BCSE Game Theory 07-02

Continuous Strategies: Cournot, Bertrand, and Stackelberg

Author

Nov. 18, 2025

Today's Agenda

Continuous Strategy Games

Today's Objectives

- Extend game theory to continuous strategy spaces (quantities, prices).
- Analyse Cournot competition (simultaneous quantity choice).
- Analyse Bertrand competition (simultaneous price choice).
- Analyse Stackelberg competition (sequential quantity choice).
- Compare equilibrium outcomes across different market structures.
- Apply backward induction to continuous-strategy games.

Where Lecture 07-02 Starts

- Lecture 07-01 covered discrete-strategy extensive-form games and SPNE.
- Many economic applications involve continuous strategies (quantities, prices, investments).
- We now extend SPNE analysis to continuous strategy spaces.
- Our focus: oligopoly models where firms choose quantities or prices.
- Key insight: backward induction works with continuous strategies using calculus.

Review: SPNE with Continuous Strategies

SPNE with Continuous Strategies

- **Subgame Perfect Nash Equilibrium** still requires Nash equilibrium in every subgame.
- With continuous strategies, we use calculus to find best responses.
- First-order conditions (FOCs) identify optimal choices.
- Backward induction: solve follower's problem first, then leader's problem.
- The key difference: strategies are functions (e.g., $q_2(q_1)$) rather than discrete actions.

Best Response Functions

Definition: Best Response Function

For player i, the best response function $BR_i(s_{-i})$ gives the strategy that maximises i's payoff given opponents' strategies s_{-i}.

- With continuous strategies, best responses are typically functions, not single actions.
- Example: In quantity competition, firm 2's best response is $q_2^*(q_1)$, a function of firm 1's quantity.
- SPNE requires that each player's strategy is a best response to others' strategies.

Cournot Competition

Cournot Model: Setup

- Two firms, $i = 1, 2$, choose quantities $q_1, q_2 \geq 0$ simultaneously.
- Market price: $P(Q) = a - bQ$ where $Q = q_1 + q_2$ is total quantity.
- Firm i's cost: $C_i(q_i) = cq_i$ (constant marginal cost c).
- Firm i's profit: $\pi_i(q_i, q_j) = P(Q)q_i - C_i(q_i) = (a - b(q_i + q_j) - c)q_i$.
- This is a simultaneous-move game with continuous strategies.

Cournot: Firm 1's Best Response

- Firm 1 maximises $\pi_1(q_1, q_2) = (a - b(q_1 + q_2) - c)q_1$.
- First-order condition:

$$\frac{\partial \pi_1}{\partial q_1} = a - bq_2 - 2bq_1 - c = 0$$

- Solving for q_1:

$$q_1^*(q_2) = \frac{a - c - bq_2}{2b} = \frac{a - c}{2b} - \frac{q_2}{2}$$

- This is firm 1's best response function: $BR_1(q_2)$.

Cournot: Firm 2's Best Response

- By symmetry, firm 2's best response is:

$$q_2^*(q_1) = \frac{a - c - bq_1}{2b} = \frac{a - c}{2b} - \frac{q_1}{2}$$

- Best response functions are downward-sloping: quantities are strategic substitutes.
- If firm 1 increases q_1, firm 2's optimal q_2 decreases.

Cournot Nash Equilibrium

- Nash equilibrium requires mutual best responses:

$$q_1^* = \frac{a-c}{2b} - \frac{q_2^*}{2}$$
$$q_2^* = \frac{a-c}{2b} - \frac{q_1^*}{2}$$

- Solving the system:

$$q_1^* = q_2^* = \frac{a-c}{3b}$$

- Total quantity: $Q^* = \frac{2(a-c)}{3b}$.
- Market price: $P^* = a - bQ^* = \frac{a+2c}{3}$.
- Each firm's profit: $\pi_i^* = \frac{(a-c)^2}{9b}$.

