BCSE Game Theory 07-01

Subgame Perfect Equilibrium and Information Sets

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Today's Agenda

Complete-Information Dynamics

Today's Objectives

▶ Review extensive-form game fundamentals from Lectures 06-01 and 06-02.

▶ Define subgame perfect Nash equilibrium (SPNE) formally.

▶ Understand how information sets affect subgame identification.

▶ Apply backward induction to find SPNE in games with complete information.

▶ Analyse commitment and monitoring problems using SPNE.

Where Lecture 07 Starts

▶ Lectures 06-01 and 06-02 introduced extensive-form games, backward induction, and information sets.

▶ We now formalise the equilibrium concept that captures credible play: subgame perfect Nash equilibrium.

▶ Complete information means all players know payoffs, action sets, and possible histories.

▶ Non-degenerate information sets can still arise from simultaneous moves or delayed observation, even with complete information.

▶ Our examples come from entry deterrence, capacity commitment, and procurement monitoring.

Extensive Form Game Refresher

Building on Lectures 06

What We Learned in 06-01

- **Extensive-form games** represent sequential interactions with timing, observability, and conditional choices.
- Key components: players N, histories $H \cup Z$, player function $P(h)$, action sets $A(h)$, and payoffs $u_i(z)$.

Definition: Information Set

An information set I_i for player i is a collection of decision nodes where:

1. Player i moves at all nodes in I_i.
2. Player i cannot distinguish which node in I_i has been reached.
3. The same actions are available at all nodes in I_i.
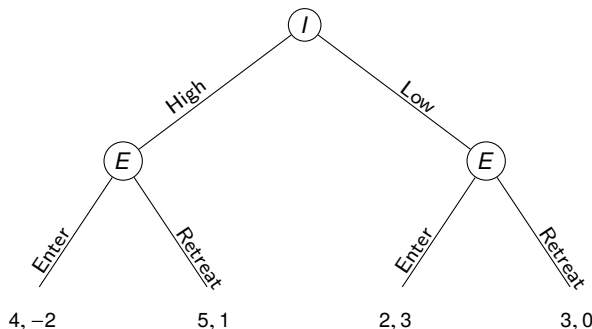

What We Learned in 06-01

▶ **Perfect information** means every decision node is in a singleton information set (all previous moves are observed).

▶ **Backward induction** solves finite perfect-information games by evaluating terminal nodes first, then working upward.

▶ Backward induction ensures **credibility**: each player's action is optimal at every node reached.

What We Learned in 06-02

▶ **Information sets** group decision nodes that a player cannot distinguish when acting.

▶ **Perfect information**: every information set is a singleton.

▶ **Imperfect information**: at least one information set contains multiple nodes.

▶ Strategies are **contingent plans** specifying an action for every information set.

▶ Converting extensive-form to normal-form reveals all Nash equilibria, including non-credible ones.

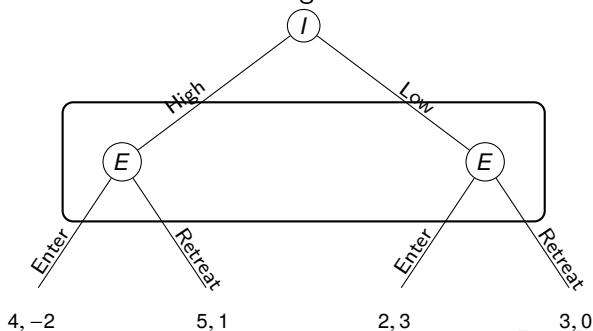▶ **Subgame perfection** filters equilibria to those that are credible in every subgame.

Perfect Information Example: Investment Game

▶ Incumbent *I* chooses High or Low investment.

▶ Entrant *E* observes the choice and decides Enter or Retreat.

▶ All moves are perfectly observed—each information set is a singleton.

Imperfect Information: Delayed Observation

▶ Suppose I's investment decision is not immediately public.
▶ Entrant E must move before learning whether High or Low was chosen.
▶ E's information set contains both nodes—E cannot distinguish them.
▶ This creates a non-degenerate information set even though payoffs are common knowledge.

Information Sets: Formal Definition

▶ **Degenerate information set**: contains exactly one node (perfect observation).

▶ **Non-degenerate information set**: contains at least two nodes (imperfect observation).

▶ Complete information means payoffs are known, but observation may still be imperfect.

Why Information Sets Matter

▶ They determine what players know when making decisions.

▶ They affect which subgames can be identified.

