Game Theory 01-02

A Language to Describe Games

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Today's Goals (01-02)

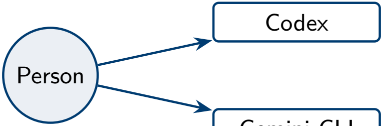
- Connect individual preferences to utility representations we can reuse across many decision stories.
- Translate narratives (teamwork, bidding, security) into the normal-form vocabulary: players, strategies, payoffs, and IESDS.
- Test the language with quick scenarios so we can critique algorithms, contracts, or policies like a game theorist-in-practice.

Decision Making for a Single Person

Preferences

Game theory studies strategic interaction, but it starts with single-person decisions.

In economics, we often represent how desirable an option is with numbers (money or utility). The goal is not to measure emotions precisely; we only need consistent choices.



Preferences (cont.)

We will:

- define what a preference is and which properties we want;
- see when preferences can be represented by a utility function;
- remember that basic utility is ordinal (it ranks options) rather than cardinal, which matches how we compare app features, products, or job offers.

Preferences

Definition: Binary relation

Let X be a set of options. A binary relation $\succ$ on X assigns to each pair $(x, y) \in X^2$ whether x is strictly preferred to y ($x \succ y$) or not ($x \not\succ y$).

Examples of X: $\{\mathrm{MNT}, \mathrm{MCSE}, \mathrm{MBA}\}$, or $\{\mathrm{work}, \mathrm{sleep}\}$.

We read $x \succ y$ as "x is strictly preferred to y" . (We will later derive a weak preference and indifference from $\succ$.)

Preferences

Not every relation models sensible choices. We impose at least:

Definition: Asymmetry

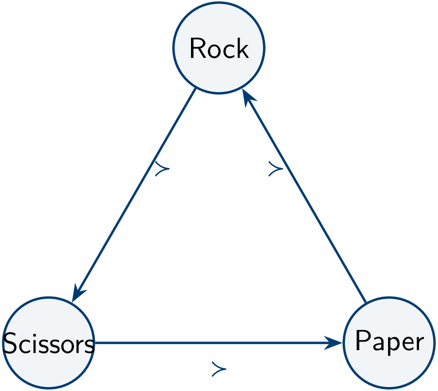
For all $x, y \in X$, if $x \succ y$ then $y \not\succ x$.

Interpretation: There are no two-way strict preferences (no cycles of length 2). We treat $\succ$ as the strict part of preference; ties will be captured by indifference later.

Story: Imagine a student who first tells the admission office "I strictly prefer MBA to MCSE," and later, for the same offer set, asserts "I strictly prefer MCSE to MBA." Asymmetry rules out that contradiction: we cannot promise to choose x over y and also y over x in the same decision. If context matters (different semesters or scholarship terms), we model those as distinct options instead of allowing $x \succ y$ and $y \succ x$ at the same time.

Preferences

Rock – paper – scissors (rock $\succ$ scissors, scissors $\succ$ paper, paper $\succ$ rock) does not yield a "most preferred" element. We want to rule out such cycles.



Preferences

Definition: Negatively transitive

For all $x, y, z \in X$, if $x \not\succ y$ and $y \not\succ z$, then $x \not\succ z$.

Intuition: If you do not strictly prefer x to y, and you do not strictly prefer y to z, then you should not strictly prefer x to z either. This prevents strict-preference cycles.

Preferences

Definition: Preference relation

A binary relation $\succ$ is a preference relation if it is **asymmetric** and **negatively transitive**.

Consequences (sketch):

- Transitivity of $\succ$: if $x \succ y$ and $y \succ z$, then $x \succ z$.
- Irreflexivity: $x \not\succ x$:
- Acyclicity: if $x_1 \succ x_2 \succ \cdots \succ x_n$ then $x_1 \neq x_n$.

Preferences

Definition: Indifference $\sim$

Given a strict preference $\succ$, the Indifference $x \sim y$ holds if and only if if and only if $x \nsucc y$ and $y \nsucc x$.

Definition: Weak preference $\succeq$

Given a strict preference relation $\succ$, the weak preference $\succeq$ is defined as follows: for any alternatives x and y, we write $x \succeq y$ if and only if $x \succ y$ or $x \sim y$.

From Preference to Utility Function

Definition: Utility function

A function $U\colon X \to \mathbb{R}$ represents a strict preference $\succ$ if for all $x, y \in X$ we have $x \succ y \iff U(x) > U(y)$. Any such function is called a utility function for $\succ$.

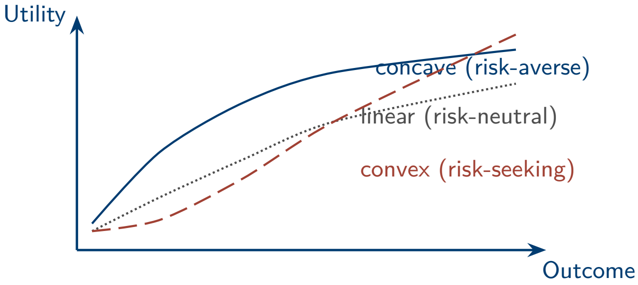
If $x_1 \succ x_2 \succ \cdots \succ x_n$ orders X, then $U(x_i) = 2^{-i}$ defines $U\colon X \to [0,1]$ that represents $\succ$ (i.e., $x \succ y \iff U(x) > U(y)$).

