1A

Machine Learning II
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1A-1.

Let $X \sim B(1, p)$ and $Y \sim Exp(\lambda)$ with mass $f_X(x) = p^x \cdot (1-p)^{1-x},\ x \in \{0,1\}$
and density $f_Y(y) = \lambda e^{-\lambda y}, with\ y \in [0, \infty)$.

The joint pdf is then given by

$$f_{X,Y}(x,y) = \begin{cases} (1-p) \cdot \lambda e^{-\lambda y} & , \quad iff\ x = 0 \\ p \cdot \lambda e^{-\lambda y} & , \quad iff\ x = 1 \end{cases},$$

which is a mixture of a continuous Y and discrete X.

Proof

$$f_{X,Y}(x,y) = \underbrace{f_X(x) f_Y(y)}_{X,Y\ independence} = \begin{cases} (1-p) \cdot \lambda e^{-\lambda y} & , \quad iff\ x = 0 \\ p \cdot \lambda e^{-\lambda y} & , \quad iff\ x = 1 \end{cases}.$$

Additionally, $f_{X,Y}$ is a valid density function, because:

(i) $f_{X,Y} \geq 0,\ \forall x, y \in \{0,1\} \times \mathbb{R}^+$.

(ii) $f_{X,Y}(x,y) = \int_0^\infty \sum_{i=0}^1 f(x = i, y)\, dy = \int_0^\infty (1-p) \cdot \lambda e^{-\lambda y} dy + \int_0^\infty p \cdot \lambda e^{-\lambda y} dy = 1.$

1A-2.

We have the following constraints:

Given the above restrictions R, we have to prove / disprove:

$R \Longrightarrow P(B, C|X) = P(B|X) \cdot P(C|X)$

Solution: Transform the joint probability $\mathrm{P}(B, C|X)$ into
$P(B, C|X) = P(C|B, X) \cdot P(B|X)$.

Excluding the degenerate case $P(B|X) = 0$, this leads to:

$$
\begin{aligned}
& P(B,C|X) = P(B|X)\cdot P(C|X) \\
\Leftrightarrow\quad & P(C|B,X)\cdot P(B|X) = P(B|X)\cdot P(C|X) \\
\Leftrightarrow\quad & P(C|B,X) = P(C|X) \\
\Leftrightarrow\quad & \frac{P(C,B,X)}{P(B,X)} = P(C|X).
\end{aligned}
$$

The above only holds iff $P(C|B,X) = P(C|X)$, which is generally not true, so conditional independence of A,B and A,C is not sufficient for transitivity.

1A-4.

Definition of the events:
$G = Person\ is\ guilty,$
$T = Person\ passes\ the\ test.$

(i) The negations $\bar{T}, \bar{G}$ can be read as *not*.

$$P(G|\bar{T}) = \frac{P(\bar{T}|G)\cdot P(G)}{P(\bar{T})} = \frac{\frac{5}{6}\cdot\frac{1}{3}}{\frac{7}{18}} = \frac{5}{7},$$

with

$$P(\bar{T}|G) = \frac{5}{6},$$

$$P(G) = \frac{1}{3},$$

$$P(\bar{T}) = P(G)\cdot P(\bar{T}|G) + P(\bar{G})\cdot P(\bar{T}|\bar{G}) = \frac{1}{3}\cdot\frac{5}{6} + \frac{2}{3}\cdot\frac{1}{6} = \frac{7}{18}$$

(ii)

$$P(G|\bar{T},\bar{T}) = \frac{P(\bar{T},\bar{T}|G)}{P(\bar{T},\bar{T})} = \underbrace{\frac{P(\bar{T}|G)\cdot P(\bar{T}|G)}{P(\bar{T},\bar{T})}}_{conditional\ independence} = \frac{\left(\frac{5}{6}\right)^2}{\frac{1}{3}\cdot\left(\frac{5}{6}\right)^2 + \frac{2}{3}\cdot\left(\frac{1}{6}\right)^2} = 0.\overline{925}.$$

Using the conditional independence of $P(\bar{T},\bar{T}|G)$ and independence of testing $P(\bar{T},\bar{T}|G)$.

1A-5.

$$E[X] = \sum_{i=1}^{6} i\cdot P(X=i) = (1+2+3)\cdot\frac{1}{12} + (4+5)\cdot\frac{1}{6} + 6\cdot\frac{5}{12} = 4.5.$$

$$
\begin{aligned}
Var[X] = E[X^2] - E[X]^2 &= \left[(1+4+9)\cdot\frac{1}{12} + (16+25)\cdot\frac{1}{6} + 36\cdot\frac{5}{12}\right] - 4.5^2 = \frac{271}{12} - 4.5^2 \\
&= 2.\bar{3}\,.
\end{aligned}
$$

$E[X_1 + E_2] = 2 \cdot E[X_1] = 9.$

1A-6.

$$
\begin{aligned}
E[(X - E[X])(Y - E[Y])] &= E\big[XY - XE[Y] - YE[X] + E[X]E[Y]\big] \\
&= E[XY] - E[Y]E[X] - E[X]E[Y] + E[X]E[Y] \\
&= \underbrace{E[XY] - E[X]E[Y]}_{*} \\
&= 0.
\end{aligned}
$$

$$
E[XY] = \int_{y_0}^{y_1}\int_{x_0}^{x_1} xy \cdot f_{X,Y}(x,y) dx\, dy = \int_{y_0}^{y_1} y \int_{x_0}^{x_1} x \cdot \underbrace{f_X(x) f_y(y)}_{independence}\, dx\, dy
$$

$$
= \int_{y_0}^{y_1} y f_y(y) \int_{x_0}^{x_1} x\, f_X(x) dx\, dy = \int_{y_0}^{y_1} y f_y(y) \cdot E[X]\, dy = E[X] \int_{y_0}^{y_1} y f_y(y)\, dy
$$

$$
= E[X]E[Y].
$$

Utilizing that $*$ $E[XY] = E[X]E[Y]$ if X, Y are independent.