GENERAL ASSEMBLY

INTRO TO DATA SCIENCE

LECTURE 8: BAYESIAN INFERENCE

LAST TIME:

- LOGISTIC REGRESSION

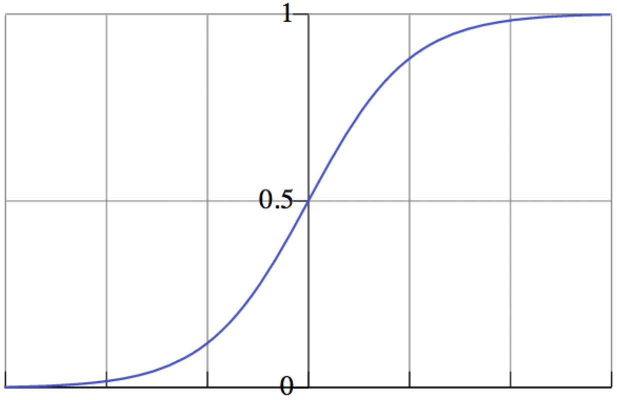
QUESTIONS?

I. REVIEW LOGISTIC REGRESSION
II. PROBABILITY
III. BAYESIAN INFERENCE

I. LOGISTIC REGRESSION

The logistic function*:*

$$E(y|x) = \pi(x) = \frac{e^{\alpha+\beta x}}{1+e^{\alpha+\beta x}}$$



The **logit function** *is an important transformation of the logistic function. Notice that it returns the linear model!*

$$g(x) = ln(\frac{\pi(x)}{1-\pi(x)}) = \alpha + \beta x$$

I. INTRO TO PROBABILITY

Q: What is a **probability***?*

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The probability of event A *is denoted* $P(A)$.

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The probability of the sample space $P(\Omega)$ *is 1.*

Q: What is a probability distribution?

A: A function that assigns probability to each event in the sample space.

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A distribution can be discrete or continuous
Ex:
Discrete – Uniform distribution

$$X \sim \{1, \dots, N\} \quad - \quad P(X = x) = 1/N$$

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Ex:

Continuous – Normal distribution – N(u, o)

$X \sim N(0, 1)$ - $P(X = x) = 0$

Q: What is expected value?

A: It is the average value of a random variable – one that represents the most common value

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For discrete distributions

$$E(X) = \Sigma\ x * p(x)$$

For continuous distributions

$$E(X) = \text{integral}\ (\ x * p(x)\ )$$

Linda is 31 years old, single, outspoken and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

Which is more probable?

1) *Linda is a bank teller.*
2) *Linda is a bank teller and active in the feminist movement.*

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A: With the **joint probability** *of* A *and* B*, written* $P(AB)$.

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Notice, with this we can also write P(AB) = P(A|B) * P(B).

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Using the definition of the conditional probability, we can also write:

$$P(A|B) = P(AB) \ / \ P(B) = P(A) \ \rightarrow \qquad P(AB) = P(A) * P(B)$$

II. BAYESIAN INFERENCE

This result is called **Bayes' theorem***. Here it is again:*

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This result is called **Bayes' theorem**. *Here it is again:*

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Some facts:

- This is a simple algebraic relationship using elementary definitions.

- It's interesting because it's kind of a "wormhole" between two different "interpretations" of probability.

- It's a very powerful computational tool.

Each term in this relationship has a name, and each plays a distinct role in any probability calculation (including ours).

$$P(\text{class } C \mid \{x_i\}) = \frac{P(\{x_i\} \mid \text{class } C) \cdot P(\text{class } C)}{P(\{x_i\})}$$

This term is the **likelihood function**. *It represents the joint probability of observing features* $\{x_i\}$ *given that that record belongs to class* C.

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We can observe the value of the likelihood function from the training data.

This term is the **prior probability** *of* C*. It represents the probability of a record belonging to class* C *before the data is taken into account.*

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The value of the prior is also observed from the data.

This term is the **normalization constant.** *It doesn't depend on* C, *and is generally ignored until the end of the computation.*

$$P(\text{class } C \mid \{x_i\}) = \frac{P(\{x_i\} \mid \text{class } C) \cdot P(\text{class } C)}{P(\{x_i\})}$$

Maximum likelihood estimator (MLE):

*What parameters **maximize** the likelihood function?*

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Maximum a posteriori estimate (MAP):

What parameters ***maximize*** *the likelihood function* ***AND*** *prior?*

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Problem:

We observe the following coin flips:

HTHH

What is $P(X = \text{Heads})$?

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What is P(X = Heads) ? 3/4, Why?

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What is $P(X = \text{Heads})$? 4/7, Why?

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HTHHTHT

Maximum likelihood estimator (MLE):
*What parameters **maximize** the likelihood function?*
Let P(X = Heads) = q, and write Bayes Theorem

P(q | observations) = P (observations | q) * P (q) / constant

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P(q | observations) = P (observations | q) * P (q) / constant

P(observations | q) = Binomial Distribution

P(q) = ????

Binomial Distribution:

$$\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

P (HTHHTHT | q) = P (X = 4, n = 7) =

= (7 choose 4) * q^4 * (1-q) ^ 3

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After optimizing, the **MLE is 4/7**

A prior distribution is known as **conjugate prior** if its from the same family as the posterior for a certain likelihood function

For the binomial distribution, the conjugate prior is the **Beta distribution**

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

$$= \frac{1}{\mathrm{B}(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

The **MAP estimate** is the value that maximizes both the likelihood function and prior – the product of the two.

