EECS 16B Summer 2020 | Designing Information Devices and Systems II UC Berkeley | Note 1

1 Introduction

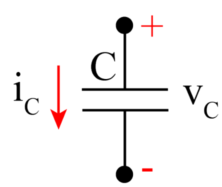
So far we learned that the key enabler of all the digital electronics around us such as smart phones, laptops, etc. is the Transistors! Now that we know how digital circuits and CMOS gates operate, we can ask many questions like:

- How fast a computer operates? (How fast a CMOS gate switch?)
- Why smart phones ran out of the battery quickly?! (How much energy do CMOS gates consume when switching (or even when not)?
- What limits the complexity of processing can be done with this circuits? How much space do they take up?

We will tackle the first two question regarding switching speed and energy in this lecture, but first we need a quick review on capacitors.

2 Capacitors

Any time we have two conductive materials separated by a dielectric (i.e. an insulator), we have the potential to store electrical charge across the two conductors. This is called a **capacitance** (the device is a **capacitor**) and this way we can store energy. Examples are touch-screen pixels from EE16A labs, two metal wires close to each other, etc.



A capacitor's symbol and its voltage/current is shown above. The following is always true about the capacitors:

(a) The stored electrical charge can be derived as : $q_C(t) = Cv_C(t) \Rightarrow i_C(t) = C\frac{dv_C(t)}{dt}$ [Amps] (This hold since by definition $i = \frac{dq}{dt}$)

(b) $v_C(t)$ can never change instantly (i.e. it can never be discontinuous). To see why, examine 1; if $v_C(t)$ is discontinuous, then $\frac{dv_C(t)}{dt} \to \infty$ at then discontinuity and $i_c \to \infty$ which is not feasible (why?).

(c) The energy stored in a capacitor at any instant in time is: $U_C = \frac{1}{2}Cv_C{}^2(t)$ [Joules]

3 Capacitors & Transistors

Until now transistors have been modeled as switches that change state based on the voltage applied at their gates. However, it turns out that whenever we make a transistor, there are always capacitances associated with the nodes. This is unavoidable and unwanted and arises from solid state physics (explained in the last lecture transistor's physics section).

We model transistors as having some resistance and some characteristic capacitance from their gates.

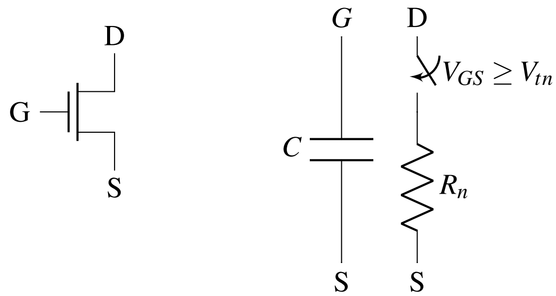


Figure 1: NMOS Transistor Model

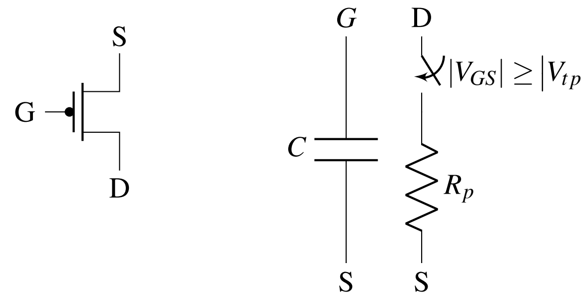
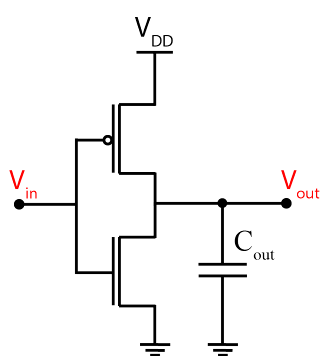


Figure 2: PMOS Transistor Model

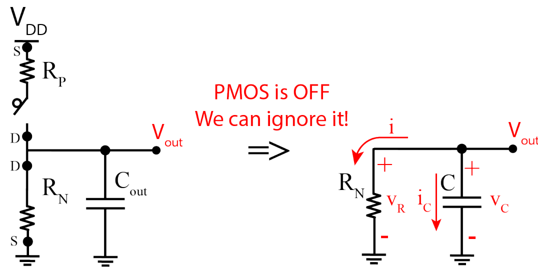
4 RC Circuits

Let's look at the inverter illustrated below. We add an output capacitance which may represent the capacitance of the wire or the gate capacitance of the next inverter, if we were to chain these inverters. We'll also assume that $V_{in} = 0\,\text{V}$ and thus $V_{out} = V_{DD}$ and the transistor has been in this state for a long time (since $t = -\infty$).



Now, at $t = 0$, we instantly switch V_{in} to V_{DD}. We know, eventually $V_{out} = 0V$. *if there were no capacitance, the output would change instantly because the equations have no concept of time!* but considering the capacitance (C_{out}), things slow down, let's see why?

