Game Theory 08-02

Infinite Repeated Games, Folk Theorems, and Monitoring

BCSE Game Theory

Nov. 25, 2025

Learning Goals

- Introduce the infinite repeated game model and the discounted payoff evaluation.
- Define continuation payoffs, dynamic programming, and the Bellman equation for enforcement.
- Present the Folk theorem for perfect monitoring and its implications for equilibrium payoffs.
- Examine imperfect public monitoring and why standard triggers can reduce average payoffs as players become more patient.

Infinite Repetition

Strategy and payoffs

Why infinite repetition?

- Finite repetitions still unravel because the last period often provides no future punishment.
- An infinite horizon eliminates a final date, allowing more room for threats and cooperation.
- We now need to track discounted sums and recursive values to characterise equilibria.

Infinite repeated game definition

- The stage game G is played in every period $t = 1, 2, \ldots$ with no predetermined end.
- Each player chooses actions based on the entire history of past signals or profiles.
- The strategy set includes plans for every conceivable history, including off-path events.

Discounted payoff formula

- Players evaluate infinite streams using a discount factor $\delta \in (0, 1)$.
- Normalised payoff:

$$U_i = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(s^t).$$

- The prefactor $(1 - \delta)$ keeps payoffs bounded and comparable as $\delta \to 1$.
- **Interpretations of** δ:
  - Inverse of growth rate (Net Present Value discount rate)
  - Probability game continues: $\delta = \Pr(\text{game continues to next period})$
  - Pure time preference: higher δ means more patient

Histories and strategies

- The history at time t is $h^t = (s^1, \dots, s^{t-1})$. Infinite histories run forever, but strategies only depend on finite prefixes.
- Strategies map any finite history into an action: $\sigma_i(h^t) \in S_i$.
- Perfect recall ensures players can condition on all past actions or signals.

Strategy categories

- Stationary strategies depend only on the current public signal or last action profile.
- Trigger strategies switch state once a deviation/signal occurs and stay in punishment thereafter.
- Finite automaton strategies keep a compact history summary, which suffices for most repeated-game paths.

Trigger strategy concept

- A trigger strategy starts by cooperating but switches to punishment if it detects a deviation.
- The threat is typically a stage Nash equilibrium that delivers lower payoffs.
- The hope is that the future loss outweighs the temptation to deviate today.

Teamwork Game (Reminder)

	C	D
C	$(3, 3)$	$(1, 4)$
D	$(4, 1)$	$(2, 2)$

- C = Cooperate, D = Defect
- Unique Nash equilibrium: (D,D) with payoff $(2, 2)$
- (C,C) is Pareto superior with payoff $(3, 3)$

Grim trigger definition

- Play a cooperative action (e.g., (C,C)) as long as everyone cooperated before.
- Once a deviation is observed, permanently switch to the worst stage Nash equilibrium (e.g., (D,D)).
- Grim trigger relies on infinite punishment, so the weight on the future is crucial.

Comparing grim and tit-for-tat

- Grim trigger punishes forever, while tit-for-tat retaliates only once.
- Grim is harsher and requires higher δ to maintain cooperation, but it deters any deviation.
- Tit-for-tat is forgiving, making it more robust when noise or monitoring errors exist.

Incentive constraint for Grim trigger

- Let R be reward, T temptation, and P punishment payoffs in the Teamwork Game.
- **Teamwork Game values**: $R = 3$ (cooperate), $T = 4$ (defect when opponent cooperates), $P = 2$ (mutual defection).
- The inequality for cooperation is:

$$R \geq (1 - \delta)T + \delta P.$$

- Substituting: $3 \geq (1 - \delta) \cdot 4 + \delta \cdot 2 = 4 - 2\delta$.
- Rearranged: $2\delta \geq 1$, so cooperation is sustainable if $\delta \geq \frac{1}{2}$.
- General formula: $\delta \geq \frac{T-R}{T-P} = \frac{4-3}{4-2} = \frac{1}{2}$.

Verifying Optimality Off-Path

- Grim trigger specifies playing D forever after any deviation. Is this optimal?
- Suppose we are in the punishment phase (history includes a deviation).
- Opponent plays D forever.
- **If I play D**: Payoff is $2 + 2\delta + 2\delta^2 + \cdots = \frac{2}{1-\delta}$.
- **If I deviate to C**: Payoff is $1 + 1\delta + 1\delta^2 + \cdots = \frac{1}{1-\delta}$.
- Since $2 > 1$, playing D is strictly better than C.
- Therefore, carrying out the punishment is optimal (credible threat).
- Note: This relies on (D,D) being a stage Nash equilibrium.

