SADLER UNIT 4 CHAPTER 10

EXERCISE 10A

Q1. $\frac{dy}{dx} = 8x - 5$

$y = \frac{8x^2}{2} - 5x + c$

$\underline{y = 4x^2 - 5x + c}$

Q2. $\frac{dy}{dx} = 6\sqrt{x}$

$= 6x^{\frac{1}{2}}$

$y = \frac{6x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + c$

$= 4x^{\frac{3}{2}} + c$

$= \underline{4\sqrt{x^3} + c}$

Q3. $8y\frac{dy}{dx} = 4x - 1$

$\int 8y \, dy = \int 4x - 1 \, dx$

$\underline{4y^2 = 2x^2 - x + c}$

Q4. $3y\frac{dy}{dx} = \frac{5}{x^2}$

$\int 3y \, dy = \int \frac{5}{x^2} \, dx$

$\frac{3y^2}{2} = -5x^{-1} + c$

$\underline{\frac{3y^2}{2} = -\frac{5}{x} + c}$

Q5. $14x^2y\frac{dy}{dx} = 1$

$\int 14y \, dy = \int \frac{1}{x^2} \, dx$

$\underline{7y^2 = -\frac{1}{x} + c}$

Q6. $4x^2 \sin 2y \frac{dy}{dx} = 5$

$\int 4\sin 2y \, dy = \int \frac{5}{x^2} \, dx$

$\underline{-2\cos 2y = -\frac{5}{x} + c}$

Q7. $\frac{dy}{dx} = \frac{8x+1}{2y-3}$

$\int (2y-3) \, dy = \int (8x+1) \, dx$

$\underline{y^2 - 3y = 4x^2 + x + c}$

Q8. $\frac{dy}{dx} = \frac{x(2-3x)}{4y-5}$

$\int (4y-5) \, dy = \int x(2-3x) \, dx$

$\underline{2y^2 - 5y = x^2 - x^3 + c}$

Q9. $x^2\frac{dy}{dx} = \frac{1}{\cos y}$

$\int \cos y \, dy = \int \frac{1}{x^2} \, dx$

$\underline{\sin y = -\frac{1}{x} + c}$

Q10. $(y^2+1)^5\frac{dy}{dx} = \frac{x}{2y}$

$\int 2y(y^2+1)^5 \, dy = \int x \, dx$

$\underline{\frac{1}{6}(y^2+1)^6 = \frac{x^2}{2} + c}$

Q11. $\frac{dy}{dx} = 6x$, $(-1, 4)$

$y = 3x^2 + c$

$4 = 3 + c$

$\underline{c = 1}$ $\quad \therefore \underline{y = 3x^2 + 1}$

Q12. $6x^2y\frac{dy}{dx} = 5$, $(0.5, 1)$

$\int 6y \, dy = \int \frac{5}{x^2} \, dx$

$3y^2 = -\frac{5}{x} + c$

$3 = -10 + c$

$\underline{c = 13}$ $\quad \therefore \underline{3y^2 = -\frac{5}{x} + 13}$

Q13. $(2+\cos y)\dfrac{dy}{dx} = 2x+3, \quad (1, \frac{\pi}{2})$

$\int (2+\cos y)\,dy = \int 2x+3\,dx$

$2y + \sin y = x^2+3x+c$

$\pi + 1 = 1+3+c$

$\pi - 3 = c$

$\therefore 2y+\sin y = x^2+3x+\pi-3$

Q14. $\dfrac{dy}{dx} = \dfrac{4x(x^2+2)}{2y+3}, \quad (1, 2)$

$\int (2y+3)\,dy = \int 4x(x^2+2)\,dx$

$y^2+3y = (x^2+2)^2 + c$

$4+6 = 9+c$

$c = 1$

$\therefore y^2+3y = (x^2+2)^2+1$

Q15. $v\dfrac{dv}{ds} = 6s^2$

$\int v\,dv = \int 6s^2\,ds$

$\dfrac{v^2}{2} = 2s^3 + c$

when $v=6$, $s=2$

$\dfrac{36}{2} = 2(8)+c$

$18 = 16+c$

$c = 2$

$\therefore \dfrac{v^2}{2} = 2s^3+2$

when $s=3$,

$\dfrac{v^2}{2} = 54+2$

$v^2 = 112$

$v = \pm\sqrt{112}$, reject $-ve$.

