BCSE Game Theory 04-01

Mixed Strategies I

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Introducing Mixed Strategies

Today's Goals

- Motivate finite games where no pure Nash equilibrium exists.
- Define mixed strategies and expected payoffs.
- Learn how to compute mixed equilibria via the penalty-kick example.
- Interpret best-response correspondences graphically.

When Pure Strategies Fail

- In Matching Pennies and penalty kicks, each player wants to switch based on the other's move, so no pure Nash equilibrium exists.
- Rock–Paper–Scissors has the same cyclic dominance and therefore lacks a pure equilibrium.
- We need a framework where players randomise over actions to discuss equilibrium.

Rock–Paper–Scissors as a Normal Form Game

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	(0,0)	(-1,1)	(1,-1)
	Paper	(1,-1)	(0,0)	(-1,1)
	Scissors	(-1,1)	(1,-1)	(0,0)

- The payoff matrix fully specifies the Normal Form game for the two players.
- Symmetry implies each action wins, loses, and ties with equal frequency across the opponent's choices.

Defining Mixed Strategies

Mixed Strategies and Simplices

Definition: Mixed strategy

Player i's mixed strategy σ_i is a probability distribution over her pure strategy set S_i.

- The simplex $\Delta(S_i) \stackrel{\text{def}}{=} \{\sigma_i \in \mathbb{R}^{|S_i|}_{\geq 0} \mid \sum_{s_i \in S_i} \sigma_i(s_i) = 1\}$ collects all feasible probability vectors.
- The support $\operatorname{supp}(\sigma_i) \stackrel{\text{def}}{=} \{s_i \in S_i \mid \sigma_i(s_i) > 0\}$ is the set on which indifference must hold.
- Expected payoff for profile $\sigma \stackrel{\text{def}}{=} (\sigma_i)_{i \in N}$ is

$$U_i(\sigma) = \sum_{s \in S} \Big(\prod_{j \in N} \sigma_j(s_j) \Big) u_i(s).$$

Rock–Paper–Scissors: Three-Step Indifference Method

Procedure

1. Express $U_1(\text{Rock})$, $U_1(\text{Paper})$, $U_1(\text{Scissors})$ using Player 2's mix (q_R, q_P, q_S), and mirror the step for Player 2.
2. Enforce indifference on the support:

$$U_1(\text{Rock}) = U_1(\text{Paper}) = U_1(\text{Scissors}),$$
$$q_R + q_P + q_S = 1.$$

3. Check that the solution keeps probabilities in $[0, 1]$ and yields the same value for every supported action (zero here).

Rock–Paper–Scissors Insights

- Symmetric zero-sum: each player mixing $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ leaves the opponent indifferent.
- Expected payoff is zero; any bias lets the opponent concentrate on the winning action.
- A classic preview of solving indifference conditions, which we formalise later.

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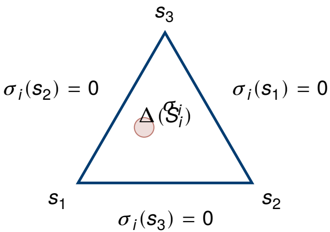
Geometry of the Simplex



- With $|S_i| = 3$, $\Delta(S_i)$ is a filled triangle whose vertices correspond to pure strategies.
- Points on an edge mix two strategies; interior points randomise over all three.
- Reading coordinates of a point clarifies which pure strategies appear in the support.

Best Responses

Definition: Best response

The best-response set to σ_{-i} is

$$BR_i(\sigma_{-i}) \stackrel{\text{def}}{=} \arg\max_{\hat{\sigma}_i \in \Delta(S_i)} U_i(\hat{\sigma}_i, \sigma_{-i}).$$

- Every pure strategy in the support yields the same expected payoff (indifference principle).
- The support satisfies $\operatorname{supp}(\sigma_i) \subseteq BR_i(\sigma_{-i})$, but the reverse inclusion need not hold.

Mixed-Strategy Nash Equilibria

Definition: Mixed-strategy Nash equilibrium

A mixed profile $\sigma^* \stackrel{\text{def}}{=} (\sigma_i^*)_{i \in N}$ is a mixed-strategy Nash equilibrium when, for every player $i \in N$,

$$\sigma_i^* \in BR_i(\sigma_{-i}^*),$$
$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\hat{\sigma}_i, \sigma_{-i}^*) \quad \forall \hat{\sigma}_i \in \Delta(S_i).$$

Equivalent characterisation

The equilibrium requirements for each $i \in N$ are equivalent to:

1. $U_i(s, \sigma_{-i}^*) = U_i(s', \sigma_{-i}^*)$ for all $s, s' \in \mathrm{supp}(\sigma_i^*)$ (indifference on the support).
2. $U_i(s, \sigma_{-i}^*) \geq U_i(\hat{s}, \sigma_{-i}^*)$ for every $s \in \mathrm{supp}(\sigma_i^*)$ and $\hat{s} \in S_i$ (no profitable deviation).

Mixed Equilibria at a Glance

- Everyone is best-responding, so no player can improve expected payoff unilaterally—indifference only has to hold on the support.
- Pure equilibria arise when a support collapses to one action, so mixed equilibria extend all earlier concepts seamlessly.

