Unifying the Dark QCD

Chaitanya Paranjape - IIT(ISM) Dhanbad, India

Abstract

This is the technical report on the project of unifying the symmetry group for "dark force" SU(N_d) with the SM gauge group SU(3) ⊗ SU(2) ⊗ U(1). We will discuss our motivation for this project, our methods of approach and the future prospects in regards to this project.

Contents

1 Introduction

The recent discovery of Higgs boson [1],[2] corroborates the strong theoretical foundations of the Standard Model. The Higgs had been predicted since long accounting for the spontaneous symmetry breaking of the electroweak sector $\mathrm{SU}(2) \otimes \mathrm{U}(1)$. This remarkable achievement lays down the Standard model as our current best theory to explain the elementary particle interactions at high energies. However, there are still many open problems which motivate us to work with extensions of Standard model in search of beyond the Standard model physics. Accounting for neutrino oscillations and dark matter are just a few of them.

The nature of dark matter is probably one of the most sought after open problems in modern physics. The existence of dark matter has been confirmed from the astronomical observations but we still haven't been able to experimentally detect any dark matter candidate so far. We know the cold dark matter energy density in the universe is $\Omega_c h^2 = 0.120 \pm 0.001$ [3] which is almost five times of the baryon energy density $\Omega_b h^2 = 0.0224 \pm 0.0001$ [3]. To explain this cold dark matter energy density, there have been many theoretical advancements introducing prominent dark matter candidates like WIMPs, Axion which are actively being searched for (Refer [6] for review on dark matter candidates). Motivated by the comparable energy densities $\Omega_c \sim 5 \cdot \Omega_b$, we will base our work on the model of "Dark QCD" introduced by Y. Bai and P. Schwaller in [7]. The energy density for the ordinary matter is mostly accounted by the theory of strong interaction(QCD), which gives rise to Baryon masses related with Λ_{QCD} confinement scale in QCD. This motivates us to believe a similar dynamics with strong couplings should exist in the dark matter sector, and in order to produce comparable energy densities, they could have Baryon masses $m_D \sim m_p$ and number densitites $n_D \sim n_B$ of the same order. The theoretical models to explore these possibilities have been treated in [7].

Therefore, to assume an asymptotic-free QCD-like dynamics in the dark sector, it has been assigned the symmetry group $\mathrm{SU}(N_d)$ identical to the color symmetry $\mathrm{SU}(N_c)$ with $N_d = N_c = 3$. Our target during this project would be to unify the symmetry group for this 'dark force' with the symmetry group associated with the Standard model of particle physics $\mathrm{SU}(3)\otimes\mathrm{SU}(2)\otimes\mathrm{U}(1)$. Please refer [8] for a review on Standard model and unification theories.

2 Grand Unified Theories :

Goal of this project is to unify the dark force symmetry group SU(3) with the Standard model symmetry group $\mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$. Quantitatively, we are interested to unify $\mathrm{SU}(3) \otimes \mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$ as a sub-algebra of some larger group G :

$$G \supset \mathrm{SU}(3) \otimes \mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$$

This is the overall concept of Grand unified theories. Since we don't observe these perfect symmetries directly, they are sometimes hidden through the mechanism of spontaneous symmetry breaking. Before we move on to Unified theories, let's first review the attributes associated with Standard model $\mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$.

The SM gauge theory involves SU(3) for strong interactions with 8 massless gauge bosons in the form of gluons. The $\mathrm{SU}(2) \otimes \mathrm{U}(1)$ is called the electroweak sector which via Higgs mechanism undergoes spontaneous symmetry breaking and is observed as the $U(1)_{em}$ of the Electromagnetic interactions. The 3 gauge bosons (W^+, W^-, Z) acquire mass through this mechanism while the photon remains mass-less. We will explore the details of the mechanism in more detail when we get to the spontaneous symmetry breaking part. The matter content is given by the 15 Weyl fermions which transform in 5 representation of the group as follows :

$$\begin{aligned}
u^\dagger &= (3, 1, 2/3) \\
d^\dagger &= (3, 1, -1/3) \\
e^\dagger &= (1, 1, -1) \\
\overline{\psi}^\dagger &= (\overline{3}, 2, -1/6) \\
\overline{l}^\dagger &= (1, 2, 1/2)
\end{aligned} \tag{1}$$

The above transformations are for right-handed fermions[1], where $\overline{\psi}_1^\dagger = \overline{d}^\dagger$, $\overline{\psi}_2^\dagger = \overline{u}^\dagger$ and $\overline{l}_1^\dagger = \overline{e}^\dagger$, $\overline{l}_2^\dagger = \overline{\nu}^\dagger$. Where, the first number corresponds to representation under SU(3), such that the **3** is fundamental representation (transforming like a triplet under SU(3)), $\overline{3}$ is it's complex conjugation and **1** is said to transform as a singlet. Similarly, the second number is the transformation under SU(2) and the third number is the Hypercharge of U(1) transformation. The origin of this numbers is related to transformation of corresponding creation operators $a^\dagger_{xr}$ under the generators of the gauge group. We say it transforms in the representation (D, d, s) if it satisfies :

$$[T_a, a^\dagger_{xr}] = a^\dagger_{yr}[T_a^D]_{yx}$$

$$[R_a, a^\dagger_{xr}] = a^\dagger_{xt}[R_a^d]_{tr}$$
$$[S, a^\dagger_{xr}] = s a^\dagger_{xr}$$