Cournot vs. Monopoly

- Monopoly quantity: $q^m = \frac{a-c}{2b}$ (maximises $(a - bq - c)q$).
- Cournot total: $Q^* = \frac{2(a-c)}{3b} > q^m$ (more output).
- Monopoly price: $P^m = \frac{a+c}{2}$.
- Cournot price: $P^* = \frac{a+2c}{3} < P^m$ (lower price).
- Competition increases output and reduces price relative to monopoly.

Stackelberg Competition

Extensive form form game of Cournot competition

Stackelberg Model: Setup

- Leader (firm 1) chooses quantity q_1 first.
- Follower (firm 2) observes q_1 and chooses q_2.
- This is a sequential game with continuous strategies.
- Market price: $P(Q) = a - b(q_1 + q_2)$.
- Costs: $C_i(q_i) = cq_i$ for both firms.
- We solve using backward induction (SPNE).

Stackelberg: Follower's Best Response

- Follower (firm 2) maximises $\pi_2(q_1, q_2) = (a - b(q_1 + q_2) - c)q_2$ given q_1.
- First-order condition:

$$\frac{\partial \pi_2}{\partial q_2} = a - bq_1 - 2bq_2 - c = 0$$

- Follower's best response:

$$q_2^*(q_1) = \frac{a - c - bq_1}{2b} = \frac{a - c}{2b} - \frac{q_1}{2}$$

- This is the same as in Cournot, but now firm 2 observes q_1 before choosing.

Stackelberg: Leader's Problem

- Leader anticipates follower's best response and maximises:

$$\pi_1(q_1, q_2^*(q_1)) = \left(a - b\left(q_1 + \frac{a - c - bq_1}{2b}\right) - c\right) q_1$$

- Simplifying:

$$\pi_1(q_1) = \left(\frac{a - c - bq_1}{2}\right) q_1 = \frac{(a-c)q_1}{2} - \frac{bq_1^2}{2}$$

- First-order condition:

$$\frac{\partial \pi_1}{\partial q_1} = \frac{a-c}{2} - bq_1 = 0$$

- Leader's optimal quantity:

$$q_1^* = \frac{a-c}{2b}$$

Stackelberg Equilibrium

- Leader chooses: $q_1^* = \frac{a-c}{2b}$.
- Follower responds: $q_2^* = \frac{a-c-bq_1^*}{2b} = \frac{a-c}{4b}$.
- Total quantity: $Q^* = q_1^* + q_2^* = \frac{3(a-c)}{4b}$.
- Market price: $P^* = a - bQ^* = \frac{a+3c}{4}$.
- Leader's profit: $\pi_1^* = \frac{(a-c)^2}{8b}$.
- Follower's profit: $\pi_2^* = \frac{(a-c)^2}{16b}$.

Stackelberg vs. Cournot

- **Leader advantage**: $q_1^* = \frac{a-c}{2b} > \frac{a-c}{3b} = q_1^{Cournot}$.
- Leader produces more than in Cournot equilibrium.
- Follower produces less: $q_2^* = \frac{a-c}{4b} < \frac{a-c}{3b} = q_2^{Cournot}$.
- Total output: $Q^* = \frac{3(a-c)}{4b} > \frac{2(a-c)}{3b} = Q^{Cournot}$ (more competition).
- Price: $P^* = \frac{a+3c}{4} < \frac{a+2c}{3} = P^{Cournot}$ (lower price).
- Leader's profit: $\pi_1^* = \frac{(a-c)^2}{8b} > \frac{(a-c)^2}{9b} = \pi_1^{Cournot}$.

Why First-Mover Advantage?

- Leader commits to a high quantity before follower moves.
- Follower's best response is to produce less (quantities are strategic substitutes).
- This commitment is credible because quantity is chosen and observed.
- Leader benefits from reducing follower's output.
- The key: sequential moves allow commitment, which changes the equilibrium.

What Does “Commit Low Output” Mean?

