So you want to know about…

Non-zero-sum Game Theory, Auctions and Negotiation

…Well what's it worth to you, eh?

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A Non-Zero Sum Game
Prisoner's Dilemma

	B Cooperates	B Defects
A Cooperates	**-1 , -1** A's payoff B's payoff	**-9 , 0** A's payoff B's payoff
A Defects	**0 , -9** A's payoff B's payoff	**-6 , -6** A's payoff B's payoff

Non-Zero-Sum means there's at least one outcome in which (A's PAYOFF + B's PAYOFF) ≠ 0

Normal Form Representation of a Non-Zero-Sum Game with *n* players

Is a set of *n* strategy spaces S_1 , S_2 …S_n
where S_i = The set of strategies available to player *i*

And *n* payoff functions

u_1 , u_2 … u_n

where

$u_i : S_1 \times S_2 \times \ldots S_n \rightarrow \Re$

is a function that takes a combination of strategies (one for each player) and returns the payoff for player *i*

PLAYER B (2)

PLAYER A (1)	C	D
C	-1 , -1	-9 , 0
D	0 , -9	-6 , -6

$n = 2$

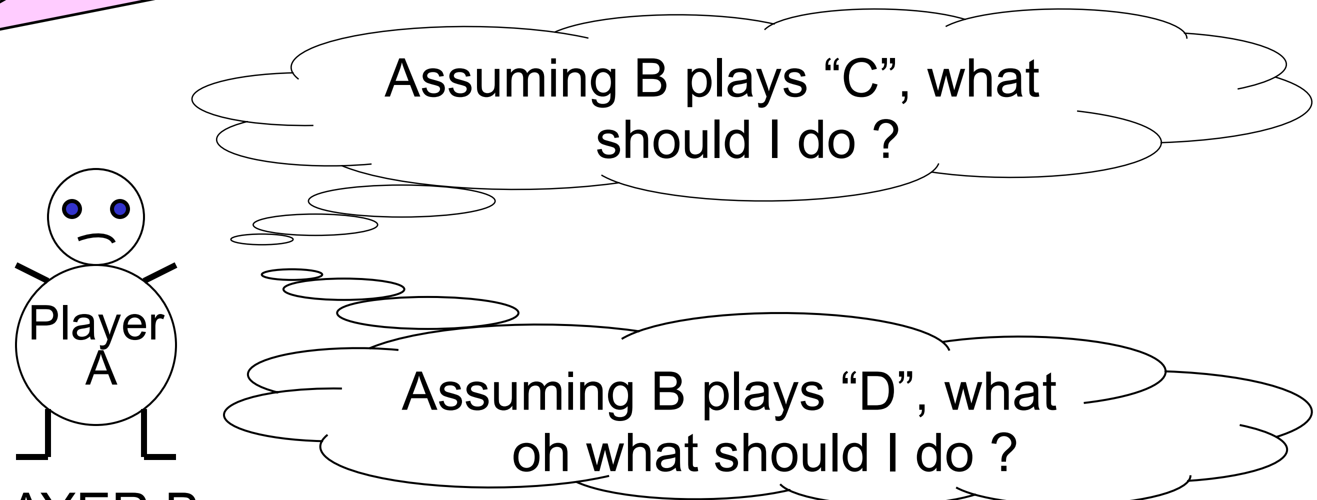
$S_1 = \{C,D\}$

$S_2 = \{C,D\}$

$u_1\ (C,C) = -1$	$u_2\ (C,C) = -1$
$u_1\ (C,D) = -9$	$u_2\ (C,D) = 0$
$u_1\ (D,C) = 0$	$u_2\ (D,C) = -9$
$u_1\ (D,D) = -6$	$u_2\ (D,D) = -6$

what would you do if you were Player A ??

Strict Domination



PLAYER B

PLAYER A	C	D
C	**-1 , -1**	**-9 , 0**
D	**0 , -9**	**-6 , -6**

If one of a player's strategies is never the right thing to do, no matter what the opponents do, then it is Strictly Dominated

"Understanding" a Game

Fundamental assumption of game theory:

Get Rid of the Strictly Dominated strategies.
They Won't Happen.

	C	D
C	-1 , -1	-9 , 0
D	0 , -9	-6 , -6

In some cases (e.g. prisoner's dilemma) this means, if players are "rational" we can predict the outcome of the game.

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→

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↓

	D
D	-6 , -6

In some cases (e.g. prisoner's dilemma) this means, if players are "rational" we can predict the outcome of the game.

Strict Domination Removal Example

		Player B			
		I	II	III	IV
Player A	I	3 , 1	4 , 1	5 , 9	2 , 6
	II	5 , 3	5 , 8	9 , 7	9 , 3
	III	2 , 3	8 , 4	6 , 2	6 , 3
	IV	3 , 8	3 , 1	2 , 3	4 , 5

So is strict domination the best tool for predicting what will transpire in a game ?

