

1 Trigo Formulae

- $\sin^2 \theta + \cos^2 \theta = 1$, $\sin 2\theta = 2 \sin \theta \cos \theta$
- $\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$, $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$
- $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$
- $\sin P + \sin Q = 2 \sin \frac{1}{2}(P + Q) \cos \frac{1}{2}(P - Q)$
- $\sin P - \sin Q = 2 \cos \frac{1}{2}(P + Q) \sin \frac{1}{2}(P - Q)$
- $\cos P + \cos Q = 2 \cos \frac{1}{2}(P + Q) \cos \frac{1}{2}(P - Q)$
- $\cos P - \cos Q = -2 \sin \frac{1}{2}(P + Q) \sin \frac{1}{2}(P - Q)$
- $a^2 = b^2 + c^2 - 2bc \cos \theta$ and $\frac{a}{\sin a} = \frac{b}{\sin b}$

2 Functions and Limits

Existence of Limits

$\lim_{x \rightarrow a} f(x)$ only exists when:

- $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$ (limit from left = right)
- For $a = \infty$ or $-\infty$, only if $f(x)$ does not oscillate

Rules of Limits

- $\lim_{x \rightarrow a} (f \pm g)(x) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ provided $\lim_{x \rightarrow a} g(x) \neq 0$
- $\lim_{x \rightarrow a} kf(x) = k \lim_{x \rightarrow a} f(x)$

Continuity

f is continuous at point $a \Leftrightarrow \lim_{x \rightarrow a} f(x) = f(a)$

L'Hôpital's Rule

Suppose:

- f and g are differentiable
- $f(a) = g(a) = 0$
- $g'(x) \neq 0$ for all $x \in I \setminus a$

$$\text{Then } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

- Use L'Hôpital's Rule for $\frac{0}{0}$ and $\frac{\infty}{\infty}$ forms.

$$\begin{aligned} \text{Common: } \lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\tan x} &= \lim_{x \rightarrow \frac{\pi}{2}} e^{\ln(\sin x) \tan x} \\ &= e^{\lim_{x \rightarrow \frac{\pi}{2}} \tan x \ln(\sin x)} = e^{\lim_{x \rightarrow \frac{\pi}{2}} \frac{\ln(\sin x)}{\cot x}} \quad (\text{now in } \frac{0}{0} \text{ form}) \end{aligned}$$

- Tips:

- Convert $0 \cdot \infty, \infty - \infty$ by algebra manip
- Convert $1^\infty, \infty^0, 0^0$ by first taking \ln

3 Derivative

The derivative of f at point a is $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$,

denoted by $f'(a)$ provided the limit exists.

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \left. \frac{dy}{dx} \right|_{x=a}$$

$f'(a)$ = slope of tangent at pt a

Some properties

- $f'(a)$ exists $\Rightarrow f(x)$ is smooth (\therefore continuous) at a
- $f'(a)$ does not exist at **discontinuity, corner, and vertical tangent**.

Since derivative is limit, if lim from left \neq right, then $f'(a)$ does not exist.

Formulae

Function	Derivative
$(f(x))^n$	$nf'(x)f(x)^{n-1}$
$\sin f(x)$	$f'(x)\cos f(x)$
$\cos f(x)$	$-f'(x)\sin f(x)$
$\tan f(x)$	$f'(x)\sec^2 f(x)$
$\cot f(x)$	$-f'(x)\csc^2 f(x)$
$\sec f(x)$	$f'(x)\sec f(x)\tan f(x)$
$\csc f(x)$	$-f'(x)\csc f(x)\cot f(x)$
$a^f(x)$	$f'(x)a^{f(x)} \ln a$

Function	Derivative	Function	Derivative
$\frac{1}{k}$	0	$\sin^{-1} f(x)$	$\frac{f'(x)}{\sqrt{1-f(x)^2}}$
$e^f(x)$	$f'(x)e^f(x)$	$\cos^{-1} f(x)$	$-\frac{f'(x)}{\sqrt{1-f(x)^2}}$
$\log_a f(x)$	$\frac{f'(x)}{f(x) \ln a}$	$\tan^{-1} f(x)$	$\frac{f'(x)}{1+f(x)^2}$
$\ln f(x)$	$\frac{f'(x)}{f(x)}$		

Rules of Differentiation

- $(kf)'(x) = kf'(x)$
- $(f \pm g)'(x) = f'(x) \pm g'(x)$
- $\frac{d}{dx} uv = u \frac{dv}{dx} + v \frac{du}{dx}$
- $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
- $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$ or $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Parametric Differentiation

