1 Trigo Formulae

- $\sin^2\theta+\cos^2\theta=1$, $\sin 2\theta=2\sin\theta\cos\theta$
- $\sin(A\pm B)=\sin A\cos B\pm\sin B\cos A$
- $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$
- $\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$, $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$
- $\cos 2\theta=cos^2\theta-sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta$
- $\sin P+\sin Q=2\sin\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$
- $\sin P-\sin Q=2\cos\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$
- $\cos P+\cos Q=2\cos\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$
- $\cos P-\cos Q=-2\sin\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$
- $a^2=b^2+c^2-2bc\cos\theta$ and $\frac{a}{\sin a}=\frac{b}{\sin b}$

2 Functions and Limits

Existence of Limits

$\lim_{x\to a}f(x)$ only exists when:

- $\lim_{x\to a^-}f(x)=\lim_{x\to a^+}f(x)$ (limit from left = right)
- For $a=\infty$ or $-\infty$, only if $f(x)$ **does not** oscillate

Rules of Limits

1. $\lim_{x\to a}(f\pm g)(x)=\lim_{x\to a}f(x)\pm\lim_{x\to a}g(x)$
2. $\lim_{x\to a}f(x)g(x)=\lim_{x\to a}f(x)\lim_{x\to a}g(x)$
3. $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$ provided $\lim_{x\to a}g(x)\neq 0$
4. $\lim_{x\to a}kf(x)=k\lim_{x\to a}f(x)$

Continuity

f is continuous at point $a\Leftrightarrow\lim_{x\to a}f(x)=f(a)$

L'Hôpital's Rule

Suppose:

1. f and g are differentiable
2. $f(a)=g(a)=0$
3. $g'(x)\neq 0$ for all $x\in I\setminus a$

Then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$

- Use L'Hôpital's Rule for $\frac{0}{0}$ and $\frac{\infty}{\infty}$ forms.
- Common: $\lim_{x\to\frac{\pi}{2}^-}(\sin x)^{\tan x}=\lim_{x\to\frac{\pi}{2}^-}e^{\ln(\sin x)^{\tan x}}$

$=e^{\lim_{x\to\frac{\pi}{2}^-}\tan x\ln(\sin x)}=e^{\lim_{x\to\frac{\pi}{2}^-}\frac{\ln(\sin x)}{\cot x}}$ (now in $\frac{0}{0}$ form)

- Tips:

1. Convert $0\cdot\infty,\infty-\infty$ by algebra manip
2. Convert $1^\infty,\infty^0,0^0$ by first taking ln

3 Derivative

The derivative of f at point a is $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, denoted by $f'(a)$ provided the limit exists.

$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\left.\frac{dy}{dx}\right|_{x=a}$

$f'(a)$ = slope of tangent at pt a

Some properties

- $f'(a)$ exists $\Rightarrow f(x)$ is smooth ($\therefore$ continuous) at a
- $f'(a)$ does not exist at **discontinuity, corner,** and **vertical tangent.**

Since derivative is limit, if lim from left ≠ right, then $f'(a)$ does not exist.

Formulae

Function	Derivative
$(f(x))^n$	$nf'(x)f(x)^{n-1}$
$\sin f(x)$	$f'(x)\cos f(x)$
$\cos f(x)$	$-f'(x)\sin f(x)$
$\tan f(x)$	$f'(x)\sec^2 f(x)$
$\cot f(x)$	$-f'(x)\csc^2 f(x)$
$\sec f(x)$	$f'(x)\sec f(x)\tan f(x)$
$\csc f(x)$	$-f'(x)\csc f(x)\cot f(x)$
$a^f(x)$	$f'(x)a^{f(x)}\ln a$

Function	Derivative
k	0
$e^f(x)$	$f'(x)e^{f(x)}$
$\log_a f(x)$	$\frac{f'(x)}{f(x)\ln a}$
$\ln f(x)$	$\frac{f'(x)}{f(x)}$

