2020-2021学年第一学期

高数A(上)期中试卷.

1. $\lim\limits_{x\to\infty}\dfrac{|x|}{x}\arctan x = \dfrac{\pi}{2}$

解: $\lim\limits_{x\to+\infty}\dfrac{|x|}{x}\arctan x = \lim\limits_{x\to+\infty}\dfrac{x}{x}\arctan x = 1\cdot\dfrac{\pi}{2} = \dfrac{\pi}{2}$

$\lim\limits_{x\to-\infty}\dfrac{|x|}{x}\arctan x = \lim\limits_{x\to-\infty}\dfrac{-x}{x}\arctan x = (-1)\cdot(-\dfrac{\pi}{2}) = \dfrac{\pi}{2}$

故 $\lim\limits_{x\to\infty}\dfrac{|x|}{x}\arctan x = \dfrac{\pi}{2}$

2. $\lim\limits_{x\to\infty}\left(\dfrac{\sin 2x}{x} + x\sin\dfrac{3}{x}\right) = 3$

解: $\lim\limits_{x\to\infty}\left(\dfrac{\sin 2x}{x} + x\sin\dfrac{3}{x}\right) = \lim\limits_{x\to\infty}\left(\underbrace{\dfrac{1}{x}}_{\text{无穷小量}}\cdot\underbrace{\sin 2x}_{\text{有界量}} + \dfrac{\sin\frac{3}{x}}{\frac{1}{x}}\right)$

$\sin\dfrac{3}{x}\sim\dfrac{3}{x}\ (x\to\infty)$.

$= 0 + \lim\limits_{x\to\infty}\dfrac{\sin\frac{3}{x}}{\frac{1}{x}} = 3$

3. 设 $0<\alpha<1$, $\lim\limits_{n\to\infty}\left[(n+n^{\alpha})^{1-\alpha} - n^{1-\alpha}\right] = 1-\alpha$

解: 原式 $= \lim\limits_{n\to\infty}\left\{\left[n(1+n^{\alpha-1})\right]^{1-\alpha} - n^{1-\alpha}\right\}$

$= \lim\limits_{n\to\infty} n^{1-\alpha}\left[(1+n^{\alpha-1})^{1-\alpha} - 1\right] \xlongequal{n=\frac{1}{t}} \lim\limits_{t\to 0}\dfrac{(1+t^{1-\alpha})^{1-\alpha}-1}{t^{1-\alpha}}$

$= \lim\limits_{t\to 0}\dfrac{(1-\alpha)t^{1-\alpha}}{t^{1-\alpha}} = 1-\alpha.$

$0<\alpha<1$, 故 $t^{1-\alpha}\to 0\ (t\to 0)$.
有 $(1+t^{1-\alpha})^{1-\alpha}-1\sim(1-\alpha)\cdot t^{1-\alpha}\ (t\to 0)$.

4. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x^2\arctan x}-1}{(1-\cos 2x)\sin x} = \dfrac{1}{4}$

解: 原式 $= \lim\limits_{x\to 0}\dfrac{\frac{1}{2}x^2\arctan x}{\frac{1}{2}\cdot(2x)^2\cdot x} = \lim\limits_{x\to 0}\dfrac{x^2\cdot x}{4x^2\cdot x} = \dfrac{1}{4}$

用到 $(1+x)^{\alpha}-1\sim\alpha x$. $1-\cos x\sim\frac{1}{2}x^2$, $\sin x\sim\arctan x\sim x$, $(x\to 0)$

5. $\lim\limits_{x\to 0}\dfrac{\left(\frac{1+\cos x}{2}\right)^x-1}{x^3}=-\frac{1}{4}$

解：原式 $=\lim\limits_{x\to 0}\dfrac{e^{x\ln\frac{1+\cos x}{2}}-1}{x^3}=\lim\limits_{x\to 0}\dfrac{x\ln\frac{1+\cos x}{2}}{x^3}$

