,dx = \dfrac{1}{a \times \frac{5}{2}}(ax-b)^{\frac{5}{2}}$

$= \dfrac{2}{5a}(ax-b)^{\frac{5}{2}}$

3 B The area of the shaded region from $x = 0$ to $x = 2$ is given by $\int_0^2 f(x) - g(x)\,dx$.
The area of the shaded region from $x = 2$ to $x = 5$ is given by $\int_2^5 g(x) - f(x)\,dx$.
The area of the shaded region is $\int_0^2 f(x) - g(x)\,dx + \int_2^5 g(x) - f(x)\,dx$.

4 B $\int_a^b c\,dx = [cx]_a^b$

$= cb - ca$

5 A $\dfrac{dy}{dx} = \dfrac{ax}{2} + 1$

$y = \dfrac{ax^2}{4} + x + c$

$x = 0, y = 1$ so $c = 1$

$y = \dfrac{ax^2}{4} + x + 1$

6 D $f'(x) = -6\sin 3x$

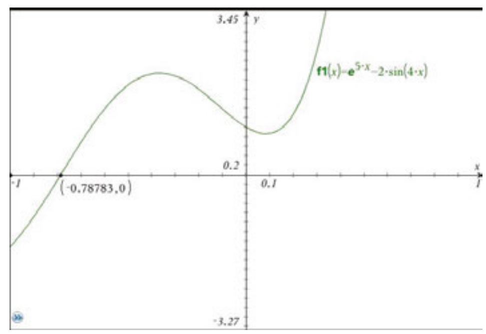
$f(x) = 2\cos 3x + c$

$f\left(\dfrac{2\pi}{3}\right) = 3$

$c = 3 - 2\cos 2\pi = 1$

$f(x) = 2\cos 3x + 1$

7 E Use a CAS calculator to plot the curve on the interval $(-1, 1)$. It shows one x intercept at $x = -0.78783$.



Hence the required area is given by

$A = \int_{-1}^{-0.78783} 2\sin 4x - e^{5x}\,dx$

$+ \int_{-0.78783}^{1} e^{5x} - 2\sin 4x\,dx$

$= 30.02$ to 2 dp

where the integrals have been evaluated using a CAS calculator.

8 C $\dfrac{dy}{dx} = ae^{-x} + 2$

$x = 0, \dfrac{dy}{dx} = 5,$ so $a + 2 = 5,$ i.e. $a = 3.$

$y = -3e^{-x} + 2x + c$

$x = 0, y = 1,$ so $-3 + c = 1,$ i.e. $c = 4.$

$y = -3e^{-x} + 2x + 4$

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If $x = 2,\ y = -3e^{-2} + 4 + 4 = -\dfrac{3}{e^2} + 8$.

9 C $R(t) = 5e^{-0.1t}$ litres/minute.
Since $R(t)$ is the **rate** of flow, it is equal to $\dfrac{dV}{dt}$ where V L is the volume of water at time t. Thus the outflow in the first 3 minutes is given by

$$\begin{aligned}\int_0^3 5e^{-0.1t}\,dt &= -\frac{5}{0.1}\left[e^{-0.1t}\right]_0^3\\ &= -50(e^{-0.3} - 1)\\ &= 12.959\ldots\end{aligned}$$

To the nearest litre, this is 13 1itres.

10 D By symmetry, the shaded regions have equal area, so the total area is given by

$$\begin{aligned}2\int_{\pi-a}^{\pi} \sin x\,dx &= 2[-\cos x]_{\pi-a}^{\pi}\\ &= 2\,(-\cos\pi + \cos(\pi - a))\\ &= 2(1 - \cos a)\end{aligned}$$

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Solutions to extended-response questions

1 a For $y = x - \dfrac{1}{x^2} = x - x^{-2}$

$\dfrac{dy}{dx} = 1 + 2x^{-3}$

When $x = 2$, $\dfrac{dy}{dx} = 1 + \dfrac{2}{23} = 1\dfrac{1}{4}$

When $x = 2, y = 2 - \dfrac{1}{4} = \dfrac{7}{4}$

The equation of the tangent is

$$y - \frac{7}{4} = \frac{5}{4}(x - 2)$$

$\therefore 4y - 7 = 5x - 10$

and $4y - 5x = -3$

b When $y = 0,\ x = \dfrac{3}{5}$ The coordinates are $\left(\dfrac{3}{5}, 0\right)$

c When $y = 0,\ x - \dfrac{1}{x^2} = 0$

implies $x^3 - 1 = 0$

i.e. $x = 1$ The coordinates are $(1, 0)$

d Required area = Area of triangle ATC – Area of should region

$$= \frac{1}{2}\left(2 - \frac{3}{5}\right) \times \frac{7}{4} - \int_1^2 x - x^{-2}\,dx$$

$$= \frac{1}{2} \times \frac{7}{5} \times \frac{7}{4} - \left[\frac{x^2}{2} + x^{-1}\right]_1^2$$

$$= \frac{49}{40} - \left(\left(2 + \frac{1}{2}\right) - \left(\frac{1}{2} + 1\right)\right)$$

$$= \frac{49}{40} - 1$$

$$= \frac{9}{40}$$

e The required ratio $= \dfrac{9}{40} : \dfrac{49}{40} = 9 : 94$

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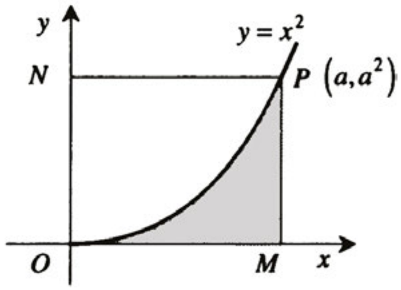
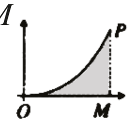
2 a



Let M, have coordinates $(a,\ 0)$

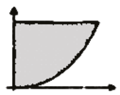
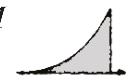
Area of OPM 

$$= \int_0^a x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{a^3}{3}$$

The coordinates of P are $(a,\ a^2)$

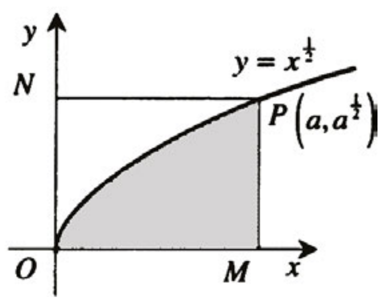
Area of OPN  = area of rectangle $OMPN$ − area of OPM 

$$= a \times a^2 - \frac{a^3}{3}$$

$$= \frac{2a^3}{3}$$

$\therefore$ The ratio of the areas $= \dfrac{2a^3}{3} : \dfrac{a^3}{3} = 2 : 1$

b



Let M have coordinates $(a,\ 0)$.

Area shaded $= \displaystyle\int_0^a x^{\frac{1}{2}}\,dx$

$$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^a$$

$$= \frac{2a^{\frac{3}{2}}}{3}$$

Area of rectangle $OMPN = a \times a^{\frac{1}{2}} = a^{\frac{3}{2}}$

$\therefore$ shaded area $= \dfrac{2}{3}$ of the area of rectangle $OMPN$.

c Let M have coordinates $(a,\ 0)$. Then the coordinates of P are $(a,\ a^n)$ Area of region enclosed by PM, the x-axis and the curve

$$= \int_0^a x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^a = \frac{a^{n+1}}{n+1}$$

Area of rectangle $OMPN = a \times a^n = a^{n+1}$

$\therefore$ Area of described region $= \dfrac{1}{n+1}$ (area of rectangle)

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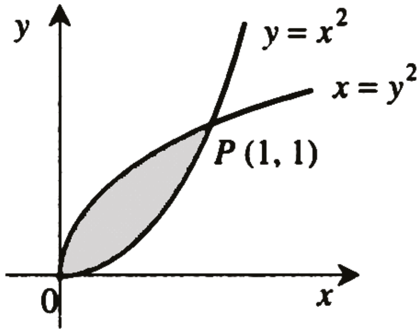
3 **a**



The parabolas intersect at (1, 1)

$$\therefore \text{ the area } = \int_0^1 x^{\frac{1}{2}} - x^2 \, dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$= \tfrac{2}{3} - \tfrac{1}{3}$$

$$= \tfrac{1}{3}$$

The area is $\dfrac{1}{3}$ square units.

b For $y = x^n$ and $y^n = x$

$x^{\frac{1}{n}} = x^n$

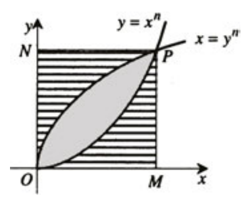
which implies $1 = x^{n-\frac{1}{n}}$

i.e. $1 = x^{\frac{n^2-1}{n}}$

$\therefore x = (1)^{\frac{n}{n^2-1}} = \pm 1$

If n is even, $x = 1$ is the only solution. All such pairs of curves intersect at (1, 1).

c The coordinates of P are (1, 1)



$$\text{Area } = \int_0^1 x^{\frac{1}{n}} - x^n \, dx$$

$$= \left[\frac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1} - \frac{x^{n+1}}{n+1}\right]_0^1$$

$$= \frac{1}{\frac{1}{n}+1} - \frac{1}{n+1}$$

$$= \frac{n}{n+1} - \frac{1}{n+1}$$

$$= \frac{n-1}{n+1} \text{ square units}$$

d Area with shading $= 1 - \left(\dfrac{n-1}{n+1}\right)$

$$= \frac{n+1-n+1}{n+1}$$

$$= \frac{2}{n+1} \text{ square units}$$

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e For $n = 10$, Area $= \dfrac{10-1}{10+1} = \dfrac{9}{11}$ square units

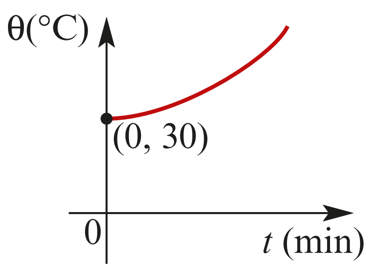
For $n = 100$, Area $= \dfrac{100-1}{100+1} = \dfrac{99}{101}$ square units

For $n = 1000$, Area $= \dfrac{1000-1}{1000+1} = \dfrac{999}{1001}$ square units

f For $\dfrac{n-1}{n+1} = 1 + \dfrac{2}{n-1}$, as $n \to \infty$, $\dfrac{n-1}{n+1} \to 1$

4 a
$$\frac{d\theta}{dt} = e^{2.6t}$$
$$\therefore \theta = \frac{1}{2.6}e^{2.6t} + c$$
$$= \frac{5}{13}e^{2.6t} + c$$
when $t = 0,\ \theta = 30$
$$\therefore c = 30 - \frac{5}{13} = \frac{385}{13}$$
$$\text{and } \theta = \frac{5}{13}e^{2.6t} + \frac{385}{13}$$
when $t = 3$
$$\theta = \frac{5}{13}e^{2.6\times 3} + \frac{385}{13} = 968.3$$
The temperature is 968.3°C after 3 minutes.

b



c When $\theta = 500$
$$500 = \frac{5}{13}e^{2.6t} + \frac{385}{13}$$
$$\therefore \quad \frac{6115}{13} \times \frac{13}{5} = e^{2.6t}$$
$$1223 = e^{2.6t}$$
$$\therefore \quad t = \frac{5}{13}\log_e(1223)$$
$$\approx 2.734$$

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The temperature is 500° after 2.734 minutes.

d The average rate of change for interval [1, 2]

$$= \frac{\theta(2) - \theta(1)}{2 - 1}$$

$$= \frac{5}{13}e^{5.2} + \frac{385}{13} - \left(\frac{5}{13}e^{2.6} + \frac{385}{13}\right)$$

$$= \frac{5}{13}\left(e^{5.2} - e^{2.6}\right)$$

$$\approx 64.5$$

The average rate of change for the interval [1, 2] is 64.5° per minute.

5 $\frac{dx}{dt} = ve^{-t}$, where $v = 5 \times 10^4$ m/s

a When $t = 0$, $\frac{dx}{dt} = 5 \times 10^4$ m/s

b $\frac{dx}{dt} = \frac{v}{e^t} = \frac{5 \times 10^4}{e^t}$

as $t \to \infty$, $\frac{dx}{dt} \to 0$

c The distance travelled between $t = 0$ and $t = 20$

$$= \int_0^{20} 5 \times 10^4 e^{-t}\, dt$$

$$= \left[-5 \times 10^4 e^{-t}\right]_0^{20}$$

$$= -5 \times 10^4 \times e^{-20} + 5 \times 10^4$$

$$= 5 \times 10^4(1 - e^{-20})\text{metres}$$

d $\frac{dx}{dt} = ve^{-t}$

$\therefore x = -ve^{-t} + c$

When $t = 0, x = 0$

$\therefore = -v + c$

i.e. $c = v$

$\therefore x = v - ve^{-t}$

$= v(1 - e^{-t})$

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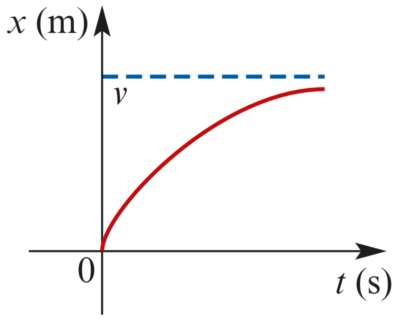
e 

6 **a** Let $y = e^{-3x} \sin 2x$

then, using the produce rule,

$\dfrac{dy}{dx} = -3e^{-3x} \sin 2x + 2e^{-3x} \cos 2x$ ①

For $y = e^{-3x} \cos 2x$

$\dfrac{dy}{dx} = 3e^{-3x} \cos 2x - 2e^{-3x} \sin 2x$ ②

b From ①

$\int\left(-3e^{-3x} \sin 2x + 2e^{-3x} \cos 2x\right) dx = e^{-3x} \sin 2x + c_1$

i.e. $-3 \int e^{-3x} \sin 2x dx + 2 \int e^{-3x} \cos 2x dx = e^{-3x} \sin 2x + c_1$③

From ②

$\int(-3e^{-3x} \cos 2x - 2e^{-3x} \sin 2x)\, dx = e^{-3x} \cos 2x + c_2$

i.e. $-3 \int e^{-3x} \cos 2x - 2 \int e^{-3x} \sin 2x dx = e^{-3x} \cos 2x + c_2$④

c Let $a = \int e^{-3x} \sin 2x dx$ and $b = \int e^{-3x} \cos 2x dx$

Then the equations can be rewritten as

$-3a + 2b = e^{-3x} \sin 2x + c_1$

$-3b - 2a = e^{-3x} \cos 2x + c_2$

Multiply ⑤ by 3 and ⑥ by 2 and add:

$-9a - 4a = 3e^{-3x} \sin 2x + 2e^{-3x} \cos 2x + (3c_1 + 2c_2)$

$\therefore -13 \int e^{-3x} \sin 2x dx = 3e^{-3x} \sin 2x + 2e^{-3x} \cos 2x + (3c_1 + 2c_2)$

i.e. $\int(e^{-3x} \sin 2x)\, dx = -\dfrac{1}{13}(3e^{-3x} \sin 2x + 2e^{-3x} \cos 2x) + C$

where $C = \dfrac{3c_1 + 3c_2}{-13}$

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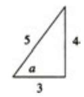
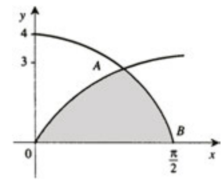
7 a





i $3\sin a = 4\cos a$

$\therefore \dfrac{3\sin a}{\cos a} = 4$

$\tan a = \dfrac{4}{3}$

ii Consider the triangle

Then $\sin(a) = \dfrac{4}{5}$ and $\cos(a) = \dfrac{3}{5}$

b Area of the shaded region $= \int_0^a 3\sin\ dx + \int_a^{\frac{\pi}{4}} 4\cos x dx$

$= [-3\cos x]_0^a + [4\sin x]_a^{\frac{\pi}{4}}$

$= -3\cos a - (-3) + 4\sin\dfrac{\pi}{2} - 4\sin a$

$= -3 \times \dfrac{3}{5} + 3 + 4 - 4 \times \dfrac{4}{5}$

$= -\dfrac{9}{5} + 7 - \dfrac{16}{5}$

$= -5 + 7$

$= 2$

Area of the shaded region = 2 square units.

8 a $y = x\log_e x$

Using the product rule gives

$\dfrac{dy}{dx} = \log_e x + x \times \dfrac{1}{x}$

$= \log_e x + 1$

Also $\int(\log_e x + 1)\,dx = x\log_e x + c$

$\therefore \int \log_e x dx + x = x\log_e x + c$

and $\int_1^e \log e x dx = [x\log\ ex - x]_1^e$

$= e\log_e e - e - (e\log_e 1 - 1)$

$= e - e - 0 + 1$

$\therefore \int_1^e \log_e x dx = 1$

b $y = x(\log_e x)^n$

Using the product rule:

$\dfrac{dy}{dx} = (\log_e x)^n + x \times \dfrac{1}{x} \times n(\log_e x)^{n-1}$

$= (\log_e x)^n + n(\log_e x)^{n-1}$

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c $I_n = \int_1^e (\log_e x)^n \, dx$, *and* $I_{n-1} = \int_1^e (\log_e x)^{n-1} \, dx$

From ⓣ

$\int (\log_e x)^n + n(\log_e x)^{n-1} \, dx = x(\log_e x)^n + c$

$\therefore I_n + nI_{n-1} = \left[x(\log_e x)^n \right]_1^e$

$= e$

d $I_3 = \int_1^3 (\log_e x)^3 \, dx$

From (c) $\quad I_3 = e - 3I_2$

$$= e - 3[e - 2I_1]$$

$$= e - 3e + 6I_1$$

$$= -2e + 6 \quad \text{by (a)}$$

9 To find the point of intersection, consider $x^2 = by$ and $y^2 = ax$

as a simultaneous pair.

$\therefore y = \dfrac{x^2}{b}$ and $\left(\dfrac{x^2}{b}\right)^2 = ax$

which implies $x^3 = ab^2$

and $x = a^{\frac{1}{3}} b^{\frac{2}{3}}$ Substitute for x in

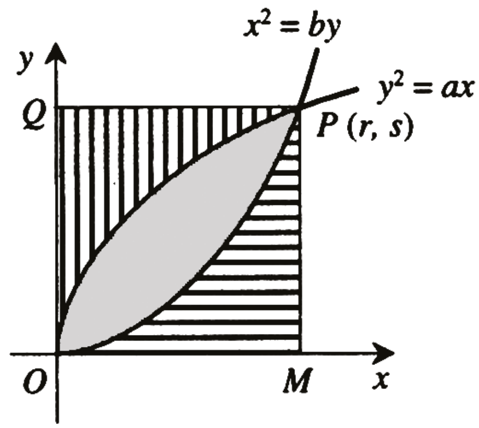
$y^2 = ax$

$$\therefore y^2 = a\left(a^{\frac{1}{3}} b^{\frac{2}{3}}\right)$$

$$= a^{\frac{1}{3}} b^{\frac{2}{3}}$$

$$\therefore y = a^{\frac{2}{3}} b^{\frac{1}{3}}$$

$$\therefore r = a^{\frac{1}{3}} b^{\frac{2}{3}} \text{ and } s = a^{\frac{2}{3}} b^{\frac{1}{3}}$$



The area of the region shaded horizontally

$$= \int_0^{a^{\frac{1}{3}} b^{\frac{2}{3}}} \frac{x^2}{b} \, dx$$

$$= \frac{1}{b} \left[\frac{x^3}{3} \right]_0^{a^{\frac{1}{3}} b^{\frac{1}{3}}}$$

$$= \frac{1}{3b} \times a \times b^2$$

$$= \frac{ab}{3}$$

The area of the region shaded vertically

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$$= \int_0^{a^{\frac{2}{3}}b^{\frac{1}{3}}} \frac{y^2}{a}\,dy$$

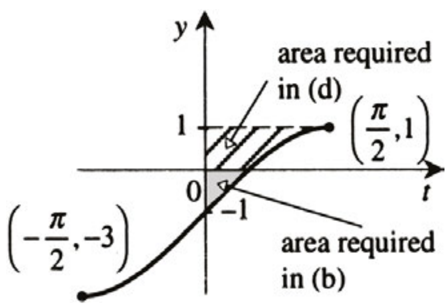
$$= \left[\frac{y^3}{3a}\right]_0^{a^{\frac{2}{3}}b^{\frac{1}{3}}}$$

$$= \frac{ab}{3}$$

The area of rectangle $OMPQ = a^{\frac{2}{3}}b^{\frac{1}{3}} \times a^{\frac{1}{3}}b^{\frac{2}{3}} = ab$

$\therefore$ All three regions have area $\dfrac{ab}{3}$

10 a



b $\displaystyle\int_0^{\frac{\pi}{6}} 2\sin x - 1\,dx = [-2\cos x - x]_0^{\frac{\pi}{6}}$

$$= -2\cos\frac{\pi}{6} - \frac{\pi}{6} - (-2\cos 0 - 0)$$

$$= -2 \times \frac{\sqrt{3}}{2} - \frac{\pi}{6} + 2$$

$$= 2 - \sqrt{3} - \frac{\pi}{6}$$

c For the inverse of $f(x) = 2\sin x - 1$

consider $\quad x = 2\sin y - 1$

$\dfrac{(x+1)}{2} = \sin y$

and $y = \sin^{-1}\left(\dfrac{x+1}{2}\right)$

i.e. $f^{-1}(x) = \sin^{-1}\left(\dfrac{x+1}{2}\right)$

The domain of f^{-1} = range of $f = [-3, 1]$

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d $\displaystyle\int_0^1 f^{-1}(x)\,dx = \text{ area of rectangle } - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} f(x)\,dx$

$$= \frac{\pi}{2} - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\sin x - 1)\,dx$$

$$= \frac{\pi}{2} - [-2\cos x - x]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + [2\cos x + x]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + \left[0 + \frac{\pi}{2} - \left(2\cos\frac{\pi}{6} + \frac{\pi}{6}\right)\right]$$

$$= \frac{\pi}{2} + \frac{\pi}{2} - \sqrt{3} - \frac{\pi}{6}$$

$$= \frac{5\pi}{6} - \sqrt{3}$$

The 'integral' command of a CAS could be used in this question.

11 a For $y = e^{\frac{x}{10}}(10 - x)$

$$\frac{dy}{dx} = \tfrac{1}{10}e^{\frac{x}{10}}(10 - x) - e^{\frac{x}{10}}$$

$$= e^{\frac{x}{10}}\left(1 - \frac{x}{10} - 1\right)$$

$$= -\frac{x}{10}e^{\frac{x}{10}}$$

For $y = \sqrt{100 - x^2} = \left(100 - x^2\right)^{\frac{1}{2}}$

$$\frac{dy}{dx} = -x(100 - x^2)^{-\frac{1}{2}}$$

$$= \frac{-x}{(100 - x^2)^{\frac{1}{2}}}$$

b When $x = 0$, $\dfrac{dy}{dx} = 0$ for both functions

c When $x = 10$, $\dfrac{dy}{dx} = -e$

d $\int_0^{10} e^{\frac{x}{10}}(10 - x)\,dx = 71.828\ldots$

$\int_0^{10} \sqrt{10^2 - x^2}\;dx = 78.5398\ldots$

$\therefore$ area between the curves = 6.7118 square units

e percentage error $= \dfrac{6.7118}{25\pi} \times 100 = 8.55\%$

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f Equation of the chord is $y = 10 - x$

Area of the shaded region $= 25\pi - 50 = 28.54$ square units

g **i**
$$\frac{d}{dx}\left(e^{\frac{x}{10}}(10-x)\right) = \tfrac{1}{10}e^{\frac{x}{10}}(10-x) - e^{\frac{x}{10}}$$

$$\therefore \tfrac{1}{10}\int_0^{10} e^{\frac{x}{10}}(10-x)\,dx = \left[e^{\frac{x}{10}}(10-x)\right]_0^{10} + \int_0^{10} e^{\frac{x}{10}}\,dx$$
$$= -(10) + \left[10e^{\frac{x}{10}}\right]_0^{10}$$
$$= -10 + 10e - 10$$
$$= 10e - 20$$

$$\therefore \int_0^{10} e^{\frac{x}{10}}(10-x)\,dx = 10(10e - 20)$$

ii $\therefore$ exact area of shaded region $= 25\pi - 100e + 200$ square units

12 $R(t) = 10e^{-(\frac{t}{10})}\sin\left(\frac{\pi t}{3}\right)$

a **i** $R(0) = 0$

ii $R(3) = 10e^{-\frac{3}{10}}\sin\pi = 0$

b $R'(\text{t}) = -e^{-\frac{t}{10}}\sin\left(\frac{\pi t}{3}\right) + \frac{10\pi}{3}e^{-\frac{t}{10}}\cos\left(\frac{\pi t}{3}\right)$

$= e^{-\frac{t}{10}}\left[\frac{10\pi}{3}\cos\left(\frac{\pi t}{3}\right) - \sin\left(\frac{\pi t}{3}\right)\right]$

c **i** $R'(t) = 0$ implies

$\frac{10\pi}{3}\cos\left(\frac{\pi t}{3}\right) = \sin\left(\frac{\pi t}{3}\right)$ as $e^{-\frac{t}{10}} \neq 0$

$\therefore \tan\left(\frac{\pi t}{3}\right) = \frac{10\pi}{3}$

$\therefore \frac{\pi t}{3} = \tan^{-1}\left(\frac{10\pi}{3}\right)$ or $\pi + \tan^{-1}\left(\frac{10\pi}{3}\right)$ or $2\pi + \tan^{-1}\left(\frac{10\pi}{3}\right)$ or $3\pi + \tan^{-1}\left(\frac{10\pi}{3}\right)$

$\therefore t = \frac{3}{\pi}\tan^{-1}\left(\frac{10\pi}{3}\right)$ or $3 + \frac{3}{\pi}\tan^{-1}\left(\frac{10\pi}{3}\right)$ or $6 + \frac{3}{\pi}\tan^{-1}\left(\frac{10\pi}{3}\right)$ or $9 + \tan^{-1}\left(\frac{10\pi}{3}\right)$

ii stationary points $(1.409, 8.646)$ and $(7.409, 4.745)$ loc max $(4.409, -6.405)$ and $(10.409, -3.515)$ loc min

d $R(t) = 0$ implies $\sin\left(\frac{\pi t}{3}\right) = 0$ as $10 \therefore e^{-\frac{t}{10}} \neq 0$

$\therefore \frac{\pi t}{3} = 0$ or π or 2π or $3\,\pi$ or 4π

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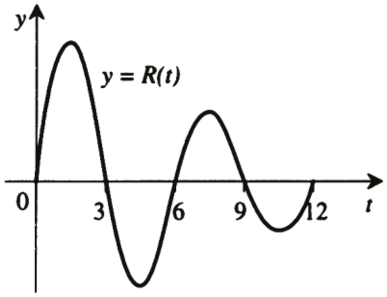
$t = 0$ or 3 or 6 or 9 or 12

e



f **i** Use a CAS calculator to find areas

$\int_0^3 R(t)\,dt \approx 16.47337 \quad \therefore \quad$ 16.47 litres flowed in

ii $\int_3^6 R(t)\,dt \approx -12.20377 \quad \therefore \quad$ 12.20 litres flowed out

iii Total amount of water in the device

$= 16.47337\ldots - 12.20377\ldots + 4$

$= 4.2695 + 4$

$= 8.2695$

There are approximately 8.27 litres in the device

g $\int_0^{30} R(t)\,dt \approx 8.9918\ldots$

$\therefore$ There are $4 + 8.9918 = 12.992$ litres in the device after 30 minutes. (Use a CAS calculator with this problem.)

13 **a** If $\cos 2x = 2\cos^2 x - 1$ and $\cos 2x = 1 - 2\sin^2 x$

$$\frac{1-\cos 2x}{1+\cos 2x} = \frac{1-(1-2\sin^2 x)}{1+(2\cos^2 x - 1)}$$

$$= \frac{2\sin^2 x}{2\cos^2 x}$$

$$= \tan^2 x$$

$$= \sec^2 x - 1$$

b $$\int_0^{\frac{\pi}{4}} \frac{1-\cos 2x}{1+\cos 2x}\,dx = \int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\,dx$$

$$= \Big[\tan x - x\Big]_0^{\frac{\pi}{4}}$$

$$= \tan\frac{\pi}{4} - \frac{\pi}{4} = 1 - \frac{\pi}{4}$$

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Chapter 12 – Revision of Chapters 9–11

Solutions to Technology-free questions

1 $y = \dfrac{x^2 - 1}{x^4 - 1}$

a $\dfrac{dy}{dx} = \dfrac{(x^4 - 1)\dfrac{d(x^2 - 1)}{dx} - (x^2 - 1)\dfrac{d(x^4 - 1)}{dx}}{(x^4 - 1)^2}$

$= \dfrac{2x(x^4 - 1) - 4x^3(x^2 - 1)}{(x^4 - 1)^2}$

$= \dfrac{2x^5 - 2x - 4x^5 + 4x^3}{(x^4 - 1)^2}$

$= \dfrac{-2(x^5 - 2x^3 + x)}{(x^4 - 1)^2}$

$= \dfrac{-2x(x^2 - 1)^2}{(x^4 - 1)^2}$

b $\dfrac{dy}{dx} = 0,$

$0 = \dfrac{-2x(x^2 - 1)^2}{(x^4 - 1)^2}$

looking at the numerator

$x = 0,\ x^2 = 1$

$x = -1, 0, 1$

looking at the denominator

$x^4 \neq 1$

$x \neq \pm 1$

$\therefore \quad x = 0$

in set notation, $\{0\}$

2 $y = (3x^2 - 4x)^4$

Let $u = 3x^2 - 4x$.

Then $y = u^4$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{dy}{dx} = 4u^3 \times (6x - 4)$

$= 8(3x - 2)(3x^2 - 4x)^3$

3 $f(x) = x^2 \log_e(2x)$

$f'(x) = 2x \log_e(2x) + x^2 \times \dfrac{1}{x}$

$= 2x \log_e(2x) + x$

4 a $f(x) = e^{2x+1}$

$f'(x) = 2e^{2x+1}$

$f'(b) = 2e^{2b+1}$

The tangent is at the point (b, e^{2b+1})

Gradient of line from the point to the origin is $\dfrac{e^{2b+1}}{b}$

$\therefore \dfrac{e^{2b+1}}{b} = 2e^{2b+1}$

$\therefore b = \dfrac{1}{2}$

b $f(b) = e^{2b+1} + k$

$f'(b) = 2e^{2b+1}$

$\therefore \dfrac{e^{2b+1} + k}{b} = 2e^{2b+1}$

$\therefore e^{2b+1} + k = 2be^{2b+1}$

$\therefore k = (2b - 1)e^{2b+1}$

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5 $y = x^{\frac{1}{3}} + c$

$\dfrac{dy}{dx} = \dfrac{1}{3}x^{-\frac{2}{3}}$

When $x = 8, y = a$

$\therefore a = 2 + c \ldots (1)$

When $x = 8, \dfrac{dy}{dx} = \dfrac{1}{12}$

$\therefore m = \dfrac{1}{12}$

When $x = 8, a = 8m - 8$

$\therefore a = \dfrac{2}{3} - 8 = -\dfrac{22}{3}$

Substitute in (1)

$-\dfrac{22}{3} = 2 + c$

$c = -\dfrac{28}{3}$

6 Average value$= \dfrac{1}{2}\displaystyle\int_0^2 \dfrac{1}{3x+1}, dx$

$= \dfrac{1}{6}\Big[\log_e(3x+1)\Big]_0^2$

$= \dfrac{1}{6}\log_e 7$

7 a $\displaystyle\int \dfrac{3}{5x-2}\,dx = \dfrac{3}{5}\log_e(5x-2) + c$

b $\displaystyle\int \dfrac{3}{(5x-2)^2}\,dx = \dfrac{3}{10-25x}$

8 a $g(x) = 3x^2 - 5f(x)$

$g'(x) = 6x - 5f'(x)$

$g'(3) = 6 \times 3 - 5f'(3x)$

$= 18 - 5 \times 5$

$g'(3) = -7$

b

$g(x) = \dfrac{3x+1}{f(x)}$

$g'(x) = \dfrac{f(x)\dfrac{d(3x+1)}{dx} - f(x)(3x+1)}{(f(x))^2}$

$= \dfrac{3f(x) - (3x+1)f'(x)}{(f(x))^2}$

$g'(3) = \dfrac{3f(3) - (9+1)f'(3)}{(f(3))^2}$

$= \dfrac{3 \times -2 - 10 \times 5}{(-2)^2}$

$= \dfrac{-6-50}{4}$

$g'(3) = -14$

c $g(x) = [f(x)]^2$

$g'(x) = 2f(x)f'(x)$

$\therefore g'(3) = 2 \times (-2) \times 5$

$= -20$

9 a $g(x) = \sqrt{x}f(x)$

$g'(x) = \sqrt{x}f'(x) + \dfrac{f(x)}{2f\sqrt{x}}$

$g'(4) = \sqrt{4}f'(4) + \dfrac{f(4)}{2\sqrt{4}}$

$= 2 \times 2 + \dfrac{6}{4}$

$g'(4) = \dfrac{11}{2} = 5\dfrac{1}{2}$

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b $g(x) = \dfrac{f(x)}{x}$

$$g'(x) = \frac{xf(x) - f(x)\dfrac{dx}{dx}}{x^2}$$

$$g'(x) = \frac{xf'(x) - f(x)}{x^2}$$

$$g'(4) = \frac{4f'(4) - f(4)}{4^2}$$

$$= \frac{4 \times 2 - 6}{16}$$

$$g'(4) = \frac{1}{8}$$

10 $f(x) = f(g(x))$

$f'(x) = \sqrt{3x+4},\ g(x) = x^2 - 1$

$\therefore\ g'(x) = 2x$

$$f'(x) = f'(g(x)) \times g'(x)$$

$$= \sqrt{3(x^2-1)+4} \times 2x$$

$$= \sqrt{3(x^2-3)+4} \times 2x$$

$$f'(x) = 2x\sqrt{3x^2+1}$$

11 $f(x) = 2x^2 - 3x + 5$

a $f'(x) = 4x - 3$

b $f'(0) = -3$

c $f'(x) = 1$
$4x - 3 = 1$
$4x = 4$
$x = 1$
in set notation$\{1\}$

12 $\dfrac{d}{dx}\left(\log_e 3f(x)\right) = \dfrac{f'(x)}{f(x)}$

$$y = \sqrt{a-x} = (a-x)^{\frac{1}{2}}$$

Let $u = a - x$.

Then$y = u^{\frac{1}{2}}$

13 $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}}(-1)$$

$$= -\frac{1}{2\sqrt{a-x}}$$

When $x = 1$, $\dfrac{dy}{dx} = -6$

$$\therefore, -\frac{1}{2\sqrt{a-1}} = -6$$

$$1 = 12\sqrt{a-1}$$

$$\frac{1}{144} = a - 1$$

$$a = \frac{145}{144}$$

14 Area of region A $= \displaystyle\int_0^1 -x^2 - x + 2\,dx$

$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_0^1$$

$$= \frac{7}{6}$$

Area of region B $=- \displaystyle\int_1^m -x^2 - x + 2\,dx$

$$= -\left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_1^m$$

$$= \frac{m^3}{3} + \frac{m^2}{2} - 2m + \frac{7}{6}$$

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$$\text{Area A} = \text{Area B}$$

$$\frac{m^3}{3} + \frac{m^2}{2} - 2m = 0$$

$$2m^3 + 3m^2 - 12m = 0$$

$$m(2m^2 + 3m - 12) = 0$$

$$m = 0 \text{ or } m = \frac{-3 \pm \sqrt{9 + 96}}{4}$$

$$m = 0 \text{ or } m = \frac{-3 \pm \sqrt{105}}{4}$$

$$\text{But } m > 1, \therefore m = \frac{-3 + \sqrt{105}}{4}$$

$$f(x) = x^3 + 3x^2 - 4$$

$$f'(x) = 3x^2 + 6x$$

15 a $f'(x) = 0 \Rightarrow 3x(x + 2) = 0$

$$\therefore x = 0 \text{ or } x = -2$$

$$f(0) = -4, f(-2) = 0$$

b $\displaystyle\int_{-2}^{2} f(x)\,dx = \left[\frac{x^4}{4} + x^3 - 4x\right]_{-2}^{2} = 0$

c $\displaystyle\int_{0}^{2} f(x)\,dx = \left[\frac{x^4}{4} + x^3 - 4x\right]_{0}^{2} = 4$

d Area $\displaystyle = -\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$

$$= \frac{19}{2}$$

16 $f(x) = \dfrac{1}{3x - 1} = (3x - 1)^{-1}$

$$f'(x) = \frac{d(3x - 1)}{dx} \times -(3x - 1)^{-2}$$

$$= \frac{-3}{(3x - 1)^2}$$

$$f'(2) = \frac{-3}{(6 - 1)^2}$$

$$f'(2) = \frac{-3}{25}$$

17 $y = 1 - x^2$

$$\frac{dy}{dx} = -2x$$

$$LHS = x\frac{dy}{dx} + 2$$

$$= x \times -2x + 2$$

$$= 2 - 2x^2$$

$$= 2(1 - x^2)$$

$$= 2y = RHS \ \ QED$$

18 $A = 4\pi r^2$

$$\frac{dA}{dr} = 8\pi r$$

$$r = 3, \frac{dA}{dr} = 8\pi \times 3$$

$$= 24\pi$$

19 $y = 1.8x^2$

$$\frac{dy}{dx} = 3.6x$$

$$\frac{dy}{dx} = 1, \ x = \frac{1}{3.6}$$

$$x = \frac{10}{36}$$

$$x = \frac{10}{36}$$

$$y = \frac{18}{10} \times \left(\frac{10}{36}\right)^2$$

$$= \frac{18 \times 10}{(36)^2}$$

$$= \frac{10}{2 \times 36}$$

$$= \frac{10}{72}$$

$$co\text{-}ords = \left(\frac{10}{36}, \frac{10}{72}\right)$$

$$\approx (0.28, 0.14)$$

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20 $$y = 3x^2 - 4x + 7$$
$$\frac{dy}{dx} = 6x - 4$$
$$\frac{dy}{dx} = 0,$$
$$6x - 4 = 0$$
$$x = \frac{2}{3}$$

21 $$y = \frac{x^2 + 2}{x^2 - 2}$$
$$y = \frac{x^2 - 2 + 4}{x^2 - 2}$$
$$y = 1 + \frac{4}{x^2 - 2}$$
$$y = 1 + 4(x^2 - 2)^{-1}$$
$$\frac{dy}{dx} = 2x \times -4(x^2 - 2)^{-2}$$
$$= \frac{-8x}{(x^2 - 2)^2}$$

22 $$z = 3y + 4,$$
$$y = 2x - 1$$
$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$
$$= 3 \times 2$$
$$= 6$$

23 $$y = (5 - 7x)^9$$
$$\frac{dy}{dx} = -7 \times 9(5 - 7x)^8$$
$$= -63(5 - 7x)^8$$

24 $$y = 3x^{\frac{1}{3}}$$
$$\frac{dy}{dx} = \frac{3}{3} = x^{-\frac{2}{3}}$$
$$= x^{-\frac{2}{3}}$$
$$x = 27,$$
$$\frac{dy}{dx} = (27)^{-\frac{2}{3}}$$
$$= (3)^{-2}$$
$$= \frac{1}{9}$$

25 $$y = \sqrt{5 + x^2}$$
$$\frac{dy}{dx} = 2x \times \frac{1}{2\sqrt{5 + x^2}}$$
$$= \frac{x}{\sqrt{5 + x^2}}$$
$$x = 2,$$
$$\frac{dy}{dx} = \frac{2}{\sqrt{5 + 4}}$$
$$= \frac{2}{3}$$

26 $$y = (x^2 + 3)(2 - 4x - 5x^2)$$
$$\frac{dy}{dx} = (x^2 + 3)(-4 - 10x) + (2 - 4x - 5x^2)(2x)$$
$$x = 1,$$
$$\frac{dy}{dx} = (1 + 3)(-4 - 10) + (2 - 4 - 5)(2)$$
$$= 4 \times -14 + (-7) \times 2$$
$$= 5 \times -14$$
$$= -70$$

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27 $y = \dfrac{x}{1+x^2}$

$$\frac{dy}{dx} = \frac{(1+x^2)\dfrac{dx}{dx} - x\dfrac{d(1+x^2)}{2dx}}{(1+x^2)^2}$$

$$= \frac{(1+x^2) - x \times 2x}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

$x = 1,$

$$\frac{dy}{dx} = \frac{1-1}{(1+1)^2} = 0$$

28

$$y = \frac{2+x}{x^2+x+1}$$

$$\frac{dy}{dx} = \frac{x^2+x+1\dfrac{d(2+x)}{dx} - (2+x)\dfrac{d(x^2+x+1)}{dx}}{(x^2+x+1)^2}$$

$$= \frac{(x^2+x+1) - (2+x)(2x+1)}{(x^2+x+1)^2}$$

$$= \frac{x^2+x+1-2x^2-5x-2}{(x^2+x+1)^2}$$

$$= \frac{-x^2-4x-1}{(x^2+x+1)^2}$$

$x = 0,$

$$\frac{dy}{dx} = \frac{-1}{1}$$

$$= -1$$

29 $f(x) = \dfrac{1}{2x+1}$

a

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\dfrac{1}{2x+2h+1} - \dfrac{1}{2x+1}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{h} \times \left(\frac{2x+1-(2x+2h+1)}{(2x+1)(2x+2h+1)}\right)\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{h} \times \left(\frac{-2h}{4x^2+4xh+2x+2x+2h+1}\right)\right)$$

$$= \lim_{h\to 0}\left(\frac{-2}{4x^2+4xh+4x+2h+1}\right)$$

$$= \frac{-2}{4x^2+4x+1}$$

$$= \frac{-2}{(2x+1)^2}$$

b $f'(0) = \dfrac{-2}{(1)^2}$

$= -2$

30 $f(x) = x^3 + 3x^2 - 1$

a $f'(x) = 3x^2 + 6x$

$f'(x) = 0,$

$3x^2 + 6x = 0$

$x(x+2) = 0$

$x = 0, -2$

$\textit{in set notation}, \{-2, 0\}$

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b $f'(x) > 0$

$3x^2 + 6x > 0$

$x(x+2) > 0$

$x < -2,\ x > 0$

in set notation, $R\backslash[-2, 0]$

c $f'(x) < 0$

$3x^2 + 6x < 0$

$x(x+2) < 0$

$-2 < x < 0$

in set notation, $(-2, 0)$

31 $y = \dfrac{x}{1-x}$

a
$$\begin{aligned} y &= \frac{x-1+1}{1-x} \\ y &= \frac{-(1-x)}{1-x} + \frac{1}{1-x} \\ &= -1 - \frac{1}{1-x} \\ &= -1 - (x-1)^{-1} \\ \frac{dy}{dx} &= -1 - (x-1)^{-2} \\ &= \frac{1}{(x-1)^2} \end{aligned}$$

b
$$\begin{aligned} y &= \frac{x}{1-x} \\ (1-x)y &= x \\ y - yx &= x \\ x(1+y) &= y \\ x &= \frac{y}{1+y} \\ x &= \frac{y+1}{1+y} - \frac{1}{1+y} \\ x &= 1 - \frac{1}{1+y} \\ &= 1 - (y+1)^{-1} \\ \frac{dx}{dy} &= -1 \times -(y+1)^{-2} \\ &= (y+1)^{-2} \\ \frac{dy}{dx} &= \left(\frac{dx}{dy}\right)^{-1} = ((y+1)^{-2})^{-1} \\ \frac{dy}{dx} &= (y+1)^2 \end{aligned}$$

32
$$\begin{aligned} y &= (x^2+1)^{-\frac{3}{2}} \\ \frac{dy}{dx} &= 2x \times \frac{-3}{2}(x^2+1)^{-\frac{5}{2}} \\ &= -3x(x^2+1)^{-\frac{5}{2}} \end{aligned}$$

33
$$\begin{aligned} y &= x^4 \\ \frac{dy}{dx} &= 4x^3 \\ LHS &= x \times \frac{dy}{dx} \\ &= x \times 4x^3 \\ &= 4x^4 \\ &= 4y \\ &= RHS\ \ QED \end{aligned}$$

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34 $f'(x) = 10x^4 > 0$ for all $x \neq 0$

$f(b) = 2b^5 > f(0) = 0$ for all $b > 0$

$f(b) = 2b^5 < f(0) = 0$ for all $b < 0$

35 a $\displaystyle\int_0^{\frac{\pi}{2}} 2\sin\left(\frac{x}{2}\right) dx$

$= \left[-4\cos\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}}$

$= 4 - 2\sqrt{2}$

b $\displaystyle\int_0^{\frac{3}{2}} e^{\frac{x}{2}}\, dx$

$= \left[2e^{\frac{x}{2}}\right]_0^{\frac{3}{2}}$

$= 2(e^{\frac{3}{4}} - 1)$

c $\displaystyle\int_{\frac{1}{2}}^{1} \frac{1}{2x}\, dx$

$= \left[\frac{1}{2}\log_e(x)\right]_{\frac{1}{2}}^{1}$

$= \frac{1}{2}\log_e 2$

d $\displaystyle\int_{-1}^{-\frac{1}{2}} \frac{1}{2x}\, dx$

$= -\displaystyle\int_{\frac{1}{2}}^{1} \frac{1}{2x}\, dx$

$= -\left[\frac{1}{2}\log_e(x)\right]_{\frac{1}{2}}^{1}$

$= -\frac{1}{2}\log_e 2$

e $\displaystyle\int_3^4 \frac{1}{2(x-2)^2}\, dx$

$= \frac{1}{4}$

f $\displaystyle\int_2^4 \frac{1}{(3x-2)^2}\, dx$

$= \frac{1}{20}$

36 $f'(x) = -6x^2 < 0$ for all $x \neq 0$

$f(b) = -6b^3 < f(0) = 0$ for all $b > 0$

$f(b) = -6b^3 > f(0) = 0$ for all $b < 0$

37 a $f(x) = e^{-mx+2} + 4x$

$f'(x) = -me^{-mx+2} + 4$

$f'(x) = 0 \Rightarrow e^{-mx+2} = \frac{4}{m}$

$\Rightarrow -mx + 2 = \log_e \frac{4}{m}$

$\Rightarrow x = \frac{1}{m}\left(2 - \log_e \frac{4}{m}\right)$

b $\frac{1}{m}\left(2 - \log_e \frac{4}{m}\right) < 0$

$\Leftrightarrow 2 - \log_e \frac{4}{m} < 0$

$\Leftrightarrow \log_e \frac{4}{m} > 2$

$\Leftrightarrow \frac{4}{m} > e^2$

$\Leftrightarrow m < 4e^{-2}$

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Solutions to multiple-choice questions

1 E $y = \dfrac{x^4 + x}{x^2}$

$= x^2 + x^{-1}$

$\dfrac{dy}{dx} = 2x - x^{-2}$

$= 2x - \dfrac{1}{x^2}$

2 D $f(x) = x^5 + x^3 + x$

$f'(x) = 5x^4 + 3x^2 + 1$

$f'(1) = 5 + 3 + 1 = 9$

3 E $y = (4 - 9x^4)^{\frac{1}{2}}$

$\dfrac{dy}{dx} = \dfrac{1}{2}(4 - 9x^4)^{-\frac{1}{2}} \times (-36x^3)$

$= -18x^3(4 - 9x^4)^{-\frac{1}{2}}$

4 D $y = \sin 2x + 1$

$\dfrac{dy}{dx} = 2\cos 2x$

$x = 0 \colon \dfrac{dy}{dx} = 2\cos 0 = 2$

5 B $\dfrac{dy}{dx} = (e^{x^2+1}) = e^{x^2+1} \times (2x)$

$= 2xe^{x^2+1}$

6 C $\dfrac{dy}{dx} = \left(\dfrac{1}{1+x}\right) = \dfrac{1}{dx}(1+x)^{-1}$

$= -1(1+x)^{-2}$

$= -\dfrac{1}{(1+x)^2}$

7 D grad $PQ = \dfrac{(2+h)^3 - 2^3}{(2+h) - 2}$

$= \dfrac{(2+h)^3 - 8}{h}$

8 A $\dfrac{[f(x+h) - f(x)]}{h} = \dfrac{\dfrac{3}{x+h} - \dfrac{3}{x}}{h}$

$= \dfrac{3x - 3(x+h)}{hx(x+h)}$

$= \dfrac{-3h}{hx(x+h)}$

$= -\dfrac{3}{x(x+h)}$

9 D $y = ce^{2x}$

$\dfrac{dy}{dx} = 2ce^{2x}$

$= 2c$ (when$x = 0$)

$= 11$

$\Rightarrow c = 5.5$

10 A The graph shows two local maximum and local minimum; there are no stationary point of inflexion. Here, $f'(x) = 0$ at 3 points

11 E $f(x) = 4 - e^{-2x}$

$f'(x) = 2e^{-2x}$

The graph is a decaying exponential with the x-axis as a horizontal asymptote.

Only the last graph fits.

12 B $y = bx^2 - cx = x(bx - c)$

$y = 0$ if $x = 0$ or $x = \dfrac{c}{b}$.

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Since the graph crosses the x-axis at $(4, 0), \frac{c}{b} = 4$, i.e. $c = 4b$ ①

$$\frac{dy}{dx} = 2bx - c$$

$$= 1 \; at \; (4, 0)$$

So $8b - c = 1, i.e. \; c = 8b - 1$ ②

② − ① gives $0 = 4b - 1$

$$b = \frac{1}{4}$$

$$c = 4 \times \frac{1}{4}.$$

13 B The gradient of the gives graph is zero at some negative value of x and again at $x = 0$.
The gradient goes from positive to negative to positive through these two stationary points.
Only the second graph fits.

14 E $y = (3x^4 - 2)^4$

$$\frac{dy}{dx} = 4(3x^4 - 2)^3 \times 12x^3$$

$$= 48x^3(3x^4 - 2)^3$$

15 D Since the derivations are equal, the functions differ by at most of constant.

So $g(x) = f(x) + c$

$$= 3x^2 + 2 + c$$

$$g(2) = 29$$

$$12 + 2 + c = 29$$

$$c = 15$$

$$g(x) = 3x^2 + 2 + 15$$

$$= 3x^2 + 17$$

(Alternatively, $g(x) = 3x^2 + k$ for some constant k, so only options A, B and D are posible. Substitute x = 2 in each to see which gives 29.)

16 E $f'(x) = ke^{kx} - ke^{-kx}$

$$= ke^{-kx}(e^{2kx} - 1)$$

Case(1): $k > 0$

Then $f'(x) > 0$ provided $e^{2kx} - 1 > 0$

i.e $e^{2kx} - 1$

$2kx < 0$ (Since $e^0 = 1$)

$x > 0$ (Since $k > 0$)

Case(2): $k < 0$

Then $f(x) > 0$ provided $e^{2kx} - 1 < 0$

i.e $e^{2kx} < 1$

$2kx < 0$

$x > 0$ (Since $k > 0$)

In either case, $f'(x) > 0$ for $x > 0$

17 B $y = 3x^2 + 2x - \frac{4}{x^2}$

$$= 3x^2 + 2x - 4x^{-2}$$

$$\frac{dy}{dx} = 6x + 2 + 8x^{-3}$$

$$= 6x + 2 + \frac{8}{x^3}$$

18 C Average rate $= \frac{V(4) - V(2)}{4 - 2}$

$$= \frac{45 - 15}{2}$$

$$= 15 \text{ m}^3/\text{min}$$

19

E Gradient $PQ = \frac{f(x + h) - f(x)}{(x + h) - (x)}$

$$= \frac{[(x + h)^2 - 2(x + h) + 1] - [x^2 - 2x + 1]}{h}$$

$$= \frac{x^2 + 2xh + h^2 - 2x - 2h + 1 - x^2 + 2x - 1}{h}$$

$$= \frac{2xh + h^2 - 2h}{h}$$

$$= 2x - 2 + 2h$$

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20 D The gradient of the gives graph is zero at a point in $(-3, -1)$ and again at a point in $(0, 2)$. The gradient goes from position to negative to positive through these two points.
Only the fourth graph fits.

21 B The gradient function is negative for $x < -3$ and also for $x > 2$. It is zero or positive for $-3 \leq x \leq 2$, so the function is increasing for $-3 \leq x \leq 2$.

22 E The gradient of a tangent to $y = f(x)$ at $x = a$ is given by
$\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$
If a=2, this becomes
$\lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$

23 B

$$\left(\frac{e^{2x} + e^{-2x}}{e^x}\right) = e^x + e^{-3x}$$

$$\frac{d}{dx}\left(\frac{e^{2x} + e^{-2x}}{e^x}\right) = \frac{d}{dx}(e^x + e^{-3x})$$

$$= e^x - 3e^{-3x}$$

24 D The graph of $y = -x^2 + 4x + 3$ is an invested Problem will a maximum tuning point.

$$\frac{dy}{dx} = -2x + 4$$

$$= 0 \text{ if } x = 2$$

The $y = -4 + 8 + 3 = 7$, which is the required maximum value.

25 B

$$y = 1 + e^{2x}$$

$$\frac{dy}{dx} = 2e^{2x}$$

$$= 2 \text{ when } x = 0$$

Equation of tangent is given by

$$y - 2 = 2(x - 0)$$

$$y = 2x + 2$$

26 A

$$y = x^2 - x^3$$

$$\frac{dy}{dx} = 2x - 3x^2$$

$$= x(2 - 3x)$$

$$= 0 \text{ if } x = 0, \frac{2}{3}$$

There are stationary point where $x = 0$ and $x = \dfrac{2}{3}$

27 D

$$\frac{d}{dx}\left(\frac{4x^2 + 6}{x}\right) = \frac{d}{dx}(4x + 6x^{-1})$$

$$= 4 - 6x^{-2}$$

$$= \frac{4x^2 - 6}{x^2}$$

28 D

$$f(x) = 4x^3 - 3x + 7 - 2x^{-1}$$

$$f'(x) = 12x^2 - 3 + 2x^{-2}$$

$$f'(1) = 12 - 3 + 2 = 11$$

29 A

$$f'(x) = x^2 + \frac{1}{x}$$

$$f(x) = \frac{1}{3}x^3 + \log_e x + c$$

(Since the condition has $x > 0$)

$$f(1) = \frac{1}{3} + \log_e 1 + c$$

$$= \frac{1}{3} + c$$

$$= \frac{1}{3} \text{ if } c = 0$$

$$f(x) = \frac{1}{3}x^3 + \log_e x$$

30 C

$$\int_2^3 f(x)dx = [F(x)]_2^3$$

$$= F(3) - F(2)$$

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31 C Area $= \int_{\frac{\pi}{2}}^{\pi} \sin x\, dx - \int_{\pi}^{\frac{3\pi}{2}} \sin x\, dx$

$= \int_{\frac{\pi}{2}}^{\pi} \sin x\, dx + \int_{\frac{3\pi}{2}}^{\pi} \sin x\, dx$

32 D The Straight line crosses the x-axis at $x = -1$

Area $= \int_{-1}^{2} x + 1 dx - \int_{-2}^{-1} x + 1 dx$

33 B $\frac{dy}{dx} = \frac{1}{x^2}$

$= x^{-2}$

$y = -x^{-1} + c$

$= -\frac{1}{x} + c$

$y = 2$ where $x = 1$

$2 = -1 + c$

$c = 3$

$y = -\frac{1}{x} + 3$

34 A $\int_0^{36} \frac{dx}{2x+9} = \left[\frac{1}{2}\log_e(2x+9)\right]_0^{36}$

$= \frac{1}{2}\log_e 81 - \frac{1}{2}\log_e 9$

$= \frac{1}{2}\log_e 9$

$= \frac{1}{2}\log_e 3^2$

$= \log_e 3$

So $k = 3$

35 E Area $= -\int_{-3}^{0} f(x)dx + \int_0^4 f(x)dx$

Which is not the same as any a of the first four options.

36 D $\int x^2 - \frac{1}{x^2} + \sin x\, dx$

$= \int x^2 - x^{-2} + \sin x\, dx$

$= \frac{1}{3}x^3 + x^{-1} - \cos x + c$

$= \frac{1}{3}x^3 + \frac{1}{x} - \cos x + c$

37 A Area $= \int_0^2 \frac{1}{3-x} dx$

$= [-\log_e(3-x)]_0^2$

$= -\log_e 1 + \log_e 3$

$= \log_e 3$

38

A $\int_a^b \sin 2x\, dx = \left[-\frac{1}{2}\cos 2x\right]_a^b$

$= -\frac{1}{2}\cos 2b + -\frac{1}{2}\cos 2a$

$= \frac{1}{2}(\cos 2a - \cos 2b)$

For A: Substituting $b = \frac{3\pi}{4}$ and $a = \frac{\pi}{4}$ give,

$\frac{1}{2}\left(\cos\frac{\pi}{2} - \cos\frac{3\pi}{2}\right) = \frac{1}{2}(0 - 0) = 0$ as required.

(Checking each other option shows that none of these gives zero.)

39 A Since the shaded region is below the x-axis, its area is given by $\int_0^a -f(x)dx$.

40 B $\int x^2 - \frac{1}{x}dx = \frac{1}{3}x^3 - \log_e x$

(taking $x > 0$ and letting the constant of integration be 0 since *an* antiderivative is requested.)

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41 D $f'(x) = 3x^2 + 6x - 9$

$= 3(x^2 + 2x - 3)$

$= 3(x + 3)(x - 1)$

> 0 if $x < -3$ or $x > 1$

So the function is increasing if $x < -3$ or $x > 1$

42 B $f(x) = \dfrac{\sin x}{x}$

$f'(x) = \dfrac{(x)(\cos x) - (\sin x)(1)}{x^2}$

$= \dfrac{x\cos x - \sin x}{x^2}$

43 B $y = \log_e(\cos 2x)$

$\dfrac{dy}{dx} = \dfrac{1}{\cos 2x}x(-2\sin 2x)$

$= \dfrac{2\sin 2x}{\cos 2x}$

44 C $f'(x) = \sin 2x$

$f(x) = -\dfrac{1}{2}\cos 2x + c$

$f(0) = 3$, so

$3 = -\dfrac{1}{2}\cos 0 + c$

$= -\dfrac{1}{2} + c$

$c = 3\dfrac{1}{2}$

$f(x) = -\dfrac{1}{2}\cos 2x + 3\dfrac{1}{2}$

45 D $y = 4e^{3x} - x$

$\dfrac{dy}{dx} = 12e^{3x} - 1$

$= 12 - 1 = 11$ at $x = 0$

Equation of tangent at (0,4) is

$y - 4 = 11(x - 0)$

$y = 11x + 4$

46 B $f'(x) = 3x^2 - 2x - 1$

$= (3x + 1)(x + 1)$

$= 0$ if $x = -\dfrac{1}{3}, 1$

The gradient is positive if $x < -\dfrac{1}{3}$ and negative if $-\dfrac{1}{3} < x < 1$.

So there is a local maximum at $x = -\dfrac{1}{3}$.

The gradient is negative if $-\dfrac{1}{3} < x < 1$ and positive if $x > 1$.

So there is a local minimum at $x = 1$. $f(1) = 1 - 1 - 1 + 2 = 1$ there is a local minimum at (1, 1).

47 D $\dfrac{d}{dx}\left(\dfrac{x-1}{\sqrt{x}}\right) = \dfrac{d}{dx}\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)$

$= \dfrac{1}{2}x^{-\frac{1}{2}} + \dfrac{1}{2}x^{-\frac{3}{2}}$

$= \dfrac{1}{2\sqrt{x}} + \dfrac{1}{2x\sqrt{x}}$

$= \dfrac{x+1}{2x\sqrt{x}}$

48 C $\dfrac{d}{dx}(e^{\cos x}) = e^{\cos x}x(-\sin x)$

$= -\sin x e^{\cos x}$

49 A Area $= \displaystyle\int_0^{\frac{\pi}{2}} \cos x\,dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\,dx$

$= [\sin x]_0^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$

$= (1 + 0) - (-1 - 1)$

$= 3$ square units

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50 C $x + y = 1$ so $y = 1 - x$

$$p = x^2 + xy - y^2$$
$$= x^2 + x(1 - x) - (1 - x)^2$$
$$= x^2 + x - x^2 - 1 + 2x - x^2$$
$$= -x^2 + 3x - 1$$
$$\frac{dP}{dx} = -2x + 3$$
$$= 0 \text{ if } x = \frac{3}{2}$$

and this corresponds to a maximum since the graph of P against x is an inverted parabola.

51 B

$$y = e^{-\cos x}$$
$$\frac{dy}{dx} = e^{-\cos x} \times \sin x$$
$$= \sin x e^{-\cos x}$$

where $x = \frac{\pi}{3}, \frac{dy}{dx} = \sin\frac{\pi}{3} e^{-\cos\frac{\pi}{3}}$

$$= \frac{\sqrt{3}}{2} e^{-\frac{1}{2}}$$

and this is the gradient of the tangent at $x = \frac{\pi}{3}$.

Using $m_1 m_2 = -1$

$$m_2 = -1 \div \left(\frac{\sqrt{3}}{2} e^{-\frac{1}{2}}\right)$$
$$= -\frac{2}{\sqrt{3}} e^{\frac{1}{2}}$$
$$= \frac{-2e^{\frac{1}{2}}}{\sqrt{3}}$$

52

E $\int_0^{\frac{\pi}{2}} (\cos x + \sin x)dx = [\sin x - \cos x]_0^{\frac{\pi}{2}}$

$$= (1 - 0) - (0 - 1)$$
$$= 2$$

53 C Since (1, 3) is a maximum point on the graph, $f'(1) = 0$.

The equation of the tangent is

$$y - 3 = 0\,(x - 1)$$
$$y = 3$$

54 C $y = 4 - x^2$

$$\frac{dy}{dx} = -2x$$
$$= -2 \text{ when } x = 1$$

Equation of the tangent is

$y - 3 = -2(x - 1)$

$y = -2x + 5$

55 D $P = -x^2 + 6x + 4$

$$\frac{dP}{dx} = -2x + 6$$
$$= 0 \text{ if } x = 3$$

and this corresponds to a maximum since the graph of P against x is an inverted parabola.

When $x = 3, P = -9 + 18 + 4$

$$= 13$$

The maximum value is 13.

56 B Stationary points $f'(x) = x^3 - x^2 - 1$ occur when $f'(x) = 0$.

$$f'(x) = 3x^3 - 2x$$
$$= x(3x - 2)$$
$$= 0 \text{ if } x = 0, \frac{2}{3}$$

57 D $f'(2) = \lim_{h\to 0} \frac{f(2 + h) - f(2)}{h}$

$$= \lim_{h\to 0} h^2 + 6h + 12$$
$$= 12$$

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58 C $f(x) = \log_e 3x$

$$f'(x) = \frac{1}{3x} \times 3$$

$$f'(1) = 1$$

59 B $y = x^2 e^x$

$$\frac{dy}{dx} = x^2 e^x + 2xe^x$$

$$= xe^x(x + 2)$$

$$= 0 \text{ if } x = 0, 2.$$

Note that $y \geq 0$ and when $x = 0$, $y = 0$, So the minimum value is 0.

60 E $f(x) = a\sin(3x)$

$$f'(x) = 3a\cos(3x)$$

$$f'(\pi)3a\cos 3\pi$$

$$= -3a$$

$$= 2 \text{ if } a = -\frac{2}{3}$$

61 B $\int \frac{1}{(2x-5)^{\frac{5}{2}}} dx$

$$= \int (2x-5)^{\frac{5}{2}} dx$$

$$= \frac{1}{2 \times \left(-\frac{3}{2}\right)}(2x-5)^{-\frac{3}{2}}$$

$$= -\frac{1}{3} \times \frac{1}{(2x-5)^{-\frac{3}{2}}}$$

$$= \frac{1}{3(2x-5)^{-\frac{3}{2}}}$$

Where the constant of integration is taken as 0 since an antiderivative is required.

62 D From the graph, there is a stationary point of inflexion at x = −3 and a minimum stationary point at $x = \frac{9}{4}$. A quick check of each option shows that the fourth is not true, as there are two stationary point on the graph. (Checking the other options shows each is true.)

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Solutions to extended-response questions

1 **a** $S = 50 + 30e^{-\frac{1}{3}t}$

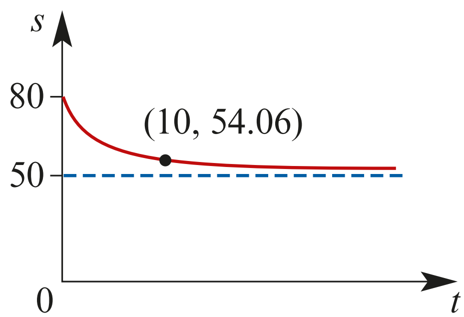
When $t = 10$

$s = 50 + 30e^{-2}$

≈ 54.06

There are 54.06 g of salt in the mixture after 10 minutes.

b



c $\dfrac{ds}{dt} = -6e^{\frac{1}{5}t}$

d Considering $s = 50 + 30e^{-\frac{1}{5}t}$

Solve for $e^{-\frac{1}{5}t}$

$\therefore \dfrac{s - 50}{30} = e^{-\frac{1}{5}t}$

Substitute in $\dfrac{ds}{dt} = -6e^{-\frac{1}{5}t}$

to yield $\dfrac{ds}{dt} = -6\left(\dfrac{S - 50}{30}\right)$

$= \dfrac{1}{5}(50 - s)$

e When $t = 0$

$s = 50 + 30e^{0}$

$= 80$

The volume of water is 100 litres.

$\therefore$ Concentration $= \dfrac{80}{100} = 0.8$g/litre

f Concentration $= \dfrac{s}{100}$

$= \dfrac{50 + 30e^{-\frac{1}{5}t}}{100}$

Concentration = 0.51 gram/litre

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implies $0.51 = \dfrac{50 + 30e^{-\frac{1}{5}t}}{100}$

$\therefore\ 51 = 50 + 30e^{-\frac{1}{5}t}$

$\dfrac{1}{30} = e^{-\frac{1}{5}t}$

and therefore

$t = -5\log_e \dfrac{1}{30}$

$= 5\log_e 30$

≈ 17.006

The concentration first reaches 0.51 g/litre after about 17 seconds.

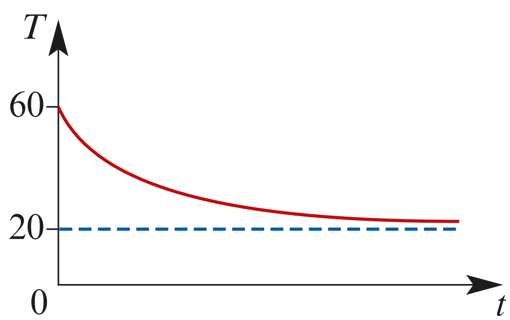
2 a $T = 40e^{-0.36t} + 20,\ t \geq 0$

When $t = 0$

$T = 60$

The initial temperature of the body was 60° C.

b



c $\dfrac{dT}{dt} = -14.4e^{-0.36t}$

d Since $T = 40e^{-0.36t} + 20$

$e^{-0.36t} = \dfrac{T - 20}{40}$

Hence $\dfrac{dT}{dt} = -14.4\left(\dfrac{T - 20}{40}\right) = -0.36(T - 20)$

3 a $f(t) = 1000e^{-0.5t}$

$f(0) = 1000$

Initially there were 1000 F-type spores.

50% of the initial number is 500.

Consider

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$500 = 1000e^{-0.5t}$

$0.5 = e^{-0.5t}$

$\therefore\ t = -2\log_e(0.5)$

$= 2\log_e 2$

≈ 1.386

It takes about 1.386 minutes to kill half of the F-type spores.

b $f(0) = 1000$ and $g(1000) = 1200$

Initially there are 1000 F-type spores and 1200 G-type spores, so there are 2200 line spores of both types.

