Discrete Comput Geom (2015) 54:728–740
DOI 10.1007/s00454-015-9715-9

Positive Center Sets of Convex Curves

Pingliang Huang[1] · **Shengliang Pan**[2] ·
Yunlong Yang[2]

Received: 22 December 2014 / Revised: 1 June 2015 / Accepted: 9 June 2015 /
Published online: 17 July 2015

Abstract In this paper we shall investigate the positive center set $\mathfrak{P}(\gamma)$ of a convex curve γ and show that $\mathfrak{P}(\gamma)$ has only one point if and only if γ is a circle; $\mathfrak{P}(\gamma)$ is a segment if and only if γ is a sausage curve; if γ is a strictly convex non-circular curve, then $\mathfrak{P}(\gamma)$ is a domain of positive area; and furthermore, if γ is a constant width curve, then $\mathfrak{P}(\gamma)$ is its inner parallel body K_{-r_1}.

Keywords Convex curve · Positive center set · Sausage curve · Constant width curve · Inner parallel body

Mathematics Subject Classification 52A10 · 52A40

1 Introduction

In order to show that the isoperimetric ratio of the evolving convex curve is decreasing under the curve shortening flow (cf. [5]), Gage [4] proved an "isoperimetric inequality" for a convex curve γ,

Editor in Charge: János Pach

Shengliang Pan
slpan@tongji.edu.cn

Pingliang Huang
huangpingliang@shu.edu.cn

Yunlong Yang
88ylyang@tongji.edu.cn

[1] Department of Mathematics, Shanghai University, Shanghai 200444, People's Republic of China

[2] Department of Mathematics, Tongji University, Shanghai 200092, People's Republic of China

$$\int_\gamma \kappa^2 \mathrm{d}s \geq \frac{\pi L}{A}, \tag{1.1}$$

where κ and L are the curvature and perimeter of γ, respectively. A is the area of the domain enclosed by γ. His argument is based on the following famous *Bonnesen inequality* (see [1]):

$$Lr - A - \pi r^2 \geq 0, \quad r_{\text{in}} \leq r \leq r_{\text{out}}. \tag{1.2}$$

The equality in (1.2) holds when $r = r_{\text{in}}$ if and only if γ is either a circle or a sausage curve and the equality in (1.2) holds when $r = r_{\text{out}}$ if and only if γ is a circle, where r_{in} and r_{out} are, respectively, the inradius and outradius of γ. There have been many proofs of (1.2), see e.g., [2,3,12,14]. A more comprehensive account of various aspects of the Bonnesen inequality can be found in [10] or [13, pp. 321–327].

To simplify the proof of his inequality (1.1), Gage [6] introduced for the first time the concept of positive center for a convex curve γ. A *positive center* of γ is an inner point of γ from which the support function $p(\theta)$ satisfies

$$Lp(\theta) - A - \pi p(\theta)^2 \geq 0, \quad \theta \in [0, 2\pi]. \tag{1.3}$$

Gage [6] proved that a convex plane curve must have at least one positive center because the center of its minimal annulus (cf. [11]) is a positive center. Gage also pointed out that many other natural "centers" of γ such as the center of mass, the centroid, and the Steiner point are not positive centers in general.

A natural question is how many positive centers a convex curve γ may have or how to describe the set of positive centers of γ. The purpose of the present paper is to answer this question to some extent.

Let E and F be two compact sets in $\mathbb{R}^2$. The Minkowski sum of E and F is defined by

$$E + F = \{x + y \mid x \in E,\ y \in F\}.$$

In particular, the Minkowski sum of a disk and a line segment is called a *sausage body* (see [8]), and its boundary is known as *sausage curve* in this paper. The following set associated with E is called the *inner parallel body of E at distance λ* (cf. [13, pp. 133–135]):

$$E_{-\lambda} = \{x \in \mathbb{R}^2 \mid x + \lambda D \subseteq E\}, \quad 0 \leq \lambda \leq r_{\text{in}}, \tag{1.4}$$

where D is the unit disk and r_{in} the inradius of E. Denote by K the domain bounded by γ and by K° its interior. For a point $c \in K$, let

$$r_{\text{in}}(c) = \max\{r \geq 0 \mid c + rD \subseteq K\}, \quad r_{\text{out}}(c) = \min\{r > 0 \mid c + rD \supseteq K\}. \tag{1.5}$$

