BCSE Game Theory 10-02

Adverse Selection, Other Auction Formats, and Matching Markets

BCSE Game Theory

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Today's Agenda

Today's Goals

- Analyze optimal reserve prices and their impact on revenue.
- Understand Adverse Selection and market failure.
- Analyze Akerlof's "Lemons" model with numerical examples.
- Study other auction formats: ascending, descending, multi-unit.
- Learn about recent auction research and applications.
- Explore matching markets and Gale-Shapley algorithm.
- Understand strategy-proofness in matching contexts.

Lecture Roadmap

1. Optimal Reserve Prices (continued from 10-01).
2. Review: Adverse Selection concept.
3. Akerlof's Lemons model: setup and analysis.
4. Numerical example: market failure.
5. Complete market collapse.
6. Solutions to Adverse Selection.
7. Other auction formats (ascending, descending, multi-unit).
8. Recent auction research.
9. Matching markets and Gale-Shapley.
10. Takeaways.

Reserve Price

What Is a Reserve Price?

Definition: Reserve Price

A reserve price r is a minimum price set by the seller. Bids below r are rejected.

- **Purpose**: Exclude low-valuation bidders, increase competition among remaining bidders.
- **Trade-off**: Higher price if sale occurs, but lower probability of sale.
- **Real-world examples**: eBay minimum bid, government procurement, art auctions.

Note

We will calculate the optimal reserve price in the next section, after establishing the basic auction mechanisms.

Why Set a Reserve Price?

- **Without reserve price**: Seller accepts any bid ≥ 0.
- **Problem**: Low-valuation bidders may win at very low prices.
- **Solution**: Set minimum price $r > 0$.
- **Benefit**:
  - Excludes low-valuation bidders.
  - Increases competition among remaining bidders.
  - Can increase expected revenue.
- **Cost**: Lower probability of sale.

Goal

Find the reserve price r^* that maximizes expected revenue (to be calculated later).

Optimal Reserve Price

Optimal Reserve Price: Setup

- Now we calculate the optimal reserve price r^* that maximizes expected revenue.
- **Setup**: n bidders, valuations $v_i \sim \text{Uniform}[0,1]$.
- **Reserve price**: $r \in [0,1]$.
- **Rule**: Only bidders with $v_i \geq r$ can participate.
- **Probability of sale**: At least one bidder has $v_i \geq r$.
- For uniform distribution: $\Pr(\text{sale}) = 1 - r^n$.

Key Question

What is the optimal r that maximizes expected revenue?

Expected Revenue with Reserve Price: Second-Price

- **Conditional on sale**: At least one bidder has $v \geq r$.
- **Payment**: $\max(r, v_{(2)})$ (reserve price or second-highest).
- **Expected revenue**:

$$E[R_{\text{2nd}}(r)] = \Pr(\text{sale}) \cdot E[\max(r, v_{(2)}) \mid \text{sale}]$$

- For uniform $[0,1]$ with n bidders:

$$E[R_{\text{2nd}}(r)] = (1 - r^n) \cdot \left[r \cdot r^{n-1} + \int_r^1 v_{(2)} f_{(2)}(v) dv \right]$$

- After calculation:

$$E[R_{\text{2nd}}(r)] = \frac{n-1}{n+1}(1 - r^{n+1}) + r(1 - r^n)$$

Why is r^* Independent of n? The Virtual Valuation Approach

- **Key Insight**: Myerson (1981) showed that optimal auction design uses virtual valuations.
- **Virtual valuation** of a bidder with value v:

$$\phi(v) = v - \frac{1 - F(v)}{f(v)}$$

- The optimal reserve price r^* satisfies: $\phi(r^*) = c$ (seller's cost).
- This gives: $r^* - \frac{1-F(r^*)}{f(r^*)} = c$.

Applying to Uniform Distribution

- For Uniform$[0, 1]$: $F(v) = v$ and $f(v) = 1$.
- Virtual valuation: $\phi(v) = v - \frac{1-v}{1} = 2v - 1$.
- Optimal reserve price condition: $\phi(r^*) = c$.

$$2r^* - 1 = c \quad \Rightarrow \quad r^* = \frac{c+1}{2}$$

- For $c = 0$: $r^* = \frac{1}{2}$ for **all** n.

