EECS 16B Designing Information Devices and Systems II
Fall 2020 UC Berkeley

Note 5

1 Introduction

This note follows up on Note 4 in which we examined **vector** differential equations and solved them using a systematic approach by changing coordinates into a basis in which the matrix A had a diagonal representation Λ. This let us examine our complicated system of differential equations as a series of n first-order differential equations.

After solving the first-order differential equations and converting the system back into standard basis coordinates, we saw that each state $x_i(t)$ was a linear combination of exponentials $e^{\lambda_1 t}, \dots, e^{\lambda_n t}$. As a result, we saw the connection between a system of differential equations and the eigenvalues of the matrix A.

In this note, we will develop more techniques to solving vector differential equations and also introduce a new device called the **inductor.** The combination of an inductor with a capacitor will create an oscillatory system with complex eigenvalues. Such oscillatory systems are the main focus of this note.

2 Guessing and Checking

We introduce the guess and check method for vector differential equations as a means of efficiency. You'll notice that we don't have to compute the eigenvectors of the matrix A and won't need diagonalization.[1]

In the previous note, we were able to solve for a system of differential equations and showed that the solution is a linear combination of exponentials $e^{\lambda_i t}$. This means that we should be able to guess a solution $x_i(t) = \alpha_1 e^{\lambda_1 t} + \dots + \alpha_n e^{\lambda_n t}$. To illustrate this, we provide an example below

2.1 Example

Consider the following system of differential equations with the inital condition $\vec{x}(0)$.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \vec{x}(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \tag{1}$$

1 The first step is to solve for the eigenvalues of the matrix A. For the sake of bervity, we won't show the steps here, but $\lambda_1 = -5, \lambda_2 = -2$.

2 Now we guess a solution $\vec{x}(t)$. As stated above, we'll pick a linear combination of $e^{\lambda_i t}$ as our guess.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} \\ \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t} \end{bmatrix}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are unknown constants that we need to solve for.

[1] The diagonalization method was used to rigorously show why the solution is in fact a linear combination of $e^{\lambda_i t}$. Without it, we wouldn't know what to guess.

3 Notice that we have four unknowns but our intial condition only gives us two equations. Therefore, we create two more equations by computing $\frac{d}{dt}\vec{x}(0)$.

Notice that our initial condition tells us that

$$\vec{x}(0) = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \beta_1 + \beta_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

To find $\frac{d}{dt}\vec{x}(0)$, we must first take the derivative of $\vec{x}(t)$.

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -5\alpha_1 e^{-5t} - 2\alpha_2 e^{-2t} \\ -5\beta_1 e^{-5t} - 2\beta_2 e^{-2t} \end{bmatrix} \implies \frac{d}{dt}\vec{x}(0) = \begin{bmatrix} -5\alpha_1 - 2\alpha_2 \\ -5\beta_1 - 2\beta_2 \end{bmatrix}$$

Then we use the fact that $\frac{d}{dt}x(t) = A\vec{x}$ from our differential equation:

$$\frac{d}{dt}\vec{x}(0) = A\vec{x}(0) = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -4x_1(0) + x_2(0) \\ 2x_1(0) - 3x_2(0) \end{bmatrix} = \begin{bmatrix} -9 \\ -3 \end{bmatrix}$$

4 Now that we have four equations and four unknowns, we can solve our system.

Solving for the α, we get

$$\begin{aligned} \alpha_1 + \alpha_2 &= 3 \\ -5\alpha_1 - 2\alpha_2 &= -9 \\ \implies \alpha_1 = 1, \alpha_2 &= 2 \end{aligned}$$

Then we solve for the β

$$\begin{aligned} \beta_1 + \beta_2 &= 3 \\ -5\beta_1 - 2\beta_2 &= -3 \\ \implies \beta_1 = -1, \beta_2 &= 4 \end{aligned}$$

5 We conclude by saying that the solution to the differential equation is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{-5t} + 2e^{-2t} \\ -e^{-2t} + 4e^{-2t} \end{bmatrix}$$

2.2 Second-Order Differential Equations

In differential equation literature, you will more often see **higher-order** differential equations as opposed to vector differential equations. A higher-order differential equation is a scalar differential equation that involves higher-order derivatives.

