MA1521 Finals Cheatsheet v1.2 (2020-10-27)
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1 Trigo Formulae

- $\sin^2\theta + \cos^2\theta = 1$, $\sin 2\theta = 2\sin\theta\cos\theta$
- $\sin(A \pm B) = \sin A\cos B \pm \sin B\cos A$
- $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$
- $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$, $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$
- $\cos 2\theta = cos^2\theta - sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
- $\sin P + \sin Q = 2\sin\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$
- $\sin P - \sin Q = 2\cos\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$
- $\cos P + \cos Q = 2\cos\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$
- $\cos P - \cos Q = -2\sin\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$
- $a^2 = b^2 + c^2 - 2bc\cos\theta$ and $\frac{a}{\sin a} = \frac{b}{\sin b}$

2 Functions and Limits

Existence of Limits

$\lim_{x\to a} f(x)$ only exists when:

- $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ (limit from left = right)
- For $a = \infty$ or $-\infty$, only if $f(x)$ **does not** oscillate

Rules of Limits

1. $\lim_{x\to a}(f \pm g)(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
2. $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x)\lim_{x\to a} g(x)$
3. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$
4. $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x)$

Continuity

f is continuous at point $a \Leftrightarrow \lim_{x\to a} f(x) = f(a)$

L'Hôpital's Rule

Suppose:

1. f and g are differentiable
2. $f(a) = g(a) = 0$
3. $g'(x) \neq 0$ for all $x \in I \setminus a$

Then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

- Use L'Hôpital's Rule for $\frac{0}{0}$ and $\frac{\infty}{\infty}$ forms.
- Common: $\lim_{x\to\frac{\pi}{2}^-}(\sin x)^{\tan x} = \lim_{x\to\frac{\pi}{2}^-} e^{\ln(\sin x)^{\tan x}}$

$= e^{\lim_{x\to\frac{\pi}{2}^-}\tan x\ln(\sin x)} = e^{\lim_{x\to\frac{\pi}{2}^-}\frac{\ln(\sin x)}{\cot x}}$ (now in $\frac{0}{0}$ form)

- Tips:

1. Convert $0\cdot\infty, \infty - \infty$ by algebra manip
2. Convert $1^\infty, \infty^0, 0^0$ by first taking $\ln$

3 Derivative

The derivative of f at point a is $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, denoted by $f'(a)$ provided the limit exists.

$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \left.\frac{dy}{dx}\right|_{x=a}$

$f'(a)$ = slope of tangent at pt a

Some properties

- $f'(a)$ exists $\Rightarrow f(x)$ is smooth ($\therefore$ continuous) at a
- $f'(a)$ does not exist at **discontinuity, corner,** and **vertical tangent.**

Since derivative is limit, if lim from left $\neq$ right, then $f'(a)$ does not exist.

Formulae

Function	Derivative
$(f(x))^n$	$nf'(x)f(x)^{n-1}$
$\sin f(x)$	$f'(x)\cos f(x)$
$\cos f(x)$	$-f'(x)\sin f(x)$
$\tan f(x)$	$f'(x)\sec^2 f(x)$
$\cot f(x)$	$-f'(x)\csc^2 f(x)$
$\sec f(x)$	$f'(x)\sec f(x)\tan f(x)$
$\csc f(x)$	$-f'(x)\csc f(x)\cot f(x)$
$a^f(x)$	$f'(x)a^{f(x)}\ln a$

Function	Derivative
k	0
$e^f(x)$	$f'(x)e^{f(x)}$
$\log_a f(x)$	$\frac{f'(x)}{f(x)\ln a}$
$\ln f(x)$	$\frac{f'(x)}{f(x)}$

Function	Derivative
$\sin^{-1} f(x)$	$\frac{f'(x)}{\sqrt{1-f(x)^2}}$
$\cos^{-1} f(x)$	$-\frac{f'(x)}{\sqrt{1-f(x)^2}}$
$\tan^{-1} f(x)$	$\frac{f'(x)}{1+f(x)^2}$

Rules of Differentiation

- $(kf)'(x) = kf'(x)$
- $(f \pm g)'(x) = f'(x) \pm g'(x)$
- $\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$
- $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
- $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ or $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

Parametric Differentiation

Given $\begin{cases} y = u(t) \\ x = v(t) \end{cases}$, we have $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

