BCSE Game Theory 09-01

Incomplete Information Static Games: Introduction

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Dec. 3, 2025

Today's Agenda

Today's Goals

- Understand what incomplete information means in game theory.
- Learn about types and why they matter for strategic decisions.
- Introduce Nature as a way to model uncertainty.
- Understand Harsanyi's transformation: how to convert incomplete information games into extensive-form games.
- Define and compute Bayesian Nash equilibrium.
- Work through three examples: market entry, price competition, and auction preparation.

Lecture Roadmap

1. Review: complete information games we've studied so far.
2. Motivation: why incomplete information matters (three examples).
3. Introducing Nature: a new concept for modeling uncertainty.
4. Harsanyi's solution: transforming incomplete information games.
5. Bayesian Nash equilibrium: definition and computation.
6. Example 1: Market entry game with unknown incumbent type.
7. Example 2: Price competition with unknown costs.
8. Example 3: Auction preparation with unknown valuations.
9. Takeaways and next steps.

Review: Complete Information Games

What We've Learned So Far

- **Complete information**: all players know the game structure, payoffs, and types.
- **Static games** (01–04): simultaneous moves, Nash equilibrium.
- **Dynamic games** (06–08): sequential moves, subgame-perfect Nash equilibrium.
- In all cases, players know everything about the game and each other.

Key Assumption

Every player knows all payoffs, all possible actions, and all relevant information about other players.

Limitations of Complete Information

- In reality, players often don't know everything about their opponents.
- Examples:
  - A startup doesn't know if an incumbent will fight or accommodate entry.
  - Firms don't know each other's production costs.
  - Bidders don't know others' valuations in an auction.
- We need a framework to analyze strategic interactions under uncertainty.

Motivation: Why Incomplete Information?

Example 1: Market Entry Game

- A startup considers entering a market dominated by an incumbent firm.
- The incumbent can be **Tough** (will fight entry) or **Weak** (will accommodate).
- The startup doesn't know which type the incumbent is.
- The startup's decision depends on the incumbent's type.
- **Question**: How should the startup decide whether to enter?

Example 1: Market Entry Payoffs

Tough Incumbent

	Fight	Accommodate
Enter	$(-4, -6)$	$(3, 5)$
Stay out	$(0, 2)$	$(0, 2)$

Weak Incumbent

	Fight	Accommodate
Enter	$(-2, -4)$	$(4, 6)$
Stay out	$(0, 2)$	$(0, 2)$

- If the startup enters, a Tough incumbent fights (payoff -4 for incumbent).
- If the startup enters, a Weak incumbent accommodates (payoff 4 for incumbent).
- The startup doesn't know which table applies.

Example 2: Price Competition

- Two firms compete in prices.
- Each firm has a cost type: **High cost** or **Low cost**.
- Each firm knows its own cost but not the opponent's cost.
- Lower-cost firms can set lower prices and capture more market share.
- **Question**: What prices should each firm set?

Example 2: Price Competition Structure

- Firm 1's cost: $c_1 \in \{c_H, c_L\}$ (known to Firm 1).
- Firm 2's cost: $c_2 \in \{c_H, c_L\}$ (known to Firm 2, unknown to Firm 1).
- Profit for firm i: $\pi_i = (p_i - c_i) \cdot q_i(p_i, p_j)$.
- If c_i is high, firm i must set a higher price to break even.
- If c_i is low, firm i can undercut the opponent.
- **Challenge**: Firm 1 doesn't know Firm 2's cost, so it can't predict Firm 2's best response.

Example 3: Auction Preparation

- Two bidders prepare for an auction.
- Each bidder has a valuation: **High valuation** or **Low valuation**.
- Each bidder knows their own valuation but not the opponent's.
- Higher-valuation bidders are willing to bid more.
- **Question**: How should bidders prepare their bidding strategies?

Example 3: Auction Payoffs

- If Bidder 1 wins with bid b_1 and valuation v_1: payoff $v_1 - b_1$.
- If Bidder 1 loses: payoff 0.
- Bidder 1's valuation: $v_1 \in \{v_H, v_L\}$ (known to Bidder 1).
- Bidder 2's valuation: $v_2 \in \{v_H, v_L\}$ (known to Bidder 2, unknown to Bidder 1).
- **Challenge**: Bidder 1 doesn't know Bidder 2's valuation, so it can't predict Bidder 2's maximum bid.
- This connects to Game Theory 10 (auction theory).

