BCSE Game Theory 10-01

Auction Theory: Incomplete Information in Practice

BCSE Game Theory

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Today's Agenda

Today's Goals

- Understand auctions as incomplete information games.
- Analyze strategic bidding in first-price and second-price auctions.
- Learn about Revenue Equivalence Theorem and its limitations.
- Understand the private values framework used in our analysis.
- Explore reserve prices and strategy-proofness.
- Connect auction theory to real-world applications.

Lecture Roadmap I

1. Introduction: What are auctions?
2. Why Auction Theory for Computer Science?
3. Basic setup: incomplete information framework.
4. First-price auction analysis.
5. Second-price auction (Vickrey auction) analysis.
6. Revenue Equivalence Theorem.
7. Strategy-proofness and revelation principle.

Introduction: What Are Auctions?

Auctions in Everyday Life

- **eBay**: Online auctions for consumer goods.
- **Google AdWords**: Advertising space auctions.
- **Government procurement**: Public works, spectrum licenses.
- **Art auctions**: Christie's, Sotheby's.
- **Stock exchanges**: IPO pricing, bond auctions.

Key Insight

Auctions are mechanisms for allocating goods when values are private information.

Auctions as Incomplete Information Games

- **Players**: Bidders (buyers) and seller.
- **Private information**: Each bidder knows their own valuation v_i.
- **Incomplete information**: Bidders don't know others' valuations.
- **Strategic choice**: How much to bid?
- **Outcome**: Who wins? What price is paid?

Connection to Game Theory 09

This is a Bayesian game: each bidder has a type (valuation), and bidders have beliefs about others' types.

Why Auction Theory for Computer Science?

- **Algorithm Design**: Auctions are algorithms. Input: Bids → Output: Allocation & Payments.
- **Platform Business**: Google, Facebook, Uber, Amazon run real-time auctions.
- **Resource Allocation**: Cloud computing (spot instances), bandwidth allocation.
- **Multi-Agent Systems**: Protocols for autonomous agents negotiation.

Key Relevance

Understanding incentives is crucial for designing robust systems. CS is not just about efficiency (P vs NP), but also about strategic behavior (Game Theory).

Basic Setup

Auction Framework

- **Bidders**: n players, indexed by $i = 1, \ldots, n$.
- **Valuation**: Each bidder i has valuation $v_i \in [\underline{v}, \overline{v}]$.
- **Type space**: v_i is bidder i's private information (type).
- **Prior distribution**: $F(v)$ is common knowledge.
- **Payoff**: If bidder i wins and pays p_i:

$$u_i = v_i - p_i$$

- If bidder i loses: $u_i = 0$.

Key Assumptions

- **Independent private values**: Each bidder's valuation is independent.
- **Risk neutrality**: Bidders maximize expected payoff.
- **Symmetry**: All bidders have the same prior distribution $F(v)$.
- **No collusion**: Bidders act independently.
- **Common knowledge**: The auction rules and prior distribution are known to all.

Private Values and Independence

- **Private Values**: Each bidder's valuation depends only on their own preferences (e.g., taste for art), not on others'.
- **Independence**: Valuations are drawn independently from $F(v)$.
- **Implication**: Knowing your own value is sufficient; you don't need to estimate others' signals to know what the item is worth to you.

Note

This framework (Independent Private Values, IPV) allows us to derive explicit solutions and is the standard starting point for auction theory.

First-Price Auction

First-Price Sealed-Bid Auction

- Each bidder submits a **sealed bid** b_i.
- The highest bidder wins.
- The winner pays their **own bid** b_i.

Strategic Challenge

- If you bid $b_i = v_i$ and win, payoff is $v_i - b_i = 0$.
- To make a profit, you must **shade your bid** $(b_i < v_i)$.
- **The Trade-off**:
  - Lower bid $\rightarrow$ Higher profit if you win.
  - Lower bid $\rightarrow$ Lower probability of winning.

