Problem 2. A Graph Coloring Game

Anatoly Kochenyuk, ⟨*team name*⟩

July 2019

Annotation
⟨*annotation text*⟩

Contents

Introduction

The main point of this problem is game theory.

In this case, game is played on an undirected graph with all vertices colored in white. Two payers, Bob and Riley, make alternating moves starting with Bob. On his first move, each player chooses any white vertex and colors it in his favourite color. On each consequent move, each player picks a white vertex connected with an edge to a vertex of his color (if there is no such vertex, the player skips his move). Then he colors it in his color. The game finishes when no white vertices remain.

Definition 0.1. *G is an undirected Graph*

Let $B(G)$ be the maximal number of blue vertices, which Bob can guarantee at the end of the game.

Let $R(G)$ be the maximal number of red vertices, which Riley can guarantee at the end of the game.

Definition 0.2. $M_B(n) = \min\limits_{G \in S(n)} B(G)$ *$S(n)$ – set of all simple undirected connected graphs with n vertices.*

Definition 0.3. *$|G|$, where G is undirected graph, equals the number of vertices in G*

1 Describe the optimal strategy for Bob and evaluate $B(G)$ where:

1.1 Tree

G is a tree. Let's define weight for each vertex as following:

Definition 1.1. *Let S_v be a set of trees that was produced by removing vertex $v \in G$ from the tree*

Weight of vertex v is $w_v = \max\limits_{s \in S} |s|$

Now, let's take a look at Bob's first move. By choosing one vertex to color, he divides the tree into several subtrees. For Riley it's only possible to color one of these subtrees, because once he chooses one of these subtrees he can't 'move' to another one, because it's a tree and the only way goes through the vertex chosen by Bob.

So Riley chooses the subtree with the maximal number of vertices. And all the other subtrees goes to Bob.

Thus, for Bob, it's beneficial to choose the vertex with the least weight for first move.

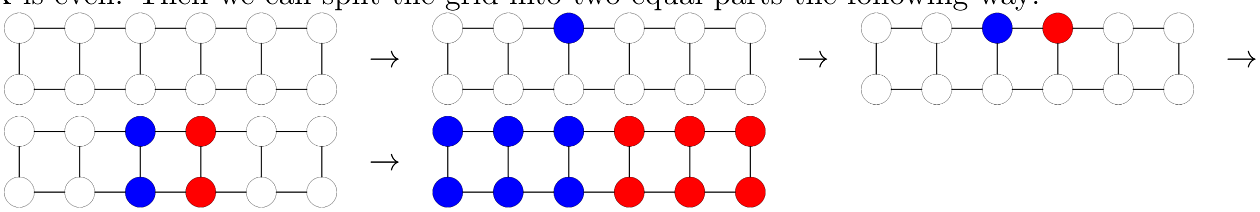
Because Riley 'takes' the subtree with the maximal number of vertices,

$$B(G) = |G| - \min_{v \in G} w_v$$

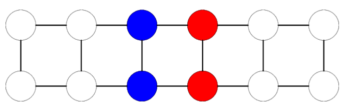
1.2 Grid

G is a grid $M \times K$. Let $M \leqslant K$. Then let's take a look at two different case:

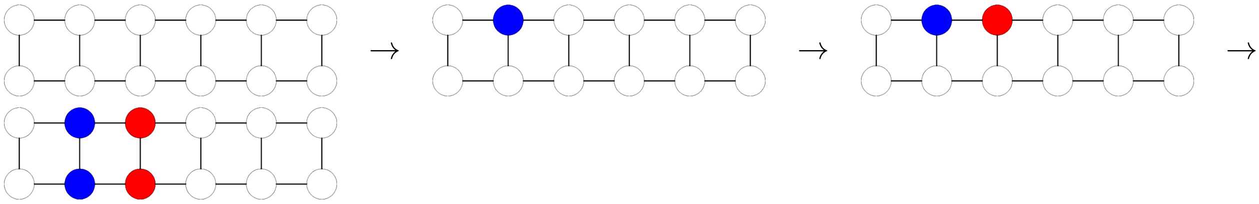
- K is even. Then we can split the grid into two equal parts the following way:



Definition 1.2. *Let **center line** be a shortest line that is placed in-between two edges of the grid, e.g. the following grid has two of them (colored vertical lines).*



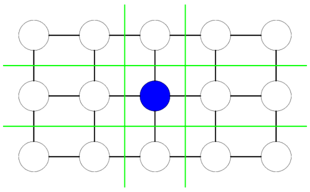
Here appears invisible line splitting the grid in half. Bob's first move must be some vertex on a center line, because either way Riley can border the grid like shown on the next pictures:

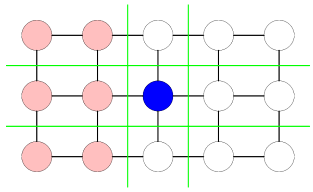


horizontal lines in the previous example could be center lines, but first move on them does not guarantee the optimal number of blue vertices.

Therefore, in this case $B(G) = min(M, K) \cdot \dfrac{max(M, K)}{2}$, where $max(M, K)$ is even

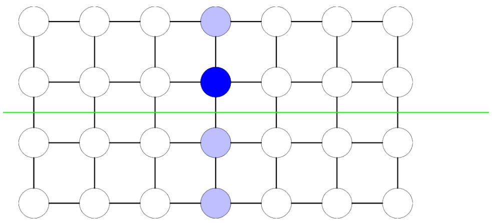
- K is odd.
  - M is odd. Then there's a center vertex. Bob's first move must be this vertex, i'll explain later why. We can draw four lines that border the grid into parts that include the vertex and the parts that don't.

    By acting symmetrically Riley can border one of these areas. We know that M *and* K are odd, then $\exists m, k : M = 2m + 1 \quad K = 2k + 1$.
    Areas of the areas are $(2m+1)\cdot(k+1)$ & $(2k+1)\cdot(m+1)$, i.e. $2mk+k+2m+1$ & $2mk+m+2k+1$.
    $M \leqslant K \Rightarrow m \leqslant k \Rightarrow m + 2mk + m + k + 1 \leqslant k + 2mk + m + k \Rightarrow$
    $\Rightarrow (2m + 1)(k + 1) \leqslant (2k + 1)(m + 1)$, so Riley will choose to take one of $2mk + k$ areas.
    if first Bob's move was not the central vertex, then several areas would expand, increasing the gain of Riley.

    Therefore $B(G) = min(M, K) \cdot \left\lceil \dfrac{max(M, K)}{2} \right\rceil$
  - M is even. Then there's a single center line. Bob's first move must be on this center line.

    And after his move Riley can make his moves symmetrically, taking the other half.
    If bob's first move was somewhere not on the center line, then Riley could border the grid so that he gets most of area.

    Therefore $B(G) = \left\lceil \dfrac{min(M, K)}{2} \right\rceil \cdot max(M, K)$, where $min(M, K)$ is even.

Let's sum it up using the fact that $M, \leqslant K$ and make some inequalities:

- K is even $\quad B(G) = M \cdot \left\lceil \dfrac{K}{2} \right\rceil = M \cdot \dfrac{K}{2} = \dfrac{M}{2} \cdot K \leqslant \left\lceil \dfrac{M}{2} \right\rceil \cdot K$
- K is odd:

– M is even $\quad B(G) = \left\lceil \frac{M}{2} \right\rceil \cdot K = \frac{M}{2} \cdot K = M \cdot \frac{K}{2} \leqslant M \cdot \left\lceil \frac{K}{2} \right\rceil$

– M is odd $\quad B(G) = M \cdot \left\lceil \frac{K}{2} \right\rceil \leqslant \left\lceil \frac{M}{2} \right\rceil \cdot K$

Proof. M, K are odd $\Rightarrow \begin{cases} M = 2m+1 & \left\lceil \frac{M}{2} \right\rceil = m+1 \\ K = 2k+1 & \left\lceil \frac{K}{2} \right\rceil = k+1 \end{cases} \quad M \leqslant K \Rightarrow m \leqslant k$

