EECS 16B Designing Information Devices and Systems II
Fall 2020 UC Berkeley

Note 5

1 Introduction

This note follows up on Note 4 in which we examined **vector** differential equations and solved them using a systematic approach by changing coordinates into a basis in which the matrix A had a diagonal representation Λ. This let us examine our complicated system of differential equations as a series of n first-order differential equations.

After solving the first-order differential equations and converting the system back into standard basis coordinates, we saw that each state $x_i(t)$ was a linear combination of exponentials $e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$. As a result, we saw the connection between a system of differential equations and the eigenvalues of the matrix A.

In this note, we will develop more techniques to solving vector differential equations and also introduce a new device called the **inductor.** The combination of an inductor with a capacitor will create an oscillatory system with complex eigenvalues. Such oscillatory systems are the main focus of this note.

2 Guessing and Checking

We introduce the guess and check method for vector differential equations as a means of efficiency. You'll notice that we don't have to compute the eigenvectors of the matrix A and won't need diagonalization.[1]

In the previous note, we were able to solve for a system of differential equations and showed that the solution is a linear combination of exponentials $e^{\lambda_i t}$. This means that we should be able to guess a solution $x_i(t) = \alpha_1 e^{\lambda_1 t} + \ldots + \alpha_n e^{\lambda_n t}$. To illustrate this, we provide an example below

2.1 Example

Consider the following system of differential equations with the inital condition $\vec{x}(0)$.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \vec{x}(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \tag{1}$$

1 The first step is to solve for the eigenvalues of the matrix A. For the sake of bervity, we won't show the steps here, but $\lambda_1 = -5, \lambda_2 = -2$.

2 Now we guess a solution $\vec{x}(t)$. As stated above, we'll pick a linear combination of $e^{\lambda_i t}$ as our guess.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} \\ \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t} \end{bmatrix}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are unknown constants that we need to solve for.

[1] The diagonalization method was used to rigorously show why the solution is in fact a linear combination of $e^{\lambda_i t}$. Without it, we wouldn't know what to guess.

3 Notice that we have four unknowns but our intial condition only gives us two equations. Therefore, we create two more equations by computing $\frac{d}{dt}\vec{x}(0)$.

Notice that our initial condition tells us that

$$\vec{x}(0) = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \beta_1 + \beta_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

To find $\frac{d}{dt}\vec{x}(0)$, we must first take the derivative of $\vec{x}(t)$.

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -5\alpha_1 e^{-5t} - 2\alpha_2 e^{-2t} \\ -5\beta_1 e^{-5t} - 2\beta_2 e^{-2t} \end{bmatrix} \implies \frac{d}{dt}\vec{x}(0) = \begin{bmatrix} -5\alpha_1 - 2\alpha_2 \\ -5\beta_1 - 2\beta_2 \end{bmatrix}$$

Then we use the fact that $\frac{d}{dt}x(t) = A\vec{x}(t)$ from our differential equation:

$$\frac{d}{dt}\vec{x}(0) = A\vec{x}(0) = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -4x_1(0) + x_2(0) \\ 2x_1(0) - 3x_2(0) \end{bmatrix} = \begin{bmatrix} -9 \\ -3 \end{bmatrix}$$

4 Now that we have four equations and four unknowns, we can solve our system.

Solving for the α, we get

$$\begin{aligned} \alpha_1 + \alpha_2 &= 3 \\ -5\alpha_1 - 2\alpha_2 &= -9 \\ \implies \alpha_1 &= 1, \alpha_2 = 2 \end{aligned}$$

Then we solve for the β

$$\begin{aligned} \beta_1 + \beta_2 &= 3 \\ -5\beta_1 - 2\beta_2 &= -3 \\ \implies \beta_1 &= -1, \beta_2 = 4 \end{aligned}$$

5 We conclude by saying that the solution to the differential equation is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-5t} + 2e^{-2t} \\ -e^{-2t} + 4e^{-2t} \end{bmatrix}$$

2.2 Second-Order Differential Equations

In differential equation literature, you will more often see **higher-order** differential equations as opposed to vector differential equations. A higher-order differential equation is a scalar differential equation that involves higher-order derivatives. Consider the differential equation

$$\frac{d^2}{dt^2}y(t) + a\frac{d}{dt}y(t) + by(t) = 0 \tag{2}$$

$$y(0) = y_0; \ \frac{d}{dt}y(0) = w_0 \tag{3}$$

This is an example of a second order differential equation. Notice how there are two initial conditions for this problem. An n^{th} order differential equation will require n initial conditions for it to have a unique solution.