Consider the differential equation

$$\frac{d^2}{dt^2}y(t) + a\frac{d}{dt}y(t) + by(t) = 0 \tag{2}$$

$$y(0) = y_0; \; \frac{d}{dt}y(0) = w_0 \tag{3}$$

This is an example of a second order differential equation. Notice how there are two initial conditions for this problem. An n^{th} order differential equation will require n initial conditions for it to have a unique solution.

2.2.1 Guess and Check

To solve this differential equation, we can either guess and check or convert it into a system of differential equations. We will start by guess the solution $y(t) = ke^{\lambda t}$.

$$y(t) = ke^{\lambda t};\ \frac{d}{dt}y(t) = k\lambda e^{\lambda t};\ \frac{d^2}{dt^2}y(t) = k\lambda^2 e^{\lambda t} \tag{4}$$

$$k\lambda^2 e^{\lambda t} + ka\lambda e^{\lambda t} + kbe^{\lambda t} = 0 \implies k(\lambda^2 + a\lambda + b) = 0 \tag{5}$$

If our initial condition is nonzero, k will be nonzero meaning we have a quadratic equation for λ similar to the characteristic polynomial of our matrix A. Since this quadratic equation has two roots λ_1 and λ_2, our solution $y(t)$ will be a linear combination of the functions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ or of the form

$$y(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} \tag{6}$$

Plugging in the initial conditions $y(0)$ and $\frac{dy}{dt}(0)$, we should be able to solve for the coefficients α_1 and α_2.

2.2.2 Converting to a Vector Differential Equation

Similar to how we converted a system of differential equations into a vector differential equation, we can also turn our second order differential equation into a **first order** vector differential equation. We will do so by defining state variables

$$x_1(t) = y(t),\ x_2(t) = \frac{d}{dt}y(t) \tag{7}$$

Taking the derivative of our states, we see that

$$\frac{d}{dt}x_1(t) = \frac{d}{dt}y(t) = x_2(t) \tag{8}$$

$$\frac{d}{dt}x_2(t) = \frac{d^2}{dt^2}y(t) = -by(t) - a\frac{d}{dt}y(t) = -bx_1(t) - ax_2(t) \tag{9}$$

Therefore, we can write this as a vector differential equation

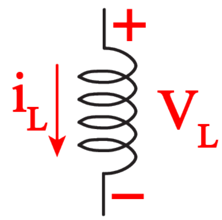
$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{10}$$

Note that the eigenvalues of the A matrix yields the exact characteristic polynomial that we found using guess and check. This is not coincidental and in fact arises since we were looking for eigenvalues of the differentiation operator $\frac{d}{dt}$.

3 Inductors

Let's introduce a new passive component, an inductor. This new component will help us design more interesting circuits and introduce oscillations within our circuits.

Inductors have a dual relationship in terms of voltage and current (I-V) as compared to capacitors (i.e., V being proportional to change in I as opposed to I being proportional to change in V). The schematic symbol of an inductor is drawn below:



(a) The unit of inductance is Henry (H).

(b) $v_L(t) = L\frac{di_L(t)}{dt}$

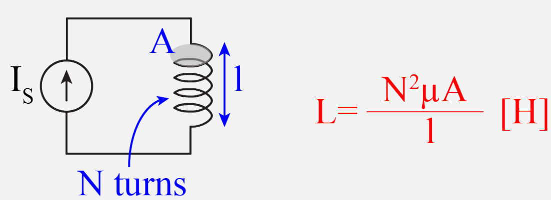
(c) $i_L(t)$ cannot change instantly.

(d) At DC steady state, inductors behave like short circuits since the current is constant meaning there is no voltage drop.