Incomplete Information: Definitions

What Is Incomplete Information?

Definition: Incomplete Information

A game has incomplete information if at least one player does not know some relevant information about other players, such as their payoffs, costs, or valuations.

- **Type**: a player's private information (cost, valuation, strength, etc.).
- **Incomplete information**: players don't know others' types.
- **Complete information**: all players know all types (what we've studied so far).

Types and Type Spaces

- Each player i has a **type** $\theta_i \in \Theta_i$.
- Θ_i is player i's **type space** (set of possible types).
- Example: $\Theta_i = \{\text{High cost}, \text{Low cost}\}$.
- A **type profile** is $\theta = (\theta_1, \ldots, \theta_n)$.
- Players know their own type but may not know others' types.

Key Insight

Incomplete information means players are uncertain about which game they are playing (which payoff matrix applies).

Prior Beliefs

- Players have **prior beliefs** about others' types.
- $p(\theta)$: probability that type profile θ occurs.
- $p_i(\theta_{-i} \mid \theta_i)$: player i's belief about others' types given their own type.
- Prior beliefs are **common knowledge**: all players know the distribution.
- Example: "The incumbent is Tough with probability μ and Weak with probability $1 - \mu$."

Introducing Nature

What Is Nature?

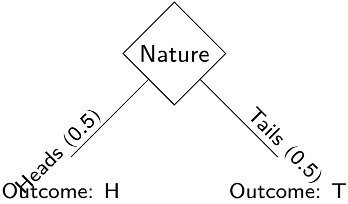
- **Nature** is a special "player" that makes random choices.
- Nature doesn't have preferences or strategic goals.
- Nature chooses types (or other random events) according to a probability distribution.
- This is the **first time** we introduce Nature in this course.

Visual Representation

Nature is represented by a **diamond-shaped node** in game trees.

Nature: Simple Example (Coin Flip)

- Nature flips a coin: Heads or Tails, each with probability 0.5.
- This is a random event, not a strategic choice.
- We use Nature to model uncertainty that is not controlled by players.



Nature Chooses Types

- In incomplete information games, Nature chooses each player's type.
- Example: Nature chooses the incumbent's type (Tough or Weak).
- Each player observes their own type but not others' types.
- The probability distribution is common knowledge.

Key Idea

Nature "plays first" by choosing types, then players observe their own type and play the game.

Why Do We Need Nature?

- Without Nature, we can't represent incomplete information in standard game forms.
- Nature allows us to convert incomplete information games into extensive-form games.
- This transformation is called **Harsanyi transformation** (after John Harsanyi, Nobel Prize 1994).
- Harsanyi showed that any incomplete information game can be represented as a complete information game with Nature.

Harsanyi's Transformation

The Problem with Incomplete Information

- In a normal-form game, we need to specify payoffs for all action profiles.
- But if types are unknown, we don't know which payoff matrix applies.
- Example: Market entry game has two different payoff matrices (Tough vs Weak).
- **Question**: How can we analyze such games?

Harsanyi's Solution

Harsanyi Transformation (1967–1968)

Any incomplete information game can be transformed into a complete information extensive-form game by:

1. Adding Nature as a player that chooses types first.
2. Having each player observe their own type (but not others' types).
3. Specifying prior beliefs as Nature's probability distribution.

- This transformation is **common knowledge**: all players know the structure.
- The resulting game has complete information (about the structure) but imperfect information (about types).

Types of Information

Definition: Complete vs Incomplete Information

- **Complete Information**: All players know the game structure and payoffs (no private information).
- **Incomplete Information**: Players do not know some relevant information about others (e.g., payoffs, types).

Definition: Perfect vs Imperfect Information

- **Perfect Information**: Players know the full history of the game when making a move (no simultaneous moves, no hidden actions).
- **Imperfect Information**: Players do not know the full history (e.g., simultaneous moves, hidden actions).

