

Unit I: $Ax = b$ and the Four Subspaces

Session 1.1: The Geometry of Linear Equations

We have a system of equations:

$$\begin{cases} 2x - y = 0 \\ -x + 2y = 3 \end{cases}$$

Row Picture

Line $2x - y = 0$ and line $-x + 2y = 0$ intersects at the point $(1, 2)$, so $(1, 2)$ is the solution of the system of equations.

Maybe I should draw a X-Y coordinates here...

Column Picture

We rewrite the system of linear equations as a single equation:

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

We see x and y as scalars of column vectors: $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and the sum $x\mathbf{v}_1 + y\mathbf{v}_2$ is called a *linear combination* of \mathbf{v}_1 and \mathbf{v}_2 .

Geometrically, we can find one copy of \mathbf{v}_1 added to two copies of \mathbf{v}_2 just equals the vector $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Then the solution should be $x = 1, y = 2$.

I will add a figure when time is available >_>

Matrx Picture

We rewrite the equations in our example as a compact form,

$$Ax = b,$$

that is

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Matrix Multiplication

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

A matrix times by a vector is just a **linear combination of the column vectors of the matrix**.

Session 1.2: An Overview of Key Ideas

Vectors

Let us take linear combinations of vectors.

Matrices

The product of a matrix and a vector is a combination of the columns of the matrix.

Subspaces

All combinations of column vectors creates a subspace. The subspaces of \mathbb{R}^3 are:

- the origin,
- a line through the origin,
- a plane through the origin,
- all of \mathbb{R}^3 .

Conclusion

- A is invertible
 - $\Leftrightarrow A\mathbf{x} = \mathbf{b}$ has the unique solution \mathbf{x} for each \mathbf{b}
 - $\Leftrightarrow A\mathbf{x} = 0$ has no non-zero solution \mathbf{x}
 - \Leftrightarrow The columns of A are *independent*

\Leftrightarrow All vectors $A\mathbf{x}$ cover the whole vector space

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

- A is not invertible

$\Leftrightarrow A\mathbf{x} = \mathbf{b}$ has a solution \mathbf{x} only for some of \mathbf{b} in the vector space

$\Leftrightarrow A\mathbf{x} = 0$ has non-zero solutions \mathbf{x}

\Leftrightarrow The columns of A are *dependent*

\Leftrightarrow All vectors $A\mathbf{x}$ lies in only a subspace of the vector space

Example: $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Session 1.3 Elimination with Matrices

Method of Elimination

We have an example $A\mathbf{x} = \mathbf{b}$,

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}.$$

Steps of Elimination:

- Step 1: subtract 3 times row 1 from row 2;
- Step 2: subtract 2 times row 2 from row 3.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \rightarrow U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 2 \\ 6 \\ -10 \end{bmatrix}$$

Thus, we can easily solve the systems of equations, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$

Elimination Matrices

The product of a matrix (3x3) and a column vector (3x1) is a column vector (3x1) that is a linear combination of the columns of the matrix.

The product of a row vector (1x3) and a matrix (3x3) is a row vector (1x3) that is a linear combination of the rows of the matrix.

For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}.$$

Multiplying on the left by a permutation matrix exchanges the rows of a matrix, while multiplying on the right exchanges the columns. For example,

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

P is a *permutation matrix* and the first and second rows of the matrix PA are the second and first rows of the matrix A .

Note, matrix multiplication is *associative* but *not commutative*.

Inverses

We have a matrix:

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which subtracts 3 times row 1 from row 2. To “**undo**” this operation we must add 3 times row 1 to row 2 using the inverse matrix:

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In fact, $E_{21}^{-1}E_{21} = I$.

Session 1.4: : Multiplication and Inverse Matrices

Four and a half ways we see matrix multiplication

We have $AB = C$. A is an $m \times n$ matrix and B is an $n \times p$ matrix, then C is an $m \times p$ matrix. We use c_{ij} to denote the entry in row i and column j of matrix C and the same denotation applies to a_{ij} and b_{ij} .

1. Row times column

$$c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$$

2. Columns

The product of matrix A and column j of matrix B equals column j of matrix C . This tells us that the columns of C are combinations of columns of A .

$$A \begin{bmatrix} | & & | & & | \\ \text{column1} & \text{column2} & \text{column3} \\ | & & | & & | \end{bmatrix} = \begin{bmatrix} | & & | & & | \\ A(\text{column1}) & A(\text{column2}) & A(\text{column3}) \\ | & & | & & | \end{bmatrix}$$

3.Rows

The product of row i of matrix A and matrix B equals row i of matrix C . So the rows of C are combinations of rows of B .

$$\begin{bmatrix} - & - & - & \text{row1} & - & - & - \\ - & - & - & \text{row2} & - & - & - \\ - & - & - & \text{row3} & - & - & - \end{bmatrix} B = \begin{bmatrix} - & - & - & (\text{row1})B & - & - & - \\ - & - & - & (\text{row2})B & - & - & - \\ - & - & - & (\text{row3})B & - & - & - \end{bmatrix}$$

4. Column times row

$$AB = \sum_{k=1}^n \begin{bmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{bmatrix} \begin{bmatrix} b_{k1} & \vdots & b_{kp} \end{bmatrix}$$

note: a typo in the MIT's lecture summary here: b_{kp} , not b_{kn} .

5. Blocks

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{bmatrix}$$

Inverses

If A is *singular* or *not invertible*,

then A does not have an inverse,

and we can find some non-zero vector \mathbf{x} for which $A\mathbf{x} = 0$

Gauss-Jordan Elimination

$$E \left[\begin{array}{c|c} A & I \end{array} \right] = \left[\begin{array}{c|c} I & E \end{array} \right]$$

If $EA = I$, then $E = A^{-1}$.