Game Theory 08-01

Finite Repeated Games and History-Based Coordination

BCSE Game Theory

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Learning Goals

- Review stage games such as the Teamwork example and understand the tension between individual incentives and cooperation.
- Define finite repeated games, histories, and how strategy profiles depend on past play.
- Apply backward induction to finite horizons and explain the unraveling argument.
- Study how stage games with multiple equilibria allow threats to sustain cooperative paths.
- Prepare for the infinite repetition setting and the dynamic programming perspective.

Setting the Stage

Stage games we repeat

Why repeat (stage) games?

- Most strategic interactions continue beyond one shot, so players care about how today's move shapes tomorrow. (e.g., Teamworks (limited periods), comptition in a market)
- Module 07 taught us SPNE in extensive-form games; repeating a stage game reuses that machinery with longer histories.
- Repetition opens new enforcement tools: reputations, threats, and history-dependent punishments that are unavailable in single-shot play.

Formal Stage Game

Definition: Stage Game

A stage game is a strategic-form game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ where:

- Players $i \in N$ choose actions $s_i \in S_i$ simultaneously.
- Payoffs $u_i(s)$ are realized immediately after the action profile $s = (s_1, \ldots, s_n)$ is selected.

- We will repeat this base game T times.
- A Nash equilibrium of G is a profile s^* where $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$ for all $s_i \in S_i$.

Stage Game Example: Teamwork

	C	D
C	$(3, 3)$	$(1, 4)$
D	$(4, 1)$	$(2, 2)$

- There is a unique Nash equilibrium: (D,D).
- (C,C) is Pareto superior, but (D,D) is the only stable outcome in a one-shot game.

Stage Game Example with Multiple Nash Equilibria

	Stag	Hare
Stag	$(4, 4)$	$(0, 3)$
Hare	$(3, 0)$	$(3, 3)$

- Both (Stag,Stag) and (Hare,Hare) are Nash equilibria.
- (Stag,Stag) is payoff dominant but requires mutual commitment.
- The Stage game alone offers no mechanism to pick the Pareto-optimal equilibrium.

Equilibrium selection cues

- Payoff dominance versus risk dominance guides players when multiple equilibria exist.
- Without repetition, players rely on focal points, norms, or communication to coordinate.
- Repetition gives us an enforcement mechanism that can privilege the desired equilibrium.

Finite Repeated Games

Definitions

Finite repeated game definition

- Let G be a stage game. The finite repeated game G^T runs T periods.
- In each period t, players simultaneously select actions s^t and observe the outcome.
- The entire sequence of actions $(s^1, \ldots, s^T)$ determines the payoffs.

History-dependent strategies

- The history before period t is $h^t = (s^1, \ldots, s^{t-1})$.
- Strategies are functions $\sigma_i^t : H^t \to S_i$ mapping the realized history into an action.
- Players possess perfect recall, so they remember all past profiles.

History notation example

- If period 1 outcome is (C,C) and period 2 is (C,D), then $h^3 = ((C, C), (C, D))$.
- Strategies can condition on such histories to switch from cooperation to punishment.
- Specifying off-path behavior is necessary to define subgame perfection.

Strategy as history mapping

- A strategy specifies what to do for every possible past, including off-path histories.
- History-dependent strategies enable punishments: a deviation can change future play.
- The set of feasible continuations is richer than the stage game alone.

Payoff evaluation in finite repetition

- Each player's total payoff is the sum of stage utilities:

$$U_i = \sum_{t=1}^{T} u_i(s^t).$$

- Equivalently, we can consider the average payoff $(1/T)\sum_t u_i(s^t)$ when comparing different strategies.
- Non-discounted sums are standard for finite horizons, but we keep discounting in mind for the limit case.

Teamwork stage payoffs

	C	D
C	$(3, 3)$	$(1, 4)$
D	$(4, 1)$	$(2, 2)$

- The unique Nash equilibrium is (D,D) because defecting is a dominant strategy.
- Cooperation cannot be sustained in the single-shot game.

Finite repeated Teamwork: short horizon

- Players play the stage game T times and observe the outcome before moving to the next period.
- The only Nash equilibrium in the stage game is (D,D).
- To find the equilibrium of the repeated game, we use backward induction.
- We start from the final period and reason backwards to the first period.

