

Game Theory 13-01

Review: Games of Complete Information

BCSE Game Theory

Jan. 13, 2026

Part 1: Static Games of Complete Information

IESDS Nash Equilibrium

Review: Strategic Form Games

1. **Dominance:** A strategy is strictly dominated if another strategy always yields a strictly higher payoff, regardless of opponents' actions.
2. **IESDS:** Rational players will never play strictly dominated strategies. We can iteratively remove them to simplify the game.
3. **Nash Equilibrium (NE):** A profile where no player has an incentive to deviate unilaterally.
 - ▶ IESDS preserves NE. If IESDS leads to a unique outcome, that is the unique NE.
 - ▶ If not, we solve for NE in the simplified game.

Exercise 1: 4x4 Game

Problem

Consider the following game between Player 1 (Row) and Player 2 (Col).

	L	C	R	X
T	2, 3	5, 2	1, 2	2, 0
M	2, 1	8, 8	3, 6	9, 3
B	3, 0	9, 6	2, 8	0, 5
Y	1, 1	0, 2	0, 1	1, 1

Exercise 1: 4x4 Game (Questions)

Task

Questions:

1. Apply **IESDS** to reduce this matrix to a 2x2 game. State clearly which strategy dominated which in each step.
2. Find all **Pure Strategy Nash Equilibria (PSNE)** in the reduced game.
3. Find the **Mixed Strategy Nash Equilibrium** in the reduced game.

Ex 1: Solution (IESDS Step 1-2)

	L	C	R	X
T	2, 3	5, 2	1, 2	2, 0
M	2, 1	8, 8	3, 6	9, 3
B	3, 0	9, 6	2, 8	0, 5
Y	1, 1	0, 2	0, 1	1, 1

1. **Row Dominance:** Compare Y with T .
 - ▶ T pays $\{2, 5, 1, 2\}$, Y pays $\{1, 0, 0, 1\}$.
 - ▶ $T > Y$ in all cases. **Eliminate** Y .
2. **Col Dominance:** Compare X with C .
 - ▶ Remaining rows T, M, B : C pays $\{1, 8, 6\}$, X pays $\{0, 3, 5\}$.
 - ▶ $C > X$ strictly. **Eliminate** X .

Ex 1: Solution (IESDS Step 3-4)

Remaining: T, M, B vs L, C, R .

	L	C	R
T	2, 3	5, 2	1, 2
M	2, 1	8, 8	3, 6
B	3, 0	9, 6	2, 8

3. Row Dominance: Compare T and B .

- ▶ $B(3, 9, 2)$ vs $T(2, 5, 1)$.
- ▶ B strictly dominates T ($3 > 2, 9 > 5, 2 > 1$). **Eliminate T .**

4. Col Dominance: Compare L with C .

- ▶ Remaining rows M, B : $C(8, 6)$ vs $L(1, 0)$.
- ▶ $C > L$ strictly. **Eliminate L .**

Ex 1: Solution (Reduced 2x2)

After eliminating Y, X, T, L , we are left with:

	C	R
M	8, 8	3, 6
B	9, 6	2, 8

Note: This 2x2 matrix matches the inputs for the Nash Equilibrium calculation in the next slide (labeled U, D vs L, R).

Ex 1: Solution (Reduced 2x2 PSNE)

Final 2x2 Game: Final 2x2 Game:

	L	R
U	9, 6	2, 8
D	8, 8	3, 6

(Using U, D vs L, R for clarity).

2. Pure Nash Equilibria:

- ▶ L : $U(9) > D(8)$. P1 picks U . P2 response to U ?
 $R(8) > L(6)$. $\rightarrow (U, R)$.
- ▶ R : $D(3) > U(2)$. **Check:** $U(2)$ vs $D(3)$. P1 picks D .
- ▶ Response to D ? $L(8) > R(6)$. $\rightarrow (D, L)$.
- ▶ Cycle: $U \rightarrow R \rightarrow D \rightarrow L \rightarrow U$.
- ▶ **No Pure Strategy Nash Equilibrium.**

Ex 1: Solution (Mixed Strategy NE)

Since there is no PSNE, we look for MSNE. Let P1 play U with prob p , P2 play L with prob q .