For $t = 5$

$f(5) = 1000e^{-0.5\times 5}$ and $g(5) = 1200e^{-0.7\times 5}$

$= 1000e^{-2.5}$ $= 1200e^{-3.5}$

Percentage of spores still alive after 5 minutes $= \dfrac{f(5)+g(5)}{f(0)+g(0)} \times \dfrac{100}{1}$

$= \dfrac{1000e^{-2.5} + 1200e^{-3.5}}{2200} \times \dfrac{100}{1}$

$= \dfrac{1000\, e^{-2.5} + 1200\, e^{-3.5}}{22}$

≈ 5.378

$\therefore$ Percentage of spores still alive after 5 minutes is 5.378. %

c Total no. of spores $= 1000e^{-0.5t} + 1200e^{-0.7t}$

i.e. $T = 1000e^{-0.5t} + 1200e^{-0.7t}$ where T is the total number of spores

$$\frac{dT}{dt} = -500e^{-0.5} - 840e^{-0.7t}$$

When $t = 5$

$$\frac{dT}{dt} = -500e^{-0.5\times 5} - 840e^{0.7\times 5}$$

$$= -500e^{-2.5} - 840e^{-3.5}$$

$$\approx -66.408$$

When $t = 5$, the rate at which the total number of spores is decreasing is 66.4 spores per minute.

d Live F-type spores = live G-type spores

When $f(t) = g(t)$

i.e. $1000e^{-0.5t} = 1200e^{-0.7t}$

which implies $e^{0.2t} = \dfrac{1200}{1000}$

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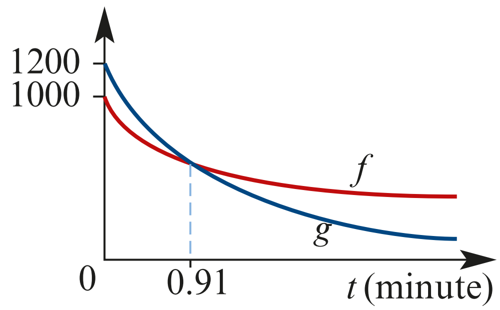
and $t = \frac{1}{0.2} \log_e(\frac{6}{5})$

$= 5 \log_e$

≈ 0.9116

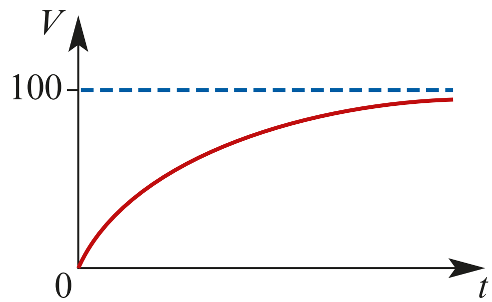
$\therefore$ the number of live F-type spores = the number of live G-type spores when $t = 0.9116$.

e number of spores



4 $V = 100(1 - e^{-0.2t})$

a



b **i** Acceleration $= \frac{dV}{dt} = 100 \times 0.2e^{-0.2t}$

$= 20e^{-0.2t} m/s^2$

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ii From $V = 100(1 - e^{-0.2t})$

$$\frac{V}{100} = 1 - e^{-0.2t}$$

$$\text{and } e^{-0.2t} = 1 - \frac{V}{100}$$

$$= \frac{100 - V}{100}$$

$$\therefore \frac{dV}{dt} = 20e^{-0.2t}$$

$$= 20\left(\frac{100 - V}{100}\right)$$

$$= \frac{1}{5}(100 - V)\text{m/s}^2$$

c when $V = 80$

$$80 = 100(1 - e^{-0.2t})$$

$$0.8 = 1 - e^{-0.2t}$$

$$e^{-0.2t} = 0.2$$

$$\therefore -0.2t = \log_e(0.2)$$

$$\text{i.e. } t = -5\log_e(0.2)$$

$$= 5\log_e 5$$

When the velocity of the body is 80 m/s, $t = 5\log_e 5 \approx 8.05$ seconds.

5 $C = 0.05x^2 + 5x + 500$

The average cost $A = \dfrac{C}{x}$

i.e. $A = 0.05x + 5 + \dfrac{500}{x}$

$$\frac{dA}{dx} = 0.05 - \frac{500}{x^2}$$

$\dfrac{dA}{dx} = 0$ implies

$$0.05 = \frac{500}{x^2}$$

i.e. $x^2 = 10\,000$

$$x = 100$$

$\dfrac{dA}{dx} > 0$ for $x > 100$

and $\dfrac{dA}{dx} < 0$ for $x < 0$

$\therefore$ a local minimum at $x = 100$

i.e. 100 units per annum minimises the average cost per unit.

6 $T = T_0e^{-kt}$

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a $\dfrac{dT}{dt} = -kT_0e^{-kt}$

$= -kT$

$\therefore \ \dfrac{dT}{dt}$ is proportional to T

b When $t = 0,\ T = 100 - 30 = 70$

i.e. $T_0 = 70$

When $t = 20,\ T = 70 - 30 = 40$

$\therefore\ 40 = 70e^{-20k}$

$\dfrac{4}{7} = e^{-20k}$

and $-20k = \log_e\left(\dfrac{4}{7}\right)$

$k = -\dfrac{1}{20}\log_e\left(\dfrac{4}{7}\right)$

$= \dfrac{1}{20}\log_e\left(\dfrac{7}{4}\right)$

$= 0.028$ (correct to 3 decimal places)

c $\dfrac{dT}{dt} = -kT_0e^{-kt}$

$= -\dfrac{70}{20}\log_e\left(\dfrac{7}{4}\right)e^{-\frac{30}{20}\log_e\left(\frac{7}{4}\right)}$

$= -\dfrac{70}{20}\log_e\left(\dfrac{7}{4}\right)e^{\log e\left(\frac{7}{4}\right)^{\frac{3}{2}}}$

$= -\dfrac{70}{20} \times \left(\dfrac{4}{7}\right)^{\frac{3}{2}}\log_e\left(\dfrac{7}{4}\right)$

≈ -0.846

The temperature is decreasing at a rate of 0.846 degrees/minute.

7 a i $p(t) = 0.2 - 0.2e^{-\frac{t}{20}} + 0.1e^{-\frac{t}{10}} t \geq 0$

$\therefore\ p(10) = 0.2 - 0.2e^{-\frac{1}{2}} + 0.1e^{-1}$

≈ 0.1155 (correct to four decimal places)

ii As $t \to \infty$

$e^{-\frac{t}{20}} \to 0$ and $e^{-\frac{t}{10}} \to 0$

$\therefore\ p(t) \to 0.2$

The proportion approaches 0.2

b $p'(t) = 0.01e^{-\frac{t}{20}} - 0.01e^{-\frac{t}{10}}$

Let $N(t)$ be the number of new cases per day

$N(t) = kp'(t)$

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$= k\left(0.01e^{-\frac{t}{20}} - 0.01e^{-\frac{t}{10}}\right)$

To find maximum, differentiate to find $N'(t)$ and solve the equation $N'(t) = 0$

$N'(t) = \left(-0.0005e^{-\frac{t}{20}} + 0.001e^{-\frac{t}{10}}\right)k$

$N'(t) = 0$ implies

$0.0005\, e^{-\frac{t}{20}} = 0.001e^{-\frac{t}{10}}$

$\therefore\ e^{\frac{t}{20}} = 2$

$\therefore\ t = 20\log_e 2 \approx 13.86$

$N'(t) < 0$ for $t > 20\log_e 2$ and $N'(t) > 0$ for $t < 20\log_e 2$

which implies a local maximum at $t = 20\log_e 2$

The number of new cases per day is a maximum when $t = 20\log_e 2$

i.e. after 13.86 days.

8 Let $\$x$ be the rent per month from each apartment.

The number of apartments occupied $= 70 - 2\dfrac{(x - 500)}{20}$

$$= \frac{700 - x + 500}{10}$$

$$= \frac{1200 - x}{10}$$

Let $\$R$ be the total revenue

then $R = \dfrac{x(1200 - x)}{10}$

$= \dfrac{1}{10}(1200x - x^2)$

$\dfrac{dR}{dx} = 0$ implies $x = 600$

this is a maximum as R is a quadratic function of x with negative coefficient of x^2 i.e.

the price per apartment to maximise monthly revenue is \$600.

9 $V = \dfrac{5 \times 10^4}{(t + 1)^2}$

a When $t = 0,\ V = 5 \times 10^4$

i.e. the initial volume of liquid is 5×10^4 m^3

b $V = (5 \times 10^4)(t + 1)^{-2}$

$\therefore\ \dfrac{dV}{dt} = -10 \times 10^4(t + 1)^{-3}$ (chain rule)

$= -\dfrac{10^5}{(t + 1)^3}$

When $t = 1$, $\dfrac{dV}{dt} = -\dfrac{10^5}{2^3} = -12\,500$

i.e. the rate of change of the volume of liquid with respect to time is $-12\,500$ m^3/day.

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c $V(4) = \dfrac{5 \times 10^4}{5^2} = \dfrac{10^4}{5} = 2000$

$V(1) = \dfrac{5 \times 10^4}{2^2} = 12\,500$

$\therefore$ the average rate of change of V with respect to t for

the interval $[1, 4] = \dfrac{2000 - 12\,500}{4 - 1}$

$= -3500$

The average rate of change for the interval [1,4] is -3500 m^3/day.

d $\dfrac{5 \times 10^4}{(t+1)^2} < 1$

$5 \times 10^4 < (t+1)^2$

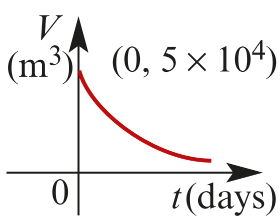
$\therefore\ t + 1 > \sqrt{5 \times 10^4};\ t > 0$

$\therefore\ t > 100\sqrt{5} - 1$

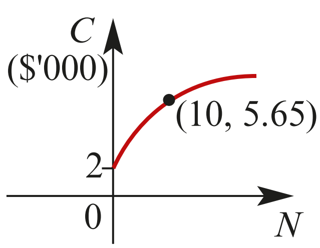
$100\sqrt{5} - 1 \approx 222.61$

$\therefore$ the amount of liquid in the pool is less than one cubic metre after $100\sqrt{5} - 1 \approx$ 222.61 days.

e



10 a



b $\dfrac{dC}{dN} = \dfrac{1}{4}.3N^2(N^3 + 16)^{-\frac{3}{4}}$ (chain rule)

$= \dfrac{3N^2}{4(N^3 + 16)^{\frac{3}{4}}}$

c The rate of change of cost in \$ 1000 s with respect to the number of bottle tops produced.

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11 Profit = Selling Price − Cost Price

Selling Price $= \dfrac{800}{p^2} \times p = \dfrac{800}{p}$

Cost Price $= \dfrac{800}{p^2} \times 2 = \dfrac{1600}{p^2}$

$\therefore$ Profit $= \dfrac{800}{p} - \dfrac{1600}{p^2}$

Let R denote profit

then $R = \dfrac{800}{p} - \dfrac{1600}{p^2}$

$$\frac{dR}{dp} = -\frac{800}{p^2} + \frac{3200}{p^3}$$

For maximum profit consider $\dfrac{dR}{dp} = 0$

$$-\frac{800}{p^2} + \frac{3200}{p^3} = 0 \; (p \neq 0)$$

$$-800p + 3200 = 0$$

$$p = 4$$

A sign diagram shows a local maximum

p	<	4	>
$\dfrac{dR}{dp}$	+	0	
sign	/	−	\

When $p = 4, R = \dfrac{800}{4} - \dfrac{1600}{4^2}$

$= 200 - 100$

$= 100$

$\therefore$ The selling price is \$4 to maximise profit and the number of items sold is $\dfrac{800}{16} = 50$

12 $y = (ax + b)^{-2}$

When $x = 0, y = \dfrac{1}{4}$

$$\frac{1}{4} = b^{-2}$$

$\therefore\ b^2 = 4$

and $b = \pm 2$

$$\frac{dy}{dx} = -2a(ax + b)^{-3}$$

When $x = 0, \dfrac{dy}{dx} = -\dfrac{3}{4}$

$\therefore\ -\dfrac{3}{4} = -2a(b)^{-3}$

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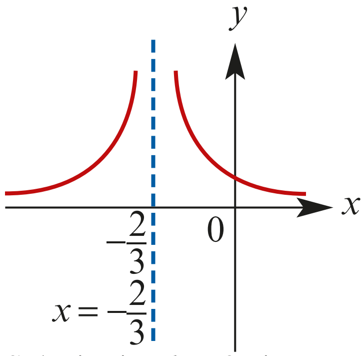

Substituting $b = 2$ gives

$-\frac{3}{4} = -2a \times \frac{1}{8}$

$\therefore \quad a = 3$

For $b = -2$

$a = 3$

$\therefore$ The possible pairs are (3, 2) and $(-3, \ -2)$

For $a = 3, \ b = 2$ (equivalently $a = -3, b = -2$)

$y = \dfrac{1}{(3x+2)^2}$

13 a Cost per hour $= 160 + \frac{1}{100}V^3$ dollars

A journey of 1000 km at 10 km/hr takes $\frac{1000}{10} = 100$ hours.

$$\begin{aligned}\therefore \text{ Cost of journey } &= (160 + \tfrac{1}{100} \times 10^3)100 \\ &= (160 + 10)100 \\ &= 17\,000\end{aligned}$$

The cost of the journey = \$17 000

b Time for a journey of 1000 km at Vkm/hr $= \frac{1000}{V}$ hours

$$\begin{aligned}\therefore \ C &= \left(160 + \frac{1}{100}V^3\right)\frac{1000}{V} \\ &= \frac{160\,000}{V} + 10V^2\end{aligned}$$

c In order to sketch the graph it is necessary to investigate stationary points.

$\frac{dC}{dV} = -\frac{160\,000}{V^2} + 20V$

$\frac{dC}{dV} = 0$ implies $20V = \frac{160\,000}{V^2}$

$\therefore \ V^3 = 8000$

i.e. $V = 20$

When $V = 20, \ C = \frac{160\,000}{20} + 10 \times 20^2$

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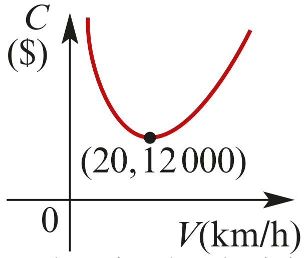
$= 12\ 000$

When $V > 20$, $\dfrac{dC}{dV} > 0$

When $0 < V < 20$, $\dfrac{dC}{dV} < 0$



$\therefore$ there is a local minimum at (20, 12 000)

For $C = \dfrac{160\ 000}{V} + 10V^2$

as $V \to \infty$, $C \to 10V^2$

as $V \to 0$, $C \to \infty$

$\therefore$ the graph is as shown here.

d From the above the most economical speed is 20 km/hr and the minimum cost is \$12 000.

e From the graph the minimum will occur when $V = 16$

$$C = \left(160 + \frac{1}{100}V^3\right)\frac{1000}{V}$$

$$= \frac{160\ 000}{V} + 10V^2$$

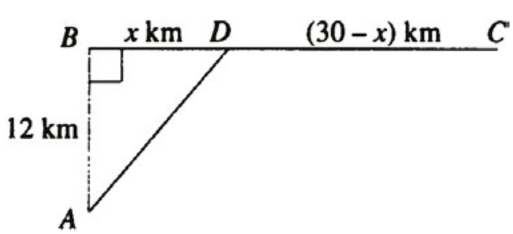
$$= \frac{160\ 000}{16} + 10 \times 16^2$$

$$= 10\ 000 + 2560$$

$$= 12\ 560$$

$\therefore$ the minimum cost is \$12 560 when the maximum speed is 16 km/hr.

14 a



Let $BD = x$, where D is the point where the camper reaches the shore.

Then $DC = 30 - x$

and by Pythagoras' Theorem

$AD = \sqrt{x^2 + 12^2}$

The camper rows at 5 km/hr. Therefore the time taken to row from

A to $D = \dfrac{\sqrt{x^2 + 144}}{5}$ hours.

The camper walks at 8 km/hr. Therefore the time taken to walk from

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D to $C = \dfrac{(30-x)}{8}$ hours.

The total time, T (hours), for the trip is given by

$$T = \frac{\sqrt{x^2+144}}{5} + \frac{30-x}{8} = \frac{(x^2+144)^{\frac{1}{2}}}{5} + \frac{30-x}{8}$$

To find the minimum time consider stationary point

$$\therefore \ \frac{dT}{dx} = \frac{2x \times \frac{1}{2}(x^2+144)^{-\frac{1}{2}}}{5} - \frac{1}{8}$$

$$= \frac{x}{5(x^2+144)^{\frac{1}{2}}} - \frac{1}{8}$$

$$\frac{dT}{dx} = 0 \text{ implies } \frac{x}{5(x^2+144)^{\frac{1}{2}}} = \frac{1}{8}$$

$$\therefore \ 8x = 5(x^2+144)^{\frac{1}{2}}$$

Squaring both sides

$$64x^2 = 25(x^2+144)$$

$$39x^2 = 25 \times 144$$

$$x^2 = \frac{25 \times 144}{39}$$

$$\therefore \ x = \frac{60}{\sqrt{39}} = \frac{60\sqrt{39}}{39} \approx 9.61$$

A gradient chart reveals a minimum

x	<	$\frac{60\sqrt{39}}{39}$	>
sign of $\frac{dT}{dx}$	−	0	+
shape	\	−	/

$\therefore$ the camper should land 9.61 km from B to minimise the time of the journey.

b If C is 24 km from B

$$T = \frac{(x^2+144)^{\frac{1}{2}}}{5} + \frac{24-x}{8}$$

$$\frac{dT}{dx} = \frac{x}{5(x^2+144)^{\frac{1}{2}}} - \frac{1}{8}$$

$\therefore$ local minimum is the same as **a**.

i.e. the camper should still row to a point 9.61 km from B.

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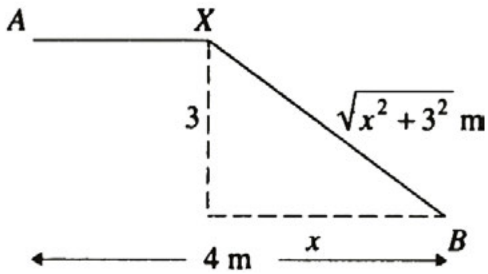
15



Let C be the cost of laying the pipe

Distance $AX = 4 - x$

The cost of laying the pipe along $AX = 10(4 - x)$ dollars

The cost of laying section $XB = 25(x^2 + 9)^{\frac{1}{2}}$

$\therefore\ C = 10(4 - x) + 25(x^2 + 9)^{\frac{1}{2}}$

To find the minimum consider $\frac{dC}{dx}$

$$\frac{dC}{dx} = -10 + 25x(x^2 + 9)^{-\frac{1}{2}}$$

$$= -10 + \frac{25x}{(x^2 + 9)^{\frac{1}{2}}}$$

$\frac{dC}{dx} = 0$ implies $10 = \frac{25x}{(x^2 + 9)^{\frac{1}{2}}}$

$\therefore\ 10(x^2 + 9)^{\frac{1}{2}} = 25x$

$2\,(x^2 + 9)^{\frac{1}{2}} = 5x$

Squaring both sides gives

$4(x^2 + 9) = 25x^2$

$4x^2 + 36 = 25x^2$

$\therefore\ 36 = 21x^2$

$\therefore\ x = \sqrt{\frac{12}{7}}$

x	<	$\sqrt{\frac{12}{7}}$	>
$\frac{dC}{dx}$	−	0	+
shape	\	−	/

A minimum occurs when $x = \sqrt{\frac{12}{7}}$

$\therefore$ Length of pipe on the surface should be $\left(4 - \sqrt{\frac{12}{7}}\right) \approx 2.7$ metres in order to minimise costs.

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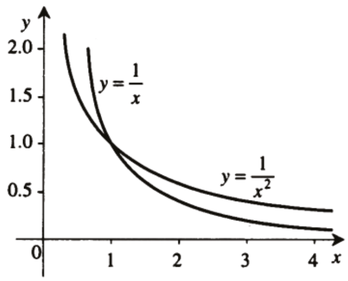
16 a



$g(x) > h(x)$

$\Leftrightarrow \frac{1}{x} > \frac{1}{x^2}$

$\Leftrightarrow x > 1$ (Multiply both sides by x^2. Note $x > 0$)

$\therefore \{x : g(x) > h(x)\} = \{x : x > 1\}$

b $g'(x) = -\frac{1}{x^2}$

$h'(x) = -\frac{2}{x^3}$

$g'(x) > h'(x)$

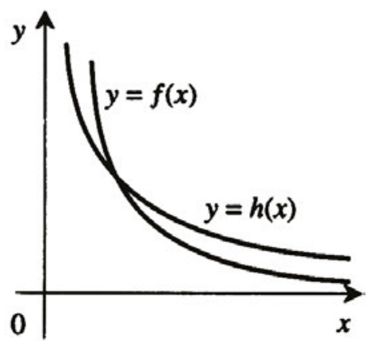
$\Leftrightarrow -\frac{1}{x^2} > -\frac{2}{x^3}$

$\Leftrightarrow -x > -2$ (Multiply both sides by x^3 : Note $x^3 > 0$)

$\therefore\ x < 2$

$\{x : g'(x) > h'(x)\} = \{x : 0 < x < 2\}$

c



$f(x) = \frac{1}{x^3}$

$h(x) = \frac{1}{x^2}$

$h(x) > f(x)$

$\Leftrightarrow \frac{1}{x^2} > \frac{1}{x^3}$

$\Leftrightarrow x > 1$ (Multiply both sides by x^3 :

Note $x^3 > 0$)

$\therefore\ \{x\colon h(x) > f(x)\} = \{x\colon x > 1\}$

$f'(x) = -\frac{3}{x^4},\ h'(x) = -\frac{2}{x^3}$

$h'(x) > f'(x)$

$\Leftrightarrow -\frac{2}{x^3} > \frac{-3}{x^4}$

(Multiply both sides by x^4: Note $x > 0$)

$-2x > -3$

$\therefore\ x < \frac{3}{2}$

$\therefore\ \left\{x\colon h'(x) > f'(x)\right\} = \left\{(x\colon 0 < x < \frac{3}{2}\right\}$

d $f_1(x) = \frac{1}{x^n},\ f_2(x) = \frac{1}{x^{n+1}}$

$f_1(x) > f_2(x)$

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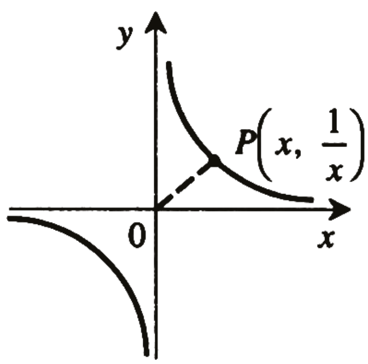
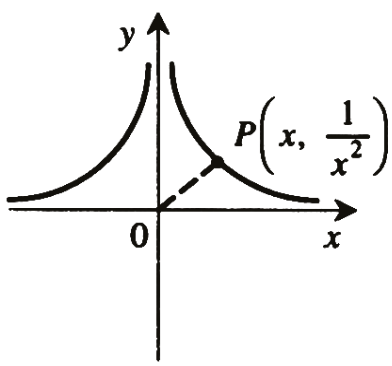
$\Leftrightarrow \frac{1}{x^n} > \frac{1}{x^{n+1}}$ (Multiply both sides by x^{n+1}: Note $x^{n+1} > 0$)
$\Leftrightarrow x > 1$
$\therefore \{x: f_1(x) > f_2(x)\} = \{x: x > 1\}$
$f_1'(x) = -\frac{n}{x^{n+1}} \quad f_2'(x) = -\frac{-(n+1)}{x^{n+2}}$
$f_1'(x) > f_2'(x)$
$\Leftrightarrow -\frac{n}{x^{n+1}} > -\frac{(n+1)}{x^{n+2}}$
(Multiplying both sides by x^{n+2} : Note $x^{n+2} > 0$)
$\therefore -nx > -(n+1)$
and $x < \frac{(n+1)}{n}$
$\therefore \{x: f_1'(x) > f_2'(x)\} = \left\{x: 0 < x < \frac{n+1}{n}\right\}$

17 a



Let $D = OP$
$D^2 = x^2 + \frac{1}{x^2}$
It is sufficient to minimise D^2 to minimise D
$\therefore \frac{d(D^2)}{dx} = 2x - \frac{2}{x^3}$
$\frac{d(D^2)}{dx} = 0$ implies $x^4 = 1$
$\therefore x = \pm 1$
A sign diagram confirms a minimum at the points $(1, 1)$ and $(-1, -1)$

b



Let $D = OP$
then $D^2 = x^2 + \frac{1}{x^4}$
$\frac{d(D^2)}{dx} = 2x - \frac{4}{x^5}$
$\frac{d(D^2)}{dx} = x$ implies $x^6 = 2$
i.e. $x = \pm\sqrt[6]{2}$
When $x = \pm\sqrt[6]{2},\ y = \frac{1}{\sqrt[3]{2}}$
As before, a sign diagram reveals minimum distance for $P\left(\sqrt[6]{2}, \frac{1}{\sqrt[3]{2}}\right)$ and $P\left(-\sqrt[6]{2}, \frac{1}{\sqrt[3]{2}}\right)$

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c Let $D = OP$

then $D^2 = x^2 + \frac{1}{x^{2n}}$

$$\frac{d(D^2)}{dx} = 2x - \frac{2n}{x^{2n+1}}$$

$$\frac{d(D^2)}{dx} = 0 \text{ implies}$$

$$2x = \frac{2n}{x^{2n+1}}$$

$\therefore$ $2x^{2n+2} = 2n$

and $x = \pm n^{\frac{1}{2n+2}}$

$\therefore$ $y = n^{-\frac{n}{2n+2}}$

18 **a** Let y m be the width of each window

$\therefore$ $6xy = 36$

$$y = \frac{6}{x}$$

The width of the wall $= 8 + 3y$

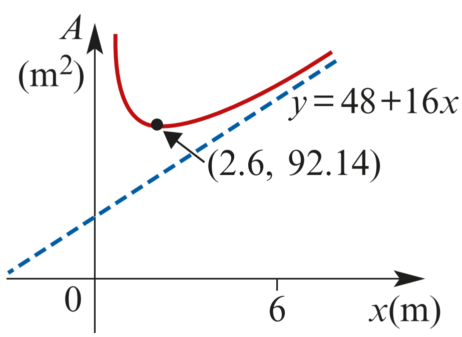
The height of the wall $= 6 + 2x$

$\therefore$ Area of the brickwork $A = (8 + 3y)(6 + 2x) - 36$

$$= \left(8 + \frac{18}{x}\right)(6 + 2x) - 36$$

$$= 48 + \frac{108}{x} + 16x$$

b



c In order to find the value of x which will give the minimum amount of brickwork, consider:

$$\frac{dA}{dx} = -\frac{108}{x^2} + 16$$

$$\frac{dA}{dx} = 0 \text{ implies } x^2 = \frac{108}{16}$$

$\therefore$ $x = \pm\frac{3\sqrt{3}}{2}$

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But $x > 0 \quad \therefore\ x = \dfrac{3\sqrt{3}}{2} \approx 2.62$

A sign diagram shows local minimum

x	$<$	$\dfrac{3\sqrt{3}}{2}$	$>$
$\dfrac{dA}{dx}$	$-$	0	$+$
shape	\	−	/

When $x = \dfrac{3\sqrt{3}}{2}$, $y = 6 \div \dfrac{3\sqrt{3}}{2} = 6 \times \dfrac{2}{3\sqrt{3}}$

$= \dfrac{4\sqrt{3}}{3} \approx 2.3$

$\therefore$ The dimensions of each window are height $\dfrac{3\sqrt{3}}{2}$ metres and width $\dfrac{4\sqrt{3}}{3}$ metres; the minimum area of brickwork is $48 + 48\sqrt{3} \approx 131.14$ metres.

d $x \geq 1$ and $y \geq 1$

implies $x \geq 1$ and $\dfrac{6}{x} \geq 1$

implies $1 \leq x \leq 6$

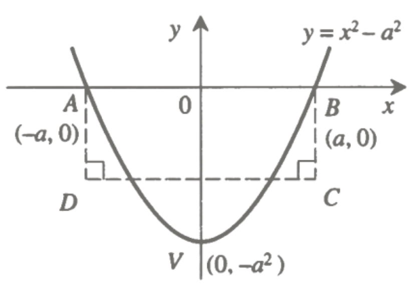
From the graph the maximum will occur at either $x = 1$ or $x = 6$

If $x = 1$, $A = 48 + 108 + 16 = 172$

If $x = 6$, $A = 48 + \dfrac{108}{6} + 16 \times 6 = 162$

$\therefore$ The maximum amount of brickwork which could be used is 172 m^2

19 a



b Area $= -\int_{-a}^{a} x^2 - a^2 dx$

$= -2\int_{0}^{a} x^2 - a^2 dx$

$= -2\left[\dfrac{x^3}{3} - a^2x\right]_0^a$

$= \dfrac{4a^3}{3}$ square units

See graph above.

Length $AB = 2a$

If the area of rectangle $ABCD$ is $\dfrac{4a^3}{3}$ square units

b $BC = \dfrac{4a^3}{3} \div 2a$

$= \dfrac{4a^3}{3} \times \dfrac{1}{2a} = \dfrac{2a^2}{3}$ units

c $\dfrac{\text{length of } BC}{\text{length of } OV} = \dfrac{2a^2}{3} \div a^2 = \dfrac{2}{3}$, a ratio of 2:3.

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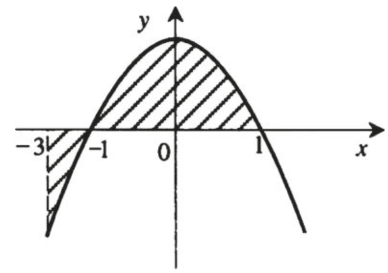
20 a $\int_{-3}^{1}(1-t^2)dt = \left[t - \frac{t^3}{3}\right]_{-3}^{1}$

$= 1 - \frac{1}{3} - \left(-3 - \frac{(-3)^3}{3}\right)$

$= \frac{2}{3} - (-3 + 9)$

$= \frac{2}{3} - (6)$

$= -5\frac{1}{3}$



b $\int_{a}^{1}(1-t^2)dt = 0$

implies $\left[t - \frac{t^3}{3}\right]_{a}^{1} = 0$

$\therefore\ 1 - \frac{1}{3} - \left(a - \frac{a^3}{3}\right) = 0$

$\frac{2}{3} - a + \frac{a^3}{3} = 0$

$\therefore\ a^3 - 3a + 2 = 0$

c From b $\int_{a}^{1}(1-t^2)\,dt = 0$ is equivalent to

$a^3 - 3a + 2 = 0$

By the factor theorem $(a-1)$ is a factor

and $a^3 - 3a + 2 = (a-1)(a^2 + a - 2) = (a-1)^2(a+2)$

$\therefore\ \int_{a}^{1}(1-t^2)\,dt = 0$ for $a = 1$ and $a = -2$

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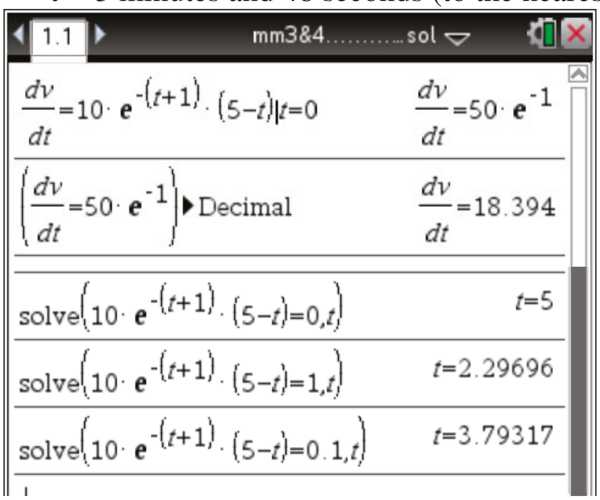
21 a i When t = 0, $\dfrac{dV}{dt} = 10e^{-1} \times 5 = 50e^{-1}$ litres/minute

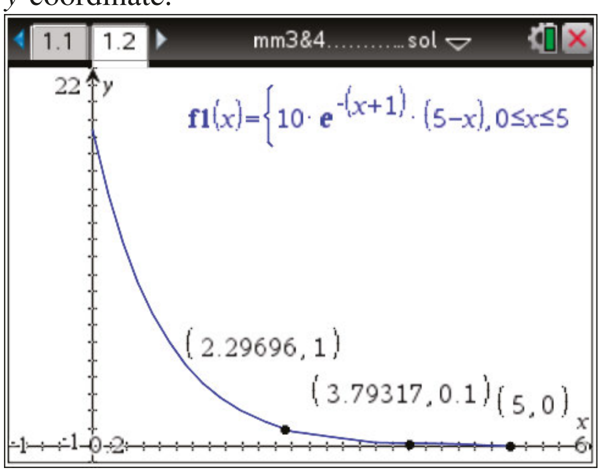
ii $\dfrac{dV}{dt} = 0$ implies $10e^{-(t+1)}(5 - t) = 0$ and therefore t = 5

iii The rate is 1 litre/minute when t = 2 minutes and 18 seconds (to the nearest second)

iv The rate is less than 0.1 litres/minute for the first time when t = 3 minutes and 48 seconds (to the nearest second)



Solving graphically. Enter $f1(x) = 10e^{-(x+1)}(5 - 1)|0 \le x \le 5$ Use the **Point On** tool (b>**Geometry**>**Points & Lines**) and edit the y-coordinate.



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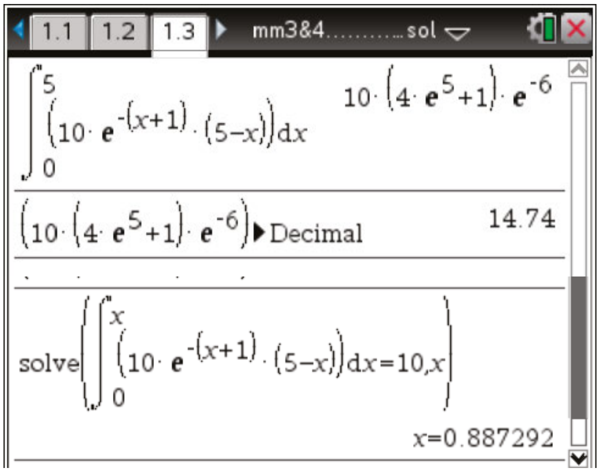
b Use **Integral** from the **Calculus** menu. There are 14.74 litres of water in the tank after 5 minutes.

c The time (to the nearest second) that there is 10 litres in the tank is 53 seconds.



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Chapter 13 – Discrete random Variables and their probability distribution

Solutions to Exercise 13A

1

1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T

2 HH1, HH2, HH3, HH4, HH5, HH6, HT1, HT2, HT3, HT4, HT5, HT6, TH1, TH2, TH3, TH4, TH5, TH6, TT1, TT2, TT3, TT4, TT5, TT6

3 a $\frac{4}{52} = \frac{1}{13}$

b $\frac{3}{4}$

c $\frac{16}{52} = \frac{4}{13}$

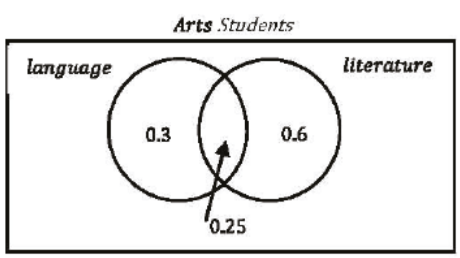
d $\frac{8}{52} = \frac{2}{13}$

4 a $\frac{3}{6} = \frac{1}{2}$

b $\frac{1}{6} + \frac{3}{6} = \frac{4}{6} = \frac{2}{3}$

5
$$\begin{aligned}\Pr(S \cup L) &= \Pr(S) + \Pr(L) - \Pr(S \cap L)\\ &= 0.7 + 0.6 - 0.5\\ &= 0.8\end{aligned}$$

6



$$\begin{aligned}&\Pr(\text{Lit} \cup \text{Lan})\\ &= \Pr(\text{Lit}) + \Pr(\text{Lan}) - \Pr(\text{Lan} \cap \text{Lit})\\ &= 0.3 + 0.6 - 0.25\\ &= 0.65\end{aligned}$$

7 a $0.05 + 0.02 - 0.003 = 0.067$

b $0.05 - 0.003 = 0.047$

8 $1 - 0.75 - 0.12 - 0.08 = 0.05 = 5\%$

9 let $\Pr(A)$ be the probability that an adult owns a car & $\Pr(B)$ be the probability that an adult is employed

$\Pr(A) = 0.7,\ \Pr(B) = 0.6$

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{0.6}{0.7} = \frac{6}{7}$$

10 a $\frac{17}{500}$

b $\frac{18}{500} = \frac{9}{250}$

c
$$\frac{30 + 45 + 33 + 39 + 17}{500} = \frac{164}{500} = \frac{41}{125}$$

d $\frac{10 + 17 + 2 + 1 + 11}{500} = \frac{41}{500}$

11 a $\Pr(\text{guns}) = \frac{130}{200} = \frac{13}{20}$

b $\Pr(\text{guns} \cap \text{male}) = \frac{70}{200} = \frac{7}{20}$

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12 **a** $\Pr(\text{head}) \approx \dfrac{114}{200} = \dfrac{57}{100}$

b $\Pr(\text{ten}) \approx \dfrac{40}{380} = \dfrac{2}{19}$

c $\Pr(\text{2 heads}) \approx \dfrac{54}{200} = \dfrac{27}{100}$

d $\Pr(\text{3 sixes}) \approx \dfrac{2}{500} = \dfrac{1}{250}$

13 $\Pr(\text{White}) = \dfrac{\text{Area of white}}{\text{Total area}}$

$$= \frac{30^2}{50^2}$$

$$= \frac{900}{2500}$$

$$= \frac{9}{25}$$

14 **a** $\Pr(\text{Green}) = \dfrac{\text{Area of green}}{\text{Total area}}$

$$= \frac{\frac{1}{2}\pi r^2}{\pi r^2}$$

$$= \frac{1}{2}$$

b $\Pr(\text{Yellow}) = \dfrac{\text{Area of yellow}}{\text{Total area}}$

$$= \frac{\frac{1}{6}\pi r^2}{\pi r^2}$$

$$= \frac{1}{6}$$

c $\Pr(\text{Not Yellow}) = 1 - \Pr(Yellow)$

$$= \frac{5}{6}$$

15

	C	C'	
T	0.32	0.13	0.45
T'	0.33	0.22	0.55
	0.65	0.35	

a $\Pr(T \cap C') = 0.13$

b $\Pr(T \cap C) = 0.32$

16

	S	S'	
D	0.25	0.15	0.40
D'	0.42	0.18	0.60
	0.67	0.33	

a $\Pr(D) = 0.4$

b $\Pr(S) = 0.67$

c $\Pr(D' \cap S') = 0.18$

17

	A	A'	
S	0.53	0.12	0.65
S'	0.18	0.17	0.35
	0.71	0.29	

a $\Pr(S') = 0.35$

b $\Pr(A \cap S'S) = 0.18$

c $\Pr(A' \cap S) = 0.12$

d $\Pr(A' \cap S') = 0.17$

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Solutions to Exercise 13B

1 a $\Pr(RR) = 0.25 \times 0.8 = 0.2$

b $\Pr(R'R') = 0.75 \times 0.9 = 0.675$

c $\Pr(R \text{ Sunday})$
$= \Pr(RR) + \Pr(R'R')$
$= 0.2 + 0.075 = 0.275$

2 a $\Pr(B|A) = \dfrac{\Pr(B \cap A)}{\Pr(A)}$
$= \dfrac{1}{6}$

b $\Pr(A|B) = \dfrac{\Pr(B \cap A)}{\Pr(B)}$
$= \dfrac{1}{3}$

3 a $\Pr(A \cap B) = \Pr(B|A)\Pr(A)$
$= 0.1 \times 0.6 = 0.06$

b $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} = \dfrac{1}{5}$

4 $\Pr(C|F) = \dfrac{\Pr(C \cap F)}{Pr(F)} = \dfrac{0.3}{0.5} = \dfrac{3}{5}$

5 Let H be the event poor harvest.
Let D be the event disease.

$$\begin{aligned}\Pr(D|H) &= \frac{\Pr(D \cap H)}{\Pr(H)}\\ &= \frac{\Pr(H|D)\Pr(D)}{\Pr(H|D')\Pr(D') + \Pr(H|D)\Pr(D)}\\ &= \frac{0.8 \times 0.3}{0.8 \times 0.3 + 0.5 \times 0.7}\\ &= \frac{24}{59}\end{aligned}$$

A second interpretation of the question which is entirely respectable.
$\Pr(D) = 0.3$

let H be a poor harvest

$$\begin{aligned}\Pr(H) &= 0.5\\ \Pr(H|D) &= 0.8\\ \Pr(H \cap D) &= \Pr(H|D) \times \Pr(D)\\ &= 0.8 \times 0.3\\ &= 0.24\\ \Pr(D|H) &= \frac{\Pr(H \cap D)}{\Pr(H)}\\ &= \frac{0.24}{0.5} = 0.48\end{aligned}$$

6 a $\dfrac{500}{1000} = \dfrac{1}{2}$

b $\dfrac{385}{1000} = \dfrac{77}{200}$

c $\dfrac{200}{385} = \dfrac{40}{77}$

d $\dfrac{200}{500} = \dfrac{2}{5}$

7 a $\Pr(S) = \dfrac{\text{total speed}}{\text{total}}$
$= \dfrac{130}{448} = \dfrac{65}{224}$

b $\Pr(F) = \dfrac{\text{total fatal}}{\text{total}} = \dfrac{115}{448}$

c Look only at the Speed column:
$\Pr(F|S) = \dfrac{42}{130} = \dfrac{21}{65}$

d Look only at the Alcohol column:
$\Pr(F|A) = \dfrac{61}{246}$

8 a $\Pr(J \cap S) = 0.8 \times 0.3 = 0.24$

b $Pr(J \cup S) = 0.8 + 0.3 - 0.24 = 0.86$

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9 $\Pr(A) = 0.6, \Pr(B) = 0.5, \Pr(C) = 0.4$

a $\Pr(A) \times \Pr(B) = 0.6 \times 0.5 = 0.3$
$A \cap B = \{1, 3, 5\} \qquad \Pr(A \cap B) = 0.3$
$\Pr(A \cap B) \neq \Pr(A) \times \Pr(B)$
$\therefore A$ and B are not independent

b $Pr(A) \times Pr(C) = 0.6 \times 0.4 = 0.24$
$A \cap C = \{2, 6\} \qquad \Pr(A \cap C) = 0.2$
$\Pr(A \cap C) \neq \Pr(A) \times \Pr(C)$
$\therefore A$ and C are not independent

c $Pr(B) \times Pr(C) = 0.5 \times 0.4 = 0.2$
$B \cap C = \{9\} \qquad \Pr(B \cap C) = 0.1$
$\Pr(B \cap C) \neq \Pr(B) \times \Pr(C)$
$\therefore B$ and C are not independent

10 $\Pr(A) = 0.5, \Pr(B) = 0.4$

a
$$\begin{aligned} Pr(A|B) &= \frac{\Pr(A \cap B)}{\Pr(B)} \\ &= \frac{\Pr(A) \times \Pr(B)}{\Pr(B)} \\ &= \Pr(A) \\ &= 0.5 \end{aligned}$$

b $\Pr(A \cap B) = \Pr(A) \times \Pr(B) = 0.2$

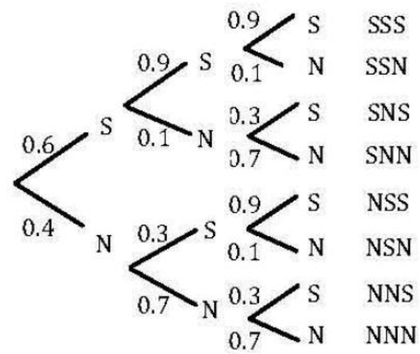
c
$$\begin{aligned} \Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\ &= 0.5 + 0.4 - 0.2 \\ &= 0.7 \end{aligned}$$

11 $0.3 \times 0.7 + 0.6 \times 0.3 = 1.3 \times 0.3 = 0.39$

12 $\dfrac{\Pr(HHH)}{1 - \Pr(TTT)} = \dfrac{\frac{1}{8}}{\frac{7}{8}} = \dfrac{1}{7}$

13 $0.03 \times 0.95 + 0.97 \times 0.02 = 0.0479$

14 $S = stop \quad N = no\ stop$



a $\Pr(SSS) = 0.6 \times 0.9 \times 0.9 = 0.486$

b $\Pr(NSN) = 0.4 \times 0.3 \times 0.1 = 0.012$

c $\Pr(SNN) = 0.6 \times 0.1 \times 0.7 = 0.042$
$\Pr(NSN) = 0.012$
$\Pr(NNS) = 0.4 \times 0.7 \times 0.3 = 0.084$
$\Pr(SNN) + \Pr(NSN) + \Pr(NNS) =$
0.138

15 a $\dfrac{4}{6} \times \dfrac{3}{5} = \dfrac{2}{5}$

b $\dfrac{2}{6} \times \dfrac{1}{5} = \dfrac{1}{15}$

c $\dfrac{4}{6} \times \dfrac{2}{5} + \dfrac{2}{6} \times \dfrac{4}{5} = \dfrac{16}{30} = \dfrac{8}{15}$

16 a $\dfrac{160}{400} = \dfrac{2}{5}$

b $\dfrac{70}{400} = \dfrac{7}{40}$

c $\dfrac{\frac{7}{40}}{\frac{2}{5}} = \dfrac{7}{16}$

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d $\dfrac{\frac{7}{40}}{\frac{150}{400}} = \dfrac{7}{15}$

17 a $\dfrac{1}{2} \times \dfrac{3}{7} + \dfrac{1}{2} \times \dfrac{2}{7} = \dfrac{5}{14}$

$$\dfrac{\frac{3}{14}}{\frac{5}{14}} = \dfrac{3}{5}$$

18 a $0.3 \times 0.75 + 0.6 \times 0.8 + 0.1 \times 0.3$

$= 0.735$

b $\dfrac{0.6 \times 0.2}{1 - 0.735} = \dfrac{0.12}{0.265} = \dfrac{24}{53}$

19 $\dfrac{3}{12} \times \dfrac{2}{11} \times \dfrac{1}{10} + \dfrac{4}{12} \times \dfrac{3}{11} \times \dfrac{2}{10}$

$+ \dfrac{5}{12} \times \dfrac{4}{11} \times \dfrac{3}{10}$

$= \dfrac{1}{220} + \dfrac{1}{55} + \dfrac{1}{22}$

$= \dfrac{1}{220} + \dfrac{4}{220} + \dfrac{10}{220}$

$= \dfrac{15}{220}$

$= \dfrac{3}{44}$

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Solutions to Exercise 13C

1 **a** discrete

b continuous

c discrete

d discrete

2 **a** continuous

b discrete

c continuous

d discrete

3 **a** $\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

b $X = 0, \{TTT\}$
$X = 1, \{HTT, THT, TTH\}$
$X = 2, \{HHT, HTH, THH\}$
$X = 3, \{HHH\}$

c $\Pr(X \geq 2) = \dfrac{4}{8} = \dfrac{1}{2}$

4 **a** Yes, since $p(x) \geq 0$ for all x, and $\Sigma p(x) = 1$

b $\Pr(X \leq 3) = 0.1 + 0.2 + 0.1 + 0.4 = 0.8$

5 **a** $\Pr(X = 3) = \Pr(RRR) = 4/9 \times 4/9 \times 4/9 = 64/729$
$\Pr(X = 2) = \Pr(RRB) + \Pr(RBR) + \Pr(BRR) = 3 \times 4/9 \times 4/9 \times 5/9 = 240/729$
$\Pr(X = 1) = \Pr(RBB) + \Pr(BBR) + \Pr(BRB) = 3 \times 4/9 \times 5/9 \times 5/9 = 300/729$
$\Pr(X = 0) = \Pr(BBB) = 5/9 \times 5/9 \times 5/9 = 125/729$

b $\Pr(X \geq 1) = 1 - \Pr(X = 0) = \dfrac{604}{729}$

c $\Pr(X > 1) = 1 - \Pr(X = 0) - \Pr(X = 1) = \dfrac{304}{729}$

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6 a $\{(1,1),(1,2),(1,3),\ldots,(6,4),(6,5),(6,6)\}$

b die 2 / die 1

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

c **i** $\Pr(Y < 5) = \dfrac{6}{36} = \dfrac{1}{6}$

ii $\Pr(Y = 3|Y < 5) = \dfrac{2}{6} = \dfrac{1}{3}$

iii $\Pr(Y \leq 3|Y < 7) = \Pr(Y \leq 3)/\Pr(Y < 7) = (3/36)/(15/36) = 3/15 = 1/5$

iv $\Pr(Y \geq 7|Y > 4) = \Pr(Y7)/\Pr(Y > 4) = (21/36)/(30/36) = 21/30 = 7/10$

v $\Pr(Y = 7|Y > 4) = \Pr(Y = 7)/\Pr(Y > 4) = (6/36)/(30/36) = 6/30 = 1/5$

vi $\Pr(Y = 7|Y < 8) = \Pr(Y = 7)/\Pr(Y < 8) = (6/36)/(21/36) = 6/21 = 2/7$

7 a die 2 / die 1

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	5
6	1	2	3	4	5	6

b $Y = 1, 2, 3, 4, 5, 6$

c
$$\begin{aligned}\Pr(Y = 1) &= 0.1 + 0.1 - 0.1 \times 0.1\\ &= 0.19\end{aligned}$$

8 a
$$\begin{aligned}\Pr(X = 2) &= \Pr(WWB) + \Pr(WBW)\\ &\quad + \Pr(BWW)\end{aligned}$$

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where B means ‘black ball drawn’ and W means ‘which ball drawn’.

$$\Pr(X = 2) = \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5} + \frac{2}{5} \times \frac{3}{5} \times \frac{2}{5} + \frac{3}{5} \times \frac{2}{5} \times \frac{2}{5}$$
$$= 3 \times \frac{12}{125}$$
$$= \frac{36}{125} = 0.288$$

b $\Pr(X = 3) = \Pr(WWW)$
$$= \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$$
$$= \frac{8}{125} = 0.064$$

c $\Pr(X \geq 2) = 0.288 + 0.064 = 0.352.$

d $\Pr(X = 3 | X \geq 2) = \dfrac{\Pr(X = 3)}{\Pr(X \geq 2)}$
$$= \frac{0.064}{0.352} = \frac{2}{11}$$
$$\approx 0.182$$

9 a $\{(1, 1), (1, 2), (1, 3), \ldots, (6, 4), (6, 5), (6, 6)\}$

b $\Pr(A) = \dfrac{1}{6}$

$\Pr(B) = \dfrac{1}{6}$

$\Pr(C) = \dfrac{15}{36}$

(counting possibilities)

$= \dfrac{5}{12}$

$\Pr(D) = \dfrac{6}{36}$

(counting possibilities)

$= \dfrac{1}{6}$

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c $\Pr(A \cap B) = \dfrac{1}{36}$

(counting possibilities)

$$\Pr(A|B) = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

(counting possibilities)

$$\Pr(A \cap C) = \frac{3}{36}$$
$$= \frac{1}{12}$$

$$\Pr(A|C) = \frac{\frac{1}{12}}{\frac{5}{12}} = \frac{1}{5}$$

$$\Pr(A \cap D) = \frac{1}{36}$$

(counting possibilities)

$$\Pr(A|D) = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

d A & B, A & D since
$\Pr(A|B) = \Pr(A)$&

$\Pr(A|D) = \Pr(A)$

10 **a** Yes, since $p(x) \geq 0$ for all x, and $\Sigma p(x) = 1$

b $\Pr(X \geq 2) = 0.2 + 0.3 = 0.5$

11 **a**, since the sum of the $p(x)$ values > 1; and **c** negative probabilities values is 0.

12 Let x be the number of black balls in the sample.

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a $\Pr(X = 0) = \left(\frac{6}{10}\right)^3 = \left(\frac{27}{125}\right)$

$\Pr(X = 3) = \left(\frac{4}{10}\right)^3 = \frac{8}{125}$

$$\begin{aligned}\Pr(X = 1) &= \frac{4}{10} \times \frac{6}{10} \times \frac{6}{10} + \frac{6}{10} \\ &\quad \times \frac{4}{10} \times \frac{6}{10} + \frac{6}{10} \times \frac{6}{10} \\ &= 3 \times \frac{18}{125} \\ &= \frac{54}{125}\end{aligned}$$

$$\begin{aligned}\Pr(X = 2) &= 1 - \frac{27}{125} - \frac{8}{125} - \frac{54}{125} \\ &= \frac{36}{125}\end{aligned}$$

x	0	1	2	3
$\Pr(X = x)$	$\frac{27}{125}$	$\frac{54}{125}$	$\frac{36}{125}$	$\frac{8}{125}$

b $\Pr(X = 0) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{1}{6}$

$\Pr(X = 3) = \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} = \frac{1}{30}$

$$\begin{aligned}\Pr(X = 1) &= \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} + \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8} \\ &\quad + \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8} \\ &= 3 \times \frac{1}{6} = \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\Pr(X = 2) &= \frac{6}{10} \times \frac{4}{9} \times \frac{3}{8} + \frac{4}{10} \times \frac{6}{9} \times \frac{3}{8} \\ &\quad + \frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} \\ &= 3 \times \frac{1}{10} = \frac{3}{10}\end{aligned}$$

x	0	1	2	3
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

13 $\Pr(X = 0) = 0.6^2 = 0.36$
$\Pr(X = 2) = 0.4^2 = 0.16$

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$\Pr(X = 1) = 1 - 0.16 - 0.36 = 0.48$

x	0	1	2
$\Pr(X = x)$	0.36	0.48	0.16

14 **a**

x	1	2	3	4	5
$\Pr(X = x)$	0.2	0.2	0.2	0.2	0.2

b $\Pr(X \geq 3) = 0.2 \times 3 = 0.6$

c $\Pr(X \leq 3 | X \geq 3) = \dfrac{0.2}{0.6} = \dfrac{1}{3}$

15 **a** $\{(1, 1), (1, 2), (1, 3), \ldots, (6,4), (6,5), (6,6)\}$

b

die 1

		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
die 2	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

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$\Pr(X = 2) = \frac{1}{36}$

$\Pr(X = 3) = \frac{2}{36} = \frac{1}{18}$

$\Pr(X = 4) = \frac{3}{36} = \frac{1}{12}$

$\Pr(X = 5) = \frac{4}{36} = \frac{1}{9}$

$\Pr(X = 6) = \frac{5}{36}$

$\Pr(X = 7) = \frac{6}{36} = \frac{1}{6}$

$\Pr(X = 8) = \frac{5}{36}$

$\Pr(X = 9) = \frac{4}{36} = \frac{1}{9}$

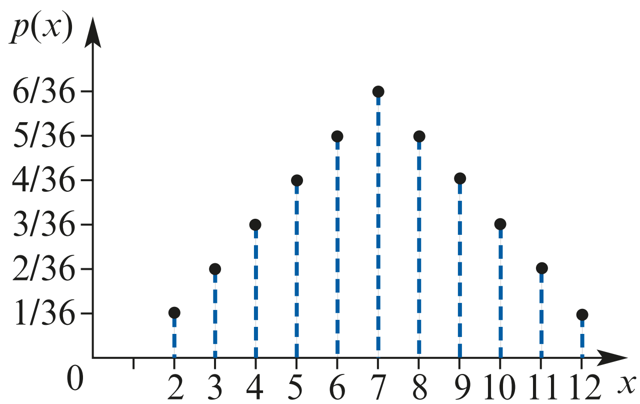
$\Pr(X = 10) = \frac{3}{36} = \frac{1}{12}$

$\Pr(X = 11) = \frac{2}{36} = \frac{1}{18}$

$\Pr(X = 12) = \frac{1}{36}$

x	2	3	4	5	6	7	8	9	10	11	12
$\Pr(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

c



d $\Pr(X > 9) = \frac{4 + 3 + 2 + 1}{36} = \frac{10}{36}$

$= \frac{5}{18}$

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e $\Pr(X \le 10 | X \ge 9) = \dfrac{\frac{7}{36}}{\frac{10}{36}} = \dfrac{7}{10}$

16 **a** $\{(1, 1), (1, 2), (1, 3), \ldots, (6, 4), (6, 5), (6, 6)\}$

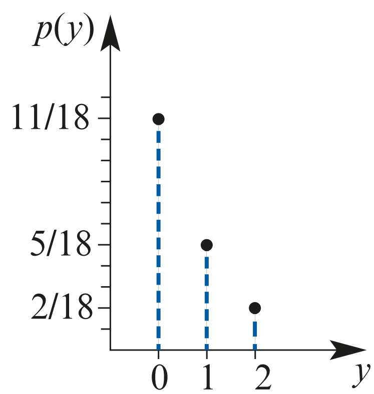
b

dice 1

die 2	1	2	3	4	5	6
1	0	0	0	0	0	0
2	0	0	1	0	1	0
3	0	1	0	1	2	0
4	0	0	1	0	1	2
5	0	1	2	1	0	1
6	0	0	0	2	1	0

y	0	1	2
$\Pr(Y = y)$	$\frac{22}{36}$	$\frac{10}{36}$	$\frac{4}{36}$

c



17 **a** $\Pr(X = 0) = \dfrac{6}{10} \times \dfrac{5}{9} = \dfrac{1}{3}$

$\Pr(X = 2) = \dfrac{4}{10} \times \dfrac{3}{9} = \dfrac{2}{15}$

$\Pr(X = 1) = 1 - \dfrac{1}{3} - \dfrac{2}{15} = \dfrac{8}{15}$

x	0	1	2
$\Pr(X = x)$	$\frac{1}{3}$	$\frac{8}{15}$	$\frac{2}{15}$

b $\Pr(X \ne 1) = \dfrac{7}{15}$

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18 centre circle $= \pi(2)^2 = 4\pi$
middle circle $= \pi(10)^2 - \pi(2)^2 = 96\pi$
outer circle $= \pi(20)^2 - \pi(10)^2 = 300\pi$

a $\Pr(X = 100) = \dfrac{4}{400} = \dfrac{1}{100}$

$\Pr(X = 20) = \dfrac{96}{400} = \dfrac{6}{25}$

$\Pr(X = 10) = \dfrac{300}{400} = \dfrac{3}{4}$

x	10	20	100
$\Pr(X = x)$	$\frac{3}{4}$	$\frac{6}{25}$	$\frac{1}{100}$

b $\Pr(Y = 200) = \dfrac{1}{100} \times \dfrac{1}{100} = \dfrac{1}{10\,000}$

$\Pr(Y = 120) = \dfrac{1}{100} \times \dfrac{6}{25} \times 2 = \dfrac{3}{625}$

$\Pr(Y = 110) = \dfrac{3}{100} \times \dfrac{3}{4} \times 2 = \dfrac{3}{200}$

$\Pr(Y = 40) = \dfrac{6}{25} \times \dfrac{6}{25} = \dfrac{36}{625}$

$\Pr(Y = 30) = \dfrac{6}{25} \times \dfrac{3}{4} \times 2 = \dfrac{9}{25}$

$\Pr(Y = 20) = \dfrac{3}{4} \times \dfrac{3}{4} = \dfrac{9}{16}$

y	20	30	40	110	120	200
$\Pr(Y = y)$	$\frac{9}{16}$	$\frac{9}{25}$	$\frac{36}{625}$	$\frac{3}{200}$	$\frac{3}{625}$	$\frac{1}{10\,000}$

19 **a** $\Pr(X = 3) = \Pr(EEE) + \Pr(NNN)$

$= \dfrac{1}{8} + \dfrac{1}{8} = \dfrac{1}{4}$

b $x = 4$,
{NEEE, ENEE, EENE, ENNN, NENN, NNEN}
$\Pr(X = 4) = 6 \times \dfrac{1}{16} = \dfrac{3}{8}$

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c $\Pr(X = 5) = 1 - \Pr(x \neq 5)$

$$= 1 - \frac{2}{8} - \frac{3}{8}$$

$$= \frac{3}{8}$$

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Solutions to Exercise 13D

1 There is:
30% chance of winning \$1 (\$2 prize less the \$1 cost to play)
10% chance of winning \$19 (\$20 prize less the \$1 cost to play)
60% chance of losing \$1 (the cost to play)
In 100 games the Expected win/loss $= 30 \times 1 + 10 \times 19 - 60 \times 1 = \$100$

2 $\begin{aligned}\text{Mean} &= 1 \times 0.1 + 3 \times 0.3 + 5 \times 0.3 \\ &\quad + 7 \times 0.3 \\ &= 0.1 + 0.9 + 1.5 + 2.1 \\ &= 4.6\end{aligned}$

$\begin{aligned}\text{Mean} &= 0.25 \times -1 + 0.25 \times 0 + 0.25 \\ &\quad \times 1 + 0.25 \times 2 \\ &= 0.5\end{aligned}$

b $\begin{aligned}\text{Mean} &= 0 \times 0.09 + 1 \times 0.22 + 2 \times 0.26 \\ &\quad + 3 \times 0.21 + 4 \times 0.13 + 5 \times 0.06 \\ &\quad + 6 \times 0.02 + 7 \times 0.01 \\ &= 0.22 + 0.52 + 0.63 + 0.52 + 0.30 \\ &\quad + 0.12 + 0.07 \\ &= 2.38\end{aligned}$

c $\begin{aligned}\text{Mean} &= 0.2 \times 0.08 + 0.3 \times 0.13 \\ &\quad + 0.4 \times 0.09 + 0.5 \times 0.19 \\ &\quad + 0.6 \times 0.7 + 0.7 \times 0.03 \\ &\quad + 0.8 \times 0.10 + 0.9 \times 0.18 \\ &= 0.569\end{aligned}$

d Mean=7

e Mean=0

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3 $\mu = \$10,000 \times 0.13 + \$5,000 \times 0.45$

$\quad + \$0 \times 0.25 - \$5,000 \times 0.15$

$= \$1,500 + \$2,250 - \$750$

$= \$3,000$

4 *assuming a payout (as opposed to a profit) of* \$5 *for a win,*

$$\mu = \frac{5}{6} \times -\$1 + \frac{1}{6} \times \$4$$

$$= -\$\frac{1}{6} = -\$0.17, \text{ i.e. a loss of 17c.}$$

5 $\mu = 0 \times 0.12 + 1 \times 0.36 + 2 \times 0.38 + 3 \times 0.14$

$= 0.36 + 0.76 + 0.42$

$= 1.54$

x	1	2	3	4	5	7	8	9	10	11	12
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$

6 $\mu = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6}$

$\quad + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 7 \times \frac{1}{36}$

$\quad + 8 \times \frac{1}{36} + 9 \times \frac{1}{36} + 10 \times \frac{1}{36}$

$\quad + 11 \times \frac{1}{36} + 12 \times \frac{1}{36}$

$$= \frac{\begin{array}{c}6 + 12 + 18 + 24 + 30 + 7 + 8 \\ + 9 + 10 + 11 + 12\end{array}}{36}$$

$$= \frac{147}{36}$$

$$= \frac{49}{12}$$

7 a $\mathrm{E}(X) = 2 \times 0.01 + 3 \times 0.25 + 4 \times 0.40$

$\quad + 5 \times 0.30 + 6 \times 0.04$

$= 0.02 + 0.75 + 1.60 + 1.50 + 0.24$

$= 4.11$

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b $\mathrm{E}(X^3) = 8 \times 0.01 + 27 \times 0.25$
$+ 64 \times 0.40 + 125 \times 0.30$
$+ 216 \times 0.04$
$= 0.08 + 6.75 + 25.60$
$+ 37.50 + 8.64$
$= 78.57$

c $\mathrm{E}(5X - 4) = 5\mathrm{E}(X) - 4 = 5 \times 4.11 - 4$
$= 16.55$

d $\mathrm{E}\left(\frac{1}{X}\right) = \frac{1}{2} \times 0.01 + \frac{1}{3} \times 0.25$
$+ \frac{1}{4} \times 0.40 + \frac{1}{5} \times 0.30 + \frac{1}{6} \times 0.04$
$= 0.255$

8 $\mathrm{E}(X) = \Sigma x \Pr(X = x) = 2.97$
E(commission) $= 2.97 \times \$2000 = \$5940$

9 **a** $p = 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16}$
$p = \frac{1}{16}$

b $\mu = 0 \times p + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 4 \times \frac{1}{8}$
$+ 8 \times \frac{1}{16}$
$= 4 \times \frac{1}{2}$
$\mu = 2$

c $\mathrm{E}(X^2) = 0 \times p + 1 \times \frac{1}{2} + 4 \times \frac{1}{4}$
$+ 16 \times \frac{1}{8} + 64 \times \frac{1}{16}$
$= 0 + \frac{1}{2} + 1 + 2 + 4$
$= \frac{15}{2}$
$\sigma^2 = \mathrm{E}(X^2) - \mathrm{E}(X)^2$
$= \frac{15}{2} - 4 = \frac{7}{2}$

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10 **a** $k + 2k + 3k + 4k + 5k + 6k = 1$

$$21k = 1$$

$$k = \frac{1}{21}$$

b $$\begin{aligned}\mu &= 1 \times k + 2 \times 2k + 3 \times 3k + 4 \times 4k \\ &\quad + 5 \times 5k + 6 \times 6k \\ &= \frac{1 + 4 + 9 + 16 + 25 + 36}{21} \\ &= \frac{91}{21} \\ &= \frac{13}{3} \approx 4.33\end{aligned}$$

c $$\begin{aligned}\sigma^2 &= \mathrm{E}(X^2) - \mu^2 \\ \mathrm{E}(X^2) &= 1 \times k + 4 \times 2k + 8 \times 3k \\ &\quad + 16 \times 4k + 25 \times 5k + 36 \times 6k \\ &= \frac{1 + 8 + 27 + 64 + 125 + 2166}{21} \\ &= \frac{441}{21} \\ &= 21 \\ \sigma^2 &= 21 - \frac{169}{9} \\ &= \frac{20}{9} \approx 2.22\end{aligned}$$

11

	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

a

x	1	2	3	4	6	8	9	12	16
$\Pr(X = x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

b **i** $$\begin{aligned}\Pr(X > 8) &= \frac{1}{16} + \frac{2}{16} + \frac{1}{16} \\ &= \frac{4}{16} = \frac{1}{4}\end{aligned}$$

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ii $\mathrm{E}(X) = 1 \times \frac{1}{16} + 2 \times \frac{2}{16}$

$+ 3 \times \frac{2}{16} + 4 \times \frac{3}{16}$

$+ 6 \times \frac{2}{16} + 8 \times \frac{2}{16}$

$+ 9 \times \frac{1}{16} + 12 \times \frac{2}{16}$

$+ 16 \times \frac{1}{16}$

$$= \frac{1 + 4 + 6 + 12 + 12 + 16 + 9 + 24 + 16}{16}$$

$$= \frac{100}{16}$$

$$= \frac{25}{4}$$

iii $\mathrm{E}(X)^2 = 1 \times \frac{1}{16} + 4 \times \frac{2}{16}$

$+ 9 \times \frac{2}{16} + 16 \times \frac{3}{16}$

$+ 36 \times \frac{2}{16} + 64 \times \frac{2}{16}$

$+ 81 \times \frac{1}{16} + 144 \times \frac{2}{16}$

$+ 256 \times \frac{1}{16}$

$$= \frac{1 + 8 + 18 + 48 + 72 + 128 + 81 + 288 + 256}{16}$$

$$= \frac{900}{16}$$

$$= \frac{225}{4}$$

$$\sigma^2 = \mathrm{E}(X^2) - \mu^2 = \frac{225}{4} - \frac{625}{16}$$

$$= \frac{275}{16}$$

12

	H	T
1	1	2
2	2	4
3	3	6
4	4	8
5	5	10
6	6	12

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a $\mu = 1 \times \frac{1}{12} + 2 \times \frac{2}{12} + 3 \times \frac{1}{12}$

$+ 4 \times \frac{2}{12} + 5 \times \frac{1}{12} + 6 \times \frac{2}{12}$

$+ 8 \times \frac{1}{12} + 10 \times \frac{1}{12} + 12 \times \frac{1}{12}$

$= \frac{1 + 4 + 3 + 8 + 5 + 12 + 8 + 10 + 12}{12}$

$= \frac{63}{12} = \frac{21}{4}$

b $\Pr(X < \mu) = \frac{7}{12}$ (*counting on the table*)

c $E(X^2) = 1 \times \frac{1}{12} + 4 \times \frac{2}{12}$

$+ 9 \times \frac{1}{12} + 16 \times \frac{2}{12}$

$+ 25 \times \frac{1}{12} + 36 \times \frac{2}{12}$

$+ 64 \times \frac{1}{12} + 100 \times \frac{1}{12}$

$+ 144 \times \frac{1}{12}$

$= \frac{1 + 8 + 9 + 32 + 25 + 72 + 64 + 100 + 144}{12}$

$= \frac{455}{12}$

$\sigma^2 = E(X^2) - \mu^2 = \frac{455}{12} - \frac{441}{16}$

$= \frac{497}{48}$

13 **a** $\mathrm{Var}(2X) = 2^2\mathrm{Var}(X) = 4 \times 16 = 64$

b $\mathrm{Var}(X + 2) = 1^2\mathrm{Var}(X) = 16$

c $\mathrm{Var}(1 - X) = (-1)^2\mathrm{Var}(X) = 16$

d $sd(3X) = \sqrt{\mathrm{Var}(3X)} = \sqrt{3^2\mathrm{Var}(X)}$

$= \sqrt{9 \times 16}$

$= 12$

14 **a** $c = 1 - 0.3 - 0.1 - 0.2 - 0.05$

$= 0.35$

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b $\mathrm{E}(X) = 1 \times c + 2 \times 0.3 + 3 \times 0.1$

$+ 4 \times 0.2 + 5 \times 0.05$

$= 0.35 + 0.6 + 0.3 + 0.8 + 0.25$

$= 2.3$

c $\mathrm{E}(X^2) = 1 \times c + 4 \times 0.3 + 9 \times 0.1$

$+ 16 \times 0.2 + 25 \times 0.05$

$= 0.35 + 1.2 + 0.9 + 3.2 + 1.25$

$= 6.9$

$\sigma^2 = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$= 6.9 - (2.3)^2$

$= 6.9 - 5.29$

$= 1.61$

$\sigma = \sqrt{\sigma^2} \approx 1.27$

d $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$

$\Pr(2.3 - 2.5 \le X \le 2.3 + 2.5)$

$\Pr(0 \le X \le 4.8)$

$= 0.35 + 0.3 + 0.1 + 0.2$

$= 0.95$

15 **a** $k + 2k + 3k + 4k + 5k = 1$

$15k = 1$

$k = \frac{1}{15}$

b $\mu = 1 \times \frac{1}{15} + 2 \times \frac{2}{15} + 3 \times \frac{3}{15}$

$+ 4 \times \frac{4}{15} + 5 \times \frac{5}{15}$

$= \frac{1 + 4 + 9 + 16 + 25}{15}$

$= \frac{55}{15}$

$\mu = \frac{11}{3} \approx 3.667$

$\sigma^2 = \mathrm{E}(X^2) - \mu^2$

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c $\mathrm{E}(X^2) = 1\frac{1}{15} + 4 \times \frac{2}{15} + 9 \times \frac{3}{15}$

$+ 16 \times \frac{4}{15} + 25 \times \frac{5}{15}$

$= \frac{1 + 8 + 27 + 64 + 125}{15}$

$= \frac{225}{15}$

$= 15$

$\sigma^2 = 15 - \frac{121}{9}$

$= \frac{14}{9} \approx 1.556$

d $\sigma = \sqrt{\sigma^2} \approx 1.25$

$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$

$= \Pr(3.67 - 2.5 \le X \le 3.67 + 2.5)$

$= \Pr(1.2 \le X \le 6.2)$

$= \frac{2}{15} + \frac{3}{15} + \frac{4}{15} + \frac{5}{15}$

$= \frac{14}{15} \approx 0.933$

16 **a** $\mathrm{E}(X) = \frac{1}{36} \times 2 + \frac{2}{36} \times 3 + \frac{3}{36} \times 4$

$+ \frac{4}{36} \times 5 + \frac{5}{36} \times 6 + \frac{6}{36} \times 7$

$+ \frac{5}{36} \times 8 + \frac{4}{36} \times 9 + \frac{3}{36} \times 10$

$+ \frac{2}{36} \times 11 + \frac{1}{36} \times 12$

$= \frac{2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12}{36}$

$= \frac{252}{36}$

$= 7$

alternatively, since we know the probability distribution is symmetrical, we also know that the mean is the central number, *i.e.* $\mathrm{E}(X) = 7$

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b $\mathrm{E}(X^2) = \dfrac{1}{36} \times 4 + \dfrac{2}{36} \times 9 + \dfrac{3}{36} \times 16$

$+ \dfrac{4}{36} \times 25 + \dfrac{5}{36} \times 36$

$+ \dfrac{6}{36} \times 49 + \dfrac{5}{36} \times 64$

$+ \dfrac{4}{36} \times 81 + \dfrac{3}{36} \times 100$

$+ \dfrac{2}{36} \times 121 + \dfrac{1}{36} \times 144$

$$= \frac{\begin{array}{l} 4 + 18 + 48 + 100 + 180 \\ + 294 + 320 + 324 + 300 \\ + 242 + 144 \end{array}}{36}$$

$= \dfrac{1974}{36}$

$= \dfrac{329}{6} \approx 54.833$

$\sigma^2 = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$= \dfrac{329}{6} - 49$

$= \dfrac{35}{6} \approx 5.83$

c $\sigma = \sqrt{\sigma^2} \approx 2.4$

$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$

$= \Pr(7 - 4.8 \le X \le 7 + 4.8)$

$= \Pr(2.2 \le X \le 11.8)$

$= 1 - \Pr(X = 2) - \Pr(X = 12)$

$= 1 - \dfrac{1}{36} - \dfrac{1}{36}$

$= \dfrac{17}{18} \approx 0.944$

17 **a** *by symmetry*, $\mathrm{E}(X) = 3$

b $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$

$\mathrm{E}(X^2) = 0 \times 0.0156 + 1 \times 0.0937$

$+ 4 \times 0.2344 + 9 \times 0.3126$

$+ 16 \times 0.2344 + 5 \times 0.0937$

$+ 36 \times 0.0156$

$= 0.0937 + 0.9376 + 2.8134$

$+ 3.7504 + 2.3425 + 0.5616$

$= 10.4992 \approx 10.5$

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$$\text{Var}(X) = 10.5 - 9$$
$$= 1.5$$

c $\sigma = \sqrt{\text{Var}(X)} \approx 1.2$

$$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$$
$$= \Pr(3 - 2.4 \le X \le 3 + 2.4)$$
$$= \Pr(0.6 \le X \le 5.4)$$
$$= \Pr(1 \le X \le 5)$$
$$= 0.0937 + 0.2344 + 0.3126$$
$$+ 0.2344 + 0.0937$$
$$= 0.9688$$

18 $Pr(c_1 \le X \le c_2) \approx 0.95$

$c_1 = \mu - 2\sigma = 50 - 10 = 40$

$c_2 = \mu + 2\sigma = 50 + 10 = 60$

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Solutions to Technology-free questions

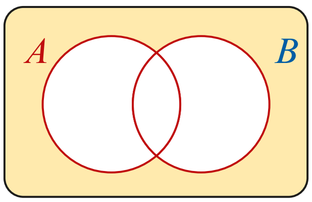
1 $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$0.7 = 0.5 + 0.2 - \Pr(A \cap B)$

$\Pr(A \cap B) = 0$

If A and B are mutually exclusive this is true.