2.2.1 Guess and Check

To solve this differential equation, we can either guess and check or convert it into a system of differential equations. We will start by guess the solution $y(t) = ke^{\lambda t}$.

$$y(t) = ke^{\lambda t};\ \frac{d}{dt}y(t) = k\lambda e^{\lambda t};\ \frac{d^2}{dt^2}y(t) = k\lambda^2 e^{\lambda t} \tag{4}$$

$$k\lambda^2 e^{\lambda t} + ka\lambda e^{\lambda t} + kbe^{\lambda t} = 0 \implies k(\lambda^2 + a\lambda + b) = 0 \tag{5}$$

If our initial condition is nonzero, k will be nonzero meaning we have a quadratic equation for λ similar to the characteristic polynomial of our matrix A. Since this quadratic equation has two roots λ_1 and λ_2, our solution $y(t)$ will be a linear combination of the functions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ or of the form

$$y(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} \tag{6}$$

Plugging in the initial conditions $y(0)$ and $\frac{dy}{dt}(0)$, we should be able to solve for the coefficients α_1 and α_2.

2.2.2 Converting to a Vector Differential Equation

Similar to how we converted a system of differential equations into a vector differential equation, we can also turn our second-order differential equation into a **first-order** vector differential equation. We will do so by defining state variables

$$x_1(t) = y(t),\ x_2(t) = \frac{d}{dt}y(t) \tag{7}$$

Taking the derivative of our states, we see that

$$\frac{d}{dt}x_1(t) = \frac{d}{dt}y(t) = x_2(t) \tag{8}$$

$$\frac{d}{dt}x_2(t) = \frac{d^2}{dt^2}y(t) = -by(t) - a\frac{d}{dt}y(t) = -bx_1(t) - ax_2(t) \tag{9}$$

Therefore, we can write this as a vector differential equation

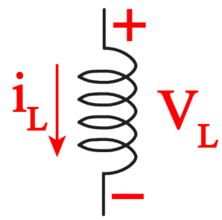
$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{10}$$

Note that the eigenvalues of the A matrix yields the exact characteristic polynomial that we found using guess and check. This is not coincidental and in fact arises since we were looking for eigenvalues of the differentiation operator $\frac{d}{dt}$.

3 Inductors

Let's introduce a new passive component, an inductor. This new component will help us design more interesting circuits and introduce oscillations within our circuits.

Inductors have a dual relationship in terms of voltage and current (I-V) as compared to capacitors (i.e., V being proportional to change in I as opposed to I being proportional to change in V). The schematic symbol of an inductor is drawn below:



(a) The unit of inductance is Henry (H).

(b) $v_L(t) = L\frac{di_L(t)}{dt}$

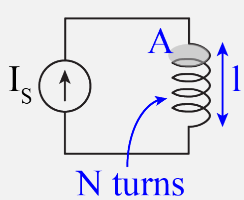
(c) $i_L(t)$ cannot change instantly.

(d) At DC steady state, inductors behave like short circuits since the current is constant meaning there is no voltage drop.

(e) The energy stored in an inductor is: $U = \frac{1}{2}L{i_L}^2(t)$

While inductors are introduced in this course only as a circuit symbol and a mathematical construct, we are not spending time on the physics behind them, they have multiple applications in the real world and you will study them in future courses.

Inductors store energy by setting up a magnetic field. In the same way that a capacitor separates charge (Q) and this leads to an $\vec{E}$ field, anytime we flow current down a conductor, this creates a magnetic field ($\vec{B}$). Likewise, the magnetic field can store energy. Their behavior can be described using Faraday's law of induction.
The magnitude of magnetic field created by a straight wire is pretty small, so we usually use other geometries if we are trying to create a useful inductance on purpose. A solenoid is a good example:



$$L = \frac{N^2 \mu A}{l} \ [H]$$

Note that the inductance (L) depends on geometry and a material property called permeability of the solenoid core material. Inductors are useful in many applications such as wireless communications, chargers, DC-DC converters, key card locks, transformers in the power grid, etc.