1. P2's Indifference:

$$\begin{aligned}E[U_2(L)] &= E[U_2(R)] \\6p + 8(1 - p) &= 8p + 6(1 - p) \\8 - 2p &= 2p + 6 \\2 &= 4p \Rightarrow p = 0.5\end{aligned}$$

2. P1's Indifference:

$$\begin{aligned}E[U_1(U)] &= E[U_1(D)] \\9q + 2(1 - q) &= 8q + 3(1 - q) \\7q + 2 &= 5q + 3 \\2q &= 1 \Rightarrow q = 0.5\end{aligned}$$

Result: Mixed Strategy NE is $((0.5, 0.5), (0.5, 0.5))$.

Part 2: Static Games of Complete Information

3-Firm Cournot Competition

Review: Dynamic Games Subgames

- ▶ **Game Tree (Extensive Form):**
 - ▶ **Nodes:** Decision points and terminal endpoints.
 - ▶ **Information Sets:** Collection of nodes where a player cannot distinguish which node allowed.
- ▶ **Subgame:** A part of the game that:
 - ▶ Starts at a singleton information set (initial node).
 - ▶ Includes all successors of that node.
 - ▶ Does not cut across information sets.

Review: Subgame Perfect Equilibrium (SPE)

Why Nash Equilibrium is insufficient?

NE allows for **non-credible threats**: optimal actions in subgames that are never reached (off-equilibrium path), which may not be rational if reached.

Subgame Perfect Equilibrium (SPE):

- ▶ A strategy profile that induces a Nash Equilibrium in **every subgame** (reached or not).
- ▶ **Backward Induction**: The algorithm to find SPE in finite games of perfect information.

Review: Cournot Competition

- ▶ **Model:** Firms compete on **Quantity** (q_i) **simultaneously**.
- ▶ **Inverse Demand:** $P(Q) = A - Q$, where $Q = \sum q_i$.
- ▶ **Nash Equilibrium:** Each firm sets q_i to maximize profit given q_{-i} .
- ▶ The intersection of reaction functions gives the Cournot-Nash Equilibrium.

Exercise 2: 3-Firm Cournot

Problem

Demand: $P = 120 - Q$, where $Q = q_1 + q_2 + q_3$. Marginal Costs: c_1, c_2, c_3 .

1. Write down the profit function for Firm 1.
2. Derive the Reaction Function (Best Response) for Firm 1, $q_1^*(q_2, q_3)$.
3. Solve for the equilibrium quantities q_1^*, q_2^*, q_3^* in terms of c_1, c_2, c_3 .
4. Calculate the numerical values if $c_1 = c_2 = c_3 = 20$.

Ex 2: Solution (1 & 2)

1. Profit Function:

$$\pi_1 = (P - c_1)q_1 = (120 - (q_1 + q_2 + q_3) - c_1)q_1$$

2. Reaction Function: FOC w.r.t q_1 :

$$\frac{\partial \pi_1}{\partial q_1} = 120 - 2q_1 - q_2 - q_3 - c_1 = 0$$

$$2q_1 = 120 - c_1 - (q_2 + q_3)$$

$$q_1^*(q_{-1}) = \frac{120 - c_1 - q_2 - q_3}{2}$$

Or,

$$q_1^* + Q = 120 - c_1$$

Ex 2: Solution (3)

By symmetry of the equations, we have:

$$q_1 + Q = 120 - c_1$$

$$q_2 + Q = 120 - c_2$$

$$q_3 + Q = 120 - c_3$$

$$4Q = 360 - \sum c_i$$

$$Q = 90 - \frac{\sum c_i}{4}$$

Now use Eq(1): $q_1 + Q = 120 - c_1 \Rightarrow q_1 = 120 - c_1 - Q$.
Substitute Q :

$$q_1^* = 120 - c_1 - \left(90 - \frac{\sum c_i}{4}\right) = 30 - c_1 + \frac{\sum c_i}{4}$$

Ex 2: Solution (4)

If $c_1 = c_2 = c_3 = 20$:

$$q_1^* = \frac{120 - 4c_1 + \sum c_i}{4} = \frac{120 - 80 + 60}{4} = 25$$

Symmetry: $q_1^* = q_2^* = q_3^* = 25$.

