Game Theory 13-01

Review: Games of Complete Information

BCSE Game Theory

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Part 1: Static Games of Complete Information

IESDS Nash Equilibrium

Review: Strategic Form Games

1. **Dominance**: A strategy is strictly dominated if another strategy always yields a strictly higher payoff, regardless of opponents' actions.
2. **IESDS**: Rational players will never play strictly dominated strategies. We can iteratively remove them to simplify the game.
3. **Nash Equilibrium (NE)**: A profile where no player has an incentive to deviate unilaterally.
   - IESDS preserves NE. If IESDS leads to a unique outcome, that is the unique NE.
   - If not, we solve for NE in the simplified game.

Exercise 1: 4x4 Game

Problem

Consider the following game between Player 1 (Row) and Player 2 (Col).

	L	C	R	X
T	2, 3	5, 2	1, 2	2, 0
M	2, 1	8, 8	3, 6	9, 3
B	3, 0	9, 6	2, 8	0, 5
Y	1, 1	0, 2	0, 1	1, 1

Exercise 1: 4x4 Game (Questions)

Task

Questions:

1. Apply **IESDS** to reduce this matrix to a 2x2 game. State clearly which strategy dominated which in each step.
2. Find all **Pure Strategy Nash Equilibria (PSNE)** in the reduced game.
3. Find the **Mixed Strategy Nash Equilibrium** in the reduced game.

Ex 1: Solution (IESDS Step 1-2)

	L	C	R	X
T	2, 3	5, 2	1, 2	2, 0
M	2, 1	8, 8	3, 6	9, 3
B	3, 0	9, 6	2, 8	0, 5
Y	1, 1	0, 2	0, 1	1, 1

1. **Row Dominance**: Compare Y with T.
   - T pays $\{2, 5, 1, 2\}$, Y pays $\{1, 0, 0, 1\}$.
   - $T > Y$ in all cases. **Eliminate** Y.
2. **Col Dominance**: Compare X with C.
   - Remaining rows T, M, B: C pays $\{1, 8, 6\}$, X pays $\{0, 3, 5\}$.
   - $C > X$ strictly. **Eliminate** X.

Ex 1: Solution (IESDS Step 3-4)

Remaining: T, M, B vs L, C, R.

	L	C	R
T	2, 3	5, 2	1, 2
M	2, 1	8, 8	3, 6
B	3, 0	9, 6	2, 8

3. **Row Dominance**: Compare T and B.
   - $B(3, 9, 2)$ vs $T(2, 5, 1)$.
   - B strictly dominates T $(3 > 2, 9 > 5, 2 > 1)$. **Eliminate** T.
4. **Col Dominance**: Compare L with C.
   - Remaining rows M, B: $C(8, 6)$ vs $L(1, 0)$.
   - $C > L$ strictly. **Eliminate** L.

Ex 1: Solution (Reduced 2x2)

After eliminating Y, X, T, L, we are left with:

	C	R
M	8, 8	3, 6
B	9, 6	2, 8

Note: This 2x2 matrix matches the inputs for the Nash Equilibrium calculation in the next slide (labeled U, D vs L, R).

Ex 1: Solution (Reduced 2x2 PSNE)

Final 2x2 Game: **Final 2x2 Game**:

	L	R
U	9, 6	2, 8
D	8, 8	3, 6

(Using U, D vs L, R for clarity).

2. Pure Nash Equilibria:

- L: $U(9) \succ D(8)$. P1 picks U. P2 response to U? $R(8) \succ L(6)$. $\rightarrow (U, R)$.
- R: $D(3) \succ U(2)$. **Check**: $U(2)$ vs $D(3)$. P1 picks D.
- Response to D? $L(8) \succ R(6)$. $\rightarrow (D, L)$.
- Cycle: $U \rightarrow R \rightarrow D \rightarrow L \rightarrow U$.
- **No Pure Strategy Nash Equilibrium**.

