BCSE Game Theory 09-02

Incomplete Information in Continuous Games

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Dec. 3, 2025

Today's Agenda

Today's Goals

- Extend incomplete information to continuous strategy spaces.
- Analyze Cournot competition with unknown costs.
- Analyze Bertrand competition with unknown costs.
- Understand when high-cost firms exit the market.
- Study Best Price Guarantee as a mechanism to reduce price competition.
- Compare complete vs incomplete information outcomes.

Lecture Roadmap

1. Review: incomplete information in finite games (09-01).
2. Extension: from discrete to continuous strategies.
3. Example 1: Incomplete information Cournot competition (symmetric case).
4. Example 2: Incomplete information Cournot competition (high-cost firm exits).
5. Example 3: Incomplete information Bertrand competition.
6. Best Price Guarantee: how it reduces price competition.
7. Comparison: complete vs incomplete information.
8. Takeaways and next steps.

Review: Incomplete Information Basics

What We Learned in 09-01

- **Incomplete information**: players don't know others' types.
- **Type**: private information (cost, valuation, etc.).
- **Nature**: chooses types randomly at the start.
- **Harsanyi transformation**: converts incomplete information games into extensive-form games.
- **Bayesian Nash equilibrium**: each type chooses a best response given beliefs.

From Discrete to Continuous Strategies

- In 09-01, we studied finite games (discrete actions: Enter/Stay out, High/Low).
- Now we extend to continuous strategies (quantities, prices).
- Methods:
  - Compute best responses using calculus (first-order conditions).
  - Solve systems of equations for equilibrium.
  - Compare outcomes under complete vs incomplete information.
- Key difference: best responses are functions, not just actions.

Example 1: Incomplete Information Cournot Competition

Cournot Competition: Setup

- Two firms compete in quantities.
- Market demand: $P(Q) = a - bQ$, where $Q = q_1 + q_2$.
- Each firm has cost type: High cost c_H or Low cost c_L (where $c_H > c_L$).
- Each firm knows its own cost but not the opponent's cost.
- Prior beliefs: Firm 1 believes Firm 2 is High cost with probability μ, Low cost with probability $1 - \mu$.
- Firm 2 has symmetric beliefs about Firm 1.

Cournot Competition: Profit Functions

- Firm i's profit: $\pi_i = (P(Q) - c_i)q_i = (a - b(q_i + q_j) - c_i)q_i$.
- Firm i chooses q_i to maximize profit.
- First-order condition:

$$\frac{\partial \pi_i}{\partial q_i} = a - 2bq_i - bq_j - c_i = 0$$

- Best response: $q_i^*(q_j) = \frac{a-c_i-bq_j}{2b}$.
- Higher cost → lower quantity (for given q_j).

Cournot Competition: Expected Best Response

- Firm 1 (with cost c_1) expects Firm 2 to choose:
  - $q_2^H = \frac{a - c_H - bq_1}{2b}$ if Firm 2 is High cost (probability μ).
  - $q_2^L = \frac{a - c_L - bq_1}{2b}$ if Firm 2 is Low cost (probability $1 - \mu$).
- Firm 1's expected profit:

$$E[\pi_1] = \mu \cdot \pi_1(q_1, q_2^H) + (1 - \mu) \cdot \pi_1(q_1, q_2^L)$$

- Firm 1 chooses q_1 to maximize expected profit.

Cournot Competition: Equilibrium Quantities

- In equilibrium, each firm type chooses a quantity that is a best response.
- Firm 1 (Low cost): $q_1^L = \frac{a - c_L - bE[q_2]}{2b}$.
- Firm 1 (High cost): $q_1^H = \frac{a - c_H - bE[q_2]}{2b}$.
- Firm 2 (Low cost): $q_2^L = \frac{a - c_L - bE[q_1]}{2b}$.
- Firm 2 (High cost): $q_2^H = \frac{a - c_H - bE[q_1]}{2b}$.
- Expected quantities: $E[q_1] = \mu \cdot q_1^H + (1 - \mu) \cdot q_1^L$, $E[q_2] = \mu \cdot q_2^H + (1 - \mu) \cdot q_2^L$.