2



$\Pr(A' \cap B') = \Pr(A \cup B)'$

$(A \cup B)'$ is the complement of $A \cup B$.

$\Pr(A \cup B) + \Pr[(A \cup B)'] = 1$

So $\Pr(A \cup B) = 1 - \Pr[(A \cup B)']$

Therefore $\Pr(A' \cup B') = \Pr[(A \cup B)']$

The intersection of the complements is the complement of the union, so the probability of the union of two sets is 1–probability of the intersection of their complements.

3 **a** $\Pr(BW \text{ or } WB) = \frac{5}{9} \times \frac{4}{9} + \frac{4}{9} \times \frac{5}{9}$

$= \frac{40}{81}$

$= \frac{5}{9} \times \frac{4}{8} + \frac{4}{9} \times \frac{5}{8}$

b $\Pr(BW \text{ or } WB) = \frac{5}{9}$

4 Require Pr(coin A| '$H\&T$' tossed)

$= \frac{\Pr(A \cap \text{'}H\&T\text{'})}{\Pr(\text{'}H\ \&\ T\text{'})}$

$\Pr(\text{selecting } A) = \Pr(\text{selecting } B) = \frac{1}{2}$

$\Pr(H|A) = 0.8,\ \Pr(T|A) = 0.2,$

$\Pr(H|B) = 0.4,\ \Pr(T|B) = 0.6$

$$\Pr(3 \cap \text{'}H\&\text{'}T) = \frac{1}{2} \times (0.8 \times 0.2 + 0.2 \times 0.8)$$

$$= 0.16$$

$$\Pr(\text{'}H\&T\text{'}) = \Pr(A \cap \text{'H\&T'}) + \Pr(B \cap \text{'}H\&T\text{'})$$

$$\Pr(A \cap \text{'}K\&T\text{'}) = \frac{1}{2} \times (0.4 \times 0.6 + 0.6 \times 0.4)$$

$$= 0.24$$

$$\text{So } \Pr(\text{'}H\&T\text{'}) = 0.16 + 0.24 = 0.40$$

$$\Pr(A|\text{'}H\&T\text{'}) = \frac{0.16}{0.40} = 0.4$$

5 $0.4p^2 + 0.1 + 0.1 + 1 - 0.6p = 1$

$0.4p^2 - 0.6p + 0.2 = 0$

$4p^2 - 6p + 2 = 0$

$2p^2 - 3p + 1 = 0$

$(2p - 1)(p - 1) = 0$

$\therefore p = \frac{1}{2} \text{ or } p = 1$

6 **a** $k + 2k + 3k + 2k + k + k = 1$

$10k = 1$

$k = 0.1$

b $E(x) = \Sigma xp(x)$

$= -k + 0 + 3k + 4k + 3k + 4k$

$= 13k$

$= 1.3$

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c $\mathrm{E}(x^2) = \Sigma x^2 p(x)$

$= k + 0 + 3k + 8k + 9k + 16k$

$= 37k = 3.7$

$\mathrm{Var}\,(x) = \mathrm{E}(x^2) - [\mathrm{E}(x)]^2$

$= 3.7 - 1.69 = 2.01$

7 **a** $\mathrm{E}(x) = \Sigma x p(x)$

$= 2 \times \frac{1}{4} + 4 \times \frac{1}{4}$

$+ 16 \times \frac{1}{4} + 64 \times \frac{1}{4}$

$= 21\frac{1}{2}(21.5)$

b $\mathrm{E}\left(\frac{1}{x}\right) = \Sigma \frac{1}{x} p(x)$

$= \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4}$

$+ \frac{1}{16} \times \frac{1}{4} + \frac{1}{64} \times \frac{1}{4}$

$= \frac{53}{256}$

c $\mathrm{E}(x^2) = \Sigma x^2 p(x)$

$= 4 \times \frac{1}{4} + 16 \times \frac{1}{4} + 256 \times \frac{1}{4}$

$+ 64^2 \times \frac{1}{4}$

$= 1 + 4 + 64 + 1024$

$= 1093$

$\mathrm{Var}\,(x) = \mathrm{E}(x^2) - [\mathrm{E}(x)]^2$

$= 1093 - (21.5)$

$= 1093 - 462.25$

$= 630.75$

$= \frac{2523}{4}$

d From past c, $\mathrm{Var}(x) = \frac{2523}{4}$.

But $2523 = 3 \times 841$

$= 3 \times 29^2$

So $\mathrm{Var}(x) = \frac{3 \times 29^2}{2^2}$

$\Rightarrow \mathrm{sd}(x) = \frac{29\sqrt{3}}{2}$

8 **a** Profit is $\$(x - 2)$ if the cylinder is ok and $-\$2$ if the cylinder is defective.

P	$x - 2$	-2
$\Pr(P = p)$	$\frac{4}{5}$	$\frac{1}{5}$

b $\mathrm{E}(P) = \Sigma p \Pr(P = p) = \frac{4}{5}(x - 2) - \frac{2}{5}$

$= \frac{4}{5}x - 2.$

c To make a profit in the long teem, require $\mathrm{E}(P) > 0$, i.e.

$\frac{4}{5}x - 2 > 0$

$\frac{4}{5}x > 2$

$x > \frac{5}{2} = 2.5$

The manufacturer should sell the cylinders for more then \$2.50.

9 **a** $\Pr(`< 30\text{'} \cap `> 1\ acc\text{'}) = \frac{470}{1000}$

$= 0.47$

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b $\Pr(\text{'} < 30\text{'} \cap \text{'} > 1\ acc\text{'})$

$$= \frac{\Pr(\text{'} < 30 \cap > 1\ acc\text{'})}{\Pr(\text{'}>1\ acc\text{'})}$$

$$= \frac{0.47}{\left(\frac{470 + 230}{100}\right)}$$

$$= \frac{0.47}{0.70}$$

$$= \frac{47}{70}$$

10

Let $I =$ 'immunised', $D =$ 'get disease'

$$\begin{aligned}\Pr(D) &= \Pr(D \cap I) + \Pr(D \cap I')\\ &= \Pr(I)\Pr(D|I) + \Pr(I')\Pr(D|I')\\ &= 0.7 \times 0.05 + 0.3 \times 0.6\\ &= 0.035 + 0.18\\ &= 0.215\end{aligned}$$

So 21.5% are expected to get the disease.

[NOTE: This is a probability way of saging: "5% of the 70% and 60% of the 30% get the disease, i.e. 3.5% + 18% = 21.5% get it]

11 $\Pr(A) = \frac{1}{2}, \Pr(B) = \frac{1}{4}, \Pr\ (A|B) = \frac{1}{6}$

a $\Pr(A \cap B) = \Pr(A|B)\Pr(B)$

$$= \frac{1}{6} \times \frac{1}{4}$$

$$= \frac{1}{24}$$

b

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ &= \frac{1}{2} + \frac{1}{4} - \frac{1}{24}\\ &= \frac{17}{24}\end{aligned}$$

c

$$\Pr(A'|B) = \frac{\Pr(A' \cap B)}{\Pr(B)}$$

$$\text{But } \Pr(B) = \Pr(A \cap B) + \Pr(A' \cap B)$$

$$\begin{aligned}\text{So } \Pr(A' \cap B) &= \Pr(B) - \Pr(A \cap B)\\ &= \frac{1}{4} - \frac{1}{24}\\ &= \frac{5}{24}\end{aligned}$$

$$\text{So } \Pr(A'|B) = \frac{\frac{5}{24}}{\frac{1}{4}} = \frac{5}{6}$$

d

$$\Pr(A|B') = \frac{\Pr(A \cap B')}{\Pr(B')}$$

$$\text{But } \Pr(A') = \Pr(A \cap B) + \Pr(A \cap B')$$

$$\begin{aligned}\text{So } \Pr(A \cap B') &= \Pr(A) - \Pr(A \cap B)\\ &= \frac{1}{2} - \frac{1}{24}\\ &= \frac{11}{24}\end{aligned}$$

$$\begin{aligned}\Pr(B') &= 1 - \Pr(B)\\ &= 1 - \frac{1}{4}\\ &= \frac{3}{4}\end{aligned}$$

$$\text{So } \Pr(A|B') = \frac{\frac{11}{24}}{\frac{3}{4}} = \frac{11}{18}$$

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Solutions to multiple-choice questions

1 A $2k + 3k + 0.1 + 3k + 2k = 1$

$10k = 0.9$

$k = 0.09$

2 D $\Pr(-3 \le X < 0) = \Pr(X = -3)$

$+ \Pr(X = -2) + \Pr(X = -1)$

$= 0.07 + 0.15 + 0.22$

$= 0.44$

3

D $E(X) = \Sigma xp(x) = \Pr(X = x))$

$= 1 \times 0.46 + 2 \times 0.26$

$+ 3 \times 0.14 + 4 \times 0.09$

$+ 5 \times 0.07$

$= 0.46 + 0.48 + 0.42 + 0.36 + 0.35$

$= 2.07$

4 E $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$= 1.69 - (1.20)^2$

$= 1.69 - 1.44$

$= 0.25$

$\mathrm{sd}(X) = \sqrt{0.25}$

$= 0.5$

5 C $E(Y) = E(3X + 10)$

$= 3E(X) + 10$

$= 3 \times 100 + 10$

$= 310$

$\mathrm{Var}(Y) = \mathrm{Var}(3X + 10)$

$= 9\,\mathrm{Var}(X)$

$= 9 \times 100$

$= 900$

6 C $E(x) = \Sigma xp(x)$

$= -p + 0 + 1 - 3p$

$= 1 - 4p$

7

B/D $a + b + 0.2 = 1 \Rightarrow a + b = 0.8 \ldots$ ①

$E(x) = -2a + 0.4 = 0.2 \ldots$ ②

From ②, $2a = 0.2$ so $a = 0.1$

Substitute in ①: $0.1 + b = 0.8$ so $b = 0.7$

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Solutions to extended-response questions

1 a

$$\Sigma \Pr(X = x) = 1$$

$$\therefore \quad c + 2c + 2c + 3c + c^2 + 2c^2 + 7c^2 + c = 1$$

$$\therefore \quad 10c^2 + 9c = 1$$

$$\therefore \quad 10c^2 + 9c - 1 = 0$$

$$\therefore \quad (10c - 1)(c + 1) = 0$$

$$\therefore \quad c = 0.1 \text{ or } c = -1$$

but $c > 0 \quad \therefore \quad c = 0.1$

b $\Pr(X \geq 5) = \Pr(X = 5) + \Pr(X = 6) + \Pr(X = 7)$

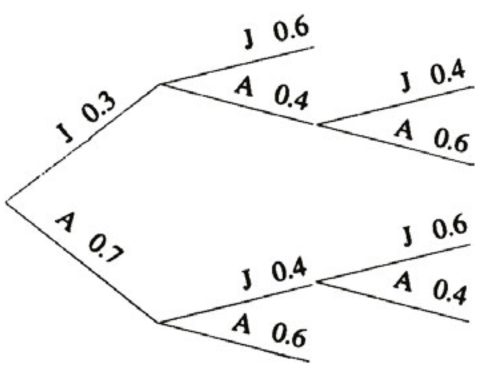
$= 10c^2 + c$

$= 10 \times (0.1)^2 + 0.1$

$= 0.2$

c If $\Pr(X \leq k) > 0.5$
then by considering cumulative probabilities
i.e. $\Pr(X \leq 2) = 0.3; \Pr(X \leq 3) = 0.5$
the minimum value of k is 4.

2 a



b i Probability of Janet winning
$= 0.3 \times 0.6 + 0.3 \times 0.4 \times 0.4 + 0.7 \times 0.4 \times 0.6$
$= 0.396$

ii Probability of Alan winning
$= 1 - 0.396$
$= 0.604$

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c **i** Let X be the number of sets played until match is complete.

$$\Pr(X = 2) = 0.3 \times 0.6 + 0.7 \times 0.6$$
$$= 0.6$$

$$\therefore \Pr(X = 3) = 0.4$$

t	2	3
$\Pr(T = t)$	0.6	0.4

ii $\mathrm{E}(X) = 2 \times 0.6 + 3 \times 0.4 = 2.4$

d $$\Pr(\text{Alan wins} \mid \text{three sets}) = \frac{\Pr(\text{Alan wins in three sets})}{\Pr(\text{Three sets})}$$
$$= \frac{0.3 \times 0.4 \times 0.6 + 0.7 \times 0.4 \times 0.4}{0.4}$$
$$= 0.46$$

3 Let \$w be the amount a player pays to play.

$\boxed{\$5}$ $\boxed{\$5}$ $\boxed{\$5}$ $\boxed{\$10}$ $\boxed{\$10}$

Let X be the possible value From 2 cards

$\therefore X = 10, 15, 20$

A score of 10 is obtained if two \$5 cards are chosen

$\therefore \Pr(X = 10) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$ (without replacement)

A score of 15 is obtained with \$10 on the first and \$5 on the second or \$5 on the first and \$10 on the second.

$$\therefore \Pr(X = 15) = \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{1}{2}$$
$$= \frac{6}{20} + \frac{3}{10}$$
$$= \frac{3}{5}$$

A score of 20 is obtained with a \$10 on each card.

$$\Pr(X = 20) = \frac{2}{5} \times \frac{1}{4}$$
$$= \frac{1}{10}$$

Let Y be the amount a player receives

$Y = 10 - w$ or $15 - w$ or $20 - w$

The probability distribution for Y is as shown

y	$10 - w$	$15 - w$	$20 - w$
$\Pr(Y = y)$	0.3	0.6	0.1

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$\therefore \ \mathrm{E}(Y) = 0.3(10 - w) + 0.6(15 - w) + 0.1(20 - w)$

$= 3 - 0.3w + 9 - 0.6w + 2 - 0.1w$

$= 14 - w$

If $\mathrm{E}(Y) = 0$, $w = 14$

i.e. The player should pay \$14 to ensure that it is a fair game.

4 Let F denote free from faults

Let N denote not free from faults (defective)

$\Pr(F|A) = 0.95 \quad \Pr(F|B) = 0.98 \quad \Pr(F|C) = 0.99$

$\Pr(A) = 0.5 \quad \Pr(B) = 0.3 \quad \Pr(C) = 0.2$

a $\Pr(A) = 0.5$

b $\Pr(N|A) = 0.05$

c $\Pr(N) = \Pr(N|A)\Pr(A) + \Pr(N|B)\Pr(B) + \Pr(N|C)\Pr(C)$

$= 0.05 \times 0.5 + 0.02 \times 0.3 + 0.01 \times 0.2$

$= 0.033$

d $\Pr(A|D) = \dfrac{\Pr(\text{produced by } A \text{ and defective})}{\Pr(\text{defective})}$

$= \dfrac{\Pr(N|A)\Pr(A)}{\Pr(N)}$

$= \dfrac{0.5 \times 0.5}{0.033}$

$= \dfrac{25}{33}$

5

p	0	1	2	3	4	5
$\Pr(P = p)$	0.39	0.27	0.16	0.12	0.04	0.02

a **i** $\mathrm{E}(P) = 0 \times 3.29 + 1 \times 0.27 + 2 \times 0.16 + 3 \times 0.12 + 4 \times 0.04 + 5 \times 0.02$

$= 0.27 + 0.32 + 0.36 + 0.16 + 0.1$

$= 1.21$

The mean number of passengers per car is 1.21

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ii $\text{Var}(P) = \text{E}(P^2) - [\text{E}(P)]^2$

$$\text{E}(P^2) = 0^2 \times 0.39 + 1^2 \times 0.27 + 2^2 \times 0.16 + 3^2 \times 0.12 + 4^2 \times 0.04 + 5^2 \times 0.02$$

$$= 0.27 + 0.64 + 1.08 + 0.64 + 0.5 = 3.13$$

$$\text{Var}(P) = 3.13 - 1.4641$$

$$= 1.6659$$

$$\text{sd}(P) = \sqrt{1.6659} = 1.2907 \text{ (correct to four decimal places)}$$

iii

$$\sigma = \text{sd}(P) = 1.2907$$

$$\mu - 2\sigma = -1.3714$$

$$\mu + 2\sigma = 3.7914$$

$$\Pr(\mu - 2\sigma \le P \le \mu + 2\sigma) = \Pr(-1.3714 \le P \le 3.7914)$$

$$= \Pr(P = 0) + \Pr(P = 1) + \Pr(P = 2) + \Pr(P = 3)$$

$$= 1 - [\Pr(P = 4) + \Pr(P = 5)]$$

$$= 0.94$$

b **i** Let T be the cost per car in dollars.

$$\Pr(T = 1) = \Pr(P = 0) = 0.39$$

$$\Pr(T = 0.40) = \Pr(P = 1) = 0.27$$

$$\Pr(T = 0) = \Pr(P = 2) + \Pr(P = 3) + \Pr(P = 4) + \Pr(P = 5) = 0.34$$

t	1	0.40	0
$\Pr(T = t)$	0.39	0.27	0.34

ii $\text{E}(T) = 1 \times 0.39 + 0.40 \times 0.27 + 0 \times 0.34$

$$= 0.39 + 0.108$$

$$= 0.498$$

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iii $\quad \mathrm{E}(T^2) = 1 \times 0.39 + 0.4^2 \times 0.27$

$$= 0.39 + 0.0432$$

$$= 0.4332$$

$$\mathrm{Var}(T) = \mathrm{E}(T^2) - [\mathrm{E}(T)]^2$$

$$= 0.4332 - 0.248004$$

$$= 0.1852$$

$$\mathrm{sd}(T) = 0.4303$$

$$\mu - 2\sigma = 0.498 - 2 \times 0.4304 = -0.3628$$

$$\mu + 2\sigma = 0.498 + 2 \times 0.4304 = 1.3588$$

$$\Pr(\mu - 2\sigma \leq T \leq \mu + 2\sigma) = \Pr(-0.3628 \leq T \leq 1.3588)$$

$$= \Pr(T = 0) + \Pr(T = 0.4) + \Pr(T = 1)$$

$$= 1$$

6 **a** $\quad \mathrm{E}(Y) = 0 \times 0.135 + 1 \times 0.271 + 2 \times 0.271 + 3 \times 0.180 + 4 \times 0.090$

$$+ 5 \times 0.036 + 6 \times 0.012 + 7 \times 0.003 + 8 \times 0.002$$

$$= 2.002$$

The mean number of sales per week is 2.002.

b $\quad \mathrm{E}(Y^2) = 6.002$

$$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - [\mathrm{E}(Y)^2] = 6.022 - 4.008004$$

$$= 2.013996 \approx 2.014$$

$$\mathrm{sd}(Y) \approx 1.419$$

c **i** Let B be the bonus paid to each salesman.

The possible values for B are 0, 100 and 200

$$\Pr(B = 0) = \Pr(Y = 0) + \Pr(Y = 1) + \Pr(Y = 2) = 0.677$$

$$\Pr(B = 100) = \Pr(Y = 3) + \Pr(Y = 4) = 0.27$$

$$\Pr(B = 200) = \Pr(Y \geq 4) = 0.053$$

The probability distribution is

b	0	100	200
$\Pr(B = b)$	0.677	0.27	0.053

ii $\quad \mathrm{E}(B) = 0 \times 0.677 + 100 \times 0.27 + 200 \times 0.053$

$$= 27 + 10.6$$

$$= 37.6$$

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The mean bonus paid is \$37.60.

7 Let P denote the percentage profit

p	40	30	20	10	0	-10	-20
$\Pr(P = p)$	0.1	0.15	0.25	0.2	0.15	0.1	0.05

a $\mathrm{E}(P) = 40 \times 0.1 + 30 \times 0.15 + 20 \times 0.25 + 10 \times 0.2 + 0 \times 0.15 - 10 \times 0.1 - 20 \times 0.05$

$= 13.5$

The mean return is 13.5%

$$\begin{aligned} \mathrm{E}(P^2) &= 1600 \times 0.1 + 100 \times 0.15 \times 400 \times 0.25 + 100 \times 0.2 + 100 \times 0.1 + 400 \times 0.05 \\ &= 445 \\ \therefore \ \mathrm{Var}(P) &= 445 - 182.25 \\ &= 262.75 \\ \therefore \ \mathrm{sd}(P) &= \sqrt{262.75} \approx 16.2\% \end{aligned}$$

b $$\begin{aligned} \Pr(13.5 - 2 \times 16.21 \le P \le 13.5 + 2 \times 16.21) &= \Pr(-18.92 \le P \le 45.92) \\ &= 1 - \Pr(P = -20) \\ &= 1 - 0.05 \\ &= 0.95 \end{aligned}$$

c $$\begin{aligned} \text{Return} &= \text{Profit–Brokerage} \\ \text{Percentage gain} &= 0.6 \times \text{Return} \\ &= 0.6(\text{Profit–Brokerage}) \end{aligned}$$

Let G be the percentage gain

Then $G = 0.6(P - 2)$

$$\begin{aligned} \therefore \quad \mathrm{E}(G) &= 0.6\mathrm{E}(P) - 1.2 \\ &= 0.6 \times 13.5 - 1.2 \\ &= 6.9\% \\ \mathrm{Var}(G) &= (0.6)^2 \ \mathrm{Var}(P) \\ &= 0.36 \times 262.75 \\ &= 94.59 \\ \mathrm{sd}(G) &\approx 9.726\% \end{aligned}$$

8 Consider the case when the promoter takes out insurance.

If it rains:

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($)Profit = 250 000 − 60 000 + 20 000

= 210 000

(assuming the $250 000 is paid by the insurance company and the $20 000 profit is added.) If it does not rain:

($)Profit = 25 0000 − 60 000

= 190 000

The probability distribution for this

p	190 000	210 000
$\Pr(P = p)$	0.67	0.33

$E(P) = 196\ 600$

Then the promoter does not take the insurance, the probability distribution is as shown below:

p	250 000	20 000
$\Pr(P = p)$	0.67	0.33

and $E(P) = 174\ 100$

∴ the promoter should buy the insurance.

9 For the tossing of two dice the sums of the values may be recorded in a table as shown

die A die B	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

The possible sums are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

The probability of obtaining a sum of 7 $= \frac{6}{36} = \frac{1}{6}$

The probability of obtaining a sum of 11 or 12 $= \frac{3}{36} = \frac{1}{12}$

The probability of any other sum $= \frac{3}{4}$

Let X be the amount obtained from game and let w be the amount obtained from obtaining a sum not equal to 7, 11 or 12

The probability distribution is as shown:

x	−10	11	w
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{3}{4}$

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$$\begin{aligned}
\mathrm{E}(X) &= -\frac{10}{6} + \frac{11}{12} + \frac{3w}{4} \\
\text{If } \mathrm{E}(X) &= 0 \\
-\frac{20}{12} + \frac{11}{12} + \frac{9w}{12} &= 0 \\
\therefore \quad \frac{9w}{12} &= \frac{9}{12} \\
w &= 1
\end{aligned}$$

There should be a payment of \$1.00 for a sum not equal to 7, 11 or 12.

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10 a i Probability that the first prototype is successful is 0.65.

ii Probability of the first not successful, but the second successful

$= 0.35 \times 0.65$

$= 0.2275$

iii Probability of the first two not successful, but the third successful

$= (0.35)^2 \times 0.65$

$= 0.0796\ 25$

iv Probability that the project is abandoned

$= (0.35)^3$

$= 0.0428\ 75$

b The following cases have to be considered:

	Cost	Probability
A First is successful	\$7 million	0.65
B First is unsuccessful but second is successful	\$10.5 million	0.2275
C First two unsuccessful but third successful	\$12.25 million	0.079625
D The project is abandoned	\$12.25 million	0.042875

Let C be the cost of the project.

C	7	10.5	12.25
$\Pr(C = c)$	0.6	0.2275	0.1225

$\therefore\ E(C) = 7 \times 0.65 + 10.5 \times 0.2275 + 12.25 \times 0.1225$

$= 8.439\ 375$

$\therefore$ the expected cost is \$8.439 375 million

c Let P denote the profit

P	$20 - 7$	$20 - 10.5$	$20 - 12.25$	-12.25
$\Pr(P = p)$	0.65	0.2275	0.079625	0.042875

$\therefore\ E(P) = 13 \times 0.65 + 9.5 \times 0.2275 + 7.75 \times 0.079\ 625 - 12.25 \times 0.042\ 875$

$\therefore$ Expected profit is \$10.703 125 million

11 If the score is 5, 6, 7, 8, 9, 10, 11 or 12. Alfred pays \$$x$ to Bertie. Therefore Alfred has $100 - x$ dollars.

If the score is 2, 3 or 4 Alfred has $100 + x + 8 = 108 + x$ dollars.

The tables gives the sum of the scores when the two die are tossed.

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die 2 / die 1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Let Y be the score.

$\Pr(Y \geq 5) = \frac{30}{36} = \frac{5}{6}$ and $\Pr(Y \leq 4) = \frac{1}{6}$

a Let A be the amount of Alfred's cash

A	$100 - x$	$108 + x$
$\Pr(A = a)$	$\frac{5}{6}$	$\frac{1}{6}$

$$\begin{aligned} \therefore\ \mathrm{E}(A) &= \frac{5}{6}(100 - x) + \frac{1}{6}(108 + x) \\ &= \frac{1}{6}(608 - 4x) \\ &= \frac{1}{3}(304 - 2x) \end{aligned}$$

b If the game is fair

$$\begin{aligned} \mathrm{E}(A) &= 100 \\ \therefore\ \frac{1}{3}(304 - 2x) &= 100 \\ 304 - 2x &= 300 \\ \therefore\ x &= 2 \end{aligned}$$

c

$$\begin{aligned} \mathrm{E}(A^2) &= 97^2 \times \frac{5}{6} + 111^2 \times \frac{1}{6} \qquad (\text{given } x = 3) \\ &= 9894\frac{1}{3} \\ \therefore\ \mathrm{Var}\,(A) &= 2894\frac{1}{3} - \left(\frac{1}{3}[298]\right)^2 \\ &= 27\frac{2}{9} \end{aligned}$$

12 Let X be the values of the die

$\Pr(X = 1) = \frac{x}{4} \quad \Pr(X = 2) = \frac{1}{4} \quad \Pr(X = 6) = \frac{1}{4}(1 - x)$

$\Pr(X = 3) = \Pr(X = 4) = \Pr(X = 5) = \frac{1}{6}$

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Table for total

2nd / 1st	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

a Let Y be the total.

$$\begin{aligned}\Pr(Y = 7) &= 2\Pr(1 \,\&\, 6) + 2\Pr(3 \,\&\, 4) + 2\Pr(5 \,\&\, 2)\\ &= 2 \times \frac{x}{4} \times \frac{1-x}{4} + 2 \times \frac{1}{6} \times \frac{1}{6} + 2 \times \frac{1}{6} \times \frac{1}{4}\\ &= \frac{x(1-x)}{8} + \frac{2}{36} + \frac{2}{24}\\ &= \frac{x(1-x)}{8} + \frac{1}{18} + \frac{1}{12}\\ &= \frac{9x(1-x) + 4 + 6}{72}\\ &= \frac{9x - 9x^2 + 10}{72}\end{aligned}$$

b Let $P = \Pr(Y = 7)$

$$\frac{dP}{dx} = \frac{9 - 18x}{72}$$

and $\dfrac{dP}{dx} = 0$ implies $x = \dfrac{1}{2}$

when $x = \dfrac{1}{2}$,

$$P = \frac{9 \times \frac{1}{2} - 9\left(\frac{1}{2}\right)^2 + 10}{72} = \frac{\frac{9}{2} - \frac{9}{4} + 10}{72} = \frac{\frac{9}{4} + 10}{72} = \frac{49}{288}$$

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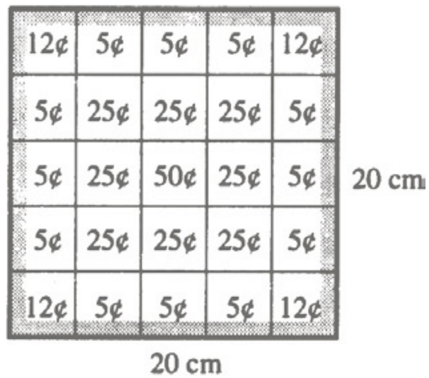
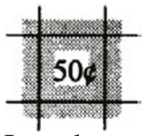
13



Coin cannot land within 1 cm of ridge.
$\therefore$ Area in which "centre" of coin can fall is
$18^2 = 324$ cm^2.



In order to land in square and not overlap another square centre must fall at a distance of > 1 from edge.
$\therefore$ Area in which "centre" of coin can fall is 4 cm^2.

a Let X be the prize money.

i $\Pr(X = 50) = \dfrac{4}{324} = \dfrac{1}{81}$

ii $\Pr(X = 25) = \dfrac{8}{81}$

iii $\Pr(X = 12) = \dfrac{4}{81}$

iv $\Pr(X = 5) = \dfrac{12}{81} = \dfrac{4}{27}$

v $\Pr(X = 0) = \dfrac{56}{81}$

b Let P be the profit.

P	$C - 50$	$C - 25$	$C - 12$	$C - 5$	C
$\Pr(P = p)$	$\frac{1}{81}$	$\frac{8}{81}$	$\frac{4}{81}$	$\frac{12}{81}$	$\frac{56}{81}$

$\mathrm{E}(P) = \dfrac{1}{81}[(C - 50) + 8(C - 25) + 4(C - 12) + 12(C - 5) + 56C] > 0$

$\Leftrightarrow 81C - 358 > 0$

$C > \dfrac{358}{81} \approx 4.4197$

Five cents will yield a profit.

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Chapter 14 – The binomial distribution

Solutions to Exercise 14A

1 a and b describe a Bernoulli sequence.

2 $n = 7, p = 0.5$

$\Pr(X = 4) = \binom{7}{4}(0.5)^4(0.5)^3 = 0.2734$

3 $n = 4, p = 0.2$

a $\Pr(X = 3) = \binom{4}{3}(0.2)^3(0.8)^1 = 0.0256$

b $\Pr(X = 4) = \binom{4}{4}(0.2)^4(0.8)^0 = 0.0016$

4 $n = 5, p = 0.4$

a $\Pr(X = 0) = \binom{5}{0}(0.4)^0(0.6)^0 = 0.0778$

b $\Pr(X = 3) = \binom{5}{3}(0.4)^3(0.6)^3 = 0.2304$

c $\Pr(X = 5) = \binom{5}{5}(0.4)^5(0.6)^5 =$ 0.01024

5 $n = 3, p = 0.5$

a $\Pr(X = x) = \binom{3}{x}(0.5)^x(0.6)^{3-x}$

$x = 0, 1, 2, 3$

b $\Pr(X = 2) = 0.375$

6 $n = 6, p = 0.48$

a $\Pr(X = x) = \binom{6}{x}(0.48)^x(0.52)^{6-x}$

$x = 0, 1, 2, 3$

b $\Pr(X = 2) = 0.2527$

7 a $\binom{6}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3 = 20 \times \frac{1}{216} \times \frac{125}{216}$

≈ 0.0536

$\binom{6}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^2 + \binom{6}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right) + \binom{6}{6}\left(\frac{1}{6}\right)^6$

b $\frac{375}{6^6} + \frac{30}{6^6} + \frac{1}{6^6} = \frac{406}{46656}$

≈ 0.0087

c $\binom{6}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3 + \binom{6}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^2$

$+ \binom{6}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right) + \binom{6}{6}\left(\frac{1}{6}\right)^6$

≈ 0.0623

8 $n = 10, p = 0.1$

a $\Pr(X = x) = \binom{10}{x}(0.1)^x(0.9)^{10-x}$

$x = 0, 1, 2, 3, \ldots, 10$

b **i** $\Pr(X = 0) = 0.3487$

ii $\Pr(X \geq 1) = 1 - \Pr(X = 0)$

$= 0.6513$

9 $n = 11, p = 0.2$

a $\Pr(X = x) = \binom{11}{x}(0.2)^x(0.8)^{11-x}$

$x = 0, 1, 2, 3, \ldots, 11$

b **i** $\Pr(X = 2) = 0.2953$

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ii $\Pr(X = 0) = 0.0859$

iii $\Pr(X \geq 1) = 1 - \Pr(X = 0)$
$= 0.9141$

10 $n = 7, p = \dfrac{1}{5}$

a $\Pr(X = x) = \dbinom{7}{x}\left(\dfrac{1}{5}\right)^x\left(\dfrac{4}{5}\right)^{7-x}$
$x = 0, 1, 2, 3, \ldots, 7$

b **i** $\Pr(X = 7) = 0.000013$

ii $\Pr(X = 0) = 0.2097$

iii $\Pr(X = 2 \text{ or } X = 3) = 0.3899$

11 $1 - \dbinom{10}{0}(0.2)^0(0.8)^{10} - \dbinom{10}{1}(0.2)^1(0.8)^9$

$= 1 - \dfrac{4^{10}}{5^{10}} - \dfrac{10 \times 1 \times 4^9}{5^{10}}$

≈ 0.624

12 $n = 7, p = \dfrac{x}{100}$

$\left(\dfrac{x}{100}\right)^6$

ii $\dfrac{6x^5(100 - x)}{100^6}$

iii $\dfrac{x^6}{100^6} + \dfrac{6x^5(100 - x)}{100^6} +$
$\dfrac{15x^4(100 - x)^2}{100^6}$

13 $1 - \dbinom{4}{0}\left(\dfrac{1}{4}\right)^0\left(\dfrac{3}{4}\right)^4$

$= 1 - \left(\dfrac{3}{4}\right)^4$

≈ 0.6836

14 *using the CAS calculator*

a 0.1156

b 0.7986

c 0.3170

15 *using the CAS calculator*
0.6791

16 *using the CAS calculator*

a 0.1123

b 0.5561

c 0.00001

d 0.00001

17 $\dbinom{6}{0}(0.4)^0(0.6)^6 + \dbinom{6}{1}(0.4)(0.6)^5$

$+ \dbinom{6}{2}(0.4)^2(0.6)^4$

$= \dfrac{3^6}{5^6} + \dfrac{6 \times 2 \times 3^5}{5^6} + \dfrac{15 \times 2^2 \times 3^4}{5^6}$

$= \dfrac{3^6 + 4 \times 3^6 + 5 \times 4 \times 3^5}{5^6}$

$= \dfrac{3 \times 3^5 + 4 \times 3^5}{5^5} = \dfrac{7 \times 3^5}{5^5} \approx 0.544$

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18 **a** $\left(\frac{1}{4}\right)^6 \approx 0.00024$

b *using the CAS calculator*
Pr(≥ 3 correct) ≈ 0.1694

19 **a** $\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^3 = \frac{6^3}{5^6} \approx 0.0138$

b $\binom{6}{3}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^3 = 20 \times \frac{6^3}{5^6} \approx 0.2765$

c *using the CAS calculator*
Pr(≥ 3) ≈ 0.8208

d $\Pr(\text{exactly } 3 | \geq 3) = \frac{\Pr(\text{exactly } 3)}{\Pr(\geq 3)} = \frac{(b)}{(c)} = \frac{0.2765}{0.8202} \cong 0.3368$

20 **a** $\left(\frac{4}{5}\right)^8 \approx 0.1678$

b *using the CAS calculator*
Pr(≥ 6 corrent) ≈ 0.00123

c Pr(8 correct | ≥ 6 correct) = $\frac{\Pr(8 \text{ correct})}{\Pr(\geq 6 \text{ correct})} = \frac{(0.2)^8}{(b)} = \frac{0.00000256}{0.00123} \approx 0.0021$

21 **a** $(0.15)^{10} \approx 0.000\,000\,006$

b $1 - (0.85)^{10} \approx 1 - 0.1969 \approx 0.8031$

c Pr(> | goal | ≥ | goal)

$$= \frac{\Pr(> | \text{goal})}{\Pr(\geq | \text{goal})}$$
$$= \frac{(b) - 10(0.15)(0.85)^9}{(b)}$$
$$= \frac{0.8031 - 0.3474}{0.8031}$$
$$\approx \frac{0.4557}{0.8031} \approx 0.5674$$

22 **a** $\left(\frac{4}{5}\right)^{20} \approx 0.0115$

b *using the CAS calculator*

$$p = 0.2,\ n = 20,$$
$$min = 10,\ max = 20$$

Pr(≥ 10 correct) ≈ 0.00259

c $\Pr(X \geq 12 | X \geq 10) = \frac{\Pr(X \geq 12)}{\Pr(X \geq 10)} \approx$ 0.0392 (using the CAS calculator)

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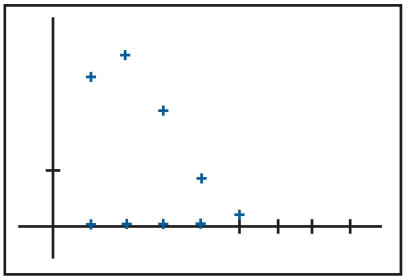
Solutions to Exercise 14B

1



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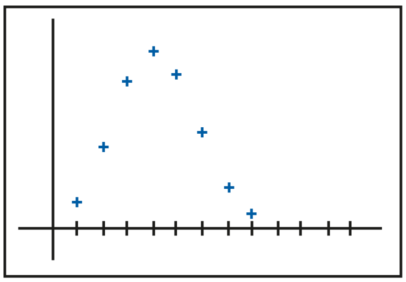
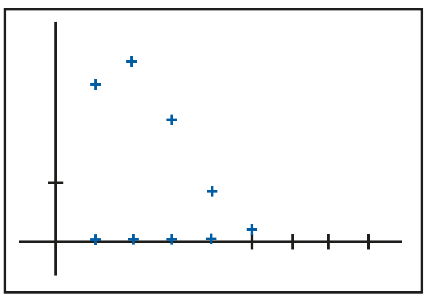


3 a b



c *the distribution in part* **b** *is a reflection of the distribution in part* **a** *in the line* $X = 5$

4 a $\mu = np = 5$

$\sigma^2 = np(1-p) = 5 \times (0.8) = 4$

b $\mu = np = 6$

$\sigma^2 = np(1-p) = 6 \times (0.4) = \dfrac{12}{5}$

c $\mu = np = \dfrac{500}{3}$

$\sigma^2 = np(1-p) = \dfrac{1000}{9}$

d $\mu = np = 8$

$\sigma^2 = np(1-p) = 8 \times \left(\dfrac{4}{5}\right) = \dfrac{32}{5}$

5 a $\mu = np = 6 \times \dfrac{1}{6} = 1$

b $\Pr(X > 1) = 1 - \Pr(X = 0) - \Pr(X = 1)$

$$= 1 - \left(\frac{5}{6}\right)^6 - 6 \times \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^5$$

$$= 1 - \frac{11}{6}\left(\frac{5}{6}\right)^5$$

$$\approx 1 - 0.7368$$

$$\approx 0.2632$$

6 $\mu = np = 50 \times \dfrac{3}{4}$

$= 37.5$ *people will survive on average*

7

$$\mu = np$$

$$\sigma^2 = np(1-p) = \mu(1-p)$$

$$\sigma^2 = 9,\ \mu = 12$$

$$9 = 12(1-p)$$

$$1 - p = \frac{3}{4}$$

$$p = \frac{1}{4}$$

$$12 = n \times \frac{1}{4}$$

$$n = 48$$

$$\Pr(X = 7) = \binom{48}{7}\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^{41}$$

$$\approx 0.0339$$

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8

$$\mu = 30,\ \sigma^2 = 21$$
$$\sigma^2 = \mu(1-p)$$
$$1 - p = \frac{21}{30}$$
$$p = \frac{9}{30} = \frac{3}{10}$$
$$\mu = np$$
$$30 = n \times \frac{3}{10}$$
$$n = 100$$
$$\Pr(X = 20) = \binom{100}{20}\left(\frac{3}{10}\right)^{20}\left(\frac{7}{10}\right)^{80}$$
$$\approx 0.0076$$

9

$$n = 20, p = 0.5$$
$$\mu = np = 10$$
$$\sigma^2 = np(1-p) = 10 \times 0.5 = 5$$
$$\sigma = \sqrt{5} \approx 2.2$$
$$\mu \pm 2\sigma \approx 10 \pm 4.4$$
$$= 5.6, 14.4$$

$\therefore$ *the probability of abtaining between* 6 *and* 14 *heads is about* 0.95

10

$$n = 200, p = 0.6$$
$$\mu = np = 200 \times \frac{6}{10} = 120$$
$$\sigma^2 = \mu(1-p) = 120 \times \frac{4}{10} = 48$$
$$\sigma = \sqrt{48} = 4\sqrt{3} \approx 6.9$$
$$\mu \pm 2\sigma \approx 120 \pm 13.8$$
$$= 106.2, 133.8$$

$\therefore$ *the probability that between* 107 *and* 133 *students will have attended a government school is about* 0.95

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Solutions to Exercise 14C

1 a $n = 5, p = 0.2$

i $\Pr(X = 0) = (0.8)^5 \approx 0.3277$

ii $\Pr(X \geq 1) = 1 - \Pr(X = 0)$
≈ 0.6723

b
$$\begin{aligned} \Pr(X \geq 1) &> 0.95 \\ 1 - \Pr(X = 0) &> 0.95 \\ \Pr(X = 0) &< 0.05 \\ (0.8)^n &< 0.05 \\ n &\approx 13.43 \end{aligned}$$
$\therefore$ *the smallest number of shots is* 14

c
$$\begin{aligned} \Pr(X \geq 1) &> 0.95 \quad \therefore \\ 1 - \Pr(X = 0 - \Pr(X = 1)) &> 0.95 \\ \Pr(X = 0) + \Pr(X = 1) &< 0.05 \\ (0.8)^n + \binom{n}{1} 0.8^{n-1} \times 0.2 &< 0.05 \\ (0.8)^n + n0.8^{n-1} \times 0.2 &< 0.05 \\ n &\approx 21.77 \end{aligned}$$
the smallest number of shots is 22

2 a i $\Pr(X = 2) = \binom{10}{2}(0.1)^2(0.9)^8$
≈ 0.1937

ii
$$\begin{aligned} \Pr(X \geq 1) &= 1 - \Pr(X = 0) \\ &= 1 - (0.9)^{10} \\ &\approx 1 - 0.3487 \\ &\approx 0.6513 \end{aligned}$$

b
$$\begin{aligned} \Pr(X \geq 1) &> 0.7 \\ 1 - \Pr(X = 0) &> 0.7 \\ \Pr(X = 0) &< 0.3 \\ (0.9)^n &< 0.3 \\ n &\approx 11.43 \end{aligned}$$
$\therefore$ *the smallest number of tickets is* 12

3 $p = 0.6$

$\Pr(X = 5) > 0.25$

$\binom{n}{5}(0.6)^5(0.4)^{n-5} > 0.25$

using CAS calculator, the minimum number of shots is 7

4
$$\begin{aligned} p &= 0.2 \\ \Pr(X = 3) &> 0.1 \\ \binom{n}{3}(0.2)^3(0.8)^{n-3} &> 0.1 \end{aligned}$$
using the CAS calculator, the minimum number of chocolates is 7

5
$$\begin{aligned} p &= 0.35 \\ \Pr(X \geq 2) &> 0.9 \\ 1 - \Pr(X = 0) - \Pr(X = 1) &> 0.9 \\ (0.65)^n + n(0.35)\,(0.65)^{n-1} &< 0.1 \end{aligned}$$
using the CAS calculator, the minimum number of games is 10

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6 $p = 0.07$

$\Pr(X > 1) > 0.8$

$1 - \Pr(X = 0) - \Pr(X = 1) > 0.8$

$(0.93)^n + n(0.07)(0.93)^{n-1} < 0.2$

using the CAS calculator, the minimum number of balls is 42

7 $p = 0.7$

$\Pr(X \geq 50) > 0.99$

using the CAS calculator the minimum number of shots is 86

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Solutions to Technology-free questions

1 X is Bi $\left(n = 4, p = \frac{1}{3}\right)$

a $\Pr(X = 0) = q^4 = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$

b $\Pr(X = 1) = \binom{4}{1} p^1 q^3 = 4\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3 =$
$\frac{32}{81}$

c $\Pr(X \leq 1) = \frac{16}{81} + \frac{32}{81} = \frac{48}{81} = \frac{16}{27}$

d $\Pr(X \geq 1) = 1 - \Pr(X = 0)$
$= 1 - \frac{16}{81} = \frac{65}{81}$

2 $n = 3, p = 0.6,\ X$ is Bi (n, p)
$\Pr(X = 2) = \binom{3}{2}(0.6)^2(0.4)$
$= 3 \times 0.36 \times 0.4$
$= 3 \times \frac{9}{25} \times \frac{2}{5}$
$= \frac{54}{125}$

3 $n = 5, p = 0.1,\ X$ is Bi (n, p)
$\Pr(X \geq 1) = 1 - \Pr(X = 0)$
$= 1 - 0.9^5$
$= 1 - \frac{9^5}{10^5}$
Now $9^2 = 81, 81^2 = 6561$ and so
$9^5 = 9 \times 6561 = 59049$
Hence $\Pr(X \geq 1) = 1 - 0.59049$
$= 0.40951$

4 $n = 20, p = 0.1,\ X$ is Bi (n, p)

a $E(X) = np = 2$

b $sd(X) = npq = 2 \times 0.9 = \frac{9}{5}$
$sd(X) = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$

5 $n = 4,\ X$ is Bi (n, p)

a Pr(no successes) $= (1 - p)^4$

b Pr(one success) $= \binom{4}{1} p^1(1 - p)^3$
$= 4p(1 - p)^3$

c Pr(at least one success)
$= 1 -$ Pr(no success)
$= 1 - (1 - p)^4$

d Pr(four successes) p^4

e
Pr(at least two successes)
$= 1 -$ Pr(zero or one success)
$= 1 - (1 - p)^4 - 4p(1 - p)^3$ (from **a**/**b**)

6 We can assume the coin is unbound,
so Pr(3 heads in 10 hours)
$= \binom{10}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7$
$= \frac{10 \times 9 \times 8}{3 \times 2 \times 1} \times \left(\frac{1}{2}\right)^{10}$
so $n = \frac{10 \times 9 \times 8}{3 \times 2 \times 1}$
$= 10 \times 3 \times 4$
$= 120$

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7 $n = 5,\ X$ is Bi (n, p)

Pr(exactly one success given

at least one success)

$$= \frac{\Pr(\text{‘1 success ‘}n\text{’} \geq 1 \text{ success ’})}{\Pr(\geq 1 \text{ success })}$$

$$= \frac{\Pr(\text{‘1 success’})}{1 - \Pr(\text{‘0 successes ’})}$$

$$= \frac{\binom{5}{1} p^1 (1-p)^4}{1-(1-p)^5}$$

$$= \frac{5p(1-p)^4}{1-(1-p)^5}$$

8 In one throw of die (assuming unbiased),

$\Pr(\text{even number uppermost}) = \frac{3}{6} = \frac{1}{2}$

$n = 5,\ p = \frac{1}{5},\ X$ is Bi $(n,\ p)$

$$\Pr(X = 3) = \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2$$

$$= \frac{5 \times 4}{2 \times 1} \times \left(\frac{1}{2}\right)^5$$

$$= \frac{5}{16}$$

9 $n = 5, p = \frac{1}{5},\ X$ is Bi (n, p)

$$\Pr(X = 3) = \binom{5}{3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^2$$

$$= 10 \times \frac{1}{125} \times \frac{16}{25}$$

$$= \frac{32}{625}$$

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Solutions to multiple-choice questions

1 D $n = 5, p = 0.6, X$ is Bi (n, p)

$$\Pr(X = 3) = \binom{5}{3}(0.6)^3(0.4)^2$$
$$= \frac{5 \times 4}{2 \times 1}(0.6)^3(0.4)^2$$
$$= 10 \times (0.6)^3(0.4)^2$$

2 A $n = 5, p = 0.35,\ X$ is Bi (n, p)

Pr(on time at least once)

$= 1 - \Pr(\text{lets all 5 days})$

$= 1 - (0.65)^5$

3 E $\Pr(\text{number} > 4) = \Pr(5 \text{ or } 6) = \frac{1}{3}$

$n = 4, p = \frac{1}{3}, X$ is Bi (n, p)

$$\Pr(X = 2) = \binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2$$
$$= \frac{4 \times 3}{2 \times 1} \times \frac{1}{9} \times \frac{4}{9}$$
$$= \frac{8}{27}$$

4 B $n = 80, p = 0.4,$

X is Bi (n, p)

$$\Pr(X < 30) = \Pr(0 \le X \le 29)$$
$$= 0.2861$$

using a CAS Calculator.

5 A $\mathrm{E}(X) = np = 18 \times \frac{1}{3} = 6$

For $(X) = np(1 - p) = 6 \times \frac{2}{3} = 4$

So $\mu = 6, \sigma^2 = 4$

6 B Since $p = 0.7$, the distribution has a long tail to the left. The greatest probablity will be near the mean, which is $10 \times 0.7 = 7$. Hence the second graph is the best representation.

7 C

$$\mu = 10 \Rightarrow np = 10 \quad \ldots ①$$
$$\sigma = 2\sigma^2 = 4$$
$$\Rightarrow 4pq = 4 \quad \ldots ②$$

$(2) \div (1) : q = \frac{4}{10} = 0.4$

$1 - p = 0.4$

$p = 0.6$

[Note that if you try to find n, you get $n = 16\frac{2}{3}$. Taking n to be 17 and p to be 0.6 gives a mean of 10.2 and variance of 4.08, i.e. the mean and variance are 10 and 4 to the nearest integer.]

8 C Use $(X)npq = np(1 - p) = 1.875n = 10 : p - p^2 = 0.1875$ Using a CAS *solve* command gives $p = 0.25$ or $p = 0.75$. (Automatically, note that $0.1875 = \frac{3}{16}$, so the quadratic can be expened as $p^2 - p + \frac{3}{16} = 0 = \left(p - \frac{1}{4}\right)\left(p - \frac{3}{4}\right) = 0$, so $p = \frac{1}{4}$ or $p = \frac{3}{4}$. Since the coin is biased towards heads, the probability of a head is 0.75.

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9 E Pr(Thomas wins at least one set)

$1 - \text{Pr(Thomas wins no set,)} =$

$1 - 0.76n$

$1 - 0.76n > 0.95$

$0.76n < 0.05$

A CAS Calculator show that

$0.76^{10} = 0.065\ldots$ and 0.76

$n = 0.048\ldots$

So that fewest number of days is 11.

(Alternatively, taking $\log_{10}$ of both side) gives $\log_{10} 0.76n < \log_{10} 0.05$

$n \log_{10} 0.76 < \log_{10} 0.05$

$$n > \frac{\log_{10} 0.05}{\log_{10} 076}$$

(Since $\log_{10} 0.76$ is negative)

so n > 10.91 . . . and hence the least number of days is 11)

10 B Pr(Thomas wins at least one set)

$= 1 - \text{Pr(no wins or one win)}$

$= 1 - 0.76n - \binom{n}{1}(0.24)^1(0.76)^{4-1}$

$= 1 - 0.76n - 0.24n\ (0.76)^{4-1}$

A CAS calculator shows that this probability i) 0.940 . . . when $n = 17$ and 0.952 . . .

when $n = 18$.

So the fewest number of day is 18.

(Note: An efficient way to use a CAS calculator is to first Define the function $f(n) = 1 - 0.76^n - 0.264(0.76)^{4-1}$ It is then a simple matter to evaluate $f(n)$ for various values of n.)

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Solutions to extended-response questions

1 a For children without disability there is an equal chance of answering A, B or C.

Let X be the number of questions out of 10 which are answered A or B. X is a binomial random variable with $n = 10$ and $p = \dfrac{2}{3}$

$\Pr(X = 10) = \left(\dfrac{2}{3}\right)^{10} = 0.0173$

The probability that the answers given by a child without either disability will be all As and Bs is 0.0173

b Pr(Answering C five or more times)

$$= \Pr(\text{Answering } A \text{ or } B \text{ 5 or less times})$$

$$= \Pr(X \le 5)$$

$$= \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)$$

$$= \left(\frac{1}{3}\right)^{10} + \binom{10}{1}\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{9} + \binom{10}{2}\left(\frac{2}{3}\right)^{2}\left(\frac{1}{3}\right)^{8} + \binom{10}{3}\left(\frac{2}{3}\right)^{3}\left(\frac{1}{3}\right)^{7}$$

$$+ \binom{10}{4}\left(\frac{2}{3}\right)^{4}\left(\frac{1}{3}\right)^{6} + \binom{10}{5}\left(\frac{2}{3}\right)^{5}\left(\frac{1}{5}\right)^{5}$$

$$= \left(\frac{1}{3}\right)^{10} + 10 \times \frac{2}{3} \times \left(\frac{1}{3}\right)^{9} + 45 \times \left(\frac{2}{3}\right)^{2} \times \left(\frac{1}{3}\right)^{8} + 120 \times \left(\frac{2}{3}\right)^{3} \times \left(\frac{1}{3}\right)^{7}$$

$$+ 210\left(\frac{2}{3}\right)^{4}\left(\frac{1}{3}\right)^{6} + 252 \times \left(\frac{2}{3}\right)^{5} \times \left(\frac{1}{3}\right)^{5}$$

$$= \left(\frac{1}{3}\right)^{10} [1 + 20 + 180 + 960 + 3360 + 8064]$$

$$= \left(\frac{1}{3}\right)^{10} [12585]$$

$$= 0.2131$$

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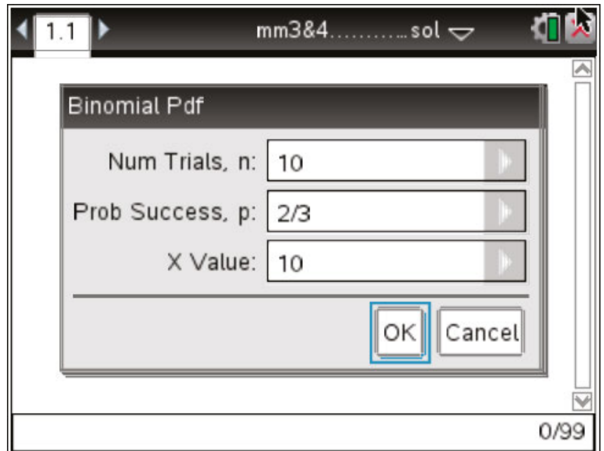
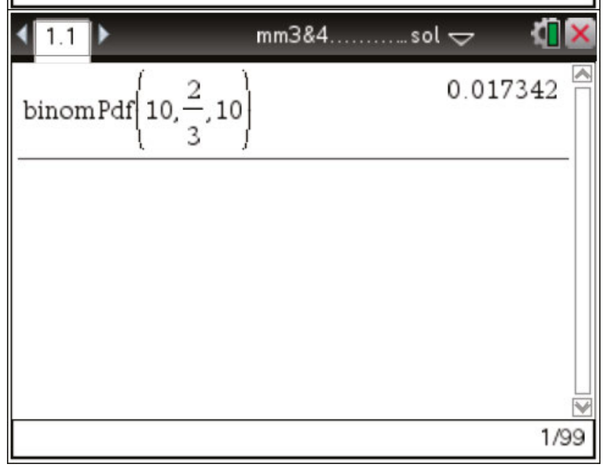
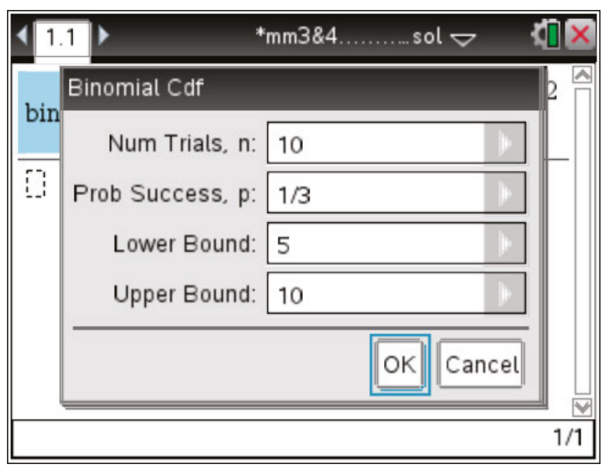
Graphic calculator techniques for question 1

a In a **Calculator** page select **Binomial Pdf** from the **Probability > Distributions** menu and complete as shown.





b For the cumulative binomial select **Binomial Cdf** from the **Probability > Distributions** menu and complete the dialogue box as shown.



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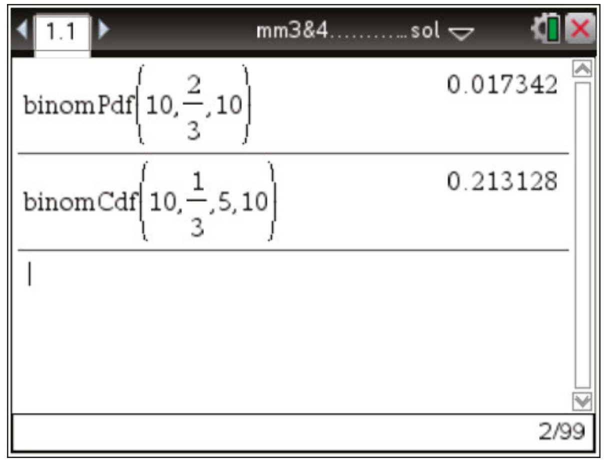
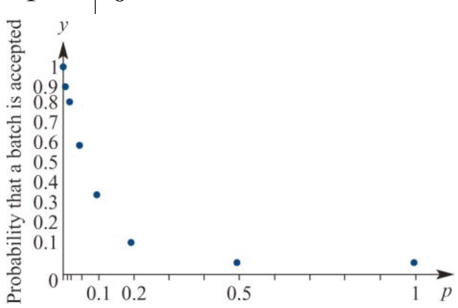


2 If the fraction of defective items is 0 the probability of acceptance is 1 If the fraction of defective items is 0.01 the probability of acceptance $(0.99)^{10} = 0.9044$ The results are recorded in the table shown:

p	probability that a batch is accepted
0	1
0.01	0.904 4
0.02	0.817 1
0.05	0.598 7
0.1	0.348 7
0.2	0.107 4
0.5	0.000 98
1	0



3 a The probability of a defective = 0.04

Let X be the number of defectives in a sample of ten.

$$\Pr(X \geq 2) = 1 - [\Pr(X = 0) + \Pr(X = 1)]$$

$$= 1 - \left[(0.96)^{10} + \binom{10}{1}(0.96)^{9}(0.04)\right]$$

$$= 0.0582$$

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b Mean number of defectives $= np = 0.04 \times 10 = 0.4$

$$\begin{aligned} \text{sd of the number of defectives} &= \sqrt{np(1-p)} \\ &= \sqrt{10 \times 0.04 \times 096} \\ &= 0.6197 \\ \mu \pm 2\sigma &= 0.4 \pm 2 \times 0.6197 \\ 0.4 + 2 \times 0.6197 &= 1.6394 \\ 0.4 - 2 \times 0.6197 &= -0.8394 \end{aligned}$$

c Yes, the claim of 4% defective must be questioned.

4 Let X be the number of Pizzas delivered late.

$\Pr(X \geq 12) = 1 - \Pr(X < 12)$

Note: X is a binomial random variable with $n = 67$ and $p = 0.1$

A table obtained through a calculator

x	$\Pr(X = x)$	$\Pr(X \leq x)$
0	0.0009	0.009
1	0.0064	0.0073
2	0.0235	0.0307
3	0.0565	0.0872
4	0.1004	0.1876
5	0.1046	0.3282
6	0.1614	0.4896
7	0.1563	0.6459
8	0.1302	0.7761
9	0.0949	0.8710
10	0.0611	0.9321
11	0.0352	0.9673

$$\begin{aligned} \therefore \quad \Pr(X \geq 12) &= 1 - 0.9673 \\ &= 0.0327 \end{aligned}$$

5 **a** **i** $p = \dfrac{1}{5}$, $n = 6$ Let X be the number of defectives.

$$\begin{aligned} \Pr(X = 3) &= \binom{6}{3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^3 \\ &= 0.0819 \end{aligned}$$

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ii $\Pr(X < 3) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)$

$= (0.8)^6 + 6 \times (0.8)^5(0.2) + {}^6C_2(0.8)^4(0.2)^2$

$= 0.9011$

b **i** Let X be the number of defectives.

$\Pr(X = 2) = \binom{6}{2}p^2(1-p)^4$

$= 15p^2(1-p)^4$

ii Let $P = 15p^2(1-p)^4$

$\frac{dP}{dp} = 30p(1-p)^4 - 60p^2(1-p)^3$

$= 30p[1-p]^3[1-3p]$

$\frac{dP}{dp} = 0 \Rightarrow p = 1 \text{ or } p = \frac{1}{3} \text{ or } p = 0$

Note when $p = 1$ or $p = 0$, $P = 0$

$\therefore \quad p = \frac{1}{3}$ gives a maximum probability.

6 **a** Mean value $= 1 \times \frac{53}{200} + 2 \times \frac{65}{200} + 3 \times \frac{45}{200} + 4 \times \frac{18}{200} + 5 \times \frac{2}{200}$

$= 2$

b $np = 2,\ n = 6 \therefore \quad p = \frac{1}{3}$

c $\therefore$ Probability distribution is as shown:

x	0	1	2	3	4	5	6
$\Pr(X = x)$	$\frac{64}{729}$	$\frac{64}{243}$	$\frac{80}{243}$	$\frac{160}{729}$	$\frac{20}{243}$	$\frac{4}{243}$	$\frac{1}{729}$

Multiple the probabilities in the table to obtain the theoretical frequencies.

Theoretical frequencies are as follows:

x	0	1	2	3	4	5	6
Theoretical no. of occurrences	17.56	52.68	65.84	43.90	16.46	3.29	0.274

7 **a** Let X be the number of faulty articles in as asample of size 10.

Then X is Bi $(n = 10,\ \text{p} = 0.05)$

Pr(batch accepted after first sample) $= \Pr(X < 2)$

$= \Pr(X = 0) \times \Pr(X = 1)$

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Using a CAS calculator given 0.9138616 = 0.9139 correct to 4 decimal places.

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b Batch is rejected if 3 or more faulty articles or if there are exactly 2 faulty articels and then a second sample of size 10 contains any faulty articles.