▶ They influence which equilibrium concepts apply.

▶ Non-degenerate information sets require beliefs and sequential rationality (to be formalised later).

▶ For now, we focus on complete information games where payoffs are common knowledge.

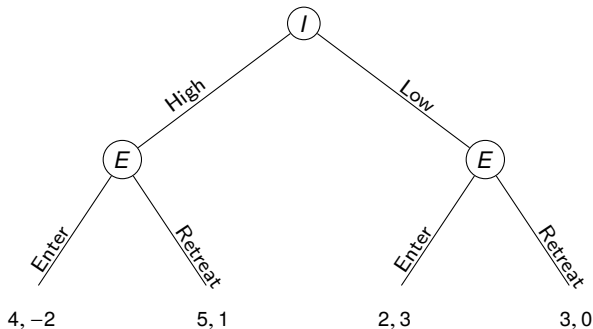# Subgames and Subgame Perfect Equilibrium

Subgame: Formal Definition

Definition: Subgame
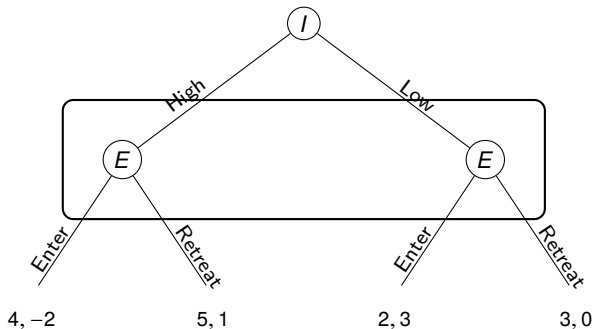
A subgame is a portion of the game tree that:

1. Begins at a single decision node whose information set is a singleton.
2. Contains all successors of that node.
3. Does not break any information sets (if a node is in the subgame, all nodes in its information set must be in the subgame).

▶ The starting node must be a singleton information set—otherwise players wouldn't know which branch they're in.

▶ The entire game is always a subgame (starting at the root).

▶ Terminal nodes are trivial subgames (no further decisions).

Identifying Subgames: Investment Game

- ▶ The entire game is a subgame (root node).
- ▶ Each of E's decision nodes starts a subgame (singleton information sets).
- ▶ Terminal nodes are trivial subgames.
- ▶ Total: 5 subgames (1 full game + 2 at E's nodes + 2 terminal subgames).

Subgames with Imperfect Information

- ▶ The entire game is a subgame (root node).
- ▶ E's information set is non-degenerate—cannot start a subgame there.
- ▶ Only the root and terminal nodes are subgames.
- ▶ Total: 3 subgames (1 full game + 2 terminal subgames).

Subgame Perfect Nash Equilibrium: Definition

Definition: Subgame Perfect Nash Equilibrium (SPNE)

A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a subgame perfect Nash equilibrium if:

1. s^* is a Nash equilibrium in the entire game.

2. For every subgame, the restriction of s^* to that subgame is a Nash equilibrium in the subgame.

▶ SPNE requires Nash equilibrium play in **every** subgame, not just the whole game.

▶ This eliminates non-credible threats and promises.

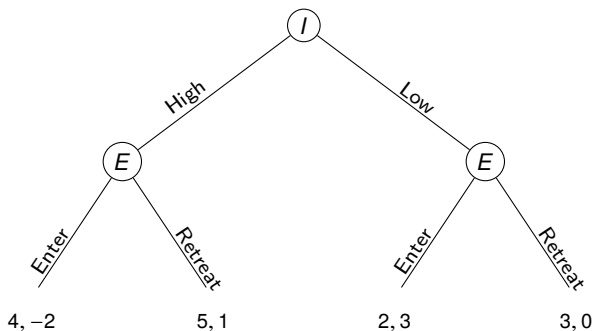▶ Backward induction in perfect-information games yields the unique SPNE (when payoffs are generic).

Why SPNE Matters

▶ **Credibility**: Players' strategies must be optimal at every node, even off the equilibrium path.

▶ **Refinement**: SPNE is a refinement of Nash equilibrium— every SPNE is a NE, but not every NE is SPNE.

▶ **Predictive power**: SPNE rules out equilibria supported by non-credible threats.

▶ **Backward induction**: In perfect-information games, backward induction finds the SPNE.

