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Unit I: $A\boldsymbol{x} = \boldsymbol{b}$ and the Four Subspaces

Session 1.1: The Geometry of Linear Equations

We have a system of equations:

$$\begin{cases} 2x - y = 0 \\ \\ -x + 2y = 3 \end{cases}$$

Row Picture

Line $2x - y = 0$ and line $-x + 2y = 0$ intersects at the point $(1, 2)$, so $(1, 2)$ is the solution of the system of equations.

> Maybe I should draw a X-Y coordinates here...

Column Picture

We rewrite the system of linear equations as a single equation:

$$x \begin{bmatrix} 2 \\ \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 3 \end{bmatrix}$$

We see x and y as scalars of column vectors: $\boldsymbol{v_1} = \begin{bmatrix} 2 \\ \\ -1 \end{bmatrix}$ and $\boldsymbol{v_2} = \begin{bmatrix} -1 \\ \\ 2 \end{bmatrix}$, and the sum $x\boldsymbol{v_1} + y\boldsymbol{v_2}$ is called a *linear combination* of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$.

Geometrically, we can find one copy of $\boldsymbol{v_1}$ added to two copies of $\boldsymbol{v_2}$ just equals the vector $\begin{bmatrix} 0 \\ \\ 3 \end{bmatrix}$. Then the solution should be $x = 1, y = 2$.

> I will add a figure when time is available >_>

Matrix Picture

We rewrite the equations in our example as a compact form,

$$A\boldsymbol{x} = \boldsymbol{b},$$

that is

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Matrix Multiplication

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

A matrix times by a vector is just **a linear combination of the column vectors of the matrix**.

Session 1.2: An Overview of Key Ideas

Vectors

Let us take linear combinations of vectors.

Matrices

The product of a matrix and a vector is a combination of the columns of the matrix.

Subspaces

All combinations of column vectors creates a subspace. The subspaces of $\mathbb{R}^3$ are:

- the origin,
- a line through the origin,
- a plane through the origin,
- all of $\mathbb{R}^3$.

Conclusion

- A is invertible

  $\Leftrightarrow$ $A\boldsymbol{x} = \boldsymbol{b}$ has the unique solution $\boldsymbol{x}$ for each $\boldsymbol{b}$

  $\Leftrightarrow$ $A\boldsymbol{x} = 0$ has no non-zero solution $\boldsymbol{x}$

  $\Leftrightarrow$ The columns of A are *independent*

  $\Leftrightarrow$ All vectors $A\boldsymbol{x}$ cover the whole vector space

  Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

- A is not invertible

  $\Leftrightarrow$ $A\boldsymbol{x} = \boldsymbol{b}$ has a solution $\boldsymbol{x}$ only for some of $\boldsymbol{b}$ in the vector space

  $\Leftrightarrow$ $A\boldsymbol{x} = 0$ has non-zero solutions $\boldsymbol{x}$

  $\Leftrightarrow$ The columns of A are *dependent*

  $\Leftrightarrow$ All vectors $A\boldsymbol{x}$ lies in only a subspace of the vector space

  Example: $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Session 1.3: Elimination with Matrices

Method of Elimination

We have an example $A\boldsymbol{x} = \boldsymbol{b}$,

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}.$$

Steps of Elimination:

- Step 1: subtract 3 times row 1 from row 2;
- Step 2: subtract 2 times row 2 from row 3.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \to U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix} \to \cdots \to \begin{bmatrix} 2 \\ 6 \\ -10 \end{bmatrix}$$

Thus, we can easily solve the systems of equations, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

Elimination Matrices

The product of a matrix (3x3) and a column vector (3x1) is a column vector (3x1) that is a linear combination of the columns of the matrix.

The product of a row vector (1x3) and a matrix (3x3) is a row vector (1x3) that is a linear combination of the rows of the matrix.

For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}.$$

Multiplying on the left by a permutation matrix exchanges the rows of a matrix, while multiplying on the right exchanges the columns. For example,

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

P is a *permutation matrix* and the first and second rows of the matrix PA are the second and first rows of the matrix A.

Note, matrix multiplication is *associative* but *not commutative.*

Inverses

We have a matrix:

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which subtracts 3 times row 1 from row 2. To "**undo**" this operation we must add 3 times row 1 to row 2 using the inverse matrix:

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In fact, $E_{21}^{-1}E_{21} = I$.