Strict Domination doesn't capture the whole picture

	I	II	III
I	0 , 4	4 , 0	5 , 3
II	4 , 0	0 , 4	5 , 3
III	3 , 5	3 , 5	6 , 6

What strict domination eliminations can we do?

What would <u>you</u> predict the players of this game would do?

Nash Equilibria

$S_1^* \in S_1, S_2^* \in S_2, \ldots S_n^* \in S_n$

are a NASH EQUILIBRIUM iff

$$\forall i \quad S_i^* = \arg\max_{S_i} u_i\left(S_1^*, S_2^*, \ldots S_{i-1}^*, S_i\ , S_{i+1}^* \ldots S_n^*\right)$$

Nash Equilibria

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	$\mathrm{I_b}$	$\mathrm{II_b}$	$\mathrm{III_b}$
$\mathrm{I_a}$	0 4	4 0	5 3
$\mathrm{II_a}$	4 0	0 4	5 3
$\mathrm{III_a}$	3 5	3 5	6 6

($\mathrm{III_a}$,$\mathrm{III_b}$) is a N.E. because

$$u_1(\mathrm{III_a}, \mathrm{III_b}) = \max \begin{bmatrix} u_1(\mathrm{I_a}, \mathrm{III_b}) \\ u_1(\mathrm{II_a}, \mathrm{III_b}) \\ u_1(\mathrm{III_a}, \mathrm{III_b}) \end{bmatrix}$$

$$\text{AND} \quad u_2(\mathrm{III_a}, \mathrm{III_b}) = \max \begin{bmatrix} u_2(\mathrm{III_a}, \mathrm{I_b}) \\ u_2(\mathrm{III_a}, \mathrm{II_b}) \\ u_3(\mathrm{III_a}, \mathrm{III_b}) \end{bmatrix}$$

- If (S_1^*, S_2^*) is an N.E. then player 1 won't want to change their play given player 2 is doing S_2^*
- If (S_1^*, S_2^*) is an N.E. then player 2 won't want to change their play given player 1 is doing S_1^*

Find the NEs:

-1 -1	-9 0
0 -9	-6 -6

0 4	4 0	• 3
• 0	0 4	• 3
3 5	• 5	6 6

- Is there always at least one NE ?
- Can there be more than one NE ?

Example with no NEs among the pure strategies:

	S_1	S_2
S_1		
S_2		

Example with no NEs among the pure strategies:

	S_1		S_2	
S_1	0	1	1	0
S_2	1	0	0	1

2-player mixed strategy Nash Equilibrium

The pair of mixed strategies (M_A , M_B) are a Nash Equilibrium iff

- M_A is player A's best mixed strategy response to M_B

AND

- M_B is player B's best mixed strategy response to M_A

Fundamental Theorems

- In the n-player pure strategy game $G=\{S_1\ S_2 \cdots S_n;\ u_1\ u_2 \cdots u_n\}$, if iterated elimination of strictly dominated strategies eliminates all but the strategies $(S_1^*, S_2^* \cdots S_n^*)$ then these strategies are the unique NE of the game
- Any NE will survive iterated elimination of strictly dominated strategies
- [Nash, 1950] If *n* is finite and S_i is finite $\forall i$, then there exists at least one NE (possibly involving mixed strategies)

The "What to do in Pittsburgh on a Saturday afternoon" game

Pat enjoys football

Chris enjoys hockey

Pat and Chris are friends: they enjoy spending time together

Pat \ Chris	H	F
H	1 2	0 0
F	0 0	2 1

- Two Nash Equilibria.
- How useful is Game Theory in this case??
- Why this example is troubling…

INTERMISSION

(Why) are Nash Equilibria useful for A.I. researchers?

Will our algorithms ever need to play…

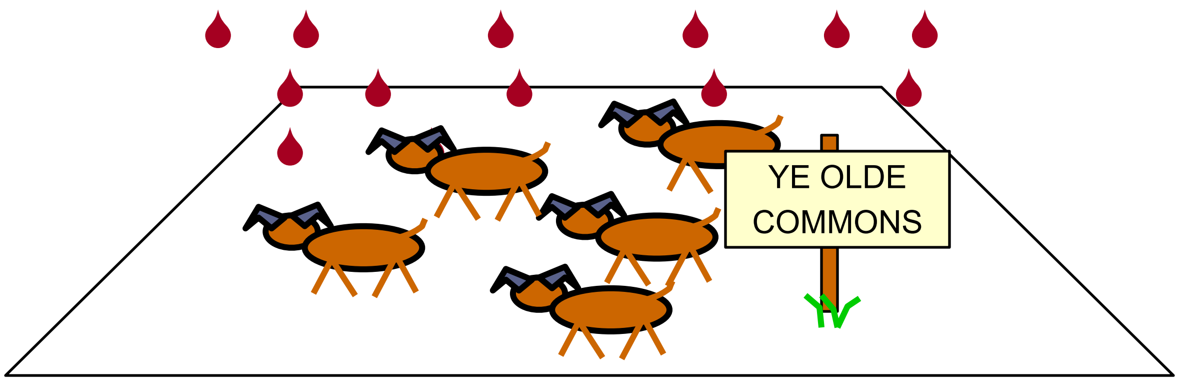
Prisoner's Dilemma?