$\Pr(X \geq 3) = 0.0115036$

$\Pr(X = 2) = 0.0746348$

In a second sample, $\Pr(\geq 1 \text{ faulty articles}) = 1 - \Pr(\text{o faulty articles})$

$$= 1 - 0.95^{\circ}$$

$$= 0.4012631$$

$$\begin{aligned}\Pr(\text{batch rejected}) &= 0.0115036 + 0.0766348 \times 0.4012631\\ &= 0.0414517\\ &= 0.04145 \text{ correct to 4 significant figures.}\end{aligned}$$

c Either 10 which are tested or, if 2 of the sample of 10 are faulty, a second 10 (giving a total of 20) are tested.

Let $p' = \Pr(2 \text{ faulty article in first sample})$, so

$1 - p' = \Pr(0, 1, 3, \ldots 10 \text{ faculty articles in first sample})$.

Thein if y = number of articles tested, this gives:

y	10	20
$\Pr(y - q)$	$1 - p'$	p'

$E(Y) = 10(1 - p') + 20p' = 10p' + 10$

From past b, $p' = 0.0746 \Rightarrow E(Y) = 10(0.0746) + 10 = 10.746$

8 a Let X be the number of people with a birthday in January.

$$\Pr(X = 2) = \binom{6}{2}\left(\frac{1}{12}\right)^2\left(\frac{11}{12}\right)^4 = 0.0735$$

b Let Y be the number of people with a birthday in January.

$$\Pr(Y \geq 1) = 1 - \Pr(Y = 0) = 1 - \left(\frac{11}{12}\right)^8$$

$$= 0.5015$$

c Let Z be the number of people with a birthday in January.

$$\Pr(Z \geq 1) = 1 - \Pr(Z = 0) = 1 - \left(\frac{11}{12}\right)^N$$

$$1 - \left(\frac{11}{12}\right)^N > 0.9$$

$$\Leftrightarrow \left(\frac{11}{12}\right)^N < 0.1$$

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$$\Leftrightarrow N\log_e\left(\frac{11}{12}\right) < \log_e(0.1)$$

$$\Leftrightarrow N > \frac{\log_e(0.1)}{\log_e\left(\frac{11}{12}\right)}$$

$$\Leftrightarrow N > 26.46304$$

$\therefore$ Least value of $N = 27$.

9 For a two-engine plane

Let X be the number of engines which will fail.

The plane will successfully complete its journey if 0 or 1 engines fail.

$$\begin{aligned}\Pr(X = 0) + \Pr(X = 1) &= (1 - q)^2 + 2q(1 - q)\\ &= 1 - 2q + q^2 + 2q - 2q^2\\ &= 1 - q^2\end{aligned}$$

For a four-engine plane:

Let Y be the number of engines which will fail.

The plane will successfully complete its journey if 0, 1 or 2 engines fail.

$$\begin{aligned}\Pr(Y = 0) + \Pr(Y = 1) + \Pr(Y = 2) &= (1 - q)^4 + 4q(1 - q)^3 + 6q^2(1 - q)^2\\ &= (1 - q)^2[(1 - q)^2 + 4q(1 - q) + 6q^2]\\ &= (1 - q)^2[1 - 2q + q^2 + 4q - 4q^2 + 6q^2]\\ &= (1 - q)^2[1 + 2q + 3q^2]\end{aligned}$$

To find when a two-engine plane is to be preferred to a one-engine consider the inequality

$$1 - q^2 > (1 - q)^2(1 + 2q + 3q^2)$$

$$(1 - q)(1 + q) > (1 - q)^2(1 + 2q + 3q^2)$$

$$\therefore \quad (1 + q) > (1 - q)(1 + 2q + 3q^2)$$

$$(1 + q) > 1 + 2q + 3q^2 - q - 2q^2 - 3q^3q > q + q^2 - 3q^3$$

$$0 > q^2 - 3q^3$$

$$0 > q^2(1 - 3q)$$

$$\therefore \quad \frac{1}{3} \le q \le 1$$

A two-engine plane it to be preferred to a four-engine plane for $\frac{1}{3} \le q \le 1$

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Chapter 15 – Continuous random variables and their probability distributions

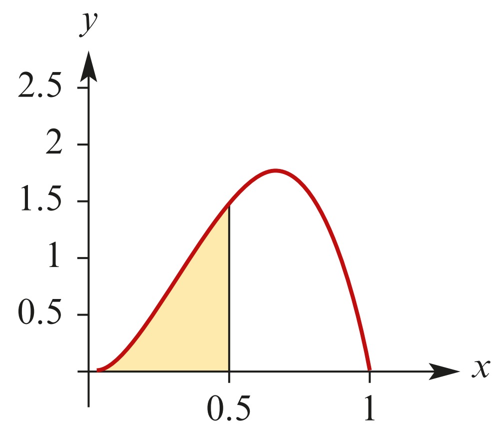
Solutions to Exercise 15A

1 $f(x)\begin{cases}\dfrac{24}{x^3} & 3 \le x \le 6\\ 0 & x < 3 \text{ or } x > 6\end{cases}$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_3^6 \frac{24}{x^3}\,dx$$
$$= \left[\frac{-12}{x^2}\right]_3^6$$
$$= \frac{-12}{6} + \frac{12}{3}$$
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$
$$f(x) \ge 0$$
$$\therefore \textit{ is a probability density function}$$

2
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$
$$\int_0^2 x^2 + kx + 1\,dx = 1$$
$$\left[\frac{x^3}{3} + \frac{kx^2}{2} + x\right]_0^2 = 1$$
$$\frac{8}{3} + \frac{4k}{2} + 2 = 1$$
$$2k = \frac{-11}{3}$$
$$k = \frac{-11}{6}$$

3 a and c



b

$$\Pr(X < 0.5) = \int_0^{0.5} 12x^2 - 12x^3\,dx$$
$$= \left[4x^3 - 3x^4\right]_0^{0.5}$$
$$= \frac{4}{8} - \frac{3}{16}$$
$$= \frac{8}{16} - \frac{3}{16}$$
$$= \frac{5}{16}$$

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4 a $$f(y) = \begin{cases} ke^{-y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f(y)\,dy = 1$$

$$\int_{0}^{\infty} ke^{-y}\,dy = 1$$

consider

$$\lim_{a\to\infty} \int_{0}^{a} ke^{-y}\,dy = 1$$

$$\lim_{a\to\infty} [-ke^{-y}]_0^a = 1$$

$$\lim_{a\to\infty} (-ke^{-a} + ke^0) = 1$$

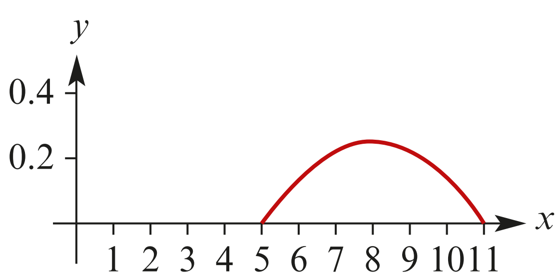
$$-k \lim_{a\to\infty} (e^{-a}) + k = 1$$

$$k = 1$$

b $$f(y) = \begin{cases} e^{-y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$$\begin{aligned} \Pr(Y \leq 2) &= \int_0^2 e^{-y}\,dy \\ &= [-e^{-y}]_0^2 \\ &= -e^{-2} + e^0 \\ &= 1 - \frac{1}{e^2} \approx 0.865 \end{aligned}$$

5 a



b $$\begin{aligned} \Pr(T \leq 7) &= \int_5^7 \frac{1}{36}(t-5)(11-t)\,dt \\ &= \frac{-1}{36}\int_5^7 t^2 - 16t + 55\,dt \\ &= \frac{-1}{36}\left[\frac{t^3}{3} - 8t^2 + 55t\right]_5^7 \\ &= \frac{1}{36}\left(\left(\frac{125}{3} - 200 + 275\right)\right. \\ &\quad \left. - \left(\frac{343}{3} - 392 + 385\right)\right) \\ &= \frac{1}{36}\left(\frac{350}{3} - \frac{322}{3}\right) \\ &= \frac{28}{108} \\ &= \frac{7}{27} \approx 0.259 \end{aligned}$$

c

$$\begin{aligned} \Pr(T < 7 | T > 5.5) &= \frac{\Pr(5.5 < T < 7)}{\Pr(T > 5.5)} \\ &= \frac{\int_{5.5}^{7} \frac{1}{36}(t-5)(11-t)\,dt}{\int_{5.5}^{11} \frac{1}{36}(t-5)(11-t)\,dt} \\ &\approx 0.244 \end{aligned}$$

d

$$\begin{aligned} \Pr(T < 7 | T < 10) &= \frac{\Pr(T < 7)}{\Pr(T < 10)} \\ &= \frac{\int_{5}^{7} \frac{1}{36}(t-5)(11-t)\,dt}{\int_{5}^{10} \frac{1}{36}(t-5)(11-t)\,dt} \\ &\approx 0.28 \end{aligned}$$

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6 a

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{7}^{17} k\sin\left(\frac{1}{10}\pi(x-7)\right)dx = 1$$

$$\left[\frac{-10}{\pi}k\cos\left(\frac{\pi}{10}(x-7)\right)\right]_7^{17} = 1$$

$$\frac{-10}{\pi}k\cos(\pi) + \frac{10}{\pi}k\cos(0) = 1$$

$$\frac{10}{\pi}k + \frac{10}{\pi}k = 1$$

$$k = \frac{\pi}{20} \quad QED$$

b i

$$\Pr(X \geq 16) = \int_{7}^{17} \frac{\pi}{20}\sin\left(\frac{\pi}{10}(x-7)\right)dx$$

$$= \left[\frac{-1}{2}\cos\left(\frac{\pi}{10}(x-7)\right)\right]_7^{17}$$

$$= \frac{-1}{2}\cos(\pi) + \frac{1}{2}\cos\left(\frac{9\pi}{10}\right)$$

$$\approx 0.024$$

ii $\Pr(12 \leq X \leq 13)$

$$\int_{12}^{13} \frac{\pi}{20}\sin\left(\frac{\pi}{10}(x-7)\right)dx$$

$$= \left[\frac{-1}{2}\cos\left(\frac{\pi}{10}(x-7)\right)\right]_{12}^{13}$$

$$= \frac{-1}{2}\cos\left(\frac{\pi}{10}\right) + \frac{1}{2}\cos\left(\frac{5\pi}{10}\right)$$

$$= \frac{-1}{2}\cos\left(\frac{3\pi}{5}\right)$$

$$\approx 0.155$$

7 a

$$\int_{-\infty}^{\infty} f(t)\,dt = 1$$

$$\int_{0}^{\infty} ke^{\left(\frac{-t}{200}\right)}\,dt = 1$$

Consider

$$\lim_{a\to\infty}\int_{0}^{a} ke^{\left(\frac{-t}{200}\right)}\,dt = 1$$

$$k \times \lim_{a\to\infty}\left[-200e^{\left(\frac{-t}{200}\right)}\right]_0^a = 1$$

$$k \times \lim_{a\to\infty}\left(-200e^{\left(\frac{-a}{200}\right)} + 200e^0\right) = 1$$

$$200k \times \lim_{a\to\infty}\left(1 - e^{\left(\frac{-a}{200}\right)}\right) = 1$$

$$200k = 1$$

$$k = \frac{1}{200}$$

$$= 0.005$$

b $\Pr(T \geq 1000) = \int_{1000}^{\infty} \frac{1}{200}e^{\left(\frac{-t}{200}\right)}dt$

Consider

$$\lim_{a\to\infty}\int_{1000}^{a} \frac{1}{200}e^{\left(\frac{-t}{200}\right)}dt$$

$$= \lim_{a\to\infty}\left[-e^{\left(\frac{-t}{200}\right)}\right]_{1000}^a$$

$$= \lim_{a\to\infty}\left(-e^{\left(\frac{-a}{200}\right)} + e^{-5}\right)$$

$$= \frac{1}{e^5}$$

$$\therefore \Pr(T \geq 1000) = \frac{1}{e^5} \approx 0.007$$

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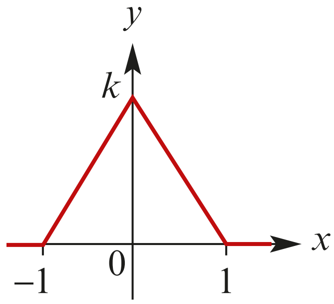
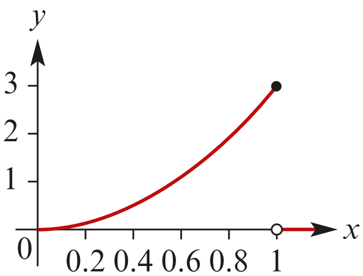
8 a



b
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$
$$\int_{-1}^{0} k(1+x)\,dx + \int_{0}^{1} k(1-x)\,dx = 1$$
$$\left[x + \frac{x^2}{2}\right]_{-1}^{0} + \left[x - \frac{x^2}{2}\right]_{0}^{1} = \frac{1}{k}$$
$$0 - \left(-1 + \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right) - 0 = \frac{1}{k}$$
$$\frac{1}{2} + \frac{1}{2} = \frac{1}{k}$$
$$k = 1$$

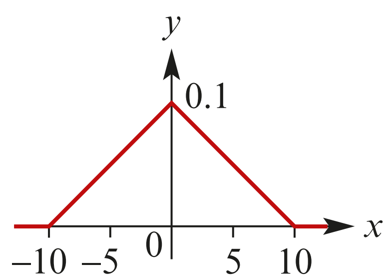
c $\Pr\left(\frac{-1}{2} \le X \le \frac{1}{2}\right)$
$$= \int_{\frac{-1}{2}}^{\frac{1}{2}} f(x)\,dx$$
$$= \int_{\frac{-1}{2}}^{0} 1 + x + \,dx + \int_{0}^{\frac{1}{2}} 1 - x\,dx$$
$$= \left[x + \frac{x^2}{2}\right]_{\frac{-1}{2}}^{0} + \left[x - \frac{x^2}{2}\right]_{0}^{\frac{1}{2}}$$
$$= 0 - \left(-\frac{1}{2} + \frac{1}{8}\right) + \left(\frac{1}{2} - \frac{1}{8}\right) - 0$$
$$= \frac{3}{8} + \frac{3}{8}$$
$$= \frac{3}{4}$$

9 a



b $$\Pr(0.25 < X < 0.75) = \int_{\frac{1}{4}}^{\frac{3}{4}} 3x^2\,dx$$
$$= \left[x^3\right]_{\frac{1}{4}}^{\frac{3}{4}}$$
$$= \frac{27}{64} - \frac{1}{64}$$
$$= \frac{13}{32} \approx 0.406$$

10 a



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b $\Pr(-1 \le X < 1)$

$$= \int_{-1}^{1} f(x)\,dx$$

$$= \int_{-1}^{0} \frac{1}{100}(10 + x)\,dx + \int_{0}^{1} \frac{1}{100}(10 - x)\,dx$$

$$= \frac{1}{100}\left[10x + \frac{x^2}{2}\right]_{-1}^{0} + \frac{1}{100}\left[10x - \frac{x^2}{2}\right]_{0}^{1}$$

$$= \frac{1}{100}\left(0 - \left(-10 + \frac{1}{2}\right)\right) + \left(10 - \frac{1}{2}\right) - 0$$

$$= \frac{19}{100} = 0.19$$

11 a $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{1000}^{\infty} \frac{k}{x^2}\,dx = 1$$

Consider

$$\lim_{a\to\infty} \int_{1000}^{a} \frac{k}{x^2}\,dx = 1$$

$$\lim_{a\to\infty}\left[\frac{-k}{x}\right]_{1000}^{a} = 1$$

$$\lim_{a\to\infty}\left(\frac{-k}{a} + \frac{k}{1000}\right) = 1$$

$$\frac{k}{1000} = 1$$

$$k = 1000$$

b $\Pr(X \ge 2000) = \int_{2000}^{\infty} \dfrac{1000}{x^2}\,dx$

Consider

$$\lim_{a\to\infty} \int_{2000}^{a} \frac{1000}{x^2}\,dx = \lim_{a\to\infty}\left[\frac{-1000}{x}\right]_{2000}^{a}$$

$$= \lim_{a\to\infty}\left(\frac{-1000}{a} + \frac{1000}{2000}\right)$$

$$= \frac{1}{2}$$

$\therefore\quad \Pr(X \ge 2000) = \dfrac{1}{2}$

12 a $\Pr(X \ge 1.5) = \displaystyle\int_{1.5}^{\infty} f(x)\,dx$

$$= \int_{1.5}^{2} 2\left(1 - \frac{1}{x^2}\right)dx$$

$$= \left[2\left(x + \frac{1}{x}\right)\right]_{1.5}^{2}$$

$$= 2\left(\left(2 + \frac{1}{2}\right) + \left(\frac{3}{2} + \frac{2}{3}\right)\right)$$

$$= \frac{2}{3}$$

b $\Pr(X \le 1.8 \mid X \ge 1.5)$

$$= \frac{\Pr(1.5 \le X \le 1.8)}{\Pr(X \ge 1.5)}$$

$$= \frac{\int_{1.5}^{1.8} 2\left(1 - \frac{1}{x^2}\right)dx}{\frac{2}{3}}$$

$$= \frac{3}{2}\left[2\left(x + \frac{1}{x}\right)\right]_{1.5}^{1.8}$$

$$= 3\left(\left(\frac{9}{5} + \frac{5}{9}\right) - \left(\frac{3}{2} + \frac{2}{3}\right)\right)$$

$$= \frac{17}{30}$$

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13 a $\Pr(X \geq 8) = \int_8^\infty f(x)\,dx$

$= \int_8^\infty \frac{1}{5}e^{\frac{-x}{5}}\,dx$

Consider

$$\lim_{a\to\infty}\int_8^a \frac{1}{5}e^{\frac{-x}{5}}\,dx = \lim_{a\to\infty}\left[-e^{\frac{-x}{5}}\right]_8^a$$

$$= \lim_{a\to\infty}\left(-e^{\frac{-a}{5}} + e^{\frac{-8}{5}}\right)$$

$$= e^{\frac{-8}{5}}$$

$\therefore \quad \Pr(X \geq 8) = e^{\frac{-8}{5}} \approx 0.202$

b $\Pr(X \geq 12 \mid X \geq 8) = \dfrac{\Pr(X \geq 12)}{\Pr(X \geq 8)}$

$= \dfrac{\int_{12}^\infty \frac{1}{5}e^{\frac{-x}{5}}\,dx}{e^{\frac{-8}{5}}}$

Consider

$$\frac{\lim_{a\to\infty}\int_{12}^a \frac{1}{5}e^{\frac{-x}{5}}\,dx}{e^{\frac{-8}{5}}} = \frac{\lim_{a\to\infty}\left[-e^{\frac{-x}{5}}\right]_{12}^a}{e^{\frac{-8}{5}}}$$

$$= e^{\frac{8}{5}}\lim_{a\to\infty}\left(-e^{\frac{-a}{5}} + e^{\frac{-12}{5}}\right)$$

$$= e^{\frac{8}{5}} \times e^{\frac{-12}{5}}$$

$$= e^{\frac{-4}{5}}$$

$\therefore \quad \Pr(x \geq 12 \mid X \geq 8) = e^{\frac{-4}{5}} \approx 0.449$

14 a $\Pr(X \leq 0.5 \mid = 0.45)$

$\int_{-\infty}^{0.5} f(x)\,dx$

$\int_{-1}^{0} 0.2\,dx + \int_{1}^{0.5} 0.2 + 1.2x\,dx$

$$= \frac{1}{5} + \left[\frac{x}{5} + \frac{3x^2}{5}\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{5} + \left(\frac{1}{10} + \frac{3}{20}\right)$$

$$= \frac{9}{20} = 0.45$$

b $\Pr(X > 0.5 \mid X > 0.1)$

$$= \frac{\Pr(X > 0.5)}{\Pr(X > 0.1)}$$

$$= \frac{1 - (a)}{\int_{0.1}^\infty f(x)\,dx}$$

$$= \frac{0.55}{\int_{0.1}^1 \frac{1}{5} + \frac{6}{5}x\,dx}$$

$$= \frac{11}{4\int_{0.1}^1 1 + 6x\,dx}$$

$$= \frac{11}{4\left[x + 3x^2\right]_{0.1}^1}$$

$$= \frac{11}{4\left((1+3) - \left(\frac{1}{10} + \frac{3}{100}\right)\right)}$$

$$= \frac{11}{\frac{387}{25}}$$

$$= \frac{275}{387} \approx 0.711$$

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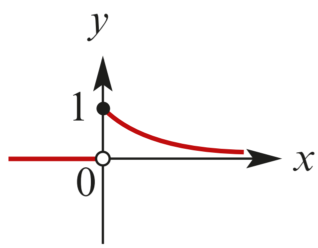
15 **a**



b **i** $\Pr(X < 0.5) = \int_0^{0.5} e^{-x}\,dx$

$= [-e^{-x}]_0^{0.5}$

$= -e^{-0.5} + e^0$

$= 1 - \dfrac{1}{\sqrt{e}}$

$= 1 - e^{-\frac{1}{2}}$

ii $\Pr(X \geq 1)$

$\int_1^{\infty} e^{-x}\,dx$

Consider

$$\lim_{a\to\infty} \int_1^{a} e^{-x}\,dx = \lim_{a\to\infty} [-e^{x}]_1^{a}$$

$$= \lim_{a\to\infty} (-e^{-a} + e^{-1})$$

$$= \frac{1}{e}$$

$\therefore \quad \Pr(X \geq 1) = \dfrac{1}{e} = e^{-1}$

iii

$$\Pr(X \geq 1 \,|\, X > 0.5) = \frac{\Pr(X \geq 1)}{\Pr(X > 0.5)}$$

$$= \frac{\frac{1}{e}}{\frac{1}{\sqrt{e}}}$$

$$= \frac{1}{\sqrt{e}}$$

$$= e^{-\frac{1}{2}}$$

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Solutions to Exercise 15B

1 a $f(x) = 2x,\ 0 < x < 1$

$$\begin{aligned}\mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)\,dx \\ &= \int_0^1 2x^2\,dx \\ &= \left[\frac{2}{3}x^3\right]_0^1 \\ &= \frac{2}{3}\end{aligned}$$

b $f(x) = \dfrac{1}{2\sqrt{x}},\ 0 < x < 1$

$$\begin{aligned}\mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)\,dx \\ &= \int_0^1 \frac{\sqrt{x}}{2}\,dx \\ &= \left[\frac{1}{3}x^{\frac{3}{2}}\right]_0^1 \\ &= \frac{1}{3}\end{aligned}$$

c $f(x) = 6x - 6x^2,\ 0 < x < 1$

$$\begin{aligned}\mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)\,dx \\ &= \int_0^1 6x^2 - 6x^3\,dx \\ &= \left[2x^3 - \frac{3}{2}x^4\right]_0^1 \\ &= 2 - \frac{3}{2} = \frac{1}{2}\end{aligned}$$

d $f(x) = \dfrac{1}{x^2},\ x \geq 1$

$$\begin{aligned}\mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)\,dx \\ &= \int_0^1 \frac{1}{x}\,dx\end{aligned}$$

consider

$$\begin{aligned}\lim_{a\to\infty}\int_1^a \frac{1}{x}\,dx &= \lim_{a\to\infty}[\ln|x|]_1^a \\ &= \lim_{a\to\infty}(\ln a - \ln 1) \\ &= \lim_{a\to\infty} \ln a\end{aligned}$$

$\therefore$ E(X) *does not exist*

2 a 1

b 2.097

c 1.132

d 0.4444

3 a

$$\begin{aligned}\mu &= \int_{-\infty}^{\infty} x f(x)\,dx \\ &= \int_0^1 2x^4 - x^2 + x\,dx \\ &= \left[\frac{2}{5}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2\right]_0^1 \\ &= \frac{2}{5} - \frac{1}{3} + \frac{1}{2} \\ &= \frac{12 - 10 + 15}{30} \\ &= \frac{17}{30} \approx 0.567\end{aligned}$$

b

$$\begin{aligned}\Pr(X \leq \mu) &= \int_{-\infty}^{\mu} f(x)\,dx \\ &= \int_0^{\frac{17}{30}} 2x^3 - x + 1\,dx \\ &= \left[\frac{1}{2}x^4 - \frac{1}{2}x^2 + x\right]_0^{\frac{17}{30}} \\ &= \frac{1}{2}\left(\frac{17}{30}\right)^4 - \frac{1}{2}\left(\frac{17}{30}\right)^2 + \frac{17}{30} \\ &\approx 0.458\end{aligned}$$

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4 $f(x) = \frac{1}{2\pi} + \frac{1}{2\pi}\cos x,\ -\pi \le x \le \pi$

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\pi}^{\pi} \frac{x}{2\pi} + \frac{x\cos x}{2\pi}\,dx$$

using the CAS calculator = 0

Alternatively, notice that the function is symmetrical about the y-axis, so the mean must be 0.

5 $\int_{-\infty}^{\infty} f(y)\,dy = 1$

$$\int_{0}^{B} Ay\,dy = 1$$

$$\left[\frac{A}{2}y^2\right]_0^B = 1$$

$$\frac{AB^2}{2} = 1$$

① $AB^2 = 2$

$$\mu = \int_{-\infty}^{\infty} y f(y)\,dy$$

$$2 = \int_{0}^{B} Ay^2\,dy$$

$$2 = \left[\frac{A}{3}y^3\right]_0^B$$

② $AB^3 = 6$

$\frac{②}{①} \Rightarrow B = 3$

Sub in ① $\Rightarrow$ $A(3)^2 = 2$

$$A = \frac{2}{9}$$

6 a $\mathrm{E}\left(\frac{1}{X}\right) = \int_{-\infty}^{\infty} \frac{1}{x} f(x)\,dx$

$$= \int_{0}^{1} 12x - 12x^2\,dx$$

$$= [6x^2 - 4x^3]_0^1$$

$$= 6 - 4 = 2$$

b $\mathrm{E}(e^x) = \int_{-\infty}^{\infty} e^x f(x)\,dx$

$$= \int_{0}^{1} 12x^2 e^x (1 - x)\,dx$$

using the CAS calculator = 1.858

7 a $\Pr(X \le 1)$

$$\int_{0}^{1} e^{-x}\,dx = \left[-e^{-x}\right]_0^1$$

$$= -e^1 + e^{-0}$$

$$= 1 - \frac{1}{e} \approx 0.632$$

b $\Pr(1 \le X \le 2) = \int_{1}^{2} e^{-x}\,dx$

$$= [-e^{-x}]_1^2$$

$$= -e^{-2} + e^{-1}$$

$$= \frac{e - 1}{e^2} \approx 0.233$$

c $\int_{0}^{m} e^{-x}\,dx = \frac{1}{2}$

$$[-e^{-x}]_0^m = \frac{1}{2}$$

$$-e^{-m} + e^0 = \frac{1}{2}$$

$$1 - e^{-m} = \frac{1}{2}$$

$$e^{-m} = \frac{1}{2}$$

$$e^m = 2$$

$$m = \ln 2 \approx 0.693$$

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8 **a** $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 k\,dx = 1$$

$$[kx]_0^1 = 1$$

$$k = 1$$

b $\int_0^m 1\,dx = \frac{1}{2}$

$$[m]_0^1 = \frac{1}{0}$$

$$m = \frac{1}{2}$$

9 $f(x) = 5(x-1)^4, 0 \le x \le 1$

$$\int_0^m 5(x-1)^4\,dx = \frac{1}{2}$$

$$[(x-1)^5]_0^m = \frac{1}{2}$$

$$(m-1)^5 - (-1)^5 = \frac{1}{2}$$

$$(m-1)^5 = -\frac{1}{2}$$

$$m - 1 = \left(\frac{1}{2}\right)^{\frac{1}{5}}$$

$$m = 1 - \left(\frac{1}{2}\right)^{\frac{1}{5}} \approx 0.\,1294$$

10 $\int_0^m \frac{1}{4}e^{\frac{-x}{4}}\,dx = \frac{1}{2}$

$$\left[-e^{\frac{-x}{4}}\right]_0^m = \frac{1}{2}$$

$$-e^{\frac{-m}{4}} + e^0 = \frac{1}{2}$$

$$e^{\frac{-m}{4}} = \frac{1}{2}$$

$$e^{\frac{m}{4}} = 2$$

$$\frac{m}{4} = \ln 2$$

$$m = 4\ln 2 \approx 2.773 \text{ minutes}$$

11 **a** $\mu = \int_{-\infty}^{\infty} xf(x)\,dx$

$$= \int_0^1 x^2\,dx + \int_1^2 2x - x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} + \left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)$$

$$= \frac{1}{3} + 4 - \frac{10}{3}$$

$$= 1$$

b $\int_0^m f(x)\,dx = \frac{1}{2}$

if $m \le 1$,

$$\int_0^m x\,dx = \frac{1}{2}$$

$$\left[\frac{x^2}{2}\right]_0^m = \frac{1}{2}$$

$$\frac{m^2}{2} = \frac{1}{2}$$

$$m^2 = 1$$

$$m = 1$$

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12 $f(x) = 30x^4 - 30x^5,\ 0 < x < 1$

a $$\begin{aligned}\mu &= \int_{-\infty}^{\infty} x f(x)\,dx \\ &= \int_0^1 30x^5 - 30x^6\,dx \\ &= \left[5x^6 - \frac{30}{7}x^7\right]_0^1 \\ &= 5 - \frac{30}{7}\end{aligned}$$

$$\mu = \frac{5}{7} \approx 0.714$$

b $$\begin{aligned}\int_{-\infty}^{m} f(x)\,dx &= \frac{1}{2} \\ \int_0^m 30x^4 - 30x^5\,dx &= \frac{1}{2} \\ \left[6x^5 - 5x^6\right]_0^m &= \frac{1}{2} \\ s^{6m^5} - 5m^6 &= \frac{1}{2}\end{aligned}$$

using the CAS calculator

$m \approx 0.736$

$\mu = 0.714 < m$ QED

13 $f(x) = \dfrac{\pi}{20}\sin\left(\dfrac{\pi}{10}(x-7)\right),\ 7 \le x \le 17$

$$\begin{aligned}\int_{-\infty}^{m} f(x)\,dx &= \frac{1}{2} \\ \int_7^m \frac{\pi}{20}\sin\left(\frac{\pi}{10}(x-7)\right)dx &= \frac{1}{2} \\ \left[\frac{-1}{2}\cos\left(\frac{\pi}{10}(x-7)\right)\right]_7^m &= \frac{1}{2} \\ -\cos\left(\frac{\pi}{10}(m-7)\right) + \cos 0 &= 1 \\ \cos\left(\frac{\pi}{10}(m-7)\right) &= 0 \\ \frac{\pi}{10}(m-7) &= \frac{\pi}{2} \\ m - 7 &= 5 \\ m &= 12\end{aligned}$$

14 **a** $$\begin{aligned}\mu &= \int_{-\infty}^{\infty} x f(x)\,dx \\ &= \int_{-1}^{0} \frac{x}{5}\,dx + \int_0^1 \frac{x}{5} + \frac{6x^2}{5}\,dx \\ &= \left[\frac{x^2}{10}\right]_{-1}^{0} + \left[\frac{x^2}{10} + \frac{2x^3}{5}\right]_0^1 \\ &= 0 - \frac{1}{10} + \frac{1}{10} + \frac{2}{5} - 0 \\ &= \frac{2}{5}\end{aligned}$$

b $\displaystyle\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$

if $m \le 0$,

$$\begin{aligned}\int_{-1}^{m} \frac{1}{5}\,dx &= \frac{1}{2} \\ \left[\frac{x}{5}\right]_{-1}^{m} &= \frac{1}{2} \\ \frac{m}{5} + \frac{1}{5} &= \frac{1}{2} \\ m &> 0\end{aligned}$$

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$$\therefore \int_{-1}^{0} \frac{1}{5}\,dx + \int_{0}^{m} \frac{1}{5} + \frac{6}{5}x\,dx = \frac{1}{2}$$

$$\frac{1}{5} + \left[\frac{x}{5} + \frac{3x^2}{5}\right]_0^m = \frac{1}{2}$$

$$\frac{m}{5} + \frac{3m^2}{5} - \frac{3}{10} = 0$$

$$3m^2 + m - \frac{3}{2} = 0$$

$$m = \frac{-1 \pm \sqrt{1+18}}{6}$$

since $m > 0$,

$$m = \frac{-1 + \sqrt{19}}{6}$$

15 a $\frac{d}{dx}(kxe^{-kx}) = ke^{-kx} - k^2xe^{-kx}$

$$\int kxe^{-kx}\,dx = \frac{-1}{k}\int -k^2xe^{-kx} + ke^{-kx} - ke^{-kx}\,dx$$

$$= \frac{-1}{k}\int ke^{-kx} - k^2xe^{-kx}\,dx + \int e^{-kx}\,dx$$

$$= \frac{-1}{k}(kxe^{-kx}) + \frac{-1}{k}e^{-kx}$$

$$= -xe^{-kx} - \frac{-1}{k}e^{-kx}$$

$$= -\frac{(kx+1)}{k}e^{-kx}$$

b $\mu = \int_{-\infty}^{\infty} xf(x)\,dx$

$$= \int_{0}^{\infty} \frac{x}{\lambda}e^{\frac{-x}{\lambda}}\,dx$$

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Consider

$$\lim_{a\to\infty}\int_1^a x\left(\frac{1}{\lambda}\right)e^{-\left(\frac{1}{\lambda}\right)x}\,dx$$

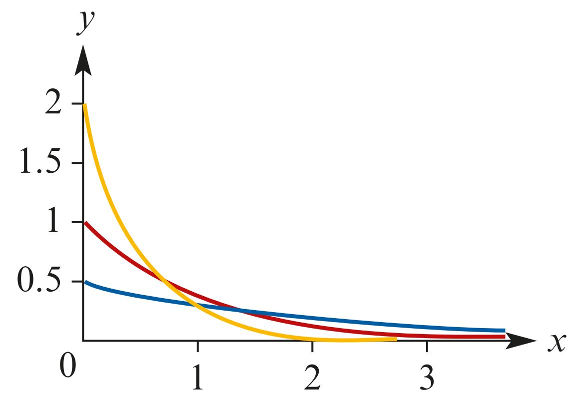
$$a = \lim_{a\to\infty}\left[xe^{-\frac{1}{\lambda}x} - \lambda e^{-\frac{1}{\lambda}x}\right]_0^a$$

$$= \lim_{a\to\infty}\left(-e^{-\frac{a}{\lambda}}(a+\lambda) + e^0(0+\lambda)\right)$$

$$= \lambda + \lim_{a\to\infty}\left(-(a+\lambda) - e^{-\frac{a}{\lambda}}\right)$$

$\mu = \lambda$

c



d $y = e^{-x}$ is dilated by factor $\dfrac{1}{\lambda}$ from the x-axis and by factor λ from the y-axis

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Solutions to Exercise 15C

1
$$
\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)\,dx \\
&= \int_0^1 2x^2\,dx \\
&= \left[\frac{2}{3}x^3\right]_0^1 \\
&= \frac{2}{3}
\end{aligned}
$$

$$(\mathrm{E}(X))^2 = \frac{4}{9}$$

$$
\begin{aligned}
\mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx \\
&= \int_0^1 2x^3\,dx \\
&= \left[\frac{1}{2}x^4\right]_0^1 \\
&= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\sigma^2 &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 \\
&= \frac{9}{18} - \frac{8}{18} = \frac{1}{18}
\end{aligned}
$$

$$\sigma = \frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{6}$$

2 (384, 416)

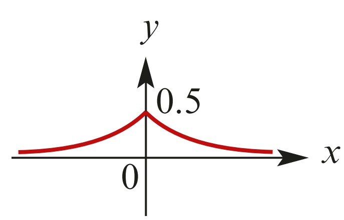
3 a
$$
\begin{aligned}
\int_{-\infty}^{a} x f(x)\,dx &= \frac{1}{4} \\
\int_0^a 3x^2\,dx &= \frac{1}{4} \\
[x^3]_0^a &= \frac{1}{4} \\
a^3 &= \frac{1}{4} \\
a &= \left(\frac{1}{4}\right)^{\frac{1}{3}} \approx 0.630
\end{aligned}
$$

b
$$
\begin{aligned}
\int_{-\infty}^{b} f(x)\,dx &= \frac{3}{4} \\
\int_0^b 3x^2\,dx &= \frac{3}{4} \\
[x^3]_0^b &= \frac{3}{4} \\
b^3 &= \frac{3}{4} \\
b &= \left(\frac{3}{4}\right)^{\frac{1}{3}} \approx 0.909
\end{aligned}
$$

c

$$
\begin{aligned}
\textit{the interquartile range} &= b - a \\
&= \left(\frac{3}{4}\right)^{\frac{1}{3}} - \left(\frac{1}{4}\right)^{\frac{1}{3}} \\
&\approx 0.279
\end{aligned}
$$

4 a



b
$$\int_{-\infty}^{a} f(x)\,dx = \frac{1}{4}$$

$$\int_{-\infty}^{a} 0.5e^x\,dx = \frac{1}{4},\ \textit{since } a < 0.$$

consider

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$$\lim_{k\to-\infty}\int_k^a \frac{1}{2}e^x\,dx = \frac{1}{4}$$

$$\lim_{k\to-\infty}[e^x]_k^a = \frac{1}{2}$$

$$\lim_{k\to-\infty}(e^a - e^k) = \frac{1}{2}$$

$$e^a = \frac{1}{2}$$

$$a = \ln\frac{1}{2}$$

$$a = -\ln 2$$

$$\int_b^\infty \frac{1}{2}e^{-x}\,dx = \frac{1}{4},\ \textit{since } b > 0$$

$$\int_b^\infty e^{-x}\,dx = \frac{1}{2}$$

consider

$$\lim_{h\to\infty}\int_b^h e^{-x}\,dx = \frac{1}{2}$$

$$\lim_{h\to\infty}[-e^{-x}]_b^h = \frac{1}{2}$$

$$\lim_{h\to\infty}(-e^{-h} + e^{-b}) = \frac{1}{2}$$

$$e^{-b} = \frac{1}{2}$$

$$-b = \ln\frac{1}{2}$$

$$b = \ln 2$$

$$\begin{aligned}\textit{the interquartile range} &= b - a\\ &= \ln 2 - (-\ln 2)\\ &= 2\ln 2 \approx 1.386\end{aligned}$$

5 **a** $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_1^9 \frac{k}{x}\,dx = 1$$

$$[k\ln x]_1^9 = 1$$

$$k\ln 9 - k\ln 1 = 1$$

$$k\ln 9 = 1$$

$$k = \frac{1}{\ln 9}$$

b $\displaystyle\mu = \int_{-\infty}^{\infty} xf(x)\,dx$

$$\int_1^9 \frac{1}{\ln 9}\,dx = \frac{8}{\ln 9} = \frac{4}{\ln 3} \approx 3.641$$

$$\begin{aligned}\mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx\\ &= \int_1^9 \frac{x}{\ln 9}\,dx\\ &= \left[\frac{x^2}{4\ln 3}\right]_1^9\\ &= \frac{81}{4\ln 3} - \frac{1}{4\ln 3}\\ &= \frac{20}{\ln 3}\end{aligned}$$

$$\begin{aligned}\sigma^2 &= \mathrm{E}(X^2) - \mu^2\\ &= \frac{20}{\ln 3} - \frac{16}{(\ln 3)^2}\\ &= \frac{20\ln 3 - 16}{(\ln 3)^2} \approx 4.948\end{aligned}$$

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6 a $$\int_{-\infty}^{a} f(x)\,dx = \frac{1}{4}$$

$$\int_{0}^{a} 2 - 2x\,dx = \frac{1}{4}$$

$$[2x - x^2]_0^a = \frac{1}{4}$$

$$2a - a^2 = \frac{1}{4}$$

$$a^2 - 2a + \frac{1}{4} = 0$$

$$a = \frac{2 \pm \sqrt{4-1}}{2}$$

$$a = 1 - \frac{\sqrt{3}}{2},$$

since $0 \le a \le 1$

$$\int_{-\infty}^{b} f(x)\,dx = \frac{3}{4}$$

$$\int_{0}^{b} 2 - 2x\,dx = \frac{3}{4}$$

$$\Big[2x - x^2\Big]_0^b = \frac{3}{4}$$

$$2b - b^2 = \frac{3}{4}$$

$$b^2 - 2b + \frac{3}{4} = 0$$

$$b = \frac{2 \pm \sqrt{4-3}}{2}$$

$$b = 1 \pm \frac{1}{2}$$

$$b = \frac{1}{2},\ \textit{since } 0 \le b \le 1$$

$$= b - a$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2} - 1$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}$$

$$\approx 0.366$$

b $$\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$= \int_0^1 2x - 2x^2\,dx$$

$$= \left[x^2 - \frac{2}{3}x^3\right]_0^1$$

$$= 1 - \frac{2}{3}$$

$$= \frac{1}{3}$$

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^1 2x^2 - 2x^3\,dx$$

$$= \left[\frac{2}{3}x^3 - \frac{1}{2}x^4\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{2}$$

$$= \frac{1}{6}$$

$$\sigma^2 = \mathrm{E}(X^2) - \mu^2$$

$$= \frac{1}{6} - \frac{1}{9}$$

$$= \frac{1}{18}$$

7 $$\int_{-\infty}^{a} f(x)\,dx = \frac{1}{4}$$

$$\int_0^a 2xe^{-x^2}\,dx = \frac{1}{4}$$

$$\Big[e^{-x2}\Big]_0^a = \frac{1}{4}$$

$$-e^{-a^2} + e^0 = \frac{1}{4}$$

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$$e^{-a^2} = \frac{3}{4}$$

$$-a^2 = \ln\frac{3}{4}$$

$$a = +\sqrt{\ln\frac{4}{3}} \approx 0.5364,$$

since $a > 0$

$$\int_{-\infty}^{b} f(x)\,dx = \frac{3}{4}$$

$$\int_{0}^{b} 2xe^{-x^2}\,dx = \frac{3}{4}$$

$$\left[e^{-x^2}\right]_0^b = \frac{3}{4}$$

$$e^{-b^2} + e^0 = \frac{3}{4}$$

$$e^{-b^2} = \frac{1}{4}$$

$$-b^2 = \ln\frac{1}{4}$$

$$b = +\sqrt{\ln 4} \approx 1.1774,$$

since $b > 0$

the interquartile range $= b - a$

≈ 0.641

8 **a** $\displaystyle\int_{-\infty}^{a} f(x)\,dx = \frac{1}{4}$

$$\int_{0}^{a} \frac{x}{2}\,dx = \frac{1}{4}$$

$$\int_{0}^{a} 2x\,dx = 1$$

$$\left[x^2\right]_0^a = 1$$

$$a^2 = 1$$

$a = 1,$ *since* $0 \le a \le 2$

$$\int_{-\infty}^{b} f(x)\,dx = \frac{3}{4}$$

$$\int_{0}^{b} \frac{x}{2}\,dx = \frac{3}{4}$$

$$\int_{0}^{b} 2x\,dx = 3$$

$$\left[x^2\right]_0^b = 3$$

$$b^2 = 3$$

$b = \sqrt{3},$ *since* $0 \le b \le 2$

the interquartile range $= b - a$

$= \sqrt{3} - 1$

≈ 0.732

b $\displaystyle\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$

$$= \int_{0}^{2} \frac{x^2}{2}\,dx$$

$$= \left[\frac{x^3}{6}\right]_0^2$$

$$= \frac{8}{6} - = \frac{4}{3}$$

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{0}^{2} \frac{x^3}{3}\,dx$$

$$= \left[\frac{x^4}{8}\right]_0^2$$

$$= \frac{16}{8} = 2$$

$$\sigma^2 = \mathrm{E}(X^2) - \mu^2$$

$$= 2 - \frac{16}{9} = \frac{2}{9}$$

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9 **a**

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^{10} kx(100 - x^2)\,dx = 1$$

$$k\int_0^{10} 100x - x^3\,dx = 1$$

$$k\left[50x^2 - \frac{x^4}{4}\right]_0^{10} = 1$$

$$k(5000 - 2500) = 1$$

$$k = \frac{1}{2500}$$

$$= 0.0004$$

b

$$\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$= \int_0^{10} \frac{100x^2 - x^4}{2500}\,dx$$

$$= \frac{1}{2500}\left[\frac{100}{3}x^3 - \frac{x^5}{5}\right]_0^{10}$$

$$= \frac{1}{2500}\left(\frac{100000}{3} - \frac{100000}{5}\right)$$

$$= \frac{1}{25}\left(\frac{2000}{15}\right)$$

$$= \frac{80}{15}$$

$$= \frac{16}{3}$$

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^{10} \frac{1}{2500}(100x^3 - x^5)\,dx$$

$$= \frac{1}{2500}\left[25x^4 - \frac{1}{6}x^6\right]_0^{10}$$

$$= \frac{1}{2500}\left(250000 - \frac{1000000}{6}\right)$$

$$= \frac{1}{25}\left(\frac{15000 - 10000}{6}\right)$$

$$= \frac{5000}{150}$$

$$= \frac{100}{3}$$

$$\sigma^2 = \mathrm{E}(X^2) - \mu^2$$

$$= \frac{100}{3} - \frac{256}{9}$$

$$= \frac{44}{9}$$

$$\sigma = \frac{2\sqrt{11}}{3} \approx 2.21$$

10 **a**

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$k\int_{-a}^{a} a^2 - x^2\,dx = 1$$

$$k\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = 1$$

$$k\left(\left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right) = 1$$

$$\frac{4}{3}a^3k = 1$$

$$k = \frac{3}{4a^3}$$

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b $\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$= \int_{-a}^{a} \frac{3x}{4a} - \frac{3x^3}{4a^3}\, dx$

$= \left[\frac{3x^2}{8a} - \frac{3x^4}{16a^3}\right]_{-a}^{a}$

$= \left(\frac{3a^2}{8a} - \frac{3a^4}{16a^3}\right) - \left(\frac{3a^2}{8a} - \frac{3a^4}{16a^3}\right)$

$= 0$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$= \int_{-a}^{a} \frac{3x^2}{4a} - \frac{3x^4}{4a^3}\, dx$

$= \left[\frac{x^3}{4a} - \frac{3x^5}{20a^3}\right]_{-a}^{a}$

$= 2\left(\frac{a^3}{4a} - \frac{3a^5}{20a^3}\right)$

$= 2\left(\frac{a^2}{4} - \frac{3a^2}{20}\right)$

$= \frac{5a^2}{10} - \frac{3a^2}{10}$

$= \frac{a^2}{5}$

$\sigma^2 = E(X^2) - \mu^2$

$= \frac{a^2}{5} - 0$

$= \frac{a^2}{5}$

but $\sigma = 2$

$\therefore \quad \sigma^2 = 4$

$\frac{a^2}{5} = 4$

$a^2 = 20$

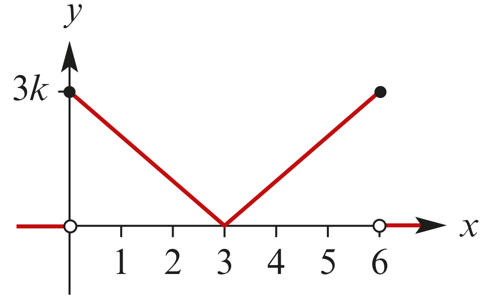
$a = \pm\sqrt{20}$

$s^a = \pm 2\sqrt{5}$

note: the way the question is stated implies that a is positive,

$\therefore \quad a = 2\sqrt{5}$

11 a



b

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_0^3 k(3-x)\, dx + \int_3^6 k(3-x)\, dx = 1$$

$$k\left(\left[3x - \frac{x^2}{2}\right]_0^3 + \left[\frac{x^2}{2} - 3x\right]_3^6\right) = 1$$

$$k\left(\left(\left(9 - \frac{9}{2}\right) - 0\right) + \left(\frac{36}{2} - 18\right) - \left(\frac{9}{2} - 9\right)\right) = 1$$

$$k \times 9 = 1$$

$$k = \frac{1}{9}$$

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c

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx$$
$$= \int_0^3 \frac{x}{9}(3-x)\,dx + \int_3^6 \frac{x}{9}(x-3)\,dx$$
$$= \int_0^3 \frac{x}{3} - \frac{x^2}{9}\,dx + \int_3^6 \frac{x^2}{9} - \frac{x}{3}\,dx$$
$$= \left[\frac{x^2}{6} - \frac{x^3}{27}\right]_0^3 + \left[\frac{x^3}{27} - \frac{x^2}{6}\right]_3^6$$
$$= \left(\left(\frac{9}{6} - \frac{27}{27}\right) - 0\right) + \left(\frac{216}{27} - \frac{36}{6}\right)$$
$$- \left(\frac{27}{27} - \frac{9}{6}\right)$$
$$= \frac{1}{2} + 2 + \frac{1}{2}$$
$$= 3 \quad QED$$

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$
$$= \int_0^3 \frac{x^2}{3} - \frac{x^3}{9}\,dx$$
$$+ \int_3^6 \frac{x^3}{9} - \frac{x^2}{3}\,dx$$
$$= \left[\frac{x^3}{9} - \frac{x^4}{36}\right]_0^3 + \left[\frac{x^4}{36} - \frac{x^3}{9}\right]_3^6$$
$$= \left(\left(\frac{27}{9} - \frac{81}{36}\right) - 0\right)$$
$$+ \left(\frac{6^4}{36} - \frac{216}{9}\right) - \left(\frac{81}{36} - \frac{27}{9}\right)$$
$$= 3 - \frac{9}{4} + 36 - 24 - \frac{9}{4} + 3$$
$$= 18 - \frac{9}{2}$$
$$= \frac{27}{2}$$

$$\sigma^2 = \mathrm{E}(X^2) - \mu^2$$
$$= \frac{27}{2} - 9$$
$$= \frac{9}{2} = 4.5$$

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Solutions to Exercise 15D

1 $E(X) = 4$

$$C = 300X + 100$$

$$\begin{aligned} E(C) &= E(300X + 100) \\ &= 300E(X) + 100 \\ &= 1300 \end{aligned}$$

2 **a**

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x)\, dx \\ &= \int_0^1 \frac{3x^3}{2} + x^2\, dx \\ &= \left[\frac{3x^4}{8} + \frac{x^3}{3}\right]_0^1 \\ &= \frac{3}{8} + \frac{1}{3} = \frac{17}{24} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\, dx \\ &= \int_0^1 \frac{3x^4}{2} + x^3\, dx \\ &= \left[\frac{3x^5}{10} + \frac{x^4}{4}\right]_0^1 \\ &= \frac{3}{10} + \frac{1}{4} = \frac{11}{20} \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E(X^2) - E(X)^2 \\ &= \frac{11}{20} - \frac{17^2}{24^2} \\ &\approx 0.048 \end{aligned}$$

b

$$\begin{aligned} E(V) &= E(100 - 1.5X) \\ &= 100 - 1.5E(X) \\ &= 100 - \frac{3}{2} \times \frac{17}{24} \\ &= \frac{4800}{48} - \frac{51}{48} \\ &= \frac{1583}{16} = \$98.94 \end{aligned}$$

$$\begin{aligned} \text{Var}(V) &= \text{Var}(100 - 1.5X) \\ &= 2.25\text{Var}(X) \\ &\approx 0.1086 \end{aligned}$$

$$\text{sd}(V) = \sqrt{\text{Var}(V)} = \$0.33$$

3 **a**

$$\begin{aligned} E(3X) &= \int_{-\infty}^{\infty} 3x f(x)\, dx \\ &= \int_{-1}^1 \frac{9x^3}{2}\, dx \\ &= \left[\frac{9x^4}{8}\right]_{-1}^1 \\ &= \frac{9}{8} - \frac{9}{8} = 0 \end{aligned}$$

$$\begin{aligned} E(9X^2) &= \int_{-\infty}^{\infty} 9x^2 f(x)\, dx \\ &= \int_{-1}^1 \frac{27x^4}{2}\, dx \\ &= \left[\frac{27x^5}{10}\right]_{-1}^1 \\ &= \frac{27}{10} + \frac{27}{10} = \frac{27}{5} = 5.4 \end{aligned}$$

$$\begin{aligned} \text{Var}(3X) &= E(9X^2) - E(3X)^2 \\ &= \frac{27}{5} \end{aligned}$$

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b $$\begin{aligned}\mathrm{E}(3-X) &= 3-\mathrm{E}(X)\\ &= 3-\frac{1}{3}\mathrm{E}(3X)\\ &= 3-0=3\\ \mathrm{Var}(3-X) &= \mathrm{Var}\,(X)\\ &= \frac{1}{9}\mathrm{Var}\,(3X)\\ &= \frac{3}{5}=0.6\end{aligned}$$

c 1, 5.4

d Let $g(x)$ be the function required
we know $g(x) = ax^2$ and since
$-1 \le x \le 1,\ -3 \le 3x \le 3$

$$\therefore \quad g(x) = \begin{cases} ax^2 & -3 \le 3x \le 3 \\ 0 & x < -3 \text{ or } x > 3 \end{cases}$$

$$\int_{-\infty}^{\infty} g(x)\,dx = 1$$

$$\int_{-3}^{3} ax^2\,dx = 1$$

$$\left[a\frac{x^3}{3}\right]_{-3}^{3} = 1$$

$$9a - (-9a) = 1$$

$$a = \frac{1}{18}$$

$$\therefore \quad g(x) = \begin{cases} \dfrac{x^2}{18} & -3 \le x \le 3 \\ 0 & x < -3 \text{ or } x > 3 \end{cases}$$

e $h(x) = \begin{cases} \dfrac{(x-1)^2}{18} & -2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$

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Solutions to Exercise 15E

1 a $F(x) = \int_0^x f(t)\,dt$

$= \int_0^x \frac{1}{5}\,dt$

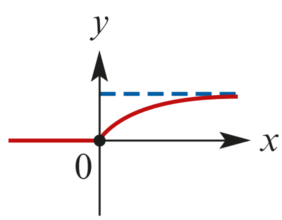
$= \left[\frac{t}{5}\right]_0^x$

$$F(x) = \begin{cases} \frac{x}{5} & \text{if } 0 < x \le 5 \\ 0 & \text{if } x \le 0 \\ 0 & x \ge 5 \end{cases}$$

b $\Pr(X \le 3) = F(3)$

$= \frac{3}{5}$

2 a



b $\Pr(X \ge 2) = 1 - \Pr(X < 2)$

$= 1 - F(2)$

$= 1 - (1 - e^{-4})$

$= e^{-4}$

c

$\Pr(X \ge 2 \mid X < 3) = \frac{\Pr(X \ge 2)}{\Pr(X < 3)}$

$= \frac{e^{-4}}{F(3)}$

$= \frac{e^{-4}}{1 - e^{-9}} \approx 0.0183$

3 a $F(6) = 1$

$k(6)^2 = 1$

$k = \frac{1}{36}$

b

$\Pr\left(\frac{1}{2} \le X \le 1\right) = \Pr(X \le 1) - \Pr\left(X < \frac{1}{2}\right)$

$= F(1) - F\left(\frac{1}{2}\right)$

$= \frac{1}{36}(1) - \frac{1}{36}\left(\frac{1}{4}\right)$

$= \frac{1}{48}$

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Solutions to Technology-free questions

1 a
$$k\int_1^{\sqrt{2}} x\,dx = 1$$
$$k\left[\frac{1}{2}x^2\right]_1^{\sqrt{2}} = 1$$
$$k\left(\frac{1}{2}\times 2 - \frac{1}{2}\times 1\right) = 1$$
$$\frac{1}{2}k = 1$$
$$k = 2$$

b
$$\Pr(1 < X < 1.1) = \int_1^{1.1} 2x\,dx$$
$$= \left[x^2\right]_1^{1.1}$$
$$= 1.21 - 1$$
$$= 0.21$$

c
$$\Pr(1 < X < 1.2) = \int_1^{1.2} 2x\,dx$$
$$= \left[x^2\right]_1^{1.2}$$
$$= 1.44 - 1$$
$$= 0.44$$

2
$$\int_0^1 (a + bx^2)\,dx = 1$$
$$\left[ax + \frac{1}{3}bx^3\right]_0^1 = 1$$
$$a + \frac{1}{3}b = 1 \qquad (1)$$

$\mathrm{E}(X) = \frac{2}{3}$, *so*:
$$\int_0^1 x(a + bx^2)\,dx = \frac{2}{3}$$
$$\int_0^1 x(ax + bx^3)\,dx = \frac{2}{3}$$
$$\left[\frac{1}{2}ax^2 + \frac{1}{4}bx^4\right]_0^1 = \frac{2}{3}$$
$$\frac{1}{2}a + \frac{1}{4}b = \frac{2}{3} \qquad (2)$$
$$2\times(2) - (1): \left(\frac{1}{2} - \frac{1}{3}\right)b = \frac{4}{3} - 1$$
$$\frac{1}{6}b = \frac{1}{3}$$
$$b = 2$$

Substituting in (1) gives $a = \frac{1}{3}$.

3 The graph of $\frac{1}{2}\sin x,\ 0 \le x \le \pi$, has x-intercepts at $(0,0)$ and $(\pi, 0)$ and a maximum at $\left(\frac{\pi}{2}, \frac{1}{2}\right)$.
Also, the graph is symmetrical about the live $x = \frac{\pi}{2}$, so the area under the curve from 0 to $\frac{\pi}{2}$ is $\frac{1}{2}$.
$\left(\text{Alternatively, solve } \int_0^m \frac{1}{2}\sin x\,dx = \frac{1}{2}.\right)$
(Note that the symmetry also implies that the mean is $\frac{\pi}{2}$.)

4 a
$$\Pr(1 < x < 3) = \frac{1}{4}(3 - 1) - \frac{1}{4}(1 - 1)$$
$$= \frac{1}{2}$$

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b $\Pr(X > 2 \mid 1 < X < 3)$

$$= \frac{\Pr(X > 2 \cap \mid < X < 3)}{\Pr(1 < X < 3)}$$

$$= \frac{\Pr(2 < X < 3)}{\Pr(1 < X < 3)}$$

$$= \frac{\Pr(X < 3) - \Pr(X < 2)}{\left(\frac{1}{2}\right)}$$

$$= \frac{\frac{1}{4}(3-1) - \frac{1}{4}(2-1)}{\frac{1}{2}}$$

$$= \frac{1}{2}(2-1)$$

$$= \frac{1}{2}$$

c

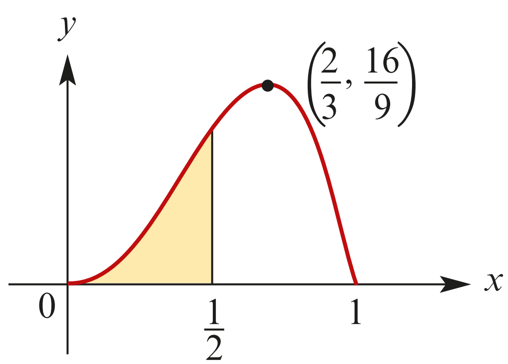
$$\Pr(X > 4 \mid X > 2) = \frac{\Pr(X > 4 \cap X > 2)}{P(X > 2)}$$

$$= \frac{\Pr(X > 4)}{\Pr(X > 2)}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}}$$

$$= \frac{1}{3}$$

5 a



b $P(X < 0.5) = \int_0^{0.5} (12x^2 - 12x^3)\,dx$

$$= \left[4x^3 - 3x^4\right]_0^{0.5}$$

$$= 4 \times \frac{1}{8} - 3 \times \frac{1}{16}$$

$$= \frac{5}{16}$$

6 a $k\int_0^1 (x^2 - x^3)\,dx = 1$

$$k\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = 1$$

$$k\left(\frac{1}{3} - \frac{1}{4}\right) = 1$$

$$\frac{1}{12}k = 1$$

$$k = 12$$

(Note that this agrees with the function given in Qn. 5 above.)

b $f(x) = 12x^2 - 12x^3$

$$f'(x) = 24x - 36x^2$$

$$= 12x(2 - 3x)$$

$$= 0 \text{ if } x = 0 \text{ } or \text{ } x = \frac{2}{3}$$

Since the graph of $y = f(x)$ is that of a negative cabic with x intercepts $(0, 0)$ and $(1, 0)$, $x = \frac{2}{3}$ must correspond to a local maximum. So the mode is $\frac{2}{3}$.

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$$\Pr\left(X < \frac{2}{3}\right) = \int_0^{\frac{2}{3}} (12x^2 - 12x^3)\,dx$$
$$= \left[4x^3 - 3x^4\right]_0^{\frac{2}{3}}$$
$$= 4 \times \frac{8}{27} - 3 \times \frac{16}{81}$$
$$= \frac{16}{27}$$

c $$\Pr\left(X < \frac{1}{3} \middle| X < \frac{2}{3}\right) = \frac{\Pr\left(X < \frac{1}{3}\right)}{\Pr\left(X < \frac{2}{3}\right)}$$
$$\Pr\left(X < \frac{1}{3}\right) = \left[4x^3 - 3x^4\right]_0^{\frac{1}{3}}$$
$$= \frac{1}{9}, \text{ so}$$
$$\Pr\left(X < \frac{1}{3} \middle| X < \frac{2}{3}\right) = \frac{\frac{1}{9}}{\frac{16}{27}} = \frac{3}{16}$$

7 **a** $$\Pr(X < 0.2) = \int_0^{0.2} 3x^2\,dx$$
$$= \left[x^3\right]_0^{0.2}$$
$$= 0.008$$

b $$\Pr(X < 0.2 | X < 0.3) = \frac{\int_0^{0.2} 3x^2\,dx}{\int_0^{0.3} 3x^2\,dx}$$
$$= \frac{0.008}{0.027}$$
$$= \frac{8}{27}$$

8 $$\Pr(X < m) = \int_0^m \frac{\pi}{4}\cos\left(\frac{\pi x}{4}\right)dx$$
$$= \left[\sin\left(\frac{\pi x}{4}\right)\right]_0^m$$
$$= \sin\left(\frac{\pi m}{4}\right)$$
$$\sin\left(\frac{\pi m}{4}\right) = 0.5$$
$$\frac{\pi m}{4} = \frac{\pi}{6}$$
$$m = \frac{2}{3}$$

9 **a** $$\mathrm{E}(X) = \int_0^4 \frac{x(x+2)}{16}\,dx$$
$$= \frac{1}{16}\int_0^4 x^2 + 2x\,dx$$
$$= \frac{1}{16}\left[\frac{x^3}{3} + x^2\right]_0^4$$
$$= \frac{7}{3}$$

b $$\int_0^a \frac{(x+2)}{16}\,dx = \frac{5}{32}$$
$$\int_0^a (x+2)\,dx = \frac{5}{2}$$
$$\left[\frac{x^2}{2} + 2x\right]_0^a = \frac{5}{2}$$
$$\frac{a^2}{2} + 2a = \frac{5}{2}$$
$$a^2 + 4a - 5 = 0$$
$$(a+5)(a-1) = 0$$
$$a = -5 \text{ or } a = 1$$
$$\therefore a = 1$$

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10 **a** $\int_{-1}^{1} c(1-x^2)\,dx = c\left[x - \frac{x^3}{3}\right]_{-1}^{1}$

$= \frac{4c}{3}$

For PDF $\quad \frac{4c}{3} = 1$

$\therefore c = \frac{3}{4}$

b 0

11 $\int_0^1 n(1-x)^{n-1}\,dx = \left[\frac{-n(1-x)^n}{n}\right]_0^1$

$= 1$

12 **a** $\int_0^m \frac{1}{x}\,dx = \left[\log_e(x)\right]_1^m$

$= \log_e m$

For median

$\log_e m = \frac{1}{2}$

$m = e^{\frac{1}{2}}$

For the interquartile range:

b $\log_e m = \frac{1}{4}$

$m = e^{\frac{1}{4}}$

c $\log_e m = \frac{3}{4}$

$m = e^{\frac{3}{4}}$

$\text{IQR} = e^{\frac{3}{4}} - e^{\frac{1}{4}} \approx 0.833$

13 For a continuous variable X,
$P(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$
Here, $\mu = 330$ and $\sigma = 5$,
$\mu + 2a = 330 + 10 = 340$

$\mu - 2\sigma = 330 - 10 = 320$
so $(320, 340)$ is the required (approximate) interval for 95% of cans.

14 Since the variance is 4, the standard deviation is $\sqrt{4} = 2$.
$\mu + 2\sigma = 250 + 4 = 254$

$\mu - 2\sigma = 250 - 4 = 246$
so $(2 \times 6, 254)$ is an (approximate) 95% interval.

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Solutions to multiple-choice questions

1 B The second graph partly lies below the x-axis. Since $f(x) \geq 0$ for all x, this could not requirement a probability density function.

2 D An antiderivative of $4x$ is $2x^2$. If the domain is of the from $0 < x < a$, then

$$\int_0^a 4x\,dx = \left[2x^2\right]_0^a$$
$$= 2a^2$$
$$= 1$$
$$\Rightarrow a = \frac{1}{\sqrt{2}} \text{ (since } a > 0\text{)},$$

so option **D** fits.
(Note that the above shows options $A \rightarrow C$ are not posible.

For option E: $\displaystyle\int_{\frac{1}{\sqrt{2}}}^{\frac{2}{\sqrt{2}}} 4x\,dx$

$$= \left[2x^2\right]_{\frac{1}{\sqrt{2}}}^{\frac{2}{\sqrt{2}}}$$
$$= 2\left(\frac{4}{2}\right) - 2\left(\frac{1}{2}\right)$$
$$= 3$$

3 D $$\int_0^L \frac{1}{2}\sin x\,dx = \left[-\frac{1}{2}\cos x\right]_0^k$$
$$= -\frac{1}{2}\cos k + \frac{1}{2}$$
$$= 1$$
$$\text{if } \cos k = -1$$
$$k = \pi$$

4 A $$\Pr(X \leq 1.3) = \int_1^{1.3} \frac{3}{4}(x^2 - 1)\,dx$$
$$= \left[\frac{1}{4}x^3 - \frac{3}{4}x\right]_1^{1.3}$$
$$\approx 0.0743$$

5 E $$\mathrm{E}(X) = \int_1^2 x \times \frac{3}{4}(x^2 - 1)\,dx$$
$$= \int_1^2 \left(\frac{3}{4}x^3 - \frac{3}{4}x\right)dx$$
$$= \left[\frac{3}{16}x^4 - \frac{3}{8}x^2\right]_1^2$$
$$= \frac{27}{16}$$

6 B $$\mathrm{E}(x^2) = \int_1^2 x^2 \times \frac{3}{4}(x^2 - 1)\,dx$$
$$= \int_1^2 \left(\frac{3}{4}x^4 - \frac{3}{4}x^2\right)dx$$
$$= \left[\frac{3}{20}x^5 - \frac{1}{4}x^3\right]_1^2$$
$$= \frac{29}{10}$$
$$\mathrm{var}(X) = \mathrm{E}(X)^2 - [\mathrm{E}(X)]^2$$
$$= \frac{29}{10} - \left(\frac{27}{16}\right)^2$$
$$= \frac{67}{1280}$$

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7 C $\displaystyle\int_0^m \frac{x^3}{4}\,dx = \frac{1}{2}$

$$\left[\frac{x^4}{16}\right]_0^m = \frac{1}{2}$$

$$\frac{m^4}{16} = \frac{1}{2}$$

$$m^4 = 8$$

$$m = \sqrt[4]{8}$$

$$\approx 1.6818$$

8 E The graph of $y = f(x)$ is that of a positive cubic which touches the x-axis at (2, 0). The end points are (1, 0) and (3, 3). Click for a local maxium turning point.

$$f(x) = \frac{3}{2}(x-1)(x-2)^2$$

$$= \frac{3}{2}(x^3 - 5x^2 + 8x - 6)$$

$$f'(x) = \frac{3}{2}(3x^2 - 10x + 8)$$

$$= \frac{3}{2}(x-2)(3x-4)$$

$$= 0 \text{ if } x = 2 \text{ } or \text{ } x\frac{4}{3}.$$

Then $x = \dfrac{4}{3}$ must correspond to a local maximum turning point.

$$f\left(\frac{4}{3}\right) = \frac{3}{2}\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)^2$$

$$= \frac{2}{9}$$

But $f(3) = 3,$ so the mode is 3 $\left(\text{NOT } \dfrac{4}{3}!\right)$

9

C $\displaystyle E(X) = \int_0^{20} x \times \frac{x}{40\,000}(400 - x^2)\,dx$

$$= \frac{32}{3}$$

(using a CAS calculator)

Then the expected consulations time for three patients is $3 \times \dfrac{32}{3} = 32$ min.