Equilibrium Bidding Strategy: General Case

- **Problem**: Finding the optimal bid requires guessing others' strategies.
- **Setup**: n bidders, valuations $v_i \sim \text{Uniform}[0, 1]$ (i.i.d.).
- **Symmetric Equilibrium**: Assume all bidders use same function $b = \beta(v)$.

Result (Derivation Next)

For n bidders with uniform valuations, the Bayesian Nash Equilibrium strategy is:

$$b^*(v) = \frac{n-1}{n} v$$

Example ($n = 2$): Bid half your valuation ($v/2$).

Deriving Equilibrium Bid: n Bidders

Step 1: Probability of Winning

- Suppose bidder i bids b.
- Wins if $b > \beta(v_j)$ for all $j \neq i \iff \beta^{-1}(b) > v_j$.
- Prob of winning: $F(\beta^{-1}(b))^{n-1}$. For uniform: $[\beta^{-1}(b)]^{n-1}$.

Envelope Theorem (Tool)

The **envelope theorem** helps us find the optimal $\beta(v)$ by relating the derivative of the value function to the partial derivative of the objective function.

Envelope Theorem: Statement

Envelope Theorem

The envelope theorem states that the derivative of the **optimal value function** with respect to a parameter equals the partial derivative of the **objective function** (Lagrangian), evaluated at the optimal choice.

- In auction theory, we use it to derive the equilibrium bidding function $\beta(v)$.
- It relates the rate of change of the maximum expected payoff to the partial derivative of the payoff function.

Linear Bidding Strategy Derivation

- **Guess**: Linear strategy $\beta(v) = \alpha v$.
- Bidder i chooses b_i to maximize $E[\pi_i] = \left(\frac{b_i}{\alpha}\right)^{n-1}(v_i - b_i)$.
- **FOC**: $(n-1)b_i^{n-2}(v_i - b_i) - b_i^{n-1} = 0$.
- Solution: $b_i = \frac{n-1}{n} v_i$.
- Thus $\alpha = \frac{n-1}{n}$ is consistent.

Advanced Problem: General Distribution

Question

What if valuations are drawn from a general distribution $F(v)$ (not necessarily Uniform)?

- **Setup**: Symmetric equilibrium $\beta(v)$, strictly increasing.
- **Inverse Bid**: Let $\phi(b) = \beta^{-1}(b)$, so $v = \phi(b)$.
- **Probability of Winning**: $\Pr(\text{win}) = F(\phi(b))^{n-1}$.
- **Optimization**: Maximize expected payoff:

$$\max_b [F(\phi(b))]^{n-1}(v - b)$$

- **First-Order Condition (FOC)**:

$$(n-1)F(\phi(b))^{n-2}f(\phi(b))\phi'(b)(v-b) - F(\phi(b))^{n-1} = 0$$

Solution for General Distribution

- At equilibrium $b = \beta(v)$, we have $\phi(b) = v$ and $\phi'(b) = 1/\beta'(v)$.
- The FOC becomes a differential equation:

$$\frac{d}{dv}[F(v)^{n-1}\beta(v)] = (n-1)vF(v)^{n-2}f(v)$$

- Integrating from $\underline{v}$ to v (with $\beta(\underline{v}) = \underline{v}$):

$$\beta(v) = v - \frac{\int_{\underline{v}}^{v} F(t)^{n-1}dt}{F(v)^{n-1}}$$

Intuition

Bid equals your valuation minus a "shading term" that depends on the distribution shape and number of bidders n. As $n \to \infty$, the integral term vanishes, and $\beta(v) \to v$.

Order Statistics: Highest Valuation

- Let $v_{(1)}$ denote the highest valuation among n bidders.
- For Uniform$[0, 1]$: The CDF of $v_{(1)}$ is $F_{(1)}(v) = v^n$.
- Expected value:

$$E[v_{(1)}] = \int_0^1 v \cdot nv^{n-1} dv = n \int_0^1 v^n dv = \frac{n}{n+1}$$

- **Examples**: For $n = 2$: $E[v_{(1)}] = \frac{2}{3}$. For $n = 3$: $E[v_{(1)}] = \frac{3}{4}$.