- **Second derivative**

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ then do implicit differentiation w.r.t x

- **Polar equation** $(r = a\theta)$: $x = r\cos\theta, y = r\sin\theta$

Implicit Differentiation

Differentiate w.r.t. to var, then multiply by $\frac{d\text{<var>}}{dx}$ Common: $y = x^x \iff \ln y = x\ln x$

Higher Order Derivatives

The n-th derivative is denoted by $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$

Maxima and Minima

- $f(c)$ **is Local Maximum** if $f(c) \geq f(x)$ for x near c
- $f(c)$ **is Local Minimum** if $f(c) \leq f(x)$ for x near c
- $f(c)$ **is abs maximum** if $f(c) \geq f(x)\forall x \in$ domain
- $f(c)$ **is abs minimum** if $f(c) \leq f(x)\forall x \in$ domain
- **Critical Point** :

Let f be a function with domain D. An interior point (not end-point) c in D is called a **Critical Point** of f if $f'(c) = 0$ of $f'(c)$ does not exist.

- **Method to find extreme values of** f :

Check critical points of f, end-points of domain D

Method to Find Local Extreme values

A function may not have a local extreme at a critical pt. Check using 1st/2nd derivative tests.

- **1st Derivative Test** :

Assume $c \in (a,b)$ is a critical point of f

1. $f'(x) > 0$ for $x \in (a,c)$ and $f'(x) < 0$ for $x \in (c,b)$, then f is a **local maximum**
2. $f'(x) < 0$ for $x \in (a,c)$ and $f'(x) > 0$ for $x \in (c,b)$, then f is a **local minimum**

- **2nd Derivative Test** :

$f'(c) = 0\begin{cases} f''(c) < 0 \iff f \text{ has local max at } c \\ f''(c) > 0 \iff f \text{ has local min at } c \end{cases}$

Note: if $f'(c) = 0$ and $f''(c) = 0$ then 2nd derivative test fails. Use 1st derivative test.

Method to Find Absolute Extreme Values

1. Find all critical points c in the interior
2. Evaluate $f(c)$, where c is a critical or end point
3. The largest and smallest of these values will be abs max & min respectively

Increasing and Decreasing Functions

Test for Monotonic Functions ($f: I$ (interval) $\to \mathbb{R}$):

- $f'(x) > 0$ for any x in $I \Rightarrow f$ is **increasing** on I
- $f'(x) < 0$ for any x in $I \Rightarrow f$ is **decreasing** on I

Concavity

$\begin{cases} f''(x) < 0 \Leftrightarrow f'(x) \text{ is decreasing} \Leftrightarrow \text{Concave Down} \\ f''(x) > 0 \Leftrightarrow f'(x) \text{ is increasing} \Leftrightarrow \text{Concave Up} \end{cases}$

Points of Inflection

Let $f: I \to \mathbb{Z}$ and $c \in I$.

c is a pt of inflection of f if f is continuous at c and the concavity of f changes at c.

In another word: c is pt of inflection $\to f''(c) = 0$ (but not the reverse – c is a pt of inflection only if $f''(c)$ crosses from (+) to (-) and vice versa.)

4 Integration

Indefinite Integral

Denoted by $\int f(x)dx = F(x) + C$

Geometrical Interpretation

All curves $y = F(x) + C$ s.t. their slopes at x are $f(x)$

Rules of Indefinite Integration

1. $\int kf(x)dx = k\int f(x)dx$
2. $\int -f(x)dx = -\int f(x)dx$
3. $\int[f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Integral Formulae

Function	Integral
$\int\cot x dx$	$\ln(\sin x) + C$
$\int\sec x\tan x dx$	$\sec x + C$
$\int\csc x\cot x dx$	$\csc x + C$
$\int\sec^2 x dx$	$\tan x + C$
$\int\csc^2 x dx$	$-\cot x + C$

