EECS 16B Designing Information Devices and Systems II
Fall 2020 UC Berkeley

Note 7

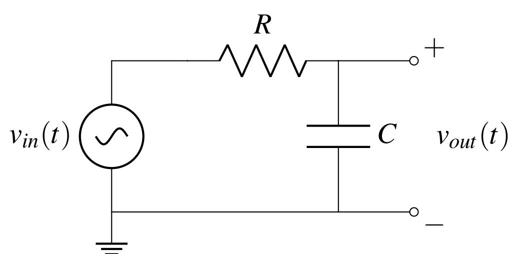
Introduction

In the previous note, we developed the method of Phasor Analysis to analyze the steady state behavior of sinusoidal voltages and currents. In this note, we will consider some applications of these circuits, and begin to explore techniques for designing these circuits to fit a set of requirements called a **filter**.

1 Transfer Functions

When analyzing circuits in the phasor domain, we have always told you what the input voltage to the circuit was. However, sometimes we have many input sinusoids and we would like to see how the circuit generically responds to a sinusoid input of frequency ω. We want to see how an input sinusoid "transfers" into an output sinusoid. How do we do this?

Let's start with a simple RC circuit.



In the phasor domain, the impedance of the capacitor is $Z_C = \frac{1}{j\omega C}$ and the impedance of the resistor is $Z_R = R$. Because we treat impedances the same as resistances, this circuit looks like a voltage divider in the phasor domain. This lets us write $\widetilde{V}_{out}$ as

$$\widetilde{V}_{out} = \frac{Z_C}{Z_R + Z_C}\widetilde{V}_{in} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}\widetilde{V}_{in} = \frac{1}{j\omega RC + 1}\widetilde{V}_{in}$$

The ratio of the output and input is called the **transfer function** or **frequency response** of the system.

$$H(\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} = \frac{1}{1 + j\omega RC}$$

Note that a transfer function can only be defined in the phasor domain.

Now, given an arbitrary input sinusoid, if we multiply it by the frequency response, we can get the output sinusoid. What this allows us to do is to model any arbitrary circuit as a single-input-single-output black box. The transfer function completely defines how our circuit works at a given frequency ω. Let's take a look at some examples to understand what this means.

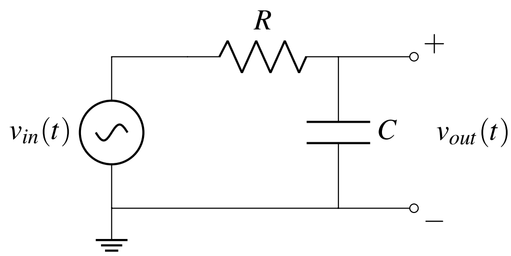
2 First Order Filters

In this section, we will develop the concept of a **filter** using our knowledge of transfer functions. The key idea here is that a filter lets signals at some frequencies through and will block out other frequencies.

2.1 Low-Pass Filters

Let's say we had some high frequency noise greater than 100 kHz in our input signal that we would like to attenuate using a filter. If we were to design an ideal filter to get rid of this noise, we would like all frequencies below 100 kHz to be let through while blocking out any frequencies above 100 kHz. Mathematically this would mean $H(\omega) = 1$ for all $\omega < 100\,\text{kHz}$ and $H(\omega) = 0$ for $\omega \geq 100\,\text{kHz}$.

While designing this ideal filter is difficult, we could certainly design a low-pass filter using an *RC* circuit.



Recall from the previous example that

$$H(\omega) = \frac{1}{1 + j\omega RC}$$

Now how can we show that this filter is in fact a low-pass filter?

As a quick intuition check, if $\omega = 0$, $H(\omega) = 1$ and if $\omega \to \infty$, then $H(\omega) \to 0$. Therefore, our filter seems to behave as expected, but what happens in between 0 and ∞? Let's take a look at a couple of values around $\omega_c = \frac{1}{RC}$.

ω	$H(\omega)$	$\lvert H(\omega)\rvert$	$\angle H(\omega)$
$0.1\omega_c$	$\frac{1}{1+0.1j}$	0.995	-6°
ω_c	$\frac{1}{1+j}$	0.71	-45°
$10\omega_c$	$\frac{1}{1+10j}$	0.1	-84°

This should show that $\omega_c = \frac{1}{RC}$ is a very important frequency to look at since this is around the frequency where the behavior of the filter starts to qualitatively change.[1] In fact, this is so important that we call this the **cutoff** or **corner** frequency. Below this frequency, the filter seems to let everything through, while much above this frequency, the filter blocks everything.