$=\lim\limits_{x\to 0}\dfrac{\ln(1+\cos x)-\ln 2}{x^2}\overset{\text{洛必达}}{=}\lim\limits_{x\to 0}\dfrac{\frac{1}{1+\cos x}\cdot(-\sin x)}{2x}=-\frac{1}{4}$

6. $\lim\limits_{n\to\infty}\left[\dfrac{2}{2^n+1}+\dfrac{2^2}{2^n+2}+\cdots+\dfrac{2^n}{2^n+n}\right]=2$

解：$2^n\le 2^n+k\le 2^n+n$. $(k=1,2,\cdots,n)$. 则有

$\dfrac{1}{2^n+n}\le\dfrac{1}{2^n+k}\le\dfrac{1}{2^n}$, $\sum\limits_{k=1}^{n}\dfrac{2^k}{2^n+n}\le\sum\limits_{k=1}^{n}\dfrac{2^k}{2^n+k}\le\sum\limits_{k=1}^{n}\dfrac{2^k}{2^n}$

$\lim\limits_{n\to\infty}\dfrac{2+2^2+\cdots+2^n}{2^n+n}=\lim\limits_{n\to\infty}\dfrac{\frac{2(1-2^n)}{1-2}}{2^n+n}=2$

$\lim\limits_{n\to\infty}\dfrac{2+2^2+\cdots+2^n}{2^n}=\lim\limits_{n\to\infty}\dfrac{\frac{2(1-2^n)}{1-2}}{2^n}=2$

故由夹逼定理可知原式 $=2$

7. 设 $f(x)=\begin{cases}\dfrac{\sqrt[3]{1-ax}-1}{x}, & x>0\\ (1-x)^{\frac{1}{x}}, & x<0\end{cases}$，$x=0$ 为 $f(x)$ 的可去间断点

则常数 $a=-\frac{3}{e}$.

解：$\lim\limits_{x\to 0^-}f(x)=\lim\limits_{x\to 0^-}(1-x)^{\frac{1}{x}}=\lim\limits_{x\to 0^-}\left\{[1+(-x)]^{-\frac{1}{x}}\right\}^{\frac{-x}{x}}=\frac{1}{e}$.

$\lim\limits_{x\to 0^+}f(x)=\lim\limits_{x\to 0^+}\dfrac{\sqrt[3]{1-ax}-1}{x}=\lim\limits_{x\to 0^+}\dfrac{\frac{1}{3}(-ax)}{x}=-\frac{a}{3}$.

故由 $\frac{1}{e}=-\frac{a}{3}$ 得 $a=-\frac{3}{e}$

8. 设 $f(x)=\sin x$, $g(x)=\begin{cases} x-\pi, & x\le 0 \\ x+\pi, & x>0 \end{cases}$, 则 $f(g(x))$ 在点 $x=0$ 连续

解: $f(g(x))=\sin(g(x))=\begin{cases} \sin(x-\pi), & x\le 0 \\ \sin(x+\pi), & x>0. \end{cases}$

$f(g(0))=\sin(0-\pi)=0$, $\lim\limits_{x\to 0^-} f(g(x))=\lim\limits_{x\to 0^-}\sin(x-\pi)=0$

$\lim\limits_{x\to 0^+} f(g(x))=\lim\limits_{x\to 0^+}\sin(x+\pi)=0$. 故 $\lim\limits_{x\to 0} f(g(x))=f(g(0))$

即在 $x=0$ 处连续.