Function	Derivative
$\sin^{-1}f(x)$	$\frac{f'(x)}{\sqrt{1-f(x)^2}}$
$\cos^{-1}f(x)$	$-\frac{f'(x)}{\sqrt{1-f(x)^2}}$
$\tan^{-1}f(x)$	$\frac{f'(x)}{1+f(x)^2}$

Rules of Differentiation

- $(kf)'(x)=kf'(x)$
- $(f\pm g)'(x)=f'(x)\pm g'(x)$
- $\frac{d}{dx}uv=u\frac{dv}{dx}+v\frac{du}{dx}$
- $\left(\frac{f}{g}\right)'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
- $\frac{d}{dx}f(g(x))=f'(g(x))g'(x)$ or $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

Parametric Differentiation

Given $\begin{cases}y=u(t)\\x=v(t)\end{cases}$, we have $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

- **Second derivative**
$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$ then do implicit differentiation w.r.t x
- **Polar equation** ($r=a\theta$): $x=r\cos\theta, y=r\sin\theta$

Implicit Differentiation

Differentiate w.r.t. to var, then multiply by $\frac{d<var>}{dx}$ Common: $y=x^x\iff\ln y=x\ln x$

Higher Order Derivatives

The n-th derivative is denoted by $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$

Maxima and Minima

- $f(c)$ **is Local Maximum** if $f(c)\geq f(x)$ for x near c
- $f(c)$ **is Local Minimum** if $f(c)\leq f(x)$ for x near c
- $f(c)$ **is abs maximum** if $f(c)\geq f(x)\forall x\in$ domain
- $f(c)$ **is abs minimum** if $f(c)\leq f(x)\forall x\in$ domain
- **Critical Point** :

Let f be a function with domain D. An interior point (not end-point) c in D is called a **Critical Point** of f if $f'(c)=0$ of $f'(c)$ does not exist.

- **Method to find extreme values of** f :

Check critical points of f, end-points of domain D

Method to Find Local Extreme values

A function may not have a local extreme at a critical pt. Check using 1st/2nd derivative tests.

- **1st Derivative Test** :

Assume $c\in(a,b)$ is a critical point of f

1. $f'(x)>0$ for $x\in(a,c)$ and $f'(x)<0$ for $x\in(c,b)$, then f is a **local maximum**
2. $f'(x)<0$ for $x\in(a,c)$ and $f'(x)>0$ for $x\in(c,b)$, then f is a **local minimum**

- **2nd Derivative Test** :

$f'(c)=0\begin{cases}f''(c)<0\iff f\text{ has local max at }c\\f''(c)>0\iff f\text{ has local min at }c\end{cases}$

Note: if $f'(c)=0$ and $f''(c)=0$ then 2nd derivative test fails. Use 1st derivative test.

Method to Find Absolute Extreme Values

1. Find all critical points c in the interior
2. Evaluate $f(c)$, where c is a critical or end point
3. The largest and smallest of these values will be abs max & min respectively

Increasing and Decreasing Functions

Test for Monotonic Functions ($f:I$ (interval) $\to\mathbb{R}$):

- $f'(x)>0$ for any x in $I\Rightarrow f$ is **increasing** on I
- $f'(x)<0$ for any x in $I\Rightarrow f$ is **decreasing** on I

Concavity

$\begin{cases}f''(x)<0\Leftrightarrow f'(x)\text{ is decreasing}\Leftrightarrow\text{Concave Down}\\f''(x)>0\Leftrightarrow f'(x)\text{ is increasing}\Leftrightarrow\text{Concave Up}\end{cases}$

Points of Inflection

Let $f:I\to\mathbb{Z}$ and $c\in I$.

c is a pt of inflection of f if f is continuous at c and the concavity of f changes at c.

In another word: c is pt of inflection $\to f''(c)=0$ (but not the reverse – c is a pt of inflection only if $f''(c)$ crosses from (+) to (-) and vice versa.)