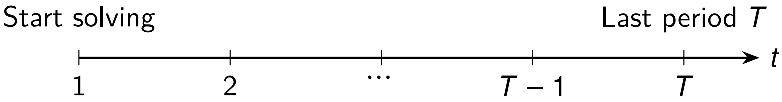
Solving Finite Repeated Games

Backward induction & Unraveling

Backward induction recipe

1. Solve the last period T: players play a stage Nash equilibrium.
2. Given the equilibrium payoffs of period T, solve period $T - 1$ with the continuation payoffs fixed.
3. Repeat until reaching period 1. Each period is solved as a stage game with continuation payoffs added.
4. The resulting profile is a subgame perfect equilibrium for the repeated game.

Timeline of backward induction



- The analysis begins at the right-hand side (period T) and moves backward.
- The final period's equilibrium determines the continuation payoffs used earlier.

Unraveling definition

- Unraveling occurs when the unique equilibrium in the last period forces that equilibrium earlier.
- If every continuation game has a unique subgame perfect equilibrium, cooperation collapses backward to $t = 1$.
- The finite horizon prevents players from using out-of-equilibrium threats that rely on future cooperation beyond T.

Unraveling in the Teamwork Game

- Period T: (D,D) is the only Nash equilibrium, so both defect.
- Period $T-1$: knowing that T is doomed, immediate defection is optimal (no future threat remains).
- By induction, every period contains defection; cooperation cannot be sustained.

Theorem: Unique Equilibrium in Finite Repeated Games

Theorem: Unique Equilibrium Payoff

Let G be a stage game with a unique Nash equilibrium s^* yielding payoff vector $u(s^*)$. Then every subgame perfect equilibrium of the finite repeated game G^T yields the average payoff vector $u(s^*)$.

- **Proof sketch**: By backward induction.
- Period T: Players must play s^* (unique Nash equilibrium), yielding $u(s^*)$.
- Period $T-1$: Given that period T yields $u(s^*)$, the continuation payoff is fixed. Players again play s^*.
- By induction, every period $t \in \{1, \dots, T\}$ plays s^*, so the average payoff is $u(s^*)$.

General unraveling condition

- Whenever the continuation game has a unique subgame perfect equilibrium, the same path repeats in earlier periods.
- Unique last-period equilibrium removes any leverage for threats in period $T - 1$, and so on.
- We need multiple equilibrium or viable punishments to break the chain of unraveling.

Teamwork repeated: building threats

- We can use the inferior equilibrium (D,D) as a punishment for deviating from (C,C).
- The punishment path must be credible: after a deviation, players switch to a Nash equilibrium of the remaining game.
- The history of play determines whether cooperation continues or the punishment path (D,D) is triggered.

Threat structure for Teamwork

- Start by playing (C,C) for $t = 1, \dots, T - 1$.
- If any player deviates, revert to (D,D) for the remaining periods.
- (D,D) is a stage Nash equilibrium, so the threat is subgame perfect.

Punishment path as credible equilibrium

- The punishment path itself must be an equilibrium of the continuation game.
- (D,D) remains an equilibrium after a deviation, so it can serve as a credible punishment.
- Without such a stage equilibrium, the punishment loses credibility.

Incentive inequality for cooperation

- The short-term gain from deviating must not exceed the loss from the punishment phase.
- For Teamwork: deviating to D yields 3 instead of 4, but the punishment replaces future $(4, 4)$ payoffs with $(2, 2)$.
- Choosing the horizon and punishment length appropriately keeps the inequality in favor of cooperation.

Verifying Subgame Perfection

- SPNE requires the strategy to be optimal in **every** subgame, not just on the equilibrium path.
- **On-path (Cooperation)**: Checked by the incentive inequality (future loss > current gain).
- **Off-path (Punishment)**: If a deviation occurs, players switch to (D,D).
  - Is playing D optimal given the opponent plays D?
  - Yes, because (D,D) is a stage Nash equilibrium.
  - Best response to D is D ($2 > 1$).
  - Thus, players have no incentive to deviate from the punishment.
- This confirms the threat is credible and the strategy is an SPNE.

Example play path with punishment

1. Periods 1 to t^*: both players choose Work, believing cooperation continues.
2. If a deviation occurs at t^*, both switch to D for $t^* + 1, \ldots, T$.
3. The deviator's short-term gain is offset by the guaranteed lower payoff in the punishment phase.

Multi-equilibrium stage games give better targets

- Stage games like the Stag-Hare offer both payoff-dominant and risk-dominant equilibria.
- Repetition enables us to target the Pareto-superior equilibrium by threatening to play the safety equilibrium in case of deviation.
- The ability to switch between equilibria is what expands the repeated equilibrium set.