10 A Let s be the minimum score for an 'A'.

Then Pr $(X \geq S) = 0.10$ or equivalently Pr $(X < S) = 0.909$.

Hence $\displaystyle\int_0^s \frac{\pi}{100}\sin\left(\frac{\pi x}{50}\right)dx = 0.90$

$$\left[-\frac{1}{2}\cos\left(\frac{\pi x}{50}\right)\right]_0^s = 0.90$$

$$-\frac{1}{2}\cos\left(\frac{\pi s}{50}\right) + \frac{1}{2} = 0.90$$

$$\cos\left(\frac{\pi s}{50}\right) = -0.80$$

$$\frac{\pi s}{50} = \cos^{-1}(-0.80)$$

$$s = \frac{50}{\pi}\cos^{-1}(-0.80)$$

$$\approx 39.8$$

so the minimum score required is closest to 40.

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Solutions to extended-response questions

1 $f(x) = \begin{cases} \dfrac{a}{100}\left(1 - \dfrac{x}{100}\right) & \text{if } 100 \le x \le 1000 \\ 0 & \text{otherwise} \end{cases}$

a $\int_{100}^{1000} f(x)\,dx = \left[\dfrac{a}{100}\left(x - \dfrac{x^2}{200}\right)\right]_{100}^{1000} = -\dfrac{81a}{2}$

For f to be a probability density function $-\dfrac{81a}{2} = 1$ and hence $a = -\dfrac{2}{81}$

b $\mathrm{E}(x) = \int_{100}^{1000} xf(x)\,dx = \left[-\dfrac{2}{8100}\left(\dfrac{x^2}{2} - \dfrac{x^3}{300}\right)\right]_{100}^{1000} = 700$ hours

c The cumulative probability density function

$= \int_{100}^{x} f(t)\,dt = \dfrac{1}{810\,000}(x^2 - 200x + 10\,000)$

Median = 736.4 hours

2 $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x - x^2 & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$

a $\Pr(X > 0.5) = 1 - \Pr(X \le 0.5)$

$= 1 - F(0.5)$

$= \dfrac{1}{4}$

b $\Pr(X < a) = 0.8$. This can be written as

$F(a) = 0.8$ which implies,

$2a - a^2 = 0.8$

Solving the quadratic for a gives $a = \dfrac{5 - \sqrt{5}}{5}$ or $a = \dfrac{5 + \sqrt{5}}{5}$

But $0 < a < 1$. Therefore $a = \dfrac{5 - \sqrt{5}}{5}$.

c The corresponding probability density function is $f(x) = 2 - 2x$

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$$\begin{aligned} \mathrm{E}(X) &= \int_0^1 xf(x)\,dx \\ &= \int_0^1 2x - 2x^2\,dx \\ &= \left[x^2 - \frac{2x^3}{3}\right]_0^1 \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \mathrm{E}\left(\sqrt{X}\right) &= \int_0^1 \sqrt{x}f(x)\,dx \\ &= \int_0^1 2x^{\frac{1}{2}} - 2x^{\frac{3}{2}}\,dx \\ &= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{4}{5}x^{\frac{5}{2}}\right]_0^1 \\ &= \frac{8}{15} \end{aligned}$$

3 a $f(x) = \begin{cases} \dfrac{\pi}{20}\cos\left(\dfrac{\pi}{10}(x-6)\right) & \text{if } 1 \le x \le 11 \\ 0 & \text{if } x < 1 \text{ or } x > 11 \end{cases}$

For the median:

Let q_1 be the first quartile and q_3 be the third quartile. Then

$\int_1^{q_1} f(x)\,dx = 0.25$ and $\int_1^{q_3} f(x)\,dx = 0.75$

From the earlier calculation, $\int_1^{q_1} f(x)\,dx = \frac{1}{2}\left(\sin\left(\frac{\pi}{10}(a-6)\right)+1\right)$.

$$\begin{aligned} \text{For } q_1\text{:} \frac{1}{2}\left(\sin\left(\frac{\pi}{10}(q_1-6)+1\right)\right) &= 0.25 \\ \sin\left(\frac{\pi}{10}(q_1-6)\right) &= -0.5 \\ \frac{\pi}{10}(q_1-6) &= -\frac{\pi}{6} \\ q_1 - 6 &= \frac{-5}{3} \\ q_1 &= \frac{13}{3} \end{aligned}$$

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$$\text{For } q_3 : \frac{1}{2}\left(\sin\left(\frac{\pi}{10}(q_3 - 6) + 1\right) = 0.75\right.$$

$$\sin\left(\frac{\pi}{10}(q_3 - 6)\right) = 0.5$$

$$\frac{\pi}{10}(q_3 - 6) = \frac{\pi}{6}$$

$$q_3 - 6 = \frac{5}{3}$$

$$q_3 = \frac{23}{3}$$

$$\text{Interqurtile range} = q_3 - q_1$$

$$= \frac{23}{3} - \frac{13}{3} = \frac{10}{3}$$

b The graph of $y = f(x)$ is symmetrical about the line $x = 6$, so the mean is 6, Alternatively:

$$E(X) = \int_1^{11} x f(x)\, dx = \frac{\pi}{20}\int_1^{11} x \cos\left(\frac{\pi}{10}(x - 6)\right) dx$$

Since the integrand is not a 'standard' function fn integration, use a CAS calculator for its evaluation. This gives $E(X) = 6$.

For the variance, first find $E(X^2)$.

$$E(X^2) = \int_1^{11} x^2 f(x)\, dx = \frac{\pi}{20}\int_1^{11} x^2 \cos\left(\frac{\pi}{10}(x - 6)\right) dx$$

Using the integration command of a CAS calculates gives $E(X^2) \approx 40.7358$ Hence

$$\text{var}(X) = E(X^2) - [E(X)]^2$$

$$\approx 40.7358 - 36$$

$$= 4.736 \text{ correct to 3 decimal places.}$$

$$\int_1^a f(x)\, dx = 0.5$$

$$\frac{\pi}{20}\int_1^a \cos\left(\frac{\pi}{10}(x - 6)\right) dx = 0.5$$

$$\frac{\pi}{20} \times \frac{10}{\pi}\left[\sin\left(\frac{\pi}{10}(x - 6)\right) dx\right]_1^a = 0.5$$

$$\sin\left(\frac{\pi}{10}(a - 6)\right) - \sin\left(-\frac{\pi}{2}\right) = 1$$

$$\sin\left(\frac{\pi}{10}(a - 6)\right) = 0$$

$$a = 6$$

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4 a $\Pr(4 \le Y \le 5)$

$$\int_4^5 f(y)\,dy\ (not\ f(x)\,dx) = \frac{2}{25}\int_4^5 y - 1\,dy$$
$$= \frac{2}{25}\left[\frac{1}{2}y^2 - y\right]_4^5$$
$$= \frac{7}{25}$$

b
$$\mathrm{E}(Y) = \int_1^6 yf(y)\,dy$$
$$= \frac{2}{25}\int_1^6 y(y-1)\,dy$$
$$= \frac{2}{25}\left[\frac{1}{3}y^3 - \frac{1}{2}y^2\right]_1^6$$
$$= \frac{2}{25}\times\left(54 + \frac{1}{6}\right) = \frac{13}{3}$$

Also $\mathrm{E}(X) = 8 \times 0.6 = 4.80$

The expected money received $= \$\left(4.80 + 4 \times \frac{13}{3}\right) = \$22.13$

5
$$\mathrm{E}(X-c)^2 = \int_2^4 (x^2 - 2cx + c^2)f(x)\,dx$$
$$= \frac{1}{2}\int_2^4 x^2 - 2cx + c^2)(x-2)\,dx$$
$$= \frac{1}{2}\int_2^4 (x^3 - (2c+2)^2 + (c^2 4c)x - 2c^2)\,dx$$
$$= \frac{1}{3}(3c^2 - 20c + 34)$$

If $\mathrm{E}(X-c)^2 = \frac{2}{3}$

implies $3c^2 - 20c + 34 = 2$

$3c^2 - 20c + 32 = 0$

$(3c-8)(c-4) = 0$

Therefore $c = \frac{8}{3}$ or $c = 4$

6 $f(x) = \begin{cases} k(x-1)(3-x) & \text{if } 1 \le x \le 3 \\ 0 & \text{if } x < 1 \text{ or } x > 3 \end{cases}$

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a $$\int_1^3 f(x)\,dx = 1$$

$$\text{Therefore } k\int_1^3 -x^2 + 4x - 3\,dx = 1$$

$$\text{Hence } k = \frac{3}{4}$$

b $$\begin{aligned} \mathrm{E}(X) &= \frac{3}{4}\int_1^3 xf(x)\,dx \\ &= \frac{3}{4}\int_1^3 x(-x^2 + 4x - 3)\,dx \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathrm{var}\,(X) &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \\ &= \frac{3}{4}\int_1^3 xf(x)\,dx - 4 \\ &= \frac{21}{5} - 4 \\ &= \frac{1}{5} = 0.2 \end{aligned}$$

c $\Pr\,(X > 2.5) = \frac{3}{4}\int_{2.5}^3 -x^2 + 4x - 3\,dx = \frac{5}{32}$

(Note: the integral, in this question have been evaluated using the 'Integral' command of a CAS calculator.)

7 $f(x) = \begin{cases} \dfrac{k}{12(x-1)^3} & \text{if } 0 \le x \le 4 \\ 0 & \text{if } x < 0 \text{ or } x > 4 \end{cases}$

a $$\int_0^4 f(x)\,dx = 1 \Rightarrow \frac{k}{12}\int_0^4 \frac{1}{(x+1)^3}\,dx = 1$$

$$\frac{k}{12}\left[-\frac{1}{2(x+1)^2}\right]_0^k = 1$$

$$\frac{k}{12}\left(-\frac{1}{50} + \frac{1}{2}\right) = 1$$

$$\frac{k}{25} = 1$$

$$k = 25$$

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b $\mathrm{E}(X+1) = \dfrac{25}{12}\displaystyle\int_0^4 \dfrac{1}{(x+1)^2}\,dy$

$$= \frac{25}{12}\left[-\frac{1}{(x+1)}\right]_0^4$$

$$= \frac{25}{12}\left(-\frac{1}{5}+1\right)$$

$$= \frac{5}{3}$$

$$\mathrm{E}(X+1) = \mathrm{E}(X) + 1$$

$$\mathrm{E}(X) = \mathrm{E}(X+1) - 1$$

$$= \frac{5}{3} - 1 = \frac{2}{3}$$

d $P(X \le c) = c$

$$\frac{25}{12}\int_0^c \frac{1}{(x+1)^3}\,dx = c$$

$$-\frac{25}{24}\left[\frac{1}{(x+1)^2}\right]_0^c = c$$

$$-\frac{25}{24}\left(\frac{1}{(c+1)^2} - 1\right) = c$$

Using the 'solve' command of a CAS calculator gives $c = \dfrac{-13}{6}$ or $c = 0$ or $c = \dfrac{2}{3}$.

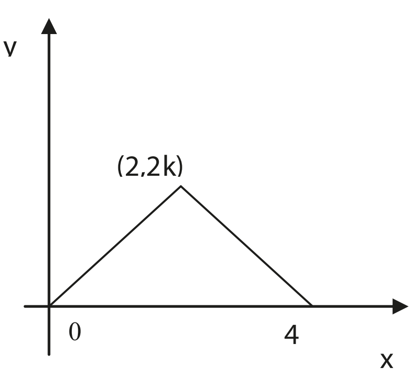
But $c > 0$, so $c = \dfrac{2}{3}$.

8 $f(x) = \begin{cases} kx & \text{if if } 0 \le x < 2 \\ k(4-x) & \text{if } 2 \le x < 4 \\ 0 & \text{if } x < 0 \text{ or } x > 4 \end{cases}$

a The area of the triangle $= \dfrac{1}{2} \times 4 \times 2k$

$$= 4k$$

Therefore $k = \dfrac{1}{4}$



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b Since the graph of $y = f(x)$ is symmetrical about $x = 2$, $\mathrm{E}(X) = 2$.

$$\begin{aligned}\text{var}(X) &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \\ &= \frac{1}{4}\int_0^2 x^3\,dx + \frac{1}{4}\int_2^4 x^2(4-x)\,dx - 4 \\ &= \frac{14}{3} - 4 \\ &= \frac{2}{3}\end{aligned}$$

c
$$\begin{aligned}\Pr(|X-\mu| < 1) &= \Pr(|X-2| < 1) \\ &= \Pr(1 < X < 3) \\ &= \frac{1}{4}\int_1^2 x\,dx + \frac{1}{4}\int_2^3 (4-x)\,dx \\ &= \frac{3}{4}\end{aligned}$$

d $\Pr(X > a) = 0.6$ or equivalently $\Pr(X \le a) = 0.4$.

Since the graph of $y = f(x)$ is symmetrical about $x = 2$, $\Pr(X \le 2) = 0.5$.

Hence $0 < a < 2$.

$$\begin{aligned}\int_0^a \frac{1}{4}x\,dx &= 0.4 = \frac{4}{10} \\ \left[\frac{1}{8}x^2\right]_0^a &= \frac{2}{5} \\ \frac{1}{8}a^2 &= \frac{2}{5} \\ a^2 &= \frac{16}{5} \\ a &= \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5} \quad (a > 0).\end{aligned}$$

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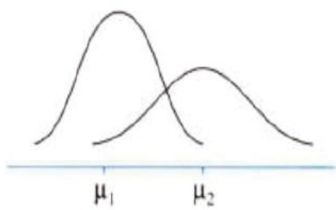
Chapter 16 – The normal distribution

Solutions to Exercise 16A

1



2 *(c) appears to be the only normally distributed curve*

3 a *using CAS calculator, integral* = 1

b i $E(X) = \int_{-\infty}^{\infty} xf(x)dx$

$$= \int_{-\infty}^{\infty} \frac{x}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2} dx$$

ii using CAS calculator, integral=2

c i $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$

$$= \int_{-\infty}^{\infty} \frac{x}{3\sqrt{2\pi}}$$

$$\times\ e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2} dx$$

ii *using CAS calculator, integral* = 13

iii $\sigma = \sqrt{\sigma^2}$

$$= \sqrt{E(X^2) - [E(X)]^2}$$

$$= \sqrt{13 - 4}$$

$$= 3$$

4 a *using CAS calculator, integral* = 1

b i $E(X) = \int_{-\infty}^{\infty} xf(x)dx$

$$= \int_{-\infty}^{\infty} \frac{x}{5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+4}{5}\right)^2} dx$$

ii *using CAS calculator, integral* = −4

c i $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$

$$= \int_{-\infty}^{\infty} \frac{x^2}{5\sqrt{2\pi}}$$

$$\times\ e^{-\frac{1}{2}\left(\frac{x+4}{5}\right)^2} dx$$

ii $E(X^2) = 41$

iii $\sigma = \sqrt{\sigma^2}$

$$= \sqrt{E(X^2) - [E(X)]^2}$$

$$= \sqrt{41 - 16}$$

$$= 5$$

5 $f(x) = \dfrac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-3}{10}\right)^2}$

a $\mu = 3$

$\sigma = 10$

$\left(\textit{read off} \left(\dfrac{x-3}{10}\right)\right.$

$\left.\textit{section of the equation}\right)$

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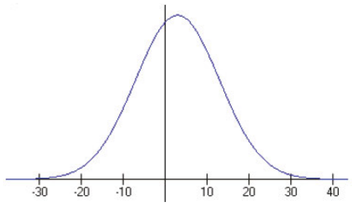
b



6 $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+3}{1}\right)^2}$

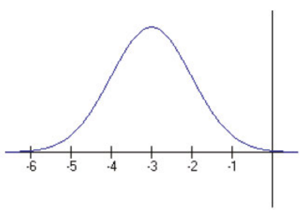
a $\mu = -3,$

$\sigma = 1$

$\left(\textit{read off} \left(\dfrac{x+3}{1} \right) \right.$

$\left. \textit{section of the equation} \right)$

b



7 $f(x) = \dfrac{1}{9\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+0}{3}\right)^2}$

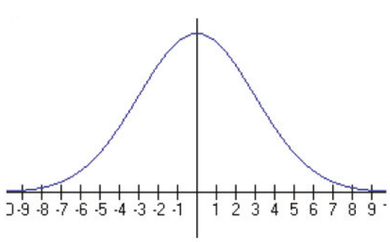
a $\mu = 0$

$\sigma = 3$

$\left(\textit{read off} \left(\dfrac{x+0}{3} \right) \right.$

$\left. \textit{section of the equation} \right)$

b



8 **a** *translation* $+3$ *along the x-axis*
$(\mu = 3)$
dilation factor 2 from the y-axis
$(\sigma = 2)$
dilation factor $\dfrac{1}{2}$ *from the x-axis*

b *translation* $+3$ *along the x-axis*
$(\mu = 3)$
dilation factor $\dfrac{1}{2}$ *from the y-axis*
$\left(\sigma = \dfrac{1}{2}\right)$
dilation factor 2 *from the x-axis*

c *translation* -3 *along the x-axis*
$(\mu = -3)$
dilation factor 2 *from the y-axis*
$(\sigma = 2)$
dilation factor $\dfrac{1}{2}$ *from the x-axis*

9 **a** *translation* -3 *along the x-axis*
$(\mu = 3)$
dilation factor 2 *from the x-axis*
dilation factor $\dfrac{1}{2}$ *from the y-axis*
$(\sigma = 2)$

b *translation* -3 *along the x-axis*
$(\mu = 3)$
dilation factor $\dfrac{1}{2}$ *from the x-axis*
dilation factor 2 *from the y-axis*
$\left(\sigma = \dfrac{1}{2}\right)$

c *translation* $+3$ *along the x-axis*
$(\mu = -3)$
dilation factor 2 *from the x-axis*
dilation factor $\dfrac{1}{2}$ *from the y-axis*
$(\sigma = 2)$

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Solutions to Exercise 16B

1 **a** 16%

b 16 %

c 2.5 %

d 2.5 %

2 **a**
$$\mu \approx 135$$
$$3\sigma \approx 15$$
$$\sigma \approx 5$$

b
$$\mu \approx 10$$
$$3\sigma \approx 4$$
$$\sigma \approx \frac{4}{3}$$

3 **a** $\approx 68\%$

b $\approx \dfrac{100\% - 68\%}{2} = 16\%$

c $\approx \dfrac{100\% - 99.7\%}{2} = 0.15\%$

4 $\mu - 2\sigma$ *and* $\mu + 2\sigma$
$27.3 - 6.2$ *and* $27.3 + 6.2$
answer:
21.1 *and* 33.5

5 *one*; 95; 99.7; *three*

6 $\approx \dfrac{1 - 0.95}{2} = 0.025$, i.e. 2.5%

7 **a** $\approx \dfrac{1 - 0.68}{2} = 0.16$, i.e. 16%

b $\approx \dfrac{1 - 0.68}{2} = 0.16$ i.e. 16%

8 **a** $\approx 68\%$

b $\approx \dfrac{100\% - 68\%}{2} = 16\%$

c $\approx \dfrac{100\% - 95\%}{2} = 2.5\%$

9 **a** $\approx 95\%$

b $\approx \dfrac{100\% - 68\%}{2} = 16\%$

c $= 50\%$, *since the mean = the median for normal distributions*

d $\approx 99.7\%$

10 **a** $\dfrac{160 - 160}{8} = 0$

b $\dfrac{150 - 160}{8} = -1.25$

c $\dfrac{172 - 160}{8} = 1.5$

11 **a** $\dfrac{256 - 270}{10} = -1.4$

b $\dfrac{281 - 270}{10} = 1.1$

c $\dfrac{305 - 270}{10} = 3.5$

12 *Michael has a score of* $\dfrac{85 - 78}{5} = 1.4$ *standard deviations*
Cheryl has a score of $\dfrac{27 - 18}{6} = 1.5$ *standard deviations*
$\therefore$ *Cheryl performed better*

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13 *Biology score is* $\dfrac{77 - 68.5}{4.9} \approx 1.73$ *standard deviations*

History score is $\dfrac{79 - 75.3}{4.1} \approx 0.90$ *standard deviations*

$\therefore$ *the student did better in Biology*

14 a

Mary:

French: $\dfrac{19 - 15}{4} = 1$

English: $\dfrac{42 - 35}{8} = 0.875$

Mathematics: $\dfrac{20 - 20}{5} = 0$

Steve:

French: $\dfrac{21 - 23}{4} = -0.5$

English: $\dfrac{39 - 42}{3} = -1$

Mathematics: $\dfrac{23 - 18}{4} = 1.25$

Sue:

French: $\dfrac{15 - 15}{5} = 0$

English: $\dfrac{42 - 35}{10} = 0.7$

Mathematics: $\dfrac{19 - 20}{5} = -0.2$

b **i** *Mary*

ii *Mary*

iii *Steve*

c *if all the subjects are weighted equally, Mary is the best student, since her total standardised mark is higher*

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Solutions to Exercise 16C

1 **a** 0.9772

b 0.9938

c 0.9938

d 0.9943

e 0.0228

f 0.0668

g 0.3669

h 0.1562

2 **a** 0.9772

b 0.6915

c 0.9938

d 0.9003

e 0.0228

f 0.0099

g 0.0359

h 0.1711

3 **a** 0.6826

b 0.9544

c 0.9974
These results are vary close to the '68%–95%–99.7%' rule.

4 **a** 0.0214

b 0.9270

c 0.0441

d 0.1311

5 $c = 1.2816$

6 $c = 0.6745$

7 $c = 1.96$

8 -1.6449

9 -0.8416

10 -1.2816

11 -1.9600

12 **a** 0.9522

b 0.7977

c 0.0478

d 0.1547

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13 **a** 0.9452

b 0.2119

c 0.9452

d 0.1571

14 **a** 9.2897

b 8.5631

15 **a** $c = 10$

b $k = 15.88$

16 **a** $a = 0.994$

b $b = 1.96$

c $c = 2.968$

17 **a** 0.7161

b 0.0965

c $\Pr(x < 26 \mid 25 < x < 27)$

$$= \frac{\Pr('x < 26' \cap '15 < x < 27)}{\Pr(25 < x < 27)}$$

$$= \frac{\Pr(25 < x < 26)}{0.096\cdots}$$

$$= 0.5204$$

d $c = 33.5143$

e $k = 13.02913$

f $c_1 = 8.28; c_2 = 35.72$
(assumed symmetrical about the mean)

18 **a** 0.9772

b $\Pr(x < 11 \mid x < 13)$

$$= \frac{\Pr('x < 11 \mid ' \cap 'x < 13')}{\Pr(x < |3)}$$

$$= \frac{\Pr(x < 11)}{\Pr(x < 13)}$$

$$= \frac{0.9772}{0.9999}$$

$$= 0.9772$$

c 10.822

d 9.5792

e $c_2 = 10.98; c_1 = 9.02$
(assumed symmetrical about the mean)

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Solutions to Exercise 16D

1 a i 0.2525

ii 0.0478

iii Pr(IQ > 130 | IQ > 110)

$= \dfrac{\Pr(\text{IQ} > 130)}{\Pr(\text{IQ} > 110)}$

$= \dfrac{0.0227\cdots}{0.2524\cdots}$

$= 0.0901$

b 124.7

2 a i 0.7340

ii 0.8944

iii Pr(> 170 | between | 68 &174)

$= \dfrac{\Pr(\text{between } 170\ \&174)}{\Pr(\text{between } 168\ \&174)}$

$= \dfrac{0.0655\cdots}{0.1185\cdots}$

$= 0.5531$

b 170.25 cm

c 153.267 cm

3 a i 0.0766

ii 0.9998

iii 0.1531

b 57.3

4 a 10.56%

b 78.51%

5 *mean* = 1.55 kg; *sd* = 0.194 kg

6 a 36.9%

b 69

7 a 0.0228

b 0.0005

c If Y is the number with heights exceeding 190 cm, then Y is Binomial with $n = 10, P = 0.02275\ldots$

$\Pr(Y \geq 2) = \Pr(2 \leq Y \leq 10)$

$= 0.0206$

using a CAS calculator's 'bimom CAS' function.

8 a $\Pr(X < 295) = 0.05$

$Pr(Z < \dfrac{295 - 300}{\sigma} = 0.05$

$\dfrac{5}{\sigma} = -1.6449$

$\sigma = 3.04$ grams

b $\Pr(X < 340) = 0.02$

$Pr(Z < \dfrac{340 - \mu}{5} = 0.02$

$\dfrac{340 - \mu}{5} = -2.0537$

$\mu = 350.27$ grams

9 1004 ml

10 a *small* 0.1587

medium 0.7745

large 0.0668

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b Expected cost

$= 100 \times \$(2.80 \times 0.1587 + 3.50$

$\times\ 0.7745 + 5.00 \times 0.0688)$

$= \$348.92$

11 **a** **i** 0.1169

ii 17.7

b 0.0284

12 **a** 0.0228, 0.1587

b Let x be the amount of chemical in a type A call, so x is normal with mean 10 and sd 1 Let Y be the amount of chemical in a type 1 cell, so y is normal with mean 14 and sd 2

$$\Pr(x < c) = \Pr(y > c)$$

$$\Pr\left(\frac{x-10}{1} < \frac{c-10}{1}\right) = \Pr\left(\frac{y-14}{2} > \frac{c-14}{2}\right)$$

$$\text{i.e. } \Pr(z < c - 10) = \Pr\left(z > \frac{c-14}{2}\right)$$

where z has a standard normal distribution. Since the graph of $y = f(z)$ is symmetrical about the y-axis, the number $c - 10\,\text{cm}\ \frac{c-14}{2}$ are equidistant from the original.

$$\text{Hence } \frac{c-14}{2} = -(c-10)$$

$$c - 14 = -2c + 20$$

$$3c = 34$$

$$c = \frac{34}{3}.$$

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Solutions to Exercise 16E

1 $n = 100, p = \frac{1}{6}$
$\mu = np = \frac{100}{6} \approx 16.667$
$\sigma = \sqrt{np(1-p)} \approx 3.727$
$\Pr(X > 10) = 0.9632$ calculator

2 $n = 300, p = 0.5$
$\mu = np = 150$
$\sigma = \sqrt{np(1-p)} \approx 8.660$
$\Pr(X > 156) = 0.2442$ calculator

3 $n = 100, p = 0.1$
$\mu = np = 10$
$\sigma = \sqrt{np(1-p)} = 3$

a $\Pr(X \geq 15) = 0.0478$ calculator

b $\Pr(X \leq 15) = 0.2525$ calculator

4 $n = 400, p = 0.4$
$\mu = np = 16$
$\sigma = \sqrt{np(1-p)} \approx 6.898$

a $\Pr(10 \leq X < 20) = 0.7834$ calculator

b $\Pr(X \geq 15) = 0.0108$ calculator

5 $n = 200, p = 0.4$
$\mu = np = 80$
$\sigma = \sqrt{np(1-p)} \approx 6.928$
$\Pr(X < 76) = 0.2819$ calculator

6 $n = 25, p = 0.25$
$\mu = np = 6.25$
$\sigma = \sqrt{np(1-p)} \approx 2.165$

a $\Pr(X \geq 10) = 0.0416$ calculator

b $\Pr(12 \leq X \leq 14) = 0.0038$ calculator

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Solutions to Technology-free questions

1 a $\Pr(Z > a) = 1 - \Pr(Z \le a)$
$= 1 - p$

b $\Pr(Z < -a) = \Pr(Z > a)$
$= 1 - p$

c $\Pr(-a \le Z \le a) = \Pr(Z \le a)$
$- \Pr(Z < -a)$
$= P - (1 - p)$
$= 2p - 1$

2 a $\Pr(X < 3) = \Pr\left(\dfrac{X-4}{1} < \dfrac{3-4}{1}\right)$
$= \Pr(Z < -1)$
So $\quad a = -1$

b $\Pr(X > 5) = \Pr\left(\dfrac{X-4}{1} > \dfrac{5-4}{1}\right)$
$= \Pr(Z > 1)$
So $\quad b = 1$

c $\Pr(x > 4) = \Pr(Z > 0)$
$= 0.5$

3 $(x, y) \to \left(\dfrac{x-\mu}{\sigma}, \sigma y\right)$
Sine $\mu = 8$ and $\sigma = 3$, then
$(x, y) \to \left(\dfrac{x-8}{3}, 3y\right)$

4 a $\Pr(x < a | x < b)$
$= \dfrac{\Pr('x < a' \cap 'x < b')}{\Pr(x < b)}$
$= \dfrac{\Pr(x < a)}{\Pr(x < b)}$
$= \dfrac{q}{p}$

b $\Pr(X < 2\mu - a) = \Pr\left(\dfrac{x-\mu}{\sigma} < \dfrac{\mu - a}{\sigma}\right)$
$= \Pr\left(Z < \dfrac{\mu - a}{\sigma}\right)$
$= \Pr\left(Z > \dfrac{a - \mu}{\sigma}\right)$
$= 1 - \Pr\left(Z - \dfrac{a-\mu}{\sigma}\right)$
Also, $\quad \Pr(X < a) = q$
$\Pr\left(\dfrac{x-\mu}{\sigma} < \dfrac{a-\mu}{\sigma}\right) = q$
$\Pr\left(Z < \dfrac{a-\mu}{\sigma}\right) = q$
Hence $\Pr(X < 2\mu - a)$
$= 1 - \Pr\left(Z < \dfrac{a-\mu}{\sigma}\right)$
$= 1 - q$

c $\Pr(X > b \mid X > a)$
$= \dfrac{\Pr('x > b' \cap 'x > a')}{\Pr(x > a)}$
$= \dfrac{\Pr(x > b)}{\Pr(x > a)}$
$= \dfrac{1 - \Pr(x < b)}{1 - \Pr(x < a)}$
$= \dfrac{1 - p}{1 - q}$

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5 **a** $\Pr(X < 5) = \Pr\left(\frac{x-4}{2} < \frac{5-4}{2}\right)$

$= \Pr\left(Z < \frac{1}{2}\right)$

b $\Pr(X < 3) = \Pr\left(\frac{x-4}{2} < \frac{3-4}{2}\right)$

$= \Pr\left(Z < -\frac{1}{2}\right)$

c $\Pr(X > 5) = \Pr\left(Z > \frac{1}{2}\right)$

d $\Pr(3 < X < 5) = \Pr\left(-\frac{1}{2} < Z < \frac{1}{2}\right)$

$= \Pr\left(-\frac{1}{2} < Z < 1\right)$

e $\Pr(3 < X < 6) = \Pr(-\frac{1}{2} < Z < 1)$

6 **a** $\Pr(X < 2.55) = \Pr\left(Z < \frac{2.55 - 2.5}{0.05}\right)$

$= \Pr(Z < 1)$

$= 0.84$

b $\Pr(X < 2.5) = 0.5 \quad \text{since } \mu = 2.5$

c $\Pr(X < 2.45) = \Pr\left(Z < \frac{2.45 - 2.5}{0.05}\right)$

$= \Pr(Z < -1)$

$= \Pr(Z > 1)$

$= 1 - \Pr(Z < 1)$

$= 0.16$

d $\Pr(2.45 < X < 2.55)$

$= \Pr(-1 < Z < 1)$

$= \Pr(Z < 1) - \Pr(Z < -1)$

$= 0.84 - 0.16$

$= 0.68$

7 **a** $\Pr(\mathrm{W} > 505) = \Pr\left(Z > \frac{505 - 500}{5}\right)$

$= \Pr(Z > 1)$

$= 1 - \Pr(Z < 1)$

$= 1 - 0.84$

$= 0.16$

b $\Pr(500 < \mathrm{W} < 505) = \Pr(0 < Z < 1)$

$= \Pr(Z < 1)$

$= -\Pr(Z < 0)$

$= 0.84 - 0.5$

$= 0.34$

c $\Pr(\mathrm{W} > 505 | \mathrm{W} > 500)$

$= \frac{\Pr(W > 505)}{\Pr(W > 500)}$

$= \frac{0.16}{0.5}$

$= 0.32$

d $\Pr(W > 510) = \Pr\left(Z > \frac{510 - 500}{5}\right)$

$= \Pr(Z > 2)$

$= 1 - \Pr(Z < 2)$

$= 1 - 0.98$

$= 0.02$

8 **a** $\Pr(X < 6.5) = \Pr(Z < 0.5)$

$= 0.69$

b $\Pr(6 < X < 6.5) = \Pr(0 < Z < 0.5)$

$= 0.69 - 0.5$

$= 0.19$

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c $\Pr(6.5 < X < 7)$

$= \Pr(0.5 < Z < 1)$

$= \Pr(Z < 1) - \Pr(Z < 0.5)$

$= 0.84 - 0.69$

$= 0.15$

d $\Pr(5 < X < 7) = \Pr(-1 < Z < 1)$

$= \Pr(Z < 1)$

$\quad - \Pr(Z < -1)$

$= 0.84 - (1 - 0.84)$

$= 0.84 - 0.16$

$= 0.68$

9 The standardised scores are as follows.

Test A: $\dfrac{62 - 50}{11} = \dfrac{12}{11} = 1.0909\cdots$

Test B: $\dfrac{64 - 48}{17} = 1$

Test C: $\dfrac{73 - 63}{8} = \dfrac{10}{8} = 1.25$

So the best test was test C and the worst test was test *B*.

10 **a** $\Pr(X > 10) = \Pr(Z > 0) = 0.5$

b $\Pr(X > 13) = \Pr(Z > b)$

$\therefore \Pr(Z > \dfrac{13 - 10}{2}) = \Pr(Z < b)$

$\therefore \Pr(Z > 1.5) = \Pr(Z < b)$

$\therefore \Pr(Z < -1.5) = \Pr(Z < b)$

$\therefore b = -1.5$

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Solutions to multiple-choice questions

1 A The graph is symmetrical about the line $x = 4$, so $\mu = 4$.
Almost all of the distribution lies between -5 and 13, i.e. 18 unit, so

$$6\sigma = 18$$
$$\sigma = 3$$

2 C Use the *normCdf* command of a CAS calculator with lower bound 1.45 and upper bound ∞. This gives 0.0735 count to 4 decimal places.

3 B Use the *invNom* command of a CAS calculator with *Area* set to 0.25. This gives -0.6745 correct to 4 decimal places.

4 B X has mean 12 and variance 9, so the standard deviation is 3.

$$\begin{aligned}\Pr(X > 15) &= \Pr\left(\frac{x-12}{3} > \frac{15-12}{3}\right)\\ &= \Pr(Z > 1)\end{aligned}$$

5 E Use the *normCdf* command if a CAS calculator with lower bound 110, upper bound ∞, $\mu = 102$ and $\sigma = 3$. This gives 0.00383, so the parentage is about 0.38%

6 E 10 goals is 6 below the mean of 16, and this is 3 standard deviations below the mean. similarity, 22 goals is 3 standard deviations above the mean.
So from 10 to 22 corresponds to $\mu \pm 3\sigma$, and this corresponds to approximately 99.7%.

7 C $(x, y) \to \left(\dfrac{x-\mu}{\sigma}, \sigma y\right)$
Here $\mu = 6$ and $\sigma = 3$, so
$(x, y) \to \left(\dfrac{x-6}{3}, 3y\right)$

8 D The given information means that $\Pr(X > k) = 0.20$
where x is normally distributed with $\mu = 100$ and $\sigma = 14$.
This can be re-written in the form $\Pr(X \leq k) = 0.80$.
Use the *invNorm* command of a CAS calculator with *Area* set to 0.80 and the values 100 and 14 for the mean and standard deviation respectively.
This gives $k = 111.8$, correct to one decimal place.

9 A Angie's standardised scores are follows.

$$\text{Mathematics: } \frac{72-72}{5} = 0$$
$$\text{Indonesian: } \frac{57-59}{4} = -1$$
$$\text{Politics: } \frac{68-64}{4} = 1$$

So her best subject was Politics, followed by Mathematics and then Indonesian.

10 D Choosing equal areas in each tail means $\Pr(X < c_1) = 0.05$, and $\Pr(X < c_2) = 0.95$
(so that $\Pr(c_1 < x < c_2) = 0.90$).
Use the *invNom* command of a CAS calculator with *Area* and 0.95 respectively and the values 11.3 for the mean and 2.9 for the standard deviation. This gives $c_1 = 6.53$ and $c_2 = 16.07$, correct to two decimal places.

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11 D $\mu = ?, \sigma = 0.005$

$\Pr(X > 1) \approx 0.999$

$\therefore \Pr\left(Z > \frac{1 - \mu}{0.005}\right) \approx 0.999$

$\therefore \frac{1 - \mu}{0.005} = 3.0902$

$\therefore \mu = 1.015$

12 C $\mu = 272, \sigma = ?$

$\Pr(X < 260) \approx 0.091$

$\therefore \Pr\left(Z < \frac{260 - 272}{\sigma}\right) \approx 0.091$

$\therefore \frac{12}{\sigma} = -1.3346$

$\therefore \sigma \approx 8.99$

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Solutions to extended-response questions

1 $\mu = 50, \sigma = 10$

Let X be the score.

For the top 10% consider

$$\Pr(X > k_1) = 0.1$$

$$\therefore \Pr(X \le k_1) = 0.9$$

$$\Pr\left(Z \le \frac{k_1 - 50}{10}\right) = 0.9$$

$$\therefore \frac{k_1 - 50}{10} = 1.2816$$

$$\therefore k_1 = 10 \times 1.2816 + 50$$

$$= 12.816 + 50 = 62.816$$

$\therefore$ 63 and above indicate high aptitude.

For the next 20% consider

$$\Pr(X > k_2) = 0.3$$

$$\therefore \Pr(X \le k_2) = 0.7$$

$$\therefore \Pr\left(Z \le \frac{k_2 - 50}{10}\right) = 0.7$$

$$\frac{k_2 - 50}{10} = 0.5244$$

$$\therefore k_2 = 50 + 5.244 = 55.244$$

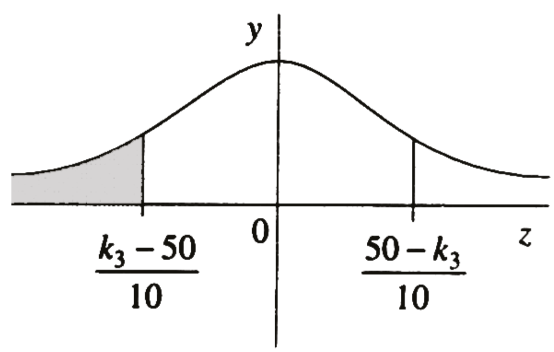
$\therefore$ Scores from 56 to 62 indicate moderate aptitude.

For the middle 40% consider

$$\Pr(X > k_3) = 0.7$$

$$\therefore \Pr(X \le k_3) = 0.3$$

$$\Pr\left(Z \le \frac{k_3 - 50}{10}\right) = 0.3$$



$\therefore$ From the diagram

$$\Pr\left(Z \le \frac{50 - k_3}{10}\right) = 0.7$$

$$\therefore \frac{50 - k_3}{10} = 0.5244$$

$$\therefore k_3 = 50 - 5.244 = 44.756$$

$\therefore$ Scores from 45 to 55 indicate average aptitude.

For the category of little aptitude consider

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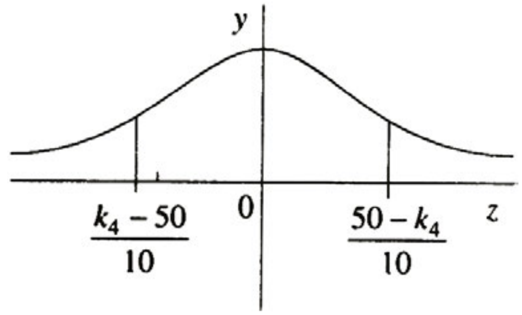
$$\Pr(X > k_4) = 0.1$$
$$\therefore\ \Pr\left(Z > \frac{k_4 - 50}{10}\right) = 0.1$$
$$\therefore\ \Pr\left(Z \le \frac{50 - k_4}{10}\right) = 0.9$$
$$\frac{50 - k_4}{10} = 1.2816$$
$$50 - k_4 = 12.816$$
$$\text{and } k_4 = 10 - 12.816$$
$$= 37.184$$



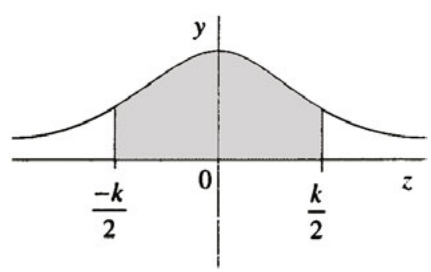
Scores from 37 to 44 indicate little aptitude.
Scores less than 37 indicate no aptitude.
i.e. Scores 63 and above indicate high aptitude.
Scores from 56 to 62 indicate moderate aptitude.
Scores from 45 to 55 indicate average aptitude.
Scores from 37 to 44 indicate little aptitude.
Scores < 37 indicate no aptitude.

2 $\Pr(\mu - k \le X \le \mu + k) = 0.95$

$$\mu = 10 \text{ and } \sigma = 2$$

$\therefore\ \Pr(10 - k \le X \le 10 + k) = 0.95$
and transforming to the standard normal
$\Pr\left(\frac{10 - k - 10}{2} \le Z \le \frac{10 + k - 10}{2}\right) = 0.95$
i.e. $\Pr\left(-\frac{k}{2} \le Z \le \frac{k}{2}\right) = 0.95$
The graph of the standard normal with the region being considered is as shown:

$$\therefore\ \Pr\left(Z \le \frac{k}{2}\right) - \Pr\left(Z \le -\frac{k}{2}\right) = 0.95$$
$$\Pr\left(Z \le \frac{k}{2}\right) - \left[1 - \Pr\left(Z \le \frac{k}{2}\right)\right] = 0.95$$
$$2\Pr\left(Z \le \frac{k}{2}\right) - 1 = 0.95$$
$$\Pr\left(Z \le \frac{k}{2}\right) = \frac{1.95}{2}$$
$$\text{i.e. } \Pr\left(Z \le \frac{k}{2}\right) = 0.975$$
$$\therefore\ \frac{k}{2} = 1.96$$
$$\text{and } k = 3.92$$



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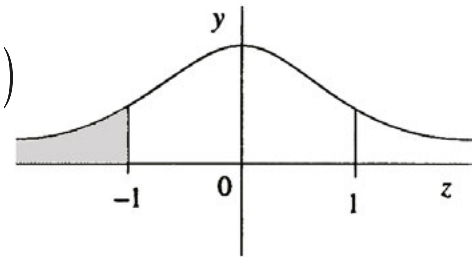
3 $\mu = 60\,000, \sigma = 5000$

a **i** Let X be the mileage for a tyre

$$\Pr(X \leq 55\,000) = \Pr\left(Z \leq \frac{55\,000 - 60\,000}{5000}\right)$$

where Z is the standard normal variable



$$= \Pr\left(Z \leq -\frac{5000}{5000}\right)$$

$$= \Pr(Z \leq -1)$$

$$= 1 - 0.8413$$

$$= 0.1587$$

The proportion of the tyres which last less than 55 000 kilometres is 0.1587 or 15.87%

ii $\Pr(50\,000 \leq X \leq 74\,000) = \Pr\left(\frac{50\,000 - 60\,000}{5000} \leq Z \leq \frac{74\,000 - 60\,000}{5000}\right)$

$$= \Pr(-2 \leq Z \leq 2.8)$$

The required region is shown:

$\Pr(-2 \leq Z \leq 2.8)$

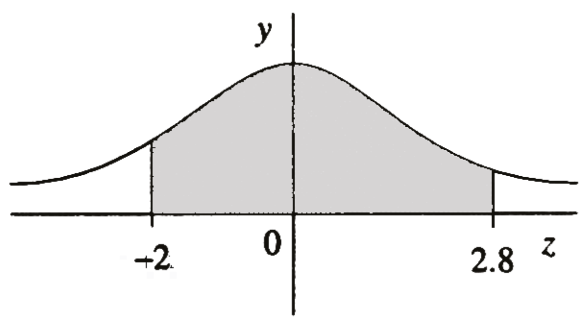
$= \Pr(Z \leq 2.8) - \Pr(Z \leq -2)$

$= \Pr(Z \leq 2.8) - [1 - \Pr(Z \leq 2)]$

$= \Pr(Z \leq 2.8) + \Pr(Z \leq 2) - 1$

$= 0.99744 + 0.97725 - 1$

$= 0.9747$



The proportion of tyres which last less than 74 000 kilometres but more than 50 000 is 0.9746 or 97.46%

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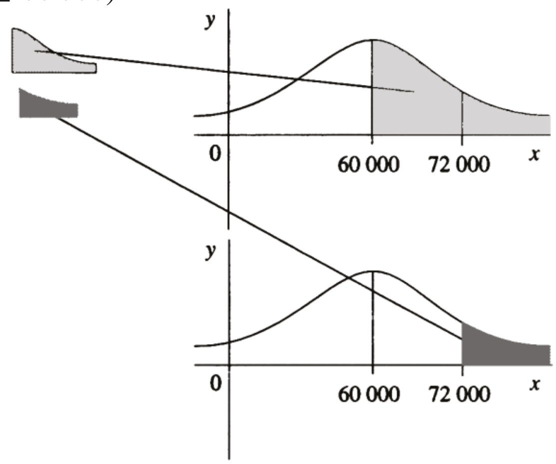
iii $\Pr(X \geq 72\,000 \mid X \geq 60\,000) = \dfrac{\Pr(X \geq 72\,000)}{\Pr(X \geq 60\,000)}$ (conditional probability)

The diagrams show that the required probability is given by Area divided by Area and transforming to the standard normal



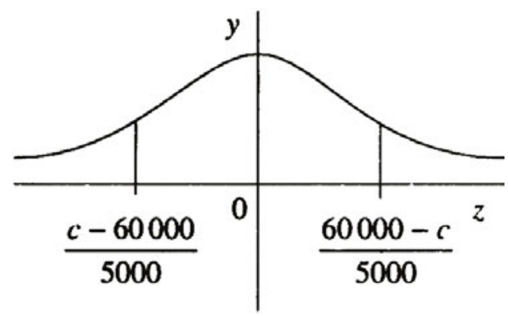
$$\frac{\Pr(X \geq 72\,000)}{\Pr(X \geq 60\,000)} = \frac{\Pr(Z \geq 2.4)}{\Pr(Z \geq 0)}$$
$$= \frac{1 - \Pr(Z < 2.4)}{0.5}$$
$$= \frac{1 - 0.9918}{0.5}$$
$$= 0.0164$$

b $\Pr(X \geq c) = 0.9$

Transforming to the standard normal

$$\Pr\left(Z \geq \frac{c - 60\,000}{5000}\right) = 0.9$$

A graph of the standard normal curve helps:



$$\therefore\ \Pr\left(Z \geq \frac{c - 60\,000}{5000}\right) = \Pr\left(Z \leq \frac{60\,000 - c}{5000}\right)$$
$$\therefore\ \Pr\left(Z \leq \frac{60\,000 - c}{5000}\right) = 0.9$$
$$\therefore\ \frac{60\,000 - c}{5000} = 1.2816$$
$$\therefore\ 60\,000 - 5000 \times 1.2816 = c$$
$$\therefore\ c = 53\,592$$

The company's advertising manager can claim that 90% of their tyres last more than 53 592 kilometres.

c $\Pr(X \geq 72\,000) = \Pr\left(Z \geq \dfrac{72\,000 - 60\,000}{5000}\right)$

$$= \Pr\left(Z \geq \frac{12\,000}{5000}\right)$$
$$= \Pr(Z \geq 2.4)$$
$$= 1 - \Pr(Z < 2.4)$$
$$= 1 - 0.9918$$
$$= 0.0082$$

The probability of one tyre lasting more than 72 000 kilometres is 0.0082

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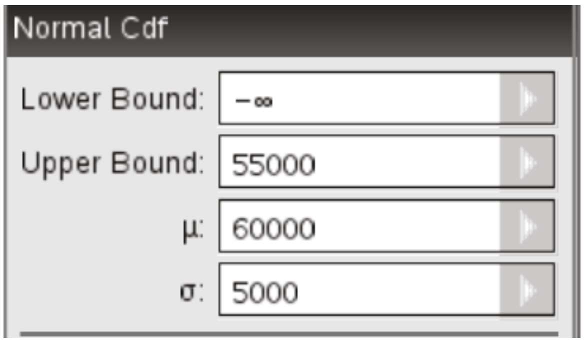
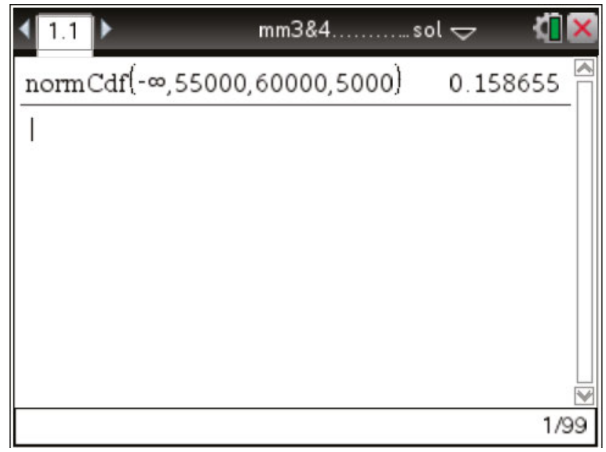
The probability of 5 tyres lasting longer than 72 000 kilometres is $(0.0082)^5 \approx 3.7 \times 10^{-11}$

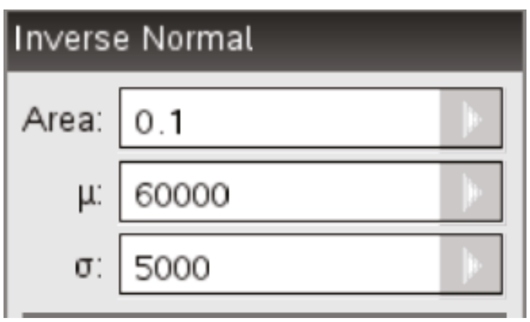
Graphic calculator techniques for question 3

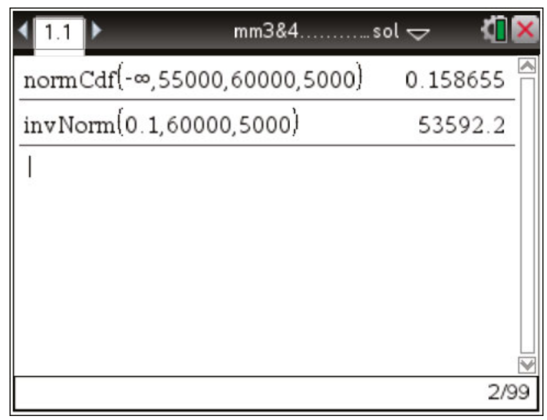
a **i** Choose **Normal Cdf** from the **Probability>Distributions** menu. Complete as shown.





ii $\Pr(X \geq c) = 0.9 \Leftrightarrow \Pr(X \leq c) = 0.1$
Choose **Inverse Normal** from the **Probability>Distributions** menu. Complete as shown.





4 $\mu = 15, \sigma = 0.75$

a Let X be the number of litres/100 km used

$$\begin{aligned}\Pr(X \geq 18) &= \Pr\left(Z \geq \frac{18-15}{0.75}\right)\\ &= \Pr(Z \geq 4)\\ &= 0.0000317 = 3.17 \times 10^{-5}\end{aligned}$$

b The manufacturer's claim is false.

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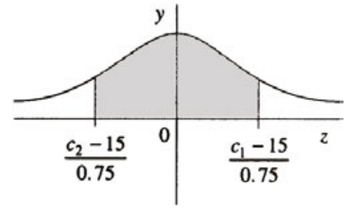
c It is assumed, as in the text, that c_1 and c_2 are symmetric about the mean



$$\Pr(c_1 \leq X \leq c_2) = 0.95$$

$$\therefore \Pr\left(\frac{c_1 - 15}{0.75} \leq Z \leq \frac{c_2 - 15}{0.75}\right) = 0.95$$

$$\text{By symmetry } \frac{c_2 - 15}{0.75} = \frac{15 - c_1}{0.75}$$

$$\therefore \Pr\left(\frac{c_1 - 15}{0.75} \leq Z \leq \frac{15 - c_1}{0.75}\right) = 0.95$$

$$\therefore \Pr\left(Z \leq \frac{15 - c_1}{0.75}\right) - \left[\Pr\left(Z \leq \frac{15 - c_1}{0.75}\right) - 1\right] = 0.95$$

$$\text{i.e. } 2\Pr\left(Z \leq \frac{15 - c_1}{0.75}\right) = 0.95 + 1$$

$$\Pr\left(Z \leq \frac{15 - c_1}{0.75}\right) = 0.975$$

$$\therefore \frac{15 - c_1}{0.75} = 1.96$$

$$\therefore c_1 = 15 - 0.75 \times 1.96$$

$$= 13.53$$

$$\text{and by symmetry } c_2 = 15 + 0.75 \times 1.96$$

$$= 16.47$$

5 Let L be the useful life of a fluorescent tube

$$\mu = 600, \sigma = 4$$

$$\Pr(L \geq 605) = \Pr\left(Z \geq \frac{605 - 600}{4}\right)$$

$$= \Pr\left(Z \geq \frac{5}{4}\right)$$

$$= \Pr(Z \geq 1.25)$$

$$= 1 - \Pr(Z \leq 1.25)$$

$$= 1 - 0.8944$$

$$= 0.1056$$

The situation of ten tubes is described by a binomial distribution.

Let X be the number of tubes in a box which last longer than 605 hours.

X is the random variable of a binomial distribution with $n = 10$ and $p = 0.1056$

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$$\begin{aligned}\Pr(X \geq 3) &= 1 - [\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)] \\ &= 1 - [0.3275 + 0.3868 + 0.2055] \\ &= 1 - 0.9198 \\ &= 0.0802\end{aligned}$$

The probability of at least three tubes in a randomly selected box lasting longer than 605 hours is 0.0802

6 Let L be the amount (mg) for a lethal dose
$\mu = 110, \sigma = 20$
Let D be the amount (mg) for a surgical anaesthesia
$\mu = 50, \sigma = 10$
Let c mg be the dose such that 90% of patients need less than this amount for surgical anaesthesia
i.e. $\Pr(D \leq c) = 0.9$
Transforming to the standard normal

$$\begin{aligned}\Pr\left(Z \leq \frac{c - 50}{10}\right) &= 0.9 \\ \frac{c - 50}{10} &= 1.2816 \\ \therefore\ c &= 10 \times 1.2816 + 50 \\ &= 12.816 + 50 \\ &= 62.816\end{aligned}$$

To find what percentage of patients would be killed by these amounts consider

$$\begin{aligned}\Pr(L \leq 62.816) &= \Pr\left(Z \leq \frac{62.816 - 110}{20}\right) \\ &= \Pr(Z \leq -2.3592) \\ &= 1 - \Pr(Z \leq 2.3592) \\ &= 1 - 0.9908 \\ &= 0.0092\end{aligned}$$

i.e. 0.92% of patients would be killed by a dose of 62.816 mg or less.

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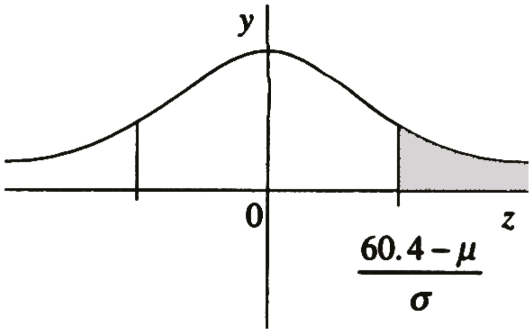
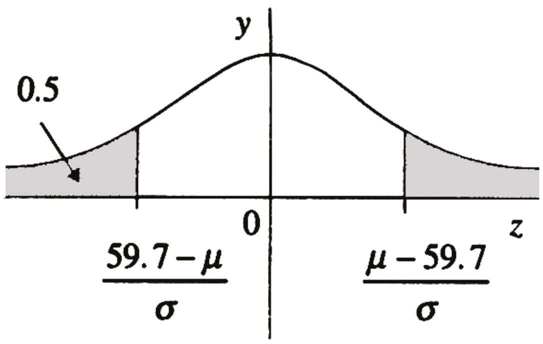
7 **a** Let X be the length of the dimension

$\Pr(X > 60.4) = 0.03$

$\Pr(X < 59.7) = 0.05$





$\Pr\left(Z < \dfrac{60.4-\mu}{\sigma}\right) = 0.97 \qquad \Pr\left(Z < \dfrac{\mu-59.7}{\sigma}\right) = 0.95$

$\therefore \dfrac{60.4-\mu}{\sigma} = 1.88079$ and $\dfrac{\mu-597}{\sigma} = 1.64485$

$\therefore 60.4-\mu = 1.88079\sigma$ ① and $\mu - 59.7 = 1.64485\sigma$ ②

Add equations ① and ②

$0.7 = 3.52564\sigma$

$\therefore \sigma = 0.19854,$ i.e. $\sigma = 0.2,$ correct to one decimal plea.

Substitute in ①

$60.4 - \mu = 1.88079\sigma$

$\therefore \mu = 60.4 - 1.88079\sigma$

$= 60.02658,$ i.e. $\mu = 60.0,$ correct to one decimal plea.

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b $\Pr(X > 60.3) + \Pr(X < 59.6) = \Pr\left(Z > \dfrac{60.3 - 60.02658}{0.19854}\right)$

$$+ \Pr\left(Z < \frac{59.6 - 60.02658}{0.19854}\right)$$

$$= \Pr(Z > 1.37715) + \Pr(Z < -2.14858)$$

$$= 1 - \Pr(Z < 1.37715) + 1 - \Pr(Z < 2.14858)$$

$$= 2 - \Pr(Z < 1.37715) - \Pr(Z < 2.14858)$$

$$= 2 - 0.915767 - 0.98416$$

$$= 0.1$$

These the percentage of rejects is 10%.

8 Let H denote the hardness of the metal

$\mu = 70$ and $\sigma = 3$

a $\Pr(65 \le H \le 75) = \Pr\left(\dfrac{65 - 70}{3} \le Z \le \dfrac{75 - 70}{3}\right)$

$$= \Pr\left(-\frac{5}{3} \le Z \le \frac{5}{3}\right)$$

$$= \Pr\left(Z \le \frac{5}{3}\right) - \Pr\left(Z \le -\frac{5}{3}\right)$$

$$= \Pr\left(Z \le \frac{5}{3}\right) - \left[1 - \Pr\left(Z \le \frac{5}{3}\right)\right]$$

$$= 2\Pr\left(Z \le \frac{5}{3}\right) - 1$$

$$= 0.9044$$

The probability that a randomly chosen specimen has acceptable hardness is 0.9044.

b $\Pr(70 - c \le H \le 70 + c) = 0.95$

implies $\Pr\left(\dfrac{70 - c - 70}{3} \le Z \le \dfrac{70 + c - 70}{3}\right) = 0.95$

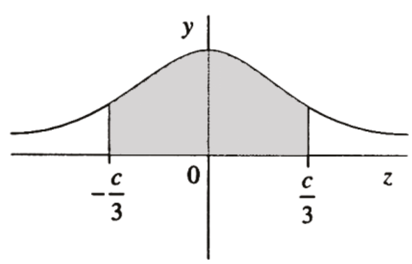
$$\Pr\left(-\frac{c}{3} \le Z \le \frac{c}{3}\right) = 0.95$$

$$\therefore\ 2\Pr\left(Z \le \frac{c}{3}\right) - 1 = 0.95$$

$$\Pr\left(Z \le \frac{c}{3}\right) = 0.975$$

$$\therefore\ \frac{c}{3} = 1.96$$

$$c = 5.88$$



c Let X be the number of acceptable specimens out of 10 randomly selected

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specimens.

X is a binomial random variable with $n = 10$ and $p = 0.9044$

$E(X) = np = 9.044$

The expected number of acceptable specimens is 9.044.

d $\Pr(H < 73.84) = \Pr\left(Z < \dfrac{73.84 - 70}{3}\right)$

$= \Pr\left(Z < \dfrac{3.84}{3}\right)$

$= \Pr(Z < 1.28)$

$= 0.8997$

Let X be the number of specimens out of the ten selected which have a hardness less than 73.84.

$\Pr(X \leq 8) = 1 - [\Pr(X = 9) + \Pr(X = 10)]$

$= 1 - \dbinom{10}{9}(0.8997)^9(0.1003) - (0.8997)^{10}$

$= 0.2651$ (to four decimal places)

e Let P be profit. The probability distribution for P

P	20	−5
$\Pr(P = p)$	0.9044	0.0956

$\therefore\ E(P) = 20 \times 0.9044 - 5 \times 0.0956$

$= 17.61$

The expected profit is \$17.61.

$E(P^2) = 400 \times 0.9044 + 25 \times 0.0956$

$= 364.15$

$\therefore\ \mathrm{Var}(P) = E(P^2) - [E(P)]^2$

$= 364.15 - 310.1121$

$= 54.04$

9 Let μ be the mean lifetime for a watch and σ the standard deviation.

a The mean error is 0

Let X be the error

$\Pr(-5 \leq X \leq 5) = 0.94$

$\therefore\ \Pr\left(\dfrac{-5}{\sigma} \leq Z \leq \dfrac{5}{\sigma}\right) = 0.94$

$\therefore\ 2\Pr\left(Z \leq \dfrac{5}{\sigma}\right) - 1 = 0.94$

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$$\Pr\left(Z \le \frac{5}{\sigma}\right) = \frac{1.94}{2}$$

$$\frac{5}{\sigma} = 1.8808$$

$$\therefore\ \sigma = \frac{5}{1.8808}$$

$$= 2.658$$

b Let Y be the number of watches rejected out of a batch of 10 watches.
This is a Binomial distribution with $p = 0.06$ and $n = 10$

$$\Pr(Y < 2) = \Pr(Y = 0) + \Pr(Y = 1)$$

$$= (0.94)^{10} + \binom{10}{1}(0.06)(0.94)^9$$

$$= 0.5386 + 0.3438$$

$$= 0.882$$

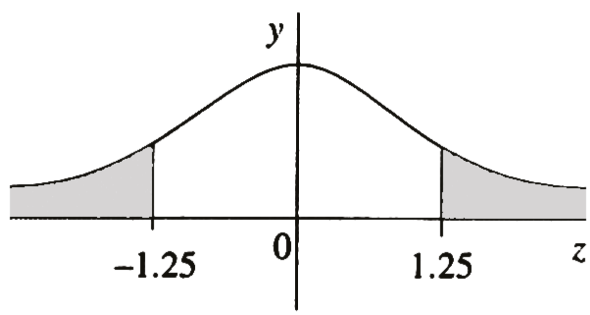
10 **a** Let X be the number of litres in a standard bottle.

$$\Pr(X < 0.75) = \Pr\left(Z < \frac{0.75 - 0.76}{0.008}\right)$$

$$= \Pr(Z < -1.25)$$

$$= 1 - \Pr(Z < 1.25)$$

$$= 0.1056$$



b Let N be the number of bottles out of ten which contain less than 0.75 1itres. This is a binomial random variable with $n = 10$ and $p = 0.10565$.

$$\therefore\ \Pr(N \ge 3) = 1 - [\Pr(N = 0) + \Pr(N = 1) + \Pr(N = 2)]$$

$$= 1 - 0.9197$$

$$= 0.0803$$

c Let Y be the number of litres in a large bottle.
Define $W = 4X - 3Y$
We require $\Pr(W) \ge 0$
i.e. $\Pr(4X - 3Y \ge 0)$

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$$
\begin{aligned}
\text{Note: } \mathrm{E}(W) &= 4\mathrm{E}(X) - 3\mathrm{E}(Y) \\
&= 0.01 \\
\mathrm{Var}(W) &= 16\mathrm{Var}(X) + 9\mathrm{Var}(Y) \\
&= 16 \times (0.008)^2 + 9 \times (0.009)^2 \\
&= 0.001753 \\
\therefore\ \mathrm{sd}(W) &= 0.04187 \\
\therefore\ \Pr(W > 0) &= \Pr\left(Z > \frac{0.01}{0.04187}\right) \\
&= \Pr(Z > -0.23883) \\
&= \Pr(Z < 0.23883) \\
&= 0.5944
\end{aligned}
$$

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Chapter 17 – Sampling and estimation

Solutions to Exercise 17A

1 No; sample will be biased towards the type of movie being shown.

2 **a** No; biased towards shoppers.

b Randomly select a sample from telephone lists or an electoral roll.

3 No; only interested people will call, and they may call more than once.

4 **a** No; biased towards older, friendly or sick guinea pigs which may be easier to catch.

b Number guinea pigs and then generate random numbers to select a sample.

5 No; a student from a large school has less chance of being selected than a student from a small school.

7 **a** Unemployed will be under represented.

b Unemployed or employed may be under represented, depending on time of day.

c Unemployed will be over represented.

Use random sampling based on the whole population (e.g. electoral roll).

8 **a** Divide platform into a grid of 1 m^2 squares. Select squares using a random number generator to give two digits, one a vertical reference and one a horizontal reference.

b Yes, if crabs are fairly evenly distributed; otherwise, five squares may not be enough.