Harsanyi Transformation: Step by Step

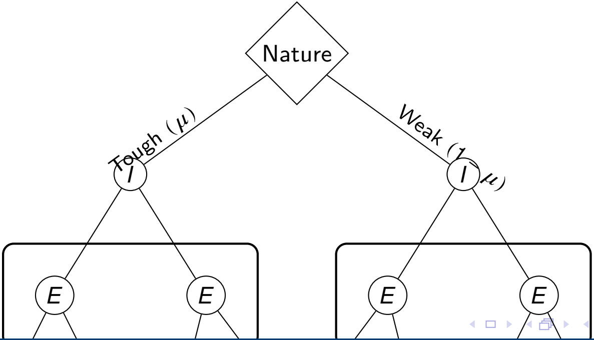
1. **Step 1**: Nature chooses type profile $\theta = (\theta_1, \dots, \theta_n)$ with probability $\mu(\theta)$.
2. **Step 2**: Each player i observes their own type θ_i (but not θ_{-i}).
3. **Step 3**: Players choose actions simultaneously (or sequentially, depending on the game).
4. **Step 4**: Payoffs are determined based on actions and the actual type profile.

Result

We now have an extensive-form game with Nature, which we can analyze using standard methods.

Market Entry Game: Harsanyi Transformation

- Nature chooses: Incumbent is Tough (probability μ) or Weak (probability $1 - \mu$).
- Incumbent observes their type (Tough or Weak).
- Entrant does **not** observe the incumbent's type (information set).
- Both players choose actions simultaneously.
- Payoffs depend on actions and the actual type.



Information Sets in Incomplete Information Games

- The entrant's information set contains nodes after "Tough" and "Weak".
- The entrant cannot distinguish which node was reached.
- This creates a **non-degenerate information set**.
- The entrant must choose the same action at both nodes (since they can't tell them apart).

Key Insight

Incomplete information creates imperfect information: players don't know which node they are at.

Bayesian Nash Equilibrium

Strategies in Incomplete Information Games

- A **strategy** for player *i* specifies an action for each possible type.
- Example: Entrant's strategy: "Enter if I believe the incumbent is Weak with probability > 0.5, Stay out otherwise."
- Incumbent's strategy: "If I'm Tough, Fight; if I'm Weak, Accommodate."
- Strategies are **type-contingent**: they depend on the player's type.

Expected Payoffs

- Since players don't know others' types, they compute **expected payoffs**.
- Player i's expected payoff from action a_i given type θ_i:

$$E_{\theta_{-i}}[u_i(a_i, a_{-i}, \theta_i, \theta_{-i}) \mid \theta_i]$$

- This expectation is over the distribution of others' types, given player i's own type.
- Example: Entrant's expected payoff from Entering depends on the probability that the incumbent is Tough vs Weak.

Bayesian Nash Equilibrium: Definition

Definition: Bayesian Nash Equilibrium

A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a Bayesian Nash equilibrium if for every player i and every type θ_i:

$$E_{\theta_{-i}}[u_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i}) \mid \theta_i] \geq E_{\theta_{-i}}[u_i(a_i, s_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i}) \mid \theta_i]$$

for all actions $a_i \in A_i$.

- Each type chooses a best response given beliefs about others' types.
- Beliefs are given by the prior distribution $\mu(\theta)$.
- This is the natural extension of Nash equilibrium to incomplete information.

Computing Bayesian Nash Equilibrium

1. **Step 1**: For each type profile, compute payoffs for all action profiles.
2. **Step 2**: For each player and each type, compute expected payoffs over others' types.
3. **Step 3**: For each player and each type, find the action that maximizes expected payoff.
4. **Step 4**: Check that the strategy profile is a best response for all types.

Key Difference from Nash Equilibrium

We compute expected payoffs because players don't know others' types.

Example 1: Market Entry Game

Market Entry: Setup

- Incumbent I has type: Tough (probability μ) or Weak (probability $1 - \mu$).
- Entrant E does not know the incumbent's type.
- If incumbent is Tough: Fight is dominant (payoff $-4 > -2$ if Enter, $0 = 0$ if Stay out).
- If incumbent is Weak: Accommodate is dominant (payoff $4 > -2$ if Enter, $0 = 0$ if Stay out).
- Entrant chooses: Enter or Stay out (without knowing the type).