Given $\begin{cases} y = u(t) \\ x = v(t) \end{cases}$, we have $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Second derivative

$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$ then do implicit differentiation w.r.t x

- Polar equation** ($r = a\theta$): $x = r \cos \theta, y = r \sin \theta$

Implicit Differentiation

Differentiate w.r.t. to var, then multiply by $\frac{d<var>}{dx}$ Common: $y = x^x \Leftrightarrow \ln y = x \ln x$

Higher Order Derivatives

The n -th derivative is denoted by $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$

Maxima and Minima

- $f(c)$ is **Local Maximum** if $f(c) \geq f(x)$ for x near c
- $f(c)$ is **Local Minimum** if $f(c) \leq f(x)$ for x near c
- $f(c)$ is **abs maximum** if $f(c) \geq f(x) \forall x \in \text{domain}$
- $f(c)$ is **abs minimum** if $f(c) \leq f(x) \forall x \in \text{domain}$

Critical Point :

Let f be a function with domain D . An interior point (not end-point) c in D is called a **Critical Point** of f if $f'(c) = 0$ or $f'(c)$ does not exist.

- Method to find extreme values of f :**

Check critical points of f , end-points of domain D

Method to Find Local Extreme values

A function may not have a local extreme at a critical pt. Check using 1st/2nd derivative tests.

- 1st Derivative Test :**

Assume $c \in (a, b)$ is a critical point of f

- $f'(x) > 0$ for $x \in (a, c)$ and $f'(x) < 0$ for $x \in (c, b)$, then f is a **local maximum**
- $f'(x) < 0$ for $x \in (a, c)$ and $f'(x) > 0$ for $x \in (c, b)$, then f is a **local minimum**

2nd Derivative Test :

$$f'(c) = 0 \begin{cases} f''(c) < 0 \Leftrightarrow f \text{ has local max at } c \\ f''(c) > 0 \Leftrightarrow f \text{ has local min at } c \end{cases}$$

Note: if $f'(c) = 0$ and $f''(c) = 0$ then 2nd derivative test fails. Use 1st derivative test.

Method to Find Absolute Extreme Values

- Find all critical points c in the interior
- Evaluate $f(c)$, where c is a critical or end point
- The largest and smallest of these values will be abs max & min respectively

Increasing and Decreasing Functions

Test for Monotonic Functions ($f : I$ (interval) $\rightarrow \mathbb{R}$):

- $f'(x) > 0$ for any x in $I \Rightarrow f$ is **increasing** on I
- $f'(x) < 0$ for any x in $I \Rightarrow f$ is **decreasing** on I

Concavity

$\begin{cases} f''(x) < 0 \Leftrightarrow f'(x) \text{ is decreasing} \Leftrightarrow \text{Concave Down} \\ f''(x) > 0 \Leftrightarrow f'(x) \text{ is increasing} \Leftrightarrow \text{Concave Up} \end{cases}$

Points of Inflection

Let $f : I \rightarrow \mathbb{Z}$ and $c \in I$.

c is a pt of inflection of f if f is continuous at c and the concavity of f changes at c .

In another word: c is pt of inflection $\rightarrow f''(c) = 0$ (but not the reverse - c is a pt of inflection only if $f''(c)$ crosses from (+) to (-) and vice versa.)

4 Integration

Indefinite Integral

Denoted by $\int f(x)dx = F(x) + C$

Geometrical Interpretation

All curves $y = F(x) + C$ s.t. their slopes at x are $f(x)$

Rules of Indefinite Integration

- $\int kf(x)dx = k \int f(x)dx$
- $\int -f(x)dx = - \int f(x)dx$
- $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Integral Formulae

Function	Integral
$\int \cot x dx$	$\ln(\sin x) + C$
$\int \sec x \tan x dx$	$\sec x + C$
$\int \csc x \cot x dx$	$\csc x + C$
$\int \sec^2 x dx$	$\tan x + C$
$\int \csc^2 x dx$	$-\cot x + C$
$\int x^n dx$	$\frac{x^{n+1}}{n+1} + C, n \neq -1, n \text{ rational}$
$\int \frac{1}{\sqrt{a^2 - x^2}} dx$	$\sin^{-1} \left(\frac{x}{a} \right) + C$
$\int \frac{1}{a^2 + x^2} dx$	$\frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$
$\int 1 dx = \int dx$	$x + C$
$\int e^x dx$	$e^x + C$
$\int a^x dx$	$\frac{a^x}{\ln a}$
$\int \ln x dx$	$x \ln x - x + C$
$\int \frac{1}{x} dx$	$\ln x + C$
$\int \sin kx dx$	$-\frac{\cos kx}{k} + C$
$\int \cos kx dx$	$\frac{\sin kx}{k} + C$
$\int \tan x dx$	$\ln(\sec x) + C$ or $-\ln(\sec x) + C$