4 Integration

Indefinite Integral

Denoted by $\int f(x)dx=F(x)+C$

Geometrical Interpretation

All curves $y=F(x)+C$ s.t. their slopes at x are $f(x)$

Rules of Indefinite Integration

1. $\int kf(x)dx=k\int f(x)dx$
2. $\int -f(x)dx=-\int f(x)dx$
3. $\int[f(x)\pm g(x)]dx=\int f(x)dx\pm\int g(x)dx$

Integral Formulae

Function	Integral
$\int\cot x dx$	$\ln(\sin x)+C$
$\int\sec x\tan x dx$	$\sec x+C$
$\int\csc x\cot x dx$	$\csc x+C$
$\int\sec^2 x dx$	$\tan x+C$
$\int\csc^2 x dx$	$-\cot x+C$

$\int x^n dx$	$\frac{x^{n+1}}{n+1}+C, n\neq -1, n$ rational
$\int\frac{1}{\sqrt{a^2-x^2}}dx$	$\sin^{-1}(\frac{x}{a})+C$
$\int\frac{1}{a^2+x^2}dx$	$\frac{1}{a}\tan^{-1}(\frac{x}{a})+C$
$\int 1dx=\int dx$	$x+C$
$\int e^x dx$	e^x+C
$\int a^x dx$	$\frac{a^x}{\ln a}$
$\int\ln x dx$	$x\ln x-x+C$
$\int\frac{1}{x}dx$	$\ln x+C$
$\int\sin kx dx$	$-\frac{\cos kx}{k}+C$
$\int\cos kx dx$	$\frac{\sin kx}{k}+C$
$\int\tan x dx$	$\ln(\sec x)+C$ **or** $-\ln(\sec x)+C$

Function	Integral
$\int\tan^2 x dx$	$\tan x-x+C$
$\int\sec x dx$	$\ln(\sec x+\tan x)+C$
$\int\csc x dx$	$\ln(\csc x-\cot x)+C$

Riemann (Definite) Integrals

Riemann sum on f on $[a,b]\approx\sum_{k=1}^n f(c_k)\Delta x$

Exact area $=\lim_{n\to\infty}\sum_{k=1}^n f(c_k)\Delta x$

Riemann Integral of f over $[a,b]$:

$\int_a^b f(x)dx=\lim_{n\to\infty}\sum_{k=1}^n f(c_k)\Delta x$

Rules of Definite Integrals

1. $\int_a^a f(x)dx=0$, $\int_a^b kf(x)dx=k\int_a^b f(x)dx$
2. $\int_a^b f(x)dx=-\int_b^a f(x)dx$
3. $\int_a^b[f(x)\pm g(x)]=\int_a^b f(x)\pm\int_a^b g(x)$
4. If $f(x)\geq g(x)$ on $[a,b]$, then $\int_a^b f(x)dc\geq\int_a^b g(x)dx$
If $f(x)\geq 0$ on $[a,b]$, then $\int_a^b f(x)dx\geq 0$
5. If f is continuous on the interval joining a,b and c,
then $\int_a^b f(x)dx+\int_b^c f(x)dx=\int_a^c f(x)dx$

Fundamental Thm of Calculus

$F'(x)=f(x)$ If F is an antiderivative of f on $[a,b]$, then $\int_a^b F'(x)dx=\int_a^b f(x)dx=F(b)-F(a)$

x‘ Let f be continuous on $[a,b]$. Then $\frac{d}{dx}\int_a^x f(t)dt=f(x)$

Note the 2 x's: on $\frac{d}{dx}$ and $\int_a^x$ and $f(t)$ is indep of x

1. $\frac{d}{dx}\int_0^2 t^2dt=0$, $\frac{d}{dx}\int_0^x\sin\sqrt{t}dt=\sin\sqrt{x}$
2. $\frac{d}{dx}\left(\int_1^{x^4}\frac{t}{\sqrt{t^3+2}}dt\right)=\frac{d}{dx^4}\left(\int_1^{x^4}\frac{t}{\sqrt{t^3+2}}dt\right)\frac{dx^4}{dx}$
$=\frac{x^4}{\sqrt{(x^4)^3+2}}(4x^3)=\frac{4x^7}{\sqrt{x^12+2}}$
3. $\frac{d}{dx}\int_x^a f(t)dt=-\frac{d}{dx}\int_a^x f(t)dt$
4. $\frac{d}{dx}\int_{x^2}^{x^4}f(t)dt=\frac{d}{dx}\int_a^{x^4}f(t)dt-\frac{d}{dx}\int_a^{x^2}f(t)dt$