9 No; a parent's chance of selection depends on how many children they have at the school.

10 Not a random sample; only interested people will call, and they may call more than once.

11 People who go out in the evenings will not be included in the sample.

12 **a** All students at this school

b $p = 0.35$

c $\hat{p} = 0.42$

13 **a** 0.22

b $\hat{p}$

Solutions to Exercise 17B

1 **a** $p = \frac{5}{10} = \frac{1}{2}$

b $0, \frac{1}{3}, \frac{2}{3}, 1$

c $\Pr(\hat{P} = 0) = \frac{\binom{5}{0}\binom{5}{3}}{\binom{10}{3}} = \frac{1}{12}$

$\Pr(\hat{P} = \frac{1}{3}) = \frac{\binom{5}{2}\binom{5}{1}}{\binom{10}{3}} = \frac{5}{12}$

$\Pr(\hat{P} = \frac{2}{3}) = \frac{\binom{5}{2}\binom{5}{1}}{\binom{10}{3}} = \frac{5}{12}$

$\Pr(\hat{P} = 1) = \frac{\binom{5}{3}\binom{5}{0}}{\binom{10}{3}} = \frac{1}{12}$

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

d $\Pr(\hat{P} > 0.5) = \frac{5}{12} + \frac{1}{12} = \frac{1}{2}$

2 **a** $p = \frac{12}{20} = \frac{3}{5}$

b Values of $\hat{P}$: $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$

c

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$
$\Pr(\hat{P} = \hat{p})$	0.0036	0.0542	0.2384
$\hat{p}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$\Pr(\hat{P} = \hat{p})$	0.3973	0.2554	0.0511

d $\Pr(\hat{P} > 0.7) = 0.2554 + 0.0511$
$= 0.3065$

e $\Pr(\hat{P} < 0.7|\hat{P} > 0) = \dfrac{\Pr(0 < \hat{P} < 0.7)}{\Pr(\hat{P} > 0)} = 0.6924$

3 a $p = 0.5$

b Values of $\hat{P}$: $0, \frac{1}{2}, \frac{2}{3}, 1$

c

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	0.1	0.4	0.4	0.1

d $\Pr(\hat{P} > 0.25) = 0.9$

4 a $p = 0.4$

b Values of $\hat{P}$: $0, \frac{1}{3}, \frac{2}{3}, 1$

c

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

d $\Pr(\hat{P} > 0.5) = \dfrac{1}{3}$

e $\Pr(\hat{P} < 0.5|\hat{P} > 0)$
$= \dfrac{\Pr(0 < \hat{P} < 0.5)}{\Pr(\hat{P} > 0)} = \dfrac{4}{5}$

5 a $p = 0.5$

b Values of $\hat{P}$: $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

c

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

d $\Pr(\hat{P} > 0.7) = \dfrac{5}{16}$

6 **a** Values of $\hat{P}$: $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$

b

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{32}$	$\frac{1}{32}$

c $\Pr(\hat{P} < 0.4) = \frac{3}{16}$

d $\Pr(\hat{P} > 0 | \hat{P} < 0.8)$

$= \dfrac{\Pr(0 < \hat{P} < 0.8)}{\Pr(\hat{P} < 0.8)} = \dfrac{25}{26}$

7 **a** Values of $\hat{P}$: $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

b

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

c $\Pr(\hat{P} > 0.5 | \hat{P} > 0)$

$= \dfrac{\Pr(\hat{P} > 0.5)}{\Pr(\hat{P} > 0)} = \dfrac{17}{369}$

8

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

$$\mathrm{E}(X) = 0 \times \frac{1}{16} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{2} \times \frac{3}{8} + \frac{3}{4} \times \frac{1}{4} + 1 \times \frac{3}{16} = 0.5$$

$$\mathrm{E}(X^2) = 0^2 \times \frac{1}{16} + \left(\frac{1}{4}\right)^2 \times \frac{1}{4} + \left(\frac{1}{2}\right)^2 \times \frac{3}{8} + \left(\frac{3}{4}\right)^2 \times \frac{1}{4} + \left(1\right)^2 \times \frac{1}{16} = \frac{5}{16}$$

$$\therefore \mathrm{Var}(X) = \left(\frac{5}{16}\right) - \left(\frac{1}{2}\right)^2 = \frac{1}{16}$$

$$\therefore \mathrm{sd}(x) = \frac{1}{4}$$

9

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{32}$	$\frac{1}{32}$

$$\mathrm{E}(X) = 0 \times \frac{1}{32} + \frac{1}{5} \times \frac{5}{32} + \frac{2}{5} \times \frac{5}{16} + \frac{3}{5} \times \frac{5}{16} + \frac{4}{5} \times \frac{5}{32} + 1 \times \frac{1}{32} = 0.5$$

$$E(X^2) = 0^2 \times \frac{1}{32} + \left(\frac{1}{5}\right)^2 \times \frac{5}{32} + \left(\frac{2}{5}\right)^2 \times \frac{5}{16} + \left(\frac{3}{5}\right)^2 \times \frac{5}{16} + \left(\frac{4}{5}\right)^2 \times \frac{5}{32} + \left(1\right)^2 \times \frac{1}{32} = 0.340176$$

$\therefore \mathrm{Var}(X) = 0.340176 - \left(\frac{1}{2}\right)^2 = 0.050176$

$\therefore \mathrm{sd}(X) = 0.224$

$\mu = 0.5,\ \sigma = 0.224$

10

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

$$E(X) = 0 \times \frac{256}{625} + \frac{1}{4} \times \frac{256}{625} + \frac{3}{4} \times \frac{16}{625} + 1 \times \frac{1}{625} = 0.2$$

$$E(X^2) = 0^2 \times \frac{256}{625} + \left(\frac{1}{4}\right)^2 \times \frac{256}{625} + \left(\frac{3}{4}\right)^2 \times \frac{16}{625} + \left(1\right)^2 \times \frac{1}{32} = 0.08$$

$\therefore \mathrm{Var}(X) = 0.08 - \left(0.02\right)^2 = 0.04$

$\therefore \mathrm{sd}(X) = 0.2$

$\mu = 0.2,\ \sigma = 0.2$

11 $n = 30, p = 0.4, \mu = p = 0.3, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.3 \times 0.7}{30}} = 0.084$

$\mu = 0.3,\ \sigma = 0.084$

12 $n = 100, p = 0.4, \mu = p = 0.4, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.4 \times 0.6}{100}} = 0.049$

$\mu = 0.4,\ \sigma = 0.049$

13 $n = 100, p = 0.2, \mu = p = 0.2, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.2 \times 0.8}{100}} = 0.04\ \mu = 0.2,$

$\sigma = 0.04$

14 **a** $p = 0.65, n = 20$

$\Pr(\hat{P} = 0.65) = \Pr(X = 13) = 0.1844$

b $\mu = 0.65, \sigma = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.65 \times 0.35}{20}} = 0.1066$

$\mu - \sigma = 0.543$

$\mu + \sigma = 0.757$

$$\begin{aligned}\Pr(0.543 < \hat{P} < 0.757) &= \Pr(10.86 < X < 15.14)\\ &= \Pr(11 \le X \le 15)\\ &= 0.7600\end{aligned}$$

c $\mu - 2\sigma = 0.4368$

$\mu + 2\sigma = 0.8632$

$$\begin{aligned}\Pr(0.4368 < \hat{P} < 0.8632) &= \Pr(8.74 < X < 17.26)\\ &= \Pr(9 \le X \le 17)\\ &= 0.9683\end{aligned}$$

Solutions to Exercise 17C

1 $p = 0.5, n = 50$

$\mu = 0.5, \sigma = \sqrt{\dfrac{p(1-p)}{n}} =$

$\sqrt{\dfrac{0.5 \times 0.5}{50}} = 0.0707$

$\Pr(\hat{P} < 0.46) \approx \Pr(Z \leq \dfrac{0.46 - 0.5}{0.7070}) =$

$Pr(Z \leq -0.5658)$

The calculation can also be done directly with calculator:

$\Pr(\hat{P} < 0.46) \approx 0.2858$

2 $p = 0.12, n = 300$

$\mu = 0.12, \sigma = \sqrt{\dfrac{p(1-p)}{n}}$

$= \sqrt{\dfrac{0.12 \times 0.88}{300}} = 0.018762$

The calculation can also be done directly with calculator:

$\Pr(\hat{P} > 0.1) \approx 0.8568$

3 $p = 0.5, n = 25$

$\mu = 0.5, \sigma = \sqrt{\dfrac{p(1-p)}{n}}$

$= \sqrt{\dfrac{0.5 \times 0.5}{25}} = 0.1$

The calculation can also be done directly with calculator:

$\Pr(\hat{P} > 0.6) \approx 0.1587$

4 $p = 0.1, n = 200$

$\mu = 0.1, \sigma = \sqrt{\dfrac{p(1-p)}{n}}$

$= \sqrt{\dfrac{0.1 \times 0.9}{200}} = 0.0212$

The calculation can also be done directly with calculator:

$\Pr(\hat{P} > 0.15) \approx 0.0092$

5 $p = 0.3, n = 50$

$\mu = 0.3, \sigma = \sqrt{\dfrac{p(1-p)}{n}}$

$= \sqrt{\dfrac{0.3 \times 0.7}{50}} = 0.0648$

The calculation can also be done directly with calculator:

$\Pr(\hat{P} < 0.2) \approx 0.0614$

6 $p = 0.6, n = 100$

$\mu = 0.6, \sigma = \sqrt{\dfrac{p(1-p)}{n}}$

$= \sqrt{\dfrac{0.6 \times 0.4}{100}} = 0.0490$

a $\Pr(\hat{P} < 0.8) \approx 1$

b $\Pr(0.6 < \hat{P} < 0.8) \approx 0.5$

c $\Pr(0.7 < \hat{P} < 0.8 | \hat{P} > 0.6)$

$= \dfrac{\Pr(0.7 < \hat{P} < 0.8)}{\Pr(\hat{P} > 0.6)} \approx 0.0412$

7 $p = 0.5, n = 100$

$\mu = 0.5, \sigma = \sqrt{\dfrac{p(1-p)}{n}}$

$= \sqrt{\dfrac{0.5 \times 0.5}{100}} = 0.05$

The calculation can also be done directly with calculator:

$\Pr(0.4 < \hat{P} < 0.6) \approx 0.9545$

8 $p = 0.1, n = 1000$

$\mu = 0.1, \sigma = \sqrt{\dfrac{p(1-p)}{n}}$

$= \sqrt{\dfrac{0.1 \times 0.9}{1000}} = 0.0095$

a $\Pr(0.08 < \hat{P} < 0.12) \approx 0.9650$

b $\Pr(0.08 < \hat{P} < 0.12 | \hat{P} > 0.10) = \dfrac{\Pr(0.08 < \hat{P} < 0.12)}{\Pr(\hat{P} < 0.12)} \approx 0.9650$

9 $p = 0.52, n = 400$

a $\hat{p} = \dfrac{230}{400} = 0.575$

b $\mu = 0.52, \sigma = \sqrt{\dfrac{p(1-p)}{n}} = 0.0350$

$\Pr(\hat{P} \geq 0.575) \approx 0.0139$

10 $p = 0.9, n = 250$

a $\hat{p} = \dfrac{212}{250} = 0.848$

b $\mu = 0.9, \sigma = \sqrt{\dfrac{p(1-p)}{n}} = 0.0190$

$\Pr(\hat{P} \leq 0.848) \approx 0.0031$

c Yes, because the chance of the battery lasting only this short period of time is very small if the manufacturers claim is correct.

Solutions to Exercise 17D

1 **a** 0.08

b $(0.0268, 0.1332)$

2 **a** 0.192

b $(0.1432, 0.2408)$

3 **a** 0.2

b $(0.1216, 0.2784)$

4 $(0.28395, 0.3761)$

5 **a** $(0.4761, 0.5739)$

b $(0.5095, 0.5405)$

c The second interval is narrower because the sample size is larger

6 **a** $(0.8035, 0.8925)$

b $(0.8839, 0.8621)$

c The second interval is narrower because the sample size is larger

7 $M = 0.02, \hat{p} = 0.8$

$n = \left(\dfrac{1.96}{0.02}\right)^2 \times 0.8 \times 0.3 = 1536.64$

Since n must be an integer larger than the calculated value to ensure the margin of error is no more than 0.02, $n = 1537$

8 $M = 0.05, \hat{p} = 0.2$ $n =$

$\left(\dfrac{1.96}{0.05}\right)^2 \times 0.2 \times 0.8 = 245.86$

Since n must be an integer larger than the calculated value to ensure the margin of error is no more than 0.05, $n = 246$

9 $p^* = 0.30$

$M = 0.03$

$n = \left(\dfrac{1.96}{0.03}\right)^2 \times 0.3 \times 0.7 = 896.37$

Since n must be an integer larger than the calculated value to ensure the margin of error is no more than 0.03, $n = 897$

b $M = 0.02, \left(\dfrac{1.96}{0.02}\right)^2 \times 0.3 \times 0.7 = 2017$

c Reducing margin of error by 1% requires the sample size to be more than doubled

10 **a** $p^* = 0.3, M = 0.02$ $n =$

$\left(\dfrac{1.96}{0.02}\right)^2 \times 0.3 \times 0.7 = 2016.94 \approx 2017$

b $p^* = 0.5, M = 0.02$

$n = \left(\dfrac{1.96}{0.02}\right)^2 \times 0.5 \times 0.5 \approx 2401$

c **i** $p^* = 0.3, n = 2401$

$M = 1.96\sqrt{\dfrac{0.3 \times 0.7}{2401}} \approx 1.8$

The margin of error is less than 2%

ii $p^* = 0.5, n = 2017$

$M = 1.96\sqrt{\dfrac{0.5 \times 0.5}{2017}} \approx 2.2$

The margin of error is greater than 2%

d 2401, as this ensures that M is 2% or less, whoever is correct

11 90%: $(0.5194, 0.6801)$,
95%: $(0.5034, 0.6940)$,
99%: $(0.4738, 0.7262)$; Interval width increases as confidence level increases

12 90%: $(0.5111, 0.5629)$,
95%: $(0.5061, 0.5679)$,
99%: $(0.4964, 0.5776)$; Interval width increases as confidence level increases

Solutions to Technology-free questions

1 a All employees of the company

b $p = 0.35$

c $\hat{p} = 0.40$

2 a No; only people already interested in yoga

b Use electoral roll

3 a $\dfrac{k}{100}$

b $\hat{p} \pm 1.96\sqrt{\dfrac{\hat{p}(1-\hat{p}}{n}}$

$$= \frac{k}{100} \pm 1.96\sqrt{\frac{\frac{k}{100}(1-\frac{k}{100}}{100}}$$

$$= \frac{k}{100} \pm \frac{1.96\sqrt{k(100-k)}}{1000}$$

4 a $\hat{p} = 0.9$

b $M = 1.96\sqrt{\dfrac{\hat{p}(1-\hat{p}}{100}}$

$$= 1.96 \times \sqrt{\frac{0.9 \times 0.1}{n}}$$

$$= 1.96 \times \frac{0.3}{\sqrt{n}}$$

$$\therefore M = \frac{0.588}{\sqrt{n}}$$

c Margin of error would decrease by a factor of $\sqrt{2}$

5 a $40 \times 0.95 = 38$

b $\Pr(Y = 40) = \binom{40}{40}(0.95)^{40}(0.05)^0 = (0.95)^{40}$

6 a $50 \times 0.95 = 45$

b

$$\begin{aligned}\Pr(Y \geq 49) &= \Pr(Y = 49) + \Pr(Y = 50)\\ &= \binom{50}{49}(0.1)^1(0.9)^{49} + \binom{50}{50}(0.1)^0(0.9)^{50}\\ &= 5(0.9)^{49} + (0.9)^{50}\\ &= 5.9(0.9)^{49}\end{aligned}$$

7 a $\hat{p} = 0.60$

b $M = 0.10$

c Increase sample size

Solutions to multiple-choice questions

1 **B** This class is a sample of the whole school population, so any statistics determined from this sample is called a sample statistic.

2 **C** When the statistics is calculated from the whole population it is known as a population parameter.

3 **D** All we can say about a 95% confidence interval is that 95% of such intervals will capture the true mean. Statement B is a common incorrect interpretation of a confidence interval.

4 **E** $M = 1.96\sqrt{\dfrac{\hat{p}(1-\hat{p}}{100}}$

$= 1.96 \times \sqrt{\dfrac{0.3 \times 0.7}{50}}$

$= 0.1270$

5 **C** $\hat{p} = \dfrac{4}{50} = 0.08$
95% CI= (0.005, 0155) Use of calculator

6 **E** $\hat{p} = \dfrac{14}{88} = 0.08$
95% CI = (0.083, 0.236) Use of calculator

7 **B** As to be more confidence of capturing the true mean the interval will be wider.

8 **E** I the centre of a confidence interval is a sample parameter not a population parameter
II the bigger the margin of error the bigger the confidence interval
III a point estimate is a single value estimate like $\hat{p}$
IV the sample proportion a point estimate

9 **C** Since the width of the confidence interval is inversely proportional to the square root of the sample size, increasing the sample size by a factor of 4 decreases the width by a factor of 2.

10 **E** $M = 0.03$ $n = \left(\dfrac{1.96}{0.03}\right)^2 \times 0.3 \times 0.7 = 896.37 \approx 897$

11 **A** See definitions

12 **B** A sampling distribution is the distribution of a sample statistic, and as such shows how this statistic varies from sample to sample.

13 **C** $\hat{p} = 0.78, n = 100$
95% CI= (0.6988, 0.8682)

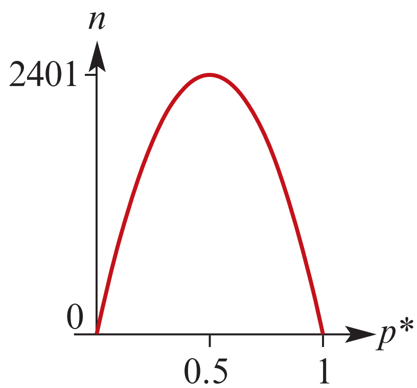
14 **D** The width of a confidence interval will decrease if the sample size is increased, or if the level of confidence is decreased.

Solutions to extended-response questions

1 **a** $n = \left(\frac{1.96}{M}\right)^2 p^*(1 - p^*) \qquad 0 \le p^* \le 1$

$= \left(\frac{1.96}{0.02}\right)^2 p^*(1 - p^*)$

$= 9604^2 p^*(1 - p^*)$



b From the graph, the maximum occurs when $p^* = 0.5$

c If they use the maximum samples size (2401) then they will ensure the margin of error stays within the desired range of ±2%

2 $p = 0.6, n = 100$

$\mu = \hat{p} = 0.6, \sigma = \sqrt{\frac{\hat{p}(1 - \hat{p}}{1n}} = 0.0490$

a $\Pr(\hat{P} > 0.65) = 0.1537$

b $\Pr(0.5 < \hat{P} < 0.65) = 0.8257$

3 **a** $\hat{p} = 0.57, n = 100$

95% CI = (0.4730, 0.6670)

b **i** $\Pr(Y = 5) = \binom{5}{5}(0.95)^5(0.05)^0 = 0.7738$

ii $\Pr(Y = 0) = \binom{5}{0}(0.95)^0(0.05)^5 = 0.0000003$

iii $\Pr(Y \le 4) = 0.2262$

iv $0.95 \times 5 = 4.75$

c $n = 500$

$X = 57 + 67 + 72 + 55 + 60 = 311$

$\hat{p} = \dfrac{311}{500} = 0.622$

CI = (0.5795, 0.6645)

4 **a** $p = \dfrac{500}{N}$

b $\hat{p} = \dfrac{60}{400} = 0.15$

c $\dfrac{500}{N} = 0.15 \; N \approx \dfrac{500}{0.15} = 3333.33 \approx 3333$

d 95% CI for $\hat{p}$

$$0.15 - 1.96\sqrt{\frac{0.15 \times 0.85}{400}} < p < 0.15 + 1.96\sqrt{\frac{0.15 \times 0.85}{400}}$$

$$0.15 - 1.96\sqrt{\frac{0.1275}{400}} < p < 0.15 + 1.96\sqrt{\frac{0.1275}{400}}$$

e $0.1150 < \dfrac{500}{N} < 0.1850$

$5.4056 < \dfrac{N}{500} < 8.6951$

$2703 < N < 4348$

Chapter 18 – Revision of chapters 13–17

Solutions to Technology-free questions

1 a

$$\int_{\frac{3}{2}}^{\frac{5}{2}} k\cos(\pi x)\,dx = \left[\frac{k}{\pi}\sin \pi x\right]_{\frac{3}{2}}^{\frac{5}{2}}$$

$$= \frac{k}{\pi}\left(\sin\frac{5\pi}{2} - \sin\frac{3\pi}{2}\right)$$

$$= \frac{2k}{\pi}$$

Since area= 1, $\frac{2k}{\pi} = 1 \Rightarrow k = \frac{\pi}{2}$

b

$$\int_{\frac{3}{2}}^{m} \cos(\pi x)\,dx = \left[\frac{\pi}{2}\times\frac{1}{\pi}\sin(\pi x)\right]_{\frac{3}{2}}^{m}$$

$$= \frac{1}{2}\left(\sin(m\pi) - \sin\frac{3\pi}{2}\right)$$

$$= \frac{1}{2}\left(\sin(m\pi) + 1\right)$$

For the median;

$$\frac{1}{2}\left(\sin(m\pi) + 1\right) = 0$$

$$m\pi = 0 \text{ or } m\pi = 2\pi$$

$$\therefore m = 2$$

$$(\text{since } \frac{3}{2} < m < \frac{5}{2})$$

c $\Pr\left(X < \frac{7}{4}\middle| X < 2\right) = \dfrac{\Pr\left(X < \frac{7}{4}\right)}{\Pr(X < 2)}$

$$= \frac{2 - \sqrt{2}}{2}$$

d $\Pr\left(X > \frac{9}{4}\middle| X > \frac{7}{4}\right) = \dfrac{\Pr\left(X > \frac{9}{4}\right)}{\Pr(X > \frac{7}{4})}$

$$= 3 - 2\sqrt{3}$$

2 a $\Pr(X > 3|X > 2) = \dfrac{\Pr(X > 3)}{\Pr(X > 2)}$

$$= \frac{0.1}{0.5}$$

$$= \frac{1}{5}$$

b $\Pr(X > 1|X \le 2) = \dfrac{\Pr(2 \le X \le 3)}{\Pr(X \le 2)}$

$$= \frac{0.4}{0.9}$$

$$= \frac{4}{9}$$

c $E(X) = \Sigma x \Pr(X = x) = 1.7$

d $E(X^2) = \Sigma x^2 \Pr(X = x) = 4.9$

$\mathrm{Var}(X) = 4.9 - 1.7^2 = 2.01$

3 a

$$\int_0^6 kx(6 - x)\,dx = k\left[3x^2 - \frac{x^3}{3}\right]_0^6$$

$$= k(108 - 72) \quad = 36k$$

$36k = 1 \Rightarrow k = \frac{1}{36}$

b $\frac{1}{36}\int_0^4 x(6 - x)\,dx = \frac{20}{27}$

c

$$\int_0^m \frac{x}{36}(6 - x)\,dx = 0.5$$

$$\frac{x}{36}\left[3x^2 - \frac{x^3}{3}\right]_0^m = 0.5$$

$$3m^2 - \frac{1}{3}m^3 - 18 = 0$$

$$9m^2 - m^3 - 54 = 0$$

$$-(m - 3)(m^2 - 6m - 18) = 0$$

$\therefore m = 3$

Other solutions are outside $[0, 6]$

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d $\dfrac{1}{36}\displaystyle\int_0^6 x^2(6-x)\,dx = 3$ Symmetry can be used for this.

e

$$\begin{aligned}\Pr(X < 2 | X < 3) &= \frac{\Pr(X < 2)}{\Pr(X < 3)}\\ &= \frac{\frac{1}{36}\int_0^2 x(6-x)\,dx}{\frac{1}{36}\int_0^3 x(6-x)\,dx}\\ &= \frac{\int_0^2 x(6-x)\,dx}{\int_0^3 x(6-x)\,dx}\\ &= \frac{\frac{28}{3}}{18}\\ &= \frac{14}{27}\end{aligned}$$

f

$$\begin{aligned}\Pr(X > 2 | X < 4) &= \frac{\Pr(2 < X < 4)}{\Pr(X < 4)}\\ &= \frac{\frac{1}{36}\int_2^4 x(6-x)\,dx}{\frac{1}{36}\int_0^4 x(6-x)\,dx}\\ &= \frac{\int_2^4 x(6-x)\,dx}{\int_0^4 x(6-x)\,dx}\\ &= \frac{\frac{52}{3}}{\frac{80}{3}}\\ &= \frac{13}{20}\end{aligned}$$

4 a $\Pr(RG) = \dfrac{3}{8} \times \dfrac{2}{7} = \dfrac{3}{28}$

b $\Pr(RG) + \Pr(GR) = \dfrac{3}{28} + \dfrac{3}{28} = \dfrac{3}{14}$

c $\Pr(G_2|R_1) + \Pr(B_2|R_1) + \Pr(Y_2|R_1) = \dfrac{2}{7} + \dfrac{2}{7} + \dfrac{1}{7} = \dfrac{5}{7}$

d $\Pr(R_1' \cap R_2') = \dfrac{5}{8} \times \dfrac{4}{7} = \dfrac{5}{14}$

e $\Pr(R_1 \cap R_2) + \Pr(B_1 \cap B_2) + \Pr(G_1 \cap G_2) = \dfrac{3}{14}$

5 $\Pr(A) = \dfrac{4}{7}, \Pr(B) = \dfrac{1}{3}, \Pr(A' \cap B) = ?$

a $\Pr(A' \cap B) + \Pr(A \cap B) = \Pr(B)$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\therefore$

$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B)$

$\therefore \Pr(A \cap B) = \dfrac{4}{7} + \dfrac{1}{3} - \dfrac{5}{7} = \dfrac{4}{21}$

Also

$\Pr(A' \cap B) + \Pr(A \cap B) = \Pr(B)$

$\therefore \Pr(A' \cap B) = \dfrac{1}{3} - \dfrac{4}{21} = \dfrac{1}{7}$

b $\Pr(A' \cap B) = \Pr(B) = \dfrac{1}{3}$

6 a

$$\begin{aligned}\Pr(A \cap B) &= \Pr(B|A)\Pr(A)\\ &= \frac{1}{5} \times \frac{3}{4}\\ &= \frac{3}{20}\end{aligned}$$

b $\Pr(B) = 1 - \Pr(A \cap B) - \Pr(A \cap B)$

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$$\begin{aligned}\Pr(A' \cap B') &= \Pr(B'|A')\Pr(A')\\ &= \frac{4}{7} \times \frac{1}{4}\\ &= \frac{1}{7}\\ \Pr(A \cap B') &= \Pr(A) - \Pr(A \cap B)\\ &= \frac{3}{4} - \frac{3}{20}\\ &= \frac{3}{10}\\ \Pr(B) &= 1 - \frac{3}{10} - \frac{1}{7}\\ &= \frac{39}{70}\end{aligned}$$

c $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$

$$= \frac{3}{20} \times \frac{70}{39}$$
$$= \frac{7}{26}$$

7 $n = 5, p = 0.45$

a $\Pr(X = 2) = \binom{5}{2}\left(\frac{9}{20}\right)^2\left(\frac{11}{20}\right)^3$
(≈ 0.3369)

b Pr(Janet and only one friend) =
$\frac{9}{20} \times \binom{4}{1}\left(\frac{9}{20}\right)\left(\frac{11}{20}\right)^3$ (≈ 0.2995)

8 a

$$\begin{aligned}\Sigma \Pr(X = x) &= 1\\ \therefore a + 0.3 + 0.1 + 0.2 + b &= 1\\ \therefore a + b &= 0.4 \ldots (1)\\ \mathrm{E}(X) = \Sigma x \Pr(X = x) &= 2.34\\ \therefore a + 0.6 + 0.3 + 0.8 + 5b &= 2.34\\ \therefore a + 5b &= 0.64 \ldots (2)\\ (2) - (1)&\\ 4b &= 0.24\\ b &= 0.06\\ \text{From(1) } a &= 0.34\end{aligned}$$

b $\mathrm{E}(X^2) = 6.54$
$\mathrm{Var}(X) = 1.0644$

9 a $\Pr(\text{win}) = 0.7 \times 0.9 + 0.3 \times 0.4$
$= 0.63 + 0.12$
$= 0.75$

b

$$\begin{aligned}\Pr(\text{Fully fit}|\text{ Did not win}) &= \frac{\Pr(\text{Fully fit} \cap \text{Did not win})}{\Pr(\text{Did not win})}\\ &= \frac{0.1 \times 0.7}{0.25}\\ &= \frac{7}{25}\end{aligned}$$

10 a

$$\int_a^{2a} (x - a)(2a - x)\,dx = \left[-\frac{x^3}{3} + \frac{3ax^2}{2} - 2xa^2\right]_a^{2a}$$
$$= \frac{a^3}{6}$$

Since the area = 1

$$\frac{a^3}{6} = 1$$
$$\therefore a^3 = 6$$

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b $\mathrm{E}(X) = \int_a^{2a} x(x-a)(2a-x)\,dx$

$$= \left[-\frac{x^4}{4} + \frac{3ax^3}{3} - (xa)^2\right]_a^{2a}$$

$$= \frac{a^4}{4}$$

$$= \frac{6^{\frac{4}{3}}}{4}$$

11 $\mu = 40, \sigma = 2$

$Pr(36 < X < 44) = q$

$Pr(X > 44) = \dfrac{1-q}{2}$

12 $\int_a^0 2(1-x)\,dx = \dfrac{3}{4}$ since

$$\left[2(x - \frac{x^2}{2})\right]_0^a = \frac{3}{4}$$

$$2a - a^2 = \frac{3}{4}$$

$$8a - 4a^2 = 3$$

$$4a^2 - 8a + 3 = 0$$

$$(2a-1)(2a-3) = 0$$

$$a = \frac{1}{2} \text{ or } a = \frac{3}{2}$$

$0 \le x \le 1, a = \dfrac{1}{2}$

13 $n = 3, =?$

a $\Pr(X = 0) = (1-p)^3$

b $\Pr(X = 0) = p^3$

$$(1-p)^3 = 8p^3$$

$$1 - p = 2p$$

$$1 = 3p$$

$$p = \frac{1}{3}$$

14 **a** $\hat{p} = \dfrac{a+b}{2}$

b M = half the width of the interval

$= \dfrac{b-a}{2}$

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Solutions to multiple-choice questions

1 E $\Pr(\text{both green}) = \frac{4}{16} \times \frac{3}{15}$

$= \frac{1}{4} \times \frac{1}{5}$

$= \frac{1}{20}$

2 D $\Pr(\text{six correct}) = \left(\frac{1}{2}\right)^6$

$= \frac{1}{64}$

$= 0.0156$

3 D $4c^2 + 5c^2 + 4c^2 + 3c^2 = 1$

$16c^2 = 1$

$c^2 = \frac{1}{16}$

$c = \pm\frac{1}{4}$

$= \pm 0.25$

of the option a available, only option D fits.

4 C If X is the number of spins if takes to get a '3', then it could take 1 spin or 2 spins or . . . .; there is no theoretical upper limit. So the sample space is $\{1, 2, 3, 4, \ldots\}$.

5 A $E(X) = \Sigma x \Pr(X = x)$

$= 4 \times 0.3 + 6 \times 0.2$

$+ 7 \times 0.1 + 9 \times 0.4$

$= 1.2 + 1.2 + 0.7 + 3.6$

$= 6.7$

6 B $E(X^2) = \Sigma x^2 \Pr(X = x)$

$= 16 \times 0.3 + 36 \times 0.2$

$+ 49 \times 0.1 + 81 \times 0.4$

$= 4.8 + 7.2 + 4.9 + 32.4$

$= 49.3$

$\text{var}(X) = E(X^2) - [E(X)]^2$

$= 49.3 - 6.7^2$

$= 4.41$

7 E The values of X are 4, 6, 7, 9. Since $Y = 2X - 1$, the corresponding values of Y are 7, 11, 13, 17. The probabilities are uncharged, so the fifth option fits.

8 D If $Z = aX + b$, then

$\text{var}(Z) = a^2\, \text{var}(x)$

Here, $a = -1$ and $b = 4$, so

$\text{var}(Z) = (1-)^2\, \text{var}(X)$

$= \text{var}(X)$

$= 4.41$

9 E The required probably is 0.46 (Alternatively use a tree diagram)

10 C $\text{Var}(X) = E(X^2) - [E(X)]^2$

$= 202 - 11^2$

$= 202 - 121$

$= 81$

$\text{sd}(X) = 9$

11 B 95% of scores, assuming an approximate normal distribution, will lie in the internal $(\mu - 2\sigma,\ \mu + 2\sigma)$.

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$\mu - 2\sigma = 50 - 20 = 30$

$\mu - 2\sigma = 50 + 20 = 70$

So the required interval is (30, 70).

12 D Pr(at last 2 heads)

$$= {}^3C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 + \left(\frac{1}{3}\right)^3$$

$$= 3 \times \frac{1}{8} + \frac{1}{8}$$

$$= \frac{4}{8}$$

$$= \frac{1}{2}$$

13 C $\mathrm{E}(X) = np$

$$= 400 \times 0.1$$

$$= 40$$

14 B If a die is rolled until a six is obtained the sample space is $\{1, 2, 3, \ldots\}$. This can not be a binominal variable since there is no theoretical limit to the number of rolls.

15 E $\mathrm{var}(X) = np(1 - p)$

$$= 900 \times 0.2 \times 0.8$$

$$= 9 \times 16$$

$$\mathrm{sd}(X) = 3 \times 4$$

$$= 12$$

16 E $\mathrm{var}(X) = np(1 - p)$

$$= 42p(1 - p)$$

$$= 9.4248$$

$$p(1 - p) = 0.2244$$

Use the *solve* command of a CAS calculator, giving $p = 0.34$ or $p = 0.66$. (Alternatively, solve the equation formula)

17 A If p is the probability of success, then Pr(5 sucesses) $= \binom{7}{5}p^5(1 - p)^2$. But 5 successes is the same as 2 fails. So this represents the probability of exactly two failures.

18 A Pr(4 females) $= {}^{10}C_4(0.2)^4(0.8)^6$

$$\approx 0.0881$$

19 D $\Pr(\geq 1) = 1 - (0 \text{ at home})$

$$= 1 - (0.4)^5$$

$$= 0.9898$$

20 D $\int_0^2 \left(kx^3 + \frac{3}{4}x\right)dx = 1$

$$\left[\frac{1}{4}kx^4 + \frac{3}{8}x^2\right]_0^2 = 1$$

$$4k + \frac{3}{2} = 1$$

$$4k = -\frac{1}{2}$$

$$k = -\frac{1}{8}$$

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21 **C** $\Pr(X \le 2) = \frac{1}{9}\int_0^2 (4x - x^2)dx$

$$= \frac{1}{9}\left[2x^2 - \frac{1}{3}x^3\right]_0^2$$

$$= \frac{1}{9}\left(8 - \frac{8}{3}\right)$$

$$= \frac{16}{29}$$

$$\approx 0.5926$$

22 **E** $\frac{8}{3}\int_0^m (1 - x)dx = \frac{1}{2}$ where m is the median.

$$\frac{8}{3}\left[x - \frac{x^2}{2}\right]_0^m = \frac{1}{2}$$

$$m - \frac{1}{2}m^2 = \frac{3}{16}$$

Solving this quadratic with a CAS 'Solve' command (or by use of the quadratic formula) gives $m \approx 0.209$ or $m \approx 1.791$

But $0 < m < \frac{1}{2}$, so $m \approx 0.209$

23 **B** $\mathrm{E}(X) = \int_1^2 2x\left(1 - \frac{1}{x^2}\right)dx$

$$= \int_1^2 \left(2x - \frac{2}{x}\right)dx$$

$$= \left[x^2 - 2\log_e x\right]_1^2$$

$$= (4 - 2\log_e 2) - (1 - 0)$$

$$= 3 - 2\log_e 2$$

$$\approx 1.614$$

24 **C** $\Pr(-1.0 < Z < 0)$

$$= \frac{1}{2}\Pr(-1.0 < Z < 1.0)$$

$$\approx \frac{1}{2} \times 0.68$$

$$= 0.34$$

25 **D** From the definition of standard deviation, it is always positive for any distribution, including.
(Checking the other options:
A: a mean can be negative
B: values for any normal distribution be any number in the interval $(-\infty, \infty)$
C: the area is *exactly* 1
E: the standard deviation could be greater than the mean (it is for a standard normal distribution))

26 **C** $\Pr(X > 2.6) \approx 0.1151$, using the 'normCdf' command of a CAS calculator. (You do not need to standardise.)

27 **E** $\Pr(X < -2), \approx 0.0228$, using the 'normCdf' command of a CAS calculator. (In this case, you might note that:

$$\Pr(X < -2) = \Pr\left(Z < \frac{-2-2}{2}\right)$$

$$= \Pr(Z < -2)$$

$$\approx \frac{1}{2} \times 0.05 = 0.025$$

using the 2σ limits. the only close option is the last option.)

28 **A** Since $\sigma^2 = 0.4,\ \sigma = \sqrt{0.4}$
$\Pr(X > -2.73) = 1$, using the 'normCdf' command of a CAS calculator.

29 **B** Since $\sigma^2 = 4,\ \sigma = 2$.
$\Pr(1 < X < 2.5) \approx 0.2902$, using the 'normCdf' command of a CAS calculator. (If you mistakenly used 4 for σ, you would get 0.1484, which is not one of the option!)

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30 **E** If X is the amount of cordial in a cup, then X is normal with $\mu = 50$ and $\sigma = 2$.
$\Pr(X > c) = 0.90$, so
$\Pr(X < c) = 0.10$, giving
$c = 47.44$ mL using the 'invnorm' command of CAS calculator.

31 **C** If X cm is the length of a lock of cheese, then X is normal with $\mu = 10$ and $\sigma = \sqrt{0.5}$.
$\Pr(X < c) = 0.95$, giving $c \approx 11.16$ cm using the 'invNorm' command of a CAS calculator.

32 **A** $\Pr(\mu - k < x < \mu + k) = 0.7$

$$\Pr\left(-\frac{k}{\sigma} < \frac{X-\mu}{\sigma} < \frac{k}{\sigma}\right) = 0.7$$

$$\Pr\left(-\frac{k}{\sigma} < Z < \frac{k}{\sigma}\right) = 0.7$$

Thus an area of 0.3 remains in the two tail, or 0.15 in each tail.

$$\text{So } \Pr\left(Z < \frac{k}{\sigma}\right) = 0.7 + 0.15$$

$$= 0.85$$

Using the 'invNorm' command of a CAS calculator shown that
$\Pr(Z < 1.03643) = 0.85$

$$\Rightarrow \frac{k}{\sigma} = 1.03643$$

Now $\sigma^2 = 2.25$, so $\sigma = 1.5$.
Hence $k = 1.555$. to 3 decimal places.
(Note that the value of the mean μ is not actually needed.)

33 **B** If X kg is the weight of a pocket, then X is normal $\mu = 1$.
More than 0.05 kg underweight means $X < 0.95$ and 3% are underweight.
$\Pr(X < 0.95) = 0.03$

$$\Pr\left(Z < \frac{-0.05}{\sigma}\right) = 0.03$$

Using the 'invNorm' command of a CAS calculator shows that
$\Pr(Z < -1.88079) = 0.03$

$$\Rightarrow -\frac{0.05}{\sigma} = -1.88079$$

$$\sigma = \frac{0.05}{1.88079}$$

$$\approx 0.027$$

34 **B** The graphs have the some centre so $\mu_1 = \mu_2$.
The lower graph is more spread out then the upper graph so $\sigma_1 > \sigma_2$.

35 **B** The standard deviation is $\sqrt{25} = 5$.
About 68% represent, ± 1 standard deviation from the mean of 173.
$173 - 5 = 168$
$173 + 5 = 178$
So the interval is (168, 178)

36 **B** $n = 200, p = 0.38$
95% CI = (0.313, 0.447) (Calculator)

37 **B** Increasing the level of confidence means that the interval will be wider

38 **C** Only statement II is correct

39 **B** Since the width of the confidence interval is inversely proportional to the square root of the sample size, decreasing the sample size by a factor of 2 will increase the width of the interval by a factor of $\sqrt{2}$

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Solutions to extended-response questions

1 In the following E denotes the event occurring, N the event not occurring.
Three trials are considered first.

a **i** The outcomes to consider are

(E, E, E) $\quad \Pr(E, E, E) = 0$

(E, E, N) $\quad \Pr(E, E, N) = 0$

(E, N, E) $\quad \Pr(E, N, E) = \frac{1}{2} \times 1 \times \frac{1}{2}$

(N, E, E) $\quad \Pr(N, E, E) = 0$

(N, N, E)

(N, E, N) $\quad$ Note: Remember the event cannot occur in consecutive trials.

(E, N, N)

(N, N, N)

$\therefore$ Probability of it occurring just twice $= \Pr\{(E, N, E)\} = \frac{1}{4}$

ii Consider the following outcomes

(E, E, N, N) $\quad \Pr(E, E, N, N) = 0$

(E, N, E, N) $\quad \Pr(E, N, E, N) = \frac{1}{2} \times 1 \times \frac{1}{2} \times 1 = \frac{1}{4}$

(E, N, N, E) $\quad \Pr(E, N, N, E) = \frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

(N, E, E, N) $\quad \Pr(N, E, E, N) = 0$

(N, E, N, E) $\quad \Pr(N, E, N, E) = \frac{1}{2} \times \frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{8}$

(N, N, E, E) $\quad \Pr(N, N, E, E) = 0$

$\therefore$ Pr(the event occurs exactly twice) $= \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

b For 5 trials there are 10 possible outcomes to consider

(E, E, N, N, N) $\quad \Pr(E, E, N, N, N) = 0$

(E, N, E, N, N) $\quad \Pr(E, N, E, N, N) = \frac{1}{2} \times 1 \times \frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{8}$

(E, N, N, E, N) $\quad \Pr(E, N, N, E, N) = \frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{2} \times 1 = \frac{1}{8}$

(E, N, N, N, E) $\quad \Pr(E, N, N, N, E) = \frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$

(N, E, E, N, N) $\quad \Pr(N, E, E, N, N) = 0$

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$(N, E, N, E, N) \quad \Pr(N, E, N, E, N) = \frac{1}{2} \times \frac{1}{2} \times 1 \times \frac{1}{2} \times 1 = \frac{1}{8}$

$(N, E, N, N, E) \quad \Pr(N, E, N, N, E) = \frac{1}{2} \times \frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$

$(N, N, E, E, N) \quad \Pr(N, N, E, E, N) = 0$

$(N, N, E, N, E) \quad \Pr(N, N, E, N, E) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{16}$

$(N, N, N, E, E) \quad \Pr(N, N, N, E, E) = 0$

$\therefore \quad \Pr(\text{the event occurs exactly twice}) = \frac{9}{16}$

2 Let X be the number of sixes obtained in 5 tosses of a die

$$\begin{aligned}
\Pr(\text{an even number of sixes}) &= \Pr(X = 0) + \Pr(X = 2) + \Pr(X = 4) \\
&= \left(\frac{5}{6}\right)^5 + 10\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 + 10\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right) \\
&= \left(\frac{1}{6}\right)^5 [5^5 + 10.5^3 + 10.5] \\
&= \left(\frac{1}{6}\right)^5 [4425] \\
&= 0.5692
\end{aligned}$$

$$\begin{aligned}
\Pr(\text{an odd number of sixes}) &= 1 - \Pr(\text{an even number of sixes}) \\
&= 0.4309
\end{aligned}$$

Let Y be the amount won by Katia. The probability distribution is as shown

y	1	$-x$
$\Pr(Y = y)$	0.4309	0.5691

The game is fair if $E(Y) = 0$

i.e. if $1 \times 0.4308 - 0.5691x = 0$

This implies $x = 0.7570$

Therefore Mikki should receive 76 cents from Katia if there is an even number of sixes.

3 a Let x be the daily demand

Let s be the number of newspapers stocked

If the demand is less than the number stocked

$P = 0.75x - 0.5s$

If the demand is greater than the number stocked

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$$P = 0.25s - 0.25(x - s)$$
$$= 0.5s - 0.25x$$

$\left(\begin{array}{l}\text{Note: the newspaper seller is considered}\\ \text{to lose money by not ordering enough.}\end{array}\right)$

$$\therefore \quad P = \begin{cases} 0.75x - 0.5s & x \le s \\ 0.5s - 0.25x & x > s \end{cases}$$

b Using the result of **a** a probability distribution for P is obtained with $s = 26$

p	5	5.75	6.50	6.25	6	5.75	5.50
$\Pr(P = p)$	0.05	0.10	0.10	0.25	0.25	0.15	0.10

The computations are as follows

$x = 24 \qquad p = 0.75 \times 24 - 0.5 \times 26 = 5$

$x = 25 \qquad p = 0.75 \times 25 - 0.5 \times 26 = 5.75$

$x = 26 \qquad p = 0.75 \times 26 - 0.5 \times 26 = 6.5$

$x = 27 \qquad p = 0.5 \times 26 - 0.25 \times 27 = 6.25$

etc.

Reorganising the table

p	5	5.50	5.75	6	6.25	6.50
$\Pr(P = p)$	0.05	0.10	0.25	0.25	0.25	0.1

$$\therefore \quad \mathrm{E}(P) = 5 \times 0.05 + 5.50 \times 0.10 + 5.75 \times 0.25 + 6 \times 0.25 + 6.25 \times 0.25 + 6.25 \times 0.1$$
$$= 5.95$$

The expected profit is \$5.95.

c $\mathrm{E}(P) = \sum_{x=24}^{s} (0.75x - 0.5s)p(x) + \sum_{x=s+1}^{30} (0.5s - 0.25x)p(x)$

d The newspaper seller should stock 27 (computation not shown).

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4 $\mathrm{E}(Z) = \mathrm{E}\left(\dfrac{X}{\sigma} - \dfrac{\mu}{\sigma}\right)$

$= \mathrm{E}\left(\dfrac{X}{\sigma}\right) - \sigma$

$= \dfrac{1}{\sigma}\mathrm{E}(X) - \dfrac{\mu}{\sigma}$ (Using $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$)

$= \dfrac{\mu}{\sigma} - \dfrac{\mu}{\sigma}$

$= 0$

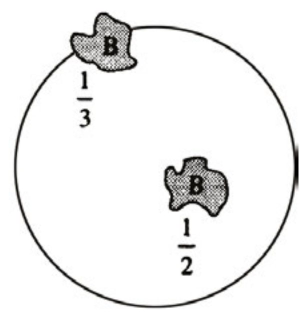
$\mathrm{Var}(Z) = \mathrm{Var}\left(\dfrac{X - \mu}{\sigma}\right)$

$= \dfrac{1}{\sigma^2}\mathrm{Var}(X)$ (Using $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$)

$= \dfrac{\sigma^2}{\sigma^2}$

$= 1$

5 a i Probability bean bag lands outside $= 1 - \dfrac{1}{3} - \dfrac{1}{2} = \dfrac{1}{6}$



ii Probability of two consecutive throws landing outside the circle $= \dfrac{1}{6} \times \dfrac{1}{6} = \dfrac{1}{36}$

iii Probability of first on the rim and second inside the circle $= \dfrac{1}{3} \times \dfrac{1}{2} = \dfrac{1}{6}$

b Let X be the score.

X	0	5	10
$\Pr(X = x)$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{2}{5}$

i With two shots to score a 20 requires two 10's. $\therefore$ Probability of score 20 $= \dfrac{4}{25}$

ii In other to score 10 the score could have resulted through 0 and 10 or 10 and 0 or 5 and 5.

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$\therefore\quad$ Probability of score 10 $= \dfrac{1}{2} \times \dfrac{2}{5} + \dfrac{1}{2} \times \dfrac{2}{5} + \dfrac{1}{10} \times \dfrac{1}{10}$

$$= \frac{1}{5} + \frac{1}{5} + \frac{1}{100}$$

$$= \frac{2}{5} + \frac{1}{100}$$

$$= \frac{41}{100}$$

c For Jane to score a 10:

- It can be a ten from 2 shots (bean bag of Anne; outside).
  Probability of this $= \dfrac{41}{100} \times \dfrac{1}{6} = \dfrac{41}{600}$
- It can be a ten from one throw (bean bag of Anne: rim)
  $$\frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$$

$\therefore\quad$ Probability of a ten $= \dfrac{41}{600} + \dfrac{2}{15} = \dfrac{121}{600}$

6 a $\mu = 400,\ \sigma = 50$

Let X be the lifetime of a light globe

$$\Pr(X > 375) = \Pr\left(Z > \frac{375 - 400}{50}\right)$$

$$= \Pr(Z > -0.5)$$

$$= \Pr(Z < 0.5)$$

$$= 0.6915$$

b Let Y be the number of light globes which will last more than 375 hours when selected from a box of 10.

This is Binomial with $n = 10$ and $p = 0.6915$

$$\Pr(Y \geq 9) = \Pr(Y = 9) + \Pr(Y = 10)$$

$$= \binom{10}{9} (0.6915)^9 (0.3085) + (0.6915)^{10}$$

$$= 0.1365$$

The probability that at least 9 of the globes in a randomly selected box will last more than 375 hours is 0.1365.

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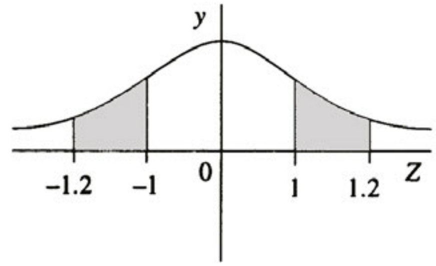
7 $\mu = 80\,000,\ \sigma = 20\,000$

a Let X be the distance travelled annually



$\Pr(56\,000 \le X \le 60\,000)$

$= \Pr\left(\dfrac{56\,000 - 80\,000}{20\,000} \le Z \le \dfrac{60\,000 - 80\,000}{20\,000}\right)$

$= \Pr(-1.2 \le Z \le -1)$

From the graph it can be seen

$\Pr(-1.2 \le Z \le -1)$

$= \Pr(1 \le Z \le 1.2)$

$= \Pr(Z \le 1.2) - \Pr(Z \le 1)$

$= 0.8849 - 0.8413$

$= 0.0436$

The probability that a randomly selected taxi will travel between 50 000 km and 60 000 km is 0.0436.

b Pr(Below 48 000 or above 96 000)

$= \Pr(X \le 48\,000) + \Pr(X \ge 96\,000)$

$= \Pr\left(Z \le \dfrac{48\,000 - 80\,000}{20\,000}\right) + \Pr\left(Z \ge \dfrac{96\,000 - 80\,000}{20\,000}\right)$

$= \Pr(Z \le -1.6) + \Pr(Z \ge 0.8)$

$= 1 - \Pr(Z \le 1.6) + 1 - \Pr(Z \ge 0.8)$

$= 2 - \Pr(Z \le 1.6) - \Pr(Z \le 0.8)$

$= 2 - 0.9452 - 0.7881$

$= 0.2667$

The percentage of taxis which travel below 48 000 km or have 96 000 km is 26.67%

c $\Pr(48\,000 \le X \le 96\,000)$

$= 1 - [\Pr(X \le 48\,000) + \Pr(X \ge 96\,000)]$

$= 1 - 0.2667$

$= 0.7333$

Let Y be the number of taxis out of the 250 which will travel between 48 000 and 96 000 km.

Y is a Binomial random variable with $n = 250$ and $p = 0.7333$

$\therefore\quad \mathrm{E}(Y) = np = 183.325$

i.e. 183 taxis out of the 250 are expected to travel between 48 000 and 96 000 km.

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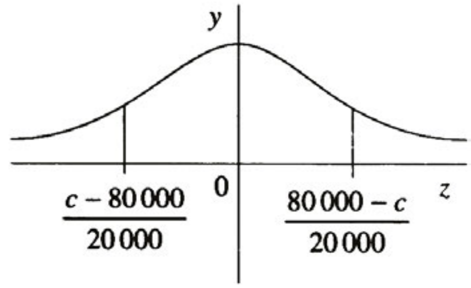
d Let c be such that

$$\Pr(X \geq c) = 0.85$$



Then $\quad \Pr\left(Z \geq \dfrac{c - 80\,000}{20\,000}\right) = 0.85$

From the graph

$$\Pr\left(Z \leq \frac{80\,000 - c}{20\,000}\right) = 0.85$$

$$\therefore \quad \frac{80\,000 - c}{20\,000} = 1.03643$$

$$\therefore \quad c = 80\,000 - 20\,000 \times 1.03643$$

$$= 59\,271$$

85% of taxis travel at least 59 271 kilometres.

8 a i Let X be the weight of cereal in a box

$9\mu = 505,\ \sigma = 5$

$\Pr(X \leq 500)$

$$= \Pr\left(Z \leq \frac{500 - 505}{5}\right)$$

$$= \Pr(Z \leq -1)$$

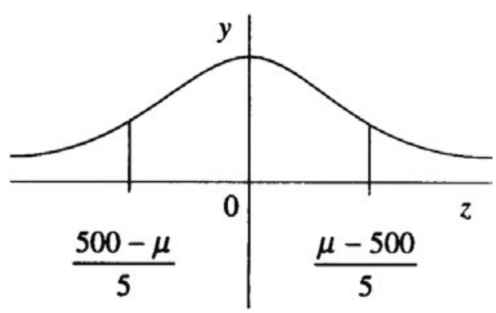
$$= 1 - \Pr(Z \leq 1)$$

$$= 1 - 0.8413$$

$$= 0.1587$$

ii $\mu = ?,\ \sigma = 5$

$$\Pr(X \leq 500) = 0.1$$



implies $\Pr\left(Z \leq \dfrac{500 - \mu}{5}\right) = 0.1$

From the graph

$\Pr\left(Z \leq \dfrac{500 - \mu}{5}\right)$

$$= \Pr\left(Z \geq \frac{\mu - 500}{5}\right)$$

$$= 1 - \Pr\left(Z \leq \frac{\mu - 500}{5}\right)$$

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$$\therefore \quad 1 - \Pr\left(Z \le \frac{\mu - 500}{5}\right) = 0.01$$

$$0.99 = \Pr\left(Z \le \frac{\mu - 500}{5}\right)$$

$$2.3263 = \frac{\mu - 500}{5}$$

$$\therefore \quad \mu = 5 \times 2.3263 + 500$$

$$= 511.63$$

b Let Y be the number of boxes under weight. Y is a Binomial random variable with $n = 5,\ p = 0.158655$

$\Pr(Y > 1) = 1 - (\Pr(Y = 0) + \Pr(Y = 1))$

$$= 1 - (0.841345)^5 - \binom{5}{1}(0.158655)(0.841345)^4$$

$$= 0.1809$$

9 a i $$f(y) = \begin{cases} k(y - 8) & \text{if } 8 \le y \le 12 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_5^{12} f(y)dy = \left[k\left(\frac{1}{2}y^2 - 8y\right)\right]_8^{12} = 8k$$

For f to be a probability density function $8k = 1$.

Hence $k = \dfrac{1}{8}$.

ii X is normally distributed with mean = 10 and standard deviation = 2

$\Pr(X < 11) = 0.6695$

$$\Pr(Y < 11) = \int_8^{11} f(y)\, dy$$

$$= \left[k\left(\frac{1}{2}y2 - 8y\right)\right]_8^{11}$$

$$= \frac{9}{16}$$

$$= 0.5625$$

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iii $\mathrm{E}(X) = 10$ and

$$\begin{aligned}\mathrm{E}(Y) &= \int_8^{12} y f(y)\, dy \\ &= \frac{1}{8}\int_8^{12} y(y-8)dy \\ &= \frac{32}{3} \\ &\approx 10.67\end{aligned}$$

Machine 1 is quicker or average.

b Probability of a widget being produced in less than 10 seconds

$= \Pr$ (less than 10 s | machine 1)Pr(machine 1)

+ Pr(less than 10 s | machine 2)Pr(machine 2)

$= 0.5 \times 0.6 + 0.25 \times 0.4$

$= 0.4$

Pr(machine 1 | less than ten seconds to produce)

$$= \frac{\text{Pr(Machine 1 and less than 10 stop roduce)}}{\text{Pr(less than 10 stop roduce)}}$$

$$= \frac{3}{4}$$

10

x	1	2	3	4	5	6
$\Pr(X = x)$	c	$\frac{c}{2}$	$\frac{c}{3}$	$\frac{c}{4}$	$\frac{c}{5}$	$\frac{c}{6}$

a Since we have a probability distribution

$$c + \frac{c}{2} + \frac{c}{3} + \frac{c}{4} + \frac{c}{5} + \frac{c}{6} = 1$$

$$\therefore \quad c\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right) = 0$$

$$\therefore \quad \frac{49c}{20} = 1$$

$$\therefore \quad c = \frac{20}{49}$$

b
$$\begin{aligned}\mathrm{E}(X) &= c + 2 \times \frac{c}{2} + 3 \times \frac{c}{3} + 4 \times \frac{c}{4} + 5 \times \frac{c}{5} + 6 \times \frac{c}{6} \\ &= \frac{20}{49}(1 + 1 + 1 + 1 + 1 + 1) \\ &= \frac{120}{49}\end{aligned}$$

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c

x^2	1	4	9	16	25	36
$\Pr(X = x)$	c	$\frac{c}{2}$	$\frac{c}{3}$	$\frac{c}{4}$	$\frac{c}{5}$	$\frac{c}{6}$

$$\begin{aligned}\therefore \quad \mathrm{E}(X^2) &= c + 2c + 3c + 4c + 5c + 6c \\ &= c(1 + 2 + 3 + 4 + 5 + 6) \\ &= 21c \\ &= \frac{21 \times 20}{49} \\ &= \frac{60}{7}\end{aligned}$$

$$\begin{aligned}\therefore \quad \mathrm{Var}(X) &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \\ &= \frac{60}{7} - \left(\frac{120}{49}\right)^2 \\ &= \frac{6180}{2401}\end{aligned}$$

11 a i
$$f(x) = \begin{cases} kx(100 - x^2) & \text{if } 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^{10} f(x)dx = \left[k\left(50x^2 - \frac{x^4}{4}\right)\right]_0^{10} = 2500k$$

For f to be the function of a probability density $k = \dfrac{1}{2500}$

ii $\mathrm{E}(X) = \int_0^{10} xf(x)dx = \dfrac{1}{2500}\int_0^{10} x^2(100 - x^2)dx = \dfrac{16}{3}$

iii $\Pr(X > 3) = \int_3^{10} f(x)dx == \left[k\left(50x^2 - \dfrac{x^4}{4}\right)\right]_3^{10} = 0.8281$

iv $\Pr(X > 3 \mid X < 7) = \dfrac{\Pr(3 < X < 7)}{\Pr(X < 7)} = \dfrac{\int_3^7 f(x)dx}{\int_0^7 f(x)dx} = 0.7677$

b This is a binomial distribution. Let W be the number of moviegoers who have to queue for more than 3 minutes

$$\Pr(W \ge 5) = ?$$

In this situation $n = 10$ and $p = 0.8281$

Using a calculator gives $\Pr(W \ge 5) = 0.9971$

12 a T is the temperature at which the sensor fails.

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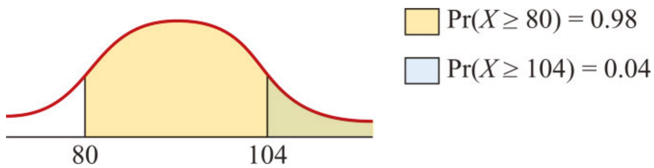
i



ii $\Pr(T \geq 80) = 0.98$

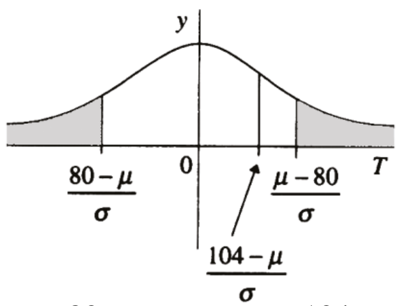
$\Pr(T \geq 104) = 0.04$

$\therefore \quad \Pr\left(Z \geq \dfrac{80-\mu}{\sigma}\right) = 0.98$

$\Pr\left(Z \geq \dfrac{104-\mu}{\sigma}\right) = 0.04$

$\therefore \quad \Pr\left(Z \leq \dfrac{\mu-80}{\sigma}\right) = 0.98$

and $\Pr\left(Z \leq \dfrac{104-\mu}{\sigma}\right) = 0.96$



$\therefore \quad \dfrac{\mu-80}{\sigma} = 2.0537$ and $\dfrac{104-\mu}{\sigma} = 1.7507$

$\therefore \quad \mu - 80 = 2.0537\sigma$ (1) and $104 - \mu = 1.750\sigma$ (2)

Add (1) and (2) $24 = (2.0537 + 1.7507)\sigma$

$6.3084 = \sigma$

$\therefore \quad$ From (1) $\mu = 80 + 2.0537 \times 6.3084 = 92.956$

b **i** $\mu = 94.5$ and $\sigma = 5.7$

$$\Pr(T \leq 100) = \Pr\left(Z \leq \frac{100-94.5}{5.7}\right)$$

$$= \Pr\left(Z \leq \frac{5.5}{5.7}\right)$$

$$= \Pr\left(Z \leq \frac{55}{57}\right)$$

$$= 0.8327$$

$\therefore \quad \Pr(T \geq 100) = 1 - 0.8327 = 0.1673$

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16.73% of sensors will operate in boiling water.

ii $\Pr(T \geq c) = 0.99$

$\therefore \quad \Pr\left(Z \geq \dfrac{c - 94.5}{5.7}\right) = 0.99$

$\therefore \quad \Pr\left(Z \leq \dfrac{c - 94.5}{5.7}\right) = 0.01$

$\therefore \quad \dfrac{c - 94.5}{5.7} = -2.3263$

$\therefore \quad c = 81.2398$

They should quote 81°C as a temperature for the sensors to work.

13 a i Probability that the plane is early $= 1 - (0.5 + 0.3) = 0.2$

ii Probability that it does not arrive late $= 1 - 0.3 = 0.7$

iii Probability that the plane is on time on three consecutive days $= (0.5)^3 = 0.125$

iv Late on Monday but one time all the remaining days $= 0.3 \times (0.5)^4 = \dfrac{3}{160}$

b i Let X be the number of times late

$\Pr(X = 1) = {}^5C_1(0.3)(0.7)^4 = 0.36015$

ii Let Y be the number of times early

$\Pr(Y = 2) = {}^5C_2(0.2)^2(0.8)^3 = \dfrac{128}{625}$

c $\Pr(X > 2) = \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)$

$= {}^5C_3(0.3)^3(0.7)^2 + {}^5C_4(0.3)^4(0.7) + (0.3)^5$

$= 0.1323 + 0.02835 + 0.00243$

$= 0.16308$

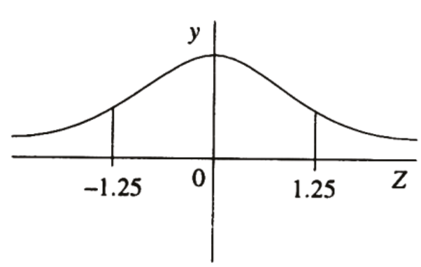
14 a $\Pr(X \leq 985) = \Pr\left(Z \leq \dfrac{985 - 1000}{12}\right)$

$= \Pr\left(Z \leq \dfrac{-15}{12}\right)$

$= \Pr(Z \leq -1.25)$

$= 1 - \Pr(Z \leq 1.25)$

$= 0.1056$



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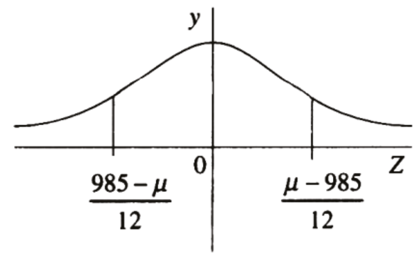
b

$$\Pr(X \le 985) = 0.01$$



$$\therefore \quad \Pr\left(Z \le \frac{985-\mu}{12}\right) = 0.01$$

$$\therefore \quad 1 - \Pr\left(Z \le \frac{\mu-985}{12}\right) = 0.01$$

$$\Pr\left(Z \le \frac{\mu-985}{12}\right) = 0.99$$

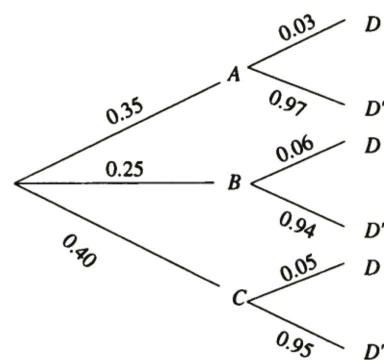
$$\therefore \quad \frac{\mu-985}{12} = 2.3263$$

$$\therefore \quad \mu = 12 \times 2.3263 + 985$$

$$\therefore \quad = 1012.92$$

The machine should be set at 1012.92

15 In the tree diagram A, B and C are the machines. D is defective. D′ is not defective.



a **i** $\Pr(A \cap D) = 0.35 \times 0.03 = 0.0105$

ii $\Pr(D) = \Pr(A \cap D) + \Pr(B \cap D) + \Pr(C \cap D)$

$= 0.0105 + 0.25 \times 0.06 + 0.40 \times 0.05$

$= 0.0455$

b $\Pr(C \mid D) = \dfrac{\Pr(C \cap D)}{\Pr(D)} = \dfrac{0.4. \times 0.05}{0.0455} = 0.4396$

c $\Pr(A \cup B \mid D') = \dfrac{\Pr((A \cup B) \cap D')}{\Pr(D')} = \dfrac{\Pr(A \cap D') + \Pr(B \cap D')}{\Pr(D')} = \dfrac{0.5745}{0.9545} = \dfrac{1149}{1909}$

16 **a** **i** $\mu = \mathrm{E}(X) = \Sigma x \Pr(X = x)$

$= 4.25$

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ii $\sigma = \sqrt{\text{Var}(X)}$

$$\begin{aligned}\text{Var}(X) &= \text{E}(X^2) - [\text{E}(X)]^2\\ &= 1 \times 0.02 + 4 \times 0.03 + 9 \times 0.04 + 16 \times 0.45 + 25 \times 0.45 - (4.25)^2\\ &= 0.02 + 0.12 + 0.36 + 7.2 + 11.25 - (4.25)^2\\ &= 18.95 - 18.0625\\ &= 0.8875\end{aligned}$$

$\therefore \quad \sigma = \sqrt{0.8875} = 0.9421$

iii $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$

$= \Pr(2.366 \le X \le 6.134)$

$= 0.94$

iv $\Pr(X \ge 4) = 0.45 + 0.45$

$= 0.9$

b **i** Binomial

ii Expected number of working games in box

$= \text{E}(Y)$

$= 20 \times 0.9$

$= 18$

iii $\text{Var}(Y) = 20 \times 0.9 \times 0.1$

$= 1.8$

$\sigma = \sqrt{1.8}$

$= 1.342$

iv $\Pr(Y \ge 19) = \Pr(Y = 19) + \Pr(Y = 20)$

$= 0.27017\ldots + 0.12158\ldots$

$= 0.3917$ (correct to 4 decimal places)

17 **a** **i** $\Pr(\text{Black}) = \dfrac{n-3}{n}$

ii $\Pr(\text{White}) = \dfrac{3}{n}$

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b $\Pr(B_1|W_2) = \dfrac{\Pr(B_1 \cap W_2)}{\Pr(W_2)}$

$$= \frac{\dfrac{n-3}{n} \times \dfrac{n-3}{n+1}}{\dfrac{3}{n} \times \dfrac{n-2}{n+1} + \dfrac{n-3}{n} \times \dfrac{n-3}{n+1}}$$

$$= \frac{(n-3)^2}{n^2 - 3n + 3}$$

18 **a** $n = 1000, X = 100$ CI= $(0.0814, 0.1186)$

b $m = 800, Y = 80$ CI= $(0.0792, 0.1208)$

c width female = 0.0372
width male = 0.0416
The confidence interval is narrower because the sample size for the females is larger.

d 900 of each sex

e
$$\frac{\hat{p}_1(1-\hat{p}_1)}{n} = \frac{\hat{p}_2(1-\hat{p}_2)}{m}$$
$$\frac{0.1 \times 0.9}{1000} = \frac{\hat{p}_2(1-\hat{p}_2)}{800}$$
$$0.072 = \hat{p}_2(1-\hat{p_2})$$

$\hat{p}_2 = 0.078$ or 0.922

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Chapter 19 – Revision of Chapters 1–18

Solutions to Technology-free questions

1 $f(g(x)) = f(3x+1)$
$= (3x+1)^2 + 6$
$= 9x^2 + 6x + 7$

2 Infinitely many solutions if the determined of the coefficients matrix is zero, i.e.
$$\begin{vmatrix} l_c & 3 \\ 4 & (l_c + 2) \end{vmatrix} = 0$$
$l_c(l_c+2) - 12 = 0$
$l_c^2 + 2l_c - 12 = 0$
$(l_c+1)^2 - 13 = 0$
$l_c = -1 \pm \sqrt{13}$

3 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$
$= \begin{bmatrix} 2x \\ -3y \end{bmatrix}$
$x' = 2x$ and $y' = -3y$
$x = \frac{1}{2}x'$ and $y = -\frac{1}{3}y'$
$y = \frac{1}{x}$ becomes $-\frac{1}{3}y' = \dfrac{1}{\frac{1}{2}x'}$
i.e. $y' = -\frac{6}{x'}$ or in terms of x, y; $y = -\frac{6}{x}$
Reflection in x-axis, dilation by factor 2 from y-axis, dilation by factor 3 from x-axis, OR (using the final rule) reflection in x-axis, then dilation by factor 6 from the x-(or y-) axis.

