Unit I: $A\boldsymbol{x} = \boldsymbol{b}$ and the Four Subspaces

Session 1.1: The Geometry of Linear Equations

We have a system of equations:

$$\begin{cases} 2x - y = 0 \\ -x + 2y = 3 \end{cases}$$

Row Picture

Line $2x - y = 0$ and line $-x + 2y = 0$ intersects at the point $(1, 2)$, so $(1, 2)$ is the solution of the system of equations.

Maybe I should draw a X-Y coordinates here >_>

Column Picture

We rewrite the system of linear equations as a single equation:

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

We see x and y as scalars of column vectors: $\boldsymbol{v_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\boldsymbol{v_2} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and the sum $x\boldsymbol{v_1} + y\boldsymbol{v_2}$ is called a *linear combination* of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$.

Geometrically, we can find one copy of $\boldsymbol{v_1}$ added to two copies of $\boldsymbol{v_2}$ just equals the vector $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Then the solution should be $x = 1, y = 2$.

I will add a figure when time is available >_>

Matrix Picture

We rewrite the equations in our example as a compact form,

$$A\boldsymbol{x} = \boldsymbol{b},$$

that is

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Matrix Multiplication

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

A matrix times by a vector is just **a linear combination of the column vectors of the matrix**.

Session 1.2: An Overview of Key Ideas

Vectors

Let us take linear combinations of vectors.

Matrices

The product of a matrix and a vector is a combination of the columns of the matrix.

Subspaces

All combinations of column vectors creates a subspace. The subspaces of $\mathbb{R}^3$ are:

- the origin,
- a line through the origin,
- a plane through the origin,
- all of $\mathbb{R}^3$.

Conclusion

- A is invertible

  $\Leftrightarrow$ $A\boldsymbol{x} = \boldsymbol{b}$ has the unique solution $\boldsymbol{x}$ for each $\boldsymbol{b}$

  $\Leftrightarrow$ $A\boldsymbol{x} = 0$ has no non-zero solution $\boldsymbol{x}$

  $\Leftrightarrow$ The columns of A are *independent*

  $\Leftrightarrow$ All vectors $A\boldsymbol{x}$ cover the whole vector space

  Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

- A is not invertible

  $\Leftrightarrow$ $A\boldsymbol{x} = \boldsymbol{b}$ has a solution $\boldsymbol{x}$ only for some of $\boldsymbol{b}$ in the vector space

  $\Leftrightarrow$ $A\boldsymbol{x} = 0$ has non-zero solutions $\boldsymbol{x}$

  $\Leftrightarrow$ The columns of A are *dependent*

  $\Leftrightarrow$ All vectors $A\boldsymbol{x}$ lies in only a subspace of the vector space

  Example: $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Session 1.3: Elimination with Matrices

Method of Elimination

We have an example $A\boldsymbol{x} = \boldsymbol{b}$,

$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}$.

Steps of Elimination:

- Step 1: subtract 3 times row 1 from row 2;
- Step 2: subtract 2 times row 2 from row 3.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \rightarrow U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 2 \\ 6 \\ -10 \end{bmatrix}$$

Thus, we can easily solve the systems of equations, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

Elimination Matrices

The product of a matrix (3x3) and a column vector (3x1) is a column vector (3x1) that is a linear combination of the columns of the matrix.

The product of a row vector (1x3) and a matrix (3x3) is a row vector (1x3) that is a linear combination of the rows of the matrix.

For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}.$$

Multiplying on the left by a permutation matrix exchanges the rows of a matrix, while multiplying on the right exchanges the columns. For example,

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

P is a *permutation matrix* and the first and second rows of the matrix PA are the second and first rows of the matrix A.

Note, matrix multiplication is *associative* but *not commutative*.

Inverses

We have a matrix:

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which subtracts 3 times row 1 from row 2. To "**undo**" this operation we must add 3 times row 1 to row 2 using the inverse matrix:

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In fact, $E_{21}^{-1}E_{21} = I$.

Session 1.4: Multiplication and Inverse Matrices

Four and a half ways we see matrix multiplication

We have $AB = C$. A is an $m \times n$ matrix and B is an $n \times p$ matrix, then C is an $m \times p$ matrix. We use c_{ij} to denote the entry in row i and column j of matrix C and the same denotation applies to a_{ij} and b_{ij}.