Integration Methods

- **Integration by Substitution** :

Use the form $\int f(g(x))dg(x)$ OR use a dummy variable to get to a form in the Integral Formulae (taking into account chain rule)

Integral	Sub	Use identity
a^2-u^2	$u=a\sin\theta$	$1-\sin^2\theta=\cos^2\theta$
a^2+u^2	$u=a\tan\theta$	$1+\tan^2\theta=\sec^2\theta$
u^2-a^2	$u=a\sec\theta$	$sec^2\theta-1=\tan^2\theta$

- **Integration by Part** :

$\int uv'dx=uv-\int u'vdx$

Choose u by LIATE (Logarithmic, Inverse trigo, Algebraic, Trigo, Exponential)

Area between 2 curves

$A=\int_a^b(g(x)-f(x))dx$ provided $g(x)$ is above $f(x)$

Volume of a solid

Volume (around x-axis) $=\int_a^b\pi y^2dx$

5 Series

Geometric Series

$\sum_{r=1}^n ar^{n-1}=a\frac{1-r^n}{1-r}$

$\sum_{r=1}^\infty ar^{n-1}=\frac{a}{1-r}$ if $|r|<1$, diverges otherwise

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Rules on Series
$\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$, $\sum(ka_n) = k\sum a_n$

Ratio Test
$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho$, the series $\begin{cases}\text{converges if} & \rho<1\\ \text{diverges if} & \rho>1\\ \text{no conclusion if} & \rho=1\end{cases}$

p-series
$\sum_{n=1}^{\infty}\frac{1}{n^p}\begin{cases}\text{diverges} & 0\le p\le 1\\ \text{converges} & p>1\end{cases}$

Radius of convergence (R)
Use the **Ratio Test** to find **range** of convergence of **Power Series** about $x=a$, $\sum_{n=0}^{\infty}c_n(x-a)^n$

1. $R=0$, converges only at a
2. $R=h$, converges in $(a-h, a+h)$ but diverges outside
3. $R=\infty$, converges at every x

Differentiation and Integration of Power Series
Let $f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n, a-h<x<a+h$ where h is Radius of Convergence, then for $a-h<x<a+h$,

$f'(x)=\sum_{n=0}^{\infty}\frac{d}{dx}(c_n(x-a)^n)=\sum_{n=1}^{\infty}nc_n(x-a)^{n-1}$

$f''(x)=\sum_{n=1}^{\infty}nc_n\frac{d}{dx}(x-a)^{n-1}=\sum_{n=2}^{\infty}n(n-1)c_n(x-a)^{n-2}$

Note lower bound of sum increases by 1

$\int_0^x f(x)dx=\int_0^x\sum_{n=0}^{\infty}c_n(x-a)^n=\sum_{n=0}^{\infty}c_n\frac{(x-a)^{n+1}}{n+1}$

The radius of convergence is h after diff and integ

Taylor Series of f at a
$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$

MacLaurin Series
Taylor series of f at 0, i.e. $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$

List of common MacLaurin Series
1. $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n, -1<x<1, R=1$
2. $\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^nx^n, -1<x<1, R=1$
3. $\frac{1}{1+x^2}=\sum_{n=0}^{\infty}x^2n, -1<x<1, R=1$
4. $ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}, -1<x<1, R=1$
5. $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n+1}}{(2n+1)!}, -\infty<x<\infty, R=\infty$
6. $\cos x=\sum_{n=1}^{\infty}\frac{(-1)^nx^2n}{(2n)!}, -\infty<x<\infty, R=\infty$
7. $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}, -\infty<x<\infty, R=\infty$
8. $\tan^{-1}x=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}, -1\le x\le 1, R=1$
9. $\frac{1}{(1-x)^2}=\sum_{n=1}^{\infty}nx^{n-1}, -1<x<1, R=1$