Check: Total $Q = 75$. Price $P = 120 - 75 = 45$. $MC = 20$.

MR for firm 1? $P(Q) = (120 - q_2 - q_3) - q_1 = 70 - q_1$.

$TR = (70 - q_1)q_1 \Rightarrow MR = 70 - 2q_1$.

$MR = MC \Rightarrow 70 - 2q_1 = 20 \Rightarrow q_1 = 25$. **Correct.**

Part 3: Dynamic Games of Complete Information

2-Firm Stackelberg Competition

Exercise 3: 2-Firm Stackelberg

Problem: Dynamic Quantity Competition

Demand: $P = 120 - Q$, where $Q = q_1 + q_2$. Costs $c_1 = c_2 = 20$.

Timing:

1. **Stage 1:** Firm 1 (Leader) chooses $q_1 \geq 0$.
2. **Stage 2:** Firm 2 (Follower) observes q_1 and chooses q_2 .

Questions:

1. Solve for the Subgame Perfect Equilibrium (SPE).
2. Compare the Leader's profit to the Follower's. Does moving first help?

Ex 3: Solution (Stage 2: Follower)

Backward Induction: Start from the last stage (Stage 2). Given q_1 , Firm 2 maximizes profit on residual demand:

$$\pi_2 = (P - c_2)q_2 = ((120 - q_1) - q_2 - 20)q_2 = (100 - q_1 - q_2)q_2$$

FOC w.r.t q_2 :

$$\frac{\partial \pi_2}{\partial q_2} = 100 - q_1 - 2q_2 = 0$$

Follower's Best Response function:

$$q_2^*(q_1) = \frac{100 - q_1}{2}$$

Ex 3: Solution (Stage 1: Leader)

Firm 1 anticipates the follower's response. Total Output

$$Q = q_1 + q_2^* = q_1 + \frac{100 - q_1}{2} = \frac{100 + q_1}{2}. \text{ Price}$$

$$P = 120 - Q = 120 - \frac{100 + q_1}{2} = \frac{140 - q_1}{2}. \text{ Firm 1 Profit}$$

$$\pi_1 = (P - 20)q_1:$$

$$\pi_1 = \left(\frac{140 - q_1}{2} - 20 \right) q_1 = \frac{100 - q_1}{2} q_1$$

Max w.r.t q_1 :

$$\frac{d\pi_1}{dq_1} = \frac{100 - 2q_1}{2} = 0 \Rightarrow q_1^* = 50$$

Ex 3: Results & First Mover Advantage

Equilibrium Quantities:

- ▶ Leader: $q_1^* = 50$.
- ▶ Follower: $q_2^* = \frac{100-50}{2} = 25$.

Profits:

- ▶ Total $Q = 75$. Price $P = 120 - 75 = 45$.
- ▶ Margin $P - c = 45 - 20 = 25$.
- ▶ Leader: $\pi_1 = 25 \times 50 = 1250$.
- ▶ Follower: $\pi_2 = 25 \times 25 = 625$.

Conclusion

First Mover Advantage: The Leader commits to a high quantity (50), forcing the Follower to produce less (25). Compare to symmetric Cournot: each would produce $\frac{100}{3} \approx 33.3$.

Part 4: Capacity Constraints

Bertrand Paradox Resolution

Exercise 4: Bertrand with Capacity

Problem

Two firms compete in prices. Demand $D(p) = 100 - p$.

- ▶ Firm A: MC $c_A = 10$. Capacity $K_A = 30$.
- ▶ Firm B: MC $c_B = 12$. Unlimited Capacity. (**Asymmetric Costs**).