Row times column

$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Columns

The product of matrix A and column j of matrix B equals column j of matrix C. This tells us that the columns of C are combinations of columns of A.

$$A \begin{bmatrix} | & | & | \\ column1 & column2 & column3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A(column1) & A(column2) & A(column3) \\ | & | & | \end{bmatrix}$$

Rows

The product of row i of matrix A and matrix B equals row i of matrix C. So the rows of C are combinations of rows of B.

$$\begin{bmatrix} --- & row1 & --- \\ --- & row2 & --- \\ --- & row3 & --- \end{bmatrix} B = \begin{bmatrix} --- & (row1)B & --- \\ --- & (row2)B & --- \\ --- & (row3)B & --- \end{bmatrix}$$

Column times row

$$AB = \sum_{k=1} n \begin{bmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{bmatrix} \begin{bmatrix} b_{k1} & \cdots & b_{kp} \end{bmatrix}$$

note: I fixed a typo in the orginal MIT's lecture summary here: b_{kp} instead of b_{kn}.

Blocks

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{bmatrix}$$

Inverses

If A is *singular* or *not invertible*,

then A does not have an inverse,

and we can find some non-zero vector $\boldsymbol{x}$ for which $A\boldsymbol{x} = 0$

Gauss-Jordan Elimination

$$E\left[\begin{array}{c|c} A & I \end{array}\right] = \left[\begin{array}{c|c} I & E \end{array}\right]$$

If $EA = I$, then $E = A^{-1}$.

Session 1.5: Factorization into $A = LU$

Inverse of a product

$$(AB)^{-1} = B^{-1}A^{-1}$$

Transpose of a product

$$(AB)^T = B^T A^T, \quad (A^T)^{-1} = (A^{-1})^T$$

$A = LU$

We can use elimination to convert A into an upper triangular matrix U, that is $EA = U$, and further we can also convert this to a factorization $A = LU$ in which $L = E^{-1}$.

For example, in a three dimensional case, if $E_{32}E_{31}E_{21}A = U$ then $A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U = LU$. Suppose E_{31} is the identity matrix and E_{32} and E_{21} are as shown below:

$$\begin{array}{c} E_{32} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \end{array} \begin{array}{c} E_{21} \\ \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{array}{c} E \\ \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix} \end{array}.$$

Here $L = E^{-1} = E_{21}^{-1}E_{32}^{-1}$:

$$
\begin{array}{ccccc}
E_{21}^{-1} & E_{32}^{-1} & & L \\
\begin{bmatrix} 1 & 0 & 0 \\ \underline{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} &
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \underline{5} & 1 \end{bmatrix} & = &
\begin{bmatrix} 1 & 0 & 0 \\ \underline{2} & 1 & 0 \\ 0 & \underline{5} & 1 \end{bmatrix}
\end{array}
$$

Notice the 0 in row three column one of L, where E had a 10. The factorization $A = LU$ is preferable to the statement $EA = U$ because the combination of row subtractions does not have the effect on L that it did on E.

If there are no row exchanges, the multipliers from the elimination matrices are copied directly into L.

Cost of elimination

If we define a typical operation is to multiply one row and then subtract it from another, then the total number of operations needed to factor $n \times n$ A into LU is on the order of n^3:

$$1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = \sum_{i=1}^{n} i^2 \approx \int_0^n x \,\mathrm{d}x = \frac{1}{3}n^3.$$

While we're factoring A we're also operating on $\boldsymbol{b}$. That costs about n^2 operations, which is hardly worth counting compared to $1/3n^3$.

Row exchanges

The inverse of any permutation matrix P is $P^{-1} = P^T$.

There are $n!$ different ways to permute the rows of an $n \times n$ matrix (including the permutation that leaves all rows unfixed) so there are $n!$ permutation matrices. These matrices form a *multiplicative group.*

Session 1.6: Transpose, Permutation, Vector Spaces $\mathbb{R}^n$

Permutations

Nothing new here.

Transposes

$$(A^T)_{ij} = A_{ji}$$

Given any matrix R the product $R^T R$ is always *symmetric*, which means the transpose of a matrix equals itself, because $(R^T R)^T = R^T (R^T)^T = R^T R$.

Vector spaces

Closure

A collection of vectors has to satisfy two conditions:

1. closed under addition, which means the sum of any two vectors in the collection lies again in the collection,
2. closed under multiplication by any real numbers, that is to say that multiplying any vector in the collection by any real number will not give a vector beyond the collection,

or, to put it in another way, closed under linear combinations.

s.t. we call the collection a *vector space*.

Subspaces

A vector space that is contained inside of another vector space is called a *subspace* of that space. For example,

the subspaces of $\mathbb{R}^2$ are:

1. all of $\mathbb{R}^2$,
2. any line through the zero vector and
3. the zero vector alone.

Every subspace must contain the zero vector.

Column space

Given a matrix A, all the linear combinations of the columns of A form a subspace. This is the *column space* $C(A)$.

For example, if $A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$, the column space of A is the plane through the origin in $\mathbb{R}^3$ containing $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$.