Where the T_a are the 8 generators of the SU(3), R_a are 3 generators of the SU(2) and S is the lone generator of U(1) gauge group. The Cartan generators for SU(3) are T_3, T_8, for SU(2) it is the R_3 (weak iso-spin generator) and for U(1) it is the S generator. Adding them up it tells us that the Standard Model gauge group has a rank 4, and therefore, any group that we an hypothesize to contain SM as its sub-algebra, must at least have rank 4. This just being the initial requirement, additionally, the required gauge group must also be able to have the required transformation algebra when decomposed from its ir-reducible representations. For example, for the above mentioned right handed particles of standard model, their transformation algebra under the action of SM gauge group can be stated in one line as follows :

$$(1, 1, -1) \oplus (3, 1, 2/3) \oplus (\overline{3}, 2, -1/6) \oplus (1, 2, 1/2) \oplus (3, 1, -1/3) \tag{2}$$

For left handed particles, just take the complex conjugation (use $2 = \overline{2}$) :

$$(1, 1, 1) \oplus (\overline{3}, 1, -2/3) \oplus (3, 2, 1/6) \oplus (1, 2, -1/2) \oplus (\overline{3}, 1, 1/3) \tag{3}$$

Therefore, the larger unifying gauge group we hope to achieve, must have its ir-reducible representations decomposable to transform according to the sub-algebra given by equations 2,3. For a detailed review on this topic, review the book on “Lie algebras in particle physics” by H. Georgi [9].

The first natural gauge group to consider the unifying theory was SU(5), with rank 4 and having ir-reducible representations that can decomposed to equations 2,3. This SU(5) gauge group was proposed by H. Georgi and S. Glashow [**?**] and it was one of the first Unified theory that had been studied extensively and has been under the experimental scrutiny since long. It predicts proton decay and provides answers to some of the inadequacies of Standard model, but the experimental evidence has ruled it out as the viable GUT. Nonetheless, it is a perfect start to study the structure of GUTs and use that knowledge to our problem.

[1]The † indicates the creation operator for corresponding particle.

One can go ahead and write the branching rules for totally anti-symmetric ir-reps[2](basic ir-reps) of SU(5) to decompose them to SM algebra : $\mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$ as follows :

$$5 \to (1, 2, 1/2) \oplus (3, 1, -1/3)$$

$$\overline{5} \to (1, 2, -1/2) \oplus (\overline{3}, 1, 1/3)$$

$$10 \to (1, 1, 1) \oplus (\overline{3}, 1, -2/3) \oplus (3, 2, 1/6)$$

$$\overline{10} \to (1, 1, -1) \oplus (3, 1, 2/3) \oplus (\overline{3}, 2, -1/6)$$

Therefore, we can observe that the transformation for right handed particles in SM can be given as transformation under SU(5) as $\overline{10} \oplus 5$ which agrees with equation 2. Similarly, the left handed particles transform as $\overline{5} \oplus 10$ under SU(5), agreeing with equation 3. Which justifies our choice of SU(5) as the unifying gauge group for SM.

For our project, we aim to follow the similar concepts for GUTs and provide a consistent gauge group for unifying the "dark force" symmetry group with the SM gauge group.

3 Outline of the project

From the example of SU(5), we have two natural thoughts for approach to our problem. First being, directly trying to test some intuitive choices for gauge groups with rank ≥ 6 and identifying whether they can provide suitable branching rules for decomposition of ir-reps to the required gauge group : $\mathrm{SU}(3) \otimes \mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$. In this approach, we are firstly not concerned with matter fields for representation and we may choose to accommodate them accordingly, to satisfy the right transformation properties.

Secondly, we can directly introduce two dark quarks in our model $u_D^\dagger$ & $\overline{u}_D^\dagger$, transforming in the $\mathrm{SU}(3) \otimes \mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$ gauge group according to following representation :

$$u_D^\dagger = (3, 1, 1, 0) \quad \overline{u}_D^\dagger = (\overline{3}, 1, 1, 0)$$

The motivation being, dark matter doesn't interact electromagnetically and should tranform as singlet under SU(3) color and SU(2). Thus, we can write the other SM particles transformation properties in this gauge group, and write down a similar equation like the equation 2. Then, we may test the decomposition of ir-reps of candidate gauge groups to accomodate the right transformation property we require. We may need to account for additional matter fields in this approach as well which we can take care of accordingly.

Both approaches are in the crude stage right now, and I will be focusing on finding the right gauge group and testing out all the possibilities. Simultaneously, I will be focusing on studying the group algebra and representation theory in detail from the book "Lie algebras in particle physics" by H. Georgi [9], and I hope to make progress on the project along the way. I have also provided my own solutions to problems of first few chapters as an indication for the progress.

4 Outlook

We look forward to provide a consistent gauge group for our problem and based on our results, we will try to explore phenomenological relevance including possible experimental signatures as well.

[2] From now on we will abbreviate ir-reducible representations as ir-reps.

Method 1 :

Decompose the $8 \oplus \overline{28}$ of $\mathrm{SU}(8) \to \mathrm{SU}(5) \otimes \mathrm{SU}(3) \otimes \mathrm{U}'(1)$

$$8 \to (1,3)(-5) \oplus (5,1)(3)$$
$$\overline{28} \to (1,3)(10) \oplus (\overline{5},\overline{3})(2) \oplus (\overline{10},1)(-6)$$

We can see it provides 5, $\overline{10}$ in SU(5) which should correspond to SM transformation properties with dark singlets[3]. And also has some terms with dark triplets and SM singlets. However there is a term with transformation properties under both, dark and ordinary symmetry group. Nonetheless, let's go ahead and further decompose the SU(5) to the SM gauge group $\mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$. We obtain following :

$$(1,3)(-5) \to (1,1,3)(0)(-5)$$

Note that the numbers correspond to transformations according to :
(SU(3) color, SU(2) weak, SU(3) dark)(U(1) Hypercharge)(U'(1) Dark charge). Thus, the final decomposition with all terms is :

$$\begin{aligned}
(1,3)(-5) &\to (1,1,3)(0)(-5)\\
(5,1)(3) &\to (1,2,1)(-3)(3) \oplus (3,1,1)(2)(3)\\
(1,3)(10) &\to (1,1,3)(0)(10)\\
(\overline{5},\overline{3})(2) &\to (1,2,\overline{3})(3)(2) \oplus (\overline{3},1,\overline{3})(-2)(2)\\
(\overline{10},1)(-6) &\to (1,1,1)(6)(-6) \oplus (3,1,1)(-4)(-6) \oplus (\overline{3},2,1)(1)(-6)
\end{aligned}$$

