BCSE Game Theory 03-02

Finite Games and Canonical Examples

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Finite Games

Today's Goals

- Understand the building blocks of finite normal-form games.
- Review key properties and insights from canonical two-player games.
- Recognise typical phenomena that arise as the number of players increases.

Economic Story Vocabulary

- When we talk about markets, **quantity** refers to the number of units a firm produces or sells.
- The **price** is the amount consumers pay per unit; it may fall when total quantity rises.
- A **homogeneous product** means consumers view the firms' output as interchangeable (no brand differences).
- A firm's **payoff** represents profit or utility after subtracting costs; we encode it directly in the payoff matrix.
- Phrases such as "low" or "high" output are modelling shortcuts—think of them as discrete choices that approximate real production decisions.

Reminder

You do not need prior economics background: every new market concept will be introduced with a brief explanation when it first appears.

Definition of a Finite Game

Definition: Finite normal-form game

A game $G \stackrel{\text{def}}{=} (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ consists of

- a finite player set $N \stackrel{\text{def}}{=} \{1, \dots, n\}$;
- for each player i, a finite set of pure strategies S_i;
- payoff functions $u_i : \prod_{j \in N} S_j \to \mathbb{R}$.

- The set of strategy profiles is $S \stackrel{\text{def}}{=} \prod_{i \in N} S_i$.

All games we study in this lecture fit into this framework.

Notation Refresher

- $S_{-i} \stackrel{\text{def}}{=} \prod_{j \neq i} S_j$: the strategies of everyone except player i.
- A strategy profile is $s \stackrel{\text{def}}{=} (s_i, s_{-i})$; the best-response set is $BR_i(s_{-i}) \stackrel{\text{def}}{=} \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$.
- A Nash equilibrium is a profile with $s_i \in BR_i(s_{-i})$ for every i.

Canonical Two-Player Examples

Teamwork

Example: Teamwork Game

		Binh	
		Work Hard	Slack Off
Anh	Work Hard	(3, 3)	(1, 4)
	Slack Off	(4, 1)	(2, 2)

- Joint cooperation yields the highest payoff $(3, 3)$ in the top-left cell.
- Any unilateral switch to Slack Off tempts a player to free-ride on their partner.
- IESDS removes Work Hard, leaving Slack Off and the unique Nash equilibrium $(\text{Slack Off}, \text{Slack Off})$.

Example: Quantities in Competition

		Company 2: Low	Company 2: High
Company 1	Low	(3, 3)	(1, 4)
	High	(4, 1)	(2, 2)

- Two firms supply a homogeneous product to the market.
- Producing more allows higher sales but pushes the market price down.
- This mirrors the Cournot competition structure.

Example: Arms Race Game

		Country 2	
		Disarm	Arm
Country 1	Disarm	(3, 3)	(1, 4)
	Arm	(4, 1)	(2, 2)

- Two countries choose between arming and disarming.
- Mutual disarmament delivers the highest peace dividend $(3, 3)$.
- A unilateral armament grants a short-term edge $(4, 1)$, yet the equilibrium remains $(2, 2)$.

Example: Oil Extraction Game

		Firm 2	
		Conserve	Drill Hard
Firm 1	Conserve	(3, 3)	(1, 4)
	Drill Hard	(4, 1)	(2, 2)

- Two firms coordinate extraction from a shared oil field.
- Mutual restraint preserves the resource and splits revenue $(3, 3)$.
- If one drills aggressively it enjoys a short-term gain $(4, 1)$, but mutual over-drilling drags payoffs to $(2, 2)$.

Prisoner's Dilemma

		Prisoner 2	
		Cooperate	Defect
Prisoner 1	Cooperate	(3, 3)	(1, 4)
	Defect	(4, 1)	(2, 2)

- Defect strictly dominates; the unique Nash equilibrium is $(\text{Defect}, \text{Defect})$.
- The teamwork story is widely known as the Prisoner's Dilemma.
- We will keep calling it the Teamwork Game to avoid the misleading prison imagery.

Teamwork Game

		Binh: Work Hard	Binh: Slack Off
Anh	Work Hard	(3, 3)	(1, 4)
	Slack Off	(4, 1)	(2, 2)

The Teamwork Game has three hallmarks:

- Both players earn more by cooperating than they do when both shirk.
- Each individual still has an incentive to defect, so cooperation unravels.
- Social desirability and private incentives are misaligned.