Saturday Afternoon?

Nash Equilibria Being Useful

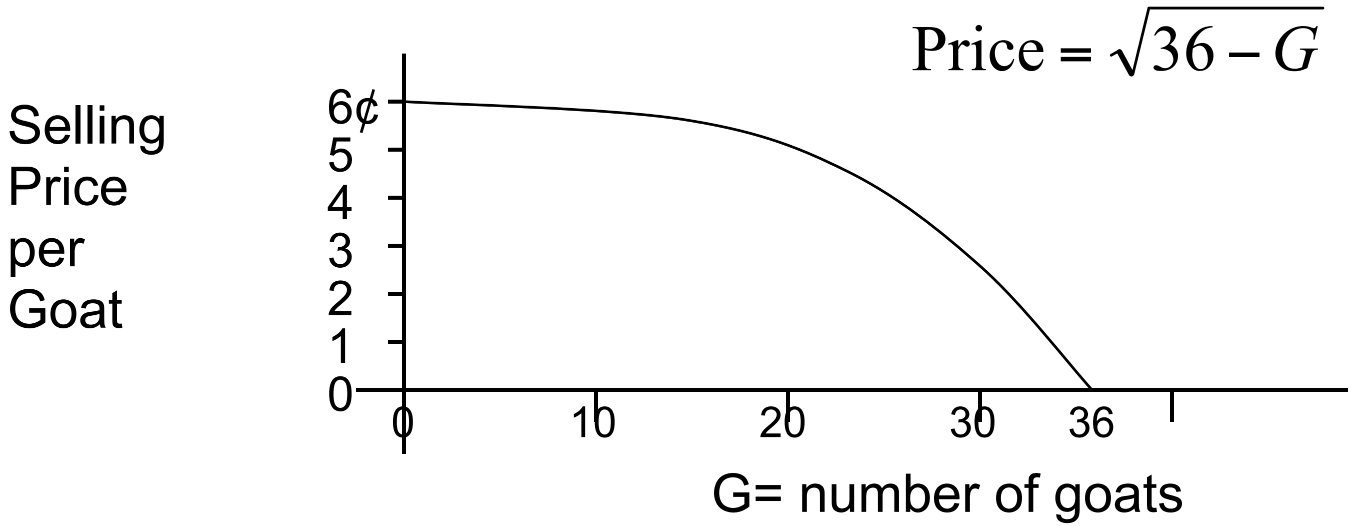
THE TRAGEDY OF THE

Commons



- You graze goats on the commons to eventually fatten up and sell
- The more goats you graze the less well fed they are
- And so the less money you get when you sell them

Commons Facts



How many goats would one rational farmer choose to graze?

What would the farmer earn?

What about a group of *n* individual farmers?

Answering this…

…is good practice for answering this

n farmers

i'th farmer has an infinite space of strategies

$$g_i \in [\ 0\ ,\ 36\ ]$$

An outcome of

$$(\ g_1\ ,\ g_2\ ,\ g_3 \cdots ,\ g_n\ )$$

will pay how much to the i'th farmer?

n farmers

i'th farmer has an infinite space of strategies

$$g_i \in [\,0\,,\,36\,]$$

An outcome of

$$(\,g_1\,,\,g_2\,,\,g_3 \cdots,\,g_n\,)$$

will pay how much to the i'th farmer?

$$g_i \times \sqrt{36 - \sum_{j=1}^{n} g_j}$$

Let's Assume a pure Nash Equilibrium exists.

Call it $\left(g_1^*, g_2^*, \boxed{?} g_n^*\right)$

What can we say about g_1^* ?

$$g_i^* = \arg\max_{g_i} \left[\begin{array}{l} \text{Payoff to farmer i, assuming} \\ \text{the other players play} \\ \left(g_1^*, g_2^*, \boxed{?} g_{i-1}^*, g_{i+1}^*, \boxed{?} g_n^*\right) \end{array} \right]$$

For Notational Convenience,

write $G_{-i}^* = \sum_{j \neq i} g_j^*$

THEN

$$g_i^* = \arg\max_{g_i} \left[\qquad \qquad \qquad \qquad \qquad \right]$$

What?