Rep term	SU(3) color	SU(2) weak	SU(3) dark	U(1) Y_W	U′(1) Y_D	Plausible matter?
$(1,1,3)(0)(-5)$	1	1	3	0	-5	dark quark-1
$(1,2,1)(-3)(3)$	1	2	1	-3	3	Anti-lepton SM doublet
$(3,1,1)(2)(3)$	3	1	1	2	3	d quark SM
$(1,1,3)(0)(10)$	1	1	3	0	10	dark quark-2
$(1,2,\overline{3})(3)(2)$	1	2	$\overline{3}$	3	2	Charged weak-dark doublet
$(\overline{3},1,\overline{3})(-2)(2)$	$\overline{3}$	1	$\overline{3}$	-2	2	Charged color-dark triplet
$(1,1,1)(6)(-6)$	1	1	1	6	-6	SM electron
$(3,1,1)(-4)(-6)$	3	1	1	-4	-6	SM u quark
$(\overline{3},2,1)(1)(-6)$	$\overline{3}$	2	1	1	-6	SM anti-quark doublet

Table 1: Method 1 decomposition

[3]Note that this corresponds to right handed particles set.

Method 2 :

Decompose the basic ir-reps $\mathrm{SU}(8) \to \mathrm{SU}(6) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$:

$$
\begin{aligned}
8 &\to (1,2)(-3) \oplus (6,1)(1) \\
28 &\to (1,1)(-6) \oplus (6,2)(-2) \oplus (15,1)(2) \\
56 &\to (6,1)(-5) \oplus (15,2)(-1) \oplus (20,1)(3) \\
70 &\to (15,1)(-4) \oplus (\overline{15},1)(4) \oplus (20,2)(0) \\
\overline{56} &\to (\overline{6},1)(5) \oplus (\overline{15},2)(1) \oplus (20,1)(-3) \\
\overline{28} &\to (1,1)(6) \oplus (\overline{6},2)(2) \oplus (\overline{15},1)(-2) \\
\overline{8} &\to (1,2)(3) \oplus (\overline{6},1)(-1)
\end{aligned}
$$

But before we decompose $\mathrm{SU}(6) \to \mathrm{SU}(3) \otimes \mathrm{SU}(3) \otimes \mathrm{U}^{'}(1)$, we should check which representations contain the SM representations as well. It makes sense to associate $\mathrm{SU}(2) \otimes \mathrm{U}(1)$ as the electro-weak sector and let SU(6) contain the color and dark symmetry groups. Thus, to contain the SM particle contents, we should pick the decompositions such that they include following content (read off from the table 1) :

$$(\#,2)(-3) \oplus (\#,1)(2) \oplus (\#,1)(6) \oplus (\#,1)(-4) \oplus (\#,2)(1)$$

Or the complex conjugation of it :

$$(\#,2)(3) \oplus (\#,1)(-2) \oplus (\#,1)(-6) \oplus (\#,1)(4) \oplus (\#,2)(-1)$$

We can observe that if we try to include these from the $\mathrm{SU}(8) \to \mathrm{SU}(6) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$, we will have to include almost all of the basic ir-reps, giving rise to un-necessary complexity and just lot of extra matter field transformations. Thus, we can hold off the investigation of the particular decomposition $\mathrm{SU}(8) \to \mathrm{SU}(6) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$ as it may lead to un-necessary complexity with lots of additional representations.

5 Lie Groups (Chapter 2) problems :

2A :

The matrix A is given as :

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

By calculating A^2 and A^3 we observe that A follows relationship $A^{m+2} = A^m$ for all positive integers m. Therefore, exponential Taylor expansion can be expressed in A and A^2 matrices only.

$$e^{i\alpha A} = \sum_{n=0}^{n=\infty} \frac{(i\alpha A)^n}{n!} = \big(\cdots\big)A + \big(\cdots\big)A^2$$

where, the coefficients for A and A^2 can be collected to identify with the Taylor expansions of hyperbolic functions.

$$\sinh x = x + \frac{x^3}{3!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \cdots$$

Finally, with all of the algebra, we can express $e^{i\alpha A}$ as folows :

$$e^{i\alpha A} = \big(i\sin\alpha\big)A + \big(\cos\alpha\big)A^2 = \begin{pmatrix} \cos\alpha & 0 & i\sin\alpha \\ 0 & 0 & 0 \\ i\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

2B :

Assuming A and B are independent of α we define,

$$f(\alpha) = e^{i\alpha A} B e^{-i\alpha A}$$

Because A obviously commutes with itself, we can write,

$$\frac{d}{d\alpha} e^{i\alpha A} = (iA)e^{i\alpha A} = e^{i\alpha A}(iA)$$

Therefore, we can now calculate

$$\frac{d}{d\alpha} f(\alpha) = i \cdot \Big(e^{i\alpha A} AB e^{-i\alpha A} - e^{i\alpha A} BA e^{-i\alpha A}\Big)$$

$$\frac{d}{d\alpha} f(\alpha) = i \cdot \Big(e^{i\alpha A}[A, B] e^{-i\alpha A}\Big)$$

If $[A, B] = B$ then we get a simple differential equation :

$$\frac{df(\alpha)}{d\alpha} = i \cdot \Big(e^{i\alpha A} B e^{-i\alpha A}\Big) = i \cdot f(\alpha)$$

Solution of which is given by,

$$f(\alpha) = Ce^{i\alpha}$$

But we know $f(0) = B$. Therefore, we can finally conclude that

$$f(\alpha) = e^{i\alpha A} B e^{-i\alpha A} = e^{i\alpha} B$$

2C :

3rd order expansion would require a bit extra hand work and lot of care, therefore I chose to skip it for now, considering I have got the gist of the expansion.