Finding SPNE: Backward Induction Algorithm

1. Start at terminal nodes: record payoffs.
2. Move to parent nodes: for each node, find the action that maximises the current player's payoff.
3. Replace subgames with their equilibrium payoffs.
4. Continue upward until reaching the root.
5. The sequence of chosen actions defines the SPNE strategy profile.

▶ This algorithm works for finite perfect-information games.
▶ It ensures the resulting profile is a Nash equilibrium in every subgame.
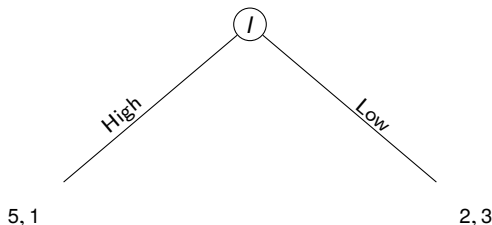
SPNE in Investment Game: Step 1

- ▶ Start at E's nodes (final decision nodes).
- ▶ After High: Enter gives E payoff -2, Retreat gives 1. E chooses Retreat.
- ▶ After Low: Enter gives E payoff 3, Retreat gives 0. E chooses Enter.

SPNE in Investment Game: Step 2

▶ After solving E's subgames, we can reduce the game tree.

▶ I now faces a simpler decision: High → $(5, 1)$ vs. Low → $(2, 3)$.

▶ I prefers High (payoff $5 > 2$).

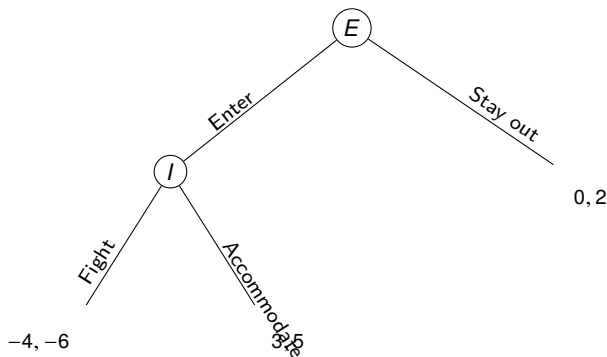▶ This completes the backward induction analysis.

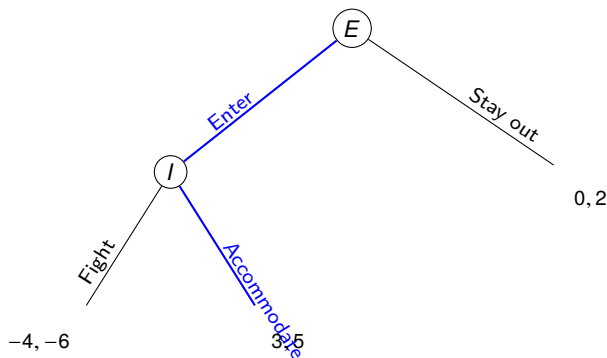# SPNE in Investment Game: Reduced Tree

▶ SPNE: *I* chooses High, *E* chooses (Retreat after High, Enter after Low).

▶ Outcome: High investment, *E* retreats, payoffs $(5, 1)$.

Entry Deterrence Game

▶ Entrant *E* decides whether to Enter or Stay out.

▶ If entry occurs, Incumbent *I* chooses Fight or Accommodate.

▶ Perfect information: all moves are observed.

SPNE in Entry Deterrence: Backward Induction

- ▶ Step 1: At I's node after entry, compare Fight (-6) vs. Accommodate (5). I chooses Accommodate.
- ▶ Step 2: E compares Enter (leading to Accommodate, payoff 3) vs. Stay out (payoff 0). E chooses Enter.
- ▶ SPNE: E enters, I accommodates, payoffs $(3, 5)$.
- ▶ Blue branches show the optimal responses along the SPNE path.

Non-Credible Threats Eliminated

▶ The profile (Stay out, Fight) is a Nash equilibrium in the normal form.

▶ But it is not SPNE: if entry occurred, I would deviate to Accommodate.

▶ SPNE requires that I's strategy be optimal at I's node, even though entry doesn't occur in equilibrium.

▶ This is the key difference between NE and SPNE: credibility at all nodes.

Commitment Games

Quantity Commitment Example

- Leader *L* chooses production level: High production or Low production.
- Follower *F* observes *L*'s choice and chooses: High production or Low production.
- Perfect information: all moves are observed.