Let's Assume a pure Nash Equilibrium exists.

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Let's Assume a pure Nash Equilibrium exists.

Call it

$$\left(g_1^*, g_2^*, \boxed{?}\, g_n^*\right)$$

What can we say about

$$g_i^* = \arg\max_{g_i} \left[\begin{array}{l} \text{Payo} \\ \text{the c} \\ (g_1^*, \ldots \end{array} \right.$$

For Notational Conven

write $\quad G_{-i}^* = \sum_{j \neq i} g_j^*$

THEN

$$g_i^* = \arg\max_{g_i} \left[g_i \sqrt{36 - g_i - G_{-i}^*} \right]$$

g^*_i must satisfy

$$\frac{\partial}{\partial g_i^*} g_i^* \sqrt{36 - g_i^* - G_{-i}^*} = 0$$

therefore

$$\frac{36 - G_{-i}^* - \frac{3}{2} g_i^*}{\sqrt{36 - g_i^* - G_{-i}^*}} = 0$$

We have *n* linear equations in *n* unknowns

$g_1^* = 24 - 2/3(\quad g_2^* + g_3^* + \quad \cdots g_n^*)$

$g_2^* = 24 - 2/3(g_1^* + \quad g_3^* + \quad \cdots g_n^*)$

$g_3^* = 24 - 2/3(g_1^* + g_2^* + \quad g_4^* \cdots g_n^*)$

$\vdots \qquad \vdots \qquad \vdots$

$g_n^* = 24 - 2/3(g_1^* + \quad \cdots g_{n-1}^*)$

Clearly all the g_i^*'s are the same (Proof by "it's bloody obvious")

Write $g^* = g_1^* = \cdots g_n^*$

Solution to $g^* = 24 - 2/3(n-1)g^*$ is: $g^* = \dfrac{72}{2n+1}$

Consequences

At the Nash Equilibrium a rational farmer grazes

$\frac{72}{2n+1}$ goats.

How many goats in general will be grazed? Trivial algebra gives: $36 - \frac{36}{2n+1}$ goats total being grazed

[as n --> *infinity* , #goats --> 36]

How much profit per farmer? $\frac{432}{(2n+1)^{3/2}}$

1.26¢ if
24 farmers

How much if the farmers could all cooperate?

$$\frac{24*\mathrm{sqrt}(12)}{n} = \frac{83.1}{n}$$

3.46¢ if
24 farmers

The Tragedy

The farmers act "rationally" and earn 1.26 cents each.

But if they'd all just got together and decided "one goat each" they'd have got 3.46 cents each.

Is there a bug in Game Theory?

in the Farmers?

in Nash?

Would you recommend the farmers hire a police force?

Recipe for Nash-Equilibrium-Based Analysis of Such Games

- Assume you've been given a problem where the *i*'th player chooses a real number x_i
- Guess the existence of a Nash equilibrium $(x_1^*, x_2^* \cdots x_n^*)$
- Note that , ∀i,

$$x_i^* = \arg\max_{x_i} \begin{bmatrix} \text{Payoff to player } i \text{ if player } i \\ \text{plays "} x_i \text{" and the } j\text{'th player} \\ \text{plays } x_j^* \text{ for } j \neq i \end{bmatrix}$$

- Hack the algebra, often using "at x_i^* we have $\frac{\partial}{\partial x_i}$ Payoff + 0 "

INTERMISSION

Does the Tragedy of the Commons matter to us when we’re building intelligent machines?

Maybe repeated play means we can learn to cooperate??

