# Game Theory 10-03

## Exercises: Auctions and Matching Markets

BCSE Game Theory

Dec. 9, 2025

# Exercise Session

## Auction Theory & Matching Markets

# Answer on Google Slides

https://sites.google.com/vju.ac.vn/bcse-gt

- Submit one PDF per team.
- Show all calculations for auction problems.
- For matching, verify stability by checking all pairs.

**Notes**

1. For Q1-Q3, remember to check equilibrium conditions.
2. For Q4, compare profits with and without reserve price.
3. For Q5, check all blocking pairs systematically.

# Q1. First-Price Auction Equilibrium

## Q1. Bidding Strategy Derivation

Consider a First-Price Auction with $n = 3$ bidders. Valuations are i.i.d. uniform on $[0, 10]$.

1. Using the equilibrium bidding formula $b^*(v) = \frac{n-1}{n}v$, calculate the bid for a bidder with valuation $v = 6$.
2. If this bidder wins, what is their expected profit?
3. Calculate the expected revenue for the seller (use $E[v_{(1)}] = \frac{n}{n+1} \cdot 10$ for the highest valuation).
4. **Bonus**: If the distribution were Uniform$[0, 20]$ instead, how would the equilibrium bid change for $v = 12$?

# Q2. Revenue Equivalence Theorem

## Q2. Comparing Auction Revenues

Consider $n = 2$ bidders with valuations $v_i \sim \text{Uniform}[0, 1]$.

1. **First-Price Auction**:
   - Equilibrium bid: $b^*(v) = \frac{1}{2}v$.
   - Calculate expected revenue: $E[R_{1\text{st}}] = E[\frac{1}{2}v_{(1)}]$ where $E[v_{(1)}] = \frac{2}{3}$.
2. **Second-Price Auction**:
   - Winner pays second-highest valuation.
   - Calculate expected revenue: $E[R_{2\text{nd}}] = E[v_{(2)}]$ where $E[v_{(2)}] = \frac{1}{3}$.
3. Verify that $E[R_{1\text{st}}] = E[R_{2\text{nd}}]$.
4. **Bonus**: Explain intuitively why the revenues are equal despite different payment rules.

# Q3. Optimal Reserve Price

## Q3. Reserve Price Analysis

A seller faces $n = 2$ bidders with valuations $v_i \sim \text{Uniform}[0, 1]$. The seller considers setting a reserve price $r$.

1. **Without Reserve** ($r = 0$):
   - Expected revenue: $E[R] = \frac{1}{3}$ (from Q3).
2. **With Reserve** ($r = 0.5$):
   - Probability of sale: $\Pr(\text{sale}) = 1 - r^n = 1 - 0.25 = 0.75$.
   - Expected revenue (given formula): $E[R] = \frac{n-1}{n+1}(1 - r^{n+1}) + r(1 - r^n)$.
   - Calculate $E[R]$ for $r = 0.5$.
3. Compare the two revenues. Does the reserve price increase revenue?
4. **Discussion**: Why might a seller prefer $r = 0.5$ even though it reduces the probability of sale?

# Q4. Matching Market Stability

## Q4. Gale-Shapley Algorithm

Three students $\{S_1, S_2, S_3\}$ and three schools $\{A, B, C\}$ have the following preferences:

| **Students** | 1st | 2nd | 3rd |
|---|---|---|---|
| $S_1$ | $A$ | $B$ | $C$ |
| $S_2$ | $B$ | $A$ | $C$ |
| $S_3$ | $A$ | $B$ | $C$ |

| **Schools** | 1st | 2nd | 3rd |
|---|---|---|---|
| $A$ | $S_2$ | $S_1$ | $S_3$ |
| $B$ | $S_1$ | $S_3$ | $S_2$ |
| $C$ | $S_2$ | $S_1$ | $S_3$ |

1. Run the Gale-Shapley algorithm with students proposing. Show each round.
2. Verify the resulting matching is stable (no blocking pairs).
3. **Bonus**: What matching would result if schools proposed instead?