BCSE Game Theory 04-02

Mixed Strategies II

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Today's Objectives

Today's Goals

- Connect best-response correspondences with fixed-point theorems to see why finite games admit mixed equilibria.
- Systematise the steps for setting up and solving mixed-strategy equations.
- Compute mixed equilibria for Rock–Paper–Scissors and the Volunteer Dilemma.
- Analyse how asymmetric payoffs or strategy constraints shift equilibrium probabilities.

Recap of 04-01

- Mixed strategies admit both an individual randomisation view and a population-frequency view.
- We solved 2×2 mixed equilibria via indifference plus the probability-sum constraint.
- Best-response diagrams locate the mixed equilibrium at the intersection of reaction curves.
- Today we formalise these intuitions, prove existence, and extend to richer games.

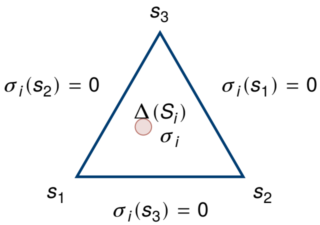
Notation Refresher

- Pure strategies: S_i is player i's finite action set.
- Mixed strategies:

$$\Delta(S_i) \stackrel{\text{def}}{=} \left\{\sigma_i \in \mathbb{R}_{\geq 0}^{|S_i|} \,\middle|\, \sum_{s_i \in S_i} \sigma_i(s_i) = 1\right\}.$$

- Profile space: $X \stackrel{\text{def}}{=} \prod_{i \in N} \Delta(S_i)$; closed and bounded $\Rightarrow$ compact.
- Support: $\operatorname{supp}(\sigma_i) \stackrel{\text{def}}{=} \{s_i \in S_i \mid \sigma_i(s_i) > 0\}$ is where indifference constraints apply.
- Cardinality: $|A|$ denotes the number of elements in a finite set A.

Geometry of the Simplex



- With three pure strategies the simplex is a filled triangle.
- Vertices are pure strategies, edges mix two strategies, and interior points mix all three.
- Visualising the simplex clarifies which strategies appear in the support.

Fixed-Point Approach

Fixed-Point Vocabulary (1/2)

- **Closed set**: contains its limit points, keeping optima inside the feasible region.
- **Compact set**: closed and bounded—essential for Brouwer/Kakutani style results.
- **Convex set**: line segments stay in the set, so we can mix strategies without leaving it.

- In our setting $X = \prod_i \Delta(S_i)$ is compact and convex, so the geometry prerequisites are already satisfied.

Fixed-Point Vocabulary (2/2)

- **Continuous map**: small input changes produce small output changes—needed for Brouwer's theorem.
- **Upper hemicontinuous correspondence**: the graph is closed, ensuring best responses do not jump erratically.
- **Convex-valued correspondence**: mixtures of best responses remain best responses, a property guaranteed by linear payoffs.

- Verifying these conditions for BR_i is the routine step before invoking Kakutani to obtain a fixed point.

Properties of Best-Response Correspondences

- $\Delta(S_i)$ is a compact, convex simplex because each player has finitely many pure strategies.
- The best-response correspondence BR_i is never empty: a continuous function attains a maximum on $\Delta(S_i)$.
- $BR_i(\sigma_{-i})$ is convex: expected payoff is linear in strategies, so convex combinations preserve equal payoffs.
- BR_i has a closed graph and is therefore upper hemicontinuous —the key requirement for applying Kakutani.

Brouwer Fixed-Point Theorem

Theorem: Brouwer

Any continuous map $f : K \to K$ on a compact, convex set $K \subseteq \mathbb{R}^m$ has a fixed point.

- Intuition: define $\Phi(x) \stackrel{\text{def}}{=} \max_i (f_i(x) - x_i)$ and minimise Φ over K.
- If the minimum $\Phi^* \stackrel{\text{def}}{=} \min_{x \in K} \Phi(x)$ were positive, we could move slightly in a direction that lowers Φ, contradicting optimality.

