BCSE Game Theory 03-01

Nash Equilibrium

Author

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Today's Goals

- Summarise when IESDS becomes unreliable in finite games.
- Understand the idea of a best response and its formal definition.
- Derive the definition and intuition of Nash equilibrium from best responses.
- Practise computing Nash equilibria using familiar payoff tables.

Notation Refresher

Notation for Today

We reuse the standard notation introduced earlier in the course.

- $N \stackrel{\text{def}}{=} \{1, \dots, n\}$: player set.
- S_i: strategy set of Player i.
- $S \stackrel{\text{def}}{=} \prod_{i \in N} S_i$: space of strategy profiles.
- $S_{-i} \stackrel{\text{def}}{=} \prod_{j \neq i} S_j$: strategies of everyone except i.
- $u_i : S \to \mathbb{R}$: payoff function for Player i.
- $BR_i : S_{-i} \to 2^{S_i}$: best-response correspondence for Player i.

$$S \stackrel{\text{def}}{=} S_1 \times S_2 \times \cdots \times S_n,$$
$$S_{-i} \stackrel{\text{def}}{=} S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n.$$

Review: IESDS

- IESDS stands for Iterated Elimination of Strictly Dominated Strategies.
- Repeatedly delete any strictly dominated strategy for any player.
- If every player is left with a single strategy, predictions are sharp.
- Nonetheless, the procedure does not always yield a unique or easy prediction.

Why prediction is difficult?

- Many games have no strictly dominated strategies to delete.
- Even after elimination, multiple candidate profiles may remain.

Example: Two-Step Iterated Elimination

Player 2

Player 1		L	C	R
	U	(4, 3)	(5, 1)	(6, 2)
	M	(2, 1)	(8, 4)	(3, 6)
	D	(3, 0)	(9, 6)	(2, 8)

IESDS

		Player 2		
		L	C	R
Player 1	U	(4, 3)	(5, 1)	(6, 2)
	M	(2, 1)	(8, 4)	(3, 6)
	D	(3, 0)	(9, 6)	(2, 8)

- Player 2's strategy C is strictly dominated by R, so delete C.
- Player 1's strategies M and D are strictly dominated by U, so delete M and D.
- In the reduced game, C is dominated by L; only (U, L) survives.

Challenges in IESDS

- Many games contain no strictly dominated strategies to delete at all.

		Binh: Vietnam Style	Binh: Japan Style
Anh	Vietnam Style	(3, 3)	(1, 1)
	Japan Style	(1, 1)	(2, 2)

Beyond IESDS

Nash equilibrium

Beyond IESDS

- We want to identify states where everyone chooses a best response to those beliefs.
- The tools for that are best responses and Nash equilibrium.

Definition: Best response

A strategy $s_i \in S_i$ is player i's best response to (beliefs about) opponents' strategies $s_{-i} \in S_{-i}$ if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \quad \text{for all } s_i' \in S_i.$$

Intuition for Best Responses

- A best response maximises a player's payoff given the opponent's strategy.
- Think of chess or sports: "if the opponent does this, my best reply is that."
- If the strategy profile set S is finite, we can list the candidate best responses for every opponent action.

Example: Teamwork Game

		Binh	
		Work Hard	Slack Off
Anh	Work Hard	(3, 3)	(1, 4)
	Slack Off	(4, 1)	(2, 2)

Best Responses for Anh

		Binh: Work Hard	Binh: Slack Off
Anh	Work Hard	(3, 3)	(1, 4)
	Slack Off	(4, 1)	(2, 2)

Best Responses for Anh

- If Binh chooses Work Hard, Slack Off yields Anh a payoff of 4 and is optimal.
- If Binh chooses Slack Off, Slack Off still yields the highest payoff (2).
- Hence $BR_{\text{Anh}}(\text{Work Hard}) = \{\text{Slack Off}\}$.
- And $BR_{\text{Anh}}(\text{Slack Off}) = \{\text{Slack Off}\}$.

Best Responses for Binh

		Binh: Work Hard	Binh: Slack Off
Anh	Work Hard	(3, 3)	(1, 4)
	Slack Off	(4, 1)	(2, 2)

Best Responses for Binh

- If Anh plays Work Hard, Binh's best response is Slack Off (payoff 4).
- If Anh plays Slack Off, Slack Off remains optimal (payoff 2).
- Therefore $BR_{\text{Binh}}(\text{Work Hard}) = \{\text{Slack Off}\}$.
- And $BR_{\text{Binh}}(\text{Slack Off}) = \{\text{Slack Off}\}$.

What Is an Equilibrium?

Consider a strategy profile $s \stackrel{\text{def}}{=} (s_1, s_2, \dots, s_n)$ and the best-response correspondences $BR_i : S_{-i} \to 2^{S_i}$ satisfying $s_i \in BR_i(s_{-i})$ for every player i.

- A strategy profile where everyone is already playing a best response.
- Everyone is best-responding simultaneously.
- No player can profit from deviating unilaterally.
- Such a fixed point is the building block of a Nash equilibrium.

The situation appears naturally in markets, teamwork, and many strategic settings.

Definition: Nash Equilibrium

Such a profile is what we call a Nash equilibrium.