(e) The energy stored in an inductor is: $U = \frac{1}{2}L{i_L}^2(t)$

While inductors are introduced in this course only as a circuit symbol and a mathematical construct, we are not spending time on the physics behind them, they have multiple applications in the real world and you will study them in future courses.

Inductors store energy by setting up a magnetic field. In the same way that a capacitor separates charge (Q) and this leads to an $\vec{E}$ field, anytime we flow current down a conductor, this creates a magnetic field ($\vec{B}$). Likewise, the magnetic field can store energy. Their behavior can be described using Faraday's law of induction.
The magnitude of magnetic field created by a straight wire is pretty small, so we usually use other geometries if we are trying to create a useful inductance on purpose. A solenoid is a good example:



Note that the inductance (L) depends on geometry and a material property called permeability of the solenoid core material. Inductors are useful in many applications such as wireless communications, chargers, DC-DC converters, key card locks, transformers in the power grid, etc.

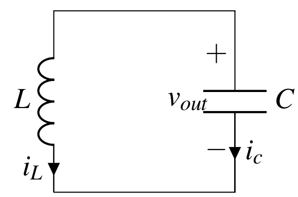
Concept Check: The current across the inductor cannot change instantaneously. Why?
Solution: If our current changes instantaneously, then $\frac{d}{dt}I_L \to \infty$, and from equation ((b)) the voltage across the inductor $V_L \to \infty$, which is not possible. Hence, our current cannot change instantaneously

4 LC Tank

Let's take a look at a circuit with an inductor and capacitor in parallel. This is commonly known as an LC tank, whose matrix will have purely imaginary eigenvalues.

In the following circuit, we have an inductor $L = 10\,\text{nH}$ and capacitor $C = 10\,\text{pF}$ in parallel.

Let $I_L(0) = 50\,\text{mA}$ and $V_c(0) = 0\,\text{V}$:



First we define states $x_1 = I_L$ and $x_2 = V_c$. Then since the inductor and capacitor are in parallel:

$$V_L = V_c \tag{11}$$

KCL gives:

$$I_L = -I_c = -C\frac{dV_c}{dt} \implies \frac{dV_c}{dt} = -\frac{1}{C}i_L \tag{12}$$

$$V_L = V_c = L\frac{dI_L}{dt} \implies \frac{dI_L}{dt} = \frac{1}{L}v_{out} \tag{13}$$

Putting it into matrix form, as before:

$$\frac{d}{dt}\begin{bmatrix} V_c \\ I_L \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} V_c \\ I_L \end{bmatrix} \tag{14}$$

Finding the eigenvalues:

$$\det\left(\begin{bmatrix} -\lambda & -\frac{1}{C} \\ \frac{1}{L} & -\lambda \end{bmatrix}\right) = \lambda^2 + \frac{1}{LC} = 0 \tag{15}$$

$$\therefore \lambda_{1,2} = 0 \pm j\frac{1}{\sqrt{LC}} \tag{16}$$

Now we will solve the differential equation using guess and check and start by guessing the following solution:

$$\vec{x} = \begin{bmatrix} I_L \\ V_c \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} \\ \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t} \end{bmatrix}$$

Then we take the derivative and evaluate at $t = 0$ to find $\frac{d}{dt}\vec{x}(0)$

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} \lambda_1\alpha_1 e^{\lambda_1 t} + \lambda_2\alpha_2 e^{\lambda_2 t} \\ \lambda_1\beta_1 e^{\lambda_1 t} + \lambda_2\beta_2 e^{\lambda_2 t} \end{bmatrix} \quad \frac{d}{dt}\vec{x}(0) = \begin{bmatrix} \lambda_1\alpha_1 + \lambda_2\alpha_2 \\ \lambda_1\beta_1 + \lambda_2\beta_2 \end{bmatrix}$$

and use the differential equation $\frac{d}{dt}\vec{x} = A\vec{x}$ to get the following

$$\frac{d}{dt}\vec{x}(0) = A\vec{x}(0) = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} 5 \cdot 10^{-3} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{50 \cdot 10^{-3}}{L} \end{bmatrix}$$