9. 当 $x\to 0$ 时 $\sqrt{x\sin x+\cos x}-1 \sim \ln(1+kx^2)$, 则 $k=\frac{1}{4}$

解: $\sqrt{x\sin x+\cos x}-1 \sim \ln(1+kx^2)\sim kx^2$. ($x\to 0$). 故

$k=\lim\limits_{x\to 0}\frac{\sqrt{x\sin x+\cos x}-1}{x^2}=\lim\limits_{x\to 0}\frac{[1+(x\sin x+\cos x-1)]^{\frac{1}{2}}-1}{x^2}$

$=\lim\limits_{x\to 0}\frac{\frac{1}{2}(x\sin x+\cos x-1)}{x^2} \xlongequal{\text{洛必达}} \lim\limits_{x\to 0}\frac{\frac{1}{2}\cdot(\sin x+x\cos x-\sin x)}{2x}=\frac{1}{4}$

10. 设 $f(x)$ 在点 $x=0$ 处可导 $f(0)=0$, $f'(0)=1$, 则 $\lim\limits_{x\to 0}(1+2x)^{\frac{1}{f(x)}}=e^2$

解: 原式 $=\lim\limits_{x\to 0}\left[(1+2x)^{\frac{1}{2x}}\right]^{\frac{2x}{f(x)}}=e^{\lim\limits_{x\to 0}\frac{2x}{f(x)}}$

$=e^{\lim\limits_{x\to 0}\frac{2x}{f(0)+f'(0)x+o(x)}}=e^2$

11. 设 $y=\sqrt[3]{x\sin x\sqrt{(1+x^2)}e^x}$, 则 $y'=\frac{1}{3}\sqrt[3]{x\sin x\sqrt{(1+x^2)}e^x}\left(\frac{1}{x}+\cot x+\frac{x}{1+x^2}+\frac{1}{2}\right)$

解: $\ln y=\frac{1}{3}\ln(x\sin x)+\frac{1}{6}\ln[(1+x^2)e^x]$

$\ln y=\frac{1}{3}\left[\ln x+\ln\sin x+\frac{1}{2}\ln(1+x^2)+\frac{1}{2}x\right]$ 两边对 x 求导

$\frac{1}{y}y'=\frac{1}{3}\left(\frac{1}{x}+\frac{1}{\sin x}\cdot\cos x+\frac{1}{2}\frac{1}{1+x^2}\cdot 2x+\frac{1}{2}\right)$. 整理即可.

12. $f(x)=\begin{cases}\dfrac{\ln(1+x)}{x}, & x\neq 0\\ 1, & x=0\end{cases}$，则 $f'(x)=\begin{cases}\dfrac{x-(1+x)\ln(1+x)}{x^2(1+x)}, & x\neq 0\\ -\dfrac{1}{2} & x=0\end{cases}$

解：当 $x\neq 0$ 时，$f'(x)=\left(\dfrac{\ln(1+x)}{x}\right)'=\dfrac{\frac{1}{1+x}\cdot x-\ln(1+x)\cdot 1}{x^2}=\dfrac{x-(1+x)\ln(1+x)}{x^2(1+x)}$

当 $x=0$ 时，$f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\dfrac{\frac{\ln(1+x)}{x}-1}{x}$

$=\lim\limits_{x\to 0}\dfrac{\ln(1+x)-x}{x^2}=\lim\limits_{x\to 0}\dfrac{x-\frac{x^2}{2}+o(x^2)-x}{x^2}=-\dfrac{1}{2}$

13. 设 $y=2x^3+\ln x$ 的反函数为 $x=\varphi(y)$，则 $\varphi'(2)=\dfrac{1}{7}$

解：当 $y=2$ 时，$x=1$，故 $\varphi'(2)=\dfrac{1}{f'(1)}=\dfrac{1}{6x^2+\frac{1}{x}\big|_{x=1}}=\dfrac{1}{7}$

14. 设 $f(x)=(x^2+1)\cos 2x$，则 $f^{(4)}(0)=-32$

解：$f(x)=x^2\cos 2x+\cos 2x$，$\cos 2x=1-\dfrac{1}{2!}(2x)^2+\dfrac{1}{4!}(2x)^4+o(x^4)$

故 $f(x)=x^2(1-2x^2+o(x^2))+(1-2x^2+\frac{2}{3}x^4+o(x^4))$

$=1-x^2-\frac{4}{3}x^4+o(x^4)$。由 $f^{(n)}(x_0)=n!\,a_n$ 可得

故 $f^{(4)}(0)=4!\,a_4=4!\cdot\left(-\frac{4}{3}\right)=-32$.