SPNE in Commitment Game: Analysis

- ▶ Step 1: At F's nodes, find best responses.
- ▶ After High production: High production gives F payoff 2, Low production gives 4. F chooses Low production.
- ▶ After Low production: High production gives F payoff 6, Low production gives 4. F chooses High production.
- ▶ Step 2: L compares High production $\rightarrow$ (6, 4) vs. Low production $\rightarrow$ (3, 6)

Commitment Game: SPNE Outcome

▶ After High production: F chooses Low production → $(6, 4)$.

▶ After Low production: F chooses High production → $(3, 6)$.

▶ L compares: High production yields 6, Low production yields 3.

▶ L prefers High production, leading to $(6, 4)$.

▶ The commitment is effective: L's choice of high production changes F's best response from High to Low production.

▶ SPNE: L chooses High production, F responds with Low production, payoffs $(6, 4)$.

When Commitment Works

▶ Commitment must change the follower's incentives.

▶ In this example, L's commitment to high production makes low production more attractive for F (avoiding intense competition).

▶ If L chose low production, F would respond with high production to capture market share.

▶ The key is that commitment must be **observable** and **irreversible** (or costly to reverse).

▶ In our example, commitment successfully changed F's best response, demonstrating the strategic value of moving first.

Moral Hazard with Imperfect Monitoring

Hidden-Action Setup

- ▶ Principal P designs a contract that pays W_S after a success signal S and W_F after a failure signal F.

- ▶ Agent A privately chooses effort $e \in \{e_H, e_L\}$ with costs $c_H > c_L \geq 0$.

- ▶ Signals are public but imperfect: P observes only S or F and never the effort itself.

- ▶ Success probabilities satisfy $p_H = \Pr(S \mid e_H) > p_L = \Pr(S \mid e_L)$.

- ▶ Agent is risk-averse with utility $u(W) = \ln W$, while the principal is risk-neutral over revenues.

Timeline and Payoffs

▶ Stage 1: P fixes (W_S, W_F) anticipating A's response.

▶ Stage 2: A observes the contract and selects effort e_H or e_L.

▶ Stage 3: Nature draws S or F with probabilities p_H or p_L depending on the chosen effort.

▶ Stage 4: P observes the signal, pays the contingent wage, and realises revenue R_S or R_F.

Signal Structure and Contract Payoffs

	S	F
High effort e_H	p_H	$1 - p_H$
Low effort e_L	p_L	$1 - p_L$

▶ Principal payoff from S: $R_S - W_S$; from F: $R_F - W_F$.

▶ Agent payoff from S: $\ln W_S - c(e)$; from F: $\ln W_F - c(e)$.

▶ Typical assumption: $R_S > R_F$ and $W_S \geq W_F$ (bonus after success).

Signal Structure and Contract Payoffs

▶ For convenience define the principal's expected profit functions:

$$\Pi_H(W_S, W_F) \stackrel{\text{def}}{=} p_H(R_S - W_S) + (1 - p_H)(R_F - W_F),$$

$$\Pi_L(W_S, W_F) \stackrel{\text{def}}{=} p_L(R_S - W_S) + (1 - p_L)(R_F - W_F).$$

These capture the project's expected profit under high vs. low effort.

▶ Imperfect monitoring: only the signal is observable, so wages are conditioned on S or F alone.

▶ Contract must trade off risk sharing and incentives because A dislikes wage dispersion.

Agent Incentives with $\ln W$ Utility

▶ Expected utility from e_H:

$$U_H = p_H \ln W_S + (1 - p_H) \ln W_F - c_H.$$

▶ Expected utility from e_L:

$$U_L = p_L \ln W_S + (1 - p_L) \ln W_F - c_L.$$

▶ Incentive compatibility (IC) for high effort:

$$U_H \geq U_L \quad \Longleftrightarrow \quad (p_H - p_L)(\ln W_S - \ln W_F) \geq c_H - c_L.$$

▶ Because $\ln(\cdot)$ is concave, the marginal value of extra wage is larger in the failure state; risk aversion pushes wages toward smoothing.

Implementing High Effort

▶ Rearranging the IC constraint yields a bonus ratio condition:

$$\ln\left(\frac{W_S}{W_F}\right) \geq \frac{c_H - c_L}{p_H - p_L} \quad \Rightarrow \quad \frac{W_S}{W_F} \geq \exp\left(\frac{c_H - c_L}{p_H - p_L}\right).$$

▶ Higher effort costs or smaller likelihood ratios ($p_H - p_L$) require a larger success bonus.