Special Property of Uniform Distribution

The uniform distribution has a linear virtual valuation, which makes r^* independent of n. This is **not** true for all distributions!

Optimal Reserve Price: General Result

Myerson's Formula (1981)

For a risk-neutral seller with opportunity cost c, the optimal reserve price r^* satisfies:

$$r^* = c + \frac{1 - F(r^*)}{f(r^*)}$$

where F is the CDF and f is the PDF of bidder valuations.
Important: This formula does not depend on the number of bidders n!

- **Intuition**: Reserve price balances two effects:
  - Higher r increases revenue when sale occurs.
  - Higher r reduces probability of sale.
- More bidders increase expected revenue, but don't change r^*.

Application to Uniform Distribution

- For Uniform$[0, 1]$: $F(v) = v$ and $f(v) = 1$.
- Myerson's formula gives:

$$r^* = c + \frac{1 - r^*}{1} = c + 1 - r^*$$

- Solving: $2r^* = c + 1 \Rightarrow r^* = \frac{c+1}{2}$.

Special Cases

- $c = 0$: $r^* = \frac{1}{2}$ (matches our earlier result!).
- $c = 0.3$: $r^* = 0.65$.
- $c = 1$: $r^* = 1$ (seller never sells below maximum value).

Why Does r^* Not Depend on n?

- Recall: For Uniform$[0, 1]$ with $c = 0$, we have $r^* = \frac{1}{2}$ (independent of n).
- **Intuition**: The optimal reserve price balances two effects:
  1. **Higher** r: Increases revenue when sale occurs.
  2. **Higher** r: Reduces probability of sale (fewer bidders exceed r).
- For Uniform$[0, 1]$, these two effects exactly balance at $r^* = \frac{1}{2}$ regardless of n.

Key Insight

The optimal reserve price depends on the distribution of valuations, not the number of bidders. More bidders increase competition, but don't change the optimal cutoff for accepting bids.

Numerical Verification: r^* vs n

- Let's verify that $r^* = \frac{1}{2}$ maximizes revenue for different n.
- Expected revenue: $E[R(r)] = \frac{n-1}{n+1}(1 - r^{n+1}) + r(1 - r^n)$.

n	$E[R(r=0)]$	$E[R(r=0.5)]$	$E[R(r=1)]$
2	0.333	0.417	0
5	0.667	0.672	0
10	0.818	0.819	0
50	0.961	0.961	0

- **Observation**: $r^* = 0.5$ is optimal for all n, and revenue increases with n.

Optimal Reserve Price: First-Price Auction

- With reserve price r, only bidders with $v \geq r$ participate.
- Equilibrium bid function:

$$b^*(v) = \begin{cases} r + \frac{n-1}{n}(v - r) & \text{if } v \geq r \\ \text{no bid} & \text{if } v < r \end{cases}$$

- Intuition: Bidders bid reserve price plus a fraction of the surplus $(v - r)$.
- Expected revenue: $E[R_{1st}(r)] = \Pr(\text{sale}) \cdot E[b^*(v_{(1)}) \mid \text{sale}]$.
- **Result**: Same optimal reserve price r^* as second-price auction.

Revenue Equivalence with Reserve Price

Even with reserve prices, Revenue Equivalence holds (under same conditions). The optimal reserve price is the same for both formats.

Numerical Example: Reserve Price Effect

Setup: $n = 3$ bidders, uniform $[0, 1]$

Reserve Price r	$\Pr(\text{sale})$	$E[R]$
0.0	1.000	0.500
0.2	0.992	0.523
0.4	0.936	0.541
0.5	0.875	0.547
0.6	0.784	0.540
0.8	0.488	0.435

Observation

Optimal reserve price is around $r^* \approx 0.5$, maximizing expected revenue at $E[R] \approx 0.547$.

Why Reserve Price Increases Revenue

- **Direct effect**: Higher price if sale occurs.
- **Indirect effect**: Excludes low-valuation bidders.
- **Competition effect**: Remaining bidders bid more aggressively.
- **Trade-off**: Lower probability of sale.

Intuition

- Very low r: Many bidders, but low prices.
- Very high r: High price, but few bidders (low sale probability).
- Optimal r: Balances price and sale probability.