Expected Revenue: First-Price Auction

- Winner pays $\frac{n-1}{n} v_{(1)}$.
- Expected Revenue:

$$E[R_{1\text{st}}] = \frac{n-1}{n} E[v_{(1)}] = \frac{n-1}{n} \cdot \frac{n}{n+1} = \frac{n-1}{n+1}$$

Summary of First-Price

Complex strategic calculation needed. Bidders must shade bids based on competition (n).

Second-Price Auction (Vickrey Auction)

Second-Price Sealed-Bid Auction

- Each bidder submits a **sealed bid** b_i.
- The highest bidder wins.
- The winner pays the **second-highest bid**.

Motivation

Can we design an auction where bidders don't need to do complex calculations (like in First-Price)?

Dominant Strategy: Truth-Telling I

Theorem: Vickrey's Insight

In a second-price auction, bidding your true valuation v_i is a weakly dominant strategy.

Note

This is a **weakly** dominant strategy: truth-telling is at least as good as any other strategy, and sometimes strictly better. When $v_i = b^*_{-i}$, bidding v_i yields the same payoff as other bids, but never worse.

Dominant Strategy: Truth-Telling II

Proof Sketch

- Let $b^*_{-i} = \max_{j \neq i} b_j$ be the highest bid among others.
- If $b_i > b^*_{-i}$: You win, pay b^*_{-i}. Payoff: $v_i - b^*_{-i}$.
- If $b_i < b^*_{-i}$: You lose, payoff: 0.
- Bidding $b_i = v_i$ maximizes expected payoff regardless of others' bids.

Why Truth-Telling is Optimal

Case 1: $v_i > b^*_{-i}$

- Bidding $b_i = v_i$: Win, payoff $v_i - b^*_{-i} > 0$.
- Bidding $b_i < v_i$: Might lose (if $b_i < b^*_{-i}$), payoff 0.
- Bidding $b_i > v_i$: Still win, same payoff.
- **Best**: Bid $b_i = v_i$.

Case 2: $v_i < b^*_{-i}$

- Bidding $b_i = v_i$: Lose, payoff 0.
- Bidding $b_i > v_i$: Might win, but payoff $v_i - b^*_{-i} < 0$.
- Bidding $b_i < v_i$: Still lose, payoff 0.
- **Best**: Bid $b_i = v_i$.

Truth-Telling Strategy

Definition: Truth-Telling Strategy

A truth-telling strategy is one where each bidder reports their true valuation: $b_i = v_i$.

- In second-price auctions, truth-telling is optimal (weakly dominant).
- Bidders have no incentive to misreport their valuation.
- This simplifies the auction: no need to guess others' strategies.

Key Advantage

Truth-telling eliminates strategic complexity. Bidders can simply bid their true valuation without worrying about others' behavior.

What Is Adverse Selection?

Definition: Adverse Selection

A market phenomenon where information asymmetry causes high-quality goods to exit the market, leaving only low-quality goods.

- **Information asymmetry**: Sellers know quality, buyers don't.
- **Result**: Market failure or complete collapse.
- **Classic example**: Used car market (Akerlof, 1970).

Real-World Examples of Adverse Selection

- **Used car market**: Lemons problem (high-quality cars exit).
- **Insurance markets**: High-risk buyers stay, low-risk exit.
- **Labor markets**: Ability asymmetry (high-ability workers underpaid).
- **Financial markets**: Credit risk (risky borrowers remain).

Common Pattern

In all cases, information asymmetry drives out the "good" type, leaving only the "bad" type in the market.

Strategy-Proofness

Definition: Strategy-Proof

A mechanism is strategy-proof (or incentive-compatible) if truth-telling (reporting true valuation) is a weakly dominant strategy.

- **Second-price auction**: Strategy-proof (as we just proved).
- **First-price auction**: Not strategy-proof (bid shading required).
- **Benefit**: Strategy-proof mechanisms are simple and robust.
- Bidders don't need to form beliefs about others' valuations.