Definition: Nash Equilibrium

A strategy profile $s^* \stackrel{\text{def}}{=} (s_1^*, \ldots, s_n^*)$ is a Nash equilibrium if, for every player i,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \text{for all } s_i \in S_i.$$

A profile s^* is a Nash equilibrium if and only if

- every component s_i^* is a best response to s_{-i}^*.

Symbolically: $s^* \in \prod_{i \in N} BR_i(s_{-i}^*)$.

IESDS versus Best Responses

- Strictly dominated strategies never appear in a best-response set.
- However, strategies that survive IESDS need not be best responses to anything (IESDS often leaves many strategies alive).

Hence we search for intersections of best-response correspondences

Example: Battle of the Sexes

- Two partners prefer different activities—Opera for Player A, Football for Player B—but value being together more than attending alone.
- They must choose simultaneously without the chance to coordinate explicitly.

		Player B	
		Opera	Football
Player A	Opera	(2, 1)	(0, 0)
	Football	(0, 0)	(1, 2)

Equilibria in Battle of the Sexes

- (Opera, Opera) and (Football, Football) are Nash equilibria.
- Coordination is required; multiple equilibria coexist.
- IESDS cannot select between them, illustrating its predictive limits.

Example: Matching Pennies

- Two players flip a coin; Player A wins if the coins match, Player B if they differ.
- The situation also models inspection games or security scenarios where one side tries to match the other's move and the opponent prefers to mismatch.

		Player B: Heads	Player B: Tails
Player A	Heads	(1, -1)	(-1, 1)
	Tails	(-1, 1)	(1, -1)

Lessons from Matching Pennies

- Every cell leaves at least one player with an incentive to deviate.
- Some finite games have no pure-strategy Nash equilibrium at all.
- This motivates the mixed strategies that we will introduce soon.

IESDS and Nash Equilibrium

- Iteratively deleting strictly dominated strategies (IESDS) shrinks the set of plausible profiles.
- **Every pure Nash equilibrium survives IESDS.** Because each player's Nash strategy is not strictly dominated, the procedure never removes it.
- Yet not every profile that survives IESDS is a Nash equilibrium; we still need to check whether the remaining candidates are mutual best responses.
- Think of IESDS as a filter for the strategy space and the Nash equilibrium as the final fixed point.

Containment relations

$$\text{Nash equilibria} \subseteq \text{IESDS survivors} \subseteq S$$

Best-Response Table (Example)

- Using the cross-cultural teamwork game, mark each player's best response in the payoff table.
- Blue highlights Player 1's best responses, red highlights Player 2's, and purple indicates both—candidates for Nash equilibria.
- Arrows indicate where each player would move given the opponent's action (horizontal: Player 2, vertical: Player 1).

		Binh: Vietnam Style	Binh: Japan Style
Anh	Vietnam Style	(3 , 3)	(1 , 1)
	Japan Style	(1 , 1)	(2 , 2)

How to Build a Best-Response Table

1. For each player, compare payoffs across actions of the opponent.
2. Mark the strategies that deliver the highest payoff (arrows, colours, etc.).
3. Cells with marks for every player are candidates for Nash equilibria.
4. Confirm that strategies eliminated by IESDS are not still in the table.

		Binh: Vietnam Style	Binh: Japan Style
Anh	Vietnam Style	($\underline{3}$, $\underline{3}$)	(1 , 1)
	Japan Style	(1 , 1)	($\underline{2}$, $\underline{2}$)

Exercise: Draw the Best-Response Table

		Player 2	
		L	R
Player 1	U	(2, 4)	(5, 1)
	D	(1, 3)	(3, 2)

Task

Sketch the best-response table by hand and identify every Nash equilibrium.

Further Reading

- Osborne and Rubinstein (1994), Chapter 2 on static games and equilibria.
- Gibbons (1992), Chapter 1 with many best-response exercises.
- Pick two computational drills to solve before the next session to solidify the ideas.

Exercise 1: List the Best Responses

		Player 2 L	Player 2 R
Player 1	U	(3, 2)	(1, 1)
Player 1	D	(0, 0)	(2, 3)

Task

List the best-response set for Player 1 and for Player 2.

Exercise 2: Find the Nash Equilibria

		Player 2		
		L	C	R
Player 1	U	(4, 4)	(1, 3)	(0, 2)
	M	(2, 1)	(3, 5)	(1, 1)
	D	(1, 0)	(2, 2)	(2, 4)

Question

Identify every Nash equilibrium profile. List them all if there are multiple.

Exercise 3: Revisit Earlier Games

- Explain the equilibria in the teamwork game from Lecture 01-01.
- Does the 0.9 average guessing game admit a pure-strategy equilibrium?
- Revisit the matrices we analysed with IESDS and verify your conclusions.

Wrap-Up and Next Steps

Today's Takeaways

- IESDS is powerful, yet its predictions can vary with the elimination order.
- A best response is the optimal reply to the opponents' strategy profile.
- Nash equilibria arise where best responses intersect for all players.
- Some finite games still lack a pure-strategy equilibrium.

Checklist

- You can compute the best-response set for each player.
- You can explain the definition of Nash equilibrium verbally and in notation.
- You can articulate how IESDS and Nash equilibrium differ conceptually.

Coming Up

- Lecture 03-02 introduces beliefs and Pareto efficiency.
- We examine whether equilibrium concepts deliver socially desirable outcomes.
- Lecture 04 moves to the Prisoner's Dilemma and games with infinite strategies.