Feasibility region for δ

- The condition $\delta \geq (T - R)/(T - P)$ defines the patience needed for cooperation.
- As the punishment payoff P approaches R, the fraction shrinks and the inequality tightens.
- When stage payoffs are farther apart, the required δ becomes smaller.

Intuition for δ threshold

- Higher δ means future payoffs weigh more; the punishment phase becomes more costly.
- When δ is small, the loser immediately recovers from punishment, so the threat loses bite.
- The threshold $\frac{T-R}{T-P}$ captures how patient players must be before cooperation pays.

Continuation payoff definition

- After seeing history h, the continuation payoff $\gamma_i(h)$ is the discounted value of future play.
- Formally:

$$\gamma_i(h) = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(s^{t+|h|} \mid h).$$

- Continuation payoffs act as the state variable in our recursive enforcement problem.

Dynamic programming overview

- Dynamic programming breaks the infinite-horizon problem into a stage payoff plus a continuation value.
- The value function satisfies a recursive equation (Bellman equation) linking today and tomorrow.
- We use DP to ensure a given path is incentive compatible by adjusting continuation payoffs.

Bellman equation for repeated games

- The Bellman equation reads:

$$V_i(h) = \max_{a_i} \{u_i(a_i, a_{-i}) + \delta \mathbb{E}[V_i(h, a_i, a_{-i})]\},$$

- The expectation accounts for how others respond and how public signals evolve.
- Solving this equation gives us continuation payoffs that support desired equilibria.

Recursive enforcement example

- Suppose we aim to support (C,C) with a target payoff v.
- Choose a stage action delivering $u(C, C)$ and set continuation payoff γ so that $u(C, C) + \delta\gamma = v$.
- The punishment continuation payoff must satisfy the Bellman inequality for deviators.

Continuation state as future value

- The continuation payoff $\gamma_i(h)$ serves as the state variable in DP.
- When players reach history h, they compare the current payoff with the future value $\gamma_i(h)$.
- Properly chosen continuation values ensure that deviating today reduces the total payoff.

Using continuation payoffs to support paths

- To support a target path, split its payoff into the current stage reward and continuation payoff.
- Deviations must yield no higher total payoff when combined with the punishment continuation value.
- This decomposition is what makes the folk theorem constructive.

Folk Theorem

Perfect monitoring enforcement

Folk Theorem (Informal Statement)

Theorem: Folk Theorem for Infinite Repeated Games

Let G be a stage game with minmax payoff vector $(\underline{v}_1, \underline{v}_2)$. For any feasible payoff vector (v_1, v_2) with $v_i > \underline{v}_i$ for all i (strictly above minmax), there exists $\bar{\delta} < 1$ such that for all $\delta \geq \bar{\delta}$, there is a subgame perfect equilibrium of the infinitely repeated game with average payoff (v_1, v_2).

- Key: achievable payoffs are in the **interior** of the feasible region above minmax
- Boundary points (including minmax itself) can only be approximated

Implications of the Folk theorem

- The equilibrium set is huge: any individually rational payoff can be implemented.
- Cooperation can be sustained even if the stage game has a unique Nash equilibrium.
- This flexibility is obtained by carefully designing continuation payoffs and punishment paths.

Constructing SPNE via continuation payoffs

- Pick a target payoff vector v that lies strictly above the minmax values (in the interior).
- Choose stage actions to deliver the current reward, and assign continuation payoffs that satisfy incentive constraints.
- The Bellman recursion ensures these continuation payoffs themselves come from equilibrium paths.

Minmax Payoff: Definition

- **Minmax value** for player i: The minimum payoff player i can guarantee, regardless of opponent's strategy.
- **Formula**: $\underline{v}_i = \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$
- **Interpretation**:
  1. Opponent $-i$ chooses strategy to minimize i's payoff.
  2. Player i chooses best response to this worst-case scenario.
- **Example (Teamwork Game)**:
  - If Player 2 plays C: Player 1 gets $\max(3, 4) = 4$ by playing D.
  - If Player 2 plays D: Player 1 gets $\max(1, 2) = 2$ by playing D.
  - Player 2 minimizes by playing D $\Rightarrow$ Player 1's minmax $= 2$.
  - By symmetry, Player 2's minmax $= 2$.
  - Minmax vector: $(2, 2)$ (same as Nash equilibrium).

