Game Theory 13-02

Review: Games of Incomplete Information

BCSE Game Theory

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Part 1: Static Games of Incomplete Information

Auctions

Review: Bayesian Nash Equilibrium

- **Incomplete Information**: Players do not know opponents' payoff functions (types).
- **Example**: Auctions (values are private).
- **Strategy**: A function $s_i(v_i)$ mapping type to action.
- **Bayesian Nash Equilibrium (BNE)**: Each type v_i maximizes expected utility given the strategies of others.

Exercise 1: First-Price Auction (3 Bidders)

Problem

- 3 Bidders ($N = 3$).
- Private values v_i independently distributed uniform on $[0, 1]$.
- **First-Price Sealed-Bid**: Highest bidder wins and pays their bid.
- Guess a linear strategy: $b_i(v_i) = k \cdot v_i$.

Task:

1. Determine probability of winning for Bidder 1 with bid b, assuming Bidders 2 and 3 follow $b(v) = kv$.
2. Write down Bidder 1's Expected Payoff function.
3. Solve for the optimal bid b^* and find the equilibrium constant k.

Ex 1: Solution (Probability & Payoff)

1. Probability of Winning: Win if $b_1 > B_j = kv_j \implies v_j < b_1/k$ for all $j \neq 1$.

$$P(\text{Win}) = \prod_{j \neq 1} P(v_j < \frac{b_1}{k}) = \left[F\left(\frac{b_1}{k}\right) \right]^{N-1}$$

For Uniform $[0,1]$, $F(x) = x$.

$$P(\text{Win}) = \left(\frac{b_1}{k}\right)^{N-1} = \left(\frac{b_1}{k}\right)^2 \quad (N = 3)$$

2. Expected Payoff:

$$E[\pi_1(b, v_1)] = (v_1 - b_1) \cdot \left[F\left(\frac{b_1}{k}\right) \right]^{N-1}$$

Ex 1: Solution (Equilibrium)

3. Optimization: Maximize w.r.t b:

$$\frac{\partial E}{\partial b} = -[F(\cdot)]^{N-1} + (v-b)(N-1)[F(\cdot)]^{N-2} f(\cdot)\frac{1}{k} = 0$$

Substitute Uniform $\big(F(x) = x, f(x) = 1, F(\cdot) = b/k\big)$:

$$-(\frac{b}{k})^{N-1} + (v-b)(N-1)(\frac{b}{k})^{N-2}\frac{1}{k} = 0$$

Multiply by $k(b/k)^{2-N}$:

$$-b + (v-b)(N-1) = 0 \implies bN = v(N-1) \implies \mathbf{b}^* = \frac{\mathbf{N-1}}{\mathbf{N}}\mathbf{v}$$

Comparing to guess $b(v) = kv$, we see $k = \frac{2}{3}$.

Result

In an FPSB auction with N bidders, the equilibrium strategy is:

$$b^*(v) = \frac{N-1}{N}v$$

Part 2: Dynamic Games of Incomplete Information

Signaling Equilibrium Selection

Review: Perfect Bayesian Equilibrium (PBE)

In dynamic games of incomplete information, purely strategy-based NE is insufficient. We need to track **Beliefs**.

- **Belief System** μ: At every information set, players assign probabilities to nodes (types).
- **Sequential Rationality**: Strategies are optimal given beliefs μ.
- **Consistency (Bayes' Rule)**: On the equilibrium path, beliefs are derived from strategies using Bayes' Rule.

PBE = Strategy Profile + Belief System satisfying these conditions.

Review: Refining Equilibria (Intuitive Criterion)

PBE places no restriction on **off-equilibrium beliefs**, allowing "unreasonable" equilibria (e.g., pooling sustained by believing any deviator is the "worst" type).

Intuitive Criterion (Cho & Kreps):

- If a deviation is **dominated** for type L (payoff worse than equilibrium payoff no matter the belief) but **possibly advantageous** for type H, then...
- Receivers should NOT believe the deviator is L. They should believe it is H.
- This destroys equilibria relying on "if deviation, believe L".

Exercise 2: Advertising Game (Selection)

Problem

- Types H, L with prior $\Pr(H) = 0.5$.
- Payoffs (Firm):
  - With Ad + Sale: H gets $4 - c$, L gets $1 - c$.
  - Without Ad + Sale: H gets 4, L gets 1.
  - No Sale: Payoff is $-c$ (if Ad) or 0 (if No Ad).
- Consumer: Buys if $\mu(H) \geq 0.5$.

Case 1: $c = 0.8$ (Low signaling cost).
Case 2: $c = 1.5$ (High signaling cost).

Exercise 2: Advertising Game (Questions)

Task

1. For each case ($c = 0.8$ and $c = 1.5$), determine which equilibria (Pooling / Separating) can exist as PBE.
2. Explain why Low type might want to mimic High type.
3. Apply the Intuitive Criterion to refine the equilibrium selection.

Ex 2: Solution (Case 1: $c = 0.8$)

Low Signaling Cost: Both types can afford advertising.

- **Pooling (Both No Ad)**:
  - Prior $\mu = 0.5 \Rightarrow$ Consumer Buys.
  - Payoffs: $H = 4, L = 1$.
  - Deviation to Ad: Payoffs become $3.2, 0.2$ (worse).
  - **This is a PBE**.
- **Separating ($H \to$ Ad, $L \to$ No Ad)**:
  - L (No Ad) is revealed $\Rightarrow$ No Sale $\Rightarrow$ Payoff $= 0$.
  - If L mimics (Ad): Sale $\Rightarrow$ Payoff $= 1 - 0.8 = 0.2 > 0$.
  - L has incentive to deviate $\Rightarrow$ **Separation fails**.

Result: Only Pooling exists when $c < 1$.

Ex 2: Solution (Case 2: $c = 1.5$)

High Signaling Cost: Only High type can afford advertising.

- **Separating ($H \to$ Ad, $L \to$ No Ad)**:
  - H: Ad + Sale $\Rightarrow$ Payoff $= 4 - 1.5 = 2.5 > 0$.
  - L: If mimics (Ad) $\Rightarrow$ Payoff $= 1 - 1.5 = -0.5 < 0$.
  - L prefers No Ad (Payoff $= 0$) $\Rightarrow$ **Separation is stable**.
- **Pooling (Both Ad)**:
  - L: Ad + Sale $\Rightarrow$ Payoff $= 1 - 1.5 = -0.5 < 0$.
  - L prefers deviation $\Rightarrow$ **Pooling on Ad fails**.

Result: Only Separating exists when $c > 1$.

Ex 2: Intuitive Criterion Application

When multiple equilibria exist (e.g., $c = 1$, L indifferent):

- **Intuitive Criterion** helps select the "reasonable" equilibrium.
- If deviation is **equilibrium-dominated** for L but **beneficial** for H:
  - Receiver should believe deviator is H, not L.
  - This typically **eliminates pooling** equilibria.

Key Insight

- Signaling cost $c < 1$: Pooling survives (mimicking is profitable).
- Signaling cost $c > 1$: Separating survives (mimicking is too costly).
- At boundary $c = 1$: Intuitive Criterion favors Separating.