0. $\frac{1}{(1-x)^3}=\frac{1}{2}\sum_{n=2}^{\infty}n(n-1)x^{n-2}, -1<x<1, R=1$

1. $(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n, -1<x<1, R=1$

2. $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots, -1<x<1, R=1$

Taylor Polynomials
The n-th order Taylor Polynomial of f at a

$P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$

It gives a good polynomial approxn of order n

Taylor's Theorem
$f(x)=P_n(x)+R_n(x)$ where

$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for $a<c<x$.

$R_n(x)$ is **remainder of order** n or **error term**

6 Vector

Dot Product
$v_1=\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}, v_2=\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}$, $v_1\cdot v_2=x_1x_2+y_1y_2+z_1z_2$

$\cos\theta=\frac{v_1\cdot v_2}{\|v_1\|\,\|v_2\|}$, Projection of b onto $a=\frac{b\cdot a}{\|a\|^2}a$

Commut, assoc, distr, and $v_1\cdot v_1=\|v_1\|^2$

Cross Product
$v_1\times v_2=(y_1z_2-y_2z_1)\mathbf{i}-(x_1z_2-x_2z_1)\mathbf{j}+(x_1y_2-x_2y_1)\mathbf{k}$ Area of parallelogram $=\|v_1\times v_2\|=\|v_1\|\,\|v_2\|\sin\theta$

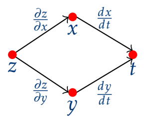
Distr, assoc, but $v_1\times v_2=-v_2\times v_1$ and $v_1\times v_1=O$

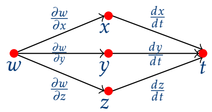
7 Functions of Several Variables

Partial Derivatives
of $z=f(x,y)$ w.r.t. x is denoted by $\left.\frac{\partial z}{\partial x}\right|_{(a,b)}$ or $f_x(a,b)$

Method: Fix the other variable (Note: $f_{xy}=f_{yx}$)

Chain Rule
$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$ AND $\frac{dw}{dt}=\frac{\partial w}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$





$z=f(x,y)$,
$x=x(t), y=y(t)$
$w=f(x,y,z)$,
$x=x(t)$,
$y=y(t), z=z(t)$

Directional Derivative
$f_x(a,b)$ is rate of change of f along direction of x-axis

Directional derivative of f at (a,b) in direction of unit vector $u=u_1\mathbf{i}+u_2\mathbf{j}$ is $D_uf(a,b)=f_x(a,b)u_1+f_y(a,b)u_2$

or $D_uf(a,b,c)=f_x(a,b,c)u_1+f_y(a,b,c)u_2+f_z(a,b,c)u_3$

$df=D_uf(a,b)\cdot dt$ (normal $\cdot$ multiplication) measures change in f (df) when we move a small distance dt, and u is the unit directional vector of the change and (a,b) is the original pt

Gradient Vector
Denoted by $\nabla f=f_x\mathbf{i}+f_y\mathbf{j}$ where

$\nabla f(a,b)\cdot u=D_uf(a,b)=\|\nabla f(a,b)\|\cos\theta$

$D_uf(a,b)>0$ and max when $\cos\theta=1\iff\theta=0°$

$D_uf(a,b)<0$ and min when $\cos\theta=-1\iff\theta=180°$

Max and Min Values
Critical Points - First Derivative Test

f has a local max or min at $(a,b)\wedge f_x$ exists $\wedge f_y$ exists $\rightarrow$ $f_x=0\wedge f_y=0$ (But not the converse)

Second Derivative Test

Disriminant $=f_{xx}(a,b)f_{yy}(a,b)-f_{xy}(a,b)^2$

1. $D>0\wedge f_{xx}(a,b)>0\rightarrow f$ has a local min at (a,b)
2. $D>0\wedge f_{xx}(a,b)<0\rightarrow f$ has a local max at (a,b)
3. $D<0\rightarrow f$ has a saddle-point at (a,b)
4. $D=0\rightarrow$ no conclusion

8 Ordinary Differential Equation (ODE)

No Crossing Principle: solution curves do not cross each other

First order ODE
There is only 1 soln for initial value problem with 1st order ODE. Intersection point of 2 curves is the initial pt.