$= 4\sqrt{7}$ m/s.

Q16. $\dfrac{dy}{dx} = -\dfrac{\sin x}{y}$

$\int y\,dy = \int -\sin x\,dx$

$\dfrac{y^2}{2} = \cos x + c, \quad (\frac{\pi}{3}, 2)$

$\dfrac{4}{2} = \dfrac{1}{2} + c$

$c = \dfrac{3}{2}$

$\therefore \dfrac{y^2}{2} = \cos x + \dfrac{3}{2}$

a) $\dfrac{a^2}{2} = \cos(\pi) + \dfrac{3}{2}$

$a^2 = 2\left(-1+\dfrac{3}{2}\right)$

$= -2+3$

$a^2 = 1$

$a = \pm 1$

$\therefore a = 1$ (as $a>0$)

b) $\dfrac{b^2}{2} = \cos\left(\frac{\pi}{6}\right) + \dfrac{3}{2}$

$b^2 = 2\left(\dfrac{\sqrt{3}}{2} + \dfrac{3}{2}\right)$

$b^2 = \sqrt{3}+3$

$b = \pm\sqrt{\sqrt{3}+3}$

$\therefore b = \sqrt{\sqrt{3}+3}, \quad (b>0)$

$\therefore \dfrac{dy}{dx} = -\dfrac{\sin\left(\frac{\pi}{6}\right)}{\sqrt{\sqrt{3}+3}}$

$= -\dfrac{1}{2\sqrt{\sqrt{3}+3}}$

Q17. $\dfrac{dV}{dt} = \dfrac{25}{2V}$

$\int V\,dV = \int \dfrac{25}{2}\,dt$

$\dfrac{V^2}{2} = \dfrac{25}{2}t + c$

$c = 200$

$$\frac{v^2}{2} = \frac{25}{2}t + 200$$

a) $\frac{v^2}{2} = \frac{25}{2}(20) + 200$

$v^2 = 2(450)$

$= 900$

$\therefore v = \pm 30$

(reject -ve)