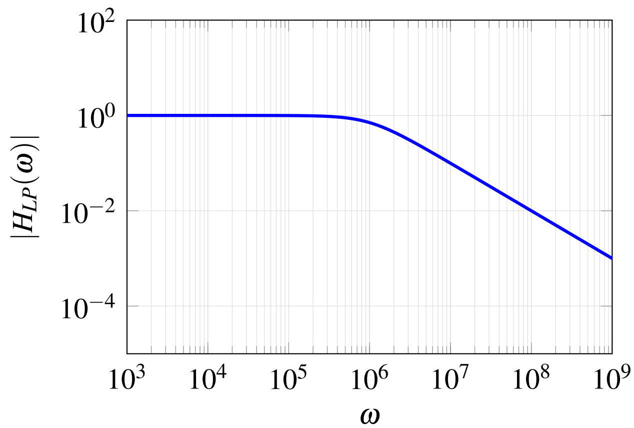
Mathematically the **cutoff frequency**, ω_c, is defined as the point at which

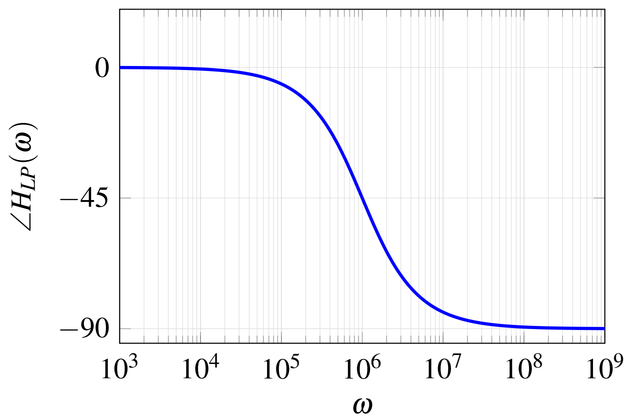
$$H(\omega_c) = \frac{\max_\omega \lvert H(\omega)\rvert}{\sqrt{2}}$$

[1] Note how this is the reciprocal of the time constant. We'll explore its significance in a later section.

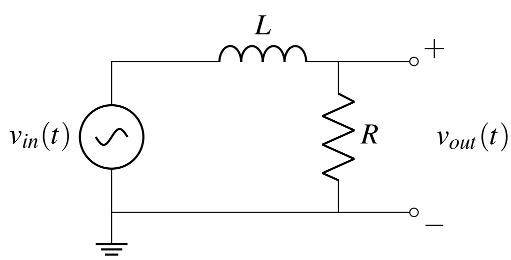
Where $\max_\omega H(\omega)$ is the maximum magnitude of $H(\omega)$ over all frequency. For a **passive circuit**, one without an external power supply, this maximum will usually be 1.

To give a visual understanding of our frequency response, we plot the magnitude and phase of the low-pass filter below. We plot on a log-log scale and we'll explore why this is the case in the next note.





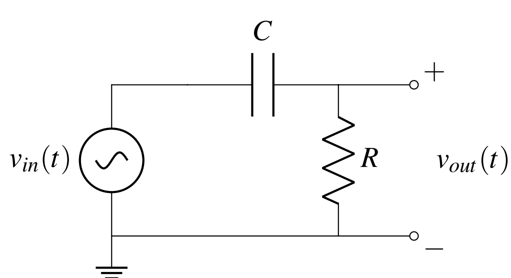
The above circuit was shown to be a low-pass filter, but there are a potpourri of other low-pass filter topologies and tradeoffs between each design. We won't look into too many of these different examples, but another example that we can analyze is an *LR* low-pass filter.



Try to compute its transfer function and find its cutoff frequency.[2]

2.2 High-Pass Filters

Now let's say we were building a sound system but the bass was too strong. We would like to filter out lower frequencies will keeping the remaining higher frequences the same. To do this, we should start thinking about how we can build a *high-pass* filter.