We can modify the definition of the positive center of γ through its *Bonnesen function*

$$B(r) = Lr - A - \pi r^2. \tag{1.6}$$

A point $c \in K^\circ$ is called a *positive center* of γ if it satisfies

$$B(r_{\text{in}}(c)) \geq 0, \quad B(r_{\text{out}}(c)) \geq 0. \tag{1.7}$$

It is clear that this definition is equivalent to that of Gage.

In this paper, "convex curve" means "closed convex plane curve"; the set of all positive centers of a convex curve γ is denoted by $\mathfrak{P}(\gamma)$, and $C(x, r)$ represents the circle with radius r and centered at x.

In the next section, we shall describe the positive center set $\mathfrak{P}(\gamma)$ of a convex curve γ. First of all, we shall show that $\mathfrak{P}(\gamma)$ is a closed convex set and therefore $\mathfrak{P}(\gamma)$ has three possibilities: it has only one point, or it is a segment or it is a domain with positive area. For the first two possibilities $\mathfrak{P}(\gamma)$ has zero area. It is clear that a circle has only one positive center which is its center and a sausage curve has a line segment as its positive center set (see Remark 2.3 and Fig. 3). Then we characterize these as the only examples of positive center sets with zero area, that is, $\mathfrak{P}(\gamma)$ has a single point if and only if γ is a circle and $\mathfrak{P}(\gamma)$ is a line segment if and only if γ is a sausage curve (see Theorems 2.6 and 2.7). Meanwhile, we provide further information about the positive center set including conditions for strict convexity, and its behavior under homotheties and outer parallels. Finally, we focus our attention on the characterization of the positive center set of a constant width curve, that is, if γ is a constant width curve, then its positive center set $\mathfrak{P}(\gamma)$ is the inner parallel body K_{-r_1} of K, where r_1 is the smaller zero of the Bonnesen function $B(r)$ in (1.6).

2 Main Results

By definition of the positive center of a convex curve γ, we have

Theorem 2.1 *The positive center set $\mathfrak{P}(\gamma)$ of a convex curve γ is a closed convex set, and for any boundary point c of $\mathfrak{P}(\gamma)$, at least one of $B(r_{\text{in}}(c)) = 0$ and $B(r_{\text{out}}(c)) = 0$ holds.*

To complete the proof of this theorem, we need the following lemma, which is a direct consequence of Gage [6, Prop. 1.6 and Thm. 1.8].

Lemma 2.2 [6] *Let γ be a convex plane curve and o the center of its minimal annulus. If $s, t \in \gamma \cap C(o, r_{\text{in}}(o))$ and $S, T \in \gamma \cap C(o, r_{\text{out}}(o))$ and the line segments $\overline{st}$ and $\overline{ST}$ satisfy $\overline{st} \cap \overline{ST} \neq \emptyset$, then there is a line l with the following properties:*

- (i) *$l \cap K$ is a line segment with o as its midpoint, where K is the domain enclosed by γ.*
- (ii) *The points s and t lie on different sides of l, and so do S and T.*

Moreover, one can get $B(r) \geq 0$, $r \in [r_{\text{in}}(o), r_{\text{out}}(o)]$.

Proof of Theorem 2.1 From Lemma 2.2, one can see that the center of the minimal annulus of γ is a positive center, that is, $\mathfrak{P}(\gamma)$ is non-empty. By definition, $\mathfrak{P}(\gamma)$ is closed and if c is a boundary point of $\mathfrak{P}(\gamma)$, then at least one of $B(r_{\text{in}}(c)) = 0$ and $B(r_{\text{out}}(c)) = 0$ holds. It now suffices to show that $\mathfrak{P}(\gamma)$ is convex. If $\mathfrak{P}(\gamma)$ has only