Function	Integral
$\int \tan^2 x dx$	$\tan x - x + C$
$\int \sec x dx$	$\ln(\sec x + \tan x) + C$
$\int \csc x dx$	$\ln(\csc x - \cot x) + C$

Riemann (Definite) Integrals

Riemann sum on f on $[a, b] \approx \sum_{k=1}^n f(c_k) \Delta x$

Exact area = $\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$

Riemann Integral of f over $[a, b]$:

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x$$

Rules of Definite Integrals

- $\int_a^a f(x)dx = 0$, $\int_a^b kf(x)dx = k \int_a^b f(x)dx$
- $\int_a^b f(x)dx = - \int_b^a f(x)dx$
- $\int_a^b [f(x) \pm g(x)] = \int_a^b f(x) \pm \int_a^b g(x)$
- If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)dx \geq 0$
- If f is continuous on the interval joining a, b and c , then $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

Fundamental Thm of Calculus

$F'(x) = f(x)$ If F is an antiderivative of f on $[a, b]$, then

$$\int_a^b F'(x)dx = \int_a^b f(x)dx = F(b) - F(a)$$

x' Let f be continuous on $[a, b]$. Then

$$\frac{d}{dx} \int_a^x f(t)dt = f(x)$$

Note the 2 x 's: on $\frac{d}{dx}$ and \int_a^x and $f(t)$ is indep of x

- $\frac{d}{dx} \int_0^x t^2 dt = 0$, $\frac{d}{dx} \int_0^x \sin \sqrt{t} dt = \sin \sqrt{x}$
- $\frac{d}{dx} \left(\int_1^{x^4} \frac{t}{\sqrt{t^3+2}} dt \right) = \frac{d}{dx^4} \left(\int_1^{x^4} \frac{t}{\sqrt{t^3+2}} dt \right) \frac{dx^4}{dx}$
 $= \frac{x^4}{\sqrt{(x^4)^3+2}} (4x^3) = \frac{4x^7}{\sqrt{x^{12}+2}}$
- $\frac{d}{dx} \int_x^a f(t)dt = - \frac{d}{dx} \int_a^x f(t)dt$
- $\frac{d}{dx} \int_x^{x^4} f(t)dt = \frac{d}{dx} \int_a^x f(t)dt - \frac{d}{dx} \int_a^{x^2} f(t)dt$

Integration Methods

- Integration by Substitution :**

Use the form $\int f(g(x))dg(x)$ OR use a dummy variable to get to a form in the Integral Formulae (taking into account chain rule)

Integral	Sub	Use identity
$a^2 - u^2$	$u = a \sin \theta$	$1 - \sin^2 \theta = \cos^2 \theta$
$a^2 + u^2$	$u = a \tan \theta$	$1 + \tan^2 \theta = \sec^2 \theta$
$u^2 - a^2$	$u = a \sec \theta$	$\sec^2 \theta - 1 = \tan^2 \theta$

- Integration by Part :**

$$\int uv' dx = uv - \int u'v dx$$

Choose u by LIATE (Logarithmic, Inverse trigo, Algebraic, Trigo, Exponential)

Area between 2 curves

$$A = \int_a^b (g(x) - f(x))dx \text{ provided } g(x) \text{ is above } f(x)$$

Volume of a solid

$$\text{Volume (around x-axis)} = \int_a^b \pi y^2 dx$$

5 Series

Geometric Series

$$\sum_{r=1}^n ar^{n-1} = a \frac{1-r^n}{1-r}$$
$$\sum_{r=1}^\infty ar^{n-1} = \frac{a}{1-r} \text{ if } |r| < 1, \text{ diverges otherwise}$$

Rules on Series

$$\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n, \quad \sum (ka_n) = k \sum a_n$$

Ratio Test

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho, \text{ the series } \begin{cases} \text{converges if} & \rho < 1 \\ \text{diverges if} & \rho > 1 \\ \text{no conclusion if} & \rho = 1 \end{cases}$$

p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{diverges} & 0 \leq p \leq 1 \\ \text{converges} & p > 1 \end{cases}$$

