Unit I: $A\boldsymbol{x} = \boldsymbol{b}$ and the Four Subspaces

Seesion 1.1: The Geometry of Linear Equations

We have a system of equations:

$$\begin{cases} 2x - y = 0 \\ -x + 2y = 3 \end{cases}$$

Row Picture

Line $2x - y = 0$ and line $-x + 2y = 0$ intersects at the point $(1, 2)$, so $(1, 2)$ is the solution of the system of equations.

Maybe I should draw a X-Y coordinates here...

Column Picture

We rewrite the system of linear equations as a single equation:

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

We see x and y as scalars of column vectors: $\boldsymbol{v_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\boldsymbol{v_2} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and the sum $x\boldsymbol{v_1} + y\boldsymbol{v_2}$ is called a *linear combination* of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$.

Geometrically, we can find one copy of $\boldsymbol{v_1}$ added to two copies of $\boldsymbol{v_2}$ just equals the vector $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Then the solution should be $x = 1, y = 2$.

I will add a figure when time is available >_>

Matrx Picture

We rewrite the equations in our example as a compact form,

$$A\boldsymbol{x} = \boldsymbol{b},$$

that is

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Matrix Multiplication

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

A matrix times by a vector is just **a linear combination of the column vectors of the matrix**.

Session 1.2: An Overview of Key Ideas

Vectors

Let us take linear combinations of vectors.

Matrices

The product of a matrix and a vector is a combination of the columns of the matrix.

Subspaces

All combinations of column vectors creates a subspace. The subspaces of $\mathbb{R}^3$ are:

- the origin,
- a line through the origin,
- a plane through the origin,
- all of $\mathbb{R}^3$.

Conclusion

- A is invertible

  $\Leftrightarrow A\boldsymbol{x} = \boldsymbol{b}$ has the unique solution $\boldsymbol{x}$ for each $\boldsymbol{b}$

  $\Leftrightarrow A\boldsymbol{x} = 0$ has no non-zero solution $\boldsymbol{x}$

  $\Leftrightarrow$ The columns of A are *independent*

$\Leftrightarrow$ All vectors $A\boldsymbol{x}$ cover the whole vector space

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

- A is not invertible

  $\Leftrightarrow$ $A\boldsymbol{x} = \boldsymbol{b}$ has a solution $\boldsymbol{x}$ only for some of $\boldsymbol{b}$ in the vector space

  $\Leftrightarrow$ $A\boldsymbol{x} = 0$ has non-zero solutions $\boldsymbol{x}$

  $\Leftrightarrow$ The columns of A are *dependent*

  $\Leftrightarrow$ All vectors $A\boldsymbol{x}$ lies in only a subspace of the vector space

  Example: $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$