Games with Hidden Information

For extra reading, see an elementary Game Theory text such as the first chapter of the book *Game Theory: Mathematical Models of Conflict* by A J Jones (Ellis Horwood, 1980)

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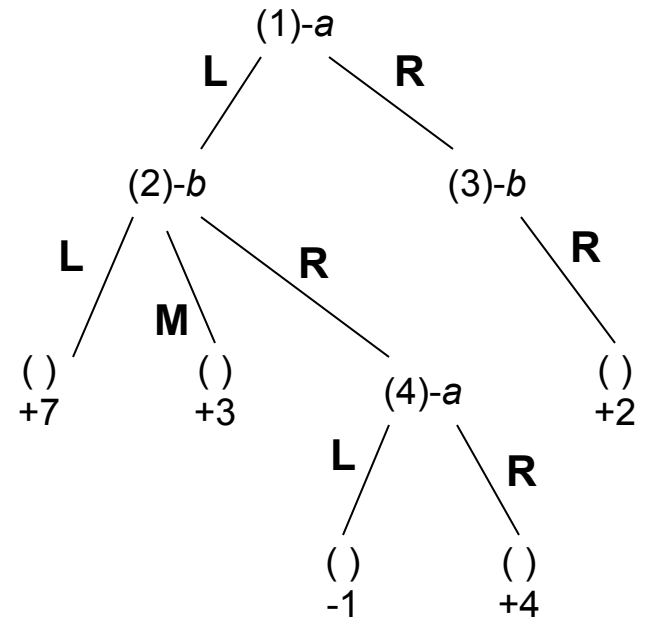**www.cs.cmu.edu/~awm**
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Overview

- Games of Hidden Information: Pure and Mixed strategies and "bluffing".

- Is this game-playing stuff useful?

- Discussion

Game Theory

A pure strategy for a player is a mapping from all the possible states that player could see to the move the player would make.

Game Theory

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Four pure strategies for A:

     A's Strategy I: $(1{\rightarrow}L , 4{\rightarrow}L)$

     A's Strategy II: $(1{\rightarrow}L , 4{\rightarrow}R)$

     A's Strategy III: $(1{\rightarrow}R , 4{\rightarrow}L)$

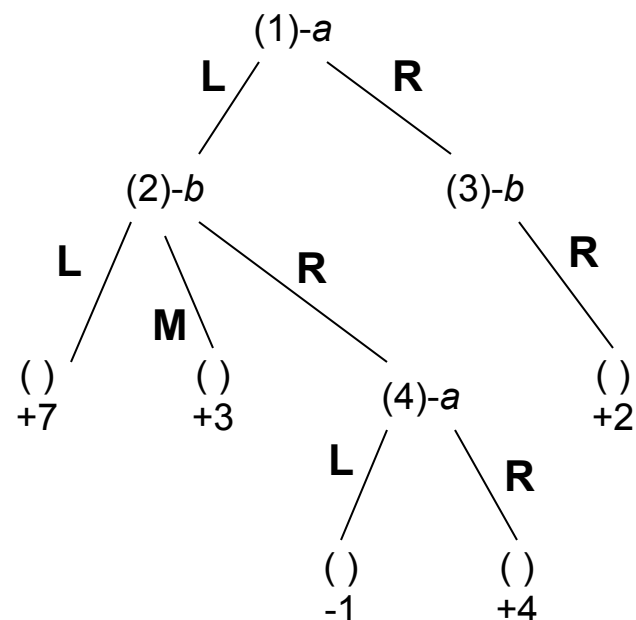   A's Strategy IV: $(1{\rightarrow}R , 4{\rightarrow}R)$

Three pure strategies for B:

     B's Strategy I: $(2{\rightarrow}L , 3{\rightarrow}R)$

     B's Strategy II: $(2{\rightarrow}M , 3{\rightarrow}R)$

     B's Strategy III: $(2{\rightarrow}R , 3{\rightarrow}R)$

In general, if each player can see N possible states, and there are b moves from each state, how many pure strategies?