Market Entry: Payoff Tables

Tough Incumbent

	Fight	Accommoda
Enter	$(-4, -6)$	$(3, 5)$
Stay out	$(0, 2)$	$(0, 2)$

Weak Incumbent

	Fight	Accommoda
Enter	$(-2, -4)$	$(4, 6)$
Stay out	$(0, 2)$	$(0, 2)$

- Tough: Fight is dominant.
- Weak: Accommodate is dominant.

Market Entry: Incumbent's Strategy

- Incumbent knows their type and chooses a best response.
- If type is Tough: choose Fight (payoff $-4 > -2$ if Enter, $0 = 0$ if Stay out).
- If type is Weak: choose Accommodate (payoff $4 > -2$ if Enter, $0 = 0$ if Stay out).
- **Incumbent's equilibrium strategy**:

$$s_I^*(\text{Tough}) = \text{Fight}, \quad s_I^*(\text{Weak}) = \text{Accommodate}$$

Market Entry: Entrant's Expected Payoff

- Entrant doesn't know the incumbent's type.
- If Entrant chooses Enter:
  - With probability μ: Incumbent is Tough and Fights → payoff -6.
  - With probability $1 - \mu$: Incumbent is Weak and Accommodates → payoff 5.
  - Expected payoff: $\mu \cdot (-6) + (1 - \mu) \cdot 5 = 5 - 11\mu$.
- If Entrant chooses Stay out:
  - Payoff is 2 regardless of type.
  - Expected payoff: 2.

Market Entry: Entrant's Best Response

- Entrant chooses Enter if:

$$5 - 11\mu > 2 \quad \Longleftrightarrow \quad \mu < \frac{3}{11} \approx 0.27$$

- Entrant chooses Stay out if:

$$5 - 11\mu < 2 \quad \Longleftrightarrow \quad \mu > \frac{3}{11}$$

- If $\mu = \frac{3}{11}$, Entrant is indifferent.
- **Entrant's equilibrium strategy**:

$$s_E^* = \begin{cases} \text{Enter} & \text{if } \mu < \frac{3}{11} \\ \text{Stay out} & \text{if } \mu > \frac{3}{11} \end{cases}$$

Market Entry: Bayesian Nash Equilibrium

Equilibrium Strategies

- Incumbent: $s_I^*(\text{Tough}) = \text{Fight}$,
  $s_I^*(\text{Weak}) = \text{Accommodate}$.
- Entrant: $s_E^* = \text{Enter if } \mu < \frac{3}{11}$, Stay out if $\mu > \frac{3}{11}$.

- This is a Bayesian Nash equilibrium: each type chooses a best response.
- The entrant's decision depends on the prior belief μ.
- If the incumbent is likely to be Tough ($\mu > 0.27$), entry is deterred.
- If the incumbent is likely to be Weak ($\mu < 0.27$), entry occurs.

Example 2: Price Competition

Price Competition: Setup

- Two firms compete in prices.
- Each firm has cost type: High cost c_H or Low cost c_L (where $c_H > c_L$).
- Each firm knows its own cost but not the opponent's cost.
- Prior beliefs: Firm 1 believes Firm 2 is High cost with probability ρ, Low cost with probability $1 - \rho$.
- Demand: $q_i = a - p_i + \frac{1}{2}p_j$ (products are substitutes).
- Profit: $\pi_i = (p_i - c_i) \cdot q_i$.

Price Competition: Best Response

- Firm i's profit: $\pi_i = (p_i - c_i)(a - p_i + \frac{1}{2}p_j)$.
- First-order condition: $\frac{\partial \pi_i}{\partial p_i} = a - 2p_i + \frac{1}{2}p_j + c_i = 0$.
- Best response: $p_i^*(p_j) = \frac{a + c_i + \frac{1}{2}p_j}{2}$.
- Higher cost $\rightarrow$ higher price.
- Firm i doesn't know p_j because it doesn't know Firm j's cost.