$\int x^n dx$	$\frac{x^{n+1}}{n+1} + C, n \neq -1, n$ rational
$\int\frac{1}{\sqrt{a^2-x^2}}dx$	$\sin^{-1}(\frac{x}{a}) + C$
$\int\frac{1}{a^2+x^2}dx$	$\frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$
$\int 1dx = \int dx$	$x + C$
$\int e^x dx$	$e^x + C$
$\int a^x dx$	$\frac{a^x}{\ln a}$
$\int\ln x dx$	$x\ln x - x + C$
$\int\frac{1}{x}dx$	$\ln x + C$
$\int\sin kx dx$	$-\frac{\cos kx}{k} + C$
$\int\cos kx dx$	$\frac{\sin kx}{k} + C$
$\int\tan x dx$	$\ln(\sec x) + C$ **or** $-\ln(\sec x) + C$

Function	Integral
$\int\tan^2 x dx$	$\tan x - x + C$
$\int\sec x dx$	$\ln(\sec x + \tan x) + C$
$\int\csc x dx$	$\ln(\csc x - \cot x) + C$

Riemann (Definite) Integrals

Riemann sum on f on $[a,b] \approx \sum_{k=1}^n f(c_k)\Delta x$

Exact area = $\lim_{n\to\infty}\sum_{k=1}^n f(c_k)\Delta x$

Riemann Integral of f **over** $[a,b]$:

$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^n f(c_k)\Delta x$

Rules of Definite Integrals

1. $\int_a^a f(x)dx = 0$, $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b[f(x) \pm g(x)] = \int_a^b f(x) \pm \int_a^b g(x)$
4. If $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)dc \geq \int_a^b g(x)dx$

   If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)dx \geq 0$
5. If f is continuous on the interval joining a, b and c, then $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

Fundamental Thm of Calculus

$F'(x) = f(x)$ If F is an antiderivative of f on $[a,b]$, then $\int_a^b F'(x)dx = \int_a^b f(x)dx = F(b) - F(a)$

x' Let f be continuous on $[a,b]$. Then $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

Note the 2 x's: on $\frac{d}{dx}$ and $\int_a^x$ and $f(t)$ is indep of x

1. $\frac{d}{dx}\int_0^2 t^2dt = 0$, $\frac{d}{dx}\int_0^x \sin\sqrt{t}dt = \sin\sqrt{x}$
2. $\frac{d}{dx}\left(\int_1^{x^4}\frac{t}{\sqrt{t^3+2}}dt\right) = \frac{d}{dx^4}\left(\int_1^{x^4}\frac{t}{\sqrt{t^3+2}}dt\right)\frac{dx^4}{dx}$

   $= \frac{x^4}{\sqrt{(x^4)^3+2}}(4x^3) = \frac{4x^7}{\sqrt{x^12+2}}$
3. $\frac{d}{dx}\int_x^a f(t)dt = -\frac{d}{dx}\int_a^x f(t)dt$
4. $\frac{d}{dx}\int_{x^2}^{x^4} f(t)dt = \frac{d}{dx}\int_a^{x^4} f(t)dt - \frac{d}{dx}\int_a^{x^2} f(t)dt$

Integration Methods

- **Integration by Substitution** :

Use the form $\int f(g(x))dg(x)$ OR use a dummy variable to get to a form in the Integral Formulae (taking into account chain rule)

Integral	Sub	Use identity
$a^2 - u^2$	$u = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$a^2 + u^2$	$u = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$u^2 - a^2$	$u = a\sec\theta$	$sec^2\theta - 1 = \tan^2\theta$

- **Integration by Part** :

$\int uv'dx = uv - \int u'vdx$

Choose u by LIATE (Logarithmic, Inverse trigo, Algebraic, Trigo, Exponential)

Area between 2 curves

$A = \int_a^b(g(x) - f(x))dx$ provided $g(x)$ is above $f(x)$

Volume of a solid

Volume (around x-axis) $= \int_a^b \pi y^2 dx$

5 Series

Geometric Series

$\sum_{r=1}^n ar^{n-1} = a\frac{1-r^n}{1-r}$

$\sum_{r=1}^\infty ar^{n-1} = \frac{a}{1-r}$ if $|r| < 1$, diverges otherwise

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Rules on Series

$\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$, $\sum(ka_n) = k\sum a_n$

Ratio Test

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho$, the series $\begin{cases}\text{converges if} & \rho < 1\\ \text{diverges if} & \rho > 1\\ \text{no conclusion if} & \rho = 1\end{cases}$

p-series

$\sum_{n=1}^{\infty}\frac{1}{n^p}\begin{cases}\text{diverges} & 0 \le p \le 1\\ \text{converges} & p > 1\end{cases}$