6 SU(2) (Chapter 3) problems :

3A :

It would be complicated to write the general procedure. I have understood the procedure involved in the last section (section 3.5 : J_3 values add), so I will choose to skip this for now. Besides, I feel comfortable with the well known singlet-triplet example with the irreducible representations of spin-0 and spin-1.

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

3B :

Let us first note down the handy relations of standard Pauli matrices corresponding to the SU(2) generators.

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$$

From the last identity we contract the indices to derive a identity that we will use,

$$u_i v_j \sigma_i \sigma_j = u_i v_j \delta_{ij} + i u_i v_j \epsilon_{ijk}\sigma_k$$

Which in vector notation reduces to,

$$(\vec{u} \cdot \vec{\sigma})(\vec{v} \cdot \vec{\sigma}) = \vec{u} \cdot \vec{v} + i(\vec{u} \times \vec{v}) \cdot \vec{\sigma}$$

Setting $\vec{u} = \vec{v} = \hat{r}$ a unit vector we get,

$$(\hat{r} \cdot \vec{\sigma})^2 = \mathbb{1}_2$$

Therefore, it is an Identity matrix for even powers and the original matrix for odd powers. We go ahead and expand the exponential $e^{ir(\hat{r}\cdot\vec{\sigma})}$ in even and odd powers as follows :

$$e^{ir(\hat{r}\cdot\vec{\sigma})} = \mathbb{1}_2 \cdot \sum_{n=0}^{n=\infty} \frac{(ir)^{2n}}{(2n)!} + (\hat{r} \cdot \vec{\sigma}) \cdot \sum_{n=0}^{n=\infty} \frac{(ir)^{2n+1}}{(2n+1)!}$$

We can replace the sums by identifying them as the expansions of cos and sin functions respectively.

$$e^{ir(\hat{r}\cdot\vec{\sigma})} = \mathbb{1}_2 \cdot \big(\cos r\big) + (\hat{r} \cdot \vec{\sigma}) \cdot \big(i \cdot \sin r\big)$$

$$e^{i(\vec{r}\cdot\vec{\sigma})} = \mathbb{1}_2 \cdot \cos(|\vec{r}|) + i \cdot \left(\frac{\vec{r} \cdot \vec{\sigma}}{|\vec{r}|}\right) \cdot \sin(|\vec{r}|)$$

3C :

Spin-1 representation matrices for SU(2) are as follows :

$$J_1 = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_2 = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The structure constant of generators is ϵ_{abc} thus the adjoint representation matrices $[T_a]_{bc} = -i\epsilon_{abc}$ are given as :

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Two matrices A and B are said to be similar if there exists a non-singular matrix P responsible for the following similarity transformation :

$$A = P \cdot B \cdot P^{-1}$$

To find the P matrix, the matrices A and B should have same set of eigenvalues such that when they are diagonalized they are identical. Their Diagonalized form can be written as :

$$D = M^{-1} \cdot A \cdot M \quad \& \quad D = N^{-1} \cdot B \cdot N$$

Where, M and N are eigenvector matrices of A and B respectively. Then the P matrix responsible for similarity transformation can be obtained with a little algebra to be $P = M \cdot N^{-1}$. We can efficiently calculate all P_i matrices relating $J_i = P_i \cdot T_i \cdot P_i^{-1}$ using following Mathematica[4] code :

```
B = {{0, 0, 0}, {0, 0, -i}, {0, i, 0}};
A = 1/Sqrt[2]*{{0, 1, 0}, {1, 0, 1}, {0, 1, 0}};
vecsB = Eigenvectors[B];
vecsA = Eigenvectors[A];
P = Transpose[vecsA].Inverse[Transpose[vecsB]] // Chop // MatrixForm
```

We obtain the corresponding P matrices as follows :

$$P_1 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & i\sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & 1 & -1 \\ -\sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad P_3 = \frac{1}{2} \cdot \begin{pmatrix} i & 1 & 0 \\ 0 & 0 & 2 \\ -i & 1 & 0 \end{pmatrix}$$

3D :

We know σ_2 and η_1 are given as follows :

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \eta_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $\sigma_2 \otimes \eta_1$ can directly be given as :

$$\sigma_2 \otimes \eta_1 = \begin{pmatrix} 0 \otimes \eta_1 & -i \otimes \eta_1 \\ i \otimes \eta_1 & 0 \otimes \eta_1 \end{pmatrix}$$

$$\sigma_2 \otimes \eta_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

3E :

Because σ and η are two different independent representations, they commute with each other. We will write the pauli matrices like matrices and use the identities to calculate.