Example: Extended Teamwork Game

	C	D	B
C	$(3,3)$	$(1,4)$	$(0,0)$
D	$(4,1)$	$(2,2)$	$(1,1)$
B	$(0,0)$	$(1,1)$	$(-1,-1)$

- C = Cooperate, D = Defect, B = Bully (punish opponent)
- B is a costly punishment strategy that hurts both players
- This game extends the Teamwork game with an additional punishment option

Extended Teamwork: Minmax Calculation

- **Player 1's minmax**: $\underline{v}_1 = \max_{s_1} \min_{s_2} u_1(s_1, s_2)$
  - If P1 plays C: $\min(3, 1, 0) = 0$
  - If P1 plays D: $\min(4, 2, 1) = 1$
  - If P1 plays B: $\min(0, 1, -1) = -1$
  - P1 maximizes by playing D $\Rightarrow \underline{v}_1 = 1$
- **Player 2's minmax** (by symmetry): $\underline{v}_2 = 1$
- **Minmax vector**: $(1, 1)$
- **Nash Equilibrium**: (D,D) with payoff $(2, 2)$
- **Key insight**: NE $(2, 2)$ strictly Pareto dominates minmax $(1, 1)$
  - Folk Theorem applies: payoffs strictly above $(1, 1)$ are achievable for high δ

Teamwork Game (Reminder)

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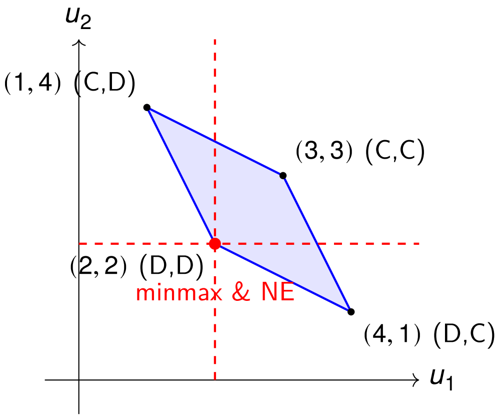
Folk Theorem: Achievable Payoff Region



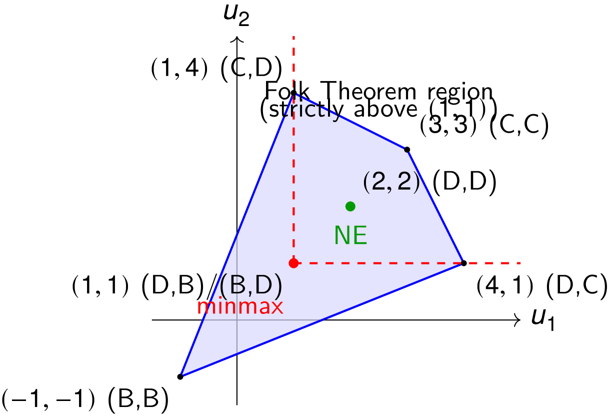
- Minmax: $(2,2)$ (same as NE). Any payoff in feasible set strictly above $(2,2)$ achievable for high δ

Example: Extended Teamwork Game

	C	D	B
C	$(3,3)$	$(1,4)$	$(0,0)$
D	$(4,1)$	$(2,2)$	$(1,1)$
B	$(0,0)$	$(1,1)$	$(-1,-1)$

- C = Cooperate, D = Defect, B = Bully (punish opponent)
- Nash equilibrium: (D,D) with payoff $(2,2)$
- Minmax: $(1,1)$ - Each player guarantees at least 1 by playing D

Extended Teamwork: Minmax vs Nash Equilibrium



- Minmax: $(1, 1)$. NE $(2, 2)$ Pareto dominates it.
- Folk Theorem: payoffs strictly above $(1, 1)$ achievable for high δ

Monitoring and signals

When triggers misfire

Public imperfect monitoring

- Players observe a public signal y_t that depends probabilistically on the action profile.
- The signal may be noisy, so identical signals can arise from both cooperation and defection.
- Imperfect monitoring complicates the detection of deviations.

Trigger strategies with noisy signals

- A trigger strategy may punish when the signal indicates defection, even if the signal is a false alarm.
- With noise, even faithful cooperation can trigger the punishment with positive probability.
- The expected time spent in punishment grows with δ, which can hurt payoffs.

High δ and reduced average payoffs

- As δ increases, each punishment spell lasts longer in present-value terms.
- False positives (due to monitoring errors) cause repeated punishments, dragging down the average payoff.
- Paradoxically, being more patient can reduce average welfare if the monitoring is too noisy.

Illustrating the noise effect

- Suppose noise forces a punishment with probability ε every period.
- The expected cost of punishment is proportional to $\frac{\delta}{1-\delta}$, which explodes as $\delta \to 1$.
- Designers therefore prefer forgiving punishments that limit the length of punishment spells.

Expected Payoff under Grim Trigger with Noise

Setup: Teamwork game with Grim trigger. Noise causes false alarm with probability ε per period.