1. $\frac{dy}{dx}=\frac{M(x)}{N(y)}\iff\int M(x)dx=\int N(y)dy$
2. $y'=f(\frac{y}{x})\Leftrightarrow$ Let $v=\frac{y}{x}, f(v)=y'\Leftrightarrow\frac{dv}{f(v)-v}=\frac{dx}{x}$
3. $y'=\frac{ax+by+c}{a_1x+b_1y+c_1}\Leftrightarrow$ Let $u=ax+by$
4. $\frac{dy}{dx}+p(x)y=Q(x)\Leftrightarrow ye^{\int p(x)dx}=\int Q(x)e^{\int p(x)dx}dx$
5. **Bernoulli eqn** $y'+p(x)y=q(x)y^n\Leftrightarrow$ Let $z=y^{1-n}\Leftrightarrow z'+(1-n)p(x)z=(1-n)q(x)$

Scenarios
Radioactive decay: $\frac{dx}{dt}=kx, x(t)=x(0)e^{-\frac{\ln 2}{\tau}t}$

Uranium-Thorium: $\frac{T}{U}=\frac{k_U}{k_T-k_U}(1-e^{-(k_T-k_U)t})k_N=\frac{\ln 2}{\tau_N}$

Cooling/Heating: $\int\frac{dT}{T-T_0}=\int kdt, T(t)-T_0=(T(0)-T_0)e^{kt}$

Retarded fall: $m\frac{dv}{dt}=mg-bv^2$,

$v=k\frac{1+ce^{-pt}}{1-ce^{-pt}}, k^2=\frac{mg}{b}, c=\frac{v(0)-k}{v(0)+k}, p=\frac{2kb}{m}$

Hyperbolic Functions
$\cosh x=\frac{e^x+e^{-x}}{2}, \sinh x=\frac{e^x+e^{-x}}{2}, \tanh x=\frac{\sinh x}{\cosh x}=\frac{1-e^{-2x}}{1+e^{-2x}}$

Second order linear ODE
Form: $\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=F(x)$

homogeneous $\Leftrightarrow F(x)=0$, else non-homogeneous

Homogeneous 2nd order linear ODE
Linearly dependent $\Leftrightarrow\forall x\exists c$ s.t. $u(x)=cv(x)$

y_1,y_2 are lin. indep. solns $\Rightarrow$ a general soln is $y=c_1y_1+c_2y_2$

y_1,y_2 are NOT lin. indep. solns $\Rightarrow y=c_1y_1+c_2y_2$ is a soln but not a general soln

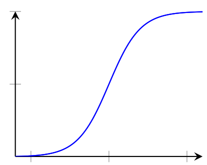
$\frac{d^2y}{dx^2}+A\frac{dy}{dx}+By=0$ has the trivial soln $y=0$ and non-trivial soln: Let $y=e^{\lambda x}$, solve $\lambda^2+A\lambda+B=0$, general soln is: (PS: Reverse is $A=-(\lambda_1+\lambda_2), B=\lambda_1\lambda_2$)

1. 2 real roots: $y=c_1e^{\lambda_1x}+c_2e^{\lambda_2x}$
2. 1 real root: $y=c_1e^{\lambda_1x}+c_2xe^{\lambda_1x}$
3. 2 complex roots $(a+ib)$: $y=e^{ax}(c_1\cos bx+c_2\sin bx)$

9 Mathematical Modelling (B = birth rate, D = death rate)

Malthusian Population Growth
$N(t)=N(0)e^{kt}$ where $k=B-D$. Conditions:

1. $k>0$ $(B>D)$: popn explosion ($e^{kt}\rightarrow\infty, N(t)\rightarrow\infty$ as $t\rightarrow\infty$)
2. $k=0$ $(B=D)$: stable ($N(t)=N(0)$ for all t)
3. $k<0$ $(B<D)$: extinction ($e^{kt}\rightarrow 0, N(t)\rightarrow 0$ as $t\rightarrow\infty$)

Logistic Growth Model


Eqn: $\frac{dN}{dt}=(B-D)N, N(0)=\hat{N}, N_\infty=\frac{B}{s}$

$\frac{dN}{dt}=(B-D)N=(B-sN)N=BN-sN^2$ where s is a small number compared to B.

$\frac{dN}{dt}=0$ when $N\approx\frac{B}{s}$ (population stops growing)

This constant $\frac{B}{s}$ is called carrying capacity, sustainable population, or logistic equilibrium population. Or that the population stabilises at $\frac{B}{s}$

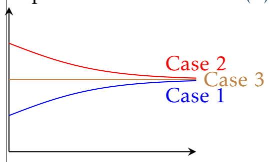
$N(t)=\frac{B}{s+\left(\frac{B}{N_0}-s\right)e^{-Bt}}=\frac{N_\infty}{1+(\frac{N_\infty}{N_0}-1)e^{-Bt}}$

$\lim_{t\to\infty}N(t)=\frac{B}{s}$

Case 1: $B-sN(t)>0\ \forall t$ **(Popn < sustainable popn)**
Logistic curve increasing

Case 2: $B-sN(t)<0$ **at all** t **(Popn > sustainable popn)**
Logistic curve decreasing

Case 3: $B-sN(t)=0$ **at all** t **(At sustainable popn)**
Population constant at $N(0)$



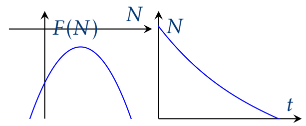
Harvesting
$\frac{dN}{dt}=BN-sN^2-E$ where E is fish caught per year.

DO NOT ATTEMPT TO SOLVE THE ODE. They will just ask to draw graph.

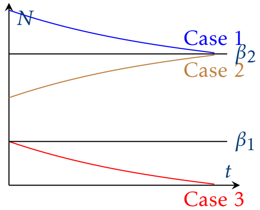
Method:

1. Let $F(N)=\frac{dN}{dt}=-sN^2+BN-E$
2. Discriminant $=B^2-4(-s)(-E)=B^2-4sE$
3. Cases:

(a) $D<0$: No equiblirium soln (Popn is decreasing to extinction)
Note: $-s<0$, shape is $\cap$, $F(N)\neq 0$



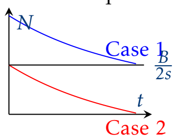
(b) $D>0$: 2 equilibrium solns
Solve $F(N)$ for β_1,β_2 where $\beta_1<\beta_2<\frac{B}{s}$
There are 3 possible cases:



$\frac{B}{s}=\beta_1+\beta_2, \frac{E}{s}=\beta_1\beta_2$

β_2 is stable ($N(0)$ slightly diff from β_2, popn will still tend to β_2). β_1 is not stable ($N(0)$ slightly diff from β_1 will not tend to β_1)

(c) $D=0$: 1 equilibrium solns



Suppose $N(0)>\frac{B}{2s}$ then max. harvesting w/o extinction $E=\frac{B^2}{4s}$

PS: more precise curves, follow the original logistic growth model graph (S-shaped) increasing: gentle-steep-gentle, decreasing: steep-gentle-steep

Additional Notes
- If the question is in powers above 2, e.g. $\frac{dN}{dt}=aN^4+bN^3+cN^2+dN+e$, the same rule about the graph still applies: the stable populations are the solutions to $aN^4+bN^3+cN^2+dN+e=0$
- If there is no harvesting, then $N=0$ is also a solution.