Radius of convergence (R)

Use the **Ratio Test** to find **range** of convergence of **Power Series** about $x = a$, $\sum_{n=0}^{\infty} c_n(x-a)^n$

1. $R = 0$, converges only at a
2. $R = h$, converges in $(a-h, a+h)$ but diverges outside
3. $R = \infty$, converges at every x

Differentiation and Integration of Power Series

Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, $a-h < x < a+h$ where h is Radius of Convergence, then for $a-h < x < a+h$,

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} (c_n(x-a)^n) = \sum_{n=1}^{\infty} n c_n(x-a)^{n-1}$$

$$f''(x) = \sum_{n=1}^{\infty} n c_n \frac{d}{dx} (x-a)^{n-1} = \sum_{n=2}^{\infty} n(n-1) c_n(x-a)^{n-2}$$

Note lower bound of sum increases by 1

$$\int_0^x f(x) dx = \int_0^x \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radius of convergence is h after diff and integ

Taylor Series of f at a

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

MacLaurin Series

$$\text{Taylor series of } f \text{ at } 0, \text{ i.e. } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

List of common MacLaurin Series

1. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, -1 < x < 1, R = 1$
2. $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, -1 < x < 1, R = 1$
3. $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} x^{2n}, -1 < x < 1, R = 1$
4. $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, -1 < x < 1, R = 1$
5. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, -\infty < x < \infty, R = \infty$
6. $\cos x = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, -\infty < x < \infty, R = \infty$
7. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, -\infty < x < \infty, R = \infty$
8. $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, -1 \leq x \leq 1, R = 1$
9. $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}, -1 < x < 1, R = 1$
10. $\frac{1}{(1-x)^3} = \frac{1}{2} \sum_{n=2}^{\infty} n(n-1) x^{n-2}, -1 < x < 1, R = 1$
1. $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, -1 < x < 1, R = 1$
2. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots, -1 < x < 1, R = 1$

Taylor Polynomials

The n -th order Taylor Polynomial of f at a

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

It gives a good polynomial approxn of order n

Calculus

$f(x) = P_n(x) + R_n(x)$ where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1} \text{ for } a < c < x.$$

$R_n(x)$ is remainder of order n or error term

6 Vector

Dot Product

$$v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, v_1 \cdot v_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$\cos \theta = \frac{v_1 \cdot v_2}{\|v_1\| \|v_2\|}, \text{ Projection of } b \text{ onto } a = \frac{b \cdot a}{\|a\|^2} a$$

Commut, assoc, distr, and $v_1 \cdot v_1 = \|v_1\|^2$

Cross Product

$$v_1 \times v_2 = (y_1 z_2 - y_2 z_1) \mathbf{i} - (x_1 z_2 - x_2 z_1) \mathbf{j} + (x_1 y_2 - x_2 y_1) \mathbf{k} \text{ Area}$$

of parallelogram $= \|v_1 \times v_2\| = \|v_1\| \|v_2\| \sin \theta$

Distr, assoc, but $v_1 \times v_2 = -v_2 \times v_1$ and $v_1 \times v_1 = 0$

7 Functions of Several Variables

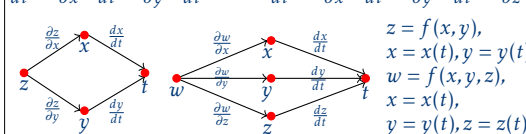
Partial Derivatives

of $z = f(x, y)$ w.r.t. x is denoted by $\frac{\partial z}{\partial x} \Big|_{(a,b)}$ or $f_x(a, b)$

Method: Fix the other variable (Note: $f_{xy} = f_{yx}$)

Chain Rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \text{ AND } \frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$



Directional Derivative

$f_x(a, b)$ is rate of change of f along direction of x -axis

Directional derivative of f at (a, b) in direction of unit vector

$$u = u_1 \mathbf{i} + u_2 \mathbf{j} \text{ is } D_u f(a, b) = f_x(a, b) u_1 + f_y(a, b) u_2$$

$$\text{or } D_u f(a, b, c) = f_x(a, b, c) u_1 + f_y(a, b, c) u_2 + f_z(a, b, c) u_3$$

$df = D_u f(a, b) \cdot dt$ (normal \cdot multiplication) measures change in f (df) when we move a small distance dt , and u is the unit directional vector of the change and (a, b) is the original pt

