# BCSE Game Theory 03-02

## Finite Games and Canonical Examples

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# Finite Games

# Today's Goals

- Understand the building blocks of finite normal-form games.
- Review key properties and insights from canonical two-player games.
- Recognise typical phenomena that arise as the number of players increases.

# Economic Story Vocabulary

- When we talk about markets, **quantity** refers to the number of units a firm produces or sells.
- The **price** is the amount consumers pay per unit; it may fall when total quantity rises.
- A **homogeneous product** means consumers view the firms' output as interchangeable (no brand differences).
- A firm's **payoff** represents profit or utility after subtracting costs; we encode it directly in the payoff matrix.
- Phrases such as "low" or "high" output are modelling shortcuts—think of them as discrete choices that approximate real production decisions.

**Reminder**

You do not need prior economics background: every new market concept will be introduced with a brief explanation when it first appears.

# Definition of a Finite Game

**Definition: Finite normal-form game**

A game $G \stackrel{\text{def}}{=} (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ consists of

- a finite player set $N \stackrel{\text{def}}{=} \{1, \dots, n\}$;
- for each player $i$, a finite set of pure strategies $S_i$;
- payoff functions $u_i : \prod_{j \in N} S_j \to \mathbb{R}$.

- The set of strategy profiles is $S \stackrel{\text{def}}{=} \prod_{i \in N} S_i$.

All games we study in this lecture fit into this framework.

# Notation Refresher

- $S_{-i} \stackrel{\text{def}}{=} \prod_{j \neq i} S_j$: the strategies of everyone except player $i$.
- A strategy profile is $s \stackrel{\text{def}}{=} (s_i, s_{-i})$; the best-response set is $BR_i(s_{-i}) \stackrel{\text{def}}{=} \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$.
- A Nash equilibrium is a profile with $s_i \in BR_i(s_{-i})$ for every $i$.

# Canonical Two-Player Examples

## Teamwork

# Example: Teamwork Game

| | | Binh | |
|---|---|---|---|
| | | Work Hard | Slack Off |
| Anh | Work Hard | (3, 3) | (1, 4) |
| | Slack Off | (4, 1) | (2, 2) |

- Joint cooperation yields the highest payoff $(3, 3)$ in the top-left cell.
- Any unilateral switch to Slack Off tempts a player to free-ride on their partner.
- IESDS removes Work Hard, leaving Slack Off and the unique Nash equilibrium $(\text{Slack Off}, \text{Slack Off})$.

# Example: Quantities in Competition

| | | Company 2: Low | Company 2: High |
|---|---|---|---|
| Company 1 | Low | (3, 3) | (1, 4) |
| | High | (4, 1) | (2, 2) |

- Two firms supply a homogeneous product to the market.
- Producing more allows higher sales but pushes the market price down.
- This mirrors the Cournot competition structure.

# Example: Arms Race Game

| | | Country 2 | |
|---|---|---|---|
| | | Disarm | Arm |
| Country 1 | Disarm | (3, 3) | (1, 4) |
| | Arm | (4, 1) | (2, 2) |

- Two countries choose between arming and disarming.
- Mutual disarmament delivers the highest peace dividend $(3, 3)$.
- A unilateral armament grants a short-term edge $(4, 1)$, yet the equilibrium remains $(2, 2)$.

# Example: Oil Extraction Game

| | | Firm 2 | |
|---|---|---|---|
| | | Conserve | Drill Hard |
| Firm 1 | Conserve | (3, 3) | (1, 4) |
| | Drill Hard | (4, 1) | (2, 2) |

- Two firms coordinate extraction from a shared oil field.
- Mutual restraint preserves the resource and splits revenue $(3, 3)$.
- If one drills aggressively it enjoys a short-term gain $(4, 1)$, but mutual over-drilling drags payoffs to $(2, 2)$.

# Prisoner's Dilemma

| | | Prisoner 2: Cooperate | Prisoner 2: Defect |
|---|---|---|---|
| Prisoner 1 | Cooperate | (3, 3) | (1, 4) |
| | Defect | (4, 1) | (2, 2) |

- Defect strictly dominates; the unique Nash equilibrium is (Defect, Defect).
- The teamwork story is widely known as the Prisoner's Dilemma.
- We will keep calling it the Teamwork Game to avoid the misleading prison imagery.

# Teamwork Game

| | | Binh: Work Hard | Binh: Slack Off |
|---|---|---|---|
| Anh | Work Hard | (3, 3) | (1, 4) |
| | Slack Off | (4, 1) | (2, 2) |

The Teamwork Game has three hallmarks:

- Both players earn more by cooperating than they do when both shirk.
- Each individual still has an incentive to defect, so cooperation unravels.
- Social desirability and private incentives are misaligned.

# Communication (Discussion)

**Communication**

In game theory we often assume, implicitly, that players can communicate extensively before the game starts.