Fig. 1 Inscribed circle

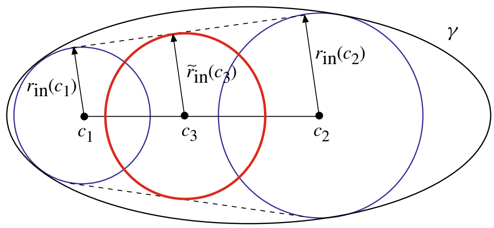
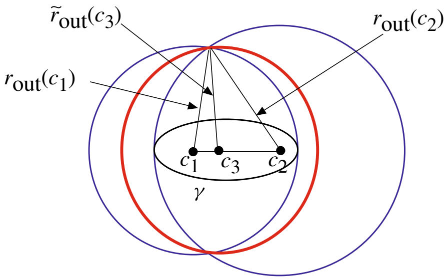


Fig. 2 Circumscribed circle



one point, its convexity is obvious. If $\mathfrak{P}(\gamma)$ has more than one point, we need to show that for two positive centers c_1 and c_2, the point c_3 on the segment $\overline{c_1c_2}$ is also a positive center of γ, that is to say, we should check that

$$r_1 \le r_{\mathrm{in}}(c_3) \le r_{\mathrm{out}}(c_3) \le r_2,$$

where r_1 and r_2 ($r_1 \le r_2$) are the two roots of the quadratic equation $B(r) = 0$.

Let $C(c_3, \widetilde{r}_{\mathrm{in}}(c_3))$ be the largest inscribed circle of the convex hull of circles $C(c_1, r_{\mathrm{in}}(c_1))$ and $C(c_2, r_{\mathrm{in}}(c_2))$, and $C(c_3, \widetilde{r}_{\mathrm{out}}(c_3))$ the circle which contains the two intersection points of the circles $C(c_1, r_{\mathrm{out}}(c_1))$ and $C(c_2, r_{\mathrm{out}}(c_2))$ (see Figs. 1, 2). Since γ is convex, γ surrounds all the circles $C(c_1, r_{\mathrm{in}}(c_1))$, $C(c_2, r_{\mathrm{in}}(c_2))$, and $C(c_3, \widetilde{r}_{\mathrm{in}}(c_3))$, and it is easy to see (from Fig. 1) that

$$\min\{r_{\mathrm{in}}(c_1), r_{\mathrm{in}}(c_2)\} \le \widetilde{r}_{\mathrm{in}}(c_3) \le r_{\mathrm{in}}(c_3). \tag{2.1}$$

In Fig. 2, each of the three circles $C(c_1, r_{\mathrm{out}}(c_1))$, $C(c_2, r_{\mathrm{out}}(c_2))$, and $C(c_3, \widetilde{r}_{\mathrm{out}}(c_3))$ contains γ. It is clear that

$$r_{\mathrm{out}}(c_3) \le \widetilde{r}_{\mathrm{out}}(c_3) < \max\{r_{\mathrm{out}}(c_1), r_{\mathrm{out}}(c_2)\}. \tag{2.2}$$

From (2.1) and (2.2) it follows that

$$r_1 \le \min\{r_{\mathrm{in}}(c_1), r_{\mathrm{in}}(c_2)\} \le r_{\mathrm{in}}(c_3) \le r_{\mathrm{out}}(c_3) \le \max\{r_{\mathrm{out}}(c_1), r_{\mathrm{out}}(c_2)\} \le r_2.$$

Thus c_3 is a point of $\mathfrak{P}(\gamma)$. $\square$

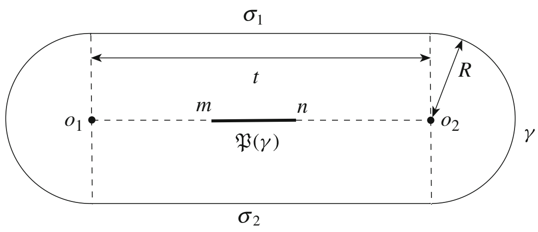


Fig. 3 Sausage curve

Remark 2.3 Theorem 2.1 and its proof tell us that the positive center set $\mathfrak{P}(\gamma)$ of a convex curve γ may have three possibilities: it has only one point, or it is a segment or it is a domain with positive area. For a circle with radius R, its Bonnesen function is $B(r) = -\pi(r-R)^2$, which is clear that its center is the only positive center. For a sausage curve (see Fig. 3), which is the Minkowski sum of a circle with radius R and a segment with length t, its Bonnesen function is $B(r) = 2(r-R)t - \pi(r-R)^2$, and its positive center set is a segment $\overline{mn}$ with length $\frac{4-\pi}{\pi}t$.

