BCSE Game Theory 11-01

Dynamic Games with Incomplete Information

BCSE Game Theory

Dec. 30, 2025

Introduction & Motivation

Course Map

	Complete Info	**Incomplete Info**
Static	Nash Equilibrium (NE)	Bayesian Nash Eq. (BNE)
Dynamic	Subgame Perfect Nash Eq. (SPNE)	**Perfect Bayesian Eq. (PBE)**

- We have covered Static (NE, BNE) and Dynamic Complete Info (SPNE).
- Today, we enter the most complex domain: **Dynamic Games with Incomplete Information**.
- **Key features**: Sequential moves + Private information.

Why is this Hard?

- In **Dynamic Complete Info** (e.g., Stackelberg), we used **Backward Induction**.
- We looked at the last subgame, found the optimal action, and moved back.
- This relies on the existence of **proper subgames**.
- **Problem**: With Incomplete Information, proper subgames often **do not exist**.
- Why? Players may not know *exactly* where they are in the game tree.
- **Key**: Information sets connect nodes → Cannot split.

Note on Harsanyi Transformation

We use Harsanyi's approach (09-02): Add Nature's move to convert incomplete info to imperfect info. But this creates information sets that prevent proper subgames!

What is a Proper Subgame?

Definition: Proper Subgame

A **proper subgame** is a part of the game tree that:

1. Starts at a **single decision node**.
2. Contains **all** successor nodes.
3. Does NOT split any **information set**.

- **Complete Information**: Every node is its own information set → Many proper subgames.
- **Incomplete Information**: Information sets span multiple nodes → Splitting problem.

Why No Proper Subgames in Entry Deterrence?

- **Harsanyi Transformation** (covered in 09-02): Add Nature's move at the start.
- Nature chooses Incumbent type (Strong or Weak).
- **Problem**: Entrant's decision nodes form an **information set**.
  - Entrant observes Price (P_L or P_H).
  - But doesn't know which type chose it.
  - Nodes following "Strong chooses P_L" and "Weak chooses P_L" are connected.
- **Cannot create subgame** starting after Incumbent's move:
  - Does not start at a singleton node (starts at information set).
  - Entrant must form **beliefs** about which node they are at.

Result The **only** proper subgame is the entire game. Backward induction cannot be applied!

Motivating Example: Limit Pricing Game

Scenario

Consider a market where an **Incumbent (I)** sets a price, and a potential **Entrant (E)** observes the price before deciding to enter.

- **Incumbent**: Can be **Strong (Low Cost)** or **Weak (High Cost)**.
- **Action**: Incumbent chooses **Limit Price (P_L)** or **Monopoly Price (P_M)**.
- **Entrant**: Observes Price (P_L or P_M), then chooses **Enter** or **Out**.
- **Information**: Entrant does NOT observe Incumbent's type (only Price).

Question: Can a Weak Incumbent mimic a Strong Incumbent to

The Strategic Situation

Timing:

1. **Nature** determines Incumbent's type (Strong or Weak).
2. **Incumbent** (Informed) chooses Price (P_L or P_M).
3. **Entrant** (Uninformed) observes Price.
4. **Entrant** decides: **Enter** or **Out**.

Incentives:

- **Strong**: Prefers P_L if it keeps Entrant out.
- **Weak**: P_L is costly, but better than Entry. Wants to hide type.
- **Entrant**: Wants to Enter ONLY against Weak Incumbent.

Why This is Challenging

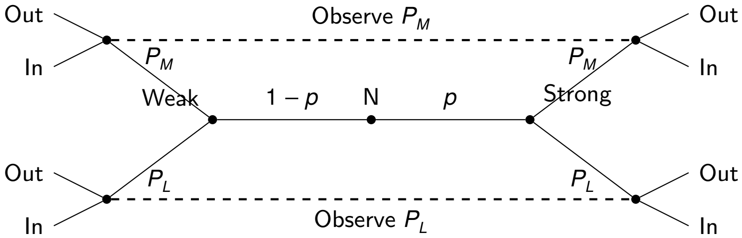
The Entrant's Dilemma:

- Entrant observes P_L. Is it a Strong firm (natural low price) or a Weak firm (bluffing)?
- Entrant must update **Beliefs** (μ) given the observed signal.

The Weak Incumbent's Strategy:

- Weak type wants to **pool** with Strong type to deter entry.
- Strong type wants to **separate** to prove they are strong.
- The outcome depends on whether the signal (P_L) is credible (costly enough).

The Game Tree



Payoffs depend on whether Entrant Enters and Incumbent Type.

Key Observation

Entrant observes the Price but not the Type. Beliefs at information sets determine the Entrant's optimal response.