Price Competition: Expected Best Response

- Firm 1 (with cost c_1) expects Firm 2 to choose:
  - $p_2^H = \frac{a+c_H+\frac{1}{2}p_1}{2}$ if Firm 2 is High cost (probability ρ).
  - $p_2^L = \frac{a+c_L+\frac{1}{2}p_1}{2}$ if Firm 2 is Low cost (probability $1-\rho$).
- Firm 1's expected profit:

$$E[\pi_1] = \rho \cdot \pi_1(p_1, p_2^H) + (1-\rho) \cdot \pi_1(p_1, p_2^L)$$

- Firm 1 chooses p_1 to maximize expected profit.

Price Competition: Equilibrium

- In equilibrium, each firm's type chooses a price that is a best response to the expected price of the opponent.
- Firm 1 (High cost): chooses p_1^H given beliefs about Firm 2.
- Firm 1 (Low cost): chooses p_1^L given beliefs about Firm 2.
- Firm 2 (High cost): chooses p_2^H given beliefs about Firm 1.
- Firm 2 (Low cost): chooses p_2^L given beliefs about Firm 1.
- All four prices are determined simultaneously (system of equations).

Price Competition: Key Insights

- Higher-cost firms set higher prices in equilibrium.
- Firms with low costs can undercut high-cost opponents.
- Uncertainty about costs leads to price dispersion.
- If costs were known (complete information), prices would be lower (more competition).
- Incomplete information can soften price competition.

Example 3: Auction Preparation

Auction Preparation: Setup

- Two bidders prepare for a first-price sealed-bid auction.
- Each bidder has valuation: High v_H or Low v_L (where $v_H > v_L$).
- Each bidder knows their own valuation but not the opponent's.
- Prior beliefs: Bidder 1 believes Bidder 2 is High valuation with probability r, Low valuation with probability $1 - r$.
- If bidder i wins with bid b_i: payoff $v_i - b_i$.
- If bidder i loses: payoff 0.

Auction Preparation: Bidding Strategy

- Higher-valuation bidders are willing to bid more.
- Bidder i's strategy: $b_i(v_i)$ (bid as a function of valuation).
- Expected payoff for Bidder 1 (with valuation v_1):

$$E[\pi_1] = \Pr(\text{win}) \cdot (v_1 - b_1) + \Pr(\text{lose}) \cdot 0$$

- Probability of winning depends on:
  - Bidder 1's bid b_1.
  - Expected bid of Bidder 2 (which depends on Bidder 2's type).

Auction Preparation: Equilibrium

- In equilibrium, each bidder type chooses a bid that maximizes expected payoff.
- Bidder 1 (High valuation): bids b_1^H given beliefs about Bidder 2.
- Bidder 1 (Low valuation): bids b_1^L given beliefs about Bidder 2.
- Bidder 2 (High valuation): bids b_2^H given beliefs about Bidder 1.
- Bidder 2 (Low valuation): bids b_2^L given beliefs about Bidder 1.
- Higher-valuation bidders bid more: $b_i^H > b_i^L$.

Auction Preparation: Key Insights

- Bidders shade their bids below their valuations (to avoid overpaying).
- The amount of shading depends on beliefs about the opponent's valuation.
- If the opponent is likely to have a high valuation, bidders bid more aggressively.
- This connects to Game Theory 10, where we'll study auction theory in detail.

Takeaways

Key Concepts

- **Incomplete information**: players don't know others' types (costs, valuations, etc.).
- **Type**: a player's private information.
- **Nature**: a special player that chooses types randomly (first introduction in this course).
- **Harsanyi transformation**: converts incomplete information games into extensive-form games with Nature.
- **Bayesian Nash equilibrium**: each type chooses a best response given beliefs about others' types.

Key Insights

- Incomplete information creates strategic uncertainty.
- Players compute expected payoffs over unknown types.
- Equilibrium strategies depend on prior beliefs.
- Incomplete information can soften competition (price competition example).
- The framework applies to many real-world situations (market entry, auctions, etc.).

Next Steps

- Game Theory 09-02: Incomplete information in continuous games (Cournot, Bertrand).
- We'll see how incomplete information affects quantity and price competition.
- Game Theory 10: Auction theory (applying incomplete information to bidding).
- We'll study different auction formats and their properties.