We first state a proposition which will be used frequently later.

Proposition 2.4 *If γ is a convex curve centrally symmetric with respect to o, then o is the center of the minimal annulus of γ and $\mathfrak{P}(\gamma)$ is a centrally symmetric domain with the same symmetric center o.*

Proof From Lemma 2.2, it follows that the center of the minimal annulus of γ is a point of $\mathfrak{P}(\gamma)$. Since γ is centrally symmetric, o is the center of its minimal annulus and for any point c of $\mathfrak{P}(\gamma)$, $r_{\text{in}}(-c) = r_{\text{in}}(c)$ and $r_{\text{out}}(-c) = r_{\text{out}}(c)$, where $-c$ is the symmetric point of c with respect to o. Thus

$$B(r_{\text{in}}(-c)) = B(r_{\text{in}}(c)) \geq 0, \quad B(r_{\text{out}}(-c)) = B(r_{\text{out}}(c)) \geq 0.$$

From the definition of positive center, one can see that $-c$ is also a point of $\mathfrak{P}(\gamma)$, that is, $\mathfrak{P}(\gamma)$ is centrally symmetric with respect to o. $\square$

Similarly, one can obtain the following proposition; the detailed proof is omitted here.

Proposition 2.5 *If γ is an axially symmetric convex curve, then $\mathfrak{P}(\gamma)$ is also axially symmetric and the symmetric axis of $\mathfrak{P}(\gamma)$ passes through the center of the minimal annulus of γ.* $\square$

The next two theorems describe the convex curve whose positive center set has only one point or is a segment.

Theorem 2.6 *The positive center set $\mathfrak{P}(\gamma)$ has only one point if and only if γ is a circle.*

Proof The "if" part is contained in Remark 2.3, and we need only to focus our attention on the "only if" part.

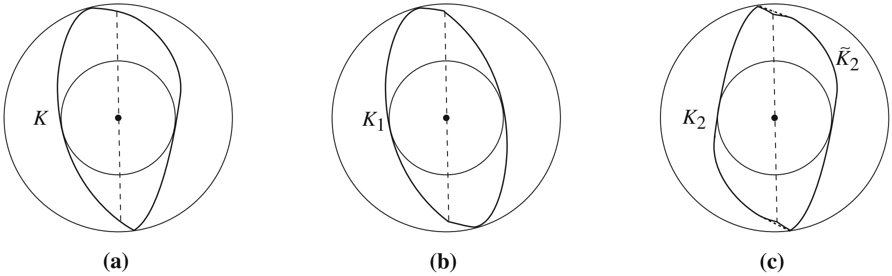


Fig. 4 Symmetrization. **a** Cut. **b** Left part. **c** Right part

If γ is centrally symmetric, the center o of its minimal annulus is its center of symmetry and $r_{\text{in}} = r_{\text{in}}(o)$, $r_{\text{out}} = r_{\text{out}}(o)$. Since o must be a positive center, it is just the only point of $\mathfrak{P}(\gamma)$. It follows from Theorem 2.1 that at least one of the following two equalities holds:

$$B(r_{\text{in}}(o)) = B(r_{\text{in}}) = Lr_{\text{in}} - A - \pi r_{\text{in}}^2 = 0,$$
$$B(r_{\text{out}}(o)) = B(r_{\text{out}}) = Lr_{\text{out}} - A - \pi r_{\text{out}}^2 = 0,$$

and thus γ is a circle or a sausage curve. While the positive center set of a sausage curve is a segment, γ must be a circle.