- A leader can deliberately cap its own quantity (or capacity) at a low level to keep market price high.
- The cap must be credible—long-term contracts, limited production lines, or regulatory quotas make it costly to expand later.
- Once the follower observes the low commitment, its best response shifts: it cannot profit by flooding the market, because doing so would drive down the shared price.
- “Commit low output” is a pricing/quantity-discipline device: soften competition and preserve margins even when moves are sequential.
- Whether the leader prefers a high or low commitment depends on the payoff trade-off (squeezing the rival vs. propping up the price).

Kreps–Scheinkman Model

Capacity and Price Competition

Kreps–Scheinkman (1983): Two-Stage Game

- **Stage 1**: Firms simultaneously choose capacities $K_1, K_2 \geq 0$.
- **Stage 2**: Firms observe capacities and simultaneously choose prices $p_1, p_2 \geq 0$.
- Production is constrained by capacity: $q_i \leq K_i$.
- This is a two-stage game with continuous strategies.
- We solve using backward induction (SPNE).

Kreps–Scheinkman: Stage 2 (Price Competition)

- Given capacities K_1, K_2, firms compete in prices.
- Market demand: $D(p) = a - bp$ where $p = \min(p_1, p_2)$.
- If $p_i < p_j$, firm i gets demand $\min(D(p_i), K_i)$.
- If $p_i = p_j$, firms split demand proportionally to capacity.
- Key insight: Capacity constraints affect price competition.
- With binding capacity constraints, price competition yields Cournot-like outcomes.

Kreps–Scheinkman: Main Result

Theorem: Kreps and Scheinkman (1983)

If firms choose capacities in stage 1 and then compete in prices in stage 2, the equilibrium outcome is the same as Cournot competition with those capacities.

- Capacity choice in stage 1 commits firms to production limits.
- Price competition in stage 2, constrained by capacities, yields Cournot quantities.
- This bridges the gap between Cournot (quantity) and Bertrand (price) competition.
- The result holds under certain assumptions about demand and cost functions.

Why Kreps–Scheinkman Matters

- Resolves the **Cournot–Bertrand paradox**: which model is more realistic?
- Shows that capacity constraints can make price competition behave like quantity competition.
- Explains why firms might invest in capacity even when competing in prices.
- Provides a microfoundation for Cournot competition: it can arise from price competition with capacity constraints.
- Important for understanding real-world markets where capacity is chosen before prices.

Kreps–Scheinkman: Equilibrium

- In equilibrium, firms choose capacities equal to Cournot quantities:
$$K_1^* = K_2^* = \frac{a-c}{3b}$$
- In stage 2, price competition with these capacities yields:
$$p_1^* = p_2^* = \frac{a+2c}{3}$$
- Quantities produced: $q_1^* = q_2^* = \frac{a-c}{3b}$ (equal to capacities).
- Profits: $\pi_1^* = \pi_2^* = \frac{(a-c)^2}{9b}$ (same as Cournot).
- Capacity investment serves as a commitment device.

Kreps–Scheinkman: Intuition

- Without capacity constraints, price competition drives prices to marginal cost (Bertrand).
- With capacity constraints, firms cannot serve all demand at low prices.
- This creates market power even in price competition.
- The equilibrium capacity choice balances the commitment value against the cost of excess capacity.
- Result: price competition with capacity constraints $\approx$ quantity competition.

Sequential Bertrand Competition

Price Leadership

(Differentiated) Bertrand (simultaneous)

- If products are differentiated, firms have market power.
- Demand for firm i: $q_i = a - bp_i + dp_j$ where $d > 0$ measures substitutability.
- Firm i's profit: $\pi_i = (p_i - c)(a - bp_i + dp_j)$.
- Best response: $p_i^*(p_j) = \frac{a+c+dp_j}{2b}$.
- Nash equilibrium: $p_1^* = p_2^* = \frac{a+c}{2b-d}$.
- With differentiation ($d < b$), prices exceed marginal cost and profits are positive.