[4]replace i with appropriate command for i in mathematica which is esc i i esc .

$$\sigma_1(\sigma_2\eta_1) = (\sigma_1\sigma_2)\eta_1 = i\sigma_3\eta_1$$

$$(\sigma_1\eta_2)(\sigma_1\eta_3) = (\sigma_1^2)(\eta_2\eta_3) = i\eta_1$$

The problem (a) is :

$$[\sigma_a, \sigma_b\eta_c] = [\sigma_a, \sigma_b]\eta_c = 2i\epsilon_{abd}\sigma_d\eta_c$$

The problem (c) is :

$$[\sigma_1\eta_1, \sigma_2\eta_2] = \sigma_1\sigma_2\eta_1\eta_2 - \sigma_2\sigma_1\eta_2\eta_1 = (i\sigma_3 \cdot i\eta_3) - (-i\sigma_3) \cdot (-i\eta_3) = 0$$

For problem (b) let's frist calculate :

$$\sigma_a\{\eta_b, \sigma_c\eta_d\} = \sigma_a\sigma_c\{\eta_b, \eta_d\} = \sigma_a\sigma_c 2\delta_{bd} = 2(\delta_{ac} + i\epsilon_{acm}\sigma_m)\delta_{bd}$$

Therefore,

$$\mathrm{Tr}\big(2(\delta_{ac} + i\epsilon_{acm}\sigma_m)\delta_{bd}\big) = 2 \cdot \mathrm{Tr}\big(\delta_{ac}\delta_{bd}\mathbb{1}_2\big) = 4\delta_{ac}\delta_{bd}$$

7 Tensor Operators (Chapter 4) problems :

First, I will note down all the different formulas which are relevant enough to refer back.

$$J^+ |j, m\rangle = \sqrt{\frac{(j-m)(j+m+1)}{2}} |j, m+1\rangle$$
$$J^- |j, m\rangle = \sqrt{\frac{(j+m)(j-m+1)}{2}} |j, m-1\rangle$$

We first write down the states for $1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$ with highest weight decomposition and acting with J^- repeatedly, and then taking the orthonormal states.

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle = |1, 1\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle$$
$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \frac{\sqrt{2}}{\sqrt{3}} \cdot |1, 0\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}} \cdot |1, 1\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$
$$\left|\frac{3}{2}, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}} \cdot |1, -1\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle + \frac{\sqrt{2}}{\sqrt{3}} \cdot |1, 0\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$
$$\left|\frac{3}{2}, -\frac{3}{2}\right\rangle = |1, -1\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$
$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}} \cdot |1, 0\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle - \frac{\sqrt{2}}{\sqrt{3}} \cdot |1, 1\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$
$$\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \frac{\sqrt{2}}{\sqrt{3}} \cdot |1, -1\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}} \cdot |1, 0\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

spin 1/2 representation of the SU(2) generators :

$$J_1 = \frac{1}{2} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad J_2 = \frac{1}{2} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad J_3 = \frac{1}{2} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$J^+ = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J^- = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

spin 1 representation of the SU(2) generators :

$$J_1 = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_2 = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$J^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad J^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Tensorial position operators in spin 1 representation : $\{r_{+1}, r_0, r_{-1}\}$ in terms of the position operators : $\{r_1, r_2, r_3\}$

$$r_{+1} = \frac{-r_1 - ir_2}{\sqrt{2}}$$

$$r_0 = r_3$$
$$r_{-1} = \frac{r_1 - ir_2}{\sqrt{2}}$$

Using the algebra $[J_a, r_b] = -i\epsilon_{acb} r_c$, we can go ahead and derive the above relationa, such that they are constrained with following algebra : (General Tensor operator transformation)

$$[J_a, O^s_l] = O^s_m [J^s_a]_{ml}$$

Let us also note down a few handy commutation relations :

$$[J^+, r_{+1}] = 0 \qquad [J^-, r_{+1}] = r_0$$
$$[J^+, r_0] = r_{+1} \qquad [J^-, r_0] = r_{-1}$$
$$[J^+, r_{-1}] = r_0 \qquad [J^-, r_{-1}] = 0$$

Finally note down the Wigner-Eckart formula to help solve problems like the problem 4A.

$$\langle J, m', \beta| O^s_l |j, m, \alpha\rangle = \delta_{m', l+m} \cdot \Big(\langle J, l+m| \cdot |s, l\rangle |j, m\rangle \Big) \cdot k_{\alpha\beta}$$

Where $k_{\alpha\beta} = \langle J, \beta| O^s |j, \alpha\rangle$ are reduced matrix elements, and the quantity inside the round brackets is the Clebsch-Gordan Coefficient. Let us see how to the above formula and other identities to solve the textbook problem. First it is given that

$$\left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| r_3 \middle| \frac{1}{2}, \frac{1}{2}, \beta \right\rangle = A$$

$$\implies \left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| r_0 \middle| \frac{1}{2}, \frac{1}{2}, \beta \right\rangle = A$$

To calculate $\left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| r_1 \middle| \frac{1}{2}, -\frac{1}{2}, \beta \right\rangle$, we first need to express it in terms of tensorial operators to be able to use the Wigner-Eckart formula. Therefore, we calculate and substitute $r_1 = \frac{-r_{+1}+r_{-1}}{\sqrt{2}}$. However, note that $[J^-, r_0] = r_{-1}$. Thus, contribution from r_{-1} is zero as J^- kills both of the states on either left or right.

$$\implies \left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| r_1 \middle| \frac{1}{2}, -\frac{1}{2}, \beta \right\rangle = \frac{-1}{\sqrt{2}} \cdot \left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| r_{+1} \middle| \frac{1}{2}, -\frac{1}{2}, \beta \right\rangle$$

Now let us use the Wigner-Eckart formula on both of the matrix elements of interest.

$$A = \left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| r_0 \middle| \frac{1}{2}, \frac{1}{2}, \beta \right\rangle = \left(\left\langle \frac{1}{2}, \frac{1}{2} \middle| \cdot |1, 0\rangle \middle| \frac{1}{2}, \frac{1}{2} \right\rangle \right) \cdot k_{\beta\alpha}$$

$$? = \left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| r_{+1} \middle| \frac{1}{2}, -\frac{1}{2}, \beta \right\rangle = \left(\left\langle \frac{1}{2}, \frac{1}{2} \middle| \cdot |1, 1\rangle \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle \right) \cdot k_{\beta\alpha}$$