If γ is not centrally symmetric, we need some symmetrization procedures as in Fig. 4. According to Lemma 2.2, we can cut the domain K enclosed by γ into two parts by a symmetry secant through o as in Fig. 4a. Denote L_i and A_i $(i = 1, 2)$ as the length and the area of the two parts, respectively. Through a symmetrization of the two parts with respect to o, one can get centrally symmetric domains K_1 and K_2 as shown in Fig. 4b, c. Meanwhile, $r_{\text{in}}(o)$ and $r_{\text{out}}(o)$ are equal in these figures.

Since K_1 is convex, from Proposition 2.4, we get

$$2L_1 r_{\text{in}}(o) - 2A_1 - \pi r_{\text{in}}^2(o) \geq 0, \quad 2L_1 r_{\text{out}}(o) - 2A_1 - \pi r_{\text{out}}^2(o) \geq 0.$$

As for K_2, as it is not necessarily convex, we consider its convex hull $\widetilde{K_2}$ and denote its perimeter and area by $\widetilde{L_2}$ and $\widetilde{A_2}$, respectively. It is clear that $\widetilde{L_2} \leq 2L_2$ and $\widetilde{A_2} \geq 2A_2$. From Proposition 2.4, this yields

$$2L_2 r_{\text{in}}(o) - 2A_2 - \pi r_{\text{in}}^2(o) \geq \widetilde{L_2} r_{\text{in}}(o) - \widetilde{A_2} - \pi r_{\text{in}}^2(o) \geq 0,$$
$$2L_2 r_{\text{out}}(o) - 2A_2 - \pi r_{\text{out}}^2(o) \geq \widetilde{L_2} r_{\text{out}}(o) - \widetilde{A_2} - \pi r_{\text{out}}^2(o) \geq 0.$$

Hence

$$B(r_{\text{in}}(o)) = Lr_{\text{in}}(o) - A - \pi r_{\text{in}}^2(o) \geq 0,$$
$$B(r_{\text{out}}(o)) = Lr_{\text{out}}(o) - A - \pi r_{\text{out}}^2(o) \geq 0.$$

If $B(r_{\text{in}}(o)) = 0$, then

$$2L_1 r_{\text{in}}(o) - 2A_1 - \pi r_{\text{in}}^2(o) = 0,$$
$$2L_2 r_{\text{in}}(o) - 2A_2 - \pi r_{\text{in}}^2(o) = \widetilde{L_2} r_{\text{in}}(o) - \widetilde{A_2} - \pi r_{\text{in}}^2(o) = 0,$$

and therefore $\widetilde{K_2} = K_2$. Since K_1 and K_2 are centrally symmetric with respect to o, $r_{\text{in}} = r_{\text{in}}(o)$, and $r_{\text{out}} = r_{\text{out}}(o)$, it follows that ∂K_1 is a circle or a sausage curve, so is ∂K_2. If either ∂K_1 is a circle and ∂K_2 is a sausage curve or ∂K_1 is a sausage curve and ∂K_2 is a circle, which contradicts that K_1 and K_2 have the same $r_{\text{in}}(o)$ and $r_{\text{out}}(o)$. If both ∂K_1 and ∂K_2 are circles or sausage curves, then γ must be a circle or a sausage curve, which is a contradiction to the assumption that γ is not centrally symmetric.

For the case of $B(r_{\text{out}}(o)) = 0$, a similar argument can also imply that γ is a circle, which is impossible. □

Theorem 2.7 *The positive center set $\mathfrak{P}(\gamma)$ is a segment if and only if γ is a sausage curve.*

Proof If γ is a sausage curve, by Remark 2.3, $\mathfrak{P}(\gamma)$ is a segment. Now we present the "only if" part. If $\mathfrak{P}(\gamma)$ is a segment $\overline{mn}$, then the center o of the minimal annulus of γ satisfies $o \in \overline{mn} \setminus \{m, n\}$ or $o \in \{m, n\}$.