Comparison: With vs Without Reserve Price

Example: $n = 5$ bidders

- **Without reserve** ($r = 0$): $E[R] = \frac{4}{6} \approx 0.667$.
- **With optimal reserve** ($r = 0.5$): $E[R] \approx 0.720$.
- **Increase**: About 8% higher revenue.

Key Insight

Reserve prices can significantly increase revenue, especially when there are many low-valuation bidders.

- **Real-world**: eBay minimum bid, government procurement, art auctions.

Trade-off: Reserve Price vs Additional Bidder

Motivating Question

Is it better to set a reserve price r with n bidders, or to set $r = 0$ and attract one more bidder ($n + 1$ total)?

Example: $n = 3$ bidders, Uniform$[0, 1]$

- **With reserve** ($r = 0.5$, $n = 3$): $E[R] \approx 0.547$.
- **Without reserve** ($r = 0$, $n = 4$): $E[R] = \frac{3}{5} = 0.600$.
- **Conclusion**: Adding one bidder is better!

- **Practical implication**: Online auctions often set very low starting prices to attract more bidders.

General Analysis: Uniform Distribution

- **Revenue without reserve, n bidders**:

$$E[R_n(r = 0)] = \frac{n-1}{n+1}$$

- **Revenue without reserve, $n + 1$ bidders**:

$$E[R_{n+1}(r = 0)] = \frac{n}{n+2}$$

- **Revenue with optimal reserve $r = 0.5$, n bidders**:

$$E[R_n(r = 0.5)] = \frac{n-1}{n+1}(1 - 0.5^{n+1}) + 0.5(1 - 0.5^n)$$

Numerical Comparison

n	$E[R_n(r=0)]$	$E[R_n(r=0.5)]$	$E[R_{n+1}(r=0)]$	Better?
2	0.333	0.458	0.500	$n+1$
3	0.500	0.547	0.600	$n+1$
4	0.600	0.609	0.667	$n+1$
5	0.667	0.653	0.714	$n+1$
10	0.818	0.773	0.833	$n+1$

Observation

For all $n \geq 2$, adding one bidder (without reserve) yields higher revenue than setting optimal reserve with n bidders.

Why Does Adding a Bidder Dominate?

- **Reserve price effect**: Excludes low-valuation bidders, increases price conditional on sale.
- **Additional bidder effect**: Increases competition, raises expected highest valuation.
- **Key difference**:
  - Reserve price: Trade-off between higher price and lower sale probability.
  - Additional bidder: Increases both competition and sale probability (no downside).
- **Mathematical insight**:

$$\frac{n}{n+2} - \frac{n-1}{n+1} = \frac{n(n+1)-(n-1)(n+2)}{(n+1)(n+2)} = \frac{2}{(n+1)(n+2)} > 0$$

- Adding a bidder always increases revenue (without reserve).

Practical Implications: Online Auctions

Online Auctions Strategy

- **Low starting prices**: Attract more bidders by reducing entry barriers.
- **No reserve (or very low)**: Maximize participation.
- **Trade-off**: Seller accepts risk of low final price to gain from increased competition.

- **Key takeaway**: Increasing n is often more valuable than optimizing r.

When to Use Reserve Prices?

- When bidder pool is fixed (cannot attract more bidders).
- When seller has high opportunity cost c (e.g., art auctions, rare items).
- When low-valuation bidders are common and high-valuation bidders are rare.

Summary

Reserve prices are useful when participation is fixed, but attracting additional bidders is generally more effective for increasing revenue.

Other Auction Formats

Ascending Auction (English Auction)

- **Format**: Price starts low, increases gradually.
- Bidders drop out when price exceeds their valuation.
- Last remaining bidder wins.
- Winner pays the price at which the second-to-last bidder dropped out.

Properties

- Equivalent to second-price auction (Revenue Equivalence).
- **Real examples**: Art auctions (Christie's, Sotheby's) - mix of private/common values.
- **Advantage**: Transparent, exciting, allows learning.

Descending Auction (Dutch Auction)

- **Format**: Price starts high, decreases gradually.
- First bidder to say "buy" wins.
- Winner pays the price at which they stopped the auction.