- Even with pre-play communication, Slack Off still maximises your own payoff, so a fully rational player sticks with Slack Off.
- Cheap talk does not authorise commitments, side payments, or any other rule-breaking behaviour after the game.
- We revisit how communication can help once we analyse repeated play and enforcement in later lectures.

# Mechanism Design

We will work through many game examples today, but keep the bigger picture in mind:

**Definition: Mechanism Design**

Mechanism design studies how to craft rules that steer the behaviour of informed agents so that desirable social objectives—efficiency, equity, stability, and the like—are achieved.

# Mechanism Design

**Definition: Transfer**

A transfer is a payment of money or resources between players, or between a mechanism and players, that reallocates payoffs without destroying total surplus.

Example: monetary payments in auctions. tax schemes.

- Contracts that involve money are not always enforceable; we must consider whether players voluntarily participate.
- When payoffs are non-transferable (e.g. the arms-race game), transfers are not an appropriate tool.

**Definition: Money burning**

Resources that are consumed rather than transferred to any agent, thereby diminishing total surplus.

# Mechanism Design: Transfers

| | |
|---|---|
| (3, 3) | (1, 4) |
| (4, 1) | (2, 2) |

| | |
|---|---|
| (3, 3) | (3, 2) |
| (2, 3) | (2, 2) |

Transfer

- Suppose only one player chooses Work Hard; adding a contract that transfers 2 units from the shirker to the worker implements $(\text{Work Hard}, \text{Work Hard})$.
- Such a transfer requires that the outcome is verifiable and observable, so the shirker can be identified and the contract enforced.
- Each player then earns 3 instead of the outside option 2, so both willingly accept the contract.

# Mechanism Design: Money Burning

| | |
|---|---|
| (3, 3) | (1, 4) |
| (4, 1) | (2, 2) |

| | |
|---|---|
| (3, 3) | (1, 2) |
| (2, 1) | (0, 0) |

Money burning

- Each student reports whether the partner chose Slack Off when submitting the assignment.
- Any student reported as slacking receives a penalty of 2, reducing total surplus but deterring shirking.

# A Catalogue of 2x2 Games

# Coordination Game (Recap 01-01)

| | | Player B | |
|---|---|---|---|
| | | Vietnam | Japan |
| Player A | Vietnam | (3, 3) | (1, 1) |
| | Japan | (1, 1) | (2, 2) |

- Both (Vietnam, Vietnam) and (Japan, Japan) are Nash equilibria.
- The Vietnam outcome yields higher payoffs for both players.

# Battle of the Sexes (Recap 03-01)

| | | Player B | |
|---|---|---|---|
| | | Opera | Football |
| Player A | Opera | (2, 1) | (0, 0) |
| | Football | (0, 0) | (1, 2) |

- The coordination game we analysed as a Nash equilibrium example in 03-01.
- Both players want to meet but have different preferences: Opera versus Football.
- $(\text{Opera}, \text{Opera})$ and $(\text{Football}, \text{Football})$ coexist as pure Nash equilibria.

# Stag Hunt (Assurance Game)

| | | Player B | |
|---|---|---|---|
| | | Stag | Rabbit |
| Player A | Stag | (4,4) | (0,3) |
| | Rabbit | (3,0) | (2,2) |

- Two hunters choose between catching a small prey (hare) on their own or coordinating to capture a stag for a larger payoff.
- A stag can only be captured in tandem; a lone attempt fails and leaves the hunter with a low payoff.

# Bank Run Coordination

| | | P2 Stay (S) | P2 Withdraw (W) |
|---|---|---|---|
| P1 | Stay (S) | (3, 3) | (0, 2) |
| | Withdraw (W) | (2, 0) | (1, 1) |

- A coordination game in which depositors move simultaneously; both $(S, S)$ and $(W, W)$ are pure Nash equilibria.
- If everyone believes deposits are safe, the good equilibrium $(S, S)$ prevails; rumours or liquidity fears push the system toward the run $(W, W)$.
- Institutions such as deposit insurance or central-bank liquidity shape which equilibrium is selected.

# Chicken race game (Hawk – Dove)

| | | Driver B | |
|---|---|---|---|
| | | Swerve | Straight |
| Driver A | Swerve | (2,2) | (1,4) |
| | Straight | (4,1) | (0,0) |

- Two drivers choose between swerving and driving straight.
- If only one drives straight, that driver gains; if both go straight, both suffer the worst outcome.
- The game has two asymmetric Nash equilibria.
- Goldenball game.

# Matching Pennies (Zero-Sum)

| | | Player B | |
|---|---|---|---|
| | | Heads | Tails |
| Player A | Heads | (1,-1) | (-1,1) |
| | Tails | (-1,1) | (1,-1) |

- Classic zero-sum structure: one player's gain of 1 is the other's loss of $-1$.
- Because neither player can anticipate the opponent's move, no pure strategy is a best response.
- This is our first example without a pure Nash equilibrium, motivating mixed strategies.