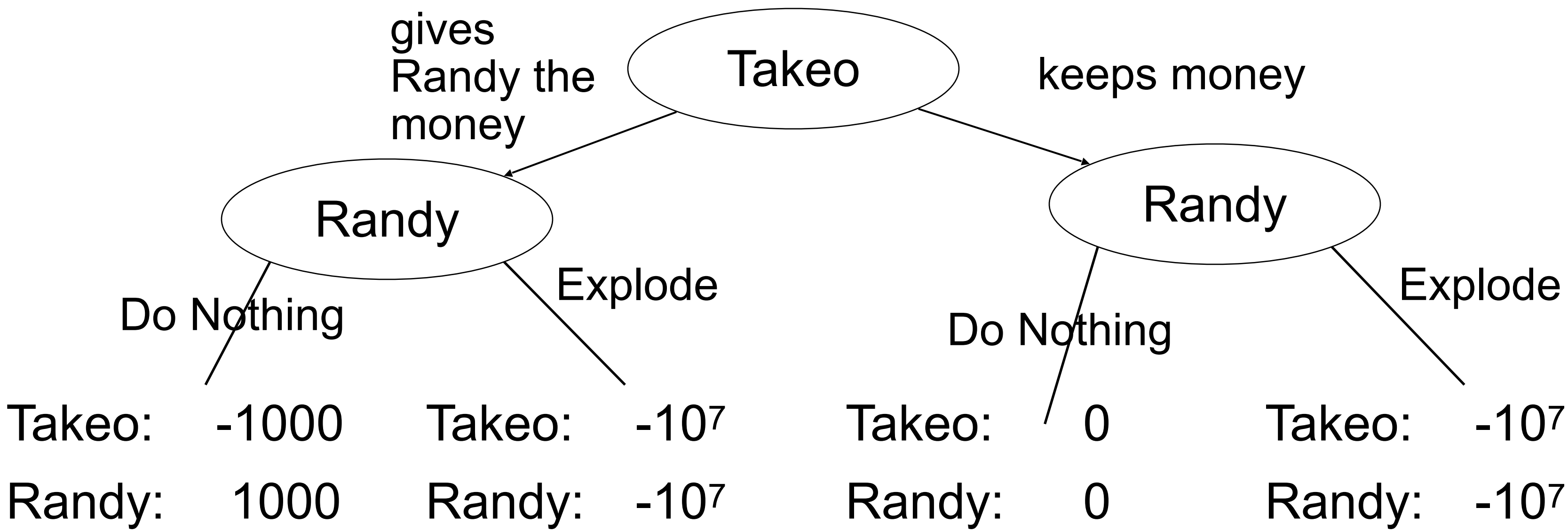
Repeated Games with Implausible Threats

Takeo and Randy are stuck in an elevator

Takeo has a $1000 bill

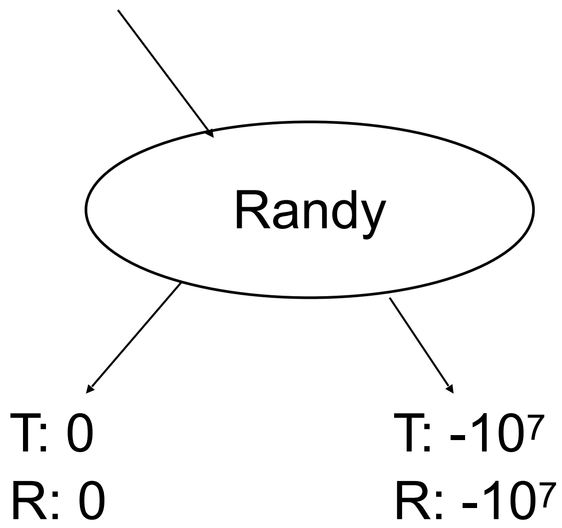
Randy has a stick of dynamite

Randy says “Give me $1000 or I’ll blow us both up.”



What should Takeo do?????

Using the formalism of Repeated Games With Implausible Threats, Takeo should Not give the money to Randy



Takeo Assumes Randy is Rational

At this node, Randy will choose the left branch

Repeated Games

Suppose you have a game which you are going to play a finite number of times.

What should you do?

2-Step Prisoner's Dilemma

GAME 1

		Player B: C	Player B: D
Player A	C	-1 , -1	-9 , 0
Player A	D	0 , -9	-6 , -6

GAME 2
(Played with knowledge of outcome of GAME 1)

		Player B: C	Player B: D
Player A	C	-1 , -1	-9 , 0
Player A	D	0 , -9	-6 , -6

Idea 1

Player A has four pure strategies

C then C
C then D
D then C
D then D

Is Idea 1 correct?

Ditto for B

Important Theoretical Result:

Assuming Implausible Threats, if the game G has a unique N.E. $(s_1^* ,\cdots s_n^*)$ then the new game of repeating G T times, and adding payouts, has a unique N.E. of repeatedly choosing the original N.E. $(s_1^* ,\cdots s_n^*)$ in every game.

If you're about to play prisoner's dilemma 20 times, you should defect 20 times.

DRAT ☹

Intermission

Game theory has been cute so far.

But depressing.

Now let's make it really work for us.

We're going to get more real.

The notation's growing teeth.

Bayesian Games

You are Player A in the following game. What should you do?

		Player B: S_1	Player B: S_2
Player A	S_1	3 ?	-2 ?
	S_2	0 ?	6 ?

Question: When does this situation arise?

Hockey lovers get 2 units for watching hockey, and 1 unit for watching football.

Football lovers get 2 units for watching football, and 1 unit for watching hockey.

Pat's a hockey lover.

Pat thinks Chris is probably a hockey lover also, but Pat is not sure.