The corresponding Clebsch-Gordan coefficients can be noted down from the above $1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$ decomposition. Taking the ratio, we finally obtain

$$\implies \left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| r_{+1} \middle| \frac{1}{2}, -\frac{1}{2}, \beta \right\rangle = -A\sqrt{2}$$

Therefore substituting back to get the required answer,

$$\implies \left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| r_1 \middle| \frac{1}{2}, -\frac{1}{2}, \beta \right\rangle = A$$

4A :

Given is a spin 1/2 operator with transformation :

$$[J_a, O_x] = \frac{O_y}{2} \cdot [\sigma_a]_{yx}$$

with $x = 1, 2$ and σ_a are the Pauli matrices. First let us note down the handy Commutation relations that follow from this :

$$[J^+, O_1] = 0 \qquad [J^+, O_2] = \frac{O_1}{\sqrt{2}}$$

$$[J^-, O_1] = \frac{O_2}{\sqrt{2}} \qquad [J^-, O_2] = 0$$

$$[J_3, O_1] = \frac{O_1}{2} \qquad [J_3, O_2] = -\frac{O_2}{2}$$

Actually, this problem is rather easy to just solve directly, so we will first do it directly without Tensorial operator description. Later, I will also do it by identifying the Tensorial operators. Given :

$$\langle 3/2, -1/2, \alpha | \, O_1 \, |1, -1, \beta\rangle = A$$

and we need to find

$$\langle 3/2, -3/2, \alpha | \, O_2 \, |1, -1, \beta\rangle = ?$$

We can use $O_2 = \sqrt{2} \cdot [J^-, O_1] = \sqrt{2} \cdot (J^- O_1 - O_1 J^-)$. However, observe that J^- kills the state on right. We have only the first term $\sqrt{2} J^- O_1$, and we make the J^- act on left state, which is equivalent to its hermitian conjugate acting on right, which will be

$$J^+ \, |3/2, -3/2, \alpha\rangle = \frac{\sqrt{3}}{\sqrt{2}} \, |3/2, -1/2, \alpha\rangle$$

Taking the hermitian conjugate we can write

$$\implies \langle 3/2, -1/2, \alpha | \, O_2 \, |1, -1, \beta\rangle = \sqrt{3} \cdot \langle 3/2, -1/2, \alpha | \, O_1 \, |1, -1, \beta\rangle = A\sqrt{3}$$

Which is our required answer.

Or, one can also identify the required spin 1/2 Tensorial Operators, as some linear combinations of O_1 and O_2. Turns out[5], the Operators are already in the standard tensorial description form[6]. We can identify $O_1 = O_{+1/2}$ and $O_2 = O_{-1/2}$ as they follow the exact commutation algebra as the Tensorial operators $O_{+1/2}, O_{-1/2}$ should. We can use the Wigner-Eckart theorem now for the matrix elements of interest.

$$A = \langle 3/2, -1/2, \alpha | \, O_{+1/2} \, |1, -1, \beta\rangle = \Big(\langle 3/2, -1/2| \cdot |1/2, 1/2\rangle \, |1, -1\rangle \Big) \cdot k_{\alpha\beta} = \frac{1}{\sqrt{3}} \cdot k_{\alpha\beta}$$

$$? = \langle 3/2, -3/2, \alpha | \, O_{-1/2} \, |1, -1, \beta\rangle = \Big(\langle 3/2, -3/2| \cdot |1/2, -1/2\rangle \, |1, -1\rangle \Big) \cdot k_{\alpha\beta} = k_{\alpha\beta}$$

Where we obtained the Clebsch-Gordan coefficients from the earlier defined $1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$ decomposition. Now we just take ratios to find

$$\implies \langle 3/2, -3/2, \alpha | \, O_{-1/2} \, |1, -1, \beta\rangle = A\sqrt{3}$$

Which is consistent with our earlier answer.

[5] Not really turns out, it is obvious because Pauli matrices are the spin 1/2 representation of J_a.

[6] Which is why I guess it was easier to just solve directly.

4B :

Given : $[J^+, (r_{+1})^2] = 0$

spin 2 representation of the SU(2) generators :

$$J^+ = \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad J^- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

We know a spin 2 tensor operator should satisfy following transformation :

$$[J^-, O_{+2}] = O_{+1}\sqrt{2}$$
$$[J^-, O_{+1}] = O_0\sqrt{3}$$
$$[J^-, O_0] = O_{-1}\sqrt{3}$$
$$[J^-, O_{-1}] = O_{-2}\sqrt{2}$$
$$[J^-, O_{-2}] = 0$$

Commuting with J^- and using $[A, BC] = [A, B]C + B[A, C]$ recursively, we can derive following relations starting from $(r_{+1})^2$:

$$[J^-, r_{+1}r_{+1}] = r_0r_{+1} + r_{+1}r_0$$
$$[J^-, r_0r_{+1} + r_{+1}r_0] = r_{-1}r_{+1} + 2(r_0)^2 + r_{+1}r_{-1}$$
$$[J^-, r_{-1}r_{+1} + 2(r_0)^2 + r_{+1}r_{-1}] = 3r_0r_{-1} + 3r_{-1}r_0$$
$$[J^-, 3r_0r_{-1} + 3r_{-1}r_0] = 6(r_{-1})^2$$
$$[J^-, 6(r_{-1})^2] = 0$$

By comparing with transformation relations for O_m operators, we can easily establish following relations :

$$O_{+2} = (r_{+1})^2$$
$$O_{+1} = \frac{r_0r_{+1} + r_{+1}r_0}{\sqrt{2}}$$
$$O_0 = \frac{r_{-1}r_{+1} + 2(r_0)^2 + r_{+1}r_{-1}}{\sqrt{6}}$$
$$O_{-1} = \frac{r_0r_{-1} + r_{-1}r_0}{\sqrt{2}}$$
$$O_{-2} = (r_{-1})^2$$