Radius of convergence (R)

Use the **Ratio Test** to find **range** of convergence of **Power Series** about $x = a$, $\sum_{n=0}^{\infty} c_n(x-a)^n$

1. $R = 0$, converges only at a
2. $R = h$, converges in $(a-h, a+h)$ but diverges outside
3. $R = \infty$, converges at every x

Differentiation and Integration of Power Series

Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, a-h < x < a+h$ where h is Radius of Convergence, then for $a-h < x < a+h$,

$f'(x) = \sum_{n=0}^{\infty}\frac{d}{dx}(c_n(x-a)^n) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

$f''(x) = \sum_{n=1}^{\infty} nc_n\frac{d}{dx}(x-a)^{n-1} = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2}$

Note lower bound of sum increases by 1

$\int_0^x f(x)dx = \int_0^x \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

The radius of convergence is h after diff and integ

Taylor Series of f at a

$f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$

MacLaurin Series

Taylor series of f at 0, i.e. $f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$

List of common MacLaurin Series

1. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, -1 < x < 1, R = 1$
2. $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n, -1 < x < 1, R = 1$
3. $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} x^2 n, -1 < x < 1, R = 1$
4. $ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}, -1 < x < 1, R = 1$
5. $\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}, -\infty < x < \infty, R = \infty$
6. $\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}, -\infty < x < \infty, R = \infty$
7. $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}, -\infty < x < \infty, R = \infty$
8. $\tan^{-1} x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}, -1 \le x \le 1, R = 1$
9. $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}, -1 < x < 1, R = 1$

0. $\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2}, -1 < x < 1, R = 1$

1. $(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n, -1 < x < 1, R = 1$

2. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots, -1 < x <$ 1, $R = 1$

Taylor Polynomials

The n-th order Taylor Polynomial of f at a

$P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$

It gives a good polynomial approxn of order n

Taylor's Theorem

$f(x) = P_n(x) + R_n(x)$ where

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for $a < c < x$.

$R_n(x)$ is **remainder of order** n or **error term**

6 Vector

Dot Product

$v_1 = \begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}, v_2 = \begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}$, $v_1 \cdot v_2 = x_1x_2 + y_1y_2 + z_1z_2$

$\cos\theta = \frac{v_1\cdot v_2}{\|v_1\|\,\|v_2\|}$, Projection of b onto $a = \frac{b\cdot a}{\|a\|^2}a$

Commut, assoc, distr, and $v_1 \cdot v_1 = \|v_1\|^2$

Cross Product

$v_1 \times v_2 = (y_1z_2 - y_2z_1)\mathbf{i} - (x_1z_2 - x_2z_1)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k}$ Area of parallellogram $= \|v_1 \times v_2\| = \|v_1\|\,\|v_2\|\,\sin\theta$

Distr, assoc, but $v_1 \times v_2 = -v_2 \times v_1$ and $v_1 \times v_1 = O$

7 Functions of Several Variables

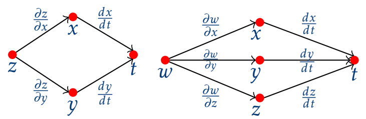
Partial Derivatives

of $z = f(x,y)$ w.r.t. x is denoted by $\left.\frac{\partial z}{\partial x}\right|_{(a,b)}$ or $f_x(a,b)$

Method: Fix the other variable (Note: $f_{xy} = f_{yx}$)

Chain Rule

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$ AND $\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$



$z = f(x,y)$,
$x = x(t), y = y(t)$
$w = f(x,y,z)$,
$x = x(t)$,
$y = y(t), z = z(t)$

Directional Derivative

$f_x(a,b)$ is rate of change of f along direction of x-axis

Directional derivative of f at (a,b) in direction of unit vector $u = u_1\mathbf{i} + u_2\mathbf{j}$ is $D_u f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2$

or $D_u f(a,b,c) = f_x(a,b,c)u_1 + f_y(a,b,c)u_2 + f_z(a,b,c)u_3$

$df = D_u f(a,b)\cdot dt$ (normal $\cdot$ multiplication) measures change in f (df) when we move a small distance dt, and u is the unit directional vector of the change and (a,b) is the original pt