Matrix forms of games

A's Strategy I: (1→L , 4→L)
A's Strategy II: (1→L , 4→R)
A's Strategy III: (1→R , 4→L)
A's Strategy IV: (1→R , 4→R)
B's Strategy I: (2→L , 3→R)
B's Strategy II: (2→M , 3→R)
B's Strategy III: (2→R , 3→R)
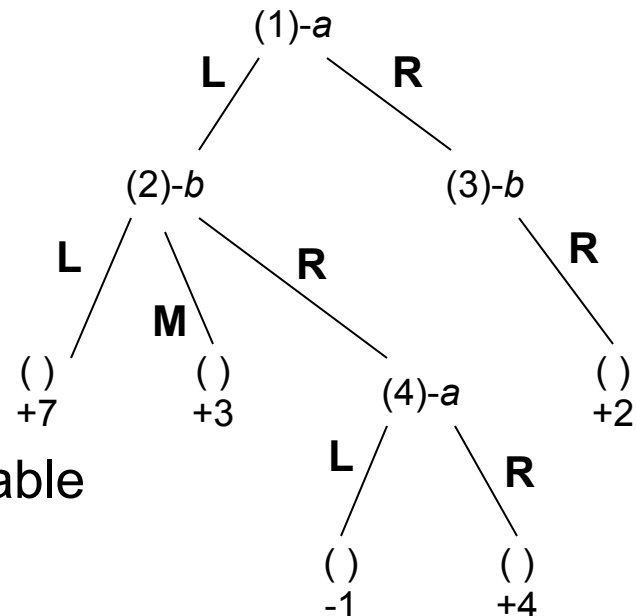
The matrix form shows the game value as a table
indexed by A's and B's strategies

	B-I	B-II	B-III
A-I	7	3	-1
A-II	7	3	4
A-III	2	2	2
A-IV	2	2	2

This matrix comprises a full
description of the game!
The rules etc. are
superfluous details!

Matrix Form Example
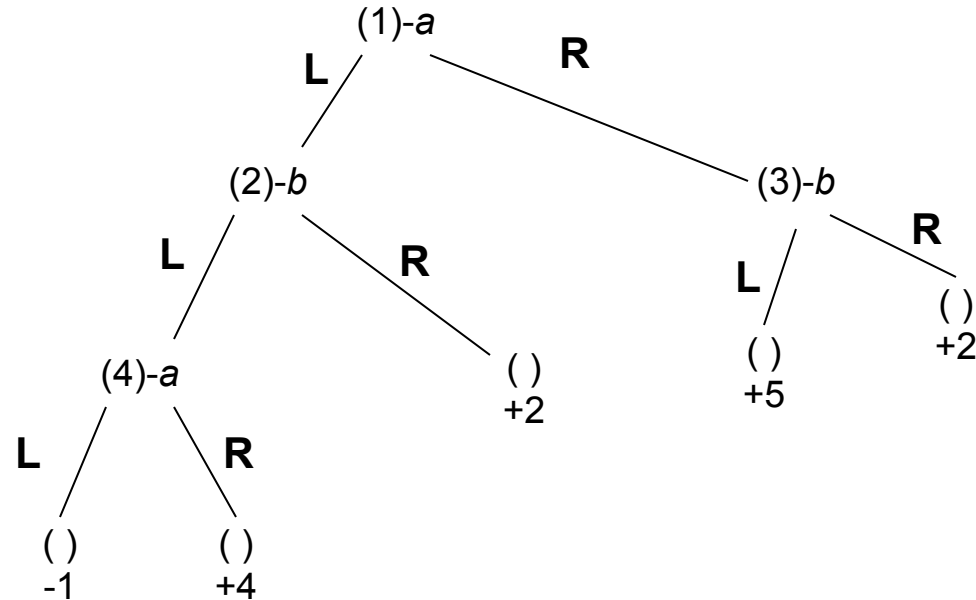
How many pure strategies does *a* have?

How many pure strategies does *b* have?

What is the shape of a Matrix form?

What is a matrix form of the above game?

Minimax with Matrix Forms

	B-I	B-II	B-III
A-I	7	3	-1
A-II	7	3	4
A-III	2	2	2
A-IV	2	2	2

A can decide from this matrix which strategy is best. For each strategy, *A* considers the worst-case counter strategy by *B*. *A* chooses the row with the maximum minimum value. For *A*, the value of the game is this value.

In this example A chooses A-II, and says game has value 3.

When *B* decides which strategy is best, *B* searches for which column has the minimum maximum value.

In this example, B chooses B-II, and says game has value 3.