Real-World Application

Strategy-proofness is a key property in mechanism design. It ensures that bidders can participate without complex strategic calculations.

Revelation Principle

Theorem: Revelation Principle

Any mechanism can be converted to an equivalent direct mechanism where truth-telling is optimal.

Intuition

- Complex mechanisms (e.g., multi-round auctions) can be simplified.
- Instead of complex bidding strategies, bidders directly report valuations.
- The mechanism designer can implement the same outcome.

Revelation Principle: Application

Key Result

If a mechanism has an equilibrium, there exists a direct mechanism with truth-telling equilibrium that yields the same outcome.

Application to Auction Theory

This principle is fundamental to mechanism design. It allows us to focus on direct mechanisms (where bidders report types) without loss of generality.

- We can analyze auctions by assuming bidders report valuations truthfully.
- The auctioneer then determines allocation and payments.
- This simplifies the analysis significantly.

Equilibrium in Second-Price Auction

- **Weakly dominant strategy equilibrium**: $b_i^* = v_i$ for all i.
- This is also a **Bayesian Nash equilibrium**.
- No bidder has incentive to deviate, regardless of beliefs about others.
- **Strategy-proof**: Truth-telling is optimal (weakly dominant).

Expected Revenue in Second-Price Auction

In a second-price auction, the seller's expected revenue is the expected value of the **second-highest valuation**: $E[R_{\text{2nd}}] = E[v_{(2)}]$.

- Winner pays the second-highest bid, which equals the second-highest valuation (since bidders bid truthfully).
- This will be compared with first-price auction revenue later.

Solving Second-Price Auction: n Bidders

- **Equilibrium**: All bidders bid truthfully: $b_i = v_i$.
- Winner: Bidder with highest valuation $v_{(1)}$.
- Payment: Second-highest valuation $v_{(2)}$.
- Expected revenue: $E[R_{\text{2nd}}] = E[v_{(2)}]$.

Order Statistics

For n i.i.d. uniform $[0, 1]$ random variables:

$$E[v_{(2)}] = \frac{n-1}{n+1}$$

- This matches the first-price auction revenue!

Deriving $E[v_{(2)}]$ for Uniform Distribution

- For n i.i.d. uniform $[0,1]$: $v_i \sim U[0,1]$.
- The second-highest order statistic $v_{(2)}$ has density:

$$f_{(2)}(x) = n(n-1)x^{n-2}(1-x)$$

- Expected value:

$$E[v_{(2)}] = \int_0^1 x \cdot n(n-1)x^{n-2}(1-x)dx$$

- Computing:

$$E[v_{(2)}] = n(n-1)\left[\frac{1}{n} - \frac{1}{n+1}\right] = \frac{n-1}{n+1}$$

Result

$E[R_{\text{2nd}}] = \frac{n-1}{n+1}$, same as first-price auction!

Revenue Equivalence Theorem

Revenue Equivalence: Comparing First-Price and Second-Price

Expected Revenues

- **First-price auction**: $E[R_{1\text{st}}] = E\left[\frac{n-1}{n} v_{(1)}\right] = \frac{n-1}{n+1}$.
- **Second-price auction**: $E[R_{2\text{nd}}] = E[v_{(2)}] = \frac{n-1}{n+1}$.
- **Result**: They are equal!

Numerical Verification

n	1st-Price	2nd-Price	Difference
2	$1/3$	$1/3$	0
3	$1/2$	$1/2$	0
5	$2/3$	$2/3$	0
10	$9/11$	$9/11$	0

Observation

For all n, both auction formats yield identical expected revenue.

Revenue Equivalence Theorem: Statement

Theorem: Revenue Equivalence

Under certain conditions, different auction formats yield the same expected revenue for the seller.

Conditions

- Private values (not common values).
- Risk neutrality.
- Symmetric bidders (same prior distribution).
- Monotonicity (higher valuation $\rightarrow$ higher bid).