Pat \ Chris	H	F
H	2 2	0 0
F	0 0	1 1

With 2/3 chance

Pat \ Chris	H	F
H	2 1	0 0
F	0 0	1 2

1/3 chance

In a Bayesian Game each player is given a type. All players know their own types but only a prob. dist. for their opponent's types

An n-player Bayesian Game has

a set of action spaces $A_1 \cdot\cdot\ A_n$

a set of type spaces $T_1 \cdot\cdot\ T_n$

a set of beliefs $P_1 \cdot\cdot\ P_n$

a set of payoff functions $u_1 \cdot\cdot\ u_n$

$P_{-i}(t_{-i}|t_i)$ is the prob dist of the types for the other players, given player i has type i .

$u_i(a_1 , a_2 \cdots a_n , t_i)$ is the payout to player i if player j chooses action a_j (with $a_j \in A_j$) (forall $j=1,2,\cdots n$) and if player i has type $t_i \in T_i$

Bayesian Games: Who Knows What?

We assume that all players enter knowing the full information about the A_i's , T_i's, P_i's and u_i's

The i'th player knows t_i, but not $t_1\ t_2\ t_3 \cdots t_{i-1}\ t_{i+1} \cdots t_n$

All players know that all other players know the above

And they know that they know that they know, *ad infinitum*

Definition: A <u>strategy</u> $S_i(t_i)$ in a Bayesian Game is a mapping from $T_i \rightarrow A_i$: a specification of what action would be taken for each type

Example

$A_1 = \{H,F\}$ $A_2 = \{H,F\}$

$T_1 = \{H\text{-love}, Flove\}$ $T_2 = \{Hlove, Flove\}$

$P_1\ (t_2 = Hlove \mid t_1 = Hlove) = 2/3$
$P_1\ (t_2 = Flove \mid t_1 = Hlove) = 1/3$
$P_1\ (t_2 = Hlove \mid t_1 = Flove) = 2/3$
$P_1\ (t_2 = Flove \mid t_1 = Hlove) = 1/3$
$P_2\ (t_1 = Hlove \mid t_2 = Hlove) = 1$
$P_2\ (t_1 = Flove \mid t_2 = Hlove) = 0$
$P_2\ (t_1 = Hlove \mid t_2 = Flove) = 1$
$P_2\ (t_1 = Flove \mid t_2 = Hlove) = 0$

$u_1\ (H,H,Hlove) = 2$ $u_2\ (H,H,Hlove) = 2$
$u_1\ (H,H,Flove) = 1$ $u_2\ (H,H,Flove) = 1$
$u_1\ (H,F,Hlove) = 0$ $u_2\ (H,F,Hlove) = 0$
$u_1\ (H,F,Flove) = 0$ $u_2\ (H,F,Flove) = 0$
$u_1\ (F,H,Hlove) = 0$ $u_2\ (F,H,Hlove) = 0$
$u_1\ (F,H,Flove) = 0$ $u_2\ (F,H,Flove) = 0$
$u_1\ (F,F,Hlove) = 1$ $u_2\ (F,F,Hlove) = 1$
$u_1\ (F,F,Flove) = 2$ $u_2\ (F,F,Flove) = 2$

(GASP, SPLUTTER)

Bayesian Nash Equilibrium

The set of strategies (s_1^* ,s_2^* ⋯ s_n^*) are a

Pure Strategy Bayesian Nash Equilibrium

iff for each player *i*, and for each possible type of *i* : $t_i \in T_i$

$s_i^*(t_i)$ =

$$\arg\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i\left(s_1^*(t_1), \ldots s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}) \ldots s_n^*(t_n)\right) \times \mathrm{P}_i\left(t_{-i} \mid t_i\right)$$

i.e. no player, in any of their types, wants to change their strategy

NEGOTIATION: A Bayesian Game

Two players:	S, (seller) and B, (buyer)
$T_s = [0,1]$	the seller's type is a real number between 0 and 1 specifying the value (in dollars) to them of the object they are selling
$T_b = [0,1]$	the buyer's type is also a real number. The value to the buyer.

Assume that at the start

$V_s \in T_s$ is chosen uniformly at random

$V_b \in T_b$ is chosen uniformly at random

The "Double Auction" Negotiation

S writes down a price for the item (g_s)

B simultaneously writes down a price (g_b)

Prices are revealed

If $g_s \geq g_b$ no trade occurs, both players have payoff 0

If $g_s \leq g_b$ then buyer pays the midpoint price $\frac{(g_s+g_b)}{2}$ and receives the item

Payoff to S : $1/2(g_s+g_b)-V_s$

Payoff to B : $V_b-1/2(g_s+g_b)$

Negotiation in Bayesian Game Notation

$T_s = [0,1]$ write $V_s \in T_s$

$T_b = [0,1]$ write $V_b \in T_b$

$P_s(V_b|V_s) = P_s(V_b)$ = uniform distribution on [0,1]

$P_b(V_s|V_b) = P_b(V_s)$ = uniform distribution on [0,1]

$A_s = [0,1]$ write $g_s \in A_s$

$A_b = [0,1]$ write $g_b \in A_b$

$u_s(P_s, P_b, V_s) =$ What?