Ex 1: Solution (Mixed Strategy NE)

Since there is no PSNE, we look for MSNE. Let P1 play U with prob p, P2 play L with prob q.

1. **P2's Indifference**:

$$\begin{aligned} E[U_2(L)] &= E[U_2(R)] \\ 6p + 8(1-p) &= 8p + 6(1-p) \\ 8 - 2p &= 2p + 6 \\ 2 &= 4p \implies p = 0.5 \end{aligned}$$

2. **P1's Indifference**:

$$\begin{aligned} E[U_1(U)] &= E[U_1(D)] \\ 9q + 2(1-q) &= 8q + 3(1-q) \\ 7q + 2 &= 5q + 3 \\ 2q &= 1 \implies q = 0.5 \end{aligned}$$

Result: Mixed Strategy NE is $((0.5, 0.5), (0.5, 0.5))$.

Part 2: Static Games of Complete Information

3-Firm Cournot Competition

Review: Dynamic Games Subgames

- **Game Tree (Extensive Form)**:
  - **Nodes**: Decision points and terminal endpoints.
  - **Information Sets**: Collection of nodes where a player cannot distinguish which node allowed.
- **Subgame**: A part of the game that:
  - Starts at a singleton information set (initial node).
  - Includes all successors of that node.
  - Does not cut across information sets.

Review: Subgame Perfect Equilibrium (SPE)

Why Nash Equilibrium is insufficient?

NE allows for **non-credible threats**: optimal actions in subgames that are never reached (off-equilibrium path), which may not be rational if reached.

Subgame Perfect Equilibrium (SPE):

- A strategy profile that induces a Nash Equilibrium in **every subgame** (reached or not).
- **Backward Induction**: The algorithm to find SPE in finite games of perfect information.

Review: Cournot Competition

- **Model**: Firms compete on **Quantity** (q_i) **simultaneously**.
- **Inverse Demand**: $P(Q) = A - Q$, where $Q = \sum q_i$.
- **Nash Equilibrium**: Each firm sets q_i to maximize profit given q_{-i}.
- The intersection of reaction functions gives the Cournot-Nash Equilibrium.

Exercise 2: 3-Firm Cournot

Problem

Demand: $P = 120 - Q$, where $Q = q_1 + q_2 + q_3$. Marginal Costs: c_1, c_2, c_3.

1. Write down the profit function for Firm 1.
2. Derive the Reaction Function (Best Response) for Firm 1, $q_1^*(q_2, q_3)$.
3. Solve for the equilibrium quantities q_1^*, q_2^*, q_3^* in terms of c_1, c_2, c_3.
4. Calculate the numerical values if $c_1 = c_2 = c_3 = 20$.

Ex 2: Solution (1 & 2)

1. **Profit Function**:

$$\pi_1 = (P - c_1)q_1 = (120 - (q_1 + q_2 + q_3) - c_1)q_1$$

2. **Reaction Function**: FOC w.r.t q_1:

$$\frac{\partial \pi_1}{\partial q_1} = 120 - 2q_1 - q_2 - q_3 - c_1 = 0$$

$$2q_1 = 120 - c_1 - (q_2 + q_3)$$

$$q_1^*(q_{-1}) = \frac{120 - c_1 - q_2 - q_3}{2}$$

Or,

$$q_1^* + Q = 120 - c_1$$

Ex 2: Solution (3)

By symmetry of the equations, we have:

$$\begin{aligned} q_1 + Q &= 120 - c_1 \\ q_2 + Q &= 120 - c_2 \\ q_3 + Q &= 120 - c_3 \\ 4Q &= 360 - \sum c_i \\ Q &= 90 - \frac{\sum c_i}{4} \end{aligned}$$