Communication (Discussion)

Communication

In game theory we often assume, implicitly, that players can communicate extensively before the game starts.

- Even with pre-play communication, Slack Off still maximises your own payoff, so a fully rational player sticks with Slack Off.
- Cheap talk does not authorise commitments, side payments, or any other rule-breaking behaviour after the game.
- We revisit how communication can help once we analyse repeated play and enforcement in later lectures.

Mechanism Design

We will work through many game examples today, but keep the bigger picture in mind:

Definition: Mechanism Design

Mechanism design studies how to craft rules that steer the behaviour of informed agents so that desirable social objectives—efficiency, equity, stability, and the like—are achieved.

Mechanism Design

Definition: Transfer

A transfer is a payment of money or resources between players, or between a mechanism and players, that reallocates payoffs without destroying total surplus.

Example: monetary payments in auctions. tax schemes.

- Contracts that involve money are not always enforceable; we must consider whether players voluntarily participate.
- When payoffs are non-transferable (e.g. the arms-race game), transfers are not an appropriate tool.

Definition: Money burning

Resources that are consumed rather than transferred to any agent, thereby diminishing total surplus.

Mechanism Design: Transfers

(3, 3)	(1, 4)
(4, 1)	(2, 2)

(3, 3)	(3, 2)
(2, 3)	(2, 2)

Transfer

- Suppose only one player chooses Work Hard; adding a contract that transfers 2 units from the shirker to the worker implements $(\text{Work Hard}, \text{Work Hard})$.
- Such a transfer requires that the outcome is verifiable and observable, so the shirker can be identified and the contract enforced.
- Each player then earns 3 instead of the outside option 2, so both willingly accept the contract.

Mechanism Design: Money Burning

(3, 3)	(1, 4)
(4, 1)	(2, 2)

(3, 3)	(1, 2)
(2, 1)	(0, 0)

Money burning

- Each student reports whether the partner chose Slack Off when submitting the assignment.
- Any student reported as slacking receives a penalty of 2, reducing total surplus but deterring shirking.

A Catalogue of 2x2 Games

Coordination Game (Recap 01-01)

		Player B: Vietnam	Player B: Japan
Player A	Vietnam	(3, 3)	(1, 1)
	Japan	(1, 1)	(2, 2)

- Both $(\text{Vietnam}, \text{Vietnam})$ and $(\text{Japan}, \text{Japan})$ are Nash equilibria.
- The Vietnam outcome yields higher payoffs for both players.

Battle of the Sexes (Recap 03-01)

		Player B	
		Opera	Football
Player A	Opera	(2, 1)	(0, 0)
	Football	(0, 0)	(1, 2)

- The coordination game we analysed as a Nash equilibrium example in 03-01.
- Both players want to meet but have different preferences: Opera versus Football.
- $(\text{Opera}, \text{Opera})$ and $(\text{Football}, \text{Football})$ coexist as pure Nash equilibria.

Stag Hunt (Assurance Game)

		Player B	
		Stag	Rabbit
Player A	Stag	(4,4)	(0,3)
	Rabbit	(3,0)	(2,2)

- Two hunters choose between catching a small prey (hare) on their own or coordinating to capture a stag for a larger payoff.
- A stag can only be captured in tandem; a lone attempt fails and leaves the hunter with a low payoff.

Bank Run Coordination

		P2	
		Stay (S)	Withdraw (W)
P1	Stay (S)	(3, 3)	(0, 2)
	Withdraw (W)	(2, 0)	(1, 1)

- A coordination game in which depositors move simultaneously; both (S, S) and (W, W) are pure Nash equilibria.
- If everyone believes deposits are safe, the good equilibrium (S, S) prevails; rumours or liquidity fears push the system toward the run (W, W).
- Institutions such as deposit insurance or central-bank liquidity shape which equilibrium is selected.

Chicken race game (Hawk – Dove)

		Driver B	
		Swerve	Straight
Driver A	Swerve	(2,2)	(1,4)
	Straight	(4,1)	(0,0)

- Two drivers choose between swerving and driving straight.
- If only one drives straight, that driver gains; if both go straight, both suffer the worst outcome.
- The game has two asymmetric Nash equilibria.
- Goldenball game.