4 a $f'(x) = 7(5x^3 - 3x)^6 \times (15x^2 - 3)$
$= 21(5x^2 - 1)(5x^2 - 3x)^6$
$= 21x^6(5x^2 - 1)(5x^2 - 3)^6$

b $f'(x) = 2e^{4x} + 2x \times 4e^{4x}$
$= 2e^{4x}(1 + 4x)$
$f'(0) = 2 \times 1 \times 1$
$= 2$

5 a
$$\frac{d}{dx}\left(x^2\log_e(2x)\right) = 2x\log_e(2x) + x^2 \times \frac{1}{x}$$
$$= x\left(1 + 2\log_e\left(2x\right)\right)$$

b
$$f'(x) = \frac{(2x+1)\cos(x) - 2\sin(x)}{(2x+1)^2}$$
$$f'\left(\frac{\pi}{2}\right) = \frac{\left(2 \times \frac{\pi}{2} + 1\right)\cos\left(\frac{\pi}{2}\right) - 1\sin\left(\frac{\pi}{2}\right)}{\left(2 \times \frac{\pi}{2} + 1\right)^2}$$
$$= -\frac{2}{(\pi+1)^2}$$

6 a $f'(x) = e^{\sin(2x)} \times 2\cos(2x)$
$= 2\cos(2x)e^{\sin(2x)}$

b $f'(x) = 3\tan(2x) + 3x \times 2\sec^2(2x)$

$$= 3\tan(2x) + 6x\sec^2(2x)$$

$$f'\left(\frac{\pi}{3}\right) = 3\tan\left(\frac{2\pi}{3}\right) + 2\pi\sec^2\left(\frac{2\pi}{3}\right)$$

$$= -3\sqrt{3} + 2\pi \times \frac{1}{\cos^2\left(\frac{2\pi}{3}\right)}$$

$$= -3\sqrt{3} + 2\pi \times 4$$

$$= 8\pi - 3\sqrt{3}$$

7 $\sin(2x) - \cos(2x) = 0$

$$\sin(2x) = \cos(2x)$$

$$\tan(2x) = 1$$

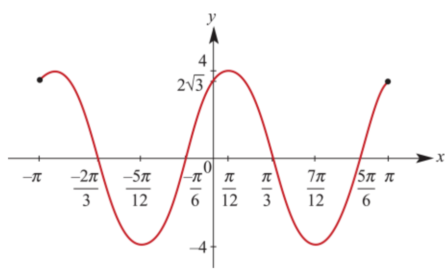
$$2\,x = \frac{\pi}{4} + n\pi$$

$$= \frac{\pi(4n+1)}{4}$$

$$x = \frac{\pi(4n+1)}{8}, n \in \mathbb{Z}$$

8 **a** Amplitude = 4, period = $\frac{2\pi}{2} = \pi$

b



9 $y = f(x) = 1 - \frac{4}{x-2}$

$x \to \pm\infty,\ y \to 1;\ x \to 2,\ y \to \pm\infty$

S, the asymptotes have equations

$x = 2$ and $y = 1$

$x = 0:\ y = 1 - \frac{4}{0-2}$

$= 1 - (-2)$

$= 3$

$y = 0:\ \frac{4}{x-2} = 1$

$= x - 2 = 4$

$x = 6$

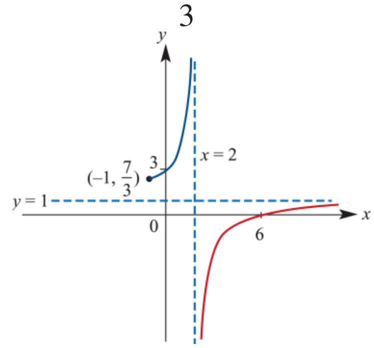
The intercepts are $(6,\ 0)$ and $(0, 3)$

check the endpoint of the domain:

$x = -1: y = 1 - \frac{4}{-1-2}$

$$= 1 + \frac{4}{3}$$

$$= \frac{7}{3}$$



10 **a** $y = 5e^{x-1} - 3$

interchange x and y and solve for y:

$$x = 5e^{y-1} - 3$$

$$5e^{y-1} = x + 3$$

$$e^{y-1} = \log_e \frac{x+3}{5}$$

$$y - 1 = \log_e\left(\frac{x+3}{5}\right)$$

$$y = f^{-1}(x) = \log_e\left(\frac{x+3}{5}\right) + 1$$

b range of $f = (-3,\ \infty)$ = domain of f^{-1}

11 $\cos\left(\frac{5x}{2}\right) = \frac{1}{2}, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

General solution is given by

$$\frac{5x}{2} = \pm\frac{\pi}{3} + 2n\pi$$

$$x = \pm\frac{2\pi}{15} + \frac{4n\pi}{15}$$

$$n = 0: x = \pm\frac{2\pi}{15}$$

$$n = 1: x = \pm\frac{2\pi}{15} + \frac{4\pi}{5}$$

$$= \frac{2\pi}{3}, \frac{14\pi}{15}\text{(outside internal)}$$

$$n = -1: x = \pm\frac{2\pi}{3} - \frac{4\pi}{5}$$

$$= -\frac{14\pi}{15}, -\frac{2\pi}{3}\text{(outside internal)}$$

Solutions are $x = -\frac{2\pi}{15}, \frac{2\pi}{15}$

12

$$g(u + v) = 5(u + v)^2$$
$$= 5(u^2 + 2uv + v^2)$$
$$g(u + v) = 5(u - v)^2$$
$$= 5(u^2 - 2uv + v^2)$$
$$g(u + v) + g(u - v) = 10(u^2 + v^2)$$
$$= 2(5u^2 + 5v^2)$$
$$= 2(9(u) + g(v))$$

13 Average value $= \frac{1}{4-0}\int_0^4 e^x\,dx$

$$= \frac{1}{4}\left[e^x\right]_0^4$$

$$= \frac{1}{4}(e^4 - 1)$$

14 a $x = 0,\ y = 6: \quad 6 = 0 + 0 + c$

$$c = 6$$

$x = -2,\ y = 0: 0 = -8a - 2b + 6$

$4a + b = 3 \quad \ldots①$

$\frac{dy}{dx} = 0$ when $x = -1$

b $\frac{dy}{dx} = 3ax^2 + b$

$= 0$ when $x = -1$, so

$3a + b = 0 \quad \ldots②$

c ①-②: $\quad a = 3$

Substitute into ②: $\quad b = -3 \times 3$

$= -9$

15 a $y = g(x) = 3 - e^{2x}$

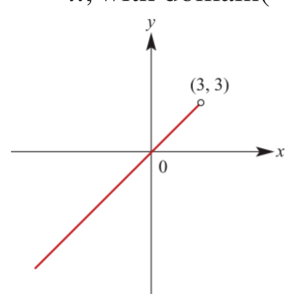
Interchange x and y and solve for y.

$$x = 3 - e^{2y}$$
$$e^{2y} = 3 - x$$
$$2\,y = \log_e(3 - x)$$
$$y = g^{-1}(x) = \frac{1}{2}\log_e(3 - x)$$

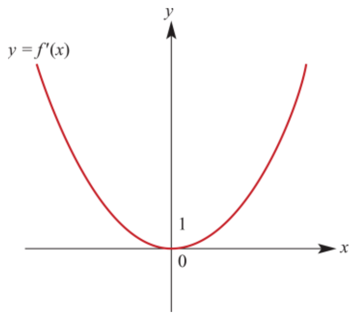
domain of g^{-1} = range of $g = (-\infty,\ 3)$

b $y = g(g^{-1}(x))$

$= x$, with domain$(-\infty,\ 3)$



16 a The graph of $y = f(x)$ is continuous and appears to be 'smooth' at (0, 1), so the derivative exists of $x = 0$ where the gradient appears to be zero. The gradient is positive for all other value of x. The graph of $y = f'(x)$ is shown below.



b $f'(x) = \begin{cases} -8x^3 & x \le 0 \\ 8x^3 & \text{otherwise} \end{cases}$

(Note that $f'(0) = 0$ as expected.)

17 $f(x) = \dfrac{1}{-3}\log_e(1 - 3x) + c$

$= -\dfrac{1}{3}\log_e(1 - 3x) + c$

18 $y = f(x) = \dfrac{3}{2x - 1} + 3$

Intercharge x and y and solve for y

$$x = \frac{3}{2y - 1} + 3$$

$$\frac{3}{2y - 1} = x - 3$$

$$\frac{2y - 1}{3} = \frac{1}{x - 3}$$

$$2y - 1 = \frac{3}{x - 3}$$

$$2y = \frac{3}{x - 3} + 1$$

$$= \frac{3 + x - 3}{x - 3}$$

$$= \frac{x}{x - 3}$$

$$y = f^{-1}(x) = \frac{x}{2(x - 3)}$$

19 $\tan(2x) = -\sqrt{3}$

$$2x = \cdots - \frac{\pi}{3} - \pi, \frac{\pi}{3}, -\frac{\pi}{3} + \pi, -\frac{\pi}{3} + 2\pi, \; \cdots$$

$$2x = \cdots - \frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}, \cdots$$

$$x = \cdots - \frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}, \cdots$$

since $x \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{4}\right) \cup \left(\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$, the solution are $x = -\dfrac{\pi}{6}, \dfrac{\pi}{3}$.

20 X is normal with mean 84 and standard derviation 6.

a $\Pr(X > 84) = \Pr(Z > 0)$

$= 0.5$

b $\Pr(78 < X < 90)$

$= \Pr\left(\dfrac{78 - 84}{6} < Z < \dfrac{90 - 84}{6}\right)$

$= \Pr(-1 < Z < 1)$

$= \Pr(Z < 1) - \Pr(Z > 1)$

$= \Pr(Z < 1) - \Pr(Z > 1)$

$= \Pr(Z < 1) - (1 - \Pr(Z < 1)$

$= 2\ \Pr(Z < 1) - 1$

$= 2 \times 0.84 - 1$

$= 0.68$

c $\Pr(X < 78 \mid X < 84)$

$= \dfrac{\Pr('X < 78' \cap' X < 84')}{\Pr(X < 84')}$

$= \dfrac{\Pr(X < 78)}{\Pr(X < 84)}$

$\Pr(X < 78) = \Pr(Z < -1)$

$= 1 - \Pr(Z < 1)$

$= 1 - 0.84$

$= 0.16$

$$\Pr(X < 84) = 0.5$$

$$\Pr(X < 78 \mid X < 84) = \frac{0.16}{0.5} = 0.32$$

21 a $\Pr(X < 3) = \int_1^3 \frac{x}{24}\,dx$

$$= \left[\frac{x^2}{48}\right]_1^3 = \frac{9-1}{48} = \frac{1}{6}$$

b $\Pr(X \geq b) = \int_b^7 \frac{x}{24}\,dx$

$$= \left[\frac{x^2}{48}\right]_b^7 = \frac{49-b^2}{48} = \frac{3}{8}$$

if $49 - b^2 = 18$

$$b^2 = 31$$

$$b = \sqrt{31}, \text{ since } b \in [1, 7]$$

22 The gradient of the tangent is $\frac{1}{3}$

Also, $\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$

$$= \frac{1}{3}$$

if $x^{-\frac{2}{3}} = 1$

$$x^{-2} = 1$$

$$x^2 = 1$$

$$x = \pm 1$$

$$x = -1,\ y = -1 : y + 1 = \frac{1}{3}(x+1)$$

$$\Rightarrow y = \frac{1}{3}x - \frac{2}{3}$$

$$x = -1,\ y = -1 : y + 1 = \frac{1}{3}(x+1)$$

$$\Rightarrow y = \frac{1}{3}x - \frac{2}{3}$$

Hence $a = \pm\frac{2}{3}$

23 a $b = 16 - 4a^2$

A = area $XYZW$

$$= 2\,ab = 2\,a(16 - 4\,a^2) = 32a - 8a^3$$

b $\frac{dA}{da} = 32 - 24a^2$

$$= 0$$

If $a^2 = \frac{32}{24} = \frac{4}{3}$

$$a = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}$$

But $a > 0$, so $a = \frac{2\sqrt{3}}{3}$, and

$A = \dfrac{128\sqrt{3}}{9}$.

(This clearly correspond, to a maximum since $a \in [0, 2]$ and $A = 0$ for $a = 0$ or $a = 2$. Alternately check the sign of the derivative.)

24 $\int_{-1}^{3} (-3x^2 + 2bx + 9)\,dx = 32$

$$\left[-x^3 + bx^2 + 9x\right]_{-1}^{3} = 32$$

$$(-27 + 9b + 27) - (1 + b - 9) = 32$$

$$8b + 8 = 32$$

$$8b = 24$$

$$b = 3$$

25 0.36

26 **a** Mean of $X = \mathrm{E}(X)$

$$= 0 \times 0.6 + | \times 0.2 + 2 \times 0.15$$

$$+ 3 \times 0.0$$

$$= 0.65$$

the mean is \$0.65.

b Pr(same amount) = Pr(0& 0 or 1&1 or 2 & 2 or 3 & 3)

$$= 0.6^2 + 0.2^2$$

$$+ 0.15^2 + 0.05^2$$

$$= 0.36 + 0.04$$

$$+ 0.0225 + 0.0025$$

$$= 0.425$$

27 The possible sequences are:

$G \rightarrow G \rightarrow R \rightarrow R$ or

$G \rightarrow R \rightarrow G \rightarrow R$ or

$G \rightarrow R \rightarrow R \rightarrow G$

where G = goes to gym, and

R = goes for run

Required probability $= 0.5 \times 0.5 \times 0.6$

$$+ 0.5 \times 0.4 \times 0.5$$

$$+ 0.5 \times 0.6 \times 0.4$$

$$= 0.15 + 0.10$$

$$+ 0.12$$

$$= 0.37$$

28 **a** Volume = area cross-section × height

$$= \frac{1}{2}x^2h$$

$$= 2000$$

$$x^2h = 4000$$

$$h = \frac{4000}{x^2}$$

b The hypotenuse of the right-angled triangle cross-section has length $\sqrt{2}x$. The surface area is made up of three vertical rectangles and two equal triangular ends.

$$A = \sqrt{2}xh + xh + xh + 2 \times \frac{1}{2}x^2$$

$$= xh(2 + \sqrt{2}) + x^2$$

$$= x \times \frac{4000}{x^2} \times (2 + \sqrt{2}) + x^2$$

$$= \frac{4000\sqrt{2} + 8000}{x} + x^2$$

c $\dfrac{dA}{dx} = -\dfrac{4000\sqrt{2} + 8000}{x^2} + 2x$

$= 0$

if $2x^3 = 4000\sqrt{2} + 8000$

i.e. $x^3 = 2000\sqrt{2} + 4000$

$= 2000(2 + \sqrt{2})$

29 a No, these people may all be at the same restaurant because they have something in common eg may be members of a tennis club, and may have better reaction times than average

b Use a random selection method such as the electoral role.

30 a $\hat{p} = 0.53$

b $\hat{p} \pm 1.96\sqrt{\dfrac{\hat{p}(1-\hat{p}}{n}}$

$= 0.53 \pm 1.96\sqrt{\dfrac{0.53 \times 0.47}{100}}$

31 a $\hat{p} = 0.37$

b $M = 1.96\sqrt{\dfrac{\hat{p}(1-\hat{p}}{n}}$

$= 1.96\sqrt{\dfrac{0.37 \times 0.63}{n}}$

c Halving n will increase the margin of error by a factor of $\sqrt{2}$

Solutions to multiple-choice questions

1 B Write the equations in matrix form:

$$\begin{bmatrix} m & -2 \\ 6 & -(m+4) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

These will be a unique solution if the coefficients matrix has a non-zero determinant, i.e.

$$\begin{vmatrix} m & -2 \\ 6 & -(m+4) \end{vmatrix} \neq 0$$

$$\begin{aligned} &= m(m+4) - (-2)(6) \neq 0 \\ -m^2 &- 4m + 12 \neq 0 \\ m^2 &+ 4m - 12 \neq 0 \\ (m-2)&(m+6) \neq 0 \\ m &\neq 2, \ -6 \end{aligned}$$

So $m \in \mathbb{R} \backslash \{-6, 2\}$

2 A Since $\sin\left(\frac{\pi}{2}\right) = 1$, then

$$2x = \frac{\pi}{2} + 2n\pi$$

$$x = n\pi + \frac{\pi}{4}$$

3 D $f(x-y) = (x-y)^3$
$\neq x^3 - y^3 = f(x) - f(y)$
Checking the other options shows each one is true.

4 B The graph of f has a sharp point at $x = -\frac{4}{5}$, so $f'\left(-\frac{4}{5}\right)$ is not defined.
Hence the graph of f' is discontinuous at $x = -\frac{4}{5}$.
Checking the other points shows that each one is true.

5 C

$$\begin{aligned} k &= \int_{-6}^{-2} \frac{2}{x}\,dx \\ &= \Big[2\log_e|x|\Big]_{-6}^{-2} \\ &= 2\log_e 2 - 2\log_e 6 \\ &= 2\log_e \frac{x}{6} \\ &= \log_e\left(\frac{1^2}{3}\right) \\ &= \log_e \frac{1}{9} \\ e^k &= \frac{1}{9} \end{aligned}$$

6 D Average value $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

$$\begin{aligned} &= \frac{1}{3-(-1)} \int_{-1}^{3} \log_e(x+2)\,dx \\ &= \frac{5\log_e 5 - 4}{4} \end{aligned}$$

using the integral command of a CAS calculator.

7 A Average value $= \dfrac{1}{\frac{\pi}{2} - 0}\displaystyle\int_0^{\frac{\pi}{2}} \sin(2x)\,dx$

$$\begin{aligned} &= \frac{2}{\pi}\left[-\frac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{2}} \\ &= \frac{2}{\pi}\left(-\frac{1}{2}\cos(\pi)\right) + \frac{1}{2}\cos(0) \\ &= \frac{2}{\pi} \end{aligned}$$

8 D $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x' \\ y' \end{bmatrix}$ so that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x' & 5 \\ y' & 1 \end{bmatrix}$$

Multiply both sides by $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$, the inverse of $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x' & -5 \\ y' & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}(x' - 5) \\ y' - 1 \end{bmatrix}$$

Hence $x = \frac{1}{3}(x' - 5)$ and $y = y' - 1$

The image of $y = x^2$ has equation

$$y^1 - 1 = \frac{1}{9}(x' - 5)^2$$

$$9y' = (x' - 5)^2 + 9$$

In terms of x and y:

9 C

$$\begin{aligned} [f(x)]^3 &= f(y) \\ (e^{3x})^3 &= e^{3y} \\ e^{3y} &= e^{9x} \\ 3y &= 9x \\ y &= 3x \end{aligned}$$

10 D

$$\begin{aligned} \Pr(X > a) &= 0.25 \\ \int_a^{\frac{\pi}{2}} \sin(2x)\,dx &= 0.25 \\ \left[-\frac{1}{2}\cos(2x)\right]_a^{\frac{\pi}{2}} &= 0.25 \\ -\frac{1}{2}\cos(\pi) + \frac{1}{2}\cos(2a) &= 0.25 - (-1) + \cos(2a) \\ &= 0.25\cos(2a) = -0.5 \\ 2a &= \frac{2\pi}{3} \\ a &= \frac{\pi}{3} \\ &\approx 1.05 \end{aligned}$$

11 C

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ \frac{dV}{dr} &= 4\pi r^2 \\ \frac{dV}{dt} &= \frac{dV}{dr}\frac{dr}{dt} \\ &= 4\pi\, r^2 \times 4 \\ &= 16\pi\, r^2 \end{aligned}$$

When $r = 2$:

$$\begin{aligned} \frac{dV}{dt} &= 16\pi \times 4 \\ &= 64\pi \text{ cm}^3/\text{min} \end{aligned}$$

12 A

$$\begin{aligned} \int_0^{2k} (1 + 2e^{\frac{x}{k}})\,dx &= 1 \\ \left[x + 2ke^{\frac{x}{k}}\right]_0^{2k} &= 1 \\ (2k + 2ke^2) - (2k) &= 1 \\ 2ke^2 &= 1 \\ k &= \frac{1}{2e^2} \\ &= \frac{1}{2}e^{-2} \end{aligned}$$

13 **A** $\Pr(X < 7.5) = \Pr(Z < \dfrac{7.5 - 8}{0.25})$

$= \Pr(Z < -2)$

$= \Pr(Z > 2)$

14 **B** $x^2 + 12x = 2kx - 2$

$x^2 + (12 - 2k)x + 2 = 0$

Quadratic has two solutions if

$(12 - 2k)^2 - 4(1)(2) > 0$

$4k^2 - 48k + 144 - 8 > 0$

$k^2 - 12k + 34 > 0$

$(k - 6)^2 - 2 > 0$

$(k - 6 - \sqrt{2})(k - 6 + \sqrt{2}) > 0$

$k < 6 - \sqrt{2}$ or $k > 6 + \sqrt{2}$

15

D $e^{4x} - 7e^{2x} + 12 = 0$

$(e^{2x} - 3)(e^{2x} - 4) = 0$

$e^{2x} = 3, 4$

$2x = \log_e 3,\ \log_e 4$

$x = \frac{1}{2}\log_e 3,\ \frac{1}{2}\log_e 4$

$= \log_e \sqrt{3},\ \log_e 2$

$\textit{Solution set} = \left\{\log_e \sqrt{3},\ \log_e 2\right\}$

16 **B** Reflection in x – axis: $-7x^{\frac{3}{2}}$

Translated 3 units right: $-7(x - 3)^{\frac{3}{2}}$

Translated 4 units down:

$-7(x - 3)^{\frac{3}{2}} - 4$

The equation of the new graph is

$y = -7(x - 3)^{\frac{3}{2}} - 4$

17 **C** $E(x) = \frac{1}{8}\int_0^4 x^2\, dx$

$= \frac{1}{8}\left[\frac{1}{3}x^3\right]_0^4$

$= \frac{1}{8}\left(\frac{64}{3}\right)$

$= \frac{8}{3}$

18 **A** Since $f(2)$ does not exist, since $\log_e 0$ is undefined, the graph of $y = f(x) = 4\log_e(x - 2)^4$ is symmetrical about the asymptote $x = 2$. For a one-one function, the domain must be restricted and for a domain of $[a, \infty)$, we must have $a > 2$ of the available options, only the first fits.

19 **A** $e^{2x+4} - 3 = e^{2(x+2)} - 3$

$= f(2(x + 2)) - 3$

So transform the graph of $y = f(x)$ using this sequences:

- dilation of factor $\frac{1}{2} = 0.5$ from the y-axis
- translations of 2 left and 3 down

The dilation can be represented by

$\begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$

The translation can be represented by

$\begin{bmatrix} -2 \\ -3 \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

20 E $f'(x) = g'(x)$, so

$f(x) = g(x) + c$

Now $f(1) = 2$ and $g(x) = -xf(x)$, so
So $f(x) = g(x) + 4$

21 A If $f(x) = \frac{1}{x}$, then

$$f\left(\frac{xy}{2}\right) = \frac{2}{xy}$$

$$= 2 \times \frac{1}{x} \times \frac{1}{y}$$

$$= 2f(x)f(y)\ (x,\ y \neq 0)$$

Checking the other options shows none fit.

22 E $E(x) = 0 \times a + 1 \times b + 2 \times 0.6$

$$= b + 1.2$$

$$= 1.6 \text{ } \mathit{if}\ b = 0.4$$

Then $a + 0.4 + 0.6 = 1 \rightarrow a = 0$

23 E In matrix form:

$$\begin{bmatrix} m-4 & 6 \\ 2 & m-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 2m-10 \end{bmatrix}$$

For no solutions, the determinant of the coefficients matrix is zero, i.e.

$(m-4)(m-3) - 12 = 0$

$$m^2 - 7m = 0$$

$$m(m-7) = 0 \rightarrow m = 0 \text{ or } m = 7$$

$m = 0$: Equations are $-4x + by = 6$ and $2x - 3y = 10$, which have no solution.

m = 7: Equations are $3x + 6y = 6$ and $2x + 4y = 4$, which are both equal to $x + 2y = 2$.

24 D $n = 1000, \hat{p} = 0.52$

95% CI = (0.489,0.551)

25 C The candidate needs more than 50 % of the vote to win. Based on the confidence interval they will get between 48.9% and 55.1% of the vote- they might win but its too close to tell.

Solutions to extended-response questions

1 a i $y = \dfrac{16x^3 + 4x^2 + 1}{2x^2}$

$$\frac{dy}{dx} = \frac{2x^2(48x^2 + 8x) - 4x(16x^3 + 4x^2 + 1)}{(2x^2)^2} \text{(quotient rule)}$$

$$= \frac{96x^4 + 16x^3 - 64x^4 - 16x^3 - 4x}{4x^4}$$

$$= \frac{32x^4 - 4x}{4x^4} = \frac{8x^3 - 1}{x^3}$$

$\dfrac{dy}{dx} = 0$ implies $8x^3 - 1 = 0$

$$x^3 = \frac{1}{8}$$

$$\therefore \quad x = \frac{1}{2}$$

$\therefore$ Stationary point at $\left(\dfrac{1}{2}, 8\right)$

ii $y = 8x + 2 + \dfrac{1}{2}x^{-2}$ (achieved by dividing by $2x^2$)

Addition of coordinates gives the shape of the graph.

To establish minimum:

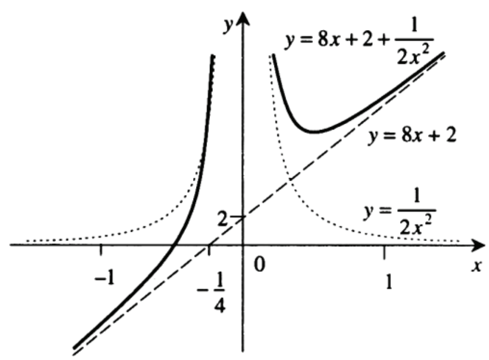
when $x = 0.25$, $y = 12$

when $x = 0.75$, $y = \dfrac{80}{9} = 8\dfrac{8}{9}$

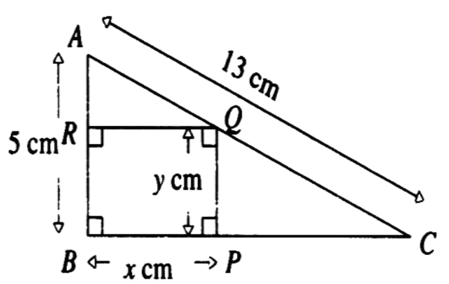
or gradient

when $x = 0.25$, $\dfrac{dy}{dx} = -56$

when $x = 0.75$, $\dfrac{dy}{dx} = \dfrac{152}{27}$



b



i $\Delta \dot{Q}PC \sim \Delta ABC$

and both are right angled triangles. By Pythagoras' Theorem

$$BC = \sqrt{13^2 - 5^2} = 12$$

$$\text{and } \frac{PC}{BC} = \frac{QP}{AB}$$

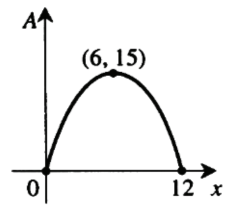
$$\therefore \quad \frac{12 - x}{12} = \frac{y}{5}$$

$$\text{and } y = \frac{60 - 5x}{12}$$

ii Area of the rectangle $A = xy = \dfrac{x(60 - 5x)}{12}$

iii The practical domain for A is $0 \le x \le 12$

By the properties of parabolas for which the coefficient of x^2 is negative, maximum point has coordinates (6, 15).



Alternately: $A = 5x - \dfrac{5x^2}{12}$

$$\frac{dA}{dx} = 5 - \frac{5x}{6}$$

$$\frac{dA}{dx} = 0, \text{implies } x = 6$$

$$\text{when } x = 6,\ A = \frac{6(60 - 5 \times 6)}{12} = 15$$

$\therefore$ maximum area is 15 cm^2

2 a

x	0	1	3
y	6	0	0

	$y = k(x - p)(x - q)$
Since $y = 6$ when $x = 0$,	$6 = kpq$ ①
Also	$0 = k(x - p)(x - q)$
implies	$x = p$ or $x = q$
hence	$p = 1$ and $q = 3$ as $p<q$
From equation ①	$k = 2$

b i For $y = m(x - p)^2(x - q)$

As before $p = 1$ and $q = 3$

Now as then $x = 2,\ y = 2$

$2 = m(2 - 1)^2(2 - 3)$

$\therefore \quad m = -2$ (Note: when $x = 0, y = 6$)

ii $y = -2(x-1)^2(x-3)$

$= -2\left[(x^2 - 2x + 1)(x - 3)\right]$

$= -2[x^3 - 2x^2 + x - 3x^2 + 6x - 3]$

$= -2[x^3 - 5x^2 + 7x - 3]$

$= -2x^3 + 10x^2 - 14x + 6$

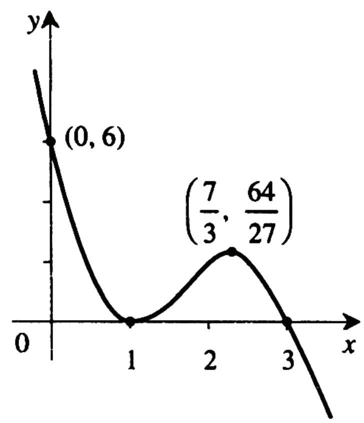
iii $\dfrac{dy}{dx} = -6x^2 + 20x - 14$

$\dfrac{dy}{dx} = 0$ implies $\quad -6x^2 + 20x - 14 = 0$

$\rightarrow -2(3x^2 - 10x + 7) = 0$

$\rightarrow (3x - 7)(x - 1) = 0$

$\rightarrow x = \dfrac{7}{3}$ or $x = 1$



When $x = 1, y = 0$, When $x = \dfrac{7}{3}$, $y = \dfrac{64}{27}$.

There is a local min at (1,0) and a local max at $\left(\dfrac{7}{3}, \dfrac{64}{27}\right)$.

Note: $\dfrac{dy}{dx} = -\dfrac{11}{2}$ when $x = \dfrac{1}{2}$

$\dfrac{dy}{dx} = 2$ when $x = 2$

$\dfrac{dy}{dx} = -1.5$ when $x = 2.5$

A gradient chart illustrates the nature of the stationary points

	$x < 1$	1	$1 < x < 2\frac{1}{3}$	$2\frac{1}{3}$	$x > 2\frac{1}{3}$
sign of $\frac{dy}{dx}$	–ve	0	+ve	0	–ve
shape	\	–	/	–	\

3 **a** $y = ax - x^2$

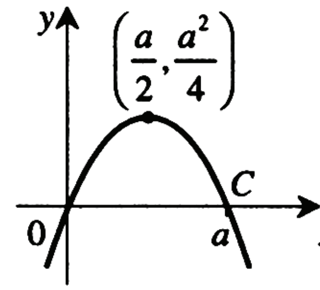
When $y = 0$

$x(a - x) = 0$

$\therefore \quad x = 0$, or $x = a$

By symmetry turning point occurs when

$$x = \frac{a}{2}$$



$x = \frac{a}{2}$ When $y = \frac{a}{2}(a - \frac{a}{2})$

$$= \frac{a^2}{4}$$

b $\int_0^a ax - x^2\,dx = \left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a$

$$= \frac{a^3}{2} - \frac{a^3}{3}$$

$$= \frac{a^3}{6}$$

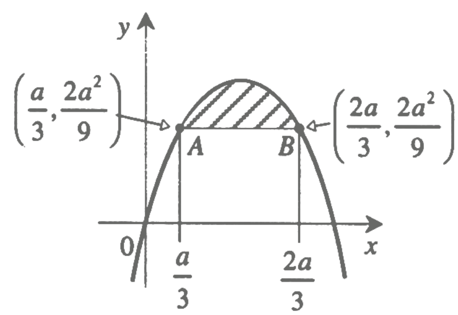
$\therefore$ The area is $\frac{a^3}{6}$ square units.

c **i** When $x = \frac{a}{3}$, $y = a \times \frac{a}{3} - \left(\frac{a}{3}\right)^2$

$$= \frac{a^2}{3} - \frac{a^2}{9}$$

$$= \frac{2a^2}{9}$$

when $x = \frac{2a}{3}$, $y = \frac{2a^2}{9}$ by symmetry



ii $\int_{\frac{1}{3}a}^{\frac{2}{3}a} ax - x^2\,dx$

$$= \left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_{\frac{1}{3}a}^{\frac{2}{3}a}$$

$$= \frac{a}{2} \times \frac{4}{9}a^2 - \frac{8a^3}{81} - \left(\frac{a}{2} \times \frac{1}{9}a^2 - \frac{1}{81}a^3\right)$$

$$= \frac{2a^3}{9} - \frac{a^3}{18} - \frac{8a^3}{81} + \frac{1}{81}a^1$$

$$= \frac{a^3}{81}\left[18 - \frac{9}{2} - 8 + 1\right]$$

$$= \frac{13a^3}{162}$$

Area of the rectangle $= \frac{a}{3} \times \frac{2a^2}{9}$

$$= \frac{12a^3}{162}$$

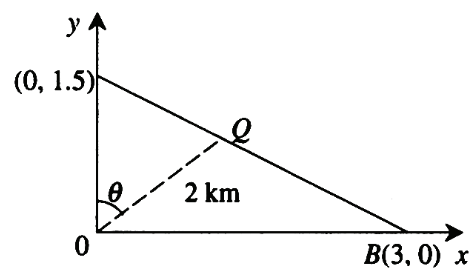
$\therefore$ required area $= \frac{a^3}{162}$ square units

Note: This area may also be found by evaluating $\int_{\frac{1}{3}a}^{\frac{2}{3}a} ax - x^2 - \frac{2a^2}{9}\,dx$

4 a Equation of line

$$y = \left(\frac{1.5 - 0}{0 - 3}\right)x + 1.5$$

$$= -\frac{1}{2}x + \frac{3}{2}$$



b **i** $y = \sin\theta + 2\cos\theta$

$\therefore \quad \frac{dy}{d\theta} = \cos\theta - 2\sin\theta$

ii $\frac{dy}{d\theta} = 0$ implies $\cos\theta = 2\sin\theta$

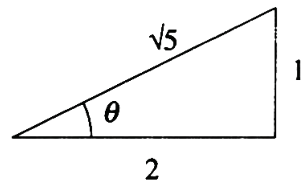
which implies $\tan\theta = \frac{1}{2}(\cos\theta \neq 0)$

$\therefore \theta = \tan^{-1}\left(\frac{1}{2}\right) \approx 26.57^0$

iii $y \approx 2.2361$ when $\theta = \tan^{-1}\left(\frac{1}{2}\right)$

$\therefore$ (26.57, 2.2361) are the coordinates of the stationary point. The following shows the exact coordinates to be $\left(\tan^{-1}(\frac{1}{2}),\ \sqrt{5}\right)$.

iv A maximum occurs when $\theta = \tan^{-1}\left(\frac{1}{2}\right)$



Note: $\sin\left(\tan^{-1}\left(\frac{1}{2}\right)\right) + 2\cos\left(\tan^{-1}\left(\frac{1}{2}\right)\right)$

$= \frac{1}{\sqrt{5}} + 2 \times \frac{2}{\sqrt{5}}$

$= \frac{5}{\sqrt{5}} = \sqrt{5}$

$\therefore \quad r = \sqrt{5}$

$\therefore \quad y = \sqrt{5}\sin(\theta + \alpha)$

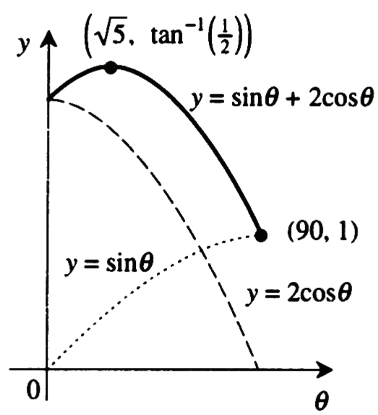
when $\theta = 0, y = 2 \quad \therefore \ \sin\ \alpha = \frac{2}{\sqrt{5}}$

$\therefore \quad \alpha = 63.435°$ [The smallest positive solution is chosen.

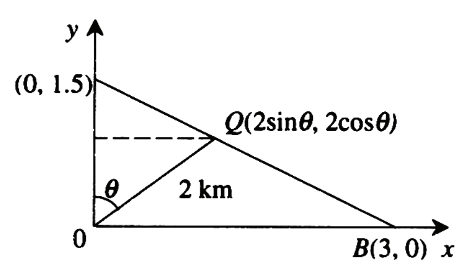
Any solution will work.]

$\therefore \quad y = \sqrt{5}\sin(\theta + 63.435)$

v



c **i** coordinates of $Q = (2\sin\theta,\ 2\cos\theta)$



ii Q is on the line with equation

$y = -\frac{1}{2}x + \frac{3}{2}$

$\therefore \quad 2\cos\theta = -\sin\theta + \frac{3}{2}$

$\therefore \quad 2\cos\theta + \sin\theta + \frac{3}{2}$

i.e. $4\cos\theta + 2\sin\theta = 3$

iii $2\cos\theta + \sin\theta = \frac{3}{2}$

From (b)(iv)

$$\sqrt{5}\sin\left(\theta + \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right) = \frac{3}{2}$$

$$\sin\left(\theta + \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right) = \frac{3}{2\sqrt{5}}$$

$$\therefore \quad \theta + \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) = \sin^{-1}\left(\frac{3}{2\sqrt{5}}\right) \text{or } 180 - \sin^{-1}\left(\frac{3}{2\sqrt{5}}\right)$$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{2}{2\sqrt{5}}\right) - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) \text{or } 180 - \sin^{-1}\left(\frac{3}{2\sqrt{5}}\right) - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

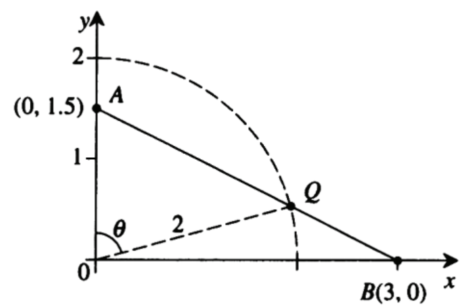
$= -21.3045$ or 74.4346

for $0° < \theta \leq 90°$, required answer $\theta = 74.4346°$

Alternative Method to find the point Q

Q can be considered to be on a circle radius 2 km centre 0.

The equation of this circle is $x^2 + y^2 = 4$



$\therefore$ Solve simultaneously the equations

$x^2 + y^2 = 4$ ①

and $y = -\frac{1}{2}x + \frac{3}{2}$ ②

Substitute from ② into ①

$\therefore \quad \left(-\frac{1}{2}x + \frac{3}{2}\right)^2 + x^2 = 4$

$\therefore \quad 9 - 6x + x^2 + 4x^2 = 16$

$5x^2 - 6x - 7 = 0$

$$\therefore x = \frac{6 \pm \sqrt{36 - 4 \times -7 \times 5}}{10}$$

$$= \frac{6 \pm \sqrt{176}}{10}$$

$$x = \frac{6 \pm 4\sqrt{11}}{10} = \frac{3 \pm 2\sqrt{11}}{5}$$

x must be positive $\therefore\ x = \dfrac{3 + 2\sqrt{11}}{5}$

i.e. $2\sin\theta = \dfrac{3 + 2\sqrt{11}}{5}$

i.e. $\sin\theta = \dfrac{3 + 2\sqrt{11}}{10}$

$\theta = 74.4346°$

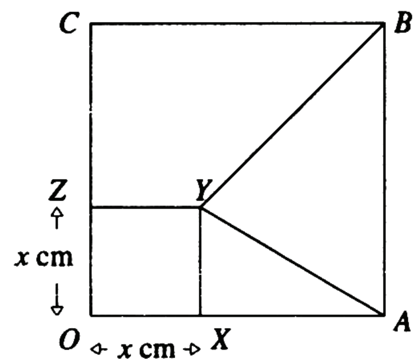
5 a i Area of $OXYZ = x^2$ cm^2

Area of $ABY = \dfrac{1}{2} \times 10 \times (10 - x)$

$= 5(10 - x)$ cm^2

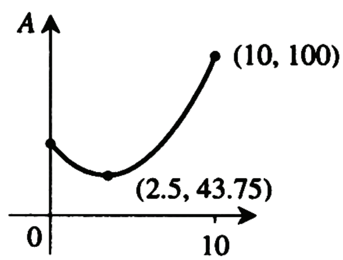
$\therefore$ total area $A = x^2 + 50 - 5x$

$= x^2 - 5x + 50$



ii domain = $(0, 10)$

iii



iv minimum area = 43.75 cm^2

b i $f(x) = \dfrac{1}{2} \times \text{base} \times \text{height}$

$= \dfrac{1}{2} \times AX \times XY$

$= \dfrac{1}{2}(10 - x)x$ domain = $(0, 10)$

ii Maximum area of $AYX = 12.5\text{cm}^2$

This occurs when $x = 5$

When $x = 5$

Area of square $OXYZ = 25\text{ cm}^2$

Area of triangle $ABY = 25\text{ cm}^2$

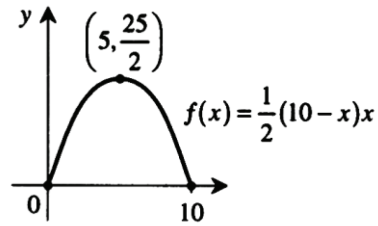
Area of trapezium $CBYZ = 37.5\text{ cm}^2$

$\therefore$ ratio of areas $AYX : OXYZ : ABY : CBYZ$

$= 12.5 : 25 : 25 : 37.5$

$= 25 : 50 : 50 : 75$

$= 1 : 2 : 2 : 3$



6 $f(t) = 1000(t^2 - 10t + 44)e^{-\frac{t}{10}} 0 \le t \le 35$

Using a CAS calculator it is interesting to graph the function for $t \in [0, 35]$.

a **i** $f'(t) = 1000(2t - 10)e^{-\frac{t}{10}} - \frac{1}{10}\left(1000\left(t^2 - 10t + 44\right)\right)e^{-\frac{t}{10}}$

$$= 100e^{-\frac{t}{10}}[20t - 100 - t^2 + 10t - 44]$$

$$= 100e^{-\frac{t}{10}}[30t - 144 - t^2]$$

$$= -100e^{-\frac{t}{10}}[t^2 - 30t + 144]$$

ii $f''(t) = -100\left[-\frac{1}{10}e^{-\frac{t}{10}}(t^2 - 30t + 144) + (2t - 30)e^{-\frac{t}{10}}\right]$

$$= 10e^{-\frac{t}{10}}\left[t^2 - 30t + 144 - 20t + 300\right]$$

$$= 10e^{-\frac{t}{10}}\left[t^2 - 50t + 444\right]$$

b **i** Increasing if $f'(t) > 0$

i.e. $-100e^{-\frac{t}{10}}[t^2 - 30t + 144] > 0$

is equivalent to $t^2 - 30t + 144 < 0$ as $-100e^{-\frac{t}{10}} < 0$ for all t

$\therefore$ $(t - 24)(t - 6) < 0$

$\therefore$ $t \in (6, 24)$

The number of unemployed was increasing for $6 < t < 24$.

ii $f''(t) < 0$

$10\,e^{-\frac{t}{10}}[t^2 - 50t + 444] < 0$

is equivalent to $t^2 - 50t + 444 < 0$

First consider the equation

$t^2 - 50t + 444 = 0$

$$t = \frac{50 \pm \sqrt{50^2 - 4 \times 444}}{2}$$

$$= \frac{50 \pm \sqrt{724}}{2}$$

$$= 25 \pm \sqrt{181}$$

$\therefore\ t^2 - 50t + 444 < 0$ for $t \in (25 - \sqrt{181}, 25 + \sqrt{181})$

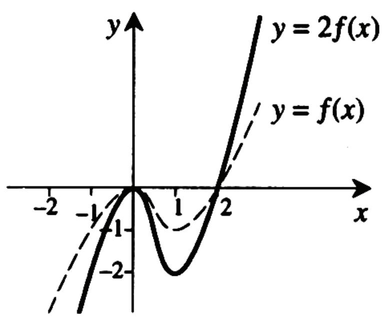
However, the domain of the function is $[0, 35]$.

So $f''(t) < 0$ for $t \in (11.546, 35)$

iii $(6.\ 24) \cap (11.546,\ 38.454)$

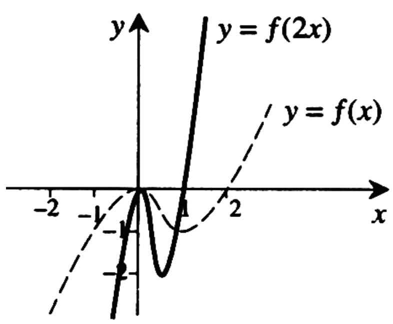
$= (11.546, 24)$

7 a i



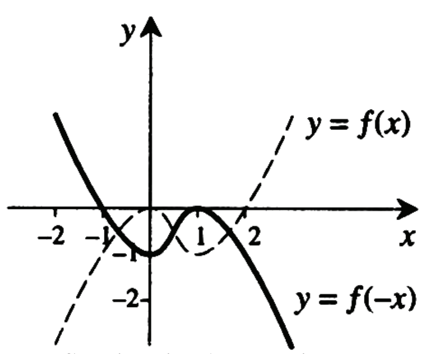
A dilation of factor 2 from the x-axis.

ii



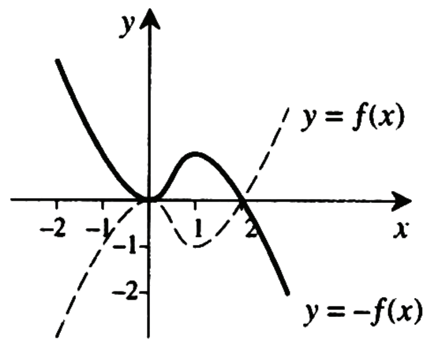
A dilation of factor $\frac{1}{2}$ from the y-axis.

iii



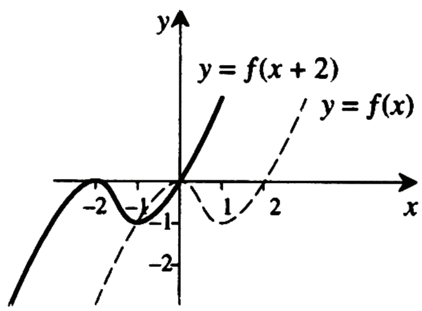
A reflection in the y-axis.

iv



A reflection in the x-axis.

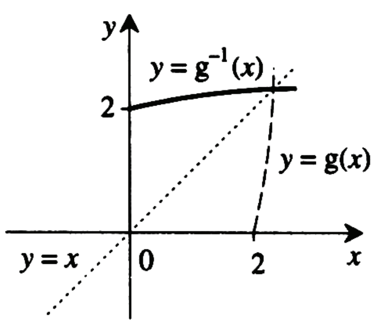
v



A translation of 2 to the left.

b f does not have an inverse function as it is not one-to-one.

c



d **i** $g(x) = x^2(x-2)$ and $g : (2,\ \infty) \to R$

$g'(x) = 2x(x-2) + x^2 = x(2x - 4 + x) = x(3x-4)$

When $x = 3g'(x) = 15$

ii $(g \circ g^{-1})'(x) = 1$

$\therefore g'\left(g^{-1}(x)\right)\left((g^{-1})'(x)\right) = 1$ (by the chain rule)

$$\therefore \left(g^{-1}\right)'(x) = \frac{1}{g'\left(g^{-1}(x)\right)}$$

$$\therefore \left(g^{-1}\right)'(9) = \frac{1}{g'\left(g^{-1}(9)\right)} = \frac{1}{g'(3)} = \frac{1}{15}$$

This can also be shown by the result $\dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}, \dfrac{dy}{dx} \neq 0$

8 a i $\cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$

Actual value, correct to three decimal places = 0.995

ii $\cos x = 0.98$

Consider the equation

$1 - \frac{1}{2}x^2 = 0.98$

$\therefore\ 2 - x^2 = 1.96$

$0.04 = x^2$

$x = \pm 0.2$

Actual value correct to three decimal places $x = \pm 0.2.00 \leftarrow (\pm 0.200)$

b i Let $f(x) = 1 - \frac{1}{2}x^2$

A reflection in the x-axis is given by

$g(x) = -f(x) = \frac{1}{2}x^2 - 1$

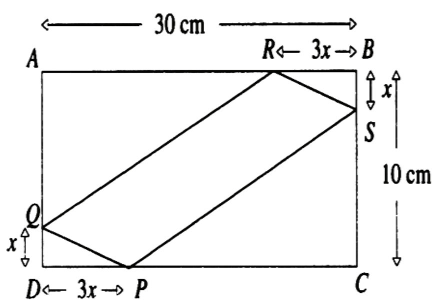
A translation of π units in the positive direction of the x-axis if given by

$h(x) = g(x - \pi) = \frac{1}{2}(x - \pi)^2 - 1$

ii $h(3) = \frac{1}{2}(3 - \pi)^2 - 1 \approx -0.98998$

(Actual $\cos(3) = -0.98999$ correct to five decimal places.)

9



a Area of a triangle RBS = area of triangle $PDQ = \dfrac{3x^2}{2}$ cm^2

Area of a triangle CPS = area of triangle $ARQ = \dfrac{1}{2} \times (30 - 3x)(10 - x)$

$\therefore$ Area of parallelogram $= 300 - 3x^2 - 3(10 - x)^2$
$= [300 - 3x^2 - 3(100 - 20x + x^2)]$
$= (60x - 6x^2)$ cm^2

b $0 < 3x < 30$ and $0 < x < 10$
$\therefore 0 < x < 10$

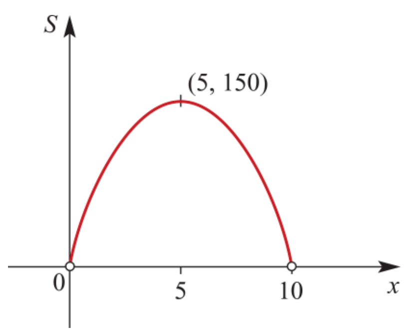
c $A = 60x - 6x^2$

and $\dfrac{dA}{dx} = 60 - 12x$

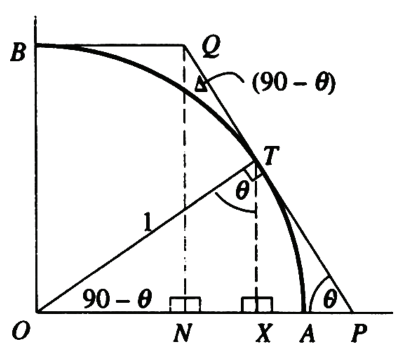
$\dfrac{dA}{dx} = 0$ implies $x = 5$

Since the expression is quadratic with negative coefficient of x^2 a local maximum at (5, 150).

d



10 a i



From triangle OTP

$\dfrac{1}{OP} = \sin\theta$

$\therefore\ OP = \dfrac{1}{\sin\theta}$

ii $BQ = OP - NP$
$NP = TP$ as ΔQNP is congment to ΔOTP

and $TP = \dfrac{1}{\tan\theta}$

$\therefore BQ = \dfrac{1}{\sin\theta} - \dfrac{1}{\tan\theta}$

$$= \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}$$
$$= \frac{1-\cos\theta}{\sin\theta}$$

b Area of the trapezium $= \frac{1}{2}\left(\frac{1-\cos\theta}{\sin\theta} + \frac{1}{\sin\theta}\right)$
$$= \frac{2-\cos\theta}{2\sin\theta}$$

c $S = \dfrac{2-\cos\theta}{2\ \sin\theta}$

$$\frac{dS}{d\theta} = \frac{\sin\theta \times 2\ \sin\theta - 2\cos\theta(2-\cos\theta)}{(2\ \sin\theta)^2}$$
$$= \frac{2\ \sin^2\theta - 4\ \cos\theta + 2\ \cos^2\theta}{(2\sin\theta)^2}$$
$$= \frac{2-4\ \cos\theta}{4\ \sin\theta}$$

d $\dfrac{dS}{d\theta} = 0$ implies, $\dfrac{2-4\cos\theta}{2-4\cos\theta = 0}$ $\quad 0 < \theta < \dfrac{\pi}{2}$

$\therefore \quad \cos\theta = \dfrac{1}{2}$

$\therefore \quad \theta = \dfrac{\pi}{3}$

and $\dfrac{dS}{d\theta} < 0$ when $\quad \theta = \dfrac{\pi}{4}$

and $\dfrac{dS}{d\theta} > 0$ when $\quad \theta = \dfrac{\pi}{2}$

$\therefore$ a minimum when $\theta = \dfrac{\pi}{3}$

When $\theta = \frac{\pi}{3}$ $\quad S = \dfrac{2 - \frac{1}{2}}{2 \times \frac{\sqrt{3}}{2}} = \dfrac{3}{2} \times \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{2}$

and $\quad AP = OP - 1$

$$= \frac{1}{\sin(\frac{\pi}{3})} - 1$$

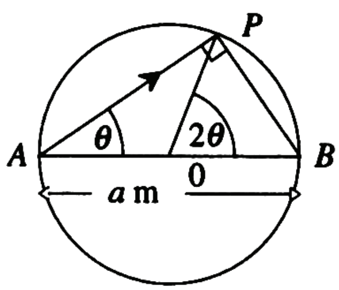
$$= \frac{1}{\frac{\sqrt{3}}{2}} - 1$$

$$= \frac{2}{\sqrt{3}} - 1$$

$$= \frac{2 - \sqrt{3}}{\sqrt{3}}$$

$$= \frac{2\sqrt{3} - 3}{3}$$

11



a **i** distance $AP = a\cos\theta$

distance $PB = \frac{a}{2} \times 2\theta = a\theta$(for arc PB)

time for $AP = \dfrac{a\cos\theta}{\frac{1}{2}} = 2a\cos\theta$

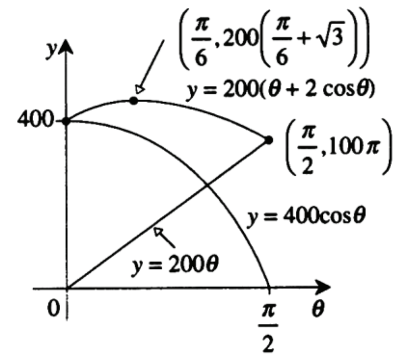
time for $PB = \dfrac{a\theta}{1} = a\theta$

$\therefore$ total time, $T = a(\theta + 2\cos\theta)$

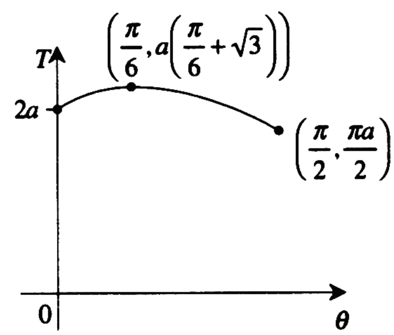
b $\dfrac{dy}{d\theta} = 200(1 - 2\sin\theta)$

$\dfrac{dy}{d\theta} = 0$ implies $\sin\theta = \dfrac{1}{2}$

$\therefore$ maximum when $\theta = \dfrac{\pi}{6}$



c The minimum value for T is $\dfrac{\pi a}{2}$. This is obtained by the dog running around outside of the lake.



12 a i $f(x) = (x - 1)\,g(x)$ and $f'(x) = (x - 1)h(x)$

$f'(x) = g(x) + (x - 1)g'(x)$ (product rule)

$\therefore\ g(x) + (x - 1)g'(x) = (x - 1)h(x)$

$\therefore\ g(x) = (x - 1)h(x) - (x - 1)g'(x)$

$= (x - 1)[h(x) - g'(x)]$

$\therefore\ (x - 1)$ is a factor of $g(x)$

ii $F(1) = 1 - k - 3 + 2k - k + 2 = 0$ where $F(x) = x^3 - kx^2 - (3 - 2k)x - (k - 2)$

$F'(x) = 3x^2 - 2kx - (3 - 2k)$

$\therefore\ F'(1) = 3 - 2k - 3 + 2k = 0$

iii By the factor theorem $x - 1$ is a factor of $F(x)$ and $F'(x)$.

$\therefore F(x) = (x - 1)g(x)$ and $F'(x) = (x - 1)h(x)$ where $g(x)$ and $h(x)$ are polynomials.

$\therefore x - 1$ is a factor of $g(x)$

$\therefore F(x) = (x - 1)^2 w(x)$ where $w(x)$ is a linear polynomial)

$(x^2 - 2x + 1)(x - p) = x^3 - kx^2 - (3 - 2k)x - (k - 2)$

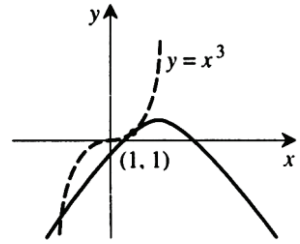
$\therefore -p = -(k - 2)$

i.e. $p = k - 2$

$\therefore F(x) = (x - 1)^2(x - (k - 2))$

and $F(x) = 0$ implies $x = 1$ or $x = k - 2$

b **i** For $y = ax^2 + bx + c$



$\frac{dy}{dx} = 2ax + b$

Given that $(1,\ 1)$ is on the parabola and the gradient is the same as $y = x^3$ at $x = 1$ we have

$a + b + c(1)\ = 1$

$2a + b = 3(2)\left(y = x^3,\ \frac{dy}{dx} = 3x^2 \text{ and when } x = 1,\ \frac{dy}{dx} = 3\right)$

From ② $\quad b = 3 - 2a$

From ① $\quad c = 1 - a - b$

$= 1 - a - (3 - 2a)$

$= 1 - a - 3 + 2a$

$= -2 + a$

$= a - 2$

ii $y = ax^2 + (3 - 2a)x + a - 2$

$y = x^3$

to find Q consider

$ax^2 + (3 - 2a)x + a - 2 = x^3$

i.e. $x^3 - ax^2 + (2a - 3)x + (2 - a) = 0$

Let $F(x) = x^3 - ax^2 + (2a - 3)x + (2 - a)$ (the polynomial of a)

$\therefore F(x) = (x - 1)^2(x - (a - 2))$

$\therefore$ The parabola meets the curve $y = x^3$ at the point $((a - 2),\ (a - 2)^3)$ and $h = a - 2$.

iii If $a - 2 = -2,\ a = 0,\ b = 3$ and $c = -2$

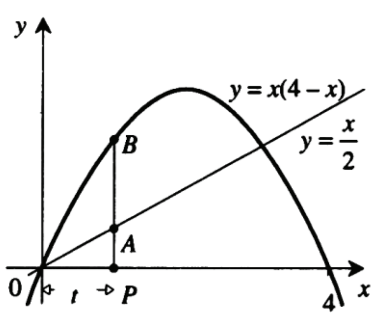
Q is the point of intersection of $y = x^3$ with the straight line $y = 3x - 2$.

Note: $y = 3x - 2$ is the equation of the tangent to the curve $y = x^3$ at the point with coordinates $(1,\ 1)$

iv If $a - 2 = -3,\ a = -1,\ b = 5$ and $c = -3$

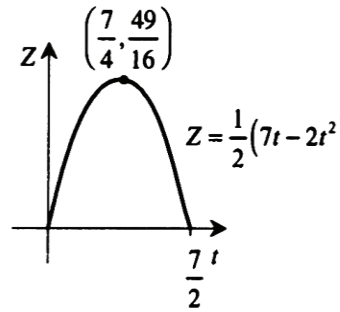
Q is the point of intersection of $y = -x^2 + 5x - 3$ and $y = x^3$

13



a coordinates of $A\left(t, \frac{t}{2}\right)$
coordinates of $B(t, t(4 - t))$
Length of $AB = Z = t(4 - t) - \frac{t}{2} = \frac{1}{2}(8t - 2t^2 - t) = \frac{1}{2}(7t - 2t^2)$

b For the intercepts consider:
$\frac{1}{2}\left(7t - 2t^2\right) = 0$
$\therefore\ t(7 - 2t) = 0$
$\therefore\quad t = 0$ or $t = \frac{7}{2}$
Note: $\left(\frac{7}{2}, \frac{7}{4}\right)$ is the point of intersection of
$y = \frac{x}{2}$ and $y = x(4 - x)$



c The maximum value of $Z = \frac{49}{16}$ and this occurs when $t = \frac{7}{4}$.

14 a Let X be the number of boys.
X is the random variable of a Binomial distribution.

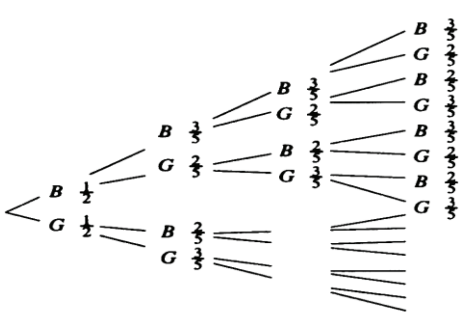
i $\Pr(X = 2) = \binom{4}{2}(0.5)^2(0.5)^2 \quad = \frac{3}{8}$

ii $\Pr(X = 1 \mid X \geq 1) = ?$

$$\Pr(X \geq 1) = 1 - \Pr(X = 0)$$
$$= 1 - (0.5)^4$$
$$= \frac{15}{16}$$

$$\Pr(X = 1 \mid X \geq 1) = \frac{\Pr(X = 1)}{\Pr(X \geq 1)} = \frac{\binom{4}{1}(0.5)^4}{\frac{15}{16}} = 4 \times \frac{1}{16} \times \frac{16}{15} = \frac{4}{15}$$

b *Child* 1 *Child* 2 *Child* 3 *Child* 4



i $\Pr(\text{all boys}) = \frac{1}{2} \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{250}$

$$\text{Pr (all girls)} = \frac{1}{2} \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{25} = \frac{27}{250}$$

$$\therefore \quad \text{Pr (same sex)} = \frac{27}{250} + \frac{27}{250} = \frac{27}{125}$$

ii $\Pr(BGBG) = \frac{1}{2} \times \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{8}{250} = \frac{4}{125}$

$\Pr(GBGB) = \frac{1}{2} \times \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{8}{250} = \frac{4}{125}$

$\therefore$ Pr(no two consecutive children will be of the same sex) $= \frac{8}{125}$

iii Two males and two females. The possible combinations are *BBGG GGBB*

BGBG Note: No. of ways of arranging $= \frac{4!}{2!\,2!} = \frac{24}{4} = 6$

GBGB

BGGB

GBBG

$\Pr(BBGG) = \frac{1}{2}\times\frac{3}{5}\times\frac{2}{5}\times\frac{3}{5} = \frac{18}{250}$

$\Pr(GGBB) = \frac{18}{250}$

$\Pr(BGBG) = \frac{4}{125}$ (see part **ii**)

$\Pr(GBGB) = \frac{4}{125}$ (see part **ii**)

$\Pr(BGGB) = \frac{1}{2} \times \frac{2}{5} \times \frac{3}{5} \times \frac{2}{5} = \frac{12}{250}$

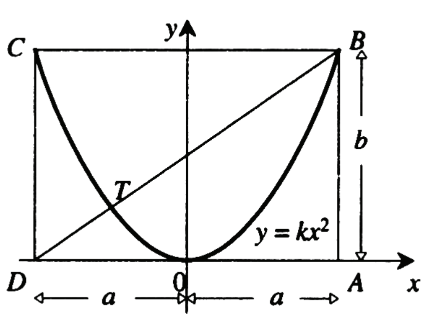
$\Pr(GBBG) = \frac{12}{250}$

$$\therefore \text{Pr(two males and two females)} = \frac{18 + 18 + 8 + 8 + 12 + 12}{250}$$

$$= \frac{76}{250}$$

$$= \frac{38}{125}$$

15 a



$y = kx^2$

$\therefore \quad b = ka^2$

$\therefore \quad k = \dfrac{b}{a^2}$

b **i** Gradient of $DB = \dfrac{b}{2a}$ and it passes through $(-a,\ 0)$

$\therefore y - 0 = \dfrac{b}{2a}(x + a)$

i.e. $y = \dfrac{b}{2a}x + \dfrac{b}{2}$

ii crosses $y = \dfrac{b}{a^2}x^2$

where $\dfrac{b}{2a}x + \dfrac{b}{2} = \dfrac{b}{a^2}x^2$

Multiply both sides by $2a^2$

$bax + ba^2 = 2bx^2$

i.e. $2bx^2 - bax - ba^2 = 0$

which implies

$2x^2 - ax - a^2 = 0$

$\therefore (2x + a)(x - a) = 0$

$\therefore x = -\dfrac{a}{2}$ or $x = a$

$\therefore$ at $T \quad x = -\dfrac{a}{2}$

and $y = \dfrac{b}{a^2}\left(\dfrac{-a}{2}\right)^2 = \dfrac{b}{4}$

$\therefore$ coordinates of T are $\left(-\dfrac{a}{2},\ \dfrac{b}{4}\right)$

c Area $= \int_{-a}^{a} b - \dfrac{b}{a^2}x^2\,dx$

$= 2b\int_0^a 1 - \dfrac{x^2}{a^2}\,dx \quad$ (by symmetry)

$= 2b\left[x - \dfrac{x^3}{3a^2}\right]_0^a$

$= 2b\left[a - \dfrac{a^3}{3a^2}\right]$

$= 2b\left[a - \dfrac{a}{3}\right]$

$= \dfrac{4}{3}\,ab$

d $S_1 = \int_{\frac{-a}{2}}^{a} \frac{b}{2a}x + \frac{b}{2} - \left(\frac{b}{a^2}x^2\right) dx$

$= b\int_{\frac{-a}{2}}^{a} \frac{x}{2a} + \frac{1}{2} - \frac{x^2}{a^2}\, dx$

$= b\left[\frac{x^2}{4a} + \frac{x}{2} - \frac{x^3}{3a^2}\right]_{-\frac{a}{2}}^{a}$

$= b\left[\left(\frac{a^2}{4a} + \frac{a}{2} - \frac{a^3}{3a^2}\right) - \left(\frac{a^2}{4} \times \frac{1}{4a} - \frac{a}{4} + \frac{a^3}{8} \times \frac{1}{3a^2}\right)\right]$

$= b\left[\frac{a}{4} + \frac{a}{2} - \frac{a}{3} - \left(\frac{a}{16} - \frac{a}{4} + \frac{a}{24}\right)\right]$

$= \frac{ba}{48}[12 + 24 - 16 - (3 - 12 + 2)]$

$= \frac{ba}{8}[20 + 7]$

$= \frac{27ba}{48} = \frac{9ba}{16}$

Now $S_2 = \frac{4}{3}ab - \frac{9ba}{16}$ from **c**

$= \frac{(64 - 27)ba}{48} = \frac{37ba}{48}$

$\therefore$ ratio $S_1 : S_2 = 27:37$

16 **a** Let X be the thickness of the washer

Let Y be the diameter of the hole

For X: $\mu = 0.25, \sigma = 0.002$

For Y: $\mu = 0.5, \sigma = 0.05$

i $\Pr(X < 0.253)$

$$= \Pr\left(z < \frac{0.253 - 0.25}{0.002}\right)$$

$$= \Pr\left(Z < \frac{3}{2}\right)$$

$$= 0.9332$$

ii $\Pr(X < 0.247)$

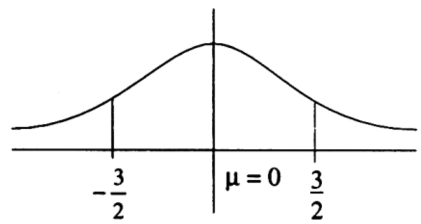
$$= \Pr\left(Z < \frac{0.247 - 0.25}{0.002}\right)$$

$$= \Pr\left(Z < -\frac{3}{2}\right)$$

$$= 1 - \Pr\left(Z < -\frac{3}{2}\right)$$

$$= 1 - 0.9332$$

$$= 0.0668$$



iii $\Pr(Y > 0.56)$

$$= \Pr\left(Z > \frac{0.56 - 0.5}{0.05}\right)$$

$$= \Pr\left(Z > -\frac{6}{5}\right)$$

$$= \Pr(Z > 1.2)$$

$$= 1 - \Pr(Z < 1.2)$$

$$= 0.1151$$

iv $\Pr(Y < 0.44)$

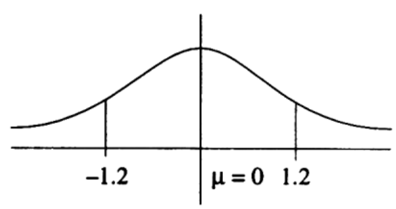
$$= \Pr\left(Z < \frac{0.44 - 0.5}{0.05}\right)$$

$$= \Pr\left(Z < -\frac{0.06}{0.05}\right)$$

$$= \Pr(Z < -1.2)$$

$$= \Pr(Z > 1.2)$$

$$= 0.1151$$



b **i** Let A be the event $0.247 < X \le 0.253$

Let B the event $0.44 \le Y \le 0.56$

$\Pr(A) = \Pr(X \le 0.253) - \Pr(X \le 0.247)$

$= 0.9332 - 0.0668$

$= 0.8664$

$\Pr(B) = \Pr(Y \le 0.56) - \Pr(Y \le 0.44)$

$= 0.8849 - 0.1151$

$= 0.7698$

$\Pr(A \cap B) = \Pr(A)\ \Pr(B)$ X and Y are independent and therefore

$= 0.8664 \times 0.7698$ A and B are independent events.

$= 0.6670$

$\therefore$ Probability of rejecting a washer is 0.333.

$\therefore$ 33.3% of washers are rejected.

ii $\Pr(A) = 0.8664$

$\therefore$ expected number of washers of acceptable thickness in a batch of 1000 is 866.4.

iii $\Pr(A \cap B') = \Pr(A)\ \Pr(B')$

$= 0.8664 \times (1 - 0.7698)$

$= 0.8664 \times 0.2302$

$= 0.1994$

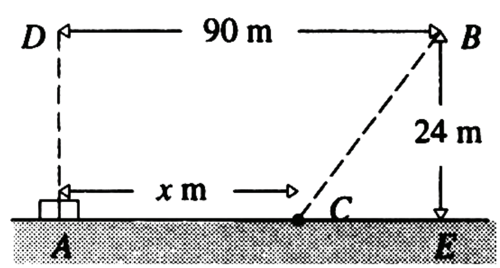
$\therefore$ Expected number with acceptable thickness but not acceptable diameter is 199.4.