Now use Eq(1): $q_1 + Q = 120 - c_1 \implies q_1 = 120 - c_1 - Q$.
Substitute Q:

$$q_1^* = 120 - c_1 - \left(90 - \frac{\sum c_i}{4}\right) = 30 - c_1 + \frac{\sum c_i}{4}$$

Ex 2: Solution (4)

If $c_1 = c_2 = c_3 = 20$:

$$q_1^* = \frac{120 - 4c_1 + \sum c_i}{4} = \frac{120 - 80 + 60}{4} = 25$$

Symmetry: $q_1^* = q_2^* = q_3^* = 25$.
Verification:

- Total $Q = 75 \implies$ Price $P = 120 - 75 = 45$. $MC = 20$.
- Firm 1's residual demand: $P(Q) = (120 - q_2 - q_3) - q_1 = 70 - q_1$.
- Total Revenue: $TR = (70 - q_1)q_1$.
- Marginal Revenue: $MR = 70 - 2q_1$.
- Optimality: $MR = MC \implies 70 - 2q_1 = 20 \implies q_1 = 25$. **Correct**.

Part 3: Dynamic Games of Complete Information

2-Firm Stackelberg Competition

Exercise 3: 2-Firm Stackelberg

Problem: Dynamic Quantity Competition

Demand: $P = 120 - Q$, where $Q = q_1 + q_2$. Costs $c_1 = c_2 = 20$.

Timing:

1. **Stage 1**: Firm 1 (Leader) chooses $q_1 \geq 0$.
2. **Stage 2**: Firm 2 (Follower) observes q_1 and chooses q_2.

Questions:

1. Solve for the Subgame Perfect Equilibrium (SPE).
2. Compare the Leader's profit to the Follower's. Does moving first help?

Ex 3: Solution (Stage 2: Follower)

Backward Induction: Start from the last stage (Stage 2). Given q_1, Firm 2 maximizes profit on residual demand:

$$\pi_2 = (P - c_2)q_2 = ((120 - q_1) - q_2 - 20)q_2 = (100 - q_1 - q_2)q_2$$

FOC w.r.t q_2:

$$\frac{\partial \pi_2}{\partial q_2} = 100 - q_1 - 2q_2 = 0$$

Follower's Best Response function:

$$q_2^*(q_1) = \frac{100 - q_1}{2}$$

Ex 3: Solution (Stage 1: Leader)

Firm 1 anticipates the follower's response. Total Output $Q = q_1 + q_2^* = q_1 + \frac{100-q_1}{2} = \frac{100+q_1}{2}$. Price $P = 120 - Q = 120 - \frac{100+q_1}{2} = \frac{140-q_1}{2}$. Firm 1 Profit $\pi_1 = (P - 20)q_1$:

$$\pi_1 = \left(\frac{140 - q_1}{2} - 20\right) q_1 = \frac{100 - q_1}{2} q_1$$

Max w.r.t q_1:

$$\frac{d\pi_1}{dq_1} = \frac{100 - 2q_1}{2} = 0 \implies \mathbf{q_1^* = 50}$$

Ex 3: Results & First Mover Advantage

Equilibrium Quantities:

- Leader: $q_1^* = 50$.
- Follower: $q_2^* = \frac{100-50}{2} = 25$.

Profits:

- Total $Q = 75$. Price $P = 120 - 75 = 45$.
- Margin $P - c = 45 - 20 = 25$.
- Leader: $\pi_1 = 25 \times 50 = 1250$.
- Follower: $\pi_2 = 25 \times 25 = 625$.

Conclusion

First Mover Advantage: The Leader commits to a high quantity (50), forcing the Follower to produce less (25). Compare to symmetric Cournot: each would produce $\frac{100}{3} \approx 33.3$.

Part 4: Capacity Constraints

Bertrand Paradox Resolution

Exercise 4: Bertrand with Capacity

Problem

Two firms compete in prices. Demand $D(p) = 100 - p$.