Credible punishment requirement

- A punishment is credible if the punishment path is a Nash equilibrium of the continuation game.
- Using (D,D) after a deviation satisfies credibility because it is a stage Nash equilibrium.
- Players must prefer sticking to the cooperative path rather than deviating and facing punishment.

Achieving True Asymmetry

- To achieve asymmetry, alternate between different equilibria strategically.
- Example for $T = 2$: Period 1 play (Stag,Stag), period 2 play (Hare,Stag).
- Player 1 gets: $(4 + 3)/2 = 3.5$.
- Player 2 gets: $(4 + 0)/2 = 2.0$.
- Note: (Hare,Stag) is not a Nash equilibrium, so this requires enforcement via threats.
- The key insight: multiple stage equilibria allow richer payoff constructions through history-dependent play.

Finite horizon strategy families

- One can design two-phase strategies: cooperate for the first m periods and play a stage equilibrium afterwards.
- The punishment can be permanent or limited to a fixed number of periods.
- The longer the horizon, the more elaborate the punishment structures we can sustain.

Tradeoff: horizon length and enforcement

- As T increases, future payoffs gain weight and deviations become less attractive.
- However, the final period still unravels unless there are alternative punishment equilibria.
- Finite repetition cannot match the richness of the infinite horizon, but it does enlarge the equilibrium set compared to the single-shot game.

Comparison: repeated NE vs stage NE

- Repeated NE can condition on histories, while stage NE cannot.
- The availability of credible punishment equilibria defines what repeated NE are feasible.
- Finite repetition allows partial cooperation even when stage cooperation is not an equilibrium.

Playing Non-Nash Action Profiles

- Repetition allows players to play action profiles that are not stage Nash equilibria.
- Example (Stag Hunt): (Hare,Stag) is not a Nash equilibrium, but can be sustained through threats.
- Period 1: Play (Stag,Stag). Period 2: Play (Hare,Stag).
- If Player 2 deviates in period 2, revert to (Hare,Hare) in future periods.
- The key: future punishment makes it rational to play non-equilibrium actions today.
- This dramatically expands the set of achievable outcomes beyond stage game equilibria.

Axelrod's Tournament

The Evolution of Cooperation

The Tournament Setup

- Robert Axelrod (1980) invited game theorists to submit programs for a repeated Prisoner's Dilemma tournament.
- Strategies played against each other in a round-robin format.
- Each match consisted of exactly 200 rounds—a finite repeated game.
- Payoffs were standard PD values (similar to our Teamwork game).

The Winner: Tit-for-Tat

- The winner was the simplest program submitted by Anatol Rapoport.
- **Tit-for-Tat (TFT)**:
  1. Cooperate on the first move.
  2. Thereafter, copy the opponent's previous move.
- It beat complex strategies that tried to be "sneaky" or "exploitative."

Why Tit-for-Tat Won? (The 4 Properties)

Axelrod identified four properties of successful strategies:

1. **Nice**: Never defect first. (Prevents unnecessary conflict).
2. **Retaliatory**: Punish defection immediately. (Discourages exploitation).
3. **Forgiving**: Return to cooperation if the opponent stops defecting. (Avoids endless vendettas).
4. **Clear**: Be easy for the opponent to understand. (Encourages cooperation).

Evolutionary Stability

- In evolutionary simulations, strategies that score poorly "die out," while successful ones replicate.
- "Nasty" strategies might win initially by exploiting naive cooperators, but they eventually run out of victims and fight each other.
- Cooperative strategies like TFT thrive when they can interact with other cooperators.
- Cooperation can emerge and persist even in a selfish world.

Exercises preview

- Compute the defection path of the Teamwork Game when $T = 4$ and identify all subgame perfect equilibria.
- Design a threat strategy for the Teamwork stage game that sustains (C,C) for $T = 5$ and compare payoffs.
- Analyze a stage game with multiple Nash equilibria and describe which equilibrium serves as punishment.

Common pitfalls

- Using punishments that are not themselves Nash equilibria, which destroys credibility.
- Ignoring off-path histories when specifying strategies.
- Assuming cooperation survives without ensuring the threat is sufficiently costly.

Takeaways & looking ahead

- Finite repetition enlarges the equilibrium set by leveraging history-dependent strategies.
- Stage games with multiple Nash equilibria provide ready-made targets for punishment paths.
- Next time we move to infinite repetition, introduce dynamic programming, and study Folk theorems.