The Problem with SPNE

- In this game, the only proper subgame is the whole game.
- Thus, **Subgame Perfect Nash Equilibrium (SPNE)** imposes no additional constraints beyond **Nash Equilibrium (NE)**.
- NE allows for **non-credible threats**.
- **Example**: Incumbent threatens "I will Fight even if I am Weak!"
  - If Entrant believes this, staying Out is optimal.
  - But is this credible? If Entrant *did* enter, Weak Incumbent would prefer Accommodate ($1 > 0$).
- SPNE cannot rule out this threat because the Entrant's decision nodes do not initiate a new subgame (due to information sets).

Why BNE is Insufficient

The Role of Beliefs:

- **Bayesian Nash Equilibrium (BNE)** handles incomplete information but focuses on ex-ante strategies.
- It does not explicitly demand that strategies remain rational **given the beliefs** at every information set reachable in the game.

The Need for Consistency:

- Entrant observes a Price (P_L or P_M) and must update their belief about the Incumbent's type.
- Rationality demands action based on these **updated beliefs**.
- BNE does not explicitly enforce this consistency or sequential rationality off the equilibrium path.
- $\rightarrow$ We need **Perfect Bayesian Equilibrium (PBE)**.

Formalizing Beliefs

Information Sets & Beliefs

Definition: System of Beliefs

A **system of beliefs** μ assigns a probability $\mu(x) \in [0, 1]$ to every node x in every information set H, such that $\sum_{x \in H} \mu(x) = 1$.

- For any information set reached, the player must have a *belief* about which node they are at (i.e., what proved to be the history).
- This belief guides their decision.

Defining PBE Components

A **Perfect Bayesian Equilibrium (PBE)** is a pair (σ, μ) consisting of:

1. A strategy profile σ (complete plan of action).
2. A system of beliefs μ.

Two Key Requirements

1. **Sequential Rationality**: Strategies must be optimal given beliefs. 2. **Consistency**: Beliefs must be consistent with strategies (via Bayes' Rule).

Requirement 1: Sequential Rationality

Sequential Rationality

Definition

A strategy profile σ is **sequentially rational** given beliefs μ if, at **every** information set H (on or off the equilibrium path), the player moving at H maximizes their expected utility, given:

1. Their beliefs μ about nodes in H.
2. The strategies of all other players from that point forward.

- This imposes "backward induction logic" even where subgames don't exist.
- "Given what I believe has happened, am I doing the best I can?"

Example: Sequential Rationality

Suppose:

- Information set H has two nodes, x_1 and x_2.
- Beliefs: $\mu(x_1) = 0.8, \mu(x_2) = 0.2$.
- Action A: Payoff 10 at x_1, 0 at x_2.
- Action B: Payoff 6 at x_1, 6 at x_2.

Expected Payoff:

- $E[U(A)] = 0.8(10) + 0.2(0) = 8$.
- $E[U(B)] = 0.8(6) + 0.2(6) = 6$.

$\implies$ Player MUST choose A. Choosing B would violate sequential rationality.

Requirement 2: Consistency

Equilibrium Path Definitions

Definition: On-Equilibrium Path

An information set is **on the equilibrium path** ("On-Path") if it is reached with positive probability under the equilibrium strategies σ^*.

Definition: Off-Equilibrium Path

An information set is **off the equilibrium path** ("Off-Path") if it is reached with probability zero under the equilibrium strategies σ^*.

Implication for Consistency:

- **On-Path**: Beliefs must be derived via Bayes' Rule.
- **Off-Path**: Bayes' Rule doesn't apply (denominator zero). Beliefs can be arbitrary (in PBE) or constrained (in

Consistency (On-Path)

Definition: Bayesian Updating

At any information set reached with **positive probability** given equilibrium strategies σ, beliefs μ must be derived using **Bayes' Rule**.

$$\mu(\text{Node } x | \text{Info Set } H) = \frac{P(\text{reaching } x | \sigma)}{P(\text{reaching } H | \sigma)}$$

- If the strategy says "Type 1 plays Left" and "Type 2 plays Right", and we observe "Left", we must believe it is Type 1.

Consistency (Off-Path)

Problem

What if an information set H is reached with **probability zero** under σ? (e.g., a deviation occurred).

- Bayes' Rule denominator is zero. It is undefined.
- **Standard PBE**: Beliefs can be **arbitrary**.
- However, they must still be specified (cannot be empty).
- The choice of off-path beliefs determines whether a deviation is profitable!

PBE Definition & Refinements

Formal Definition of PBE

Definition: Perfect Bayesian Equilibrium

A profile (σ^*, μ^*) is a PBE if:

1. σ^* is sequentially rational given μ^*.
2. μ^* is consistent with σ^* using Bayes' Rule wherever possible.

Note: Types of PBE

- **Weak PBE**: Minimal restrictions on off-path beliefs.
- **Strong PBE / Sequential Equilibrium**: Stricter consistency.