In the coin flip setting is the value that optimizes

$$P(HTHHTHT \mid q) * P(q)$$

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After optimizing, the **MAP is (4 + a -1) / (7 + a + b – 2)**

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Sample 100 people and ask if they support a politician?

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Sample 100 people and ask if they support a politician?
23 say Yes - Is the correct prediction 23/100?
What's the prior?

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Need to choose between multiple categories to present (for ads, products, news).

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You can compute response % for each category

But each should have a unique prior – **unique psuedo counts**

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$$P(\text{class } C \mid \{x_i\}) = \frac{P(\{x_i\} \mid \text{class } C) \cdot P(\text{class } C)}{P(\{x_i\})}$$

Bayes' theorem can help us to determine the probability of a record belonging to a class, given the data we observe.

source: Data Analysis with Open Source Tools, by Philipp K. Janert. O'Reilly Media, 2011.

The idea of Bayesian inference, then, is to **update** *our beliefs about the distribution of C using the data ("evidence") at our disposal.*

$$P(\text{class } C \mid \{x_i\}) = \frac{P(\{x_i\} \mid \text{class } C) \cdot P(\text{class } C)}{P(\{x_i\})}$$

Then we can use the posterior for prediction.

Q: What piece of the puzzle we've seen so far looks like it could intractably difficult in practice?

Remember the likelihood function?

$$P(\{x_i\}|C) = P(\{x_1, x_2, \ldots, x_n\})|C)$$

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Observing this exactly would require us to have enough data for every possible combination of features to make a reasonable estimate.

Q: What piece of the puzzle we've seen so far looks like it could intractably difficult in practice?

A: Estimating the full likelihood function.

Q: So what can we do about it?

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A: Make a simplifying assumption. In particular, we assume that the features x_i are conditionally independent from each other:

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Q: So what can we do about it?

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This "naïve" assumption simplifies the likelihood function to make it tractable.

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Q: Given that we can compute this value, what do we do with it?

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A: In our training phase, we 'learn' the probability of seeing our training examples under each class.

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Q: Given that we can compute this value, what do we do with it?

A: In our training phase, we 'learn' the probability of seeing our training examples under each class.

Then we use Bayes Theorem to compute P(class | inputs)

$$P(\text{class } C \mid \{x_i\}) = \frac{P(\{x_i\} \mid \text{class } C) \cdot P(\text{class } C)}{P(\{x_i\})}$$

Maximum a posteriori estimate (MAP):

What ***LABEL maximizes*** *the likelihood function* ***AND*** *prior?*

$$P(\text{class } C \mid \{x_i\}) = \frac{P(\{x_i\} \mid \text{class } C) \cdot P(\text{class } C)}{P(\{x_i\})}$$

Example: Text Classification

Does this news article talk about politics?

Training Set: Collection of New Articles

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Training Set: Collection of New Articles

Article 1: The computer contractor who exposed....
Article 2: The parents of a missing U.S. journalist in Syria...

Q: What are my features?

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A: The text in the documents.

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Q: How to I represent them?

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A: The text in the documents.

Q: How to I represent them?

A: Binary occurrence? Word counts?

the,	*computer,*	*contractor,*	*exposed,*	*parents,*	*missing,*	*Syria,*	*U.S.*
1	*1*	*1*	*1*	*0*	*0*	*0*	*0*
1	*0*	*0*	*0*	*1*	*1*	*1*	*1*

computer,	contractor,	exposed,	parents,	missing,	Syria,	U.S.
1	1	1	0	0	0	0
0	0	0	1	1	1	1

We can make some alterations

1) Drop stop words (commonly occurring words that don't have meaning)

computer,	contractor,	exposed,	parents,	missing,	Syria,	U.S.,	**POL**
1	1	1	0	0	0	0	0
0	0	0	1	1	1	1	1

Our goal is to compute compute P (POL = T | words in the text)

We need to ***learn*** *P(word | POL) i.e. P (Syria | POL)*

computer,	contractor,	exposed,	parents,	missing,	Syria,	U.S.,	**POL**
1	1	1	0	0	0	0	0
0	0	0	1	1	1	1	1

Once we've learned P(computer | POL), P(U.S. | POL) on our training set, we want to label our test set

computer,	contractor,	exposed,	parents,	missing,	Syria,	U.S.,	**POL**
1	1	1	0	0	0	0	0
0	0	0	1	1	1	1	1

The correct label, POL = True or POL = False is the one that maximize our posterior.

computer,	*contractor,*	*exposed,*	*parents,*	*missing,*	*Syria,*	*U.S.,*	***POL***
1	1	1	0	0	0	0	0
0	0	0	1	1	1	1	1

Compute probability in each class:

$$P(POL = T \mid \{x\}) = P(\{x\} \mid POL = T) * P(POL=T)$$

$$P(POL = F \mid \{x\}) = P(\{x\} \mid POL = F) * P(POL=F)$$

computer,	contractor,	exposed,	parents,	missing,	Syria,	U.S.,	**POL**
1	1	1	0	0	0	0	0
0	0	0	1	1	1	1	1

Article 2: The parents of a missing U.S. journalist in Syria...

$$P(POL = T \mid \{x\}) = P(\{x\} \mid POL = T) * P(POL=T)$$

$$= P(Syria \mid POL=T) * P(journalist \mid POL=T) * P(parents \mid POL=T) \ldots$$

$$* P(POL=T)$$