Game Theory 13-02

Review: Games of Incomplete Information

BCSE Game Theory

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Part 1: Static Games of Incomplete Information

Auctions

Review: Bayesian Nash Equilibrium

- **Incomplete Information**: Players do not know opponents' payoff functions (types).
- **Example**: Auctions (values are private).
- **Strategy**: A function $s_i(v_i)$ mapping type to action.
- **Bayesian Nash Equilibrium (BNE)**: Each type v_i maximizes expected utility given the strategies of others.

Exercise 1: First-Price Auction (3 Bidders)

Problem

- 3 Bidders ($N = 3$).
- Private values v_i independently distributed uniform on $[0, 1]$.
- **First-Price Sealed-Bid**: Highest bidder wins and pays their bid.
- Guess a linear strategy: $b_i(v_i) = k \cdot v_i$.

Task:

1. Determine probability of winning for Bidder 1 with bid b, assuming Bidders 2 and 3 follow $b(v) = kv$.
2. Write down Bidder 1's Expected Payoff function.
3. Solve for the optimal bid b^* and find the equilibrium constant k.

Ex 1: Solution (Probability & Payoff)

1. **Probability of Winning**: Win if $b_1 > B_j = kv_j \implies v_j < b_1/k$ for all $j \neq 1$. Since $v_j \sim U[0,1]$, $\Pr(v_j < x) = x$.

$$P(\text{Win}) = \left(\frac{b_1}{k}\right)^{N-1}$$

2. **Expected Payoff**:

$$E[\pi_1(b_1, v_1)] = (v_1 - b_1) \cdot \left(\frac{b_1}{k}\right)^{N-1}$$

Discussion: Bid Shading

Intuition: Why not bid v_i?

In a First-Price Auction, your profit is $v_i - b_i$.

- If you bid $b_i = v_i$, your profit is **zero** even if you win.
- To make money, you must bid **below** your value ($b_i < v_i$).
- **Trade-off**:
  - Lower bid $\rightarrow$ Higher Profit Margin (if you win).
  - Lower bid $\rightarrow$ Lower Probability of Winning.
- The equilibrium bid $b^* = \frac{N-1}{N}v$ balances this trade-off optimally.

Ex 1: Solution (Equilibrium)

3. **Optimization**: Maximize w.r.t b_1:

$$\frac{\partial E}{\partial b_1} = -\left(\frac{b_1}{k}\right)^{N-1} + (v_1 - b_1)(N-1)\left(\frac{b_1}{k}\right)^{N-2}\frac{1}{k} = 0$$

Divide by $(b_1/k)^{N-2}$ and multiply by k:

$$-b_1 + (v_1 - b_1)(N-1) = 0 \implies b_1 N = v_1(N-1) \implies b^* = \frac{N-1}{N}v_1$$

Comparing to guess $b(v) = kv$, we see $k = \frac{2}{3}$.

Result

In an FPSB auction with N bidders: $b^*(v) = \frac{N-1}{N}v$. For $N = 3$, bid $\frac{2}{3}$ of value.

Part 2: Dynamic Games of Incomplete Information

Signaling Equilibrium Selection

Review: Perfect Bayesian Equilibrium (PBE)

In dynamic games of incomplete information, purely strategy-based NE is insufficient. We need to track **Beliefs**.

- **Belief System** μ: At every information set, players assign probabilities to nodes (types).
- **Sequential Rationality**: Strategies are optimal given beliefs μ.
- **Consistency (Bayes' Rule)**: On the equilibrium path, beliefs are derived from strategies using Bayes' Rule.

PBE = Strategy Profile + Belief System satisfying these conditions.

Review: Refining Equilibria (Intuitive Criterion)

PBE places no restriction on **off-equilibrium beliefs**, allowing "unreasonable" equilibria (e.g., pooling sustained by believing any deviator is the "worst" type).

Intuitive Criterion (Cho & Kreps):

- If a deviation is **dominated** for type L (payoff worse than equilibrium payoff no matter the belief) but **possibly advantageous** for type H, then...
- Receivers should NOT believe the deviator is L. They should believe it is H.
- This destroys equilibria relying on "if deviation, believe L".

Exercise 2: Education Signaling (Spence Model)

Problem

Context:

- **Worker**: High (H) or Low (L) Ability. Prior $\Pr(H) = 0.5$.
- **Education**: Years $e \geq 0$. Cost: $c_H = 1$ (High), $c_L = 2$ (Low).
- **Employer**: Observes e, offers wage $w(e)$ based on expected productivity.
- **Productivity**: $\theta_H = 2$, $\theta_L = 1$.
- **Payoff**: Worker gets $w - c_i \cdot e$. Employer gets $\theta_i - w$.