For the first case, we choose $c_1 \in \overline{mo} \setminus \{m, o\}$ and $c_2 \in \overline{on} \setminus \{o, n\}$. From (2.1) and (2.2) and the fact that γ is not a circle, it follows that $B(r_{\text{out}}(o)) > 0$ and $B(r_{\text{in}}(o)) = 0$, which implies $r_{\text{in}}(o) = r_1$, where r_1 is the left zero of the Bonnesen function $B(r)$. To continue the above procedure, we can get $r_{\text{in}}(c_1) = r_{\text{in}}(c_2) = r_{\text{in}}(o)$. From the arbitrariness of c_1 and c_2, it deduces that there is a segment σ_1 that is a part of γ which is parallel to the segment $\overline{mn}$ and the distance between $\overline{mn}$ and σ_1 is r_1. From the fact that γ is convex and o is the center of the minimal annulus of γ, it yields that the intersection of γ and the circle $C(o, r_{\text{in}}(o))$ must be a pair of antipodal points of $C(o, r_{\text{in}}(o))$. Furthermore, there is another segment $\sigma_2 (\neq \sigma_1)$ which is parallel to $\overline{mn}$ and the distance from it to $\overline{mn}$ is also r_1; thus, $r_{\text{in}}(o) = r_{\text{in}} = r_1$, that is, $B(r_{\text{in}}(o)) = B(r_{\text{in}}) = 0$, which implies that γ is a circle or a sausage curve. But a circle has only one positive center; therefore, γ must be a sausage curve (see Fig. 3).

For the second case, at least one of $B(r_{\text{in}}(o)) = 0$ and $B(r_{\text{out}}(o)) = 0$ holds. From Proposition 2.4, γ is not centrally symmetric. By the same symmetrization procedures as in the proof of Theorem 2.6, γ is a circle or a sausage curve, which is a contradiction to the fact that $o \in \{m, n\}$. □

In order to describe the positive center set of a strictly convex curve γ, we need some basic materials from Santaló [12, I.7.5] (see also [10] or [14]) and Proposition 2.9. Define $E(r, k)$ to be the area of the set $\{x \mid \sharp\{C(x, r) \cap \gamma\} = k,\ k \in \mathbb{N}\}$. Notice that if k is odd, then the circle $C(x, r)$ must be tangent to γ at a point and $E(r, k) = 0$.

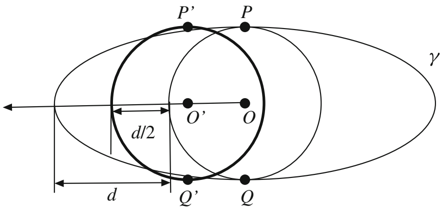
Lemma 2.8 [12] *If γ is a simple closed curve with perimeter L and enclosed area A, then*

$$4Lr = 2E(r, 2) + 4E(r, 4) + 6E(r, 6) + \cdots \tag{2.3}$$

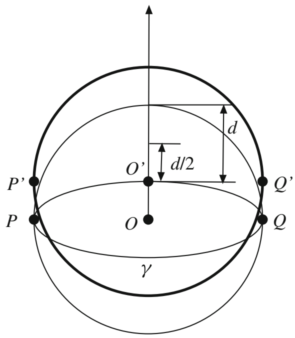
and if $r \in [r_{\text{in}}, r_{\text{out}}]$, then

$$A + Lr + \pi r^2 = E(r, 2) + E(r, 4) + E(r, 6) + \cdots . \tag{2.4}$$

Proposition 2.9 *If γ is a strictly convex non-circular curve and o is the center of its minimal annulus, then $B(r_{\text{in}}(o)) > 0$ and $B(r_{\text{out}}(o)) > 0$.*



(a) r_{in}



(b) r_{out}

Fig. 5 $E(r_{in}, 4) > 0$ and $E(r_{out}, 4) > 0$

Proof The combination of (2.3) and (2.4) yields

$$B(r) = Lr - A - \pi r^2 \geq E(r, 4), \quad r \in [r_{\text{in}}, r_{\text{out}}].$$

Firstly, for a non-circular curve, we will prove that $E(r_{\text{in}}, 4) > 0$ and $E(r_{\text{out}}, 4) > 0$ which can give us $B(r_{\text{in}}) > 0$ and $B(r_{\text{out}}) > 0$.