Cournot Competition: Solving the System

- We have 4 equations (one for each type) and 2 expectations.
- Substituting and solving:

$$q_1^L = \frac{a - c_L - b(\mu q_2^H + (1-\mu)q_2^L)}{2b}$$

$$q_1^H = \frac{a - c_H - b(\mu q_2^H + (1-\mu)q_2^L)}{2b}$$

- Similar equations for Firm 2.
- Solving this system gives equilibrium quantities for all types.

Cournot Competition: Key Results

- Low-cost firms produce more: $q_i^L > q_i^H$.
- Expected total quantity is higher than in complete information (when both are known to be low cost).
- Expected price is lower than in complete information (when both are known to be high cost).
- Incomplete information increases expected consumer surplus.
- Firms with low costs benefit from incomplete information (can produce more without revealing their advantage).

Example 2: High-Cost Firm Exits

Multiple Equilibria

Setup: Asymmetric Information

- Firm 1 has Low cost c_L (known to both firms).
- Firm 2 has cost type: High cost c_H (probability μ) or Low cost c_L (probability $1 - \mu$).
- Firm 2 knows its cost, but Firm 1 does not.
- Market demand: $P(Q) = a - bQ$, where $Q = q_1 + q_2$.
- **Key Question**: Can there be an equilibrium where the High-cost type exits ($q_2^H = 0$)?

Critical Point

Firm 1 does not know Firm 2's type, so Firm 1 must choose a **single** quantity q_1, not conditional on type.

Best Response Functions

- Firm 2's best response (if producing):

$$q_2 = \max\left(0, \frac{a - c_2 - bq_1}{2b}\right)$$

- Firm 2 (High cost) exits if $q_1 \geq \frac{a-c_H}{b}$.
- Firm 2 (Low cost) produces positive quantity if $q_1 < \frac{a-c_L}{b}$.
- Firm 1's expected profit depends on:
  - Probability μ that Firm 2 is High cost.
  - Firm 2's response to q_1 for each type.

Equilibrium 1: Both Types Enter (Pooling)

- Suppose both types of Firm 2 produce positive quantities.
- Firm 1 chooses q_1 to maximize:

$$E[\pi_1] = \mu \cdot (a - b(q_1 + q_2^H) - c_L)q_1 + (1-\mu) \cdot (a - b(q_1 + q_2^L) - c_L)q_1$$

- where $q_2^H = \frac{a - c_H - bq_1}{2b}$ and $q_2^L = \frac{a - c_L - bq_1}{2b}$.
- Solving yields equilibrium quantities (similar to Example 1).
- **Condition**: This is an equilibrium if High-cost type earns non-negative profit.

Equilibrium 2: High Type Exits (Separating)

- Suppose Firm 2 (High cost) exits: $q_2^H = 0$.
- Firm 1 chooses q_1 to maximize:

$$E[\pi_1] = \mu \cdot (a - bq_1 - c_L)q_1 + (1-\mu) \cdot (a - b(q_1 + q_2^L) - c_L)q_1$$

- where $q_2^L = \frac{a - c_L - bq_1}{2b}$.
- Firm 1 produces more than in Equilibrium 1 (no competition from High type).
- **Condition**: This is an equilibrium if:
  - High-cost type prefers to exit: $\pi_2^H(q_1) \leq 0$.
  - Low-cost type prefers to enter: $\pi_2^L(q_1) > 0$.

Multiple Equilibria: Intuition

- **Self-Fulfilling Expectations**:
  - If Firm 1 expects High type to enter → Firm 1 produces less → High type finds it profitable to enter.
  - If Firm 1 expects High type to exit → Firm 1 produces more → High type finds it unprofitable to enter.
- Firm 1's quantity choice affects Firm 2's entry decision.
- For certain parameter values, both equilibria can coexist.
- This illustrates how incomplete information can lead to multiple equilibria.

Key Insights: Exit and Entry

- High-cost firms may exit when facing aggressive competition.
- Firm 1's optimal quantity depends on beliefs about Firm 2's entry.
- Multiple equilibria arise from coordination on entry expectations.
- Consumer surplus is lower in the separating equilibrium (less competition).
- This illustrates how incomplete information affects market structure.