Properties

- Equivalent to first-price auction (Revenue Equivalence).
- **Real examples**: Dutch tulip auctions, Google IPO (2004).
- **Advantage**: Fast, efficient for perishable goods.

Note

Dutch auctions are less common today, but historically important.

Multi-Unit Auctions

- **Setting**: Multiple identical items for sale.
- **Examples**: Government bonds, electricity, spectrum licenses.
- **Two main formats**:
  1. **Uniform price auction**: All winners pay the same price.
  2. **Discriminatory price auction**: Each winner pays their bid.

Challenges

- Demand reduction (bidders shade bids to lower price).
- Allocation efficiency.
- Revenue maximization.

Combinatorial Auctions

- **Setting**: Multiple different items, bidders can bid on combinations.
- **Complementarity**: Items worth more together than separately.
- **Example**: Spectrum licenses (geographic regions), logistics routes.

Challenges

- **Computational complexity**: Finding optimal allocation is NP-hard.
- **Incentive compatibility**: Designing strategy-proof mechanisms.
- **Real applications**: FCC spectrum auctions, package delivery.

Recent Auction Research

Spectrum Auctions

- **Context**: Government sells radio spectrum licenses.
- **Format**: Simultaneous Multi-Round Auction (SMRA).
- **Features**:
  - Multiple licenses sold simultaneously.
  - Multiple rounds (bidders can adjust bids).
  - Activity rules (must bid to stay active).
- **Research**: Milgrom, Wilson, McAfee (Nobel Prize 2020).
- **Success**: Billions in revenue, efficient allocation.

Key Innovation

SMRA allows bidders to learn about values and adjust bids, reducing winner's curse.

Online Advertising Auctions

- **Context**: Google, Facebook, etc. sell ad space.
- **Format**: Generalized Second-Price (GSP) auction.
- **Features**:
  - Multiple ad slots (different click-through rates).
  - Advertiser pays the bid of the advertiser below them.
  - Not strategy-proof, but simple and practical.
- **Research**: Edelman, Ostrovsky, Schwarz (2007).
- **Alternative**: VCG mechanism (strategy-proof but complex).

Trade-off

GSP is not strategy-proof, but it's easier to understand and implement than VCG.

Procurement Auctions

- **Context**: Government or firms buy goods/services.
- **Format**: Reverse auctions (lowest price wins).
- **Multi-attribute**: Price, quality, delivery time.
- **Challenges**:
  - Common values (true cost is uncertain).
  - Quality vs price trade-off.
  - Winner's curse (winning firm may have underestimated cost).
- **Research**: Che (1993), Asker & Cantillon (2008).

Application

Designing auctions that account for quality, not just price.

Adverse Selection

What Is Adverse Selection?

Definition: Adverse Selection

A market phenomenon where information asymmetry causes high-quality goods to exit the market, leaving only low-quality goods.

- **Information asymmetry**: Sellers know quality, buyers don't.
- **Result**: Market failure or complete collapse.
- **Classic example**: Used car market (Akerlof, 1970).

Real-World Examples of Adverse Selection

- **Used car market**: Lemons problem (high-quality cars exit).
- **Insurance markets**: High-risk buyers stay, low-risk exit.
- **Labor markets**: Ability asymmetry (high-ability workers underpaid).
- **Financial markets**: Credit risk (risky borrowers remain).

Common Pattern

In all cases, information asymmetry drives out the "good" type, leaving only the "bad" type in the market.

Akerlof's Lemons Model: Setup

- **Players**: Sellers (car owners) and buyers.
- **Goods**: Used cars of two types:
  - **Peach** (high quality): Value H, proportion θ.
  - **Lemon** (low quality): Value L ($L < H$), proportion $1 - \theta$.
- **Information asymmetry**:
  - Sellers know their car's quality.
  - Buyers only know the distribution (θ).
- **Market price**: p (same for all transactions).

Complete Information Benchmark

- **Perfect information**: Buyers know each car's quality.
- **Market outcome**:
  - High-quality cars: Trade at $p = H$.
  - Low-quality cars: Trade at $p = L$.
- **Efficiency**: All mutually beneficial trades occur.
- **No market failure**: Market functions perfectly.

Key Point

With complete information, the market is efficient. Adverse Selection only occurs with incomplete information.