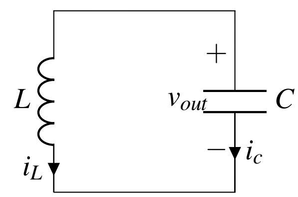
Concept Check: The current across the inductor cannot change instantaneously. Why?
Solution: If our current changes instantaneously, then $\frac{d}{dt}I_L \to \infty$, and from equation ((b)) the voltage across the inductor $V_L \to \infty$, which is not possible. Hence, our current cannot change instantaneously

4 LC Tank

Let's take a look at a circuit with an inductor and capacitor in parallel. This is commonly known as an LC tank, whose matrix will have purely imaginary eigenvalues.

In the following circuit, we have an inductor $L = 10\,\text{nH}$ and capacitor $C = 10\,\text{pF}$ in parallel.

Let $I_L(0) = 50\,\text{mA}$ and $V_c(0) = 0\,\text{V}$:



First we define states $x_1 = I_L$ and $x_2 = V_c$. Then since the inductor and capacitor are in parallel:

$$V_L = V_c \tag{11}$$

KCL gives:

$$I_L = -I_c = -C\frac{dV_c}{dt} \implies \frac{dV_c}{dt} = -\frac{1}{C}i_L \tag{12}$$

$$V_L = V_c = L\frac{dI_L}{dt} \implies \frac{dI_L}{dt} = \frac{1}{L}v_{out} \tag{13}$$

Putting it into matrix form, as before:

$$\frac{d}{dt}\begin{bmatrix} V_c \\ I_L \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} V_c \\ I_L \end{bmatrix} \tag{14}$$

Finding the eigenvalues:

$$\det\left(\begin{bmatrix} -\lambda & -\frac{1}{C} \\ \frac{1}{L} & -\lambda \end{bmatrix}\right) = \lambda^2 + \frac{1}{LC} = 0 \tag{15}$$

$$\therefore \lambda_{1,2} = 0 \pm j\frac{1}{\sqrt{LC}} \tag{16}$$

Now we will solve the differential equation using guess and check and start by guessing the following solution:

$$\vec{x} = \begin{bmatrix} I_L \\ V_c \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} \\ \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t} \end{bmatrix}$$

Then we take the derivative and evaluate at $t = 0$ to find $\frac{d}{dt}\vec{x}(0)$

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} \lambda_1\alpha_1 e^{\lambda_1 t} + \lambda_2\alpha_2 e^{\lambda_2 t} \\ \lambda_1\beta_1 e^{\lambda_1 t} + \lambda_2\beta_2 e^{\lambda_2 t} \end{bmatrix} \quad \frac{d}{dt}\vec{x}(0) = \begin{bmatrix} \lambda_1\alpha_1 + \lambda_2\alpha_2 \\ \lambda_1\beta_1 + \lambda_2\beta_2 \end{bmatrix}$$

and use the differential equation $\frac{d}{dt}\vec{x} = A\vec{x}$ to get the following

$$\frac{d}{dt}\vec{x}(0) = A\vec{x}(0) = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} 5 \cdot 10^{-3} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{50 \cdot 10^{-3}}{L} \end{bmatrix}$$

Lastly, we plug in the values for λ and solve the system of equations

$$\alpha_1 + \alpha_2 = 50 \cdot 10^{-3} \qquad \beta_1 + \beta_2 = 0$$

$$j\frac{1}{\sqrt{LC}}\alpha_1 - j\frac{1}{\sqrt{LC}}\alpha_2 = 0 \qquad j\frac{1}{\sqrt{LC}}\beta_1 - j\frac{1}{\sqrt{LC}}\beta_2 = \frac{50 \cdot 10^{-3}}{L}$$

Solving the system of equations, we get the following constants

$$\alpha_1 = 25 \cdot 10^{-2} \quad \alpha_2 = 25 \cdot 10^{-3} \quad \beta_1 = \frac{0.5\sqrt{10}}{2j} \quad \beta_2 = -\frac{0.5\sqrt{10}}{2j}$$

Therefore, the solution to the differential equation is

$$\begin{bmatrix} I_L(t) \\ V_c(t) \end{bmatrix} = \begin{bmatrix} 25 \cdot 10^{-2} e^{j\sqrt{10} \cdot 10^9 t} + 25 \cdot 10^{-2} e^{-j\sqrt{10} \cdot 10^9 t} \\ \frac{0.5\sqrt{10}}{2j} e^{j\sqrt{10} \cdot 10^9 t} - \frac{0.5\sqrt{10}}{2j} e^{-j\sqrt{10} \cdot 10^9 t} \end{bmatrix}$$