▶ Participation constraint (PC) adds a lower bound on the certainty equivalent:

$$p_H \ln W_S + (1 - p_H) \ln W_F - c_H \geq \bar{U}.$$

▶ Contract design problem: maximise P's expected revenue subject to IC and PC, recognising that A's risk aversion penalises volatile wages.

▶ Imperfect public monitoring links incentives entirely to observable signals, never to the hidden action itself.

Principal's Optimisation and Low-Effort Benchmark

▶ Using Π_H and Π_L, P's problem when targeting e_H is:

$$\max_{W_S, W_F} \Pi_H(W_S, W_F) \quad \text{s.t.} \quad U_H \geq U_L, \ U_H \geq \bar{U}, \ W_S, W_F > 0.$$

▶ If the IC constraint is expensive to satisfy (large $c_H - c_L$ or small $p_H - p_L$), the bonus spread $\frac{W_S}{W_F}$ required may wipe out the incremental surplus $\Pi_H - \Pi_L$.

▶ In that case, P may deliberately implement low effort: solve $\max_{W_S, W_F} \Pi_L(W_S, W_F)$ subject only to $U_L \geq \bar{U}$, leading to flatter wages (possibly $W_S = W_F$) and lower total pay but avoiding incentive costs.

▶ Moral hazard thus creates a region where it is optimal to accept low effort and pay a low, relatively safe wage rather than strain to induce e_H.

Additional Examples

Example: Sequential Bargaining

- ▶ Two players, A and B, bargain over a pie of size 1.
- ▶ A makes an offer: A gets x, B gets $1 - x$ (where $0 \leq x \leq 1$).
- ▶ B accepts or rejects.
- ▶ If B accepts, payoffs are $(x, 1 - x)$.
- ▶ If B rejects, both get 0 (no agreement).
- ▶ Perfect information: all moves are observed.

SPNE in Sequential Bargaining

▶ Step 1: At B's node, B accepts if $1 - x \geq 0$ (always true for $x \leq 1$).

▶ So B accepts any offer $x < 1$, and is indifferent at $x = 1$.

▶ Step 2: A chooses x to maximise payoff.

▶ A offers $x = 1$ (or arbitrarily close to 1), B accepts, payoffs $(1, 0)$ (or $(1 - \epsilon, \epsilon)$).

▶ SPNE: A gets (almost) the entire pie.

▶ This is the **ultimatum game** outcome under standard assumptions.

Cross-Cultural Differences in Ultimatum Games

- ▶ Experimental evidence shows systematic differences across cultures.
- ▶ **Western countries** (US, UK): Proposers offer 40–50%; responders reject offers below 20–30%.
- ▶ **Japan**: Offers tend to be more equal (around 50%); rejection rates similar to Western countries.
- ▶ **Vietnam**: Limited experimental data, but some studies suggest higher acceptance of unequal offers.
- ▶ These differences challenge the standard SPNE prediction of near-zero offers.

Why Do Cultural Differences Matter?

▶ **Fairness norms**: Different cultures have different notions of fairness.

▶ **Social preferences**: Concerns for others' welfare vary across societies.

▶ **Reputation effects**: In collectivist cultures, reputation may matter more.

▶ **Experimental context**: Laboratory vs. field settings may affect behavior differently.

▶ Standard game theory assumes self-interested players; real behavior incorporates social preferences.

Key Experimental Findings

▶ Roth et al. (1991): Compared bargaining in Jerusalem, Ljubljana, Pittsburgh, and Tokyo.

   ▶ Japanese proposers made more equal offers than Americans.
   ▶ Cultural differences persisted even with monetary incentives.

▶ Henrich et al. (2001): Studied 15 small-scale societies.

   ▶ Found wide variation in ultimatum game behavior.
   ▶ Market integration correlated with more "selfish" behavior.

▶ These findings suggest that cultural background affects strategic behavior beyond pure self-interest.

Key Experimental Findings

References

Roth, A. E., Prasnikar, V., Okuno-Fujiwara, M., & Zamir, S. (1991). Bargaining and market behavior in Jerusalem, Ljubljana, Pittsburgh, and Tokyo: An experimental study. *American Economic Review*, 81(5), 1068–1095.

Henrich, J., et al. (2001). In search of homo economicus: Behavioral experiments in 15 small-scale societies. *American Economic Review*, 91(2), 73–78.

Example: Two-Period Bargaining

▶ If B rejects in period 1, B makes an offer in period 2.

▶ If A accepts, payoffs are discounted: $(\delta x, \delta(1 - x))$ where $\delta \in (0, 1)$.

