Bienvenue!

Correctly Calculating min, max, and More

Walter E Brown

Correctly Calculating min, max, and More

What Can Go Wrong?

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A little about me

- B.A. (math's); M.S., Ph.D. (computer science).
- Professional programmer for over 50 years, programming in C++ since 1982.
- Experienced in industry, academia, consulting, and research:
  - Founded a Computer Science Dept.; served as Professor and Dept. Head; taught and mentored at all levels.
  - Managed and mentored the programming staff for a reseller.
  - Lectured internationally as a software consultant and commercial trainer.
  - Retired from the Scientific Computing Division at Fermilab, specializing in C++ programming and in-house consulting.
- Not dead — still doing training & consulting. (Email me!)

 4

Emeritus participant in C++ standardization

- Written ~170 papers for WG21, proposing such now-standard C++ library features as gcd/lcm, cbegin/cend, common_type, and void_t, as well as all of headers <random> and <ratio>.
- Influenced such core language features as *alias templates*, *contextual conversions*, and *variable templates*; recently worked on *requires-expressions*, operator<=>, and more!
- Conceived and served as Project Editor for *Int'l Standard on Mathematical Special Functions in C++* (ISO/IEC 29124), now incorporated into <cmath>.
- Be forewarned: Based on my training and experience, I hold some rather strong opinions about computer software and programming methodology — these opinions are not shared by all programmers, but they should be! ☺

 5

The Big Picture

The study of error …
serves as a stimulating introduction
to the study of truth.

— Walter Lippmann

About this talk

- The C++ standard library long ago selected operator < as its ordering primitive, and even spells it in several different ways (*e.g.*, std::less).
- Today, we will first demonstrate why operator < (and its aliases) must be used with care, in even seemingly simple algorithms such as max and min.
- Then we will discuss the use of operator < in other order-related algorithms, showing how easy it is to make mistakes when using the operator < primitive directly, no matter how it's spelled.
- Along the way, we will also present a straightforward technique to help us avoid such mistakes.

 7

"One of the amazing things which we … discover is that ordering is very important.

Things which we could do with ordering cannot be effectively done just with equality."

— *Alexander Stepanov*
(né *Алекса́ндр Степа́нов*)

 8

Early Attempts

Life is trying things to see if they work.
— *Ray Bradbury*

An intuitive approach ①

- As C-style function-like macros:
  - `#define MIN(a, b) ( (a) < (b) ? (a) : (b) )`
  - `#define MAX(a, b) ( (b) < (a) ? (a) : (b) )`
- Repackaged, now as functions (with one overload/type):
  - `int min ( int a, int b ) { return a < b ? a : b; }`
  - `int max ( int a, int b ) { return b < a ? a : b; }`
- Lifted, now as simple (C++20) function templates:
  - `auto min ( auto a, auto b ) { return a < b ? a : b; }`
  - `auto max ( auto a, auto b ) { return b < a ? a : b; }`

 10

An intuitive approach ②

- But those C++ templates …
  - `auto min ( auto a, auto b ) { return a < b ? a : b; }`
  - `auto max ( auto a, auto b ) { return b < a ? a : b; }`

… have a few issues:

✗ The by-value parameter passage is potentially expensive (*e.g.*, for large string arg's).

✗ When the arguments have distinct types, it's unclear what the return type should be. (Can we even compare such arg's generically? *E.g.*, consider signed vs. unsigned [forthcoming!])

✗ Major concern: are the algorithms correct for all values?

 11

The cures are mostly straightforward

- Per the std library's specification:
  - ✓ Enforce consistent types via a named type parameter.
  - ✓ Avoid expensive copies via call/return by ref-to-const.
- After these adjustments we have:

```
template< class T >
T const &
   min( T const & a, T const & b ) { return a < b ? a : b; }
```

  - And analogously for max.
- Just recall that lvalue ref's to rvalues can be subtle:
  - ✓ `auto z = min( x.calc( ), y.calc( ) );` *// copies a temporary*
  - ✗ `auto & r = min( x.calc( ), y.calc( ) );` *// dangling reference!*

 12

So What's Wrong?

[N]ever feel badly about making mistakes … as long as you … learn from them.
— *Norton Juster*

Alas, none of the code I've shown so far is right!