Gradient Vector

Denoted by $\nabla f = f_x\mathbf{i} + f_y\mathbf{j}$ where

$\nabla f(a,b)\cdot u = D_u f(a,b) = \|\nabla f(a,b)\|\cos\theta$

$D_u f(a,b) > 0$ and max when $\cos\theta = 1 \iff \theta = 0°$

$D_u f(a,b) < 0$ and min when $\cos\theta = -1 \iff \theta = 180°$

Max and Min Values

Critical Points - First Derivative Test

f has a local max or min at $(a,b) \land f_x$ exists $\land f_y$ exists $\to$ $f_x = 0 \land f_y = 0$ (But not the converse)

Second Derivative Test

Disriminant $= f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$

1. $D > 0 \land f_{xx}(a,b) > 0 \to f$ has a local min at (a,b)
2. $D > 0 \land f_{xx}(a,b) < 0 \to f$ has a local max at (a,b)
3. $D < 0 \to f$ has a saddle-point at (a,b)
4. $D = 0 \to$ no conclusion

8 Ordinary Differential Equation (ODE)

No Crossing Principle: solution curves do not cross each other

First order ODE

There is only 1 soln for initial value problem with 1st order ODE. Intersection point of 2 curves is the initial pt.

1. $\frac{dy}{dx} = \frac{M(x)}{N(y)} \iff \int M(x)dx = \int N(y)dy$
2. $y' = f(\frac{y}{x}) \Leftrightarrow$ Let $v = \frac{y}{x}, f(v) = y' \Leftrightarrow \frac{dv}{f(v)-v} = \frac{dx}{x}$
3. $y' = \frac{ax+by+c}{a_1x+b_1y+c_1} \Leftrightarrow$ Let $u = ax + by$
4. $\frac{dy}{dx} + p(x)y = Q(x) \Leftrightarrow ye^{\int p(x)dx} = \int Q(x)e^{\int p(x)dx}dx$
5. **Bernoulli eqn** $y' + p(x)y = q(x)y^n \Leftrightarrow$
   Let $z = y^{1-n} \Leftrightarrow z' + (1-n)p(x)z = (1-n)q(x)$

Scenarios

Radioactive decay: $\frac{dx}{dt} = kx$, $x(t) = x(0)e^{-\frac{\ln 2}{\tau}t}$

Uranium-Thorium: $\frac{T}{U} = \frac{k_U}{k_T - k_U}(1 - e^{-(k_T - k_U)t})k_N = \frac{\ln 2}{\tau_N}$

Cooling/Heating: $\int\frac{dT}{T-T_0} = \int kdt$, $T(t) - T_0 = (T(0) - T_0)e^{kt}$

Retarded fall: $m\frac{dv}{dt} = mg - bv^2$,

$v = k\frac{1+ce^{-pt}}{1-ce^{-pt}}, k^2 = \frac{mg}{b}, c = \frac{v(0)-k}{v(0)+k}, p = \frac{2kb}{m}$

Hyperbolic Functions

$\cosh x = \frac{e^x+e^{-x}}{2}, \sinh x = \frac{e^x+e^{-x}}{2}, \tanh x = \frac{\sinh x}{\cosh x} = \frac{1-e^{-2x}}{1+e^{-2x}}$

Second order linear ODE

Form: $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = F(x)$

homogeneous $\Leftrightarrow F(x) = 0$, else non-homogeneous

Homogeneous 2nd order linear ODE

Linearly dependent $\Leftrightarrow \forall x\exists c$ s.t. $u(x) = cv(x)$

y_1, y_2 are lin. indep. solns $\Rightarrow$ a general soln is $y = c_1y_1 + c_2y_2$

y_1, y_2 are NOT lin. indep. solns $\Rightarrow y = c_1y_1 + c_2y_2$ is a soln but not a general soln

$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$ has the trivial soln $y = 0$ and non-trivial soln: Let $y = e^{\lambda x}$, solve $\lambda^2 + A\lambda + B = 0$, general soln is: (PS: Reverse is $A = -(\lambda_1 + \lambda_2), B = \lambda_1\lambda_2$)

1. 2 real roots: $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$
2. 1 real root: $y = c_1e^{\lambda_1 x} + c_2xe^{\lambda_1 x}$
3. 2 complex roots $(a + ib)$: $y = e^{ax}(c_1\cos bx + c_2\sin bx)$