Incomplete Information: Adverse Selection

- **Buyers' expected value**:

$$E[v] = \theta H + (1 - \theta)L$$

- **Market price**: $p = E[v]$ (competitive market).
- **Sellers' decisions**:
  - High-quality sellers: $H > E[v] \rightarrow$ Don't sell (exit market).
  - Low-quality sellers: $E[v] > L \rightarrow$ Sell (stay in market).
- **Result**: Only lemons remain in the market.

Market Failure

High-quality goods exit, leaving only low-quality goods. This is Adverse Selection.

Numerical Example: Setup

Setup

- $H = 10,000$ (high-quality car value).
- $L = 2,000$ (low-quality car value).
- $\theta = 0.5$ (50% are high-quality).

Expected Value

$$E[v] = 0.5 \times 10,000 + 0.5 \times 2,000 = 6,000$$

Numerical Example: Market Outcome

- Market price: $p = 6,000$.
- High-quality sellers: $10,000 > 6,000 \rightarrow$ Don't sell.
- Low-quality sellers: $6,000 > 2,000 \rightarrow$ Sell.

Result

Only lemons ($L = 2,000$) remain in the market.

Complete Market Collapse

- **Condition**: If θ is very small, $E[v] \approx L$.
- High-quality cars exit immediately.
- Market price falls to L or below.
- **Spiral**: Lower price → More high-quality exit → Even lower price.
- **Result**: Market may collapse completely.

Example

If $\theta = 0.1$: $E[v] = 0.1 \times 10,000 + 0.9 \times 2,000 = 2,800$.
High-quality sellers exit. New $E[v] = 2,000$. Market collapses.

Solutions to Adverse Selection

1. **Signaling** (to be covered later):
   - Sellers send signals of quality (warranties, certifications).
   - Example: Education as a signal of ability.
2. **Screening**:
   - Buyers acquire information (inspection, test drive).
   - Example: Car inspection, medical exams.
3. **Regulation**:
   - Quality standards, mandatory disclosure.
   - Example: Lemon laws, food safety regulations.
4. **Reputation and warranties**:
   - Sellers build reputation, offer guarantees.
   - Example: Brand reputation, extended warranties.

Other Applications

Insurance Markets

- High-risk individuals buy insurance, low-risk exit.
- Premiums rise, more low-risk exit.
- **Solution**: Risk-based pricing, mandatory insurance.

Labor Markets

- High-ability workers exit (underpaid).
- Low-ability workers remain.
- **Solution**: Education signaling, performance evaluation.

Matching Markets

What Are Matching Markets?

- **Different from auctions**: No prices, no money transfers.
- **Two-sided matching**: Two groups need to be matched.
- **Examples**:
  - Medical residents and hospitals.
  - Students and schools.
  - Kidney donors and recipients.
  - Job seekers and employers.
- **Key feature**: Each player has preferences over the other group.

Why Study Matching?

Matching markets are another important example of strategy-proof mechanisms.

Matching vs Auctions: Key Differences

Auctions

- **One-sided**: Bidders compete for items.
- **Prices**: Money transfers determine allocation.
- **Efficiency**: Highest valuation wins (with truth-telling).
- **Strategy-proofness**: Second-price auction (Vickrey).

Matching Markets

- **Two-sided**: Both sides have preferences.
- **No prices**: Allocation based on preferences only.
- **Stability**: No pair wants to deviate from matching.
- **Strategy-proofness**: Gale-Shapley (for proposers).

Stability: Formal Definition

Definition: Stable Matching

A matching μ is stable if:

1. **Individual Rationality**: No agent prefers being unmatched to their current match.
2. **No Blocking Pairs**: There is no pair (s, h) such that:
   - Student s prefers hospital h to their current match $\mu(s)$, AND
   - Hospital h prefers student s to their current match $\mu(h)$.

Intuition

A matching is stable if no student-hospital pair would both prefer to be matched to each other instead of their current matches.

Gale-Shapley Algorithm (Deferred Acceptance)

Setup

- Two groups: **Proposers** (e.g., students) and **Acceptors** (e.g., schools).
- Each player has a **preference ranking** over the other group.
- Goal: Find a **stable matching**.