[2] $H(\omega) = \frac{1}{1+j\omega L/R}, \omega_c = \frac{R}{L}$

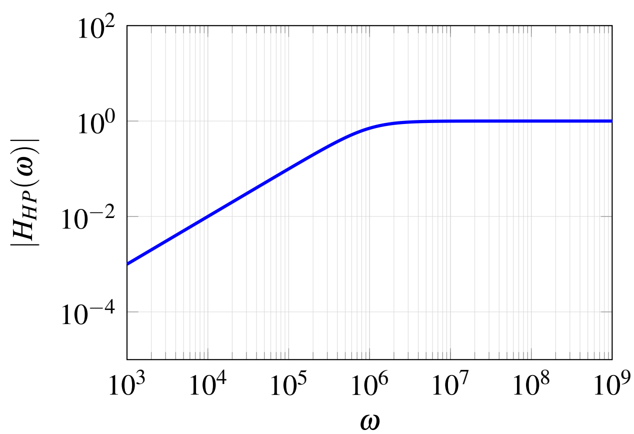
All of our principles that we have developed in the previous section apply here as well, so let's verify that the following *CR* circuit is a high-pass filter.

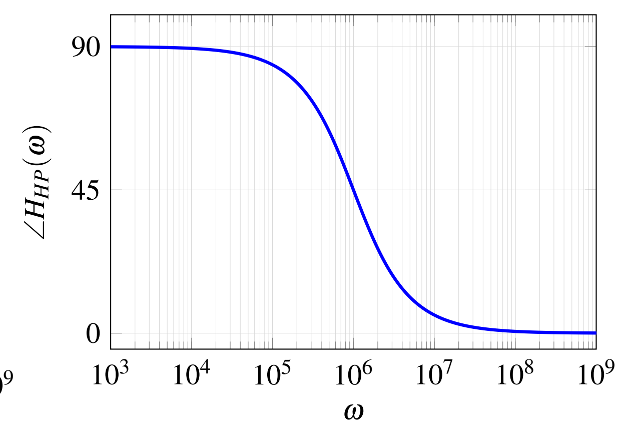
$$H(\omega) = \frac{j\omega RC}{1 + j\omega RC} \implies \omega_c = \frac{1}{RC}$$

Note that the cutoff frequency is identical to a low-pass filter with the same RC value. Let's look at some values of $H(\omega)$ around $\omega_c = \frac{1}{RC}$.

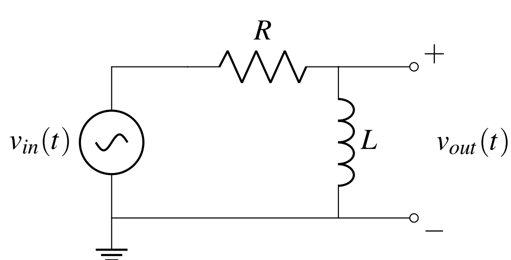
ω	$H(\omega)$	$\lvert H(\omega) \rvert$	$\angle H(\omega)$
$0.1\omega_c$	$\frac{0.1j}{1+0.1j}$	0.1	84°
ω_c	$\frac{j}{1+j}$	0.71	45°
$10\omega_c$	$\frac{10j}{1+10j}$	0.995	6°

Let's plot the magnitude and phase of this transfer function to see how it behaves over all frequencies ω.





Another example of a high-pass filter that we can look at is an *RL* filter



Try to compute its transfer function and find its cutoff frequency. [3]

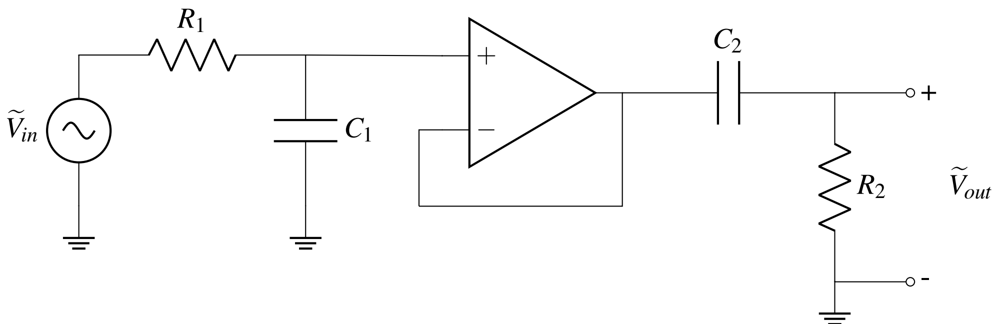
[3] $H(\omega) = \frac{j\omega L/R}{1+j\omega L/R}, \omega_c = \frac{R}{L}$

3 Second Order Filters

Using the intuition that we've gained from analyzing first order filters and their Bode plots, we will move onto more complicated examples.