- Can you identify the misbehaviors?
  - template< class T >
    T const &
    min (T const & a, T const & b) { return a < b ? a : b; }
  - template< class T >
    T const &
    max(T const & a, T const & b) { return b < a ? a : b; }
- Did you notice that each returns b when a == b?
  - Why should max and min of the same two arguments ever give the same result?
  - ("It took Stepanov 15 years to get min and max right.")

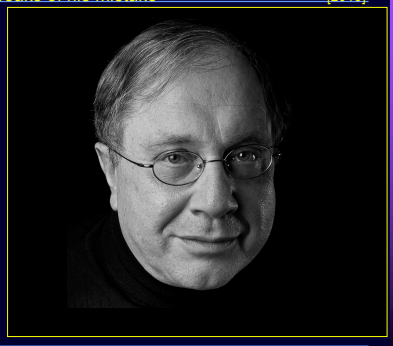
 16

To be specific, …

- … these algorithms mishandle the case of a == b!
  - "[At] CppCon 2014, Committee member Walter Brown mentioned that [std] max returns the wrong value [when] both arguments have an equal value. …
  - "*Why should it matter which value is returned?*"
- Many programmers have made similar observations:
  1. That equal values are indistinguishable, so …
  2. It ought not matter which is returned, so …
  3. This is an uninteresting case, not worth discussing.
- Alas, for min and max (and related) algorithms, such opinions are superficial and incorrect!

 17

Alex Stepanov speaks of his mistake [2013]



 18

Many types do distinguish equal values

- Bare-bones example:

```
struct student {
  string name; int id;
  inline static int registrar = 0;
  S( string n ) : name{ n }, id{ registrar++ } { }     // c'tor
  bool
    operator < ( student s )
  { return name < s.name; }                            // id is not salient
};
```

- Since each student variable has a unique id number:
  - Even equal values are distinguishable, so …
  - It can matter greatly which one is returned by min/max!

 19

How Do We Address This?

[O]nly wise men learn from their mistakes.
— Winston Churchill

A mathematics perspective

- A monotonically increasing sequence is sorted:
  - But not conversely!
  - Counterexample: a sequence of identical values is sorted, but is certainly not monotonically increasing.
- Instead, we must say:
  - That a sequence is sorted iff it is non-decreasing.
  - This allows us to have equal items in a sorted sequence.
- C++ embraces this viewpoint (see [alg.sorting.general]/5):
  - A sequence is *sorted* if, for every iterator i and non-negative integer n, *(i + n) < *i is false.

 21

An important insight

- Given two values a and b, in that order:
  - Unless we find a reason to the contrary, …
  - **min should prefer to return a,** and …
  - **max should prefer to return b.**
- *I.e.*, never should max and min return the same item:
  - When values a and b are in order, min should return a / max should return b; …
  - When values a and b are out of order, min should return b / max should return a.

 22

Even more succinctly stated

- We should always prefer algorithmic stability …
  - … especially when it costs nothing to provide it!
- Recall what we mean by stability:
  - An algorithm dealing with items' order is stable …
  - If it keeps the original order of equal items.
- *I.e.*, a stable algorithm ensures that:
  - For all pairs of equal items a and b, …
  - a will precede b in its output …
  - Whenever a preceded b in its input.

 23

Therefore, I recommend …

- For min:
  - `… { return out_of_order(a, b) ? b : a; }` *// in order ? a : b*
- For max: "Is there a reason to do otherwise?"
  - `… { return out_of_order(a, b) ? a : b; }` *// in order ? b : a*
- Where:
  - `inline bool out_of_order( … x, … y ) { return y < x; }` *// !!!*
  - `inline bool in_order( … x, … y ) { return not out_of_order(x, y); }`
  - FWIW, in my experience, out of order is the more useful.

 24

These Ideas Are Broadly Applicable

[The] principle, by which each slight variation, if useful, is preserved, [I have termed] Natural Selection.

— Charles Darwin

Analogous logic also applies elsewhere ①

```
• template< input_iterator In, output_iterator<In> Out >
  Out merge( In b1, In e1  // 1st sorted input sequence
           , In b2, In e2  // 2nd sorted input sequence
           , Out to ) {    // merged destination

    while( true )
      if      ( b2 == e2 ) return copy( b1, e1, to );
      else if ( b1 == e1 ) return copy( b2, e2, to );
      else // assert: neither sequence is empty
        *to++ = out_of_order(*b1, *b2) ? *b2++ : *b1++;
  }
```

"Prefer to take from the 1st sequence; need a reason to take from the 2nd."