Fundamental game theory result (proved by von Neumann):

In a 2-player, zero-sum game of perfect information, Minimax==Maximin. And there always exists an optimal pure strategy for each player.

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2 player zero-sum finite NONdeterministic games of perfect information

The search tree now includes states where neither player makes a choice, but instead a random decision is made according to a known set of outcome probabilities.
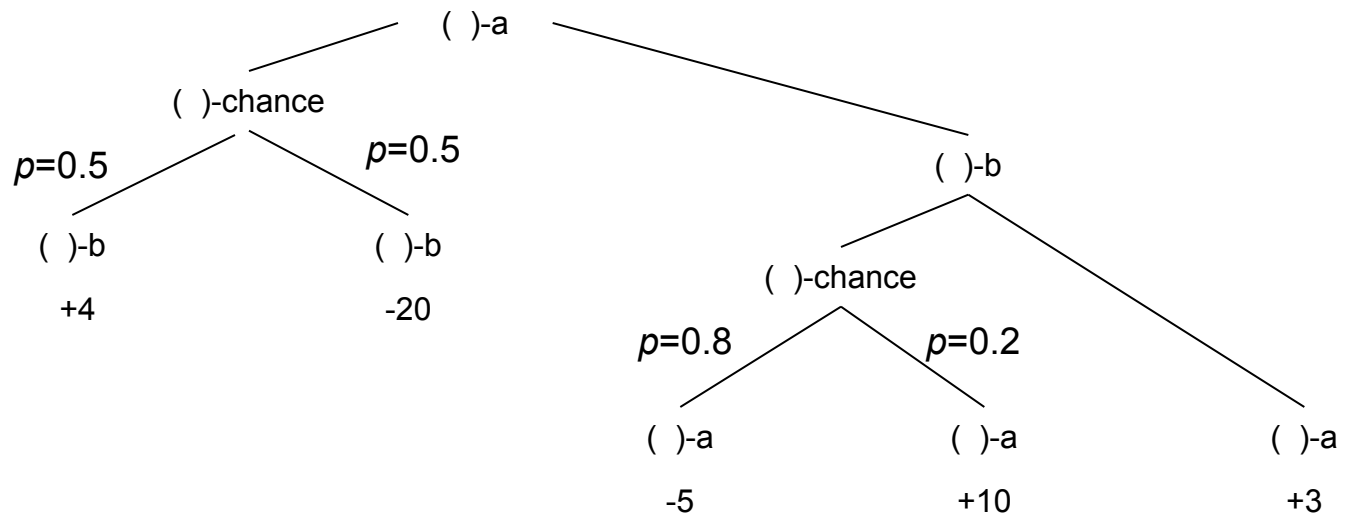
Game theory value of a state is the expected final value if both players are optimal.

Let's compute a matrix form of this!

NONdeterministic finite games: matrix forms

( )-a

( )-chance

p=0.5 p=0.5

( )-b   ( )-b

+4   -20

( )-b

( )-chance

p=0.8 p=0.2
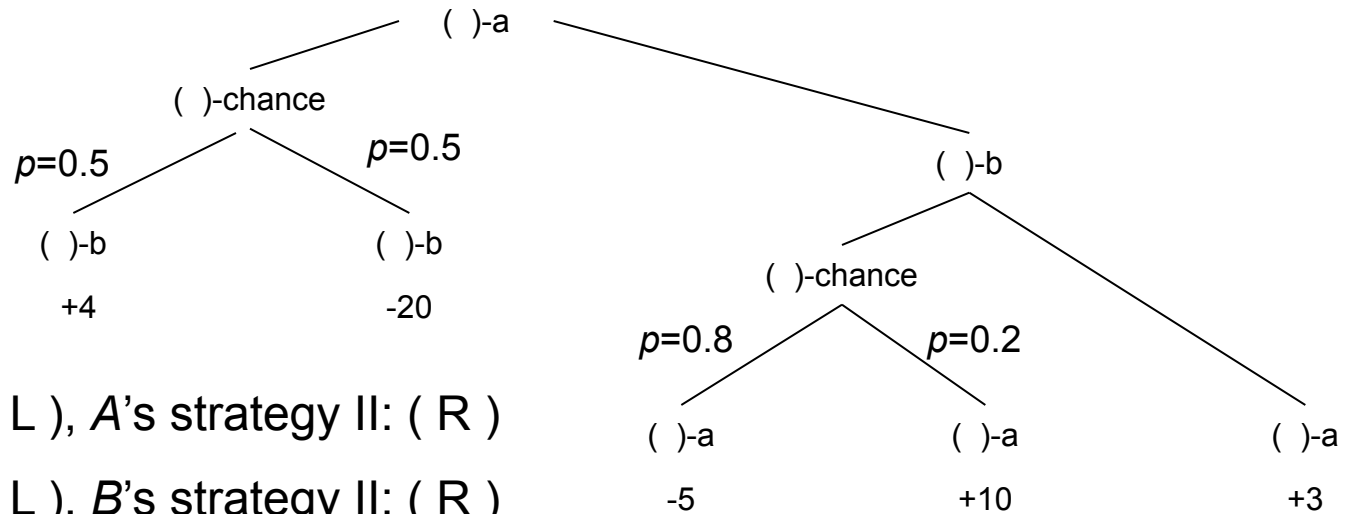
A's strategy I: (L), *A*'s strategy II: (R)