3.1 Band-Pass Filters

With the knowledge of low-pass filters that block out higher frequencies and high-pass filters that block out lower frequencies, how could we build a filter that lets a range of frequncies through? One idea could be to take the output of the low-pass filter and treat it as an input to the high-pass filter.



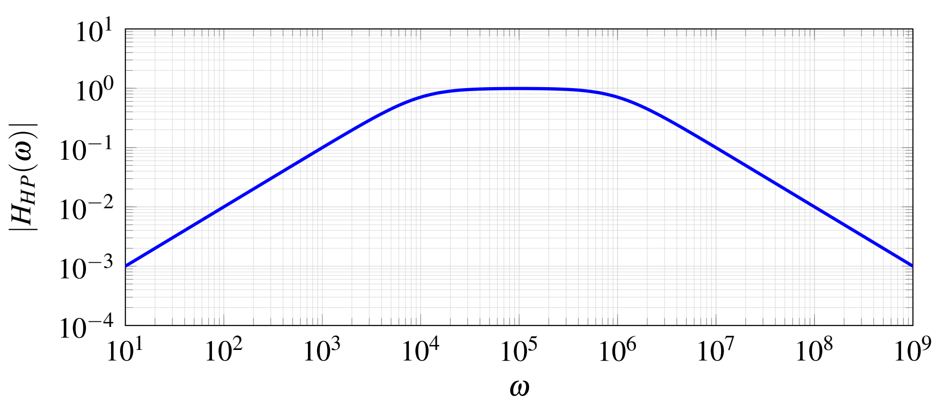
If we were to pick cutoff frequencies such that our desired range is smaller than our low-pass cutoff while being larger than our high-pass cutoff, then we could compute the transfer function of the following circuit and analyze its cutoff frequency.

The transfer function of this circuit is the product of the low-pass and high-pass transfer functions

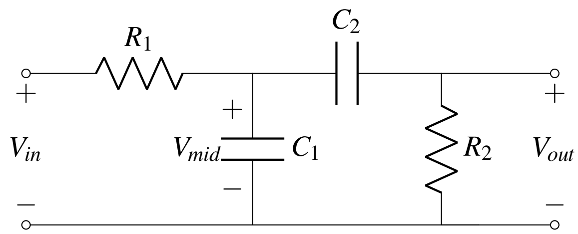
$$H_{BP}(\omega) = H_{LP}(\omega) \cdot H_{HP}(\omega) = \frac{1}{1 + j\omega R_L C_L} \cdot \frac{j\omega R_H C_H}{1 + j\omega R_H C_H}$$

To find the cutoff frequencies of this filter, we can look at the points at which $H(\omega_c) = \frac{1}{\sqrt{2}}$. However, recall that $\omega_{LP} = \frac{1}{R_L C_L}$ and $\omega_{HP} = \frac{1}{R_H C_H}$ and assuming that the low-pass and high-pass frequencies are spaced apart, we can approximate $\left|H(\omega_L)\right| \approx \frac{1}{\sqrt{2}} \cdot 1$ and $\left|H(\omega_H)\right| \approx 1 \cdot \frac{1}{\sqrt{2}}$.

Therefore, we conclude by saying that the cutoffs for the band-pass filter are identical to the individual cutoffs for the low and high-pass filter. We show a plot of $H(\omega)$ with $\omega_{LP} = 10^{-6}$ and $\omega_{HP} = 10^{-4}$ to give a visual explanation of this idea.



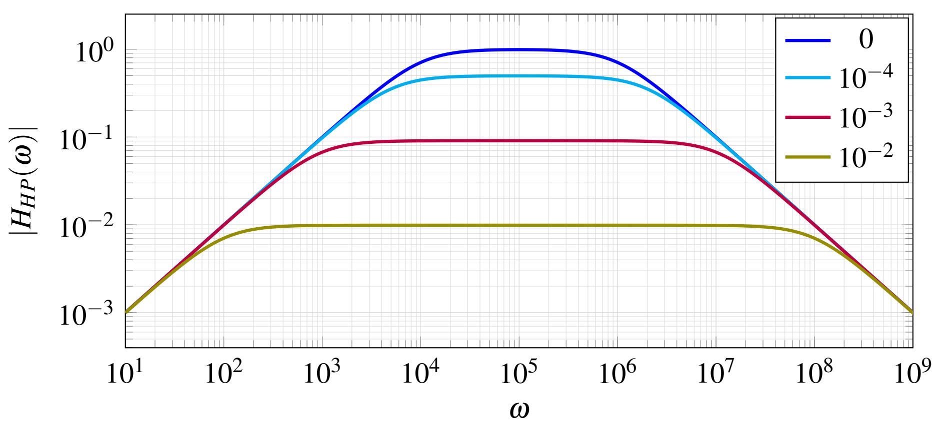
Now the band-pass filter that we built above requires the use of an op-amp. However, what would happen if we instead cascaded the two filters causing a loading effect?