Remarks

- Utility here is ordinal: any strictly increasing transform of U represents the same preference.

From Preference to Expected Utility Function

- Under uncertainty, the von Neumann – Morgenstern axioms (completeness, transitivity, continuity, independence) imply an expected utility representation; risk attitudes correspond to concavity/convexity of U and are unique up to positive affine transforms.



A Language to Describe Games

Game components

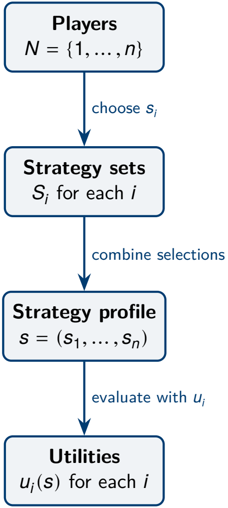
To specify strategic interaction, we need:

- **Players** $N \stackrel{\text{def}}{=} \{1, \ldots, n\}$.
- **Strategy sets** S_i for each $i \in N$; a strategy profile is $s \stackrel{\text{def}}{=} (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$.
- **Utilities** $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ assigns each profile s a payoff for player i.

Definition: Normal-form game (Strategic-form game)

A normal-form game is the triple $G \stackrel{\text{def}}{=} (N, (S_i)_{i=1}^n, (u_i)_{i=1}^n)$ consisting of the players, their strategy sets, and their payoff functions. Utilities are also called payoffs. This compact description is how we will encode bidding wars, security planning, and everyday dilemmas.

A Language to Describe Games



This pipeline is reusable: swap in your favourite story (auctions, security drills, social media) and the same trio $(N, (S_i), (u_i))$ captures it.

Players

Definition: Player

A player is a decision-making entity that selects a strategy and receives a payoff determined by the game. The set of players is $N \overset{\text{def}}{=} \{1, \dots, n\}$. We index players by $i, j \in N$.

Remarks

- Players can be individuals, firms, organizations, or algorithms —any unit making choices.
- In two-player games we often use names (e.g., Anh, Binh) instead of indices.

Strategies

Definition: Strategy

A (pure) strategy for player i is a complete plan of action specifying what i would do at every decision point. In normal form, it is represented as an element $s_i \in S_i$.

Notation and extensions

- A strategy profile is $s \stackrel{\text{def}}{=} (s_i, s_{-i}) \in S_1 \times \cdots \times S_n$, where s_{-i} denotes the strategies of all players except i.
- A mixed strategy for i is a probability distribution σ_i on S_i (randomization over pure strategies). Profiles are $\sigma \stackrel{\text{def}}{=} (\sigma_1, \ldots, \sigma_n)$. (covered later in the course)

Utilities (payoffs)

Definition: Utility (payoff) function

For each player $i \in N$, the utility function $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ assigns a real-valued payoff to every strategy profile s. Higher values indicate more preferred outcomes for i.

Remarks

- Utilities are typically ordinal in normal-form analysis (any increasing transform preserves preferences).
- With mixed strategies σ, expected payoffs are given by $u_i(\sigma) \stackrel{\text{def}}{=} \mathbb{E}_{s\sim\sigma}[\, u_i(s) \,]$ (linear in probabilities). (covered later in the course)
- Payoffs encode incentives; they can aggregate many considerations (e.g., grades, costs, time).

Players and strategies

- Players: {Anh, Binh}
- Strategies for each: {Work Hard, Slack Off}

Scenario: If at least one works hard, the team earns a good grade. Working hard is costly because it leaves less time for other courses. Slacking off saves time but risks a poor grade if both slack.

Utilities (payoffs)

Anh's utility

$$
\begin{aligned}
U_{\text{Anh}}(\text{Work Hard}, \text{Work Hard}) &= 3\\
U_{\text{Anh}}(\text{Work Hard}, \text{Slack Off}) &= 1\\
U_{\text{Anh}}(\text{Slack Off}, \text{Work Hard}) &= 4\\
U_{\text{Anh}}(\text{Slack Off}, \text{Slack Off}) &= 2
\end{aligned}
$$

Binh's utility

$$
\begin{aligned}
U_{\text{Binh}}(\text{Work Hard}, \text{Work Hard}) &= 3\\
U_{\text{Binh}}(\text{Work Hard}, \text{Slack Off}) &= 4\\
U_{\text{Binh}}(\text{Slack Off}, \text{Work Hard}) &= 1\\
U_{\text{Binh}}(\text{Slack Off}, \text{Slack Off}) &= 2
\end{aligned}
$$

Payoff matrix

		Binh: Work Hard	Binh: Slack Off
Anh	Work Hard	(3, 3)	(1, 4)
	Slack Off	(4, 1)	(2, 2)

The first number is Anh's payoff; the second is Binh's. Rows correspond to Anh's strategies; columns to Binh's.

- What is the most desirable outcome for the team?
- What is the least desirable outcome for the team?

Summary

Preferences and Game Blueprints

Summary

Preferences and Game Blueprints

- If a strict preference $\succ$ is asymmetric and negatively transitive, we can represent it with a utility function (unique up to increasing transforms), which lets us rank products, routes, or project plans.
- A normal-form game lists the players N, their strategies (S_i), and the utilities (u_i); outcomes come from the chosen profile—perfect for coding competitive scenarios.
- Payoff matrices summarize two-player normal-form games for quick intuition and analysis, and they serve as our go-to storyboard for class discussions.