$\therefore v = 30cm^3$

b) $\frac{40^2}{2} = \frac{25}{2}t + 200$

$\frac{2(800-200)}{25} = t$

$t = \frac{1200}{25}$

$t = 48$ seconds

EXERCISE 10B

Q1 $\frac{dA}{dt} = 1.5A$

$A = A_0 e^{1.5t}$

$A = 100 e^{1.5t}$

a) $A(1) = 100e^{1.5} = 448.17$

≈ 448

b) $A(5) = 100e^{1.5(5)}$

$= 180804.24$

≈ 180804

Q2 $\frac{dP}{dt} = 0.25P$

$P = P_0 e^{0.25t}$

$P = 5000e^{0.25t}$

a) $P(5) = 5000e^{0.25(5)}$

$= 17451.71$

≈ 17452

b) $P(25) = 5000e^{0.25(25)}$

$= 2590064.12$

≈ 2590064

Q3. $\frac{dQ}{dt} = -0.01Q$

$Q = Q_0 e^{-0.01t}$

$Q = 100000e^{-0.01t}$

a) $Q(20) = 100000e^{-0.01(20)}$

≈ 81873.08

≈ 81873

b) $Q(50) = 100000e^{-0.01(50)}$

$= 60653.07$

≈ 60653

Q4. $\frac{dA}{dt} = -0.08A$

$A = A_0 e^{-0.08t}$

$A = 5e^{-0.08t}$

$A(25) = 5e^{-0.08t}$

$= 0.67668$

≈ 680 grams

Q5. $\frac{dA}{dt} = -0.02A$

$A = A_0 e^{-0.02t}$

$A = 20e^{-0.02t}$

$A(50) = 20e^{-0.02(50)}$

$= 7.3576$

≈ 7.36 kg

Q6. $\frac{dA}{dt} = -0.0004A$

$t = \frac{\ln(2)}{0.0004}$

$= 1732.87$

≈ 1733 yrs

Q7. $\frac{dM}{dt} = -kM$

$t = \frac{\ln(2)}{k}$

$30 = \frac{\ln(2)}{k}$

$k = \frac{\ln(2)}{30}$

$= 0.0231$

$\therefore M = M_0 e^{-0.0231t}$

$M = e^{-0.0231t}$

a) $M(30) = 500g$

b) $M(60) = 250g$

c) $M(40) = e^{-0.0231(40)}$

$= 0.3969$

$\approx 397g$

Q8 $\frac{dM}{dt} = -kM$

$t = \frac{\ln 2}{k}$

$k = \frac{\ln 2}{t}$

$= \frac{\ln 2}{250000}$

$k = 0.000002772$

$\therefore M = 100e^{-0.000002772t}$

$M(5000) = 98.62$

$\approx 98.6\%$

Q9. $325000 = 56000e^{-k(8)}$

$\ln\left(\frac{325}{56}\right) = -8k$

$k = -0.2198$

$\approx 22\%$ p.a

Q10. $\frac{dC}{dt} = -kC$

$t = \frac{\ln(2)}{k}$

$k = \frac{\ln 2}{5700}$

$k \doteq 0.000121604$

$C = C_0 e^{-0.00012t}$

$60 = 100 e^{-0.00012t}$

$\ln(0.6) = -0.00012t$

$t = 4200.70$

≈ 4200 yrs

Q11. $\frac{dM}{dt} = -kM$

$t = \frac{\ln 2}{k}$

$30 = \frac{\ln 2}{k}$

$k = \frac{\ln 2}{30}$

$\doteq 0.02310$

$M = M_0 e^{-0.02310t}$

$1 = 15 e^{-0.02310t}$

$\frac{1}{15} = e^{-0.02310t}$

$\ln(\frac{1}{15}) = -0.02310t$

$t = 117.21$

≈ 117 yrs

Q12. $\frac{dA}{dt} = kA$

$A = A_0 e^{kt}$

$2 = e^{kt}$

$\ln 2 = kt$

$t = \frac{\ln 2}{k}$

$\ln 2 = 0.693\ldots$ //.

Q13. $\frac{dT}{dt} = -k(T-28)$

$T - 28 = (T_0 - 28)e^{-kt}$

$T_0 = 240$,

$T = 28 + 212e^{-kt}$

Some time, t, $T(t) = 135$

$135 = 28 + 212e^{-kt}$

$107 = 212e^{-kt}$

$\frac{107}{212} = e^{-kt}$

Ten mins later, $t+10$, $T(t+10) = 91$

$91 = 28 + 212e^{-k(t+10)}$

$\frac{63}{212} = e^{-kt} \cdot e^{-10k}$

$\therefore \frac{63}{212} = \frac{107}{212} e^{-10k}$

$\ln\left(\frac{63}{107}\right) = -10k$

$k = \frac{\ln\left(\frac{63}{107}\right)}{-10}$

$k = 0.05296941$

$\frac{107}{212} = e^{-0.05296941t}$

$\ln\left(\frac{107}{212}\right) = -0.05296941t$

$t = 12.91$

≈ 12.91 mins.