B's strategy I: (L), *B*'s strategy II: (R)

( )-a   ( )-a   ( )-a

-5   +10   +3

	B-I	B-II
A-1	-8	-8
A-II	-2	3

The *i*, *j*'th entry holds the **expected** amount *A* will win if *A* follows its *i*'th strategy and *B* it's *j*'th. Von Neumann's result still holds. Minimax=Maximin.

Two person zero-sum finite games, hidden information

Examples: Poker, 2-player bridge, Scrabble, Diplomacy. This gets very interesting.

Bottom line: Foundations of economic theory and "multiple agent" decision-making start here.

Bad news: The computational complexity of these foundations makes chess look like integer multiplication.

Imagine a version of mini-poker in which Red cards are bad for A and Black cards are good.

Player A is dealt a card. It is red or black with 50% probability.

A may resign if the card is red: *A loses 20c*
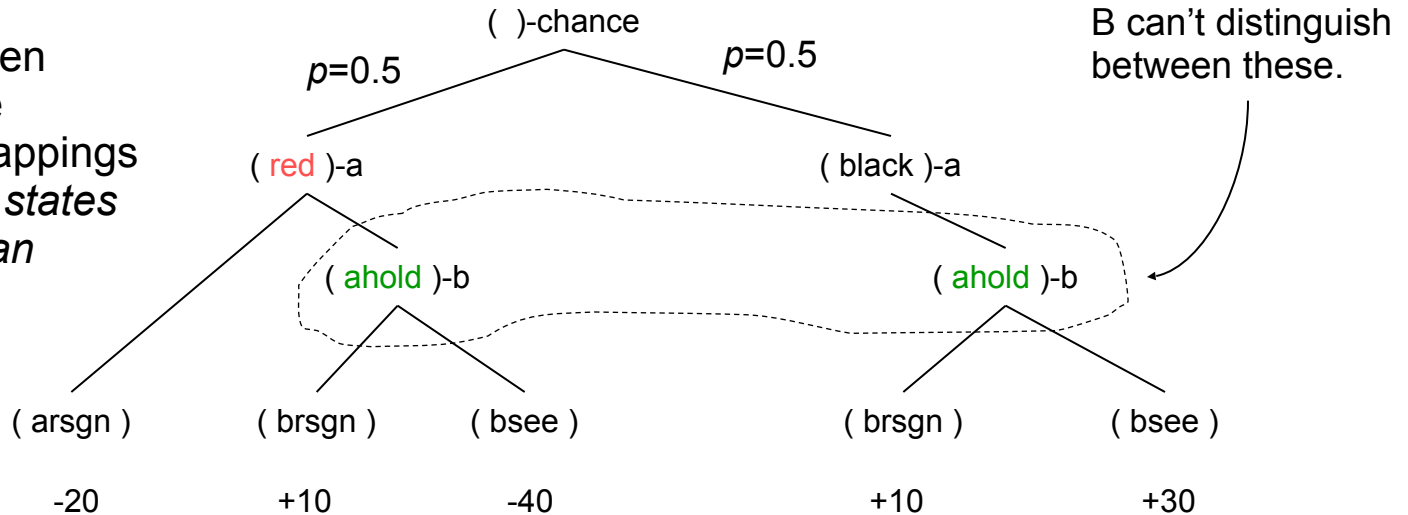Else A "holds".
B may then resign: *A wins 10c*
B may "see":
If card is red: *A loses 40c*
If card is black: *A wins 30c*

Mini-Poker Pure Strategies

In games of hidden information, pure strategies are mappings from *all possible states that the player can detect* to moves.