Worked Process for a 2×2 Game

		Column: Left	Column: Right
Row	Up	(2,-2)	(0,0)
	Down	(1,-1)	(3,-3)

- Zero-sum example where both players mix between two actions.
- Next frame walks through the indifference equations that pin down the probabilities.

Worked Process: Solving the Mix

1. Identify the support: both actions remain best responses for each player.
2. Let Column mix q on Left and Row mix p on Up. Indifference gives

$$\begin{aligned} U_{\text{Row}}(\text{Up}) &= 2q, & U_{\text{Row}}(\text{Down}) &= q + 3(1-q), \\ U_{\text{Col}}(\text{Left}) &= -2p - (1-p), & U_{\text{Col}}(\text{Right}) &= -3 + 3p. \end{aligned}$$

3. Solving yields $q = \frac{3}{4}$ and $p = \frac{1}{2}$, giving matched payoffs of zero and verifying the mixed equilibrium.

Interpreting Mixed Strategies

- **Individual view**: players do not literally roll dice, but prepare routines that randomise play to stay unpredictable.
- **Population view**: over many repetitions, play frequencies follow the equilibrium probabilities (frequency interpretation).
- Opponents learn from past frequencies, so deviating from equilibrium makes you exploitable.
- Equilibrium probabilities enforce strategic indifference—every supported pure strategy yields the same expected payoff.
- Sports and auction data often exhibit probabilities close to their equilibrium values.

Mixed Strategies in Penalty Kicks

Penalty Kicks: Baseline Model

		Goalkeeper	
		Dive Left	Dive Right
Kicker	Shoot Left	$(-1, 1)$	$(1, -1)$
	Shoot Right	$(1, -1)$	$(-1, 1)$

- The kicker scores (payoff 1) by choosing the opposite side from the keeper and is saved (payoff -1) if they match.
- The goalkeeper's payoff is the kicker's with the sign flipped.

Penalty Kicks: Normalised Payoffs

Payoffs can be rescaled by a positive affine transformation, so add 1 and halve each player's payoff.

		Goalkeeper	
		Dive Left	Dive Right
Kicker	Shoot Left	$(0, 1)$	$(1, 0)$
	Shoot Right	$(1, 0)$	$(0, 1)$

- Let p be the probability the kicker shoots left and q the probability the keeper dives left.

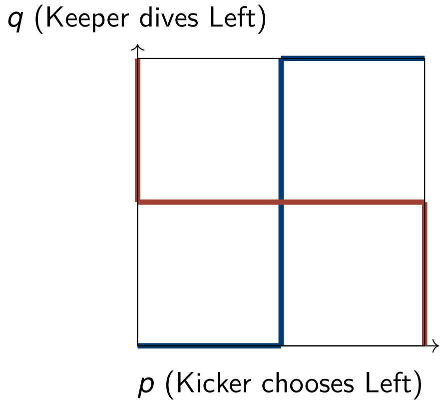
$$U_{\text{kicker}}(\text{Left}) = 1 - q,$$
$$U_{\text{kicker}}(\text{Right}) = q.$$

- The goalkeeper's indifference mirrors this calculation:

$$U_{\text{keeper}}(\text{Dive Left}) = p,$$
$$U_{\text{keeper}}(\text{Dive Right}) = 1 - p.$$

- Solving the indifference conditions with $p, q \in [0, 1]$ yields

Visualising Best Responses



Interpreting the Diagram

- Blue segments track the goalkeeper's best responses; red segments show the kicker's.
- Intersection points where both colours meet correspond to mutual best responses.
- In this game the unique intersection is $(\frac{1}{2}, \frac{1}{2})$, matching the equilibrium found algebraically.

When the Kicker Prefers the Right

		Goalkeeper	
		Dive Left	Dive Right
Kicker	Shoot Left	(0.10, -0.10)	(0.90, -0.90)
	Shoot Right	(0.95, -0.95)	(0.30, -0.30)

- Numbers report the kicker's scoring probability; the keeper's payoff is the negative of that value.

Solving for Biased Probabilities

Let q denote the probability that the keeper dives left. Indifference for the kicker requires

$$U_{\text{kicker}}(L) = 0.1q + (1-q)0.9,$$
$$U_{\text{kicker}}(R) = 0.95q + (1-q)0.3,$$

so $\frac{q}{1-q} = \frac{12}{17}$ and therefore $q^* = \frac{12}{29} \approx 0.414$.

By symmetry the keeper mixes with $p^* = \frac{13}{29} \approx 0.448$. Each side leans toward its preferred action, yet the opponent adjusts as well —mixed strategies never allow one player to be fully predictable.

Today's Summary

- Mixed strategies are indispensable for analysing finite games.
- Pure strategies in the support share the same expected payoff, letting us solve equations for equilibrium probabilities.
- The penalty-kick example derives equilibrium from indifference conditions across players.
- Next time: prove existence via fixed-point theorems and explore broader applications of mixed strategies.

Recap Checklist

- Can you reproduce the calculation that yields $(1/3, 1/3, 1/3)$ in Rock – Paper – Scissors?
- Can you solve the mixed equilibrium of a 2×2 zero-sum game using indifference and probability constraints?
- Can you explain the intersection of reaction curves in the best-response diagram?
- Can you summarise the existence-theorem keywords (convex, compact, fixed point) in your own words?