EXERCISE 10C

Q1. $\frac{dN}{dt} = 0.45N - 0.015N^2$

$N(0) = 0.5$

a) $\frac{dN}{dt} = aN - bN^2$

$\Rightarrow N = \frac{0.45}{0.015 + Ce^{-0.45t}}$

$N = \frac{450}{15 + 1000Ce^{-0.45t}}$

$0.5 = \frac{450}{15 + 1000C}$

$15 + 1000C = \frac{450}{0.5}$

$1000C = 900 - 15$

$C = \frac{885}{1000}$

$C = 0.885$

b) $N(10) = \frac{450}{15 + 885e^{-0.45(10)}}$

$= 18.12$

$\therefore \approx 18.12$ Million

Q2. $y = \frac{150\,000}{1 + Ce^{-at}}$

$300 = \frac{150000}{1 + Ce^{0}}$

$1 + C = \frac{150000}{300}$

$1 + C = 500$

$C = 499$

$y = \frac{150\,000}{1 + 499e^{-at}}$

$920 = \frac{150\,000}{1 + 499e^{-a}}$

$1 + 499e^{-a} = \frac{150000}{920}$

$499e^{-a} = \frac{150000}{920} - 1$

$e^{-a} = \frac{149080}{920} \div 499$

$e^{-a} = \frac{149080}{459080}$

$-a = \ln\left(\frac{149080}{459080}\right)$

$a = 1.1274$

$\therefore y(5) = \frac{150000}{1 + 499e^{-1.1274(5)}}$

$= 53533.13$

≈ 53500

Q3. $\frac{dy}{dx} = \frac{1}{500}y(300-y)$, $y(0) = 100$

$\int \frac{1}{y(300-y)}\,dy = \int \frac{1}{500}\,dx$

$\int \frac{1}{300y} + \frac{1}{300(300-y)}\,dy = \int \frac{1}{500}\,dx$

$\frac{1}{300}\left[\ln|y| - \ln|300-y|\right] = \frac{1}{500}x + C_1$

$\ln|y| - \ln|300-y| = \frac{3}{5}x + C_2$

$\left|\frac{y}{300-y}\right| = e^{\frac{3}{5}x + C_2}$

$\frac{y}{300-y} = \pm C_3 e^{\frac{3}{5}x}$

$\frac{300-y}{y} = C_4 e^{-\frac{3}{5}x}$

$\frac{200}{100} = C_4 e^{0}$

$C_4 = 2$

$$\frac{300-y}{y} = 2e^{-0.6x}$$
$$300-y = 2ye^{-0.6x}$$
$$300 = 2ye^{-0.6x}+y$$
$$300 = y(1+2e^{-0.6x})$$
$$y = \frac{300}{1+2e^{-0.6x}}$$

Q4. $L(0) = 51$

$$\frac{dL}{dt} = \frac{1}{500}L(200-L)$$
$$= \underbrace{\frac{2}{5}}_{a}L - \underbrace{\frac{1}{500}}_{b}L^2$$

a) $C = \frac{a}{b}$

$$= \frac{2}{5} \div \frac{1}{500}$$
$$= \frac{2\times 500}{5}$$
$$= 200$$

$\therefore$ Max length is 200 cm.

b) $$\int \frac{1}{L(200-L)}\,dL = \int \frac{1}{500}\,dt$$
$$\int \frac{1}{200L} + \frac{1}{200(200-L)}\,dL = \int \frac{1}{500}\,dt$$
$$\frac{1}{200}\left[\ln|L| - \ln|200-L|\right] = \frac{1}{500}t + c_1$$
$$\ln|L| - \ln|200-L| = 0.4t + c_2$$
$$\left|\frac{L}{200-L}\right| = e^{0.4t+c_2}$$
$$\frac{L}{200-L} = \pm c_3 e^{0.4t}$$
$$\frac{200-L}{L} = c_4 e^{-0.4t}$$
$$\frac{200-51}{51} = c_4 e^{0}$$
$$c_4 = \frac{149}{51}$$
$$200-L = \frac{149}{51}Le^{-0.4t}$$
$$10200 - 51L = 149Le^{-0.4t}$$
$$10200 = 149Le^{-0.4t} + 51L$$
$$10200 = L(149e^{-0.4t}+51)$$
$$L = \frac{10200}{51+149e^{-0.4t}}$$