We can simplify this using Euler's Formula to get our final answer.[2]

$$\begin{bmatrix} I_L(t) \\ V_c(t) \end{bmatrix} = \begin{bmatrix} 50 \cdot 10^{-2} \cos\left(\sqrt{10} \cdot 10^9 t\right) \\ 0.5\sqrt{10} \sin\left(\sqrt{10} \cdot 10^9 t\right) \end{bmatrix}$$

Figure 1 plots the above solutions for the capacitor voltage and inductor current. This system is also called an oscillator because the circuit produces a repetitive voltage waveform under the right initial conditions.

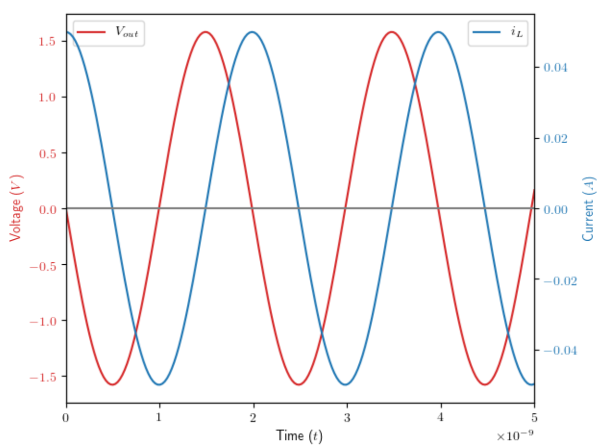


Figure 1: Voltage and Current response of LC Tank

[2]Remember that $\cos(\theta) = \frac{e^{j\theta}+e^{-j\theta}}{2}$ and $\sin(\theta) = \frac{e^{j\theta}-e^{-j\theta}}{2j}$.

From the above plots, we can see that when the capacitor is fully charged, the inductor has zero flux whereas when the inductor has full flux, the capacitor is fully discharged. What does this imply about the energy stored in the two components?

We know that, energy in the capacitor, $E_c = \frac{1}{2}CV^2 = 1.25 \times 10^{-11} \sin^2\left(\sqrt{10} \times 10^9 t\right)$ and energy in the inductor, $E_L = \frac{1}{2}LI^2 = 1.25 \times 10^{-11} \cos^2\left(\sqrt{10} \times 10^9 t\right)$. This shows that $E_{total} = E_c + E_L = 1.25 \times 10^{-11}$ is constant across all time.

Figure 2 plots these energies. As it is clear, the total energy seems to be sloshing back and forth between the inductor and capacitor.

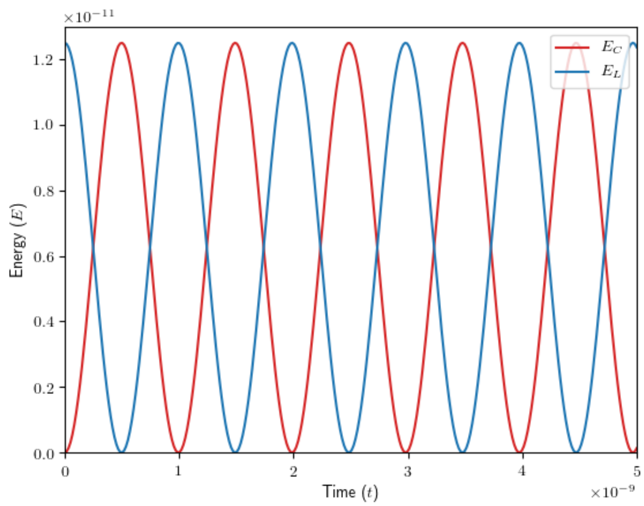
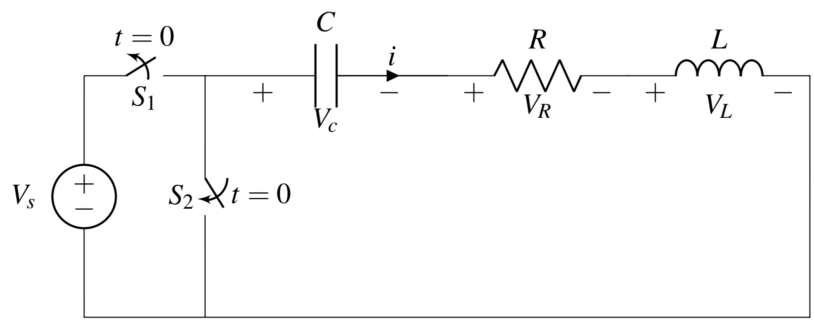


Figure 2: Energy stored in Inductor and Capacitor. Notice the sum is constant.