17 Let $AC = x$

Then $CE = 90 - x$

and $CB = \sqrt{(90-x)^2 + 24^2}$

$\therefore$ total cost $C = 100(90 - x) + 200((90 - x)^2 + 576)^{\frac{1}{2}}$



$$\frac{dC}{dx} = -100 + 200\left[\frac{1}{2} \times ((90-x)^2 + 576)^{-\frac{1}{2}} \times -2(90-x)\right]$$

$$\frac{dC}{dx} = 0 \text{ implies} \quad 100 = \frac{200(x-90)}{((90-x)^2 + 576)^{\frac{1}{2}}}$$

which implies $[(90-x)^2+576]^{\frac{1}{2}} = 2(x-90)$

$\therefore [(90-x)^2+576] = 4(x-90)^2$

$\therefore 3(x-90)^2 = 576$

$\therefore x-90 = \pm\dfrac{24}{\sqrt{3}}$

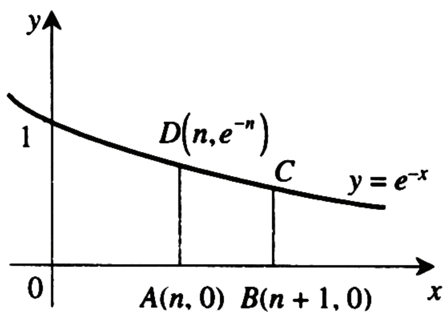
$\therefore x = 90 \pm 8\sqrt{3}$

as $0 \le x \le 90$

$x = 90 - 8\sqrt{3}$, $[x = 90 - 8\sqrt{3},\ as\ 0 \le x \le 9.]$

$\dfrac{dC}{dx} > 0$ when $x > 90 - 8\sqrt{3}$ and $\dfrac{dC}{dx} < 0$ when $x < 90 - 8\sqrt{3}$

$\therefore$ a minimum when $x = 90 - 8\sqrt{3} \approx 76.1436$ m



18 a i $\dfrac{dy}{dx} = -e^{-x}$

When $x = n$, $\dfrac{dy}{dx} = -e^{-n}$

$\therefore$ equation of tangent is $y - e^{-n} = -e^{-n}(x-n)$

$\therefore y = -e^{-n}x + e^{-n}n + e^{-n}$

ii When $y = 0$

$\therefore e^{-n}x = e^{-n}n + e^{-n}$

$x = n + 1 (e^{-n} \ne 0)$

The line DB is a segment of the tangent at D.

b i

$$\int_n^{n+1} e^{-x}\,dx = [-e^{-x}]_n^{n+1}$$

$$= -(e^{-(n+1)} - e^{-n})$$

$$= -e^{-n}(e^{-1} - 1)$$

$\therefore$ area of region $ABCD = \dfrac{1}{e^n}\left(1 - \dfrac{1}{e}\right)$

$\therefore$ The area under the curve $y = e^{-x}$ between $x = n$ and $x = n+1$ is $\dfrac{1}{e^n}\left(1 - \dfrac{1}{e}\right)$.

$$\therefore \text{ The area of the second part } = \frac{1}{e^n} - \frac{1}{e^{n+1}} - \frac{1}{2e^n}$$

$$= \frac{1}{2e^n} - \frac{1}{e^{n+1}}$$

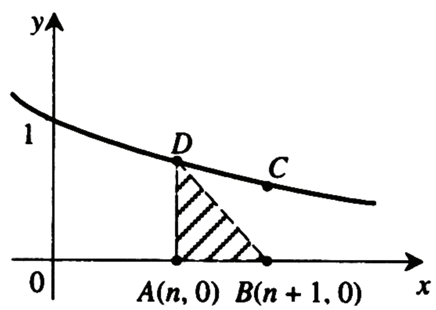
$$= \frac{1}{e^n}\left(\frac{1}{2} - \frac{1}{e}\right)$$

$$\therefore \text{ The ratio of the two parts } = \frac{1}{2e^n} : \frac{1}{e^n}\left(\frac{1}{2} - \frac{1}{e}\right)$$

$$= \frac{1}{2} : \frac{1}{2} - \frac{1}{e}$$

$$= e : e - 2$$

ii



$$\text{The shaded area } = \int_n^{n+1} -e^{-n}x + e^{-n}n + e^{-n}\,dx$$

$$= \left[-\frac{e^{-n}x^2}{2} + (e^{-n}n + e^{-n})x\right]_n^{n+1}$$

$$= \left(-\frac{e^{-n}(n+1)^2}{2} + e^{-n}(n+1)(n+1)\right) - \left(-\frac{e^{-n}n^2}{2} + e^{-n}(n+1)n\right)$$

$$= \frac{e^{-n}(n+1)^2}{2} - e^{-n}\left[-\frac{n^2}{2} + n^2 + n\right]$$

$$= \frac{e^{-n}}{2}[(n+1)^2 - (n^2 + 2n)]$$

$$= \frac{e^{-n}}{2}[n^2 + 2n + 1 - n^2 - 2n]$$

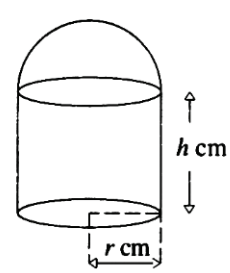
$$= \frac{e^{-n}}{2}$$

19 a i Volume of cylinder $= \pi r^2 h$

Volume of hemisphere $= \dfrac{2}{3}\pi r^3$

$\therefore$ total volume, $V = \dfrac{2}{3}\pi r^3 + \pi r^2 h$

$= \dfrac{\pi r^2}{3}(2r + 3h)$



ii Surface areas of capsule = surface area of hemisphere

$$\begin{aligned}&+ \text{curved surface of cylinder} + \text{base}\\ &= 2\pi r^2 + 2\pi rh + \pi r^2\\ &= \pi r(3r + 2h)\end{aligned}$$

b **i** If $V = \pi a^3$

$$\pi a^3 = \frac{\pi r^2}{3}(2r + 3h)$$

$$3\pi a^3 - 2\pi r^3 = 3\pi r^2 h$$

$$\therefore h = \frac{a^3}{r^2} - \frac{2}{3}r$$

$$= \frac{3a^3 - 2r^3}{3r^2}$$

ii $S = \pi r(2h + 3r)$

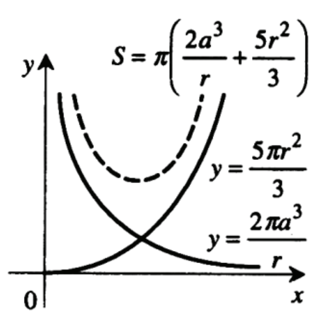
$$= \pi r\left(\frac{2 \times (3a^3 - 2r^3)}{3r^2} + 3r\right)$$

$$= \pi\left(\frac{2 \times (3a^3 - 2r^3)}{3r} + 3r^2\right)$$

$$= \pi\left(\frac{2a^3}{r} - \frac{4r^2}{3} + 3r^2\right)$$

$$= \pi\left(\frac{2a^3}{r} + \frac{5r^2}{3}\right)$$

c **i**



ii $S = \pi\left(\frac{2a^3}{r} + \frac{5r^2}{3}\right)$

$$\frac{dS}{dr} = \pi\left(-\frac{2a^3}{r^2} + \frac{10r}{3}\right)$$

$$\frac{dS}{dr} = 0 \text{ implies } \frac{2a^3}{r^2} = \frac{10r}{3}$$

$$\therefore r^3 = 0.6a^3$$
$$\therefore r = (\sqrt[3]{0.6})a$$
$$s_{\min} = \pi\left(\frac{2a^3}{\sqrt[3]{0.6}a} + \frac{5}{3}\left[\sqrt[3]{0.6}a\right]^2\right)$$
$$= \pi a^2\left(\frac{2}{\sqrt[3]{0.6}} + \frac{5}{3}(\sqrt[3]{0.6})^2\right)$$

20 **a** Let X be the cylinder diameter.

$\Pr(3 - d < X < 3 + d) = 0.75$

$$\therefore \quad \Pr\left(\frac{-d}{0.002} < Z < \frac{d}{0.002}\right) = 0.75$$
$$\therefore \quad 2\Pr\left(Z < \frac{d}{0.002}\right) - 1 = 0.75$$
$$\Pr\left(Z < \frac{d}{0.002}\right) = 0.875$$
$$\therefore \quad \frac{d}{0.002} = 1.15$$
$$d = 0.0023$$

b

q	$s - 1$	-1
$\Pr(Q = q)$	$\frac{3}{4}$	$\frac{1}{4}$

c
$$\mathrm{E}(Q) = \frac{3}{4}(s-1) - 1 \times \frac{1}{4} = \frac{3}{4}s - 1$$
$$\mathrm{E}(Q^2) = (s-1)^2 \times \frac{3}{4} + 1 \times \frac{1}{4}$$
$$\mathrm{Var}\,(Q) = (s-1)^2 \times \frac{3}{4} + \frac{1}{4} - \left(\frac{3s-4}{4}\right)^2$$
$$= \frac{3}{4}(s^2 - 2s + 1) + \frac{1}{4} - \frac{1}{16}(9s^2 - 24s + 16)$$
$$= \left(\frac{3}{4} - \frac{9}{16}\right)s^2 + \left(\frac{24}{16} - \frac{6}{4}\right)s + \frac{3}{4} + \frac{1}{4} - 1$$
$$= \left(\frac{3}{16}\right)s^2$$
$$\therefore \mathrm{sd}\,(Q) = \frac{\sqrt{3}}{4}s$$

21 Let X be the length of a worm.

$\mu = 20$ and $\sigma = 1.5$.

a $\Pr(X \geq 22) = \Pr\left(Z \geq \frac{22-20}{1.5}\right)$

$= \Pr\left(Z \geq \frac{2}{1.5}\right)$

$= \Pr(Z \geq 1.3333)$

$= 1 - \Pr(Z \leq 1.3333)$

$= 0.09121$

b $\Pr(19.5 \leq X \leq 20.5) = \Pr\left(\frac{19.5-20}{1.5} \leq Z \leq \frac{20.5-20}{1.5}\right)$

$= \Pr\left(-\frac{1}{3} \leq Z \leq \frac{1}{3}\right)$

$= \Pr(0.3333 \leq Z \leq 0.3333)$

$= 2\Pr(Z \leq 0.3333) - 1$

$= 2 \times 0.63056 - 1$

$= 0.2611$

c Let Y be the number of worms out of five of 20 cm in length, So Y has a binomial distribution with $n = 5$ and $p = 0.2611$.

$\Pr(Y = 2) = \binom{5}{2}(0.2612)^2\,(0.7392)^3$

$= 10 \times (0.2611)^2\,(0.7389)^3$

$= 10 \times 0.0682 \times 0.4039$

$= 0.275$

22 **a** $P = \frac{x^2}{90}(56 - x)\ x \in [1, 40]$

$= \frac{1}{90}(56x^2 - x^3)$

$\frac{dP}{dx} = \frac{1}{90}(112x - 3x^2)$

b **i**

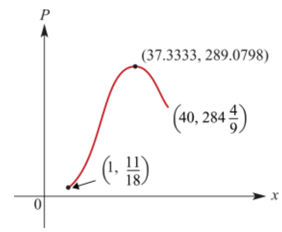
ii $P(1) = \frac{1}{90} \times (56 - 1)\ P(40) = \frac{40^2}{90} \times [56 - 40]$

$= \frac{11}{18} \qquad = 284\frac{4}{9}$

$$\frac{dP}{dx} = 0 \Rightarrow \frac{1}{90}(112x - 3x^2) = 0$$

$$\therefore x(112 - 3x) = 0$$

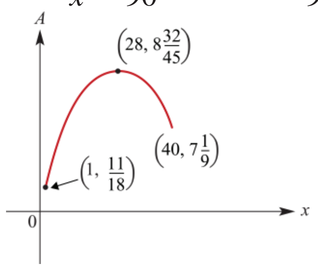
$$\therefore x = 0 \text{ or } x = 37\frac{1}{3}$$



when $x = 37\frac{1}{3}$ $P(x) = \dfrac{351232}{1215} \approx 289.0798$

The maximum value of P is 289.0798 tonnes

c **i** $A = \dfrac{1}{x} \times \dfrac{x^2}{90}(56 - x) = \dfrac{x}{90}(56 - x)$

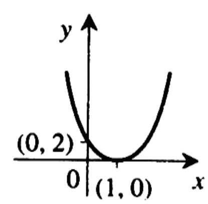


ii The maximum value of A is $8\dfrac{32}{45}$ tonnes/man, when $x = 28$.

23 $f(x) = (k + 2)x^2 + (6k - 4)x + 2$

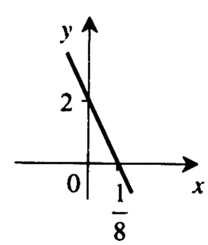
a **i** When $k = 0$

$$f(x) = 2x^2 - 4x + 2$$
$$= 2(x^2 - 2x + 1)$$
$$= 2(x - 1)^2$$



ii When $k = -2$

$$f(x) = -16x + 2$$



iii When $k = -4$

$f(x) = -2x^2 - 28x + 2$

$= -2x[x^2 + 14x - 1]$

$= -2[x^2 + 14x + 49) - 1 - 49]$

$= -2[(x + 7)^2 - 50]$

$f(0) = 2$; when $f(x) = 0$, $(x + 7)^2 = 50$

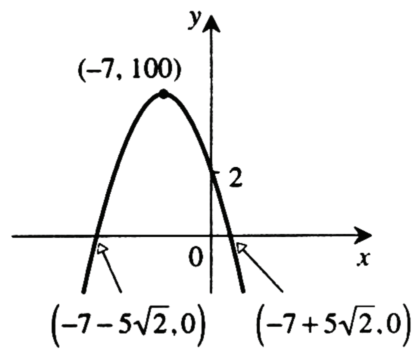
$\therefore \quad x = -7 \pm \sqrt{50}$

$= -7 \pm 5\sqrt{2}$

$f(x) = -2((x + 7)^2) + 100$

$\therefore$ axes intercepts are $(0, 2)$ $(-7 - 5\sqrt{2}, 0)$ and $(-7 + 5\sqrt{2}, 0)$

Vertex is at $(-7, 100)$



b $f'(x) = 2(k + 2)x + (6k - 4)$

$f'(x) = 0$ implies

$$x = \frac{4 - 6k}{2\,(k + 2)} = \frac{2 - 3k}{k + 2}$$

$$f\left(\frac{2 - 3k}{k + 2}\right) = (k + 2) \times \left(\frac{2 - 3k}{k + 2}\right)^2 + (6k - 4)\frac{(2 - 3k)}{k + 2} + 2$$

$$= \frac{(2 - 3k)^2}{k + 2} + 2\frac{(3k - 2)(2 - 3k)}{k + 2} + 2$$

$$= \frac{(2 - 3k)^2 - 2(2 - 3k)^2 + 2(k + 2)}{k + 2}$$

$$= \frac{-(2 - 3k)^2 + 2(k + 2)}{k + 2}$$

A check from previous results

When $k = 0$, $x = 1$, $f(1) = \dfrac{-4 + 4}{2} = 0$

When $k = -2$, f is undefined

When $k = -4$, $x = \dfrac{14}{-2} = -7$

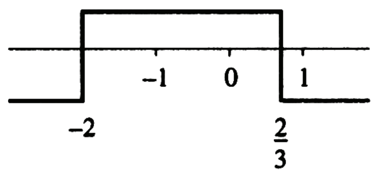
$$f(-7) = -\frac{-(2 + 12)^2 + 2(-2)}{-2}$$

$$= -\frac{-196 - 4}{-2}$$

$$= 100$$

i If $a > 0$, $\dfrac{2 - 3k}{k + 2} > 0$

Multiply both sides of the inequality by $(k + 2)^2$ $(2 - 3k)(k + 2) > 0$



A sign diagram reveals $\{k : a > 0\} = \left\{k : -2 < k < \frac{2}{3}\right\}$

ii $a = 0$ implies $k = \frac{2}{3}$

$\{k : a = 0\} = \left\{\frac{2}{3}\right\}$

iii If $b > 0$, $\dfrac{-(2-3k)^2 + 2(k+2)}{k+2} > 0$

Multiply both sides of inequality by $(k+2)^2$

$(-(2-3k)^2 + 2(k+2))(k+2) > 0$

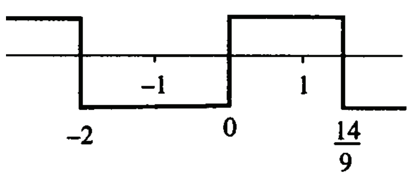
$(-(4 - 12k + 9k^2) + 2k + 4)(k+2) > 0$

$\therefore (-4 + 12k - 9k^2 + 2k + 4)(k+2) > 0$

$(14k - 9k^2)(k+2) > 0$

$k(14 - 9k)(k+2) > 0$

Consider the sign diagram



$\therefore \{k : b > 0\} = \left\{k : 0 < k < \frac{14}{9}\right\} \cup \{k : k < -2\}$

iv $\{k : b < 0\} = \{k : -2 < k < 0\} \cup \left\{k : k > \frac{14}{9}\right\}$

c f has a local maximum when $k + 2 < 0$

i.e. when $k < -2$

d For $f(x) = (k+2)x^2 + (6k-4)x + 2$

$\Delta = (6k-4)^2 - 4(k+2)2$

$= 36k^2 - 48k + 16 - 8k - 16$

$= 36k^2 - 56k$

i $f(x)$ is a perfect square if $\Delta = 0$

i.e. $36k^2 - 56k = 0$

$4k(9k - 14) = 0$

$k = 0$ or $k = \dfrac{14}{9}$

ii If there are no solutions $\Delta < 0$

i.e. $4k(9k - 14) < 0$

$0 < k < \dfrac{14}{9}$

24 a $e^{2-2x} = 2e^{-x}$

$\therefore e^2 = 2e^x$

$\therefore e^x \dfrac{e^2}{2}$

$\therefore x = \log_e\left(\dfrac{e^2}{2}\right)$

$\therefore = \log_e (e^2) - \log_e$ 2

$= 2 - \log_e 2$

b **i** $y = e^{2-2x} - 2e^{-x}$

$\dfrac{dy}{dx} = -2e^{2-2x} + 2e^{-x}$

ii $\dfrac{dy}{dx} = 0$ implies $e^{2-2x} = e^{-x}$

$\therefore e^x = e^2$

$x = 2$

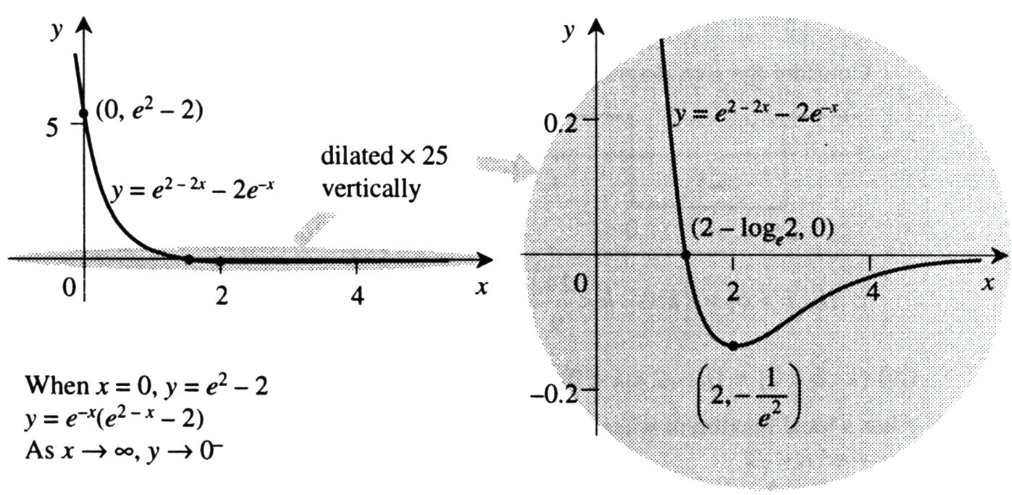
iii When $x = 2$, $y = e^{2-4} - 2e^{-2}$

$= e^{-2} - 2e^{-2}$

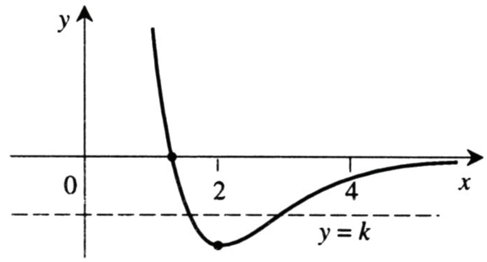
$= -\dfrac{1}{e^2}$

$\therefore$ coordinates of turning point $(2, -\frac{1}{e^2})$

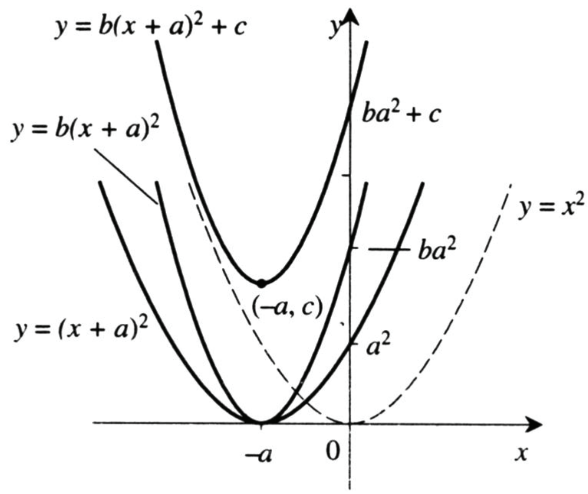
iv



c The equation $e^{2-2x} - 2e^{-x} = k$ has two distinct positive solutions for $k \in \left(-\frac{1}{e^2}, 0\right)$



25 a



b $\dfrac{3}{(x+1)^2} + 2 = \dfrac{3}{x^2 + 2x + 1} + 2$

$$= \frac{3 + 2x^2 + 4x + 2}{x^2 + 2x + 1}$$

$$= \frac{2x^2 + 4x + 5}{x^2 + 2x + 1}, x \neq -1$$

c
- A dilation of factor 3 from the x-axis
- A translation of 1 unit in the negative direction of the x-axis.
- A translation of 2 units in the positive direction of the y-axis

d $\displaystyle\int_0^1 \frac{2x^2 + 4x + 5}{x^2 + 2x + 1}\,dx = \int_0^1 \frac{3}{(x+1)^2} + 2\,dx$

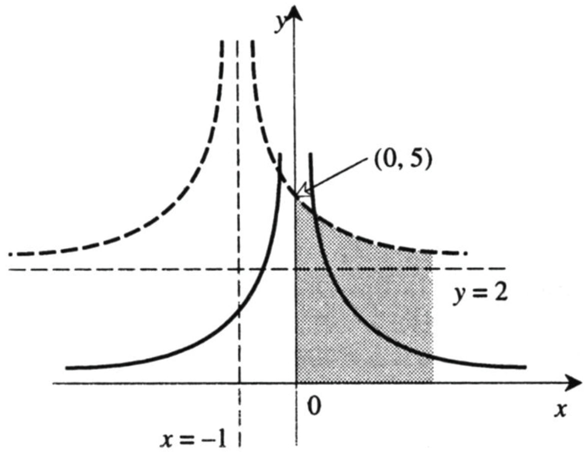
$$= \int_0^1 3(x+1)^{-2} + 2\,dx$$

$$= \left[-3(x+1)^{-1} + 2x\right]_0^1$$

$$= -\frac{3}{2} + 2 - (-3)$$

$$= \frac{7}{2}$$

e



26 **a** **i** $y = 50$

ii $y - 25 = \dfrac{25 - 0}{50 - 25}(x - 50)$

$\therefore y - 25 = x - 50$

$\therefore y = x - 25$

b $y = ax^2 + 4x + c$

$\therefore 50 = 25^2a + 100 + c \quad ①$

$25 = 50^2a + 200 + c \quad ②$

Subtract ② from ①

$25 = (25^2 - 50^2)a - 100$

$\dfrac{125}{25^2 - 50^2} = a$

$a = \dfrac{125}{75 \times -25} = \dfrac{-1}{15}$

Substitute in ①

$50 = 625 \times -\dfrac{1}{15} + 100 + c$

$\therefore c = -50 + 625 \times \dfrac{1}{15}$

$= -\dfrac{25}{3}$

$\therefore$ equation of parabola

$y = -\dfrac{1}{15}x^2 + 4x - \dfrac{25}{3}$

$= -\dfrac{1}{15}(x^2 - 60x + 125)$

c **i** area of rectangle *OABE*

$= 25 \times 50$

$= 1250$ square units

ii area of region EBC $= \int_{25}^{50} -\frac{1}{15}(x^2 - 60x + 125) - (x - 25)\,dx$

$$= -\frac{1}{15}\int_{25}^{50} x^2 - 45x - 250\,dx$$

$$= -\frac{1}{15}\left[\frac{x^3}{3} - \frac{45x^2}{2} - 250x\right]_{25}^{50}$$

$$= \frac{14375}{18}$$

iii total area $= \frac{36875}{18}$ square units

27 a Area of rectangle $PQST = (4\cos\theta + 4\cos\theta) \times 2$

$$= 16\cos\theta$$

Area of triangle $QRS = \frac{1}{2} \times 8\cos\theta \times 4\sin\theta$

$$= 16\cos\theta\ \sin\theta$$

$\therefore$ Area of metal plate $= 16(\cos\theta + \cos\theta\ \sin\theta), 0 < \theta < \frac{\pi}{2}$

b $\frac{dA}{d\theta} = 16[-\sin\theta + \sin\theta(-\sin\theta) + \cos\theta\cos\theta]$

$$= 16[-\sin\theta + \cos^2\theta - \sin^2\theta]$$

$$= 16[-\sin\theta + (1 - \sin^2\theta) - \sin^2\theta]$$

$$= 16[1 - \sin\theta - 2\sin^2\theta]$$

c $\frac{dA}{d\theta} = 0$ implies $= 16[1 - a - 2a^2] = 0$ (where $a = \sin\theta$)

$$\Leftrightarrow 2a^2 + a - 1 = 0$$

$$\Leftrightarrow (2a - 1)(a + 1) = 0$$

$$\Leftrightarrow a = \frac{1}{2} \text{ or } a = -1$$

$\therefore \sin\theta = \frac{1}{2}$ or $\sin\theta = -1$

$\theta = \frac{\pi}{6}$ since $O < Q < \frac{\pi}{2}$

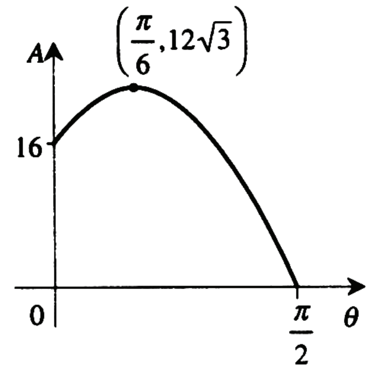
d $A(0) = 16$

$A\left(\frac{\pi}{2}\right) = 0$

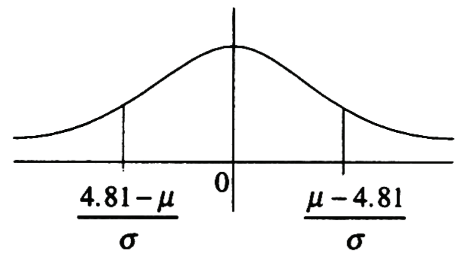
$A\left(\frac{\pi}{6}\right) = 16\left(\cos\frac{\pi}{6} + \cos\frac{\pi}{6}\sin\frac{\pi}{6}\right)$

$= 16 \times \frac{\sqrt{3}}{2} \times \frac{3}{2}$

$= 12\sqrt{3}$



28 **a** Let X be the length of the engine part.
The engine part must be between 4.81 cm and 5.20 cm.
$\Pr(X < 4.81) = 0.008$
$\Pr(X > 5.20) = 0.03$



$\Pr\left(Z < \frac{4.81-\mu}{\sigma}\right) = 0.008$

$\therefore \Pr\left(Z < \frac{\mu-4.81}{\sigma}\right) = 0.992$

$\Pr\left(Z > \frac{5.20-\mu}{\sigma}\right) = 0.03$

$\therefore \Pr\left(Z < \frac{5.20-\mu}{\sigma}\right) = 0.97$

$\therefore$ we have the equations

$\frac{\mu-481}{\sigma} = 2.41$ and $\frac{5.20-\mu}{\sigma} = 1.881$

$\mu - 4.81 = 2.41\sigma$ ① and $5.20 - \mu = 1.881\sigma$ ②

Add ① and ②

$5.20 - 4.81 = (2.41 + 1.881)\sigma$

$0.39 = 4.291\sigma$

$0.0909 = \sigma$ (correct to four decimal places)

Substitute in ①

$\mu - 4.81 = 2.41 \times 0.0909$

$\mu = 5.0290$

b Let \$C be the cost to produce a part that meets the specifications. Then with probability 0.962, the cost is \$4; with probability 0.03, the part is priced at a cost of \$(4+2)= \$6 ;with probability 0.008, the part is rejected and the process begins again: so with probability $0.008 \times 0.962 = 0.007697$, a good part is made at a cost of

$(4+4) = $8, and with probability $0.008 \times 0.03 = 0.000\,24$, a good part is made at a cost of $\$(4+4+2) = \$10$.

But with probability $(0.008)^2 = 0.000064$, it is part is rejected and the process repeats ad infinitum.

METHOD 1

C	4	6	8	10	12	> 12
$\Pr(C = c)$	0.962	0.03	0.001696	0.00024	0.0000616	insignificant

$E(C) = 4 \times 0.962 + 6 \times 0.03 + 8 \times 0.007696 + 10 \times 0.00024 + 12 \times 0.0000616 =$ 4.092707

The expected cost of producing 100 parts is $409.27.

29 **a** $\theta = 21$

$T = 21 + Ae^{-kt}$

When $t = 0,\ T = 100$

$\therefore 100 = 21 + A$

$\therefore A = 79$

$\therefore T = 21 + 79e^{-kt}$

When $t = 10, T = 84$

$\therefore 84 = 21 + 79e^{-10t}$

$\frac{63}{79} = e^{-10k}$

$\therefore -10k = \log_e \frac{63}{79}$

$\therefore k = \frac{1}{10}\log_e \frac{79}{63} \approx 0.02$

b $70 = 21 + 79e^{-kt}$

$\therefore \frac{49}{79} = e^{-kt}$

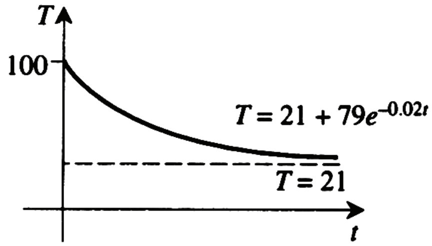
$\therefore -kt = \log_e \frac{49}{79}$

$\therefore t = -\frac{1}{k}\log_e \frac{49}{79}$

As $\frac{1}{k} = \frac{10}{\log_e \frac{79}{63}}, t = \frac{10}{\log_e\left(\frac{79}{63}\right)} \times \log_e\left(\frac{79}{49}\right) \approx 21.1$

The temperature of the kettle will be 70°C after 21.1 minutes i.e. at approximately 2.44 pm.

c



d When $t = 0,\ T = 100$

When $t = 10,\ T = 84$

$\therefore$ the average rate of change $= \dfrac{84 - 100}{10}$ °C min

$= \dfrac{-16}{10}$ °C/min

$= -1.6$ °C/min

e $\dfrac{dT}{dt} = -kAe^{-kt}$

i When $t = 6$

$$\frac{dT}{dt} = -k \times 79 \times e^{-6k}$$

$$\approx -2.0479\ ^{\circ}C/min$$

ii

$$\frac{dT}{dt} = -kAe^{-kt}$$

$$= -k(T - \theta)$$

$$= -k(60 - 21)$$

$$= -k(39)$$

$$= -39k$$

$$= -0.8826\ ^{\circ}\text{C/min}$$

30 Let X be the number of good components

a Probability of a batch being accepted

$= \Pr(X = 4) + \Pr(X = 5) \quad = \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^5$

$= 5 \times \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5$

$= 6 \times \left(\frac{1}{2}\right)^5$

$= \frac{6}{32}$

$= \frac{3}{16} = 0.1875$

b

$A(p) = \Pr(X = 4) + \Pr(X = 5)$

$= \binom{5}{4}(1 - p)^4 p + (1 - p)^5$

$= (1 - p)^4[5p + (1 - p)]$

$= (1 - p)^4[4p + 1]$

$\therefore \quad b = 4$

c $A'(p) = -4(1 - p)^3(1 + 4p) + 4(1 - p)^4$

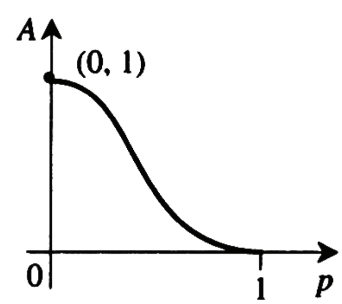
$= (1 - p)^3[-4(1 + 4p) + 4(1 - p)]$

$= (1 - p)^3[-4 - 16p + 4 - 4p]$

$= (1 - p)^3[-20p]$

$= -20p(1 - p)^3$

Note: no stationary point for $p \in (0, 1)$

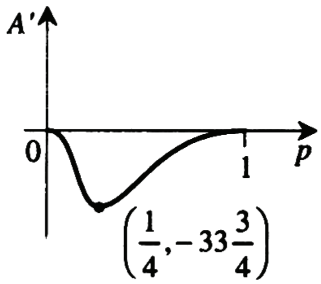


d **i** $A(p) = 0.95$: using the 'solve' command of a CAS calculate with $0 < P < 1$ gives $P \approx 0.076$

ii $A(p) = 0.05$: again using 'solve' gives $P \approx 0.657$

e **i** $A'(p) = -20p(1 - p)^3$

ii



iii

$$\begin{aligned} A''(p) &= -3(1-p)^2(-20p) - 20(1-p)^3 \\ &= -20[1-p]^2[-3p + (1-p)] \\ &= -20(1-p)^2(1-4p) \end{aligned}$$

$A''(p) = 0$ implies $p = 1$ or $p = \frac{1}{4}$ so $A'(p)$ is a minimum in $p = \frac{1}{4}$.

iv Most rapid rate of change of probabilities occurs when $p = \frac{1}{4}$.

31 $h(t) = (4.5 - 0.3t)^3$

a When $t = 0$, $h(0) = 4.5^3 = 91.125$ cm

b $h(t) \geq$ and $t \geq 0$

$\therefore (4.5 - 0.3t)^3 \geq 0$ and $t \geq 0$

equivalently $4.5 - 0.3t \geq$ and $t \geq 0$

$\therefore \frac{4.5}{0.3} \geq t$ and $t \geq 0$

$\therefore t \leq 15$ and $t \geq 0$

i.e. $t \in [0, 15]$

c $V = (0.8)^2(4.5 - 0.3t)^3$

$= 0.64(4.5 - 0.3t)^3$

d h is a 1 to 1 function

domain of h is $[0, 15]$

range of $h = [0,\ 91.125]$

Consider $x = (4.5 - 0.3y)^3$

$x^{\frac{1}{3}} = 4.5 - 0.3y$

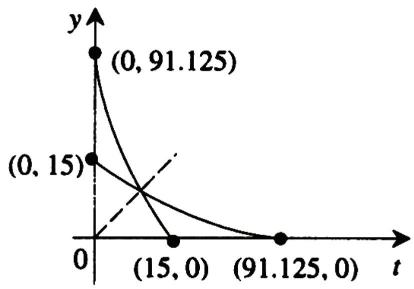
$\therefore 0.3y = 4.5 - x^{\frac{1}{3}}$

$\therefore y = 15 - \frac{10x^{\frac{1}{3}}}{3}$

$\therefore$ inverse function is $h^{-1}(t) = 15 - \dfrac{10t^{\frac{1}{3}}}{3}$

domain = [0, 91.125]

e



32 $\mu = 3$ mm

Let X be the diameter

$\Pr(X < 2.9) = 0.063$

$\Pr(X > 3.1) = 0.063$

a $\Pr\left(Z > \dfrac{3.1 - 3}{\sigma}\right) = 0.063$

$\therefore \quad \Pr\left(Z > \dfrac{0.1}{\sigma}\right) = 0.063$

$\therefore \quad \Pr\left(Z \le \dfrac{0.1}{\sigma}\right) = 0.937$

$\therefore \quad \dfrac{0.1}{\sigma} = 1.53$

$\sigma = \dfrac{0.1}{1.53}$

$= 0.06536$

b Let Y be the number of ball bearings accepted out of 8.

The probability of rejection = $\Pr(X < 2.9) + \Pr(X > 3.1)$

$= 0.063 \times 2$

$= 0.126$

For the binomial distribution, $p = 0.126$ and $n = 8$

i

$$\begin{aligned}\Pr(Y \ge 1) &= 1 - \Pr(Y = 0)\\ &= 1 - (0.874)^8\\ &= 1 - 0.34047\\ &= 0.6595\end{aligned}$$

ii $\Pr(Y = 2) = \begin{pmatrix} 8 \\ 2 \end{pmatrix}(0.126)^2(0.874)^6$

$= 0.198\ 14$

c **i** $\mu = 3.05, \sigma = 0.065\ 36$

$\Pr(X \le 2.9) + \Pr(X \ge 3.1) = \Pr\left(Z \le \dfrac{2.9 - 3.05}{0.065.36}\right) + 1 - \Pr\left(Z \le \dfrac{3.1 - 305}{0.065.36}\right)$

$= \Pr(Z \le -2.295) + 1 - \Pr(Z \le 0.765)$

$= 2 - \Pr(Z \le 2.295) - \Pr(Z \le 0.765)$

$= 2 - 0.9891 - 0.7779$

$= 0.233$

So 23.3% will now filloutside the given range.

ii $\Pr(3.05 - c \le X \le 3.05 + c) = 0.9$

$\therefore \Pr\left(\dfrac{-c}{0.06\ 536} \le Z \le \dfrac{c}{0.06\ 536}\right) = 0.9$

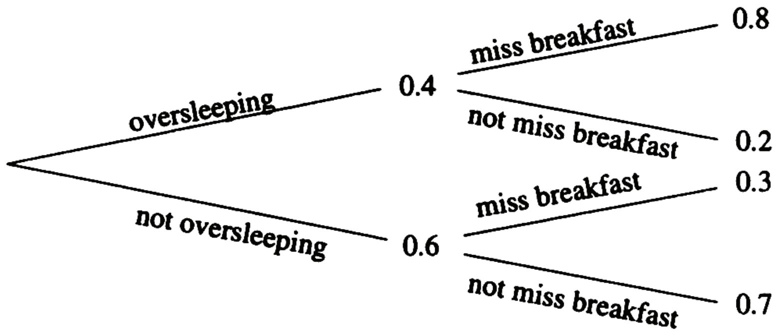
$\therefore 2\ \Pr\left(Z \le \dfrac{c}{0.06\ 536}\right) - 1 = 0.9$

$\therefore \Pr\left(Z \le \dfrac{c}{0.065\ 36}\right) = 0.95$

$\therefore \dfrac{c}{0.065\ 36} = 1.6449$

$\therefore c = 0.1075$

33



a From tree diagram

i $\Pr(\text{oversleeping} \cap \text{missing breakfast}) = 0.4 \times 0.8 = 0.32$

ii $\Pr(\text{not oversleeping} \cap \text{missing breakfast}) = 0.6 \times 0.3 = 0.18$

iii $\Pr(\text{oversleeping} \cap \text{missing breakfast}) + \Pr(\text{not oversleeping} \cap \text{missing breakfast}) =$
$0.32 + 0.18 = 0.5$

$$\Pr(\text{overslept} \mid \text{missing breakfast}) = \frac{\Pr(\text{overslept} \cap \text{missing breakfast})}{\Pr(\text{missing breakfast})}$$

b
$$= \frac{0.32}{0.5}$$
$$= 0.64$$

c This is a binomial distribution problem if it is assumed that a student's behaviour is independent of any other students behaviour.
Let X be the number of students who miss breakfast

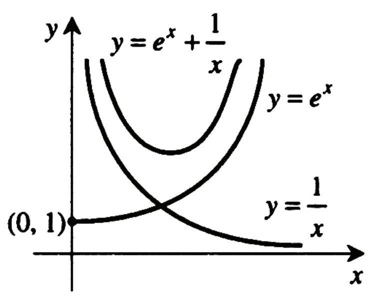
i $\Pr(X = 2) = {}^{10}C_2(0.5)^2(0.5)^8 = 0.043955$

ii $\Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - (0.5)^{10} = 0.999$

iii Probability of at least 8 not missing breakfast
$= \Pr(X \leq 2) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)$
$= (0.5)^{10} + 10 \times (0.5)^{10} + {}^{10}C_2(0.5)^2(0.8)^8$
$= \dfrac{7}{128}$

34 a

b



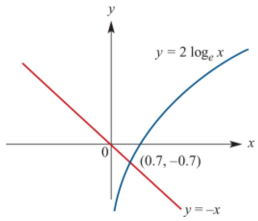
c $y = \dfrac{1}{x} + e^x$
$\dfrac{dy}{dx} = -\dfrac{1}{x^2} + e^x$

d **i** $\dfrac{dy}{dx} = 0 \Leftrightarrow -\dfrac{1}{x^2} + e^x = 0$
which implies $\dfrac{1}{x^2} = e^x$
$\therefore x^2 = e^{-x}$
$\therefore \log_e(x^2) = -x$
i.e. $2 \log_e x = -x$

ii As $x > 0,\ 2\log_e x = -x < 0$
$\therefore 2\ \log_e x < 0$

$\therefore x < 1$

$\therefore$ local minimum lies in the interval (0, 1)



iii

iv local minimum occurs when $x = 0.7$

$\therefore y = \dfrac{1}{0.7} + e^{0.7}$

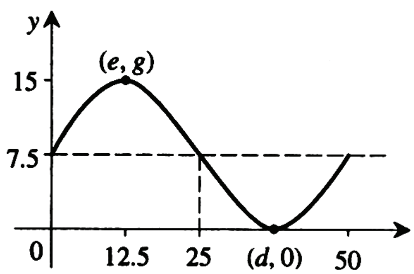
$= 3.4$ correct to one decimal place

i.e. coordinates local minimum are (0.7, 3.4)

35 **a** **i** From the diagram

amplitude $= 7.5 \therefore b = 7.5$

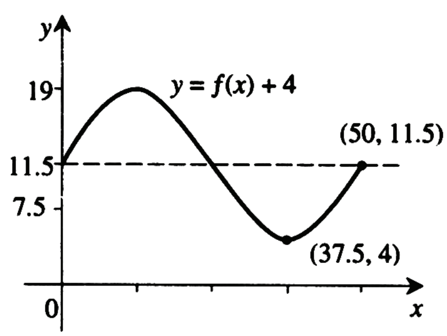
and centre is at $y = 7.5 \therefore a = 7.5$



period $= 2\pi \div \dfrac{2\pi}{50} = 50$

$\therefore m = 12.5,\ n = 15$ and $d = 37.5$

ii



b $10 = 7.5 + 7.5 \sin \dfrac{\left(2\pi x\right)}{50}$

$$\therefore \quad \frac{2.5}{7.5} = \sin\left(\frac{2\pi x}{50}\right)$$

$$\frac{1}{3} = \sin\left(\frac{2\pi x}{50}\right)$$

Let $\theta = \dfrac{2\pi x}{50}$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{1}{3}\right) \text{ or } \pi - \sin^{-1}\left(\frac{1}{3}\right)$$

$$\therefore \quad x = \frac{50}{2\pi}\sin^{-1}\left(\frac{1}{3}\right) \text{ or } \frac{50}{2\pi}\left(\pi - \sin^{-1}\left(\frac{1}{3}\right)\right)$$

$= 2.704$ or 22.296

c $g(x) = 2f\left(\dfrac{x}{5}\right) = 2\left(7.5 + 7.5\sin\left(\dfrac{2\pi}{50}\left(\dfrac{x}{5}\right)\right)\right)$

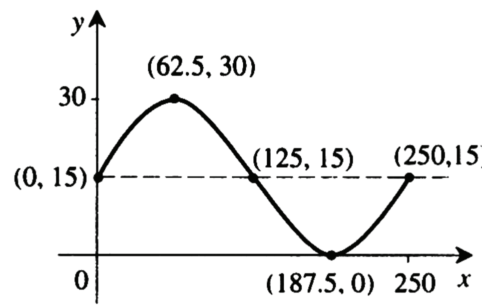
$= 15 + 15\sin\left(\dfrac{2\pi x}{250}\right)$

$= 15 + 15\sin\left(\dfrac{\pi x}{125}\right)$

$\therefore$ amplitude $= 15$

centre $y = 15$

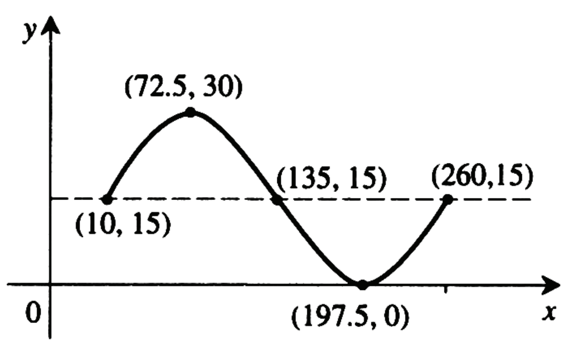
period $= 2\pi \div \dfrac{\pi}{125} = 250$



d **i** the new function has rule $h(x) = g(x - 10)$

$= 15 + 15\sin\left(\dfrac{\pi}{125}(x - 10)\right)$

ii



36 **a** $f(x) = \begin{cases} 0 & if\ x < 20 \\ k(5 - 2x) & 2 < x \leq \dfrac{5}{2} \\ 0 & x > \dfrac{5}{2} \end{cases}$

$\int_2^{\frac{5}{2}} f(x)\,dx = [k(5x - x^2)]_2^{\frac{5}{2}} = \frac{k}{4}$

For f to be a probability density function $k = 4$.

b **i** $\mathrm{E}(X) = \int_2^{\frac{5}{2}} xf(x)\,dx = 4\int_2^{\frac{5}{2}} 5x - 2x^2\,dx = \frac{13}{6}$

ii Solve $\int_0^a f(x)\,dx = 0.5$ for a

$4(5a - a^2 - (10 - 4)) = 0.5$

$8(-a^2 + 5a - 6) = 1$

$-8a^2 + 40a - 49 = 0$

Therefore $a = \frac{10 - \sqrt{2}}{4}$ as $2 < a\frac{5}{2}$

The median is $\frac{10 - \sqrt{2}}{4}$

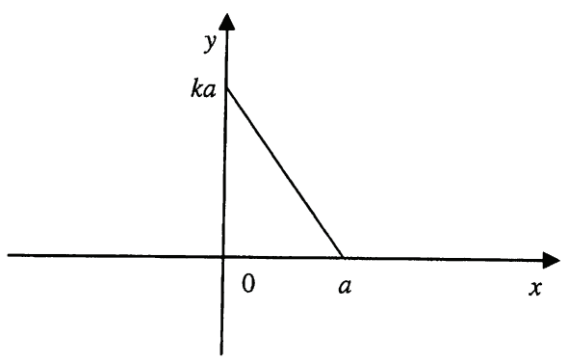
iii $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = \frac{1}{72}$

Therefore $\sigma = \frac{\sqrt{2}}{12}$

iv $\Pr(X < \mu - \sigma) = \Pr\left(x < \frac{13}{6} - \frac{\sqrt{2}}{12}\right)$

$= 0.1857$

37



a $\int_0^a f(x)\,dx = 1$

Therefore $\frac{1}{2}ka^2 = 1$

and $k = \frac{2}{a^2}$

$$\begin{aligned}E(X) &= \int_0^a x f(x)\,dx\\ &= \frac{a^3k}{6}\\ &= \frac{a}{3}\end{aligned}$$

b $$\begin{aligned}Var(\mathrm{X}) &= E(X^2) - [E(X)]^2\\ &= \int_0^a x^2 f(x)\,dx - \frac{a^2}{9}\\ &= \frac{a^4k}{12} - \frac{a^2}{9}\\ &= \frac{a^2}{18}\end{aligned}$$

c $\Pr(X > \mu + 2\sigma) = \Pr\left(x > \dfrac{a}{3} + \dfrac{2a}{3\sqrt{2}}\right) = \dfrac{6 - 4\sqrt{2}}{9}$

d Solve $\int_0^{1000} f(x)\,dx = 0.5$ for a

$a = 1000(\sqrt{2} + 2)$

38 $y = \dfrac{x}{10} - \log_e(x + 3)$, $x > -3$

a $\dfrac{dy}{dx} = \dfrac{1}{10} - \dfrac{1}{x + 3}$

and $\dfrac{dy}{dx} = 0$ implies $x + 3 = 10$. Hence $x = 7$

b $\dfrac{dy}{dx} = \dfrac{1}{10} - \dfrac{1}{x + 3} > \dfrac{1}{10}$ for $x > -3$

c The coordinates of M are $\left(7, \dfrac{7}{10} - \log_e(10)\right)$

Equation of line is $y - \left(\frac{7}{10} - \log_e(10)\right) = \frac{1}{10}(x - 7)$, i.e. $y = \frac{1}{10}x - \log_e 10$

d **i** "The line in c has gradient $\frac{1}{10}$ and hence if corners the x-axis at a point to the left of P (since the gradient of the curve $\pi < \frac{1}{10}$).
For the line, when $y = 0$,
$x = 10 \log_e 10$.
Hence the x-axis intercept at P is granter than $10 \log_e 10$."

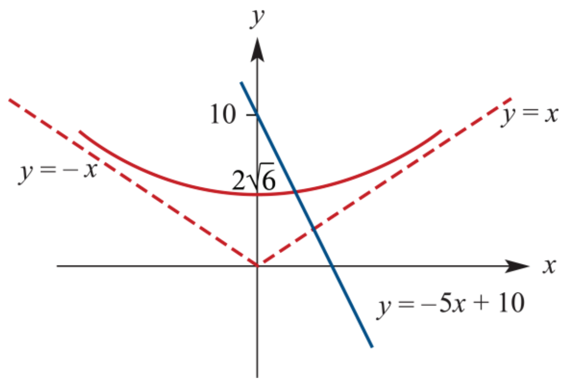
ii Using the 'solve' command of a CAS calculator shows that the intercept at P has x coordinate 36.852

39 **a** $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2 + 24}}$

b $\dfrac{dy}{dx} = 0 \rightarrow x = 0$; then $y = \sqrt{24} = 2\sqrt{6}$.
So the coordinates of the local minimum are $(0, 2\sqrt{6})$.

c $f(-x) = \sqrt{(-x)^2 + 24} = \sqrt{x^2 + 24} = f(x)$, so the function is even.

d



e When $x = 1$, $\dfrac{dy}{dx} = \dfrac{1}{5}$.
So the gradient of the normal at (1, 5) is −5.
Its equation is $y - 5 = -5(x - 1)$
$y = -5x + 10$:

f $\dfrac{dy}{dt} = 10$ at the point (5, 7):

$\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$ (chain rule)

$10 = \dfrac{x}{\sqrt{x^2 + 24}}\dfrac{dx}{dt}$

$10 = \dfrac{5}{7}\dfrac{dx}{dt}$ at (5, 7)

$\dfrac{dx}{dt} = 14$ units/second

g $\dfrac{d}{dx}\left(12\log_e\left|\sqrt{x^2 + 24} + x\right| + \dfrac{\sqrt[x]{x^2 + 24}}{2}\right)$

$$= 12 \times \frac{\dfrac{x}{\sqrt{x^2 + 24}} + 1}{\sqrt{x^2 + 24} + x} + \frac{\sqrt{x^2 + 24}}{2} + \frac{x^2}{\sqrt[2]{x^2 + 24}}$$

$$= \frac{12}{\sqrt{x^2 + 24}} + \frac{x^2 + 24}{2\sqrt{x^2 + 24}} + \frac{x^2}{2\sqrt{x^2 + 24}}$$

$$= \frac{x^2 + 24}{\sqrt{x^2 + 24}} = \sqrt{x^2 + 24} \text{ as required.}$$

h Area $= \int_2^5 \sqrt{x^2 + 24}\ dx$

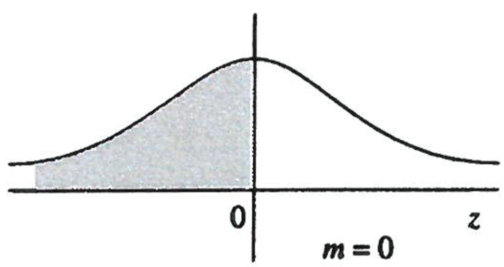
$$= \left[12 \log_e \left| \sqrt{x^2 + 24} + x \right| + \frac{x\sqrt{x^2 + 24}}{2}\right]_2^5$$

$$= \left(12 \log_e^{12} + \frac{35}{2}\right) - (12 \log_e(2\sqrt{7} + 2) + 2\sqrt{7})$$

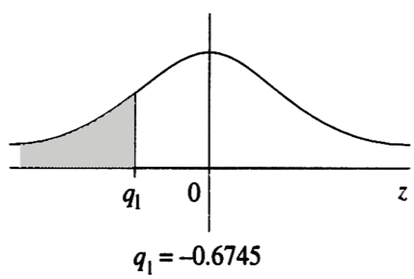
$$= 12 \log_e\left(\frac{6}{\sqrt{7} + 1}\right) - 2\sqrt{7} + \frac{35}{2}$$

$$= 12 \log_e(\sqrt{7} - 1) - 2\sqrt{7} + \frac{35}{2}$$

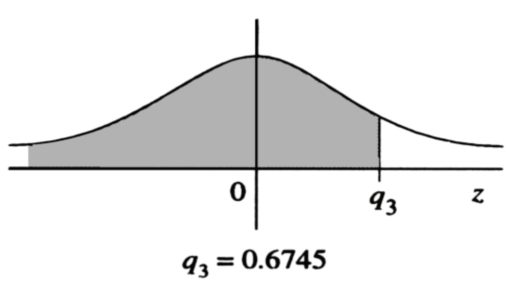
40 a i



ii



iii



iv interquartile range = 1.3490

v $\Pr(q_1 - 1.5 \times IQR < Z \leq q_3 + 1.5 \times IQR)$

$= \Pr(-0.6745 - 1.5 \times 1.3490 < Z < 0.6745 + 1.5 \times 1.3490)$

$= \Pr(-2.698 < Z < 2.698)$

$= 0.993$ or 99.3%

vi 0.7%

b **i** μ

ii $\mu - 0.6745\sigma$

iii $\mu + 0.6745\sigma$

iv 1.3490σ

v 0.993 or 99.3%

vi 0.7%

41 **a** $\int_0^1 f(x)\,dx = 1$

$\dfrac{k}{n+1} = 1$

Therefore $k = n + 1$

b $\mathrm{E}(X) = \int_0^1 xf(x)\,dx = \dfrac{n+1}{n+2}$

c $\mathrm{E}(X^2) = \int_0^1 x^2 f(x)\,dx = \dfrac{n+1}{n+3}$

$\mathrm{Var(X)} = \dfrac{n+1}{(n+2)^2(n+3)}$

d If m is the median, then

$k\int_0^m x^4\,dx = \dfrac{1}{2}$

$k\left[\dfrac{1}{n+1}x^{n+1}\right]_0^m = \dfrac{1}{2}$

$k\left(\dfrac{m^{n+1}}{n+1}\right) = \dfrac{1}{2}$

$m^{n+1} = \dfrac{n+1}{2k}$

$= \dfrac{1}{2}$ since $n = x+$

$m = n + 1\sqrt{\dfrac{1}{2}}$

e No turning pts → mode = 1.

42 a i Gradient $AB = \dfrac{\dfrac{1}{b-1} - 1}{b-2}$

$$= \frac{2-b}{(b-1)(b-2)}$$

$$= -\frac{1}{b-1}$$

$$= \frac{1}{1-b}$$

ii $g'(x) = -\dfrac{1}{(x-1)^2}$

$$= \frac{1}{1-b}$$

if $(x-1)^2 = b-1$

$x - 1 = \sqrt{b-1}$ (positive square root since $x > 1$)

$x = 1 + \sqrt{b-1}$

b i $\displaystyle\int_2^{e+1} \frac{1}{x-1}\,dx = [\log_e(x-1)]_2^{e+1}$

$$= \log_e e - \log_e 1$$

$$= 1 - 0$$

$$= 1$$

ii $\displaystyle\int_c^{1+e} \frac{1}{x-1}\,dx = 8$

$[\log_e (x-1)]_c^{1+e} = 8$

$\log_e e - log_e (c-1) = 8$

$1 - \log_e (c-1) = 8$

$\log_e (c-1) = -7$

$c - 1 = e^{-7}$

$c = 1 + e^{-7}$

c i Area of trapezium $= \dfrac{1}{2}(b-2)\left(1 + \dfrac{1}{b-1}\right)$

$$= \frac{1}{2}(b-2)\left(\frac{b}{b-1}\right)$$

$$= \frac{b(b-2)}{2(b-1)}$$

ii $$\frac{b(b-2)}{2(b-1)} = 8$$

$$b^2 - 2b = 16b - 16$$

$$b^2 - 18b + 16 = 0$$

Solving by the formula or completing the square gives $b = 9 \pm \sqrt{65}$ but $b > 2$. so $b - 9 + \sqrt{65}$.

d $$\int_2^{mn+1} \frac{1}{x-1}\,dx + \int_2^{\frac{m}{n}+1} \frac{1}{x-1}\,dx = 2$$

Now the upper terminals must be greater than 1 since we can not integrate over the discontinuity at $x = 1$. Hence:

$$[\log_e(x-1)]_2^{mn+1} + [\log_e(x-1)]_2^{\frac{m}{n}+1} = 2$$

$$\left(\log_e(mn) + \log_e(\frac{m}{n})\right) = 2 \text{ (} n \text{ positive so } m \text{ positive)}$$

$$\log_e\left[(mn) \times (\frac{m}{n})\right] = 2$$

$$\log_e m^2 = 2$$

$$m^2 = e^2$$

$$m = e \quad (m > 0)$$

43 a i Gradient $AB = \dfrac{\frac{1}{b^2} - 1}{b-1}$

$$= \frac{1 - b^2}{b^2(b-1)}$$

$$= \frac{(1-b)(1+b)}{b^2(b-1)}$$

$$= -\frac{b+1}{b^2}$$

$$f'(x) = -\frac{2}{x^3}$$

$$= -\frac{b+1}{b^2}$$

ii if $x^3 = -\dfrac{2b^2}{(b+1)}$

$$x = \left(\frac{2b^2}{b+1}\right)^{\frac{1}{3}}$$

b i Area of trapeziums $= \dfrac{1}{2}(b-1)\left(\dfrac{1}{b^2} + 1\right)$

$$S(b) = \frac{(b^2 + 1)(b - 1)}{2b^2}$$

ii $\dfrac{(b^2 + 1)(b - 1)}{2b^2} = \dfrac{10}{9}$

$9(b^3 - b^2 + b - 1) = 20b^2$

$9b^3 - 29b^2 + 9b - 9 = 0$

Using the factor theorem or a CAS calculator shows that $b - 3$ is a factor of the cubic, giving $(b - 3)(9b^2 - 2b + 3) = 0$

The quadratic has no zeroes $(B^2 - 4AC < 0)$, so $b = 3$ is the only solution.

iii
$$\begin{aligned}\int_1^b f(x)\,dx &= \int_1^b \frac{1}{x^2}\,dx\\ &= \left[-\frac{1}{x}\right]_1^b\\ &= \frac{-1}{b} + 1\\ &= 1 - \frac{1}{b}\end{aligned}$$

< 1 since $b > 1$ and so $0 < \dfrac{1}{b} < 1$

c
$$\begin{aligned}D(b) &= S(b) - \int_1^b f(x)\,dx\\ &= \frac{(b^2 + 1)(b - 1)}{2b^2} - \left(1 - \frac{1}{b}\right) \text{ from } b\ i \text{ and } b\ iii\\ &= \frac{(b^2 + 1)(b - 1)}{2b^2} - \left(\frac{b - 1}{b}\right) \qquad [t]\\ &= \frac{b - 1}{2b^2}(b^2 + 1 - 2b)\\ &= \frac{b - 1}{2b^2}(b - 1)^2\\ &= \frac{(b - 1)^3}{2b^2}\end{aligned}$$

To show that the function is strictly increasing for $b > 1$, it is sufficient to show that $D'(b) > 0$ for $b > 1$.

$$D'(b) = \frac{\left(2b^2)(3(b-1)^2\right) - \left((b-1)^3\right)\left(4b\right)}{4b^4}$$
$$= \frac{3b(b-1)^2 - 2(b-1)^3}{2b^3}$$
$$= \frac{(b-1)^2(3b - 2(b-1))}{2b^3}$$
$$= \frac{(b-1)^2(b+2)}{2b^3}$$
$$> 0 \text{ for all } b > 1$$

44 **a** $f'(x) = x^m(-ne^{-nx+n}) + mx^{m-1}e^{-nx+n}$

$$= x^{m-1}e^{-nx+n}(-nx+m)$$
$$= 0$$

if $x = 0$ or $x = \frac{m}{n}$.

So for the stationary point not at the origin, $x = \frac{m}{n}$ and then

$$f\left(\frac{m}{n}\right) = \left(\frac{m}{n}\right)^m e^{-m+n}$$

The Point with coordinates $\left(\frac{m}{n}, \left(\frac{m}{n}\right)^m e^{-m+n}\right)$ is a local maximum tuning point (by reference to the given graph or by checking the sign of the first derivative which goes from positive to negative through $x = \frac{m}{n}$).

b Find the equation of the tangent at a general point $x = a$ on the curve.

$x = a : f(a) = a^m e^{-an+n}$

$$f'(a) = a^{m-1}e^{-an+n}(-an+m)$$

using $y - y_1 = m(x - x_1)$, the equation of the tangent is

$y - a^m e^{-an+n} = a^{m-1}e^{-an+n}(-an+m)(x-a)$

The tangent passes through the origin, so $(0, 0)$ satisfies the equation.

$-a^m e^{-an+n} = a^{m-1}e^{-an+n}(-an+m)(-a)$

$$1 = (-an+m)\ (a\ m + 0,\ e^{-an+n} + 0)$$
$$an = m - 1$$
$$a = \frac{m-1}{n}$$

substitute to find the y-coordinate:

$$f(a) = f\left(\frac{m-1}{n}\right)$$
$$= \left(\frac{m-1}{n}\right)^m e^{n-m+1}$$

So the tangent at $\left(\frac{m-1}{n}, (\frac{m-1}{n})^m e^{n-m+1}\right)$ passes through the origin.
(Note: the tangent at $(0, 0)$ also passes though the origin!)

c **i** Using CAS calculator, we find that

$\int_0^\infty x^2 e^{-2x+2}\, dx = \dfrac{e^2}{4}$

So: $\dfrac{4}{e^2}\int_0^\infty x^2 e^{-2x+2}\, dx = 1$ and $k = \dfrac{4}{e^2}$

ii

$$\begin{aligned}\Pr(X < 1) &= \int_0^1 \frac{4}{e^2} x^2 e^{-2x+2}\, dx\\ &= (e^2 - 5)e^{-2} \quad \text{(using a CAS calculator)}\\ &= 1 - 5e^{-2}\\ &= 1 - \frac{5}{e^2}\end{aligned}$$

iii The made is the value for which f is a maximum. Use calculus to solve $f'(x) = 0$.

$$\begin{aligned}f'(x) &= x^2(-2e^{-2x+2}) + 2xe^{-2x+2}\, dx\\ &= 2xe^{-2x+2}(-x + 1)\\ &= 0\end{aligned}$$

if $x = 1$

So the mode is 1.

Alternatively, note that x^2e^{-2x+2} is the function from part **a** with m = 2 and n = 2
From That question, the x-coordination of the stationary point is $x = \frac{m}{n} = \frac{2}{2} = 1$ in this case.

45 **a** **i**

$$\begin{aligned}\int_0^\infty e^{-qx}\, dx &= \lim_{a\to\infty}\int_0^\infty e^{-qx}\, dx\\ &= \lim_{a\to\infty}\left[-\frac{1}{q}e^{-qa}\right]_0^a\\ &= \lim_{a\to\infty}\left(-\frac{1}{q}e^{-qa} + \frac{1}{q}\right)\\ &= 0 + \frac{1}{q}\\ &= \frac{1}{q}(\text{since} e^{-qa} \to 0 as a \to \infty)\end{aligned}$$

$$\begin{aligned}\text{Hence } \int_0^\infty ke^{-qx}\, dx &= \frac{k}{q}\\ &= 1 \text{if} k = q\end{aligned}$$

ii $\mathrm{E}(x) = \int_0^\infty x \times qe^{-qx}\,dx$

$= \frac{1}{q}$ (using a CAS calculator)

$\mathrm{E}\,(x^2) = \int_0^\infty x^2 \times qe^{-qx}\,dx$

$= \frac{2}{q^2}$(using a CAS calculator)

iii $\mathrm{var}\,(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

$$= \frac{2}{q^2} - \left(\frac{1}{q}\right)^2$$

$$= \frac{1}{q^2}$$

iv If $m = \frac{1}{2}\log_e(2)$, then

$$\int_0^m qe^{-qx}\,dx = [-e^{-qx}]_0^m$$

$$= -e^{-qm} + 1$$

$$= -e^{-\log_e^2} + 1$$

$$= -e^{-\log_e \frac{1}{2}} + 1$$

$$= -\frac{1}{2} + 1$$

$$= \frac{1}{2}$$

So m is the medium.

Alternatively, solve $\int_0^m f(x)\,dx = \frac{1}{2}$ for m.

b $$\Pr\left(X > \frac{1}{q}\log_e(3) \mid X > \frac{1}{q}\log_e(2)\right) = \frac{\Pr\left(X > \frac{1}{q}\log_e(3)\right)}{\Pr\left(X > \frac{1}{q}\log_e(2)\right)}$$

Since the median is $\frac{1}{q}\log_e(2)$ from part **a iv**, the denominator is $\frac{1}{2}$.

$$\Pr\left(X > \frac{1}{q}\log_e 3\right) = 1 - \Pr(X \le \frac{1}{q}\log_e(3))$$

$$= 1 - \int_0^{\frac{1}{q}\log_e(3)} qe^{-qx}\,dx$$

$$= 1 - [-e^{-qx}]_0^{\frac{1}{q}\log_e(3)}$$

$$= 1 + e^{\log_e^3} - 1$$

$$= e^{\log_e \frac{1}{3}}$$

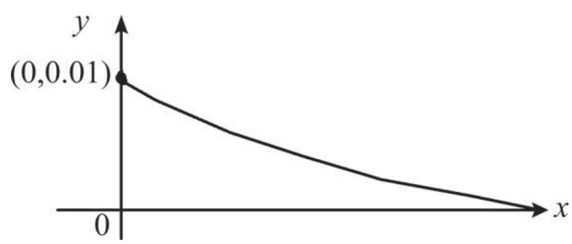
For the numerator: $= \frac{1}{3}$

So

$$\Pr(X > \frac{1}{q}\log_e(3) \mid X > \frac{1}{q}\log_e(2)) = \frac{\frac{1}{3}}{\frac{1}{2}}$$

$$= \frac{2}{3}$$

c **i** The graph of $y = f(x) = 0.01e^{-0.01x}$, $x \ge 0$, is that of an exponential function with y-axis intercept $(0, 0.01)$ and horizontal asymptote $y = 0$ (the x-axis).



ii $\Pr(X > 100) = 1 - \Pr(X \le 100)$

$$= 1 - \int_0^{100} 0.01e^{-0.01x}\,dx$$

$$= 1 - \left[-e^{-0.01x}\right]_0^{100}$$

$$= 1 + e^{-1} - 1$$

$$= e^{-1} \approx 0.37$$

iii From part **a iv**, $m = \frac{1}{0.01}\log_e(2)$

$$= 100\log_e(2) \approx 69.31$$

46 **a** 0.527

b (0.4961, 0.5580)

c For a 95% CI, $M = 1.96 \times \sqrt{\dfrac{0.527 \times 0.473}{1000}} \approx 0.0309$ Half this width is 0.0155

Thus, we need to find a such that

$$a \times \sqrt{\frac{0.527 \times 0.473}{1000}} = 0.0155$$

$a = 0.981$

To find the level of confidence associated with $a = 0.981$ we use the normal cdf function.

Level of confidence = $\Pr(-0.981 < Z < 0.981) = 0.6734$

Ie, 67.34% confidence interval

d Twice this width is 0.0618 Thus, we need to find a such that Thus, we need to find a such that

$$a \times \sqrt{\frac{0.527 \times 0.473}{1000}} = 0.0618$$

$a = 3.914$

To find the level of confidence associated with $a = 3.914$ we use the normal cdf function. Level of confidence = $\Pr(-3.914 < Z < 3.914) = 0.9999$

ie, 99.99% confidence interval