Example 3: Incomplete Information Bertrand Competition

Bertrand Competition: Setup

- Two firms compete in prices.
- Products are perfect substitutes (homogeneous goods).
- Market demand: $Q(p) = a - bp$, where $p = \min(p_1, p_2)$.
- Each firm has cost type: High cost c_H or Low cost c_L (where $c_H > c_L$).
- Each firm knows its own cost but not the opponent's cost.
- Prior beliefs: Firm 1 believes Firm 2 is High cost with probability ρ, Low cost with probability $1 - \rho$.

Bertrand Competition: Profit Functions

- If $p_i < p_j$: Firm i captures all demand, profit $\pi_i = (p_i - c_i)(a - bp_i)$.
- If $p_i > p_j$: Firm i gets no demand, profit $\pi_i = 0$.
- If $p_i = p_j$: Firms split demand, profit $\pi_i = \frac{1}{2}(p_i - c_i)(a - bp_i)$.
- Under complete information: both firms set $p = \max(c_1, c_2)$ (marginal cost pricing).
- Under incomplete information: firms don't know the opponent's cost, so pricing is more complex.

Bertrand Competition: Best Response

- Firm i (with cost c_i) expects Firm j to choose:
  - p_j^H if Firm j is High cost (probability ρ).
  - p_j^L if Firm j is Low cost (probability $1 - \rho$).
- Firm i's expected profit:

$$E[\pi_i] = \rho \cdot \pi_i(p_i, p_j^H) + (1 - \rho) \cdot \pi_i(p_i, p_j^L)$$

- Firm i chooses p_i to maximize expected profit.
- Key: Firm i must undercut the expected price of Firm j to win the market.

Bertrand Competition: Equilibrium

- In equilibrium, each firm type chooses a price that is a best response.
- Firm 1 (Low cost): $p_1^L = \text{argmax}_{p_1} E[\pi_1(p_1, p_2)]$.
- Firm 1 (High cost): $p_1^H = \text{argmax}_{p_1} E[\pi_1(p_1, p_2)]$.
- Firm 2 (Low cost): $p_2^L = \text{argmax}_{p_2} E[\pi_2(p_1, p_2)]$.
- Firm 2 (High cost): $p_2^H = \text{argmax}_{p_2} E[\pi_2(p_1, p_2)]$.
- All four prices are determined simultaneously.

Bertrand Competition: Key Results

- Low-cost firms set lower prices: $p_i^L < p_i^H$.
- Prices are higher than under complete information (when both are known to be low cost).
- Incomplete information softens price competition.
- Expected consumer surplus is lower than under complete information.
- Firms benefit from incomplete information (can charge higher prices).

Best Price Guarantee

What Is Best Price Guarantee?

Best Price Guarantee (BPG)

A firm promises: "If you find a lower price elsewhere, we will match it and refund the difference."

- Common in retail (electronics, appliances, etc.).
- Also called "price matching guarantee" or "lowest price guarantee".
- Intended to signal low prices and build customer trust.
- But it can also reduce price competition!

Best Price Guarantee: Intuition

- Without BPG: firms compete aggressively on price.
- With BPG: if Firm 1 lowers price, Firm 2 matches it (due to guarantee).
- Firm 1 gains nothing from lowering price (Firm 2 matches).
- Therefore, firms have less incentive to lower prices.
- **Result**: BPG can lead to higher equilibrium prices.

Best Price Guarantee: Incomplete Information

- Under incomplete information, BPG has additional effects.
- Firms don't know each other's costs.
- BPG can signal information about costs:
  - Low-cost firms are more willing to offer BPG (can afford to match).
  - High-cost firms may avoid BPG (cannot afford to match low prices).
- BPG can reduce uncertainty about opponent's type.
- This information effect can further reduce price competition.