For $t > 0$ the gate capacitances of the NMOS and PMOS would be instantaneously charged and the resulting circuit will be:



Notice $v_{out}(t) = v_c(t)$ and $i = -i_c = -C\frac{dv_c}{dt}$. Using KVL we know $v_c = v_R$. By substituting capacitor's current equation and ohm's law ($v_R = iR$), we have:

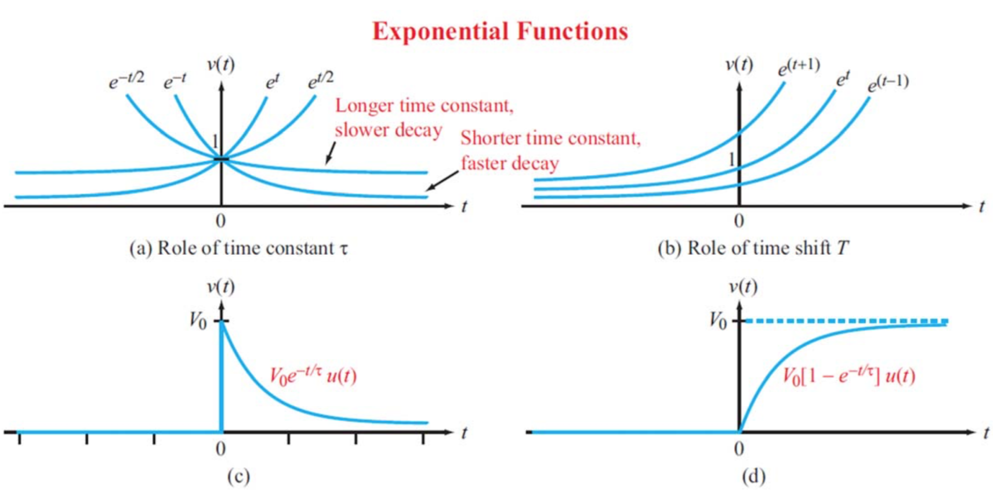
$$v_c = iR = -RC\frac{dv_c}{dt} \Rightarrow v_c + RC\frac{dv_c}{dt} = 0 \Rightarrow v_{out} + RC\frac{dv_{out}}{dt} = 0 \tag{1}$$

This is a **first order differential equation (D.E.)**. The general form is: $\frac{dx}{dt} + ax = 0$. Because in this form the right hand side (RHS) is zero, it's called a **homogenous 1st order D.E.**. The notation of derivatives are sometimes shown by dots as well ($\frac{dx}{dt} = \dot{x}$, $\frac{d^2x}{dt^2} = \ddot{x}$, and so on).

5 Solving 1st Order D.E.

As stated above these equations have a general form of: $\frac{dx}{dt} + ax = 0$. If we rewrite it in the following format: $\frac{d}{dt}(x(t)) = -ax(t)$. What will this equation remind you from linear algebra topics studied in 16A? **Eigenvalues!**. Remember the eigenvectors of a matrix A were solutions of $Av = \lambda v$ equation. In the differential equations, $\frac{d}{dt}$ can be seen as a derivative operator and thus here we are looking for eigenfunctions of the derivative operator with the eigenvalues of $-a$.

It turns out that the eigenfunctions are in the general form of an **exponential function:** $x(t) = e^{\lambda t}$. Next figure illustrates various forms of these functions for different k values ($u(t)$ denotes a step function in this figure):



The unique property of the exponential functions is that:

$$\frac{dx(t)}{dt} = \lambda \cdot e^{\lambda t} = \lambda x(t)$$

Hence a function in the form of $x(t) = Ke^{\lambda t}$ can be an answer to our 1st order D.E.:

$$\frac{dx(t)}{dt} = \lambda x(t) \Rightarrow \frac{dx(t)}{dt} = K\lambda e^{\lambda t} = \lambda K e^{\lambda t} = \lambda x(t)$$

Since this equation should hold for every value of t, the solution to $\frac{dx}{dt} + ax = 0$ is in the form Ke^{-at}, where K can be any constant. This seems to imply that we have infinitely many solutions. Luckily, a differential equation will always come with **initial conditions** which will lead to a particular solution.

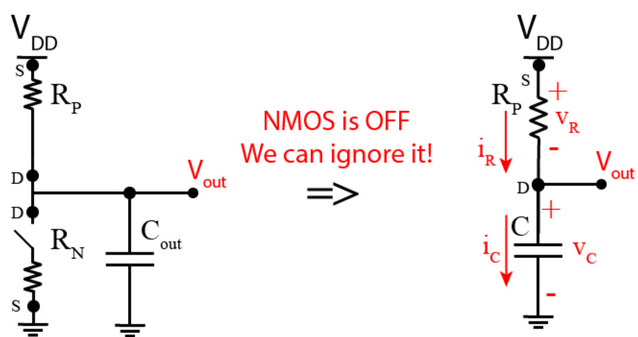
Going back to the RC equation for v_{out}, the solution for $t > 0$ will be:

$$v_{out}(t) = Ke^{-t/RC}$$

The initial condition here is $v_{out}(0) = V_{DD}$. Plugging in for $t = 0$, we see that $v_{out}(0) = K = V_{DD}$. Therefore, we conclude by saying the solution is $v_{out} = V_{DD}e^{-t/RC}$.