- Firm A: MC $c_A = 10$. Capacity $K_A = 30$.
- Firm B: MC $c_B = 12$. Unlimited Capacity. **(Asymmetric Costs)**.

Questions:

1. Consider Firm B's best response if Firm A sets a high price p_A. Should B undercut or accept residual demand?
2. Find the Nash equilibrium price p^* where B is indifferent between undercutting and residual monopoly.
3. Calculate Firm A's profit at this equilibrium.
4. Suppose Firm A expands capacity to $K_A = 100$. What is the new equilibrium and profit?

Ex 4: Solution (1: Firm B's Choice)

Suppose A sets price p_A. B has two main strategies:

1. **Residual Monopoly**: Accept A fills $K_A = 30$.
   - Residual: $D_r(p) = (100 - p) - 30 = 70 - p$.
   - Max $\pi_B^{Res} = (p - 12)(70 - p)$.
   - FOC $\implies p^* = 41$. Profit: $(41 - 12)(70 - 41) = 29 \times 29 = 841$.
2. **Price War (Undercut)**: Set $p_B = p_A - \epsilon$.
   - B captures whole market $D(p_A)$.
   - Profit $\pi_B^{Cut} \approx (p_A - 12)(100 - p_A)$.

Comparison at $p_A = 41$:
$\pi_B^{Cut}(41) = (41 - 12)(100 - 41) = 29 \times 59 = 1711$. Since $1711 > 841$, B will undercut! $p_A = 41$ is **not** an equilibrium.

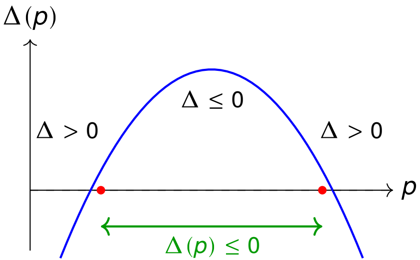
Ex 4: Graphical Analysis

Firm A must lower p_A until B prefers (or is indifferent to) Residual Monopoly.

$$\pi_B^{Cut}(p) \le \pi_B^{Res} \implies (p-12)(100-p) \le 841$$

$$-p^2 + 112p - 1200 \le 841 \implies p^2 - 112p + 2041 \ge 0$$

Profit Difference: $\Delta(p) = \pi_B^{Cut}(p) - \pi_B^{Res} = (p-12)(100-p) - 841$



Interpretation: B prefers Residual Monopoly when $p \in [22.9, 89.1]$.

Ex 4: Solution (2: Equilibrium)

Roots: $p = \frac{112 \pm \sqrt{112^2 - 4 \times 2041}}{2} = \frac{112 \pm \sqrt{12544 - 8164}}{2} = \frac{112 \pm \sqrt{4380}}{2}$.
$\sqrt{4380} \approx 66.18$. Lower stable root: $p \approx \frac{112 - 66.18}{2} = \frac{45.82}{2} \approx 22.91$.

- **Equilibrium**: $p_A = p_B \approx 22.9$.
- A sells 30 units (capacity). B sells residual.
- **A's Profit**: $\pi_A = (22.9 - 10) \times 30 = 12.9 \times 30 = \mathbf{387}$.

Ex 4: Solution (3: The Paradox)

What if A expands capacity to $K_A = 100$ (Unconstrained)?

- A has cost advantage ($10 < 12$) and sufficient capacity to serve the whole market (Max Demand ≤ 100).
- **Standard Asymmetric Bertrand**: A undercuts B to $p \approx c_B = 12$.
- A serves entire market $D(12) = 100 - 12 = 88$.
- **New Profit for A**:

$$\pi'_A = (12 - 10) \times 88 = 2 \times 88 = \mathbf{176}$$

Conclusion

Profit Drop: 387 → 176. Expansion causes a loss. Even with a cost advantage, the "Aggression" needed to fill the larger capacity destroys margins. Being "Small and Efficient" was better than "Big and Efficient" in this strategic context.