$m \leqslant k \Rightarrow 2mk + k + 2m + 1 \leqslant 2mk + 2k + m + 1 \Rightarrow (2m+1)(k+1) \leqslant (m+1)(2k+1) \Rightarrow$
$M \left\lceil \frac{K}{2} \right\rceil \leqslant \left\lceil \frac{M}{2} \right\rceil \cdot K$ ☐

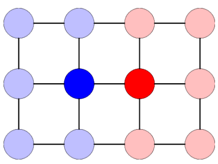
We can conclude, that $B(G) = min\left(\left\lceil \frac{M}{2} \right\rceil \cdot K \;\; M \cdot \left\lceil \frac{K}{2} \right\rceil \right)$

1.3 Torus Grid

Since all vertices on a blank torus grid are completely identical Bob can start his game with any vertex.

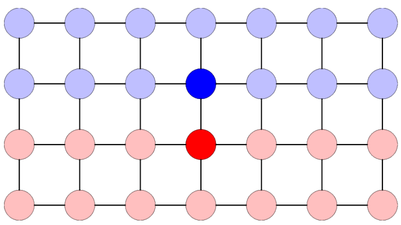
Let's consider the same cases. Suppose that G is a torus grid $M \times K$, where $M \leqslant K$:

- K is even. Then there're two center lines. We can draw an invisible splitting line between them. Riley can make his moves symmetrically to Bob's moves. So Riley can guarantee that he will get at least half of the grid. So it's maximum that Bob can get. $B(G) = M \cdot \frac{K}{2}$

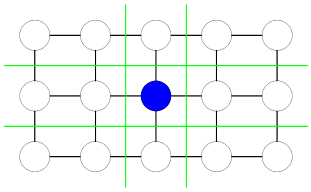
  

- K is odd

  – M is even. Then there's a line splitting the grid into two equal halves. And again Riley can act symmetrically

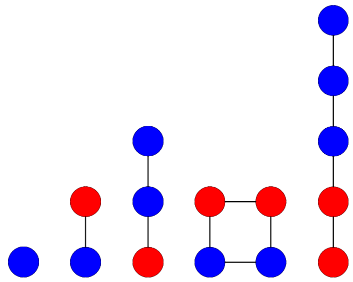
  

  – M is odd. There's a center vertex. Bob can start from it. As in Grid case we can draw four lines for symmetry Riley can act symmetrically again, but there's a line that symmetries to itself. If they will do their best to save their half and at the same time get that self-symmetrised line, they will split the graph with $\left\lceil \frac{n}{2} \right\rceil \rightarrow$ Bob and $\left\lfloor \frac{n}{2} \right\rfloor \rightarrow$ Riley.

2 Upper and lower bounds for $M_B(n)$

Let's first consider $1 \leqslant n \leqslant 5$:



$n = 1$ there's only one 1-vertexed graph and $B(G) = 1$, so $M_B(n) = 1$

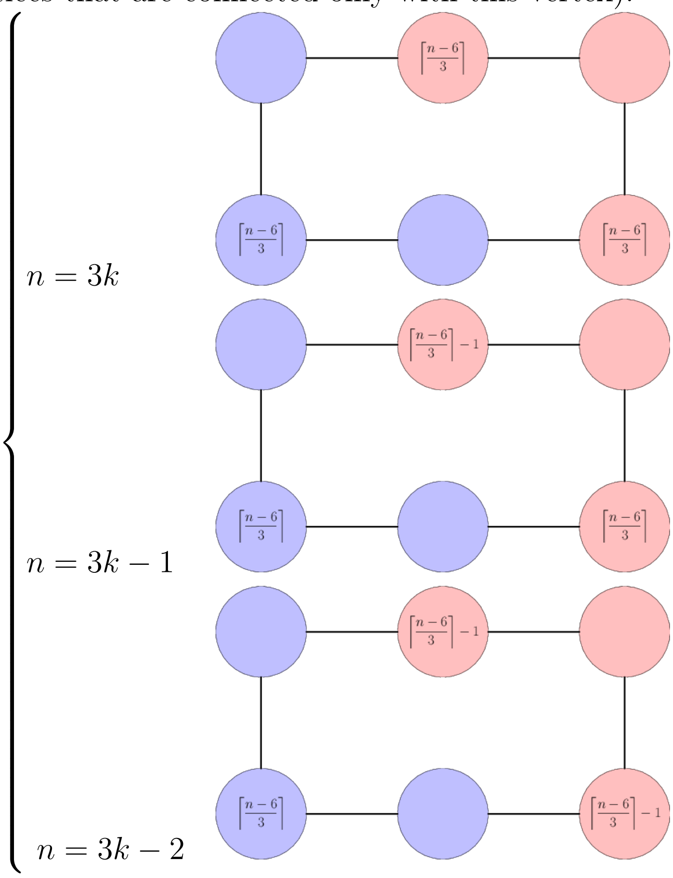
$n = 2$ $M_B(n) = 1$

$n = 3$ $M_B(G) = 2$. if all vertices are connected, then Bob can color any vertex for first move. Riley choose any of the left ones and Bob colors third one. In cases where two vertices are not linked there's a center vertex for Bob's first move and Riley can get only one of the remaining. And finally bob colors another one. So $B(G) = 2$ for all cases.

$n = 4$ $M_B(G) = 2$. You can't make $B(G)$ be 1, because always exists vertex with two connections. Riley can 'block' one but not all two, so $B(G) \geqslant 2 \Rightarrow M_B(n) = 2$

$n = 5$ $M_B(G) = 3$. It simply can be shown that there's no 5-vertexed graph with $B(G) < 3$

for the following n-s let's consider these three graphs (numbers on vertices means number of invisible vertices that are connected only with this vertex):



for $n = 3k$ in this graph $B(G) = 2 + k - 2 = k = \dfrac{n}{3}$

for $n = 3k + 1$ $\quad B(G) = \left\lceil \dfrac{n-6}{3} \right\rceil + 2 = \left\lceil \dfrac{n}{3} \right\rceil - 2 + 2 = \left\lceil \dfrac{n}{3} \right\rceil$

for $n = 3k + 2 \quad B(G) = \left\lceil \frac{n}{3} \right\rceil$

Therefore: $\begin{cases} M_B(n) \leqslant \left\lceil \frac{n}{2} \right\rceil, n < 6 \\ M_B(n) \leqslant \left\lceil \frac{n}{3} \right\rceil, n \geqslant 6 \end{cases}$

3

New Rules

For the next 3 problems consider the following modification of the game rules. Instead of choosing which vertices to color, each of the players has a pawn (blue for Bob and red for Riley), which can move from vertex to vertex connected with an edge. Pawn can't step on a vertex with the opposite color. Pawn can step back on a vertex colored in its color.

Since the game (with the new rules) can continue indefinitely, for this case let $B'(G)$ be the maximal number of blue vertices Bob can guarantee in a finite amount of moves and $R'(G)$ be the maximal number of red vertices Riley can guarantee in a finite amount of moves.

Other rules are the same:

1. 1st move – any white vertex
2. there's no white vertices – end of the game
3. you must step somewhere on your move unless there are no possible moves

4 Value of $B'(G)$ and $R'(G)$

4.1 Grid

Let G be a grid $M \times K$.

- M is even or K is even. Grid can be split into two halves by drawing a certain line. Then Riley can guarantee at least half for him by moving symmetrically, so it's maximal gain possible for Bob. $B(G) = \frac{M \cdot K}{2} = \left\lceil \frac{M \cdot K}{2} \right\rceil$
- M and K are both odd. Then there's a central vertex. If Bob's 1st move is not the central vertex, he gets less than half. If he make his first move right, he gets $\left\lceil \frac{|G|}{2} \right\rceil = \left\lceil \frac{M \cdot K}{2} \right\rceil$ (considering the case, when both players make their moves beneficially for themselves).

Therefore $B(G) = \left\lceil \frac{M \cdot K}{2} \right\rceil$

4.2 Torus Grid

Let G be a torus grid $M \times K$

- M is even or K is even. Then they can step symmetrically like in (4.1)