Rational Pig Game

		Minor: Invest	Minor: Wait
Giant	Invest	(4,-1)	(3,1)
	Wait	(1,3)	(0,0)

- Firms compete over whether to invest in a project.
- If either firm invests, the industry creates surplus worth 4.
- When the giant invests, it captures the entire surplus thanks to its bargaining power.
- If only the minor firm invests, the surplus accrues to the minor firm instead.
- Investment costs 4 for either firm; there is a positive externality of 2 on the rival whenever one party invests.

Rational Pig Game

		Minor: Invest	Minor: Wait
Giant	Invest	(3,-1)	(1,2)
	Wait	(2,1)	(0,0)

- The Nash equilibrium is $(\text{Invest}, \text{Wait})$.
- Although they share a market, the dominant and minor firms adopt different equilibrium strategies.
- In this payoff structure the minor firm ends up with the higher profit.

Volunteer Dilemma

		Player 2	
		Volunteer	Not
Player 1	Volunteer	$(b-c, b-c)$	$(b-c, b)$
	Not	$(b, b-c)$	$(0, 0)$

- The public good is provided if at least one of the two players volunteers.
- Volunteering carries cost $c > 0$, while everyone enjoys the benefit b with $b > c$.

Three-Player Coordination Game

Player 3: L

		Player 2: L	Player 2: C
Player 1	L	$(3,3,3)$	$(2,0,2)$
	C	$(0,2,2)$	$(2,2,0)$

Player 3: C

		Player 2: L	Player 2: C
Player 1	L	$(2,2,0)$	$(0,2,2)$
	C	$(2,0,2)$	$(3,3,3)$

- Players 1–3 propose either the library (L) or cafe (C) as a meeting point.
- If at least two choose the same place it becomes the meeting location; a player who ends up alone is worse off.

Matching Pennies

		Player B	
		Heads	Tails
Player A	Heads	(1,-1)	(-1,1)
	Tails	(-1,1)	(1,-1)

- Classic zero-sum structure: one player's gain of 1 is the other's loss of -1.
- Because neither player can anticipate the opponent's move, no pure strategy is a best response.
- This is our first example without a pure Nash equilibrium, motivating mixed strategies.

Matching Pennies as Penalty Kicks

		Goalkeeper	
		Dive Left	Dive Right
Kicker	Shoot Left	(-1,1)	(1,-1)
	Shoot Right	(1,-1)	(-1,1)

- The kicker scores (payoff 1) by choosing the opposite side from the keeper and is blocked (payoff -1) if they match.
- The goalkeeper's payoffs are the reverse: matching the kicker succeeds, diving opposite fails.
- Neither player can stick with a pure strategy, illustrating why mixed strategies are needed (preview of the next lecture).

Other Games to Remember

- Ultimatum Game: fairness and strategic bargaining.
- War of Attrition: war-of-nerves with exit costs.
- Rock – Paper – Scissors: three-way dominance cycle requiring mixed equilibrium.

Preparing for Mixed Strategies

- Games without pure equilibria (e.g. Matching Pennies) require finding where best-response lines intersect.
- In a best-response diagram, the intersection of the two players' response curves identifies the Nash equilibrium.
- In 04-01, 04-02, we will compute that intersection as probability distributions and connect it to the Nash existence theorem.
- Practise solving for the probabilities that leave opponents indifferent—it smooths the transition to next time.

Summary

Today's Takeaways

- Finite games are defined by finite player sets, strategy sets, and payoff functions.
- Mastering canonical two-player games sharpens your intuition for Nash equilibria.
- Multi-player settings introduce new themes such as public goods and volunteering dilemmas.

Self-Check

- Can you quickly identify whether each game has a dominant strategy?
- When multiple equilibria exist, can you describe payoff dominance versus risk dominance in words?
- For games without pure equilibria (such as Matching Pennies), can you articulate why mixed strategies are required next time?
- Can you explain the solutions to the 03-01 exercises together with the reasoning steps?

Before Next Session

- Review where the Nash equilibria and Pareto-efficient outcomes sit in each game.
- Compute the symmetric mixed equilibrium for the Volunteer Dilemma.
- Revisit how IESDS and Nash equilibrium related in lecture 03-01.

Check Your Understanding

Task

1. Using the generic Prisoner's Dilemma payoffs (T, R, P, S), derive the conditions under which (D, D) is the Nash equilibrium.
2. For the Volunteer Dilemma with $b = 5$, $c = 2$, and $n = 3$, solve for the symmetric mixed equilibrium probability.