Sequential Bertrand: Setup

- Leader (firm 1) chooses price p_1 first.
- Follower (firm 2) observes p_1 and chooses price p_2.
- This is a sequential game with continuous strategies.
- Market demand: $D(p) = a - bp$ where $p = \min(p_1, p_2)$.
- If $p_i < p_j$, firm i gets all demand; if $p_i = p_j$, firms split demand equally.
- Constant marginal cost c for both firms.
- We solve using backward induction (SPNE).

Sequential Bertrand: Follower's Best Response

- Follower (firm 2) observes p_1 and chooses p_2 to maximise profit.
- If $p_1 > c$:
  - Follower can undercut: $p_2 = p_1 - \epsilon$ captures all demand.
  - Or match: $p_2 = p_1$ splits demand equally.
  - Best response: $p_2^*(p_1) = p_1 - \epsilon$ if $p_1 > c$.
- If $p_1 = c$:
  - Follower cannot profitably undercut.
  - Best response: $p_2^*(c) = c$.
- Follower's strategy: slightly undercut if $p_1 > c$, match if $p_1 = c$.

Sequential Bertrand: Leader's Problem

- Leader anticipates follower's best response.
- If leader sets $p_1 > c$: follower chooses $p_2 = p_1 - \epsilon$ and captures all demand; leader gets zero profit.
- If leader sets $p_1 = c$: follower chooses $p_2 = c$; both get zero profit.
- Leader's profit is zero for any $p_1 \geq c$ (all are best responses in terms of profit).
- However, in SPNE, we require that follower's strategy is optimal given leader's choice.
- If $p_1 > c$, follower's unique best response is $p_2 = p_1 - \epsilon$ (captures all demand).
- If $p_1 = c$, follower is indifferent between $p_2 = c$ and $p_2 < c$ (both yield zero profit).
- The unique SPNE: $p_1^* = p_2^* = c$ (marginal cost pricing).

Sequential Bertrand: Equilibrium

- SPNE: $p_1^* = p_2^* = c$.
- Both firms earn zero profit.
- This is the same outcome as simultaneous Bertrand competition.
- Sequential moves do not help the leader in homogeneous product price competition.
- The threat of undercutting eliminates any price leadership advantage.

Why No Price Leadership Advantage?

- In quantity competition (Stackelberg), leader benefits from commitment.
- In price competition, follower can always undercut the leader.
- Undercutting is profitable as long as $p_1 > c$.
- This eliminates any market power the leader might have.
- Sequential price competition with homogeneous products yields Bertrand outcome.

Definition: Second-Mover Advantage

Definition: Second-Mover Advantage

In a sequential-move game, a follower has a second-mover advantage if acting after observing the leader allows the follower to earn a weakly higher equilibrium payoff than the leader (or than in the simultaneous-move benchmark). The advantage arises from flexibility: the follower can condition its action on the leader's move and best respond once information is revealed.

- Homogeneous-price Bertrand competition illustrates this: whichever firm moves second can undercut a high price and capture the entire market, so commitment to a high price is not profitable.
- Second-mover advantages typically occur when strategies are strategic complements (prices) or when rapid undercutting is possible, making flexibility more valuable than commitment.

Sequential Differentiated Bertrand

- If products are differentiated, price leadership can matter.
- Demand for firm i: $q_i = a - bp_i + dp_j$ where $d > 0$.
- Leader chooses p_1 first, follower observes and chooses p_2.
- Follower's best response: $p_2^*(p_1) = \frac{a+c+dp_1}{2b}$.
- Leader anticipates this and maximises:

$$\pi_1(p_1, p_2^*(p_1)) = (p_1 - c)\left(a - bp_1 + d \cdot \frac{a + c + dp_1}{2b}\right)$$

- Leader can now benefit from moving first.

Sequential Differentiated Bertrand: Equilibrium

- Solving leader's problem:

$$p_1^* = \frac{2ab + 2bc + ad - cd}{4b^2 - d^2}$$

- Follower responds:

$$p_2^* = \frac{a + c + dp_1^*}{2b}$$

- With differentiation, leader typically sets a higher price than follower.
- Leader earns higher profit than in simultaneous competition.
- Product differentiation creates market power that enables price leadership.