Lastly, we plug in the values for λ and solve the system of equations

$$\alpha_1 + \alpha_2 = 50 \cdot 10^{-3} \qquad \beta_1 + \beta_2 = 0$$

$$j\frac{1}{\sqrt{LC}}\alpha_1 - j\frac{1}{\sqrt{LC}}\alpha_2 = 0 \qquad j\frac{1}{\sqrt{LC}}\beta_1 - j\frac{1}{\sqrt{LC}}\beta_2 = \frac{50 \cdot 10^{-3}}{L}$$

Solving the system of equations, we get the following constants

$$\alpha_1 = 25 \cdot 10^{-2} \quad \alpha_2 = 25 \cdot 10^{-3} \quad \beta_1 = \frac{0.5\sqrt{10}}{2j} \quad \beta_2 = -\frac{0.5\sqrt{10}}{2j}$$

Therefore, the solution to the differential equation is

$$\begin{bmatrix} I_L(t) \\ V_c(t) \end{bmatrix} = \begin{bmatrix} 25 \cdot 10^{-2} e^{j\sqrt{10} \cdot 10^9 t} + 25 \cdot 10^{-2} e^{-j\sqrt{10} \cdot 10^9 t} \\ \frac{0.5\sqrt{10}}{2j} e^{j\sqrt{10} \cdot 10^9 t} - \frac{0.5\sqrt{10}}{2j} e^{-j\sqrt{10} \cdot 10^9 t} \end{bmatrix}$$

We can simplify this using Euler's Formula to get our final answer.[2]

$$\begin{bmatrix} I_L(t) \\ V_c(t) \end{bmatrix} = \begin{bmatrix} 50 \cdot 10^{-2} \cos\left(\sqrt{10} \cdot 10^9 t\right) \\ 0.5\sqrt{10} \sin\left(\sqrt{10} \cdot 10^9 t\right) \end{bmatrix}$$

Figure 1 plots the above solutions for the capacitor voltage and inductor current. This system is also called an oscillator because the circuit produces a repetitive voltage waveform under the right initial conditions.

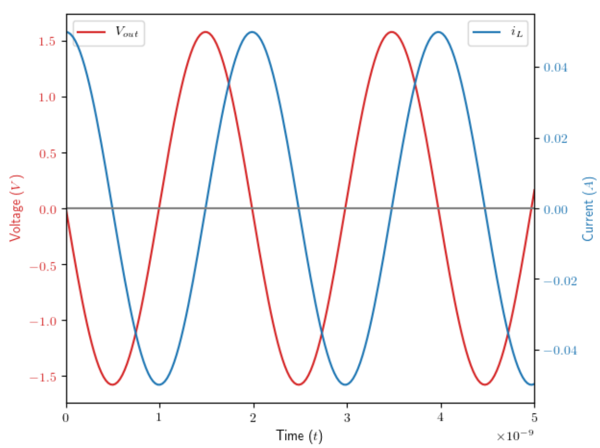


Figure 1: Voltage and Current response of LC Tank

[2] Remember that $\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$ and $\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$.

From the above plots, we can see that when the capacitor is fully charged, the inductor has zero flux whereas when the inductor has full flux, the capacitor is fully discharged. What does this imply about the energy stored in the two components?

We know that, energy in the capacitor, $E_c = \frac{1}{2}CV^2 = 1.25 \times 10^{-11} \sin^2\left(\sqrt{10} \times 10^9 t\right)$ and energy in the inductor, $E_L = \frac{1}{2}LI^2 = 1.25 \times 10^{-11} \cos^2\left(\sqrt{10} \times 10^9 t\right)$. This shows that $E_{total} = E_c + E_L = 1.25 \times 10^{-11}$ is constant across all time.

Figure 2 plots these energies. As it is clear, the total energy seems to be sloshing back and forth between the inductor and capacitor.