Algorithm

1. Each proposer proposes to their top choice.
2. Each acceptor tentatively accepts the best proposal (if any).
3. Rejected proposers propose to their next choice.
4. Repeat until no more proposals.

Example: Medical Residency Matching

Setup

- **Proposers**: Medical students.
- **Acceptors**: Hospitals.
- Each student ranks hospitals, each hospital ranks students.

Algorithm Steps

1. Students propose to their top-choice hospital.
2. Hospitals tentatively accept (keep best students up to capacity).
3. Rejected students propose to next choice.
4. Process continues until all students are matched.

Strategy-Proofness in Matching

Theorem: Gale-Shapley

The Gale-Shapley algorithm produces a stable matching. Moreover, for the proposing side, truth-telling (reporting true preferences) is a dominant strategy.

Key Insight

- Proposers (students) have no incentive to misreport preferences.
- Acceptors (hospitals) may have incentive to misreport, but the algorithm still works.
- This is a form of strategy-proofness.

Real-World Applications

National Resident Matching Program (NRMP)

- **Context**: U.S. medical students and residency programs.
- **Format**: Gale-Shapley algorithm (students propose).
- **Scale**: Tens of thousands of matches annually.
- **Success**: Stable, efficient, strategy-proof for students.

Other Applications

- **School choice**: Students and schools (Boston, NYC).
- **Kidney exchange**: Donors and recipients.
- **Job markets**: Academic job market, labor markets.

Model Extensions: Unacceptable Partners

Unacceptable Partners

- **Problem**: Some pairs are absolutely unacceptable (e.g., incompatible).
- **Solution**: Allow agents to rank some partners as "unacceptable".
- Gale-Shapley still works: agents never propose to/accept unacceptable partners.
- **Result**: Some agents may remain unmatched (but stable).

Model Extensions: Other Generalizations

- **Many-to-one matching**: Hospitals accept multiple students (capacity constraints).
- **Many-to-many matching**: Both sides can have multiple matches.
- **Couples**: Pairs of students with joint preferences (NP-hard!).
- **Incomplete preferences**: Agents don't rank all partners.

Key Point

Gale-Shapley is remarkably flexible and extends to many practical scenarios.

Computational Complexity

Gale-Shapley Algorithm

- **Time complexity**: $O(n^2)$ where n is the number of agents on each side.
- **Why?**: Each proposer makes at most n proposals, and each proposal takes $O(1)$ time.
- **Efficient**: Polynomial time, very fast in practice.

Extensions and Hardness

- **Couples problem**: Finding stable matching with couples is NP-complete (Ronn, 1990).
- **Three-sided matching**: Also NP-hard in general.
- **Practical approach**: Use heuristics or approximations for hard cases.

Why Matching Matters

- **No prices**: Can't use prices to allocate (ethical, legal reasons).
- **Two-sided**: Both sides have preferences.
- **Stability**: Important for long-term relationships.
- **Strategy-proofness**: Encourages truthful reporting.
- **Efficiency**: Maximizes welfare given constraints.

Connection to Auctions

Both auctions and matching are mechanisms for allocation under incomplete information. Strategy-proofness is a key property in both.

Takeaways

Key Concepts

Adverse Selection

- Information asymmetry causes market failure.
- High-quality goods exit, low-quality remain.
- Solutions: Signaling, screening, regulation.

Auction Formats

- Ascending (English): Equivalent to second-price.
- Descending (Dutch): Equivalent to first-price.
- Multi-unit and combinatorial: More complex, active research.

Key Insights

Recent Research

- Spectrum auctions: SMRA design (Nobel Prize 2020).
- Online advertising: GSP vs VCG trade-offs.
- Procurement: Multi-attribute auctions.

Matching Markets

- Gale-Shapley algorithm: Stable, strategy-proof for proposers.
- Real applications: Medical residency, school choice, kidney exchange.
- Connection to auctions: Both are allocation mechanisms.

Next Steps

- Game Theory 11: Signaling games (dynamic incomplete information).
- We'll study how players signal their type through actions.
- Later: Screening, mechanism design, and advanced topics.

Key Takeaway

Incomplete information creates challenges (Adverse Selection) but also opportunities (auction design, matching mechanisms) for efficient allocation.