Revenue Equivalence Theorem: Intuition

Why It Holds

- **Allocation**: Same in both auctions (highest valuation wins).
- **Expected payment**: Must be the same in equilibrium.
- In first-price: Winner pays $\frac{n-1}{n}v_{(1)}$.
- In second-price: Winner pays $v_{(2)}$.
- Expected values are equal: $E\left[\frac{n-1}{n}v_{(1)}\right] = E[v_{(2)}] = \frac{n-1}{n+1}$.
- **Key**: Different payment rules, but bidders adjust bids to yield same expected revenue.

Proof of Revenue Equivalence Theorem

Setup

- Consider two auction mechanisms M_1 and M_2 (e.g., first-price and second-price).
- Both allocate to the highest valuation bidder (efficient allocation).
- Bidders are symmetric, risk-neutral, with valuations $v_i \sim F(v)$ on $[\underline{v}, \overline{v}]$.

Key Insight

By the Revelation Principle, we can analyze direct mechanisms where truth-telling is optimal. We will derive the payment function from Incentive Compatibility (IC) constraints.

Step 1: Incentive Compatibility (IC)

IC Constraint

A bidder with true valuation v_i must prefer reporting truthfully over misreporting as v_i':

$$v_i q_i(v_i) - p_i(v_i) \geq v_i q_i(v_i') - p_i(v_i')$$

where $q_i(v)$ is the probability of winning when reporting v, and $p_i(v)$ is the expected payment.

Rearranging

$$v_i[q_i(v_i) - q_i(v_i')] \geq p_i(v_i) - p_i(v_i')$$

This must hold for all v_i, v_i'.

Step 2: Deriving Payment Differences

- Apply IC for v_i reporting as v_i':

$$v_i[q_i(v_i) - q_i(v_i')] \geq p_i(v_i) - p_i(v_i')$$

- Apply IC for v_i' reporting as v_i:

$$v_i'[q_i(v_i') - q_i(v_i)] \geq p_i(v_i') - p_i(v_i)$$

- Rearranging the second inequality:

$$v_i'[q_i(v_i) - q_i(v_i')] \leq p_i(v_i) - p_i(v_i')$$

- Combining both inequalities:

$$v_i'[q_i(v_i) - q_i(v_i')] \leq p_i(v_i) - p_i(v_i') \leq v_i[q_i(v_i) - q_i(v_i')]$$

Step 3: Monotonicity of $q_i(v)$

- From the combined inequality, if $v_i > v_i'$:

$$v_i'[q_i(v_i) - q_i(v_i')] \le p_i(v_i) - p_i(v_i') \le v_i[q_i(v_i) - q_i(v_i')]$$

- For this to hold for all $v_i > v_i'$, we need:

$$q_i(v_i) \ge q_i(v_i')$$

- **Monotonicity**: $q_i(v)$ is non-decreasing in v.
- In our auctions, $q_i(v) = F(v)^{n-1}$ is strictly increasing, satisfying this condition.

Step 4: Payment Identity

- Taking the limit as $v_i' \to v_i$ in the IC inequality:

$$\lim_{v_i' \to v_i} \frac{p_i(v_i) - p_i(v_i')}{v_i - v_i'} = q_i(v_i)$$

- This gives us:

$$\frac{dp_i(v_i)}{dv_i} = q_i(v_i)$$

- Integrating from $\underline{v}$ to v_i:

$$p_i(v_i) = p_i(\underline{v}) + \int_{\underline{v}}^{v_i} q_i(t)dt$$

Step 5: Boundary Condition

- Individual Rationality (IR): A bidder with the lowest valuation $\underline{v}$ must get non-negative surplus.
- At $v_i = \underline{v}$:

$$\underline{v} \cdot q_i(\underline{v}) - p_i(\underline{v}) \geq 0$$

- For continuous distributions with $F(\underline{v}) = 0$:

$$q_i(\underline{v}) = F(\underline{v})^{n-1} = 0$$

- Therefore: $p_i(\underline{v}) = 0$.
- Substituting into the payment identity:

$$p_i(v_i) = \int_{\underline{v}}^{v_i} q_i(t)dt$$

Step 6: Revenue Equivalence

- For both mechanisms with the same allocation rule:

$$q_i(v) = F(v)^{n-1}$$

- Payment function:

$$p_i(v) = \int_{\underline{v}}^{v} F(t)^{n-1} dt$$

- This is identical in both mechanisms!
- Expected revenue from bidder i:

$$E[p_i] = \int_{\underline{v}}^{\overline{v}} p_i(v) f(v) dv$$

- Since $p_i(v)$ is the same, expected revenue is the same.