Gradient Vector

Denoted by $\nabla f = f_x \mathbf{i} + f_y \mathbf{j}$ where

$$\nabla f(a, b) \cdot u = D_u f(a, b) = \|\nabla f(a, b)\| \cos \theta$$

$$D_u f(a, b) > 0 \text{ and max when } \cos \theta = 1 \iff \theta = 0^\circ$$

$$D_u f(a, b) < 0 \text{ and min when } \cos \theta = -1 \iff \theta = 180^\circ$$

Max and Min Values

Critical Points - First Derivative Test

f has a local max or min at $(a, b) \wedge f_x$ exists $\wedge f_y$ exists \rightarrow

$$f_x = 0 \wedge f_y = 0 \text{ (But not the converse)}$$

Second Derivative Test

$$\text{Disriminant} = f_{xx}(a, b) f_{yy}(a, b) - f_{xy}(a, b)^2$$

1. $D > 0 \wedge f_{xx}(a, b) > 0 \rightarrow f$ has a local min at (a, b)
2. $D > 0 \wedge f_{xx}(a, b) < 0 \rightarrow f$ has a local max at (a, b)
3. $D < 0 \rightarrow f$ has a saddle-point at (a, b)
4. $D = 0 \rightarrow$ no conclusion

8 Ordinary Differential Equation (ODE)

No Crossing Principle: solution curves do not cross each other

First order ODE

There is only 1 soln for initial value problem with 1st order ODE. Intersection point of 2 curves is the initial pt.

1. $\frac{dy}{dx} = \frac{M(x)}{N(y)} \iff \int M(x) dx = \int N(y) dy$
2. $y' = f\left(\frac{y}{x}\right) \iff$ Let $v = \frac{y}{x}, f(v) = y' \iff \frac{dv}{f(v)-v} = \frac{dx}{x}$
3. $y' = \frac{ax+by+c}{a_1x+b_1y+c_1} \iff$ Let $u = ax+by$

$$\frac{dy}{dx} + p(x)y = Q(x) \Rightarrow y e^{\int p(x) dx} = \int Q(x) e^{\int p(x) dx} dx$$

5. Bernoulli eqn $y' + p(x)y = q(x)y^n \iff$

$$\text{Let } z = y^{1-n} \iff z' + (1-n)p(x)z = (1-n)q(x)$$

Scenarios

$$\text{Radioactive decay: } \frac{dx}{dt} = kx, x(t) = x(0)e^{-\frac{\ln 2}{\tau} t}$$

$$\text{Uranium-Thorium: } \frac{T}{U} = \frac{k_U}{k_T - k_U} (1 - e^{-(k_T - k_U)t}) k_N = \frac{\ln 2}{\tau_N}$$

$$\text{Cooling/Heating: } \int_{T-T_0}^T \frac{dT}{T-T_0} = \int k dt, T(t) - T_0 = (T(0) - T_0)e^{kt}$$

$$\text{Retarded fall: } m \frac{dv}{dt} = mg - bv^2,$$

$$v = k \frac{1+ce^{-pt}}{1-ce^{-pt}}, k^2 = \frac{mg}{b}, c = \frac{v(0)-k}{v(0)+k}, p = \frac{2kb}{m}$$

Hyperbolic Functions

$$\cosh x = \frac{e^x + e^{-x}}{2}, \sinh x = \frac{e^x - e^{-x}}{2}, \tanh x = \frac{\sinh x}{\cosh x} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

Second order linear ODE

$$\text{Form: } \frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = F(x)$$

homogeneous $\iff F(x) = 0$, else non-homogeneous

Homogeneous 2nd order linear ODE

$$\text{Linearly dependent } \iff \forall x \exists c \text{ s.t. } u(x) = cv(x)$$

y_1, y_2 are lin. indep. solns \implies a general soln is $y = c_1 y_1 + c_2 y_2$

y_1, y_2 are NOT lin. indep. solns $\implies y = c_1 y_1 + c_2 y_2$ is a soln but not a general soln

$$\frac{d^2 y}{dx^2} + A \frac{dy}{dx} + By = 0 \text{ has the trivial soln } y = 0 \text{ and non-trivial}$$

soln: Let $y = e^{\lambda x}$, solve $\lambda^2 + A\lambda + B = 0$, general soln is: (PS: Reverse is $A = -(\lambda_1 + \lambda_2), B = \lambda_1 \lambda_2$)

1. 2 real roots: $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
2. 1 real root: $y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$
3. 2 complex roots $(a + ib)$: $y = e^{ax} (c_1 \cos bx + c_2 \sin bx)$