 26

Analogous logic also applies elsewhere ②

```
• template< class T >
  void sort2( T & a, T & b ) {
    if( out_of_order(a, b) )
      swap(a, b);
  } // postcondition: in_order(a, b)
```

`if( in_order(a, b) ) return; swap(a, b);`

```
• template< class T >  // C++20
  void sort3( T & a, T & b, T & c ) {
    if( sort2(a, b); in_order(b, c) ) return;
    if( swap(b, c); in_order(a, b) ) return;
    swap(a, b);
  }
```

- (Did you recognize bubble sort?)

 27

Algorithm logic from stackoverflow — is this correct?

```
template< class T >
void sort3( T & a, T & b, T & c ) {
   if( a < b ) {
      if( b < c ) return;
      else if( a < c ) swap(b, c);
      else { /* rotate right into order c, a, b */ }
   }
   else {
      if( a < c ) swap(a, b);
      else if( c < b ) swap(a, c);
      else { /* rotate left into order b, c, a */ }
   }
}
```

Algorithm does more work than necessary: operator < is no substitute for in order!

Algorithm isn't stable: operator < is no substitute for in order!

 28

Our main takeaways so far

By itself, operator < is not sufficient to tell us whether its operands are in order.

By itself, operator < is sufficient to tell us only whether its reversed operands are out of order.

 29

operator < Is Spelled Other Ways, Too

Sameness is tiresome; variety is pleasing.

— Mark Twain

Many std algorithms don't use operator < *per se*

- Standard library algorithms often specify an overload with an extra parameter, comp, such that:
  - comp(x, y) is called to decide ordering in lieu of x < y.
- Example:
  - ```
    template< class Fwd >
    constexpr Fwd
       is_sorted_until( Fwd first, Fwd last );   // uses operator <
    ```
  - ```
    template< class Fwd, class Compare >
    constexpr Fwd
       is_sorted_until( Fwd first, Fwd last, Compare comp );
                                   // calls comp in place of operator <
    ```

 31

About the is_sorted_until algorithm

- "*Returns*: The last iterator i in [first, last] for which the range [first, i) is sorted…. *Complexity*: Linear."
  - *I.e.*, i induces adj. partitions [first, i) and [i, last) where …
  - The former is known to be sorted and of maximal length.
- Equivalently (but better for algorithmic thinkers), without i :
  - Treat [···, first) as a partition that's known to be sorted, with an adjoining partition [first, last) in unknown order.
  - Iteratively advance first so long as *first is in sorted order with respect to its immediate predecessor (say, *prev).
  - By construction, sorted partition [···, first) has maximal length, so we simply return first (for even empty cases).

 32

My earliest operator < implementation [edited for exposition]

```
template< class Fwd >      // forward iterator
constexpr Fwd
   is_sorted_until( Fwd first, Fwd last )
{
   if( first != last )
      // init/reinit loop as if by prev = first++ :
      for( Fwd prev = first; ++first != last; prev = first )
         if( *first < *prev )      // in order? out of order?
            break;
   return first;
}
```

 33

But, as before, I prefer and recommend …

- … to use a named order predicate.
- ```
  template< class Fwd >
  constexpr Fwd
    is_sorted_until( Fwd first, Fwd last )
  {
    #define out_of_order( x, y ) ( *(y) < *(x) )
    if( first != last )
      for( Fwd prev = first; ++first != last; prev = first )
        if( out_of_order(prev, first) )
          break;
    return first;
  }
  ```

Tip: Pass the iterators (which are typically cheap to copy) rather than the dereferenced values (which may be not even copyable)!

 34

[alg.sorting.general]/2-3 [rearranged]

- "[The declaration] Compare comp is used throughout [as a parameter that denotes] an ordering relation."
  - "Compare is a function object type [whose] call operation … yields true if the first argument of the call is less than the second, and false otherwise."
  - "… comp [induces] a strict weak ordering on the values."
  - "For all algorithms that take Compare, there is a version that uses operator < instead."
- IMO, the names comp and Compare are too general:
  - I'd prefer, *e.g.*, s/comp/less than/ or s/comp/lt/ or s/comp/precedes/.