c) $$L(10) = \frac{10200}{51+149e^{-4}}$$
$$= 189.84 \text{ cm}$$

Q5. $P(0) = 160$

$$\frac{dP}{dt} = \frac{1}{5}P - \frac{1}{12500}P^2$$

a) $$\frac{dP}{dt} = \frac{1}{5}P\left(1-\frac{P}{2500}\right)$$
$$\frac{dP}{dt} = \frac{1}{12500}P(2500-P)$$
$$\int \frac{1}{P(2500-P)}\,dP = \int \frac{1}{12500}\,dt$$
$$\frac{1}{2500}\int \frac{1}{P} + \frac{1}{2500-P}\,dP = \int \frac{1}{12500}\,dt$$
$$\frac{1}{2500}\left[\ln|P| - \ln|2500-P|\right] = \frac{1}{12500}t + c_1$$
$$\ln\left|\frac{P}{2500-P}\right| = \frac{1}{5}t + c_2$$
$$\left|\frac{P}{2500-P}\right| = e^{0.2t+c_2}$$
$$\frac{P}{2500-P} = \pm c_3 e^{0.2t}$$
$$\frac{2500-P}{P} = c_4 e^{-0.2t}$$

$$2500 - P = PC_4 e^{-0.2t}$$

$$\frac{2500-160}{160} = C_4$$

$$C_4 = \frac{2340}{160}$$

$$= \frac{117}{8}$$

$$\therefore 2500 = \frac{117}{8} P e^{-0.2t} + P$$

$$20000 = 117 P e^{-0.2t} + 8P$$

$$P = \frac{20000}{8 + 117 e^{-0.2t}}$$

or $$P = \frac{2500}{1 + \frac{117}{8} e^{-0.2t}}$$

b) As $t \to \infty$, $P \to 2500$

c) $$P(10) = \frac{2500}{1 + \frac{117}{8} e^{-2}}$$

$$= 839.13$$

$$\approx 839$$

Q6. $N(0) = 200$

$$\frac{dN}{dt} = 0.8N\left(1 - \frac{N}{20000}\right)$$

$$= \frac{4}{5} N\left(\frac{20000 - N}{20000}\right)$$

$$= \frac{4}{100000} N(20000 - N)$$

$$\int \frac{1}{N(20000-N)}\, dN = \int \frac{1}{25000}\, dt$$

$$\frac{1}{20000}\left[\ln\left|\frac{N}{20000-N}\right|\right] = \frac{1}{25000} t + c_1$$

$$\ln\left|\frac{N}{20000-N}\right| = 0.8t + c_2$$

$$\frac{N}{20000-N} = \pm C_3 e^{0.8t}$$

$$\frac{20000-N}{N} = C_4 e^{-0.8t}$$

$$\frac{19800}{200} = C_4$$

$$99 = C_4$$

$$20000 - N = 99 N e^{-0.8t}$$

$$20000 = 99 N e^{-0.8t} + N$$

$$20000 = N(1 + 99 e^{-0.8t})$$

$$N = \frac{20000}{1 + 99e^{-0.8t}}$$

$$N(8) = \frac{20000}{1 + 99e^{-6.4}}$$

$$= 17174.8391$$

$$\approx 17175$$

EXERCISE 10D

No worked solutions required!
See below for notes on gradient analysis.

(A) $\frac{dy}{dx} = 0 \quad \forall (x,y) \in \mathbb{R}^2$

(B) $\frac{dy}{dx} = 0$, for $x = -3$, $x = 1$

(C) $\frac{dy}{dx} = 0$, for $x = 0$

(D) $\frac{dy}{dx} = 0$, for $x = 2$

(E) $\frac{dy}{dx} = 0$, for $x = -1$, $x = 3$

(F) $\frac{dy}{dx} = 1$, $\forall (x,y) \in \mathbb{R}^2$

(G) $\frac{dy}{dx} = 0$, $x = 0$ and only defined $x > 0$.

(H) $\dfrac{dy}{dx} \to \infty$ as $x \to \infty$

and $\dfrac{dy}{dx} \to 0$ as $x \to -\infty$.

(I) $\dfrac{dy}{dx} = 0$, $x = 3$.

(J) $\dfrac{dy}{dx} = -2$, $\forall (x,y) \in \mathbb{R}^2$