6 Non-Homoegenous D.E.

Let's go back and analyze the case when the inverter has been at the state $V_{in} = V_{DD}$ and $V_{out} = 0\,\text{V}$ for a long time and at $t = 0$, we switch $V_{in} = 0\,\text{V}$.



If we were to write a differential equation for this circuit, we'd see that

$$i_R = i_c = C\frac{dv_c}{dt} \Rightarrow \frac{V_{DD} - v_c}{R} = C\frac{dv_c}{dt} \Rightarrow v_{out} + RC\frac{dv_{out}}{dt} = V_{DD} \tag{2}$$

Notice how this differential equation is in the form $\frac{dx}{dt} + ax = b$ where b is nonzero.

One way to solve this differential equation is to guess a solution $x(t) = K_1e^{-at} + K_2$. To verify our guess, we plug it into the differential equation

$$\frac{dx}{dt} + ax = -aK_1e^{-at} + a(K_1e^{-at} + K_2) = aK_2 = b \tag{3}$$

Therefore, the guess will be valid whenever $K_2 = b/a$. It remains to find K_1 and we shall do this by plugging in the initial condition.

$$x(0) = K_1 + K_2 \implies K_1 = x(0) - \frac{b}{a} \tag{4}$$

We conclude by saying that the solution of the differential equation $\frac{dx}{dt} + ax = b$ is

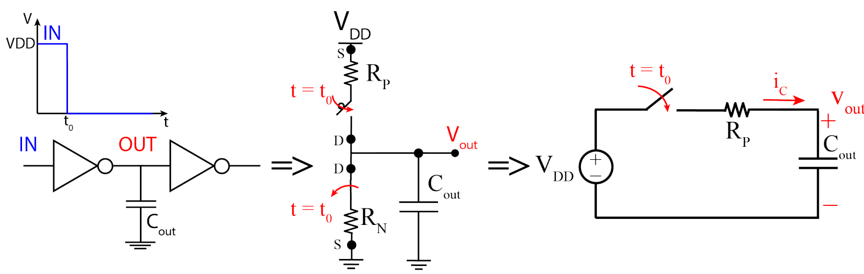
$$x(t) = \left[x(0) - \frac{b}{a}\right] e^{-at} + \frac{b}{a} \tag{5}$$

Going back to the RC equation for v_{out}, with initial condition $v_{out}(0) = 0$, the solution for $t > 0$ will be:

$$v_{out}(t) = -V_{DD}e^{-at} + V_{DD} = V_{DD}(1 - e^{-t/RC})$$

7 Energy to Switch an Inverter

Going back to the second question on the energy required for a logic gate to operate, let's see how much energy we need to switch an inverter gate (the simplest CMOS gate)?



The energy required to charge the capacitor to V_{DD} is provided by the voltage source (V_{DD}). There are two approaches to calculate it:

(a) We know the total charge to needed to charge up the capacitance to V_{DD} is $Q_C = C_{out} \cdot V_{DD}$. By definition the energy required to move the Q charge between two point with V_{DD} voltage across is $Q \cdot V_{DD}$. Thus total dissipated energy by the supply to charge the capacitor is $W_{DD} = C_{out}V_{DD}{}^2$.

(b) We know the power given by any element to the circuit is $i(t)v(t)$. Also, the energy can be calculated by integrating power over time, hence:

$$W_{DD} = \int i_{DD}(t)v_{DD}(t)\,dt = \int i_C(t)V_{DD}\,dt = V_{DD}\int \frac{dq_C(t)}{dt}\,dt = V_{DD}(q_C(\infty) - q_C(0))$$
$$\Rightarrow W_{DD} = V_{DD}(C_{out}V_{DD} - 0) = C_{out}V_{DD}{}^2$$

Question: Recall that the energy stored in a capacitor is $U_c = \frac{1}{2}CV_{DD}^2$. However, the voltage source is supplying a total of $W_{DD} = C_{out}V_{DD}{}^2$. Where does the rest of energy dissipated by the battery go?

Answer: The voltage source provides $W_{DD} = C_{out}V_{DD}{}^2$. Half of the energy is stored in the capacitor and the remaining half is dissipated across the resistor as heat.

Contributors:

- Nikhil Shinde.
- Anant Sahai.
- Aditya Arun.
- Taejin Hwang.