( )-chance

p=0.5 p=0.5

B can't distinguish between these.

(red)-a   (black)-a

(ahold)-b   (ahold)-b

(arsgn)   (brsgn)   (bsee)   (brsgn)   (bsee)

-20   +10   -40   +10   +30

For Player A there are two pure strategies:
> Strategy RESIGNER: *Resign if card = Red*
> Strategy HOLDER: *Hold if card = Red*

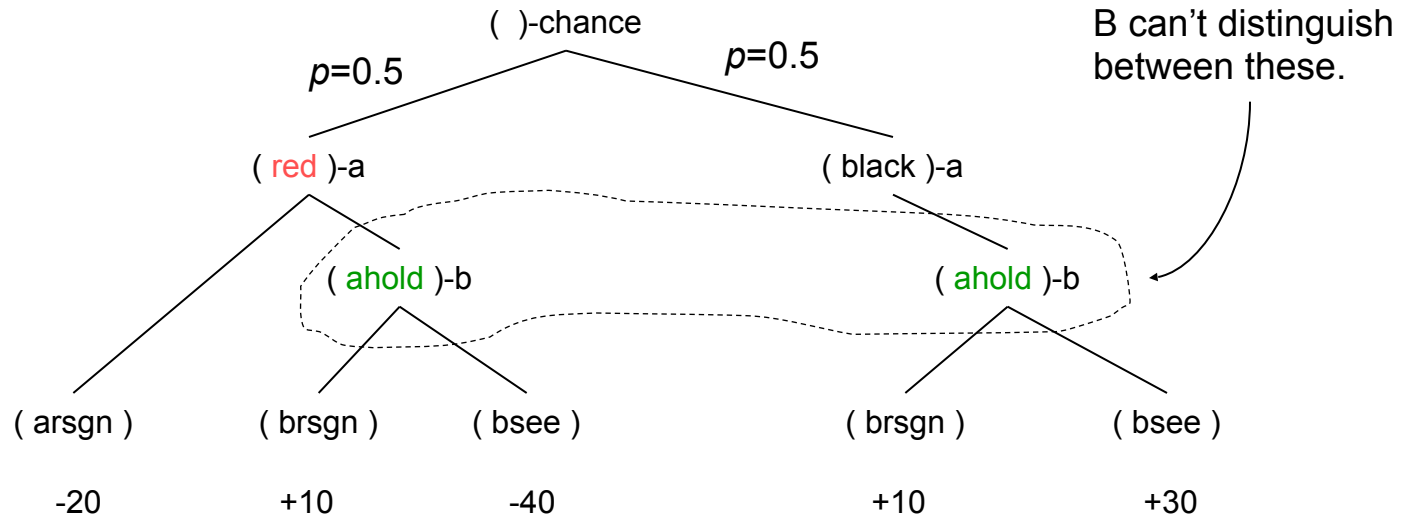For Player B there are two pure strategies:
> Strategy RESIGNER: *Resign if A holds*
> Strategy SEER: *If A holds then see*

If A is a resigner, how much will A win on average? Depends on B's strategy!

If B is a resigner, how much will A win on average? Depends on A's strategy.

Mini-Poker in Matrix Form

( )-chance

p=0.5 p=0.5

B can't distinguish between these.

(red)-a   (black)-a

(ahold)-b   (ahold)-b

(arsgn)   (brsgn)   (bsee)   (brsgn)   (bsee)

-20   +10   -40   +10   +30

The matrix form of a game shows *expected pay-offs to A* as it depends on A's and B's strategies:

	B-resigner	B-seer
A-resigner	-5	+5
A-holder	+10	-5

With hidden information, Minimax no longer necessarily equals Maximin.

A's optimal strategy depends on B's and vice versa. What can we do? What should a computer program for playing A do?

Von Neumann's game theoretic values for games of hidden information

It turns out for the game of mini-poker, the game theoretic value for A is 1c. A can expect to win 1 cent per game on average if A does the right thing.