Questions:

1. Find Pooling equilibria (both choose $e = 0$).
2. Find Separating equilibria (H chooses e^*, L chooses 0).
3. Apply Intuitive Criterion to refine equilibria.

Ex 2: Pooling Equilibrium ($e = 0$)

Candidate: Both types choose $e = 0$.

- **Employer's Belief**: $\mu(H|e = 0) = 0.5$ (prior).
- **Wage**: $w(0) = E[\theta] = 0.5 \times 2 + 0.5 \times 1 = 1.5$.
- **Payoffs**: H gets $1.5 - 0 = 1.5$. L gets $1.5 - 0 = 1.5$.

Deviation Check: Would H deviate to $e > 0$?

- If employer believes deviator is L (worst case): $w(e) = 1$.
- H's payoff from deviation: $1 - 1 \cdot e < 1.5$ for all $e > 0.5$.
- So H won't deviate if off-equilibrium belief is pessimistic.

Result: Pooling at $e = 0$ is a PBE (with pessimistic off-equilibrium beliefs).

Ex 2: Separating Equilibrium

Candidate: H chooses e^*, L chooses 0.

- **Wages**: $w(e^*) = 2$ (employer knows it's H), $w(0) = 1$ (employer knows it's L).
- **Payoffs**: H gets $2 - 1 \cdot e^* = 2 - e^*$. L gets $1 - 0 = 1$.

Incentive Constraints:

1. H prefers e^* over 0: $2 - e^* \geq 1 \implies e^* \leq 1$.
2. L prefers 0 over e^*: $1 \geq 2 - 2e^* \implies e^* \geq 0.5$.

Result: Any $e^* \in [0.5, 1]$ supports a separating PBE.

Discussion: Why Multiple Equilibria?

Intuition: Education as a Signal

- **Separating ($e^* \in [0.5, 1]$)**: Education is costly but signals ability. High-ability workers can afford it (lower cost), low-ability cannot. The exact level e^* is arbitrary as long as it's "expensive enough" to deter mimicking.
- **Pooling ($e = 0$)**: If employers are pessimistic about deviators (believe $e > 0$ means Low ability), then High types won't signal. Everyone pools at $e = 0$ and gets average wage.

Ex 2: Intuitive Criterion Application

Intuitive Criterion (Cho & Kreps):

- If deviation is **equilibrium-dominated** for type L (worse than equilibrium payoff under any belief),
- but **potentially beneficial** for type H (better under some belief),
- then receiver should believe deviator is H, not L.

Test Pooling at $e = 0$: Can H profitably deviate to $e \in [0.5, 1]$?
Consider $e = 0.6$:

- **For L**: Best case (believed H): $2 - 2(0.6) = 0.8 < 1$ (pooling).
  - Deviation is **equilibrium-dominated** for L.
- **For H**: If believed H: $2 - 0.6 = 1.4 < 1.5$ (pooling).
  - Not beneficial for H either. Try smaller e.

Try $e = 0.4$: Both types would deviate! Pooling is unstable.
Conclusion: Pooling fails IC. Separating ($e^* \in [0.5, 1]$) survives.

Ex 2: Key Insights

Summary

- **Multiple PBE**: Pooling and Separating both exist.
- **Intuitive Criterion**: Eliminates Pooling.
  - Reasoning: "Why would a Low-ability worker get expensive education?"
  - If we see deviation to $e \in [0.5, 1]$, it must be High ability.
- **Least-Cost Separating**: $e^* = 0.5$ minimizes waste (education has no productive value here, it's pure signaling).
- **Welfare**: Separating is inefficient (education is costly but doesn't increase productivity), but it's the only "reasonable" equilibrium.

Why Game Theory for Computer Science? (1/2)

Course Reflection

Game Theory provides essential tools for computational systems:

1. **Algorithm Design & Mechanism Design**
   - Auctions (Google Ads, eBay), Matching Markets (Residency, School Choice)
   - Incentive-compatible protocols (Blockchain, Voting Systems)
2. **Multi-Agent Systems & AI**
   - Nash Equilibrium in multi-agent learning (AlphaGo, Poker AI)
   - Adversarial ML: Attacker-Defender games (Security, Robustness)

Why Game Theory for Computer Science? (2/2)

3. **Network Economics & Platform Design**
   - Pricing strategies (Cloud services, SaaS)
   - Network effects and competition (Social networks, Marketplaces)
4. **Strategic Thinking**
   - Analyzing incentives, Understanding user behavior