 35

Even when we have an explicit less-than predicate …

- … I still recommend adapting it via an order predicate.
- ```
  template< class Fwd, class Compare >
  constexpr Fwd
    is_sorted_until( Fwd first, Fwd last, Compare precedes )
  {
    auto iter_out_of_order
      = [=] ( Fwd x, Fwd y ) { return precedes(*y, *x); };
    if( first != last )
      for( Fwd prev = first; ++first != last; prev = first )
        if( iter_out_of_order(prev, first) )
          break;
    return first;
  }
  ```

 36

Or we can avoid overloading

- … via a single template that has judicious default arg's:
  - ```
    template< class Fwd, class Compare = std::ranges::less >
    constexpr Fwd
      is_sorted_until( Fwd first, Fwd last, Compare lt = { } )
    {
      ⋮ // unchanged
    }
    ```
- Q1: What, exactly, is std::ranges::less?
- Q2: Do we need both a default function argument and a default template argument?

 37

Q1: What's std::ranges::less?

- It's a class declared in <functional>:
  - ```
    struct less {                          // simplified for exposition
      template< class T, class U >
      constexpr bool
        operator ( ) ( T && t, U && u ) const
      { return t < u; }                    // heterogeneous comparison
    };
    ```
  - A variable of type less is a function object, as it's callable via its operator ( ) member template.
- (There's also std::less, a template whose operator ( ) is strictly homogeneous [more later]. Many/most today seem to prefer the design of std::ranges::less.)

 38

Q2: Do algorithms need both default argument kinds?

- Review the algorithm declaration, then consider a call:
  - ```
    template< class Fwd, class Compare = std::ranges::less >
    constexpr Fwd
      is_sorted_until( Fwd first, Fwd last, Compare lt = { } ) ;
    ```
  - ```
    int a[N] = { ⋯ };
    ⋯ is_sorted_until(a+0, a+N) ⋯     // what type is Fwd?
    ```
  - Fwd is deduced as int *. Now: what type is Compare?
- It's std::ranges::less, per the default template arg:
  - (A type is never inferred from any default function arg.)
  - Enables calling code to default-construct a 3rd argument, namely std::ranges::less{ } .

 39

Q3: Why doesn't the std library use such default arg's?

- In brief, because it's prohibited (unless thusly specified):
  - "An implementation shall not declare a non-member function signature with additional default arguments." (See [global.functions]/3.)
- Why not consolidate? Because doing so is problematic:
  - "The difference between two overloaded functions and one function with a default argument can be observed by taking a pointer to function." (See N1070, 1997.)
  - Further, consider a call with a type but without a value:

```
template< class T = int >  void g( T x = { } ) { ··· }
        ⋮
g<MyType>( );  // what if MyType isn't default-constructible?
```

 40

std Disguises for operator <

Everybody's wearing a disguise....

— Bob Dylan

How many ways can std design and spell operator < ?

Name	Where found	Since	Arg. types
class template `less`	`<functional>`	C++98	T, T
specialization `less<void>`	`<functional>`	C++14	T, U
class `ranges::less`	`<functional>`	C++20	T, U
function template `cmp_less`	`<utility>` (why?)	C++20	integer I, J
overload set `isless`	`<cmath>`	C++11	arith A, B
specification `totalOrder`	IEEE 754; in spec of `<compare>`'s `strong_order`	2008; C++20	flt-pt F, F

 42

My version of std::ranges::less [edited for exposition]

```
• struct less
  {
    template< class L, class R >
    constexpr bool
      operator( ) ( L && left, R && right ) const noexcept(···)
    {
      if constexpr( are_std_integer_types<L, R> )
        return  cmp_less( left, right );        // forthcoming
      else if constexpr( are_std_arithmetic_types<L, R> )
        return  isless( left, right );          // forthcoming
      else
        return  forward<L>(left)  <  forward<R>(right);
    }
  };
```

 43

My version of std::cmp_less [edited for exposition]

```
• template< std_integer_type L, std_integer_type R >
  constexpr  bool
    cmp_less( L left, R right ) noexcept
  {
    if constexpr( signed_type<L> == signed_type<R> )
      return  left  <  right;
    else if constexpr( signed_type<L> )  // and unsigned_type<R>
      return  left  <  0  ?  true  :  as_unsigned(left) < right;
    else  // signed_type<R> and unsigned_type<L>
      return  right  <  0  ?  false  :  left  <  as_unsigned(right);
  }
```