Verification: Uniform Distribution Example

- For $v \sim \text{Uniform}[0, 1]$: $F(v) = v$, $f(v) = 1$.
- Payment formula:

$$p_i(v) = v \cdot v^{n-1} - \int_0^v t^{n-1} dt = v^n - \frac{v^n}{n} = \frac{n-1}{n} v^n$$

- Expected payment:

$$E[p_i] = \int_0^1 \frac{n-1}{n} v^n dv = \frac{n-1}{n} \cdot \frac{1}{n+1} = \frac{n-1}{n(n+1)}$$

- Expected revenue: $E[R] = n \cdot \frac{n-1}{n(n+1)} = \frac{n-1}{n+1}$.
- This matches our earlier calculations for both first-price and second-price!

When Revenue Equivalence Fails

- **Risk aversion**: First-price bidders shade more → lower revenue.
- **Asymmetric bidders**: Different distributions → different revenues.
- **Reserve prices**: Can increase revenue. (Discussed in Game Theory 10-02).
- **Common values**: In common value auctions (where true value is unknown), winner's curse can affect revenues differently, but this requires more complex analysis beyond our private values framework.

Important

Revenue Equivalence is a theoretical benchmark. Real auctions often violate the conditions. Our analysis focuses on the private values case where explicit solutions are possible.

Trade-off: Reserve Price vs Additional Bidder

Motivating Question

Is it better to set a reserve price r with n bidders, or to set $r = 0$ and attract one more bidder ($n + 1$ total)?

Example: $n = 3$ bidders, Uniform$[0, 1]$

- **With reserve** ($r = 0.5$, $n = 3$): $E[R] \approx 0.547$.
- **Without reserve** ($r = 0$, $n = 4$): $E[R] = \frac{3}{5} = 0.600$.
- **Conclusion**: Adding one bidder is better!

- **Practical implication**: Online auctions often set very low starting prices to attract more bidders.

Practical Implications: Online Auctions

Online Auctions Strategy

- **Low starting prices**: Attract more bidders by reducing entry barriers.
- **No reserve (or very low)**: Maximize participation.
- **Trade-off**: Seller accepts risk of low final price to gain from increased competition.

- **Key takeaway**: Increasing n is often more valuable than optimizing r.

When to Use Reserve Prices?

- When bidder pool is fixed (cannot attract more bidders).
- When seller has high opportunity cost c (e.g., art auctions, rare items).
- When low-valuation bidders are common and high-valuation bidders are rare.

Summary

Reserve prices are useful when participation is fixed, but attracting additional bidders is generally more effective for increasing revenue.

Takeaways

Key Concepts

Core Ideas

- Auctions are incomplete information games.
- Private values framework enables explicit solutions.
- Second-price auction: Truth-telling is dominant strategy.
- First-price auction: Bid shading is optimal.
- Revenue Equivalence Theorem: Conditions and limitations.

Advanced Topics

- Reserve prices can increase revenue.
- Asymmetric bidders break revenue equivalence.
- Strategy-proofness and revelation principle.

Key Insights

- Auction design matters for efficiency and revenue.
- Different auction formats suit different contexts.
- Theory guides practice, but real auctions are complex.
- Mechanism design is a powerful tool for market design.

Next Steps

- Game Theory 10-02: Adverse Selection and other auction formats.
- We'll study matching markets (Gale-Shapley) as another example of strategy-proofness.
- Later: Signaling, screening, and mechanism design.