Since γ is strictly convex, its largest inscribed circle is unique (denote it by C), and it is clear that there are at least two tangent points of γ and C, say P and Q, so that $\angle POQ \leq \pi$ (cf. [7, Lemma 1.11]). In the direction of the bisector of $\angle POQ$, if there is a distance d between γ and C, then C can be moved a distance say $d/2$ in the direction as shown in Fig. 5a so that C intersects γ at 4 points, and by continuity, $E(r_{\text{in}}, 4) > 0$. If there is always a tangent point in the direction of all angle bisectors, thus $d \equiv 0$, then γ must be a circle, which shows that we can produce the above progress when γ is non-circular. Similarly, we can get $E(r_{\text{out}}, 4) > 0$ (see Fig. 5b).

Next, we have to show that $B(r_{\text{in}}(o)) > 0$ and $B(r_{\text{out}}(o)) > 0$ for a strictly convex non-circular curve. If γ is centrally symmetric, then $r_{\text{in}} = r_{\text{in}}(o)$ and $r_{\text{out}} = r_{\text{out}}(o)$, and thus, $B(r_{\text{in}}(o)) = B(r_{\text{in}}) > 0$ and $B(r_{\text{out}}(o)) = B(r_{\text{out}}) > 0$. If γ is not centrally symmetric, we can proceed the same symmetrization as in the proof of Theorem 2.6 and continue to use the same notations. Since K_1 is convex, we get

$$2L_1 r_{\text{in}}(o) - 2A_1 - \pi r_{\text{in}}^2(o) > 0. \tag{2.5}$$

As for K_2, as it is not necessarily convex, we consider its convex hull $\widetilde{K_2}$ and denote its perimeter and area by $\widetilde{L_2}$ and $\widetilde{A_2}$, respectively; thus

$$2L_2 r_{\text{in}}(o) - 2A_2 - \pi r_{\text{in}}^2(o) \geq \widetilde{L_2} r_{\text{in}}(o) - \widetilde{A_2} - \pi r_{\text{in}}^2(o) > 0. \tag{2.6}$$

By (2.5) and (2.6), we have

$$B(r_{\text{in}}(o)) = L r_{\text{in}}(o) - A - \pi r_{\text{in}}^2(o) > 0.$$

A similar agrument also give us

$$B(r_{\mathrm{out}}(o)) = Lr_{\mathrm{out}}(o) - A - \pi r_{\mathrm{out}}^2(o) > 0.$$

This completes the proof. □

Now, we give a theorem which deals with the positive center set of a strictly convex curve.

Theorem 2.10 *If γ is a strictly convex curve, then $\mathfrak{P}(\gamma)$ is strictly convex and for any boundary point c of $\mathfrak{P}(\gamma)$, at least one of $B(r_{\mathrm{in}}(c)) = 0$ and $B(r_{\mathrm{out}}(c)) = 0$ holds. Furthermore, if γ is non-circular, then $\mathfrak{P}(\gamma)$ is of positive area.*

Proof First, we prove the strict convexity of $\mathfrak{P}(\gamma)$. If γ is a circle, then the result is obviously true. If γ is strictly convex and non-circular, from Theorem 2.1, it follows that $\mathfrak{P}(\gamma)$ is closed and convex. To complete the proof, it suffices to show that for two different positive centers c_1 and c_2, any point $c_3 \in \overline{c_1c_2} \setminus \{c_1, c_2\}$ is an interior point of $\mathfrak{P}(\gamma)$, that is, we should check that $B(r_{\mathrm{in}}(c_3)) > 0$ and $B(r_{\mathrm{out}}(c_3)) > 0$.

Since γ is strictly convex, (2.1) turns into

$$\min\{r_{\mathrm{in}}(c_1), r_{\mathrm{in}}(c_2)\} \leq \widetilde{r}_{\mathrm{in}}(c_3) < r_{\mathrm{in}}(c_3). \tag{2.7}$$

Combining (2.2) and (2.7) yields

$$r_1 < r_{\mathrm{in}}(c_3) < r_{\mathrm{out}}(c_3) < r_2,$$

where r_1 and r_2 are the two roots of the quadratic equation $B(r) = 0$, which implies $B(r_{\mathrm{in}}(c_3)) > 0$ and $B(r_{\mathrm{out}}(c_3)) > 0$.