 44

My version of std::isless [edited for exposition]

```
• template< std_arithmetic_type L, std_arithmetic_type R >
  constexpr  bool
    isless( L left, R right ) noexcept
  {
    using  flt  =  common_floating_point_t<L, R>;
    flt  x  =  left
       ,  y  =  right;
    return  isunordered(x, y)  ?  false     // avoid FE_INVALID
                               :  x  <  y;
  }
```

 45

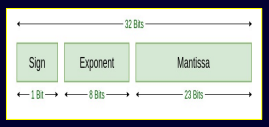
The ordering used by IEEE's totalOrder predicate

- Most- to least-negative, then least- to most-positive.
- *I.e.*, first all negative values, in the following order:
  - All negative quiet NaNs, then all negative signaling NaNs, each ordered per their payload bits.
  - Then negative infinity, then all negative normalized and denormal numbers in value order, then negative zero.
- Then all positive values, in the opposite order:
  - Positive zero, then all positive denormal and normalized numbers in value order, then positive infinity.
  - All positive signaling NaNs, then all positive quiet NaNs, each ordered per their payload bits.

 46

Now consider IEEE's floating-point layout in that light

- Relative to trad. scientific notation ±d.d… x $10^{\pm e\ldots}$, IEEE decomposes/rebases/reorders/adjusts its parts:



What if we treated these bits as a 32/64/128-bit int?

 47

My version of IEEE's totalOrder [edited for exposition]

```
template< floating_point type F >                    // assumes IEEE
constexpr bool
  total_order( F left, F right ) noexcept
{
  if( signbit(left) != signbit(right) )        // opposite sign bits
    return signbit(left);
  else {
    using int_t = big_enough_type< sizeof(F)
                                  , int32_t, int64_t, int128_t >;
    int_t x = bit_cast< int_t >( left )
        , y = bit_cast< int_t >( right );
    return signbit(x) ? in_order(y ,x)   // both have sign bit set
                      : in_order(x, y); // neither has sign bit set
  }
}
```

 48

A Bonus Algorithm

I Xeroxed a mirror.
Now I have an extra Xerox machine.

— Steven Wright

Suppose you need both *extrema*

- We could reuse min and max:

```
template< class T >
pair<T const &, T const & >
  minmax( T const & a, T const & b )
{
  return { min(a, b), max(a, b) };
}
```

- But it's cheaper to make one call to operator < than the two made within separate calls to min and to max:

```
if( out_of_order(a, b) ) return { b, a };
else                     return { a, b };
```

 50

Finally, let's consider minmax over a sequence

- Found in the <algorithm> header:

```
template< forward_iterator Fwd >
pair<Fwd, Fwd>
  minmax_element( Fwd first, Fwd last );
```

  - It returns a pair {m, M}, iterators in [first, last), such that:
    - m is the first iterator whose ∗m is smallest, while …
    - M is the last iterator whose ∗M is largest.
- Let N = distance(first, last):
  - Separate calls to min then max functions would lead to $O(N + N = 2N)$ calls to out_of_order.
  - But Pohl's 1972 algorithm needs only $O(3N/2)$ calls!

 51

Infrastructure for Ira Pohl's algorithm

- Given forward iterators f1, f2 , we'll use:
  - precedes(f1, f2) that returns *f1 < *f2 (or lt(*f1 , *f2) when there's a Compare lt).
  - out_of_order(f1, f2) that returns precedes(f2, f1) .
  - max(f1, f2) and min(f1, f2) that call out_of_order(f1, f2) .
- Let mM denote an ordered std::pair of iterators:
  - minMax(f1, f2) that makes an mM pair by returning out_of_order(f1, f2) ? mM{ f2, f1 } : mM{ f1, f2 } .
  - meld(a, b) that combines two mM pairs into one via mM{ min(a.first, b.first), max(a.second, b.second) } .

 52

The logic of Pohl's algorithm [C++20]

- Most of the code is for special cases at begin/end:

```
using mM = std::pair<Fwd, Fwd>;
Fwd prev = first;
if( prev == last or ++first == last )        // empty? singleton?
   return mM{prev, prev};
for( mM so_far = minMax(prev, first); ; )    // initial pair
   if( ++first == last )                     // nothing more to process?
      return so_far;
   else if( prev = first; ++first == last )  // final singleton?
      return meld( so_far, mM{prev, prev} );
   else                                      // general case: meld result so far w/ latest pair
      so_far = meld( so_far, minMax(prev, first) );
```

 53

Correctly Calculating min, max, and More

FIN

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