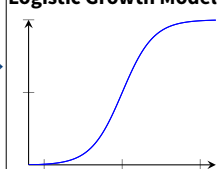
9 Mathematical Modelling (B = birth rate, D = death rate)

Malthusian Population Growth

$$N(t) = N(0)e^{kt} \text{ where } k = B - D. \text{ Conditions:}$$

1. $k > 0$ ($B > D$): popn explosion ($e^{kt} \rightarrow \infty, N(t) \rightarrow \infty$ as $t \rightarrow \infty$)
2. $k = 0$ ($B = D$): stable ($N(t) = N(0)$ for all t)
3. $k < 0$ ($B < D$): extinction ($e^{kt} \rightarrow 0, N(t) \rightarrow 0$ as $t \rightarrow \infty$)

Logistic Growth Model



$$\text{Eqn: } \frac{dN}{dt} = (B - D)N, N(0) = \hat{N}, N_\infty = \frac{B}{s}$$

$$\frac{dN}{dt} = (B - D)N = (B - sN)N = BN - sN^2 \text{ where } s \text{ is a small number compared to } B.$$

$$\frac{dN}{dt} = 0 \text{ when } N \approx \frac{B}{s} \text{ (population stops growing)}$$

This constant $\frac{B}{s}$ is called carrying capacity, sustainable population, or logistic equilibrium population. Or that the population stabilises at $\frac{B}{s}$

$$N(t) = \frac{B}{s + \left(\frac{B}{N_0} - s\right)e^{-Bt}} = \frac{N_\infty}{1 + \left(\frac{N_\infty}{N_0} - 1\right)e^{-Bt}}$$

$$\lim_{t \rightarrow \infty} N(t) = \frac{B}{s}$$

Case 1: $B - sN(t) > 0 \forall t$ (Popn < sustainable popn)

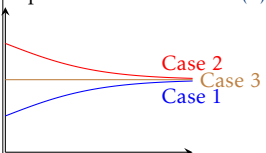
Logistic curve increasing

Case 2: $B - sN(t) < 0$ at all t (Popn > sustainable popn)

Logistic curve decreasing

Case 3: $B - sN(t) = 0$ at all t (At sustainable popn)

Population constant at $N(0)$



Harvesting

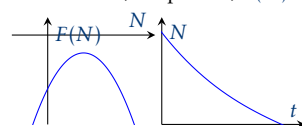
$$\frac{dN}{dt} = BN - sN^2 - E \text{ where } E \text{ is fish caught per year.}$$

DO NOT ATTEMPT TO SOLVE THE ODE. They will just ask to draw graph.

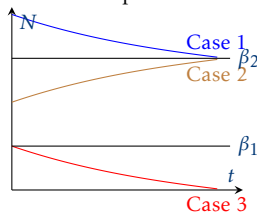
Method:

1. Let $F(N) = \frac{dN}{dt} = -sN^2 + BN - E$
2. Discriminant $= B^2 - 4(-s)(-E) = B^2 - 4sE$
3. Cases:

- (a) $D < 0$: No equilibrium soln (Popn is decreasing to extinction)
Note: $-s < 0$, shape is \cap , $F(N) \neq 0$

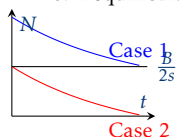


- (b) $D > 0$: 2 equilibrium solns
Solve $F(N)$ for β_1, β_2 where $\beta_1 < \beta_2 < \frac{B}{s}$
There are 3 possible cases:



$\frac{B}{s} = \beta_1 + \beta_2, \frac{E}{s} = \beta_1 \beta_2$
 β_2 is stable ($N(0)$ slightly diff from β_2 , popn will still tend to β_2). β_1 is not stable ($N(0)$ slightly diff from β_1 will not tend to β_1)

- (c) $D = 0$: 1 equilibrium solns



Suppose $N(0) > \frac{B}{2s}$ then max. harvesting w/o extinction $E = \frac{B^2}{4s}$

PS: more precise curves, follow the original logistic growth model graph (S-shaped) increasing: gentle-steep-gentle, decreasing: steep-gentle-steep

Additional Notes

- If the question is in powers above 2, e.g. $\frac{dN}{dt} = aN^4 + bN^3 + cN^2 + dN + e$, the same rule about the graph still applies: the stable populations are the solutions to $aN^4 + bN^3 + cN^2 + dN + e = 0$
- If there is no harvesting, then $N = 0$ is also a solution.