Now if we set $r_1 = \sin(\theta)\cos(\phi)$, $r_2 = \sin(\theta)\sin(\phi)$, $r_3 = \cos(\theta)$

$$\implies r_{+1} = \frac{-\sin(\theta)e^{i\phi}}{\sqrt{2}} \quad r_{-1} = \frac{\sin(\theta)e^{-i\phi}}{\sqrt{2}} \quad r_0 = \cos(\theta)$$

Then we can clearly identify our O_m operators with spherical harmonics $Y_l^m(\theta,\phi)$ with $l = 2$, with the following relation :

$$Y_2^m(\theta,\phi) = \left(\frac{1}{2}\cdot\sqrt{\frac{15}{2\pi}}\right)\cdot O_m$$

possible generalized conclusion : spin j tensor operators constructed with products of spin 1 position tensor operators, are proportional to spin j spherical harmonics.

4C :

This problem is similar to the problem 3B . In place of the Identity matrix, we need to input the said projection operator. Remember that for any projection operator P, it follows $P^n = P$, [7] for all positive integers n. We can write the format for final answer as :

$$e^{i\alpha\cdot\hat{\alpha_a}X_a} = \mathbb{1} + [\cos(\alpha)-1]\cdot(\hat{\alpha_a}X_a)^2 + i\sin(\alpha)\cdot(\hat{\alpha_a}X_a)$$

[7] Once you project onto the system, consequent identical projections will be redundant

8 Roots and weights (Chapter 6) problems :

6A :

Using $[H_i, E_\alpha] = \alpha_i E_\alpha$ and $[H_i, E_\beta] = \beta_i E_\beta$, we can calculate the direct commutator with the cartan generator H_i

$$\begin{aligned}
[H_i, [E_\alpha, E_\beta]] &= [[H_i, E_\alpha], E_\beta] + [E_\alpha, [H_i, E_\beta]] \\
&= \alpha_i [E_\alpha, E_\beta] + \beta_i [E_\alpha, E_\beta] \\
&= (\alpha + \beta)_i [E_\alpha, E_\beta] \\
[H_i, [E_\alpha, E_\beta]] &= (\alpha + \beta)_i [E_\alpha, E_\beta]
\end{aligned}$$

Therefore, we can conclude that $[E_\alpha, E_\beta] \sim E_{\alpha+\beta}$

$$\implies [E_\alpha, E_\beta] = N \cdot E_{\alpha+\beta}$$

If $\alpha + \beta$ is not a root, we won't be able to conclude the same. $(N = 0\ ?\ )$

6B :

Similar to problem 5A, we can prove that $[E_\alpha, E_{-\alpha-\beta}] \sim E_{-\beta}$ and $[E_\beta, E_{-\alpha-\beta}] \sim E_{-\alpha}$. Let's assume following :

$$\begin{aligned}
[E_\alpha, E_{-\alpha-\beta}] &= P \cdot E_{-\beta} \\
[E_\beta, E_{-\alpha-\beta}] &= Q \cdot E_{-\alpha} \\
[E_\alpha, E_\beta] &= N \cdot E_{\alpha+\beta}
\end{aligned}$$

Let's apply the Jacobi Identity to $E_\alpha, E_\beta, E_{-\alpha-\beta}$: (Also use $[E_\gamma, E_{-\gamma}] = \gamma_i H_i$

$$\begin{aligned}
0 &= [E_\alpha, [E_\beta, E_{-\alpha-\beta}]] + [E_{-\alpha-\beta}, [E_\alpha, E_\beta]] + [E_\beta, [E_{-\alpha-\beta}, E_\alpha]] \\
0 &= Q \cdot [E_\alpha, E_{-\alpha}] - N \cdot [E_{\alpha+\beta}, E_{-\alpha-\beta}] - P \cdot [E_\beta, E_{-\beta}] \\
0 &= (Q\alpha_i - N\alpha_i - N\beta_i - P\beta_i) H_i \\
0 &= \big((Q - N)\alpha_i + (-P - N)\beta_i\big) H_i
\end{aligned}$$

If α, β are linearly independent roots, then for whole quantity to be zero, their coefficients must also be zero. (H_i form a linear space)

$$\implies Q = N \quad P = -N$$

6C :

We have $H_1 = \sigma_3 = \sigma_3 \otimes \mathbb{1}_2$ and $H_2 = \sigma_3 \otimes \tau_3$, which we can evaluate as : (Refer problem 3D)

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The corresponding eigenvectors and weights are :

$$(1,0,0,0)^T \quad : (1,1)$$

$$(0,1,0,0)^T \quad : (1,-1)$$
$$(0,0,1,0)^T \quad : (-1,-1)$$
$$(0,0,0,1)^T \quad : (-1,1)$$

We will have two zero root vectors corresponding to Cartan generators. Rest roots are obtained as differences of the above weights. Ex. $(1,1)-(1,-1)=(0,2)$ and $(1,-1)-(1,1)=(0,-2)$ and so on.

$$(0,0),\ (0,0),\ (0,2),\ (0,-2),\ (2,2),\ (-2,-2),\ (2,0),\ (-2,0),\ (2,-2),\ (-2,2)$$

9 SU(3) (Chapter 7) problems :

7A :

The SU(3) generators in fundamental representation are given as $T_a = \frac{1}{2} \cdot \lambda_a$, where λ_a are the standard 8 Gell-Mann matrices. In fundamental representation :

$$\mathrm{Tr}(T_a T_b) = \frac{1}{2} \cdot \delta_{ab}$$

and thus the real anti-symmetric structure constants defining the SU(3) algebra $[T_a, T_b] = i f_{abc} T_c$ are given as :

$$f_{abc} = -2i \cdot \mathrm{Tr}([T_a, T_b] T_c)$$

The adjoint representation is then given as :

$$(F_a)_{bc} = -i \cdot f_{abc}$$

With the standard matrices, one can easily calculate $f_{147} = \frac{1}{2}$ and $f_{458} = \frac{\sqrt{3}}{2}$.