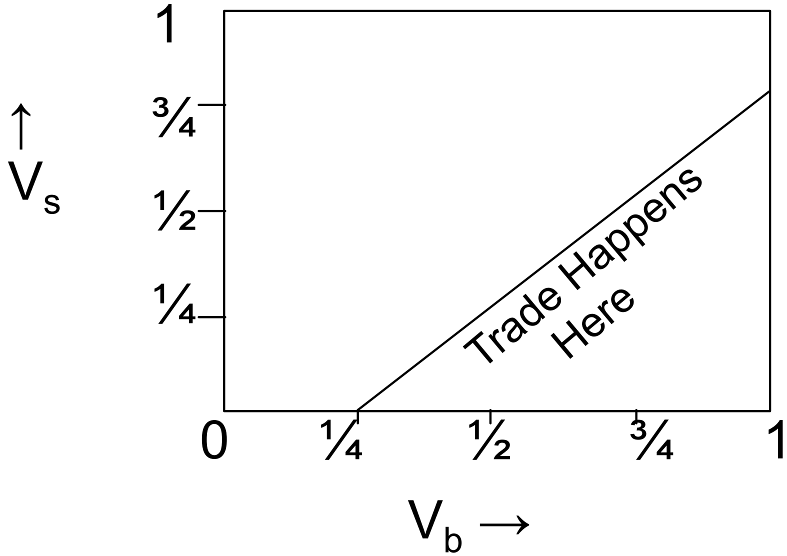
$u_b(P_s, P_b, V_b) =$ What?

Double Negotiation: When does trade occur?

…when

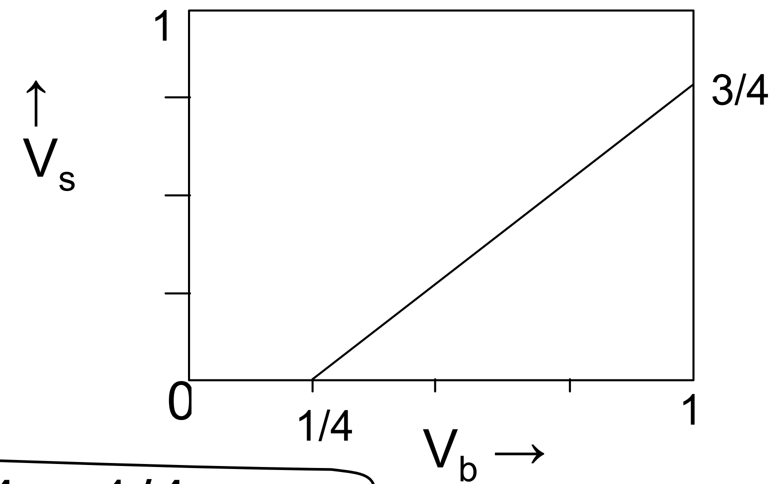
$$g_b^*(V_b) = 1/12 + 2/3\ V_b > 1/4 + 2/3\ V_s = g_s^*(V_s)$$

i.e. when $V_b > V_s + 1/4$



Prob(Trade Happens) = $1/2 \times (3/4)^2 = 9/32$

Value of Trade



$E[V_s|\text{Trade Occurs}] = 1/3 \times 3/4 = 1/4$

$E[V_b|\text{Trade Occurs}] = 1/4 + 2/3 \times 3/4 = 3/4$

If trade occurs, expected trade price is

$$1/2[g_s^*(V_s) + g_b^*(V_b)] =$$

$$1/2(1/12 + 2/3V_b + 1/4 + 2/3V_s) =$$

$$1/6 + 1/3V_b + 1/3V_s$$

Value of Trade continued…

E[profit to S | trade occurred] =

E[$1/6 + 1/3V_b + 1/3V_s – V_s$ | trade occurred] =

$1/6 + 1/3$E[V_b | trade] – $2/3$E[V_s | trade] =

$1/6 + 1/3 \times 3/4 - 2/3 \times 1/4 = 1/4$

Similar Algebra Shows: E[profit to B | trade occurred] = 1/4 also

Using This Game	**If Both Were "Honest"**
E[B's profit]= 1/4x9/32=0.07	E[B profit]=1/12=0.083
E[S's profit]= 0.07	E[S profit]=1/12=0.083

This Game seems **Inefficient**. What can be done???

Double Auction: Final Comments

- There are other Nash Equilibrium strategies.
- But the one we saw is provably most efficient.
- In <u>general</u>, even for arbitrary prob. dists. of V_s and V_b, no efficient NE's can exist.
- And no other games for this kind of trading can exist and be efficient.