Best Price Guarantee: Formal Analysis

- Suppose both firms offer BPG.
- If Firm 1 sets p_1 and Firm 2 sets p_2:
  - If $p_1 < p_2$: Firm 2 matches p_1 (due to BPG), both charge p_1.
  - If $p_1 > p_2$: Firm 1 matches p_2 (due to BPG), both charge p_2.
  - If $p_1 = p_2$: both charge $p_1 = p_2$.
- Effective price: $p = \min(p_1, p_2)$.
- Firms split the market: each gets $\frac{1}{2}$ of demand.
- Profit: $\pi_i = \frac{1}{2}(p - c_i)(a - bp)$.

Best Price Guarantee: Equilibrium

- With BPG, firms effectively choose the same price p.
- Each firm's profit: $\pi_i = \frac{1}{2}(p - c_i)(a - bp)$.
- First-order condition: $\frac{\partial \pi_i}{\partial p} = \frac{1}{2}(a - 2bp + bc_i) = 0$.
- Best response: $p_i^* = \frac{a+c_i}{2b}$.
- In equilibrium: $p = \max(p_1^*, p_2^*) = \max\left(\frac{a+c_1}{2b}, \frac{a+c_2}{2b}\right)$.
- This is higher than the competitive price (marginal cost pricing).

Best Price Guarantee: Comparison

Without BPG

- Firms compete on price.
- Equilibrium: $p = \max(c_1, c_2)$.
- Low-cost firm captures all demand.
- High competition, low prices.

With BPG

- Firms match prices.
- Equilibrium: $p = \max\left(\frac{a+c_1}{2b}, \frac{a+c_2}{2b}\right)$.
- Firms split demand.
- Reduced competition, higher prices.

Key Insight

BPG reduces price competition and leads to higher equilibrium prices, benefiting firms but harming consumers.

Best Price Guarantee: Policy Implications

- BPG may seem pro-consumer (promise of low prices).
- But in equilibrium, BPG leads to higher prices.
- This is a form of tacit collusion.
- Regulators may need to scrutinize BPG policies.
- However, BPG can also provide information benefits (signaling low costs).

Comparison: Complete vs Incomplete Information

Cournot Competition: Comparison

	Complete Information	Incomplete Information
Low-cost firm quantity	Higher	Similar or higher
High-cost firm quantity	Lower	Similar or lower
Expected total quantity	Known	Higher (on average)
Expected price	Known	Lower (on average)
Consumer surplus	Known	Higher (on average)

- Incomplete information can benefit consumers (more competition on average).
- Low-cost firms benefit from incomplete information (can produce more).
- High-cost firms may be worse off (face more competition).

Bertrand Competition: Comparison

	Complete Information	Incomplete Information
Low-cost firm price	c_L	Higher
High-cost firm price	c_H	Higher
Expected price	$\min(c_1, c_2)$	Higher
Consumer surplus	Higher	Lower
Firm profits	Lower	Higher

- Incomplete information softens price competition.
- Prices are higher than under complete information.
- Firms benefit, consumers are worse off.
- Best Price Guarantee further reduces competition.

Key Differences

- **Cournot**: Incomplete information can increase competition (higher expected quantities).
- **Bertrand**: Incomplete information reduces competition (higher prices).
- **Reason**: In Cournot, firms choose quantities (strategic substitutes).
- In Bertrand, firms choose prices (strategic complements with BPG).
- Market structure matters for how incomplete information affects outcomes.

Takeaways

Key Concepts

- Incomplete information extends naturally to continuous strategy spaces.
- Best responses are computed using calculus (first-order conditions).
- Equilibrium requires solving systems of equations.
- High-cost firms may exit the market if costs are too high.
- Best Price Guarantee can reduce price competition.

Key Insights

- Incomplete information affects market outcomes differently in Cournot vs Bertrand.
- Cournot: incomplete information can benefit consumers (more competition).
- Bertrand: incomplete information harms consumers (less competition).
- Best Price Guarantee further reduces competition in price-setting games.
- Policy implications: BPG may require regulatory scrutiny.

Next Steps

- Game Theory 10: Auction theory (applying incomplete information to bidding).
- We'll study different auction formats (first-price, second-price, etc.).
- We'll analyze how bidders' private valuations affect auction outcomes.
- We'll compare auction mechanisms and their properties.