7B :

We know the SU(2) algebra is given by $[J_i, J_j] = i\epsilon_{ijk} J_k$. In the SU(3) algebra, we can find out that the structure constant $f_{123} = f_{231} = f_{312} = +1$ and thus proving that the fundamental generators T_1, T_2, T_3 form the SU(2) sub-algebra. Specifically, one can verify that :

$$[T_1, T_2] = iT_3 \quad [T_2, T_3] = iT_1 \quad [T_3, T_1] = iT_2$$

proving the first part.

Now, observe that the these threee T_1, T_2, T_3 SU(3) generators can be written as :

$$T_a = \begin{pmatrix} S_a & 0 \\ 0 & 0 \end{pmatrix}$$

where the S_1, S_2, S_3 are the fundamental generators[8] of SU(2). Naturally, we can see that in this SU(2) sub-algebra, SU(3) triplet (3-vector) will transform as SU(2) doublet and as SU(2) singlet. S_a will act on upper two components, just like a spinor while the third component is left untouched. Refer to the formula from problem 4C to explicitely check this :

$$e^{i\alpha \cdot \hat{\alpha_a} X_a} = \mathbb{1} + [\cos(\alpha) - 1] \cdot (\hat{\alpha_a} X_a)^2 + i \sin(\alpha) \cdot (\hat{\alpha_a} X_a)$$

. The mathematical way to write this will be :

$$3 = 2 \oplus 1$$

Which in words can be stated as : Under this SU(2) sub-algebra, 3 representation of SU(3) (fundamental) can be decomposed to ir-reps of SU(2) as a direct sum of the doublet (2 or fundamental) and singlet representations of SU(2). Alternatively, one can say, under this SU(2) sub-algebra, SU(3) triplet transforms like a doublet plus singlet under SU(2). Now, Let's turn attention to transformation under adjoint representation (8 representation) of SU(3). We know that for adjoint representation we can write :

$$3 \otimes \overline{3} = 8 \oplus 1$$

[8] Generators in fundamental representation.

But, under this SU(2) sub-algebra, we know the decomposition $3 \rightarrow 2 \oplus 1$ and $\overline{3} \rightarrow 2 \oplus 1$. $(2 = \overline{2})$. Substituting and calculating the tensor product :

$$(2 \oplus 1) \otimes (2 \oplus 1) = (2 \otimes 2) \oplus (2 \otimes 1) \oplus (1 \otimes 2) \oplus (1 \otimes 1)$$

And using the adjoint representation (3) of SU(2) : $2 \otimes 2 = 3 \oplus 1$, we can write :

$$8 \oplus 1 = 3 \oplus 2 \oplus 2 \oplus 1 \oplus 1$$

Therefore, decomposition for adjoint representation under this sub-algebra is :

$$8 \rightarrow 3 \oplus 2 \oplus 2 \oplus 1$$

Stating : adjopint representaion of SU(3) is decomposed down to ir-reps of SU(2) as direct sum of 3,2,2,1 representations. Transforming like triplet plus two doublets plus a singlet.

7C :

I can prove that $\lambda_2, \lambda_5, \lambda_7$ form the SU(2) algebra with a straight-forward approach, similar to last problem. However, I am confused about the particular decomposition under this SU(2) sub-algebra. I tried infinitesimal expansion but I couldn't pick up the correct decomposition. I could use a discussion. Here all three components are mixing so I was suspecting the decomposition might as well be just $3 \rightarrow 3$ and then $8 \rightarrow 3 \oplus 5$.

References

[1] G. Aad *et al.* [ATLAS], Phys. Lett. B **716**, 1-29 (2012) doi:10.1016/j.physletb.2012.08.020 [arXiv:1207.7214 [hep-ex]].

[2] S. Chatrchyan *et al.* [CMS], Phys. Lett. B **716**, 30-61 (2012) doi:10.1016/j.physletb.2012.08.021 [arXiv:1207.7235 [hep-ex]].

[3] N. Aghanim *et al.* [Planck], Astron. Astrophys. **641**, A6 (2020) [erratum: Astron. Astrophys. **652**, C4 (2021)] doi:10.1051/0004-6361/201833910 [arXiv:1807.06209 [astro-ph.CO]].

[4] R. D. Peccei and H. R. Quinn, Phys. Rev. Lett. **38**, 1440-1443 (1977) doi:10.1103/PhysRevLett.38.1440

[5] S. Weinberg, Phys. Rev. Lett. **40**, 223-226 (1978) doi:10.1103/PhysRevLett.40.223

[6] J. L. Feng, Ann. Rev. Astron. Astrophys. **48**, 495-545 (2010) doi:10.1146/annurev-astro-082708-101659 [arXiv:1003.0904 [astro-ph.CO]].

[7] Y. Bai and P. Schwaller, Phys. Rev. D **89**, no.6, 063522 (2014) doi:10.1103/PhysRevD.89.063522 [arXiv:1306.4676 [hep-ph]].

[8] W. de Boer, Prog. Part. Nucl. Phys. **33**, 201-302 (1994) doi:10.1016/0146-6410(94)90045-0 [arXiv:hep-ph/9402266 [hep-ph]].

[9] H. Georgi, doi:10.1201/9780429499210