Double Auction Discussion

What if seller used "giant eagle" tactics?

Seller states "I'll sell it to you for price p : take it or leave it"

Exercise:

- How should* seller choose price (taking into account V_s of course) ?
- And how should* buyer choose whether to buy ?

  *(at a B.N.E.)

- **When could/should double auction technology be used?**
- **(How) can "V_s,V_b drawn randomly from [0,1]" be relaxed ?**

MULTI-PLAYER AUCTIONS

First Price Sealed Bid

Seller wants to sell an object that has no value to seller… anything seller is paid is pure profit.

There are *n* available buyers

Assumptions:

- Assume buyer *i* has a value for the object distributed uniformly randomly in [0…1]∋V_i
- Assume V_i's all independent
- Buyer *i* does not know V_j for $i \neq j$
- Buyer *i* <u>does</u> know all V_j's randomly generated from [0,1]

First Price Sealed Bid Rules

Each buyer writes down their bid.

Call buyer *i*'s bid g_i

Buyer who wrote highest bid must buy object from seller at price=bid

Question: Why is "bid = V_i" a stupid strategy ??

Auction Analysis: Back to Bayesian Nash Equils

This assumption is completely unjustified right now. Later we'll see why it was an okay assumption to make.

We'll assume that all players other than *i* do a linear strategy:

$$g_j^*(V_j) = m_j V_j \quad \text{for } j \neq i$$

Then what should *i* do ?

$$g_i^*(v_i) = \arg\max_g \mathrm{E}\begin{bmatrix} \text{Profit if} \\ \text{play } g \end{bmatrix}$$

$$g_i^*(v_i) = \arg\max_g E\begin{bmatrix}\text{Profit if}\\ \text{play } g\end{bmatrix}$$

$$= \arg\max_g E\left[\begin{matrix}\text{Profit if}\\ \text{play } g\end{matrix}\,\middle|\, i \text{ wins}\right] \text{Prob}\begin{bmatrix} g \text{ is}\\ \text{winning bid}\end{bmatrix}$$

$$= \arg\max_g \underbrace{\qquad\qquad\qquad}_{\text{what?}} \times \underbrace{\qquad\qquad\qquad}_{\text{what?}}$$

$$= g \text{ such that } (n-1)(v_i - g)g^{n-2} - g^{n-1} = 0$$

$$\Rightarrow g_i^*(v_i) = \left(1 - \frac{1}{n}\right) v_i$$

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$\boxed{?}$

$$= g \text{ such that } (n-1)(v_i - g)g^{n-2} - g^{n-1} = 0$$

$$\Rightarrow g_i^*(v_i) = \left(1 - \tfrac{1}{n}\right)v_i$$

Thus we've an N.E. because if all other players use a linear strategy then it's in *i*'s interest to do so too. Above holds $\forall i$

See, I told you the linear assumption would be okay.

First-Price Sealed Auction

At BNE all players use

$$g_i^*(V_i) = (1-1/n)V_i$$

Note: [Fact of probability]

Expected value of the largest of n numbers drawn independently from [0,1] is $\frac{n}{n+1}$

Expected profit to seller = what?

First-Price Sealed Auction

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Expected profit to seller =

Expected highest bid = what?

First-Price Sealed Auction

At BNE all players use

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Expected profit to seller =

Expected highest bid =

$$\left(1-\frac{1}{n}\right)\left(\frac{n}{n+1}\right) = 1-\frac{2}{n+1}$$

Seller likes large n

Exercise: compute expected profit to player i. Show it is $0(1/n)$.

Second-Price Sealed Bid

A different game:

Each buyer writes their bid

Buyer with highest bid must buy the object

But the price they pay is the second highest bid

- What is player *i*'s best strategy
- Why?
- What is seller's expected profit?

Auction Comments

- Second-price auction is preferred by cognoscenti
  - ➢No more efficient
  - ➢But general purpose
  - ➢And computationally better
  - ➢And less variance (better risk management)
- Auction design is interesting
  - ➢So far mostly for economics
  - ➢But soon for e-commerce etc.?
- Important but not covered here
  - ➢Expertise
  - ➢Collusion
  - ➢Combinatoric Auctions
  - ➢What if all cooperative ????

What You Should Know

Strict dominance

Nash Equilibria

Continuous games like Tragedy of the Commons

Rough, vague, appreciation of threats

Bayesian Game formulation

Double Auction

1st/2nd Price auctions

What You Shouldn't Know

- How many goats your lecturer has on his property
- What strategy Mephistopheles uses in his negotiations
- What strategy this University employs when setting tuition
- How to square a circle using only compass and straight edge
- How many of your friends and colleagues are active Santa informants, and how critical they've been of your obvious failings