5 RLC Circuits and Higher Order Differential Equations

The LC tank we studied in the previous section was a very ideal case where we assumed there was no resistor in the system. But this is rarely the case, and we will need to understand how adding this third component will modify our differential equations.

Before $t = 0$, switch S_1 is on while S_2 is off. At $t = 0$, both switches flip state (S_1 turns off and S_2 turns on):



First, let's figure out the initial conditions. Since the system had been connected to the battery for a long time, the capacitor would be at steady state meaning $v_c(0) = V_s$ and $i(0) = 0A$. From this, we can also deduce that $\frac{d}{dt}v_c = 0$. Next, let's write our branch equations:

$$i = C\frac{d}{dt}V_c,\ V_L = L\frac{d}{dt}i,\ V_R = i \cdot R \tag{17}$$

$$V_c + V_L + V_R = 0 \tag{18}$$

Using the above equations, and substituting for i from Equation (17) when needed, we can describe our system with the following differential equation:

$$\frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{V_c}{LC} = 0 \tag{19}$$

Here we have chosen the second order differential equation as means of an example. As usual, we can solve this differential equation by computing its eigenvalues and use any approach from before.[3]

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0 \tag{20}$$

$$\lambda = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \tag{21}$$

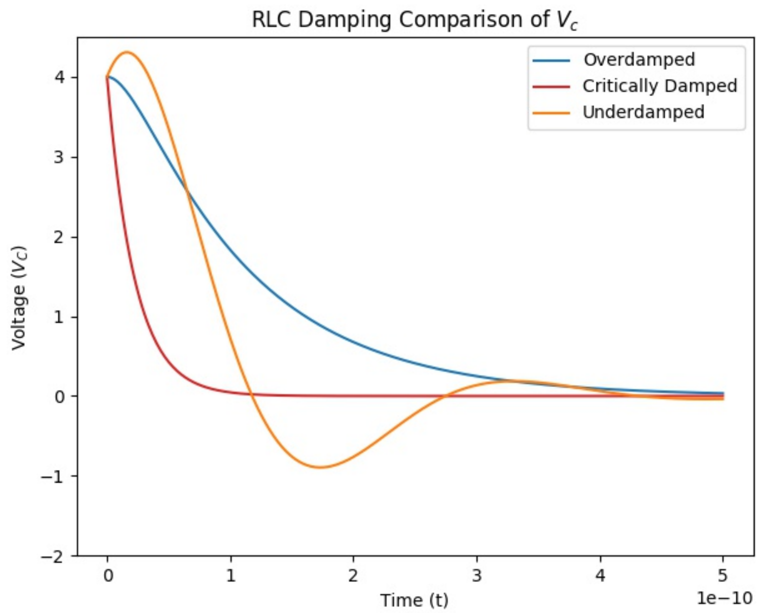
Now depending on the values of R, L, C the eigenvalues change accordingly

- When $R > 2\sqrt{\frac{L}{C}}$, the system is **overdamped** and there are distinct **purely real** eigenvalues.
- When $R < 2\sqrt{\frac{L}{C}}$, the system is **underdamped** and the two eigenvalues are **complex conjugates**
- When $R = 2\sqrt{\frac{L}{C}}$, the system is **critically damped** and there is a **single purely real** eigenvalue.

[3]Guess and check, diagonalization, changing coordinates, we are no longer emphasizing the solving process rather we would like to extrapolate information from the results.

The term **damping** refers to a system's ability to resist its natural oscillatory behavior. If $R = 0$, notice that the RLC circuit reduces to the LC tank. As the value of R increases, we'll notice the oscillations go away. This is because the damping force is stronger than the system's natural tendency to oscillate.