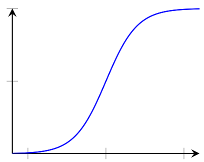
9 Mathematical Modelling (B = birth rate, D = death rate)

Malthusian Population Growth

$N(t) = N(0)e^{kt}$ where $k = B - D$. Conditions:

1. $k > 0$ $(B > D)$: popn explosion ($e^{kt} \to \infty, N(t) \to \infty$ as $t \to \infty$)
2. $k = 0$ $(B = D)$: stable ($N(t) = N(0)$ for all t)
3. $k < 0$ $(B < D)$: extinction ($e^{kt} \to 0, N(t) \to 0$ as $t \to \infty$)

Logistic Growth Model



Eqn: $\frac{dN}{dt} = (B - D)N, N(0) = \hat{N}, N_\infty = \frac{B}{s}$

$\frac{dN}{dt} = (B - D)N = (B - sN)N = BN - sN^2$ where s is a small number compared to B.

$\frac{dN}{dt} = 0$ when $N \approx \frac{B}{s}$ (population stops growing)

This constant $\frac{B}{s}$ is called carrying capacity, sustainable population, or logistic equilibrium population. Or that the population stabilises at $\frac{B}{s}$

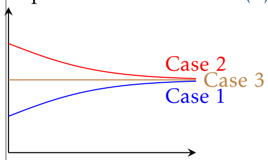
$N(t) = \frac{B}{s + \left(\frac{B}{N_0} - s\right)e^{-Bt}} = \frac{N_\infty}{1 + (\frac{N_\infty}{N_0} - 1)e^{-Bt}}$

$\lim_{t\to\infty} N(t) = \frac{B}{s}$

Case 1: $B - sN(t) > 0\ \forall t$ **(Popn < sustainable popn)**
Logistic curve increasing

Case 2: $B - sN(t) < 0$ **at all** t **(Popn > sustainable popn)**
Logistic curve decreasing

Case 3: $B - sN(t) = 0$ **at all** t **(At sustainable popn)**
Population constant at $N(0)$



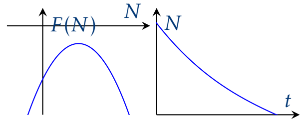
Harvesting

$\frac{dN}{dt} = BN - sN^2 - E$ where E is fish caught per year.

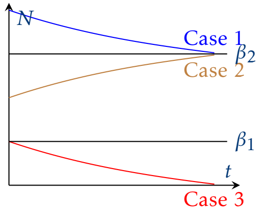
DO NOT ATTEMPT TO SOLVE THE ODE. They will just ask to draw graph.

Method:

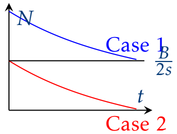
1. Let $F(N) = \frac{dN}{dt} = -sN^2 + BN - E$
2. Discriminant $= B^2 - 4(-s)(-E) = B^2 - 4sE$
3. Cases:
   (a) $D < 0$: No equiblirium soln (Popn is decreasing to extinction)
   Note: $-s < 0$, shape is $\cap$, $F(N) \neq 0$



   (b) $D > 0$: 2 equilibrium solns
   Solve $F(N)$ for β_1, β_2 where $\beta_1 < \beta_2 < \frac{B}{s}$
   There are 3 possible cases:



   $\frac{B}{s} = \beta_1 + \beta_2, \frac{E}{s} = \beta_1\beta_2$
   β_2 is stable ($N(0)$ slightly diff from β_2, popn will still tend to β_2). β_1 is not stable ($N(0)$ slightly diff from β_1 will not tend to β_1)
   (c) $D = 0$: 1 equilibrium solns



   Suppose $N(0) > \frac{B}{2s}$ then max. harvesting w/o extinction $E = \frac{B^2}{4s}$

PS: more precise curves, follow the original logistic growth model graph (S-shaped) increasing: gentle-steep-gentle, decreasing: steep-gentle-steep

Additional Notes

- If the question is in powers above 2, e.g. $\frac{dN}{dt} = aN^4 + bN^3 + cN^2 + dN + e$, the same rule about the graph still applies: the stable populations are the solutions to $aN^4 + bN^3 + cN^2 + dN + e = 0$
- If there is no harvesting, then $N = 0$ is also a solution.