1A

Machine Learning II

ID: 5684926 Tristan Scheidemann

1A-1.

Let X represent the probability of X phone calls and Y be the waiting time between different calls. In this case X would be Poisson and Y exponentially distributed.

1A-2.

We have the following constraints:

Given the above restrictions R, we have to prove / disprove:

$$R \Longrightarrow P(B,C|X) = P(B|X) \cdot P(C|X)$$

Solution: Transform the joint probability $\mathrm{P}(B,C|X)$ into
$P(B,C|X) = P(C|B,X) \cdot P(B|X)$.

Excluding the degenerate case $P(B|X) = 0$, this leads to:

$$\begin{aligned}
& P(B,C|X) = P(B|X) \cdot P(C|X) \\
\Leftrightarrow\quad & P(C|B,X) \cdot P(B|X) = P(B|X) \cdot P(C|X) \\
\Leftrightarrow\quad & P(C|B,X) = P(C|X) \\
\Leftrightarrow\quad & \frac{P(C,B,X)}{P(B,X)} = P(C|X).
\end{aligned}$$

The above only holds iff $P(C|B,X) = P(C|X)$, which is generally not true, so conditional independence of A,B and A,C is not sufficient for transitivity.

1A-4.

Definition of the events:
$G = Person\ is\ guilty,$
$T = Person\ passes\ the\ test.$

(i) The negations $\bar{T}, \bar{G}$ can be read as *not*.

$$P(G|\bar{T}) = \frac{P(\bar{T}|G) \cdot P(G)}{P(\bar{T})} = \frac{\frac{5}{6} \cdot \frac{1}{3}}{\frac{7}{18}} = \frac{5}{7},$$

with

$$P(\bar{T}|G) = \frac{5}{6},$$

$$P(G) = \frac{1}{3},$$

$$P(\bar{T}) = P(G) \cdot P(\bar{T}|G) + P(\bar{G}) \cdot P(\bar{T}|\bar{G}) = \frac{1}{3} \cdot \frac{5}{6} + \frac{2}{3} \cdot \frac{1}{6} = \frac{7}{18}$$

(ii)

$$P(G|\bar{T},\bar{T}) = \frac{P(\bar{T},\bar{T}|G)}{P(\bar{T},\bar{T})} = \underbrace{\frac{P(\bar{T}|G) \cdot P(\bar{T}|G) \cdot P(G)}{P(\bar{T},\bar{T})}}_{conditional\ independence} = \frac{\left(\frac{5}{6}\right)^2 \cdot \frac{1}{3}}{\frac{1}{3} \cdot \left(\frac{5}{6}\right)^2 + \frac{2}{3} \cdot \left(\frac{1}{6}\right)^2} = 0.\overline{925}.$$

Using the conditional independence of $P(\bar{T},\bar{T}|G)$ and independence of testing $P(\bar{T},\bar{T}|G)$.

1A-5.

$$E[X] = \sum_{i=1}^{6} i \cdot P(X = i) = (1 + 2 + 3) \cdot \frac{1}{12} + (4 + 5) \cdot \frac{1}{6} + 6 \cdot \frac{5}{12} = 4.5.$$

$$Var[X] = E[X^2] - E[X]^2 = \left[(1 + 4 + 9) \cdot \frac{1}{12} + (16 + 25) \cdot \frac{1}{6} + 36 \cdot \frac{5}{12}\right] - 4.5^2 = \frac{271}{12} - 4.5^2$$
$$= 2.\bar{3}\,.$$

$$E[X_1 + E_2] = 2 \cdot E[X_1] = 9.$$

1A-6.

$$\begin{aligned} E[(X - E[X])(Y - E[Y])] &= E\big[XY - XE[Y] - YE[X] + E[X]E[Y]\big] \\ &= E[XY] - E[Y]E[X] - E[X]E[Y] + E[X]E[Y] \\ &= \underbrace{E[XY] - E[X]E[Y]}_{*} \\ &= 0. \end{aligned}$$

$$E[XY] = \int_{y_0}^{y_1} \int_{x_0}^{x_1} xy \cdot f_{X,Y}(x,y)dx\,dy = \int_{y_0}^{y_1} y \int_{x_0}^{x_1} x \cdot \underbrace{f_X(x)f_y(y)}_{independence}\,dx\,dy$$
$$= \int_{y_0}^{y_1} yf_y(y) \int_{x_0}^{x_1} x\,f_X(x)dx\,dy = \int_{y_0}^{y_1} yf_y(y) \cdot E[X]\,dy = E[X] \int_{y_0}^{y_1} yf_y(y)\,dy$$
$$= E[X]E[Y].$$

Utilizing that $* \ E[XY] = E[X]E[Y]$ if X, Y are independent.