▶ If A rejects, both get 0.

▶ Perfect information: all moves are observed.

▶ This is a finite-horizon bargaining game with discounting.

SPNE in Two-Period Bargaining: Backward Induction

- ▶ Step 1 (Period 2): A accepts if $\delta x \geq 0$ (always true). B offers $x = 0$, A accepts, payoffs $(\delta \cdot 0, \delta \cdot 1) = (0, \delta)$.

- ▶ Step 2 (Period 1): A offers x such that B gets at least δ (otherwise B rejects).

- ▶ So A offers $x = 1 - \delta$, B gets δ, B accepts.

- ▶ SPNE: A offers $(1 - \delta, \delta)$ in period 1, B accepts.

- ▶ Outcome: agreement in period 1, payoffs $(1 - \delta, \delta)$.

Example: Stackelberg Competition

▶ Two firms, Leader L and Follower F, choose **discrete** capacities: LARGE (Q_H) or SMALL (Q_L).

▶ The LARGE option is aggressive: it expands output enough to slash the rival's margin but is itself costly.

▶ The SMALL option is conservative: it avoids cost and preserves higher prices.

▶ We calibrate the profits (in millions) so that simultaneous quantity choices yield Cournot-like payoffs, while sequential commitment exposes a sharp contrast.

▶ L moves first and the commitment is observable; F reacts after seeing L's capacity.

SPNE in Stackelberg: Follower's Best Response

▶ Payoff table (Leader rows, Follower columns):

	F: Large	F: Small
L: Large	$(1,1)$	$(9,2)$
L: Small	$(2,9)$	$(6,6)$

▶ Contrast in F's incentives: when L stays Small, F prefers Large ($9 > 6$); if L jumps to Large, F switches to Small ($2 > 1$).

▶ Observability of L's move therefore flips F's best response in a highly visible, discrete way.

SPNE in Stackelberg: Leader's Choice

▶ Simultaneous (Cournot) play: best replies intersect twice— (LARGE, SMALL) with $(9, 2)$ and (SMALL, LARGE) with $(2, 9)$—mirror images that require coordination.

▶ Sequential play eliminates the symmetry: committing to LARGE drives F to SMALL→payoffs lock in at $(9, 2)$.

▶ If L reneged to SMALL→ F would return to LARGE→payoffs slide to the follower-favoured $(2, 9)$ equilibrium.

▶ The mirrored payoff structure makes the commitment value transparent: L secures the aggressive equilibrium and avoids the adverse mirror outcome.

▶ SPNE: L chooses LARGE; F observes it and chooses SMALL.

Definition: First-Mover Advantage

Definition: First-Mover Advantage

Consider a sequential game in which a player (the leader) chooses an action before others observe it. The leader enjoys a first-mover advantage if moving first yields a strictly higher equilibrium payoff than the payoff the same player would obtain when actions are chosen simultaneously or after observing rivals. The advantage stems from credible commitment that shifts followers' best responses in a favourable direction.

▶ In Stackelberg quantity competition, the leader's commitment to a large output reduces the follower's optimal quantity, raising the leader's profit relative to Cournot.

▶ First-mover advantages require observability and limited ability to revise the initial move; otherwise, commitment cannot influence followers.

Takeaways

Key Messages

▶ **Subgame perfect Nash equilibrium** requires Nash equilibrium play in every subgame.

▶ SPNE eliminates non-credible threats and promises.

▶ **Backward induction** finds SPNE in perfect-information games.

▶ **Information sets** determine which subgames exist and what players know.

▶ Complete information (known payoffs) does not imply perfect information (observed moves).

▶ Commitment and monitoring can be analysed using SPNE, but effectiveness depends on changing incentives.

Critical Questions for Analysis

▶ What are the subgames? (Start from singleton information sets.)

▶ What is the SPNE? (Use backward induction.)

▶ Are threats credible? (Check if they're optimal at the relevant nodes.)

▶ Does commitment work? (Check if it changes best responses.)

▶ Is monitoring worth it? (Compare payoffs with and without monitoring.)

Reading and Next Steps

▶ Osborne and Rubinstein (1994), Chapters 6–7: extensive forms, subgames, and SPNE.

▶ Fudenberg and Tirole (1991), Chapter 3: dynamic games and equilibrium concepts.

▶ Next lecture (07-02) will extend these tools to more complex commitment and monitoring problems.

▶ Practice: solve SPNE in games from your project domain.