- Cooperation payoff: $R = 3$. Punishment payoff: $P = 2$.
- Expected periods until false alarm: $\frac{1}{\varepsilon}$.
- Once punishment triggered, it lasts forever.
- Expected payoff:

$$\begin{aligned}
U &= (1-\delta)\left[\sum_{t=1}^{\infty}\{\delta(1-\varepsilon)\}^{t-1}R + \delta^{t-1}\left\{1-(1-\varepsilon)^{t-1}\right\}P\right] \\
&= (1-\delta)\sum_{t=1}^{\infty}\{\delta(1-\varepsilon)\}^{t-1}(R-P) + P \\
&= \frac{1-\delta}{1-\delta(1-\varepsilon)}(R-P) + P
\end{aligned}$$

- As $\delta \to 1$: punishment phase dominates, $U \to P = 2$ (worse than cooperation $R = 3$)

Forgiving strategies as a remedy

- Generous tit-for-tat and win-stay lose-shift forgive occasional defections or noise-induced punishments.
- They reduce the time spent in punishment while still deterring persistent deviators.
- These strategies are easier to analyse with continuation payoffs that reflect lenient responses.

Continuation payoffs with public signals

- Now the continuation payoff depends on both the history and the current signal: $\gamma_i(h, y)$.
- The state updates after each signal and action profile, keeping track of where we stand in the enforcement path.
- DP methods generalise by conditioning on public information as well as past actions.

Bellman equation with signals

- The equation becomes:

$$V_i(h, y) = \max_{a_i} \left\{ u_i(a_i, a_{-i}) + \delta \sum_{y'} \pi(y' \mid a_i, a_{-i}) V_i(h, y') \right\}.$$

- The transition probability π captures the noisy public signal.
- Solving this recursion yields continuation payoffs that account for imperfect monitoring.

Alternative lenient punishments

- Generous tit-for-tat occasionally forgives defections, limiting punishment duration.
- Win-stay lose-shift cooperates after mutual cooperation and switches after a bad outcome.
- Bounded-memory strategies that forget old deviations can also reduce the cost of false alarms.
- The key is balancing deterrence with forgiveness to avoid endless punishment cycles.

Private Monitoring: Additional Challenges

- **Private monitoring**: Each player observes a private signal about others' actions, not a public outcome.
- Players cannot coordinate on a common history, making punishment harder to enforce.
- **Costly observation**: Acquiring information may require effort or resources, creating a trade-off between monitoring accuracy and cost.
- Private signals can lead to disagreement about what happened, complicating equilibrium construction.
- Folk Theorems for private monitoring require more complex belief-updating and communication protocols.

Advanced Topics in Repeated Games

- **Alternating games**: Play different stage games in even/odd periods to effectively reduce the discount factor and expand equilibrium set.
- **Review strategies**: Maintain the same action for multiple periods to accumulate information and improve signal precision before adjusting behavior.
- **Public randomization**: Use publicly observable random devices (e.g., sunspots) to coordinate on mixed-strategy equilibria or break symmetry.
- **Communication**: Allow cheap talk or costly signaling to share private information and coordinate on better outcomes.
- **Mediators**: Introduce a trusted third party to receive reports, recommend actions, and facilitate coordination in complex environments.

Takeaways & looking ahead

- Infinite repetition enables cooperation via credible threats and continuation payoffs.
- Folk theorems show that patience (high δ) expands the equilibrium set dramatically.
- Imperfect monitoring complicates enforcement and may require forgiving strategies.
- Next time we explore applications and extensions: reputation, renegotiation, and mechanism design.

Group Homework

Group Homework: The Evolution of Trust

- **Play**: The Evolution of Trust interactive game, exploring repeated game strategies and their outcomes.
- **Report (A4, one page)**: Summarize your key insights from the game, including:
  - Which strategies performed best and why
  - How the game relates to concepts from this module (Grim Trigger, Tit-for-Tat, etc.)
  - What you learned about cooperation and trust in repeated interactions
- **Contribution log**: Include a short paragraph naming who contributed what to the report.
- **Submission**: Email a single PDF per group to `y.hino@vju.ac.vn` by Nov. 30, Sunday 23:59.

Group Homework: The Evolution of Trust

Evaluation focus

1. **Content Comprehension**: Demonstrate understanding of the game's mechanics and how they relate to repeated game theory concepts covered in class.
2. **Organization & Clarity**: Present your insights in a well-structured, easy-to-follow format within the one-page limit.
3. **Critical Insight**: Identify key takeaways about cooperation, punishment, and forgiveness strategies, and explain why certain strategies succeed or fail.
4. **Application**: Connect the game's lessons to real-world scenarios or course concepts, showing depth of understanding.