Sequential Equilibrium (SE)

- Proposed by Kreps and Wilson (1982).
- **Stricter than PBE**.
- **Problem with PBE**: Off-path beliefs can be "anything". This allows implausible equilibria.
- **SE Idea**: Beliefs should not just be "anything" off-path. They should be limits of beliefs from "trembling hand" strategies.

Trembling Hand Strategies

Definition: Completely Mixed Strategy

A strategy σ_ϵ is **completely mixed** if every action is played with probability at least $\epsilon > 0$.

- **Intuition**: Players "tremble" and occasionally make mistakes.
- With trembling, **every** information set is reached with positive probability.
- Thus, Bayes' Rule applies everywhere, giving well-defined beliefs μ_ϵ.

Sequential Equilibrium Definition

Definition: Sequential Equilibrium

A pair (σ^*, μ^*) is a Sequential Equilibrium if there exists a sequence of completely mixed strategies $\{\sigma_n\}$ such that:

1. $\sigma_n \to \sigma^*$ as $n \to \infty$.
2. $\mu_n \to \mu^*$, where μ_n is derived from σ_n via Bayes' Rule.
3. σ^* is sequentially rational given μ^*.

- **Key**: Off-path beliefs must be limits of on-path beliefs from perturbed strategies.

PBE vs Sequential Equilibrium

	PBE	Sequential Equilibrium
On-path beliefs	Bayes' Rule	Bayes' Rule
Off-path beliefs	Arbitrary	Limit of trembles
Difficulty	Easier to compute	Harder to verify
Refinement	Weaker	Stronger

- **Relationship**: Every SE is a PBE, but not every PBE is an SE.
- **In Practice**: For most signaling games, PBE + "Intuitive Criterion" is sufficient.
- **In Research**: Sequential Equilibrium is often preferred for its stronger theoretical foundation, though PBE remains widely used for tractability.
- We focus on PBE for tractability in this course.

Solving PBE: The Algorithm

The 4-Step PBE Algorithm

Since we have circular dependence (Beliefs $\leftrightarrow$ Strategies), we usually solve by **Guess and Verify**:

1. **Conjecture** a strategy profile (e.g., "Try Separating", "Try Pooling on message X").
2. **Derive Beliefs**: Use Bayes' Rule consistent with your conjecture.
   - On-path: Apply Bayes' Rule directly.
   - Off-path: Specify beliefs (will check if they support equilibrium).
3. **Solve Receiver's Optimization**: Given beliefs, what is R's best response to each message?
4. **Check Sender's Optimization**: Given R's response, does S want to deviate?
   - If YES $\rightarrow$ Conjecture fails.
   - If NO $\rightarrow$ PBE found (if off-path beliefs can be justified).

Types of Equilibria in Signaling Games

Equilibrium Types: Definitions

Definition: Separating Equilibrium

A **Separating Equilibrium** is a PBE in which different types of the sender choose different actions, thereby fully revealing their type through their choice.

Definition: Pooling Equilibrium

A **Pooling Equilibrium** is a PBE in which all types of the sender choose the same action, thereby revealing no information about their type.

- In separating equilibria, the receiver can perfectly infer the sender's type.
- In pooling equilibria, the receiver learns nothing new from the sender's action.
- Both can be valid PBE depending on the game's payoff

Concrete Example: TV Advertising Game

Example: TV Advertising Game (Mankiw Model)

Scenario

- **Firm**: High (H) or Low (L) Quality. Prior $P(H) = 0.5$.
- **Action**: Advertise (A) or Not (N). **Cost of Ad = 2** for **both** types.
- **Consumer**: Buy (B) or Don't Buy (D).
- **Repeat Business**: Only High Quality generates repeat sales.
  - Initial Profit from Sale = 1.
  - Repeat Profit = 3 (if H), 0 (if L).

Example: TV Advertising Game (Payoffs)

Payoffs (Firm, Consumer): Assume C prefers H $(+1)$ and dislikes L (-1).

- **High Quality**: 1(Initial) + 3(Repeat) − Cost.
  - $U_H(A, B) = 4 - 2 = 2.$ $U_H(N, D) = 0.$
  - $U_H(N, B) = 4 - 0 = 4.$ $U_H(A, D) = -2.$
- **Low Quality**: 1(Initial) + 0(Repeat) − Cost.
  - $U_L(A, B) = 1 - 2 = -1.$ $U_L(N, D) = 0.$
  - $U_L(N, B) = 1 - 0 = 1.$ $U_L(A, D) = -2.$

Step 1: Conjecture Separating Equilibrium

- **Conjecture**: High Quality plays **Advertise (A)**, Low Quality plays **Not Advertise (N)**.
- Strategy σ_S: $H \to A, L \to N$.

Step 2: Update Beliefs

- If Consumer sees A: Must be High Quality. $\mu(H|A) = 1$.
- If Consumer sees N: Must be Low Quality. $\mu(L|N) = 1$.