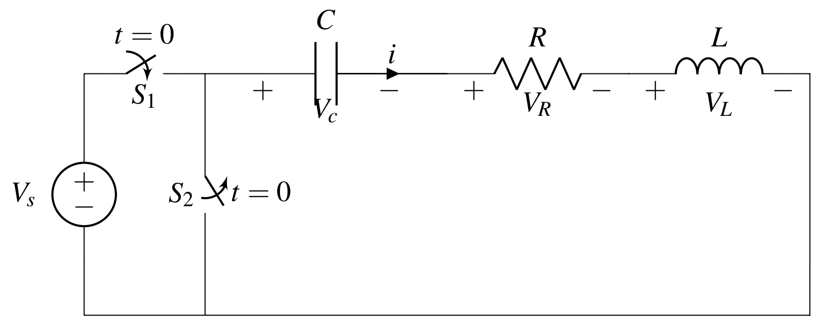
To illustrate of the effects of damping on our system, we plot the responses below for various R, L, C values.



When the system is underdamped, the response $V_c(t)$ has oscillatory behavior whereas if the system is overdamped or critically damped, there are no oscillations. In addition, notice that the critically damped case converges to steady-state much quicker than the overdamped case. This is a very desirable in controls engineering since we are able to reach steady state the quickest without any oscillations.

6 Charging an RLC Circuit

Now that we have equipped ourselves with some knowledge on eigenvalues, we will take a look at the nonhomogenous case. Consider the following circuit. Before $t = 0$, switch S_1 is off while S_2 is on. At $t = 0$, both switches flip state (S_1 turns on and S_2 turns off):



Firstly, we must find the initial conditions. Since the capacitor has been discharging for a long time, $V_c(0) = 0$ and $i_L(0) = i(0) = 0$. Next, let's write out the branch equations,

$$i = C\frac{d}{dt}V_c,\ V_L = L\frac{d}{dt}i,\ V_R = i \cdot R \tag{22}$$

$$V_c + V_L + V_R = 0 \tag{23}$$

Using the above equations, and substituting for i from Equation (17) when needed, we can describe our system with the following differential equation:

$$\frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{V_c}{LC} = V_s \tag{24}$$

A quick technique we can use to homogenize the above equation is a substitution of variables: [4] $x = V_c - V_s$, hence $\frac{d}{dt}x = \frac{d}{dt}V_c$ and $\frac{d^2}{dt^2}x = \frac{d^2}{dt^2}V_c$. Applying this substiution,

$$\frac{d^2x}{dt^2} + \frac{R}{L}\frac{dx}{dt} + \frac{x}{LC} = 0 \tag{25}$$

Looking back, Equation (19) close resembles our above equation. Hence, we will find the same eigenvalues.

$$\widetilde{\lambda_1} = -10^{10},\ \widetilde{\lambda_2} = 5 \times 10^{10} \tag{26}$$

Solving our homogenous differential equation using our method of choice, we see that the solution is

$$x(t) = -5e^{-10^{10}t} + e^{-5\times 10^{10}t} \implies V_c(t) = 4 - 5e^{-10^{10}t} + e^{-5\times 10^{10}t} \tag{27}$$

Then converting back to V_c using $V_c = x + V_s$, we see that

$$V_c(t) = 4 - 5e^{-10^{10}t} + e^{-5\times 10^{10}t} \tag{28}$$

[4] We could've also guess and checked the solution $V_c(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} + \beta$ where β is a constant. The substitution of variables approach ensures that our eigenvalues $\widetilde{\lambda}$ and λ are indeed the same.

7 Damping Ratio (Optional)

We have continually referred to the term **damping** without giving it a formal definition. It is time to introduce the meaning behind damping. For any second order differential equation of the form

$$\frac{d^2x}{dt^2}+2\zeta\omega_n\frac{dx}{dt}+\omega_n^2x=0 \tag{29}$$

ζ is defined to be the **damping ratio** and ω_n[5] is the **natural frequency** of the system.

Note that these constants will apply for physical systems outside the context of circuits as well. We will see more of these examples in a later part of the course. However, for now in the context of RLC, the natural frequency of the system is the frequency that the circuit oscillates at when undamped or $\zeta=0$. Recall that this is the specific case of the LC tank and the natural frequency will be $\omega_n=\frac{1}{\sqrt{LC}}$.