Price vs. Quantity Leadership

- **Quantity leadership** (Stackelberg): Leader benefits from commitment.
- **Price leadership** (homogeneous): No advantage; follower undercuts.
- **Price leadership** (differentiated): Leader can benefit from commitment.
- Key difference: In price competition with homogeneous products, undercutting is always profitable.
- With differentiation or capacity constraints, price leadership can be effective.

Comparison and Applications

Comparing Market Structures

	Monopoly	Cournot	Stackelberg
Total output	$\frac{a-c}{2b}$	$\frac{2(a-c)}{3b}$	$\frac{3(a-c)}{4b}$
Price	$\frac{a+c}{2}$	$\frac{a+2c}{3}$	$\frac{a+3c}{4}$
Leader profit	–	$\frac{(a-c)^2}{9b}$	$\frac{(a-c)^2}{8b}$
Follower profit	–	$\frac{(a-c)^2}{9b}$	$\frac{(a-c)^2}{16b}$

- More competition → higher output, lower price.
- Sequential moves (Stackelberg) increase competition relative to simultaneous moves (Cournot).
- Leader benefits from moving first, but total welfare increases.

When Does Stackelberg Arise?

- **Capacity commitment**: Leader invests in capacity before follower enters.
- **Technology advantage**: Leader develops technology first.
- **Market entry**: First entrant has advantage over later entrants.
- **Regulation**: Sequential licensing or approval processes.
- Key requirement: Leader's choice must be observable and (at least partially) irreversible.

Bertrand vs. Cournot: Which Is More Realistic?

- **Cournot** fits industries where capacity is chosen before production (e.g., manufacturing).
- **Bertrand** fits industries where prices are easily changed (e.g., retail, services).
- **Differentiated Bertrand** fits most real markets (products are not identical).
- Choice depends on: ease of changing prices vs. quantities, product differentiation, capacity constraints.

Example: Technology Standards

- Leader (e.g., Apple) sets a technology standard (quantity of features, quality level).
- Follower (e.g., Samsung) observes and responds.
- Stackelberg model captures this sequential decision-making.
- Leader's commitment to high quality can deter follower from competing directly.
- But follower may differentiate (differentiated Bertrand) rather than match.

Common Mistakes

- **Forgetting to substitute**: In Stackelberg, leader must account for follower's best response.
- **Sign errors**: Check that best responses have correct slopes (substitutes vs. complements).
- **Corner solutions**: Verify that equilibrium quantities/prices are non-negative.
- **Second-order conditions**: Ensure FOCs identify maxima, not minima.
- **Units**: Keep track of parameters (a, b, c) and their units.

Takeaways

Key Messages

- **Cournot** (simultaneous quantities): firms earn positive profits, output between monopoly and competitive levels.
- **Bertrand** (simultaneous prices): with homogeneous products, price equals marginal cost (zero profits).
- **Differentiated Bertrand**: firms have market power, prices exceed marginal cost.
- **Stackelberg** (sequential quantities): leader produces more, follower less, than in Cournot.
- **First-mover advantage**: commitment in sequential games can benefit the leader.
- **SPNE with continuous strategies**: use calculus and backward induction.

Reading and Next Steps

- Tirole (1988), Chapter 5: oligopoly theory and market structure.
- Gibbons (1992), Chapter 1: static games of complete information.
- Kreps and Scheinkman (1983): capacity and price competition.
- Next lectures will cover repeated games and their applications.
- Practice: solve Cournot, Bertrand, Stackelberg, and Kreps–Scheinkman models.

References

Key Papers

Kreps, D. M., & Scheinkman, J. A. (1983). Quantity precommitment and Bertrand competition yield Cournot outcomes. *Bell Journal of Economics*, 14(2), 326–337.

- This paper shows that two-stage competition (capacity then price) yields Cournot outcomes.
- Provides a microfoundation for quantity competition models.
- Important for understanding the relationship between capacity and price competition.