Step 3: Consumer's Optimal Action (Separating)

- **If** A **(Belief** $H = 1$**)**:
  - Buy: $U_C(H, A, B) = 2$.
  - Don't Buy: $U_C(H, A, D) = 0$.
  - $\implies$ **Buy**.
- **If** N **(Belief** $L = 1$**)**:
  - Buy: $U_C(L, N, B) = -1$.
  - Don't Buy: $U_C(L, N, D) = 0$.
  - $\implies$ **Don't Buy**.

Reaction: Consumer Buys if sees Ad, Doesn't Buy if no Ad.

Step 4: Check Firm's Constraints (Separating)

- **Check High Quality ($H \to A$)**:
  - Equilibrium Payoff (Play $A \to$ C Buys): $U_H = 2$.
  - Deviation (Play $N \to$ C Doesn't Buy): $U_H = 0$.
  - $2 > 0$. **OK**.
- **Check Low Quality ($L \to N$)**:
  - Equilibrium Payoff (Play $N \to$ C Doesn't Buy): $U_L = 0$.
  - Deviation (Play $A \to$ C Buys): $U_L(A, B) = -1$.
  - $0 > -1$. **OK**. Low Quality loses money even if they trick the consumer!

Conclusion: Separating Equilibrium SUCCEEDS

Separating Equilibrium ($H \to A, L \to N$) **EXISTS**. High cost of advertising screens out Low Quality.

Step 1: Try Pooling Equilibrium

- **Conjecture**: Both types **Advertise (A)**.
- Strategy σ_F: $H \to A, L \to A$.

Step 2: Update Beliefs

- If Consumer sees A: No information revealed. $\mu(H|A) = 0.5$ (Prior).
- If Consumer sees N (off-path): Need to specify. Assume $\mu(L|N) = 1$ (pessimistic).

Step 3: Consumer's Optimal Action (Pooling)

- **If A (Belief $H = 0.5, L = 0.5$)**:
  - Buy: $E[U_C] = 0.5(2) + 0.5(-1) = 0.5$.
  - Don't Buy: $U_C = 0$.
  - $\implies$ **Buy** (since $0.5 > 0$).
- **If N (Belief $L = 1$)**:
  - Buy: $U_C(L, N, B) = -1$.
  - Don't Buy: $U_C(L, N, D) = 0$.
  - $\implies$ **Don't Buy**.

Reaction: Consumer Buys if sees Ad, Doesn't Buy if no Ad.

Step 4: Check Firm's Constraints (Pooling)

- **Check High Quality ($H \to A$)**:
  - Equilibrium Payoff (Play A → C Buys): $U_H = 2$.
  - Deviation (Play N → C Doesn't Buy): $U_H = 0$.
  - $2 > 0$. **OK**.
- **Check Low Quality ($L \to A$)**:
  - Equilibrium Payoff (Play A → C Buys): $U_L = -1$.
  - Deviation (Play N → C Doesn't Buy): $U_L = 0$.
  - $0 > -1$. **L wants to deviate!**

Result

Pooling on Advertise **FAILS**. Low Quality prefers 0 to -1.

Comparison of Equilibria

	Pooling (Both $\rightarrow A$)	**Separating ($H \rightarrow A, L \rightarrow N$)**
Exists?	**NO**	**YES**
Information Revealed	-	Full
High Quality Payoff	-	2
Low Quality Payoff	Deviates ($0 > -1$)	0

- **Key Insight**: Repeat business makes advertising profitable **only** for High Quality.
- Low Quality cannot recoup the ad cost from a single sale.
- Advertising serves as a credible signal of quality (Mankiw).

Summary

Summary

- **Incomplete Information** breaks Backward Induction (no proper subgames).
- We introduce **Beliefs** (μ) and **PBE** = Strategies + Beliefs.
- **Requirements**:
  1. Sequential Rationality: Optimality given beliefs.
  2. Consistency: Correct beliefs given strategies (Bayes' Rule).
- **Sequential Equilibrium**: Stricter refinement using trembling hand.
- **Solving PBE**: Conjecture → Beliefs → R's BR → S's IC.
- **Equilibrium Types**: Separating (types reveal) vs Pooling (types hide).
- **TV Advertising Game**: Separating **SUCCEEDS**, Pooling fails (Mankiw's result).
- **Next Lecture**: Refinements (Intuitive Criterion) and Ramen-Salad Game.

Knowledge Check

1. Why doesn't SPNE work for games with Incomplete Information?
2. What are the two components of a PBE assessment?
3. What is the difference between PBE and Sequential Equilibrium?
4. What is the difference between Separating and Pooling equilibria?
5. In the TV Advertising Game, why does the Separating equilibrium fail?
6. How do off-path beliefs sustain the Pooling equilibrium?