Questions:

1. Consider Firm B's best response if Firm A sets a high price p_A . Should B undercut or accept residual demand?
2. Find the Nash equilibrium price p^* where B is indifferent between undercutting and residual monopoly.
3. Calculate Firm A's profit at this equilibrium.
4. Suppose Firm A expands capacity to $K_A = 100$. What is the new equilibrium and profit?

Ex 4: Solution (1: Firm B's Choice)

Suppose A sets price p_A . B has two main strategies:

1. **Residual Monopoly**: Accept A fills $K_A = 30$.

▶ Residual: $D_r(p) = (100 - p) - 30 = 70 - p$.

▶ Max $\pi_B^{Res} = (p - 12)(70 - p)$.

▶ FOC $\Rightarrow p^* = 41$. Profit:
 $(41 - 12)(70 - 41) = 29 \times 29 = 841$.

2. **Price War (Undercut)**: Set $p_B = p_A - \epsilon$.

▶ B captures whole market $D(p_A)$.

▶ Profit $\pi_B^{Cut} \approx (p_A - 12)(100 - p_A)$.

Comparison at $p_A = 41$:

$\pi_B^{Cut}(41) = (41 - 12)(100 - 41) = 29 \times 59 = 1711$. Since
 $1711 > 841$, B will undercut! $p_A = 41$ is **not** an equilibrium.

Ex 4: Solution (2: Equilibrium)

Firm A must lower p_A until B prefers (or is indifferent to) Residual Monopoly.

$$\pi_B^{Cut}(p) \leq \pi_B^{Res} \Rightarrow (p - 12)(100 - p) \leq 841$$

$$-p^2 + 112p - 1200 \leq 841 \Rightarrow p^2 - 112p + 2041 \geq 0$$

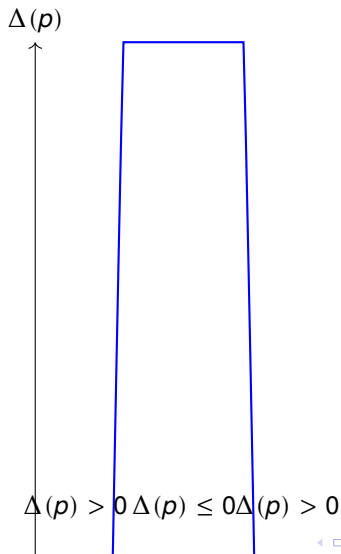
$$\text{Roots: } p = \frac{112 \pm \sqrt{112^2 - 4 \times 2041}}{2} = \frac{112 \pm \sqrt{12544 - 8164}}{2} = \frac{112 \pm \sqrt{4380}}{2}.$$

$$\sqrt{4380} \approx 66.18. \text{ Lower stable root: } p \approx \frac{112 - 66.18}{2} = \frac{45.82}{2} \approx 22.91.$$

- ▶ **Equilibrium:** $p_A = p_B \approx 22.9$.
- ▶ A sells 30 units (capacity). B sells residual.
- ▶ **A's Profit:** $\pi_A = (22.9 - 10) \times 30 = 12.9 \times 30 = 387$.

Ex 4: Graphical Analysis

Profit Difference: $\Delta(p) = \pi_B^{Cut}(p) - \pi_B^{Res} = (p - 12)(100 - p) - 841$



Ex 4: Solution (3: The Paradox)

What if A expands capacity to $K_A = 100$ (Unconstrained)?

- ▶ A has cost advantage ($10 < 12$) and sufficient capacity to serve the whole market (Max Demand ≤ 100).
- ▶ **Standard Asymmetric Bertrand:** A undercuts B to $p \approx c_B = 12$.
- ▶ A serves entire market $D(12) = 100 - 12 = 88$.
- ▶ **New Profit for A:**

$$\pi'_A = (12 - 10) \times 88 = 2 \times 88 = 176$$

Conclusion

Profit Drop: $387 \rightarrow 176$. Expansion causes a loss. Even with a cost advantage, the "Aggression" needed to fill the larger capacity destroys margins. Being "Small and Efficient" was better than "Big and Efficient" in this strategic context.