# Rational Pig Game

| | | Minor: Invest | Minor: Wait |
|---|---|---|---|
| Giant | Invest | (4,-1) | (3,1) |
| | Wait | (1,3) | (0,0) |

- Firms compete over whether to invest in a project.
- If either firm invests, the industry creates surplus worth 4.
- When the giant invests, it captures the entire surplus thanks to its bargaining power.
- If only the minor firm invests, the surplus accrues to the minor firm instead.
- Investment costs 4 for either firm; there is a positive externality of 2 on the rival whenever one party invests.

# Rational Pig Game

| | | Minor: Invest | Minor: Wait |
|---|---|---|---|
| Giant | Invest | (3,-1) | (1,2) |
| | Wait | (2,1) | (0,0) |

- The Nash equilibrium is $(\text{Invest}, \text{Wait})$.
- Although they share a market, the dominant and minor firms adopt different equilibrium strategies.
- In this payoff structure the minor firm ends up with the higher profit.

# Volunteer Dilemma

| | | Player 2 | |
|---|---|---|---|
| | | Volunteer | Not |
| Player 1 | Volunteer | $(b-c, b-c)$ | $(b-c, b)$ |
| | Not | $(b, b-c)$ | $(0,0)$ |

- The public good is provided if at least one of the two players volunteers.
- Volunteering carries cost $c > 0$, while everyone enjoys the benefit $b$ with $b > c$.

# Three-Player Coordination Game

Player 3: $L$

| | | Player 2: $L$ | Player 2: $C$ |
|---|---|---|---|
| Player 1 | $L$ | $(3,3,3)$ | $(2,0,2)$ |
| | $C$ | $(0,2,2)$ | $(2,2,0)$ |

Player 3: $C$

| | | Player 2: $L$ | Player 2: $C$ |
|---|---|---|---|
| Player 1 | $L$ | $(2,2,0)$ | $(0,2,2)$ |
| | $C$ | $(2,0,2)$ | $(3,3,3)$ |

- Players 1–3 propose either the library ($L$) or cafe ($C$) as a meeting point.
- If at least two choose the same place it becomes the meeting location; a player who ends up alone is worse off.

# Matching Pennies as Penalty Kicks

| | | Goalkeeper | |
|---|---|---|---|
| | | Dive Left | Dive Right |
| Kicker | Shoot Left | (1,-1) | (-1,1) |
| | Shoot Right | (-1,1) | (1,-1) |

- The kicker scores (payoff 1) by choosing the opposite side from the keeper and is blocked (payoff $-1$) if they match.
- The goalkeeper's payoffs are the reverse: matching the kicker succeeds, diving opposite fails.
- Neither player can stick with a pure strategy, illustrating why mixed strategies are needed (preview of the next lecture).

# Other Games to Remember

- Ultimatum Game: fairness and strategic bargaining.
- War of Attrition: war-of-nerves with exit costs.
- Rock – Paper – Scissors: three-way dominance cycle requiring mixed equilibrium.

# Preparing for Mixed Strategies

- Games without pure equilibria (e.g. Matching Pennies) require finding where best-response lines intersect.
- In a best-response diagram, the intersection of the two players' response curves identifies the Nash equilibrium.
- In 04-01, 04-02, we will compute that intersection as probability distributions and connect it to the Nash existence theorem.
- Practise solving for the probabilities that leave opponents indifferent—it smooths the transition to next time.

# Summary

# Today's Takeaways

- Finite games are defined by finite player sets, strategy sets, and payoff functions.
- Mastering canonical two-player games sharpens your intuition for Nash equilibria.
- Multi-player settings introduce new themes such as public goods and volunteering dilemmas.

# Self-Check

- Can you quickly identify whether each game has a dominant strategy?
- When multiple equilibria exist, can you describe payoff dominance versus risk dominance in words?
- For games without pure equilibria (such as Matching Pennies), can you articulate why mixed strategies are required next time?
- Can you explain the solutions to the 03-01 exercises together with the reasoning steps?

# Before Next Session

- Review where the Nash equilibria and Pareto-efficient outcomes sit in each game.
- Compute the symmetric mixed equilibrium for the Volunteer Dilemma.
- Revisit how IESDS and Nash equilibrium related in lecture 03-01.

# Check Your Understanding

**Task**

1. Using the generic Prisoner's Dilemma payoffs $(T, R, P, S)$, derive the conditions under which $(D, D)$ is the Nash equilibrium.
2. For the Volunteer Dilemma with $b = 5$, $c = 2$, and $n = 3$, solve for the symmetric mixed equilibrium probability.