Furthermore, A can even tell B in advance what A's strategy is. That information will not help B!

Mixed Strategies

The trick is that A must not use a pure strategy, but a mixed strategy, in which at the start of the game (before play) A selects which pure strategy it will use for that game at random. And there will be an optimal probability distribution for it.

For mini-poker, A must decide to be a holder with probability p and a resigner with probability $1-p$. How do we compute the optimal p?

Computing a mixed strategy for A: Guess number one: p=2/3

	B-resigner	B-seer
A-resigner	-5	+5
A-holder	+10	-5

Suppose A chooses to be a holder with probability p = 2/3. Suppose B knows this.

Which strategy will B pick?

How much will A win on average?

Computing a mixed strategy for A: Guess number two: p=1/3

	B-resigner	B-seer
A-resigner	-5	+5
A-holder	+10	-5

Suppose A chooses to be a holder with probability p = 1/3. Suppose B knows this.

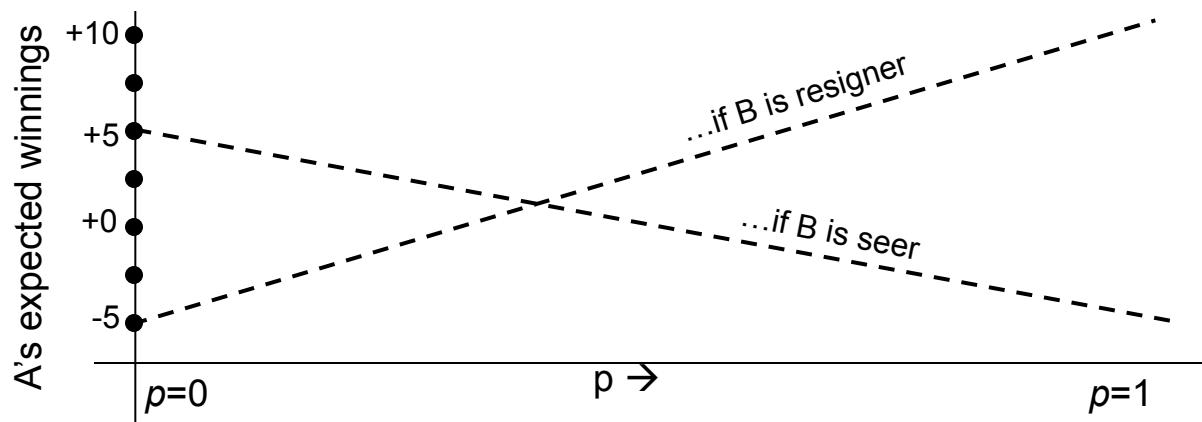Which strategy will B pick?

How much will A win on average?

Computing a mixed strategy for A: All Guesses

	B-resigner	B-seer
A-resigner	-5	+5
A-holder	+10	-5

Suppose A chooses to be a holder with probability p.

If B uses the pure strategy of resigner, A's expected profit is $15p - 5$.

If B uses the pure strategy of seer, A's expected profit is $5 - 10p$.

Suppose A knows B will always use the most annoying pure strategy. What p would A use?
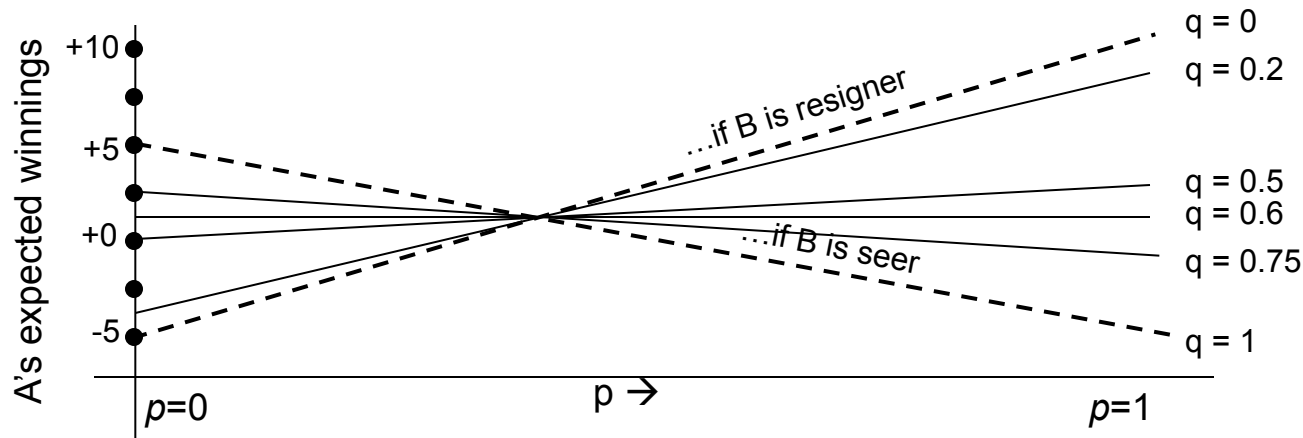
A's minimax optimal mixed strategy