This means that the damping ratio of an RLC circuit is $\zeta=\frac{R}{2}\sqrt{\frac{C}{L}}$. Connecting this back to our eigenvalues, notice that when $0<\zeta<1$, the response is underdamped whereas if $\zeta>1$, then the system is overdamped. Now what happens when $\zeta=1$? We will finally answer the question of what happens when there is a single real eigenvalue and the matrix representing the system is **not diagonalizable.**

7.1 Critical Damping

7.1.1 Repeated Eigenvalues

Our entire process of solving second order differential equations relied on the truth that a $n\times n$ matrix A has n linearly independent eigenvectors. However, if we were to have repeated eigenvalues in our system, then we cannot guarantee that A is diagonalizable. One example of a second order differential equation that is nondiagonalizable when put into matrix form is

$$\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=0 \tag{30}$$

By picking state-variables $x_1(t)=y(t)$ and $x_2(t)=\frac{dy}{dt}$, we could set up the following system of differential equations

$$\frac{d}{dt}\vec{x}(t)=A\vec{x}(t)=\begin{bmatrix}0 & 1\\ -9 & -6\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} \tag{31}$$

There is a single eigenvalue $\lambda=-3$ and the eigenspace of $A+3I$ is

$$\text{Nul}(A+3I)=\text{Nul}\begin{bmatrix}3 & 1\\ -9 & -3\end{bmatrix}=\alpha\begin{bmatrix}1\\ 3\end{bmatrix} \tag{32}$$

which is one-dimensional. This was where we left off last time since our matrix was non-diagonalizable.

Concept Check: Why did we use diagonalization when solving a system of differential equations?
Solution: Our motivation behind diagonalization was to find a basis in which the matrix A was diagonal so that we could decompose our system into n first order differential equations.

[5] The subscript 'n' in ω_n stands for "natural".

However, does our system need to be **diagonal** for us to create n first order differential equations? What if it was possible to pick a basis in which A had an **upper-triangular** representation?

$$\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} \lambda & \star \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \tag{33}$$

where $\star$ is any nonzero value. At a first glance, we are not able to uncouple the equations to create two first order equations. However, if were to solve these differential equations from the bottom up exactly like how we performed back-substitution when row-reducing, what would happen?

This would mean we first solve $\frac{d}{dt}z_2(t) = \lambda z_2(t)$ which has unique solution $z_2(t) = z_2(0)e^{\lambda t}$. Now that we have a solution $z_2(t)$, we can plug it back into our first differential equation!

$$\frac{d}{dt}z_1(t) = \lambda z_1(t) + \star \cdot z_2(0)e^{\lambda t} \tag{34}$$

Since $z_1(t)$ is a first order differential equation with an input $u(t) = ke^{\lambda t}$, referring back to Note 3, the solution is $z_1(t) = z_0 e^{\lambda t} + kte^{\lambda t}$.

7.2 But what is our basis?

At last we have developed a strategy on how to tackle the case in which A cannot be diagonalized. However, we have yet to define the basis $\{\vec{v}_1, \vec{v}_2\}$ that makes A have an upper-triangular representation. So we will now define a basis to make A have an upper triangular representation. [6]

(1) We will start by picking $\vec{v}_1 = \vec{v}$ where $\vec{v}$ is our eigenvector of A.

(2) To form a basis for $\mathbb{R}^2$, our second vector $\vec{v}_2$ can be any vector linearly independent to $\vec{v}_1$. It follows that since the null-space of $A - \lambda I$ was one-dimensional, we can pick any vector $\vec{v}_2$ not in this null-space.

(3) Since $\vec{v}_1$ and $\vec{v}_2$ form a basis, we can represent any vector in $\mathbb{R}^2$ using coordinates. For ease of calculation, we pick $\vec{v}_2$ such that $(A - \lambda I)\vec{v}_2 = \vec{v}_1$. This would mean that $A\vec{v}_2 = \vec{v}_1 + \lambda \vec{v}_2$.

(4) In matrix form we can write this as $AV = \Lambda V$ where $\Lambda = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ proving the existence of a basis V in which A is upper-triangular.

To summarize, we have shown the existence of a basis V for which A has an upper triangular representation and we have also derived a form for our solution

$$x(t) = \alpha_1 e^{\lambda t} + \alpha_2 te^{\lambda t} \tag{35}$$

[6] We do this for a 2×2 matrix, but this can be extended to an arbitrary $n \times n$ matrix through induction. We will look at the $n \times n$ case in a later note.

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