We leave the derivation as an exercise, but computing the transfer function yields

$$H(\omega) = \frac{j\omega R_H C_H}{(1 + j\omega R_L C_L)(1 + j\omega R_H C_H) + j\omega R_L C_H}$$

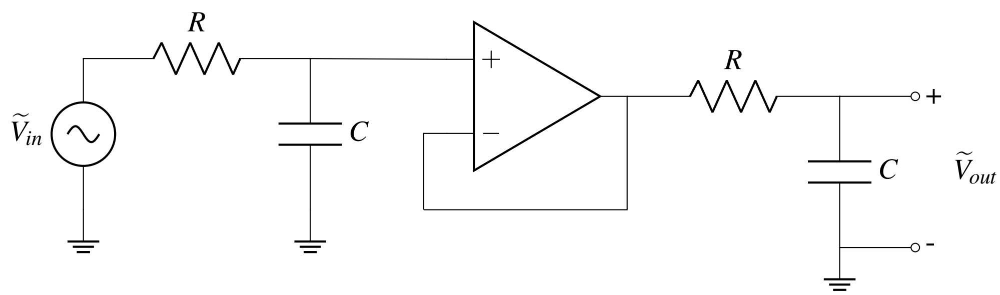
The difference due to loading is a denominator term of $j\omega R_L C_H$. Depending on how large $R_L C_H$ this could have a small or large effect on the circuit. We plot some examples of the band-pass filter with identical low and high cutoff frequencies but different $R_L C_H$ values to show this loading effect.



Note how the maximum value of $H(\omega)$ decreases as $R_L C_H$ increases. In addition, the cutoff frequencies move further and further apart from the original $\omega_{LP} = \frac{1}{R_L C_L}$ and $\omega_{HP} = \frac{1}{R_H C_H}$.

3.2 Low-Pass Filters

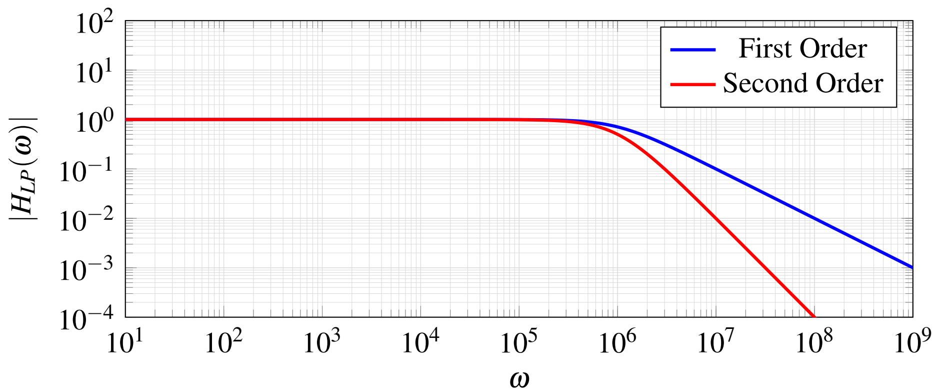
From our analysis of low-pass filters, we saw that the magnitude of the transfer function $H(\omega)$ dropped off by a factor of 10 for each decade of frequency after the cutoff ω_c. While this is a desirable effect, in the ideal case, we would like to build a filter that drops off at a quicker rate after ω_c. Therefore, let's try cascading two low-pass filters of identical cutoff with a buffer in between.