	B-resigner	B-seer
A-resigner	-5	+5
A-holder	+10	-5

The point where the two lines ($15p - 5$ and $5 - 10p$) meet is where $p = 0.4$.

And then the expected profit is 1 cent if B is a seer.

And the expected profit is 1 cent if B is a resigner.

What if B uses a mixed strategy (choose seer with prob q and resigner with prob 1-q)?

Whatever B's mixed strategy, A is still guaranteed 1 cent.

Computing A's optimal mixed strategy for a 2x2 game

(2x2 game = game with a matrix form in which A and B each have two strategies).

	B-I	B-II
A-I	m_{11}	m_{12}
A-II	m_{21}	m_{22}

- Say Player A will use strategy I with prob p.

- Compute Player A's expected gains if B uses pure strategy 1: $m_{11}p + m_{21}(1-p)$

- Compute Player A's gains if B uses pure strategy 2: $m_{12}p + m_{22}(1-p)$

- Find the p between 0 and 1 that maximizes

$$\min(m_{11}p + m_{21}(1-p) , m_{12}p + m_{22}(1-p))$$

Since the two lines are, er, linear, the optimum will be either at $p = 0$, or $p = 1$, or at the p which makes the two expressions equal.

Recipe for computing A's optimal mixed strategy for a nXm game

(nXm game = game with a matrix form in which A has n pure strategies and B has m.)

Say Player A will use strategy *1* with prob p_1.

Say Player A will use strategy *2* with prob p_2.

:

Say Player A will use strategy n with prob p_n.

Player A's expected gains if B uses pure strategy 1: $eg_1 = m_{11}p_1 + m_{21}p_2 + \ldots + m_{n1}p_n$

Player A's expected gains if B uses pure strategy 2: $eg_2 = m_{12}p_1 + m_{22}p_2 + \ldots + m_{n2}p_n$

:

Player A's expected gains if B uses pure strategy m: $eg_m = m_{1m}p_1 + m_{2m}p_2 + \ldots + m_{nm}p_n$

Choose p_1, p_2 $\cdot\cdot$ p_n to maximize

$$min(eg_1, eg_2, \ldots, eg_n)$$

subject to $\Sigma p_i = 1$ and $0 \leq p_i \leq 1$ for all i.

You are planning a meal out with your date. Unfortunately both you and your date happen to be game theorists. Worse still (and with desperate implausibility) you have diametrically opposed views as to what makes a good meal. You agree that in order to decide:

- You'll choose between Mexican and Thai food.
- Your companion will choose the location from: Atwood Street, Walnut Street, or Monroeville.
- You will choose simultaneously.

You generally like Mexican food. The Mexican at Atwood Street gives you three units of gustatory joy. The Walnut Mexican gives you just one unit. But the Monroeville Mexican is -4 units: you strongly dislike that Mall-style cuisine. As for Thai, you are not so keen. Monroeville offers reasonable Thai (one unit). Atwood you dislike (-3 units) and Walnut Street's Thai is also bad, giving you -2 units of joy.

As mentioned, your companion has opposite tastes: any joy units you have are negated from your companion's perspective. Your companion would wish to minimize your joy units as much as you wish to maximize them. Fearing your companion's game theoretic powers of analysis, what is your minimax optimal mixed strategy for selecting between Mexican and Thai dinner? And what is your expected number of gustatory joy units?

What you should know

- Understand the meaning of, and how to construct, the Matrix Normal Form of a game.

- Understand principles of decision making in games with hidden information.

- Know the recipe for solving a 2x2 game.

- Have a basic appreciation for what to do about games bigger than 2x2.