15. 设 $f(x)=\begin{cases}\cos x, & x\geq 0\\ e^{ax^2}, & x<0\end{cases}$，$f''(0)$ 存在，则 $a=-\dfrac{1}{2}$

解：$f''(0)$ 存在，故 $f''_-(0)$ 和 $f''_+(0)$ 都存在且相等.

$\cos x=1-\frac{1}{2}x^2+o(x^2)$，$e^{ax^2}=1+ax^2+o(x^2)$

$f''_-(0)=2!\cdot(a)=2!\cdot\left(-\frac{1}{2}\right)=f''_+(0)$ 故 $a=-\frac{1}{2}$.

16. $x\cos y+y-\pi=0$ 上点 $(0,\pi)$ 处的切线方程. $y=x+\pi$

解: 两边关于 x 求导. $\cos y+x\cdot(-\sin y)\cdot y'+y'=0$. 代入 $x=0$, $y=\pi$.

则 $\cos\pi+0\cdot(-\sin\pi)\,y'(0)+y'(0)=0$ 得 $y'(0)=-\cos\pi=1$.

故切线方程为 $y-\pi=1\cdot(x-0)$. 即 $y=x+\pi$

17. $\begin{cases}x=2t+\ln t\\ y=t^2+\ln t\end{cases}$, 则 $\left.\dfrac{d^2y}{dx^2}\right|_{x=2}=\dfrac{2}{9}$

解: $\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=\dfrac{d}{dx}\left(\dfrac{y'(t)}{x'(t)}\right)=\dfrac{d}{dx}\dfrac{2t+\frac{1}{t}}{2+\frac{1}{t}}=\dfrac{d}{dx}\dfrac{2t^2+1}{2t+1}$

$=\dfrac{\frac{d}{dt}\frac{2t^2+1}{2t+1}}{\frac{dx}{dt}}=\dfrac{(4t^2+4t-2)t}{(2t+1)^3}$. 故 $\left.\dfrac{d^2y}{dx^2}\right|_{x=2}=\dfrac{2}{9}$ ($x=2$ 时 $t=1$).

18. $y=\ln(x+\sqrt{1+x^2})$. 则 $dy|_{x=1}=\dfrac{1}{\sqrt{2}}dx$

解: $dy|_{x=1}=y'(1)\cdot dx=\left.\dfrac{1}{\sqrt{1+x^2}}\right|_{x=1}dx=\dfrac{1}{\sqrt{2}}dx$.

19. $f(x)=x^3$, $g(x)=x^2$ 在 $[0,1]$ 使用柯西中值定理, 则 $\xi=\dfrac{2}{3}$

解: $\dfrac{f'(\xi)}{g'(\xi)}=\dfrac{f(1)-f(0)}{g(1)-g(0)}$ 即 $\dfrac{3\xi^2}{2\xi}=\dfrac{1^3-0^3}{1^2-0^2}$ 得 $\xi=\dfrac{2}{3}$

20. $\lim\limits_{x\to0}\dfrac{e^{\tan x}-e^{\sin x}}{e^{\sin x}-e^x}=-3$

解: 原式 $=\lim\limits_{x\to0}\dfrac{e^{\sin x}}{e^x}\cdot\dfrac{e^{\tan x-\sin x}-1}{e^{\sin x-x}-1}=\lim\limits_{x\to0}\dfrac{1}{1}\cdot\dfrac{\tan x-\sin x}{\sin x-x}$.

$=\lim\limits_{x\to0}\dfrac{\sin x(1-\cos x)}{\cos x(\sin x-x)}=\lim\limits_{x\to0}\dfrac{x\cdot\frac{1}{2}x^2}{1\cdot(\sin x-x)}=\lim\limits_{x\to0}\dfrac{\frac{1}{2}x^3}{x-\frac{1}{6}x^3-x+o(x^3)}$

$=-3$.