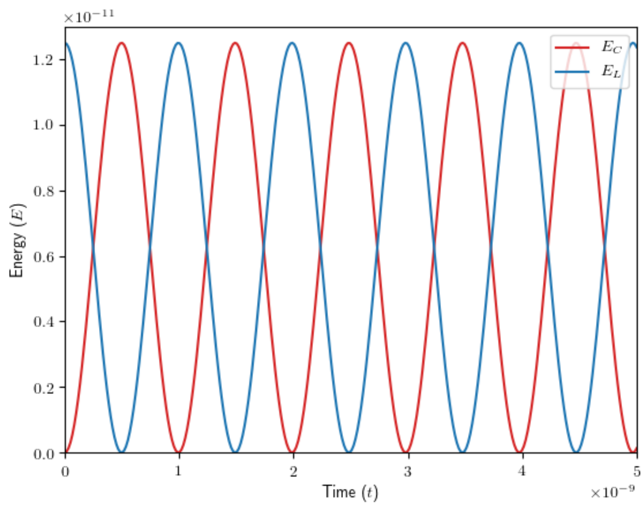
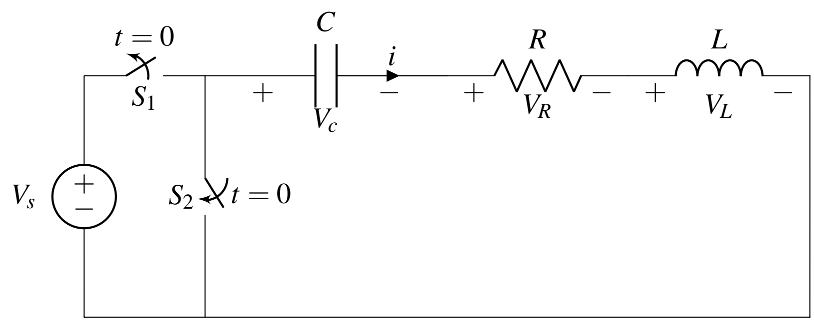


Figure 2: Energy stored in Inductor and Capacitor. Notice the sum is constant.

5 RLC Circuits and Higher Order Differential Equations

The LC tank we studied in the previous section was a very ideal case where we assumed there was no resistor in the system. But this is rarely the case, and we will need to understand how adding this third component will modify our differential equations.

Before $t = 0$, switch S_1 is on while S_2 is off. At $t = 0$, both switches flip state (S_1 turns off and S_2 turns on):



First, let's figure out the initial conditions. Since the system had been connected to the battery for a long time, the capacitor would be at steady state meaning $v_c(0) = V_s$ and $i(0) = 0A$. From this, we can also deduce that $\frac{d}{dt}v_c = 0$. Next, let's write our branch equations:

$$i = C\frac{d}{dt}V_c,\ V_L = L\frac{d}{dt}i,\ V_R = i \cdot R \tag{17}$$

$$V_c + V_L + V_R = 0 \tag{18}$$

Using the above equations, and substituting for i from Equation (17) when needed, we can describe our system with the following differential equation:

$$\frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{V_c}{LC} = 0 \tag{19}$$

Here we have chosen the second order differential equation as means of an example. As usual, we can solve this differential equation by computing its eigenvalues and use any approach from before.[3]

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0 \tag{20}$$

$$\lambda = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \tag{21}$$

Now depending on the values of R, L, C the eigenvalues change accordingly

- When $R < 2\sqrt{\frac{L}{C}}$, we will have two distinct **purely real** eigenvalues.
- When $R > 2\sqrt{\frac{L}{C}}$, we will have two eigenvalues that are **complex conjugates**
- When $R = 2\sqrt{\frac{L}{C}}$, we will have a single eigenvalue that is **purely real.**

[3] Guess and check, diagonalization, changing coordinates, we are no longer emphasizing the solving process rather we would like to extrapolate information from the results.

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