We can compute the transfer function as

$$H_{LP}(\omega) = \frac{1}{(1 + j\omega RC)^2} \tag{1}$$

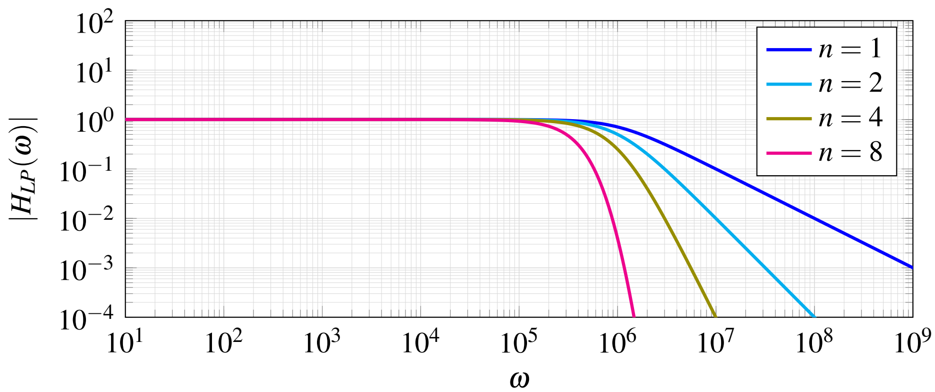
Plotting the magnitude of $H_{LP}(\omega)$, we see that $H(\omega)$ does indeed drop off at a quicker rate with slope 2 after the cutoff ω_c.



In fact, if we were to cascade even more low-pass filters, we approach an ideal low-pass filter in which

$$H(\omega) = \begin{cases} 1 & \omega < \omega_c \\ 0 & \omega \geq \omega_c \end{cases} \tag{2}$$

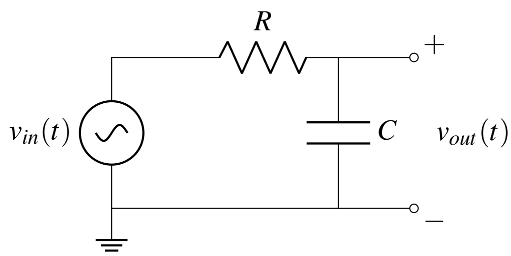
We show a plot of this effect below. For an n^{th} order filter, we see a dropoff of slope n after the cutoff. We will explore this effect in more detail in the next section.



4 Time Constant

When computing the cutoff frequency for a first order low-pass filter, we noticed that the $\omega_c = \frac{1}{RC} = \frac{1}{\tau}$. In this final section of the note, we draw the connection between time constants and cutoff frequencies.

Recall from the note on differential equations that we defined the time constant of a first-order circuit to be the point at which the response $v_c(t)$ to a constant input was $1 - e^{-1}$ away from its steady state value. With this in mind, let's try plugging in an exponential input $v_{in}(t) = V_0 e^{j\omega t}$ into an RC circuit and see what happens. [4]



The differential equation for this circuit is

$$\frac{d}{dt} v_{out}(t) = \lambda (v_{out}(t) - e^{j\omega t}) \tag{3}$$

for $\lambda = -\frac{1}{\tau}$. In Note 3 we showed that the steady state value of this differential equation is

$$v_{ss}(t) = \frac{-\lambda}{j\omega - \lambda} V_0 e^{j\omega t} \tag{4}$$

Therefore, plugging in for $\lambda = -\frac{1}{\tau}$, it follows that

$$v_{ss}(t) = \frac{1}{1 + j\omega\tau} V_0 e^{j\omega t} \tag{5}$$

Notice that $H(\omega) = \frac{1}{1+j\omega\tau}$ and the cutoff arises naturally as $\omega_c = \frac{1}{\tau}$. We can also realize that at steady state, $H(\omega)$ is in fact the eigenvalue for the differential equation with eigenfunction $e^{j\omega t}$. This is a crucial connection between differential equations and the frequency response of a linear system that you will see in later half of the course and in courses like EE120.

5 Conclusion

In this note, we were able to apply the techniques of phasor analysis to build filters that followed a specific set of constraints. Using a single resistor and a capacitor, we were able to build first-order low and high pass filters. By combining the two with a buffer, we then developed the band-pass filter in order to let a range of frequencies through.

To get a better understanding of our filters, we began to plot the magnitude and phase of the transfer function $H(\omega)$. In the next note, we'll develop a better understanding of plotting and how to approximate the magnitude and phase plots by taking advantage of the log-log scale.

[4] We should be inputting $v_{in}(t) = V_0 \cos(\omega t)$ but we choose $e^{j\omega t}$ since it provides the same result while simplifying the math.

Contributors:

- Rahul Arya.
- Anant Sahai.
- Jaijeet Roychowdhury.
- Taejin Hwang.