Kakutani Fixed-Point Theorem

Theorem: Kakutani

Let $F : X \rightrightarrows X$ be a multivalued map on a non-empty, compact, convex set $X \subseteq \mathbb{R}^m$. If

- $F(x)$ is non-empty and convex for every $x \in X$, and
- F has a closed graph (equivalently, is upper hemicontinuous),

then there exists $x^* \in X$ such that $x^* \in F(x^*)$.

Michael's Selection Theorem

Theorem: Michael, 1956

Let X be a paracompact topological space and $Y \subseteq \mathbb{R}^m$ a closed, convex subset of a Banach space. If a multivalued map $F : X \rightrightarrows Y$ is lower hemicontinuous with non-empty, closed, convex values, then there exists a continuous selection $f : X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.

- On compact subsets of Euclidean space, you may read this as: any non-empty, closed, convex, lower-hemicontinuous correspondence admits a continuous selection.
- Kakutani's proof sketch uses these selections to approximate correspondences by continuous maps and then invokes Brouwer's theorem.

Sketch of Kakutani's Theorem

- Use Michael's selection theorem to build continuous maps f_ε whose values stay within an ε-neighbourhood of $F(x)$.
- Brouwer's theorem supplies a fixed point x_ε for each f_ε.
- Compactness of X yields a convergent subsequence $x_{\varepsilon_k} \to x^*$.
- Closedness of the graph (upper hemicontinuity) ensures $x^* \in F(x^*)$.

Existence of Nash Equilibria in Finite Games

Theorem: Nash, 1950

Every finite normal-form game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ has at least one mixed-strategy Nash equilibrium.

Sketch of proof

- The product correspondence $BR(\sigma) \stackrel{\text{def}}{=} \prod_{i \in N} BR_i(\sigma_{-i})$ satisfies Kakutani's assumptions.
- Kakutani's fixed-point theorem yields $\sigma^* \in X$ with $\sigma^* \in BR(\sigma^*)$.
- Consequently each σ_i^* is a best response to σ_{-i}^*, so σ^* is a mixed-strategy Nash equilibrium.

Best-Response Roadmap: Key Checks

- Compact, convex simplices ensure the domain suits fixed-point theorems.
- Berge's maximum theorem delivers non-empty, convex, upper-hemicontinuous BR_i.
- Kakutani applied to $BR(\cdot)$ returns a profile σ^* fixed by the correspondence.
- Interpret the fixed point: each player best responds, hence σ^* is a Nash equilibrium.

Mixed-Strategy Calculations

Penalty Kick Recap

- Model the zero-sum interaction between the kicker (row player) and the keeper (column player).
- Let p denote the probability of shooting left and q the probability of diving left.
- Solving the indifference conditions $1 - 2q = 2q - 1$ and $1 - 2p = 2p - 1$ yields $p = q = \frac{1}{2}$.
- The equilibrium value is 0 (a goal with probability 0.5); any bias allows the opponent to respond and lower your payoff.
- In the 04-01 numerical example where the kicker prefers the right, the probabilities shift to $p^* = \frac{13}{29}$ and $q^* = \frac{12}{29}$.

RPS: Asymmetric Payoffs for Player 1

		Player 2		
		Rock	Scissors	Paper
Player 1	Rock	(0,0)	(1,-1)	(-1,1)
	Scissors	(-1,1)	(0,0)	(2,-2)
	Paper	(5,-5)	(-1,1)	(0,0)

- Player 1 earns $(1, 2, 5)$ when winning with Rock, Scissors, or Paper, and receives the negatives when losing.
- Let Player 2 mix (x_R, x_S, x_P) across Rock, Scissors, Paper; we derive indifference conditions from these payoffs。