Session 1.4: Multiplication and Inverse Matrices

Four and a half ways we see matrix multiplication

We have $AB = C$. A is an $m \times n$ matrix and B is an $n \times p$ matrix, then C is an $m \times p$ matrix. We use c_{ij} to denote the entry in row i and column j of matrix C and the same denotation applies to a_{ij} and b_{ij}.

1. Row times column

$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$

2. Columns

The product of matrix A and column j of matrix B equals column j of matrix C. This tells us that the columns of C are combinations of columns of A.

$$A \begin{bmatrix} | & | & | \\ column1 & column2 & column3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A(column1) & A(column2) & A(column3) \\ | & | & | \end{bmatrix}$$

3.Rows

The product of row i of matrix A and matrix B equals row i of matrix C. So the rows of C are combinations of rows of B.

$$\begin{bmatrix} --- & row1 & --- \\ --- & row2 & --- \\ --- & row3 & --- \end{bmatrix} B = \begin{bmatrix} --- & (row1)B & --- \\ --- & (row2)B & --- \\ --- & (row3)B & --- \end{bmatrix}$$

4. Column times row

$$AB = \sum_{k=1} n \begin{bmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{bmatrix} \begin{bmatrix} b_{k1} & \cdots & b_{kp} \end{bmatrix}$$

note: a typo in the MIT's lecture summary here: b_{kp}, not b_{kn}.

5. Blocks

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{bmatrix}$$

Inverses

If A is *singular* or *not invertible*,

then A does not have an inverse,

and we can find some non-zero vector $\boldsymbol{x}$ for which $A\boldsymbol{x} = 0$

Gauss-Jordan Elimination

$$E\left[\begin{array}{c|c} A & I \end{array}\right] = \left[\begin{array}{c|c} I & E \end{array}\right]$$

If $EA = I$, then $E = A^{-1}$.

Session 1.5: Factorization into $A = LU$

Inverse of a product

$$(AB)^{-1} = B^{-1}A^{-1}$$

Transpose of a product

$$(AB)^T = B^T A^T, \quad (A^T)^{-1} = (A^{-1})^T$$

$A = LU$

We can use elimination to convert A into an upper triangular matrix U, that is $EA = U$, and further we can also convert this to a factorization $A = LU$ in which $L = E^{-1}$.

For example, in a three dimensional case, if $E_{32}E_{31}E_{21}A = U$ then $A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U = LU$. Suppose E_{31} is the identity matrix and E_{32} and E_{21} are as shown below:

$$\begin{array}{ccccc} E_{32} & & E_{21} & & E \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} & & \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & = & \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix} \end{array}.$$

Here $L = E^{-1} = E_{21}^{-1}E_{32}^{-1}$:

$$\begin{array}{ccccc} E_{21}^{-1} & E_{32}^{-1} & & L \\ \begin{bmatrix} 1 & 0 & 0 \\ \underline{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \underline{5} & 1 \end{bmatrix} & = & \begin{bmatrix} 1 & 0 & 0 \\ \underline{2} & 1 & 0 \\ 0 & \underline{5} & 1 \end{bmatrix} \end{array}$$

Notice the 0 in row three column one of L, where E had a 10. The factorization $A = LU$ is preferable to the statement $EA = U$ because the combination of row subtractions does not have the effect on L that it did on E.

If there are no row exchanges, the multipliers from the elimination matrices are copied directly into L.

Cost of elimination

If we define a typical operation is to multiply one row and then subtract it from another, then the total number of operations needed to factor $n \times n$ A into LU is on the order of n^3:

$$1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = \sum_{i=1}^{n} i^2 \approx \int_0^n x\,\mathrm{d}x = \frac{1}{3}n^3.$$

While we're factoring A we're also operating on $\boldsymbol{b}$. That costs about n^2 operations, which is hardly worth counting compared to $1/3n^3$.

Row exchanges

- The inverse of any permutation matrix P is $P^{-1} = P^T$.
- There are $n!$ different ways to permute the rows of an $n \times n$ matrix (including the permutation that leaves all rows unfixed) so there are $n!$ permutation matrices. These matrices form a *multiplicative group*.