RPS: Solving the Asymmetric Mix

- Player 1's expected payoffs under (x_R, x_S, x_P) are

$$\begin{aligned} U(R) &= x_S - x_P, \\ U(S) &= -x_R + 2x_P, \\ U(P) &= 5x_R - x_S. \end{aligned}$$

- Indifference plus $x_R + x_S + x_P = 1$ yields $x_R = \frac{5}{28}$, $x_S = \frac{4}{7}$, $x_P = \frac{1}{4}$.
- Player 1 keeps Player 2 indifferent with $(p_R, p_S, p_P) = (\frac{13}{28}, \frac{11}{28}, \frac{1}{7})$ and the value $\frac{9}{28} \approx 0.321$.

RPS Case 2: Player 1 Lacks Paper

		Player 2		
		Rock	Paper	Scissors
	Rock	(0,0)	(-1,1)	(1,-1)
Player 1	Scissors	(-1,1)	(1,-1)	(0,0)

- Assumption: Player 1 can use only $\{\text{Rock}, \text{Scissors}\}$.
- Player 2 still has all three strategies and exploits the missing Paper option.
- It is obvious that player 2 will not choose Scissors dominated by Paper.
- Let p be the probability that Player 1 chooses Rock. Player 2 mixes between Rock and Paper to keep Player 1 indifferent between Rock and Scissors, solving $1 - p = p - (1 - p)$ and yielding $p = \frac{2}{3}$.

RPS Case 2: Player 1 Lacks Paper

		Player 2		
		Rock	Paper	Scissors
	Rock	(0,0)	(-1,1)	(1,-1)
Player 1	Scissors	(-1,1)	(1,-1)	(0,0)

- Indifference for Player 1's support strategies also gives $1 - q_R = 1 - q_R - q_R$, so Player 2 mixes $(q_R, q_S, q_P) = (\frac{2}{3}, 0, \frac{1}{3})$ while Player 1 uses Rock $\frac{2}{3}$ and Scissors $\frac{1}{3}$.
- The payoff drops to $-\frac{1}{3}$ for Player 1, reflecting the disadvantage from losing Paper.

Mixed Equilibria in the Volunteer Dilemma

- n players can volunteer (cost c) to secure a public good worth b to everyone.
- In a symmetric mixed equilibrium each player volunteers with probability p.
- Expected payoffs:

$$U(\text{Volunteer}) = b - c, \quad U(\text{Not}) = b\left(1 - (1-p)^{n-1}\right).$$

- Equilibrium condition:

$$b - c = b\left(1 - (1-p)^{n-1}\right) \quad \Rightarrow \quad (1-p)^{n-1} = \frac{c}{b}.$$

Volunteer Dilemma: Numerical Illustration

n	$p^*(n)$	Comment
2	0.60	Two-player case
3	0.368	Trio (baseline)
6	0.201	Larger cohort
10	0.131	Near free-riding

- Baseline $b = 5$, $c = 2$: $p^*(2) = 0.6$, $p^*(3) \approx 0.368$, $p^*(6) \approx 0.201$。
- Higher n dilutes incentives—each player hopes someone else bears the cost。
- Policy levers reduce c or raise b to sustain greater volunteering。

Wrap-Up and Looking Ahead

- Kakutani's fixed-point theorem guarantees mixed equilibria in every finite game.
- We solve for mixed strategies by combining indifference equations with the probability-sum constraint.
- Asymmetric payoffs or strategy restrictions distort equilibrium probabilities and change the value.
- Next time we apply mixed strategies to repeated play and evolutionary stability.

Recap Checklist

- Can you state Kakutani's assumptions (non-empty, convex, compact, upper hemicontinuous)?
- Can you set up the indifference-plus-probability equations for a concrete 2×2 game?
- Can you explain which conditions shift probabilities when payoffs or available strategies change?
- Can you derive the symmetric equilibrium expression $1 - (1 - p)^{n-1} = c/b$ for the Volunteer Dilemma and discuss parameter effects?