Again, by the same argument as in the proof of Theorem 2.1, for any boundary point c of $\mathfrak{P}(\gamma)$, at least one of $B(r_{\mathrm{in}}(c)) = 0$ and $B(r_{\mathrm{out}}(c)) = 0$ holds.

For a strictly convex and non-circular curve γ, from Proposition 2.9, the center o of its minimal annulus satisfies $B(r_{\mathrm{in}}(o)) > 0$ and $B(r_{\mathrm{out}}(o)) > 0$. By the continuity of $B(r)$, of $r_{\mathrm{in}}(c)$, and of $r_{\mathrm{out}}(c)$ with respect to c, $\mathfrak{P}(\gamma)$ is of positive area. □

The next proposition shows that the positive center sets of two homothetic convex curves are still homothetic.

Proposition 2.11 *If two convex curves γ_1 and γ_2 are homothetic, then $\mathfrak{P}(\gamma_1)$ and $\mathfrak{P}(\gamma_2)$ are homothetic.*

Proof Since γ_1 and γ_2 are homothetic, by translation, the center o of the minimal annulus of γ_1 coincides with that of γ_2, and o is their homothetic center. Without loss of generality, we choose o as the origin. By definition of homothetic, there exists a constant $k \geq 0$ such that $\gamma_2 = k\gamma_1$, and therefore, $L_2 = kL_1$ and $A_2 = k^2A_1$.

For any point c of $\mathfrak{P}(\gamma_1)$, it is easy to see that $r_{\mathrm{in}}(kc) = kr_{\mathrm{in}}(c)$ and $r_{\mathrm{out}}(kc) = kr_{\mathrm{out}}(c)$. Denote by $B_i(r)$ the Bonnesen function of γ_i $(i = 1, 2)$; then

$$B_2(r_{\mathrm{in}}(kc)) = kL_2r_{\mathrm{in}}(c) - A_2 - k^2\pi r_{\mathrm{in}}^2(c) = k^2B_1(r_{\mathrm{in}}(c)) \geq 0.$$

Similarly, $B_2(r_{\text{out}}(kc)) \geq 0$. Hence, kc is a point of $\mathfrak{P}(\gamma_2)$, which implies $k\mathfrak{P}(\gamma_1) \subseteq \mathfrak{P}(\gamma_2)$. In the same way, we can get $\mathfrak{P}(\gamma_2) \subseteq k\mathfrak{P}(\gamma_1)$. □

The next theorem tells us that a convex curve and its outer parallel curves have the same positive center set, which implies that the converse of Proposition 2.11 is not true.

Theorem 2.12 *If γ is a convex curve and $\widetilde{\gamma}$ is an outer parallel curve of γ, then the positive center sets of γ and $\widetilde{\gamma}$ are the same.*

Proof Let L and A be the length of γ and its enclosed area, and $\widetilde{L}$ and $\widetilde{A}$ those of $\widetilde{\gamma}$. Since $\widetilde{\gamma}$ is the outer parallel curve of γ, it is well known that

$$\widetilde{L} = L + 2\pi t, \quad \widetilde{A} = A + Lt + \pi t^2, \tag{2.8}$$

where $t > 0$; the first in (2.8) is known as the Cauchy formula and the second the Steiner formula. Denote by K and $\widetilde{K}$ the domains bounded by γ and $\widetilde{\gamma}$, respectively. For a point c of K, one gets

$$\widetilde{r}_{\text{in}}(c) = r_{\text{in}}(c) + t, \quad \widetilde{r}_{\text{out}}(c) = r_{\text{out}}(c) + t. \tag{2.9}$$

Hence, from (2.8) and (2.9), it follows that $B(\widetilde{r}_{\text{in}}(c)) = B(r_{\text{in}}(c))$ and $B(\widetilde{r}_{\text{out}}(c)) = B(r_{\text{out}}(c))$.

If c_1 is a point of $\mathfrak{P}(\gamma)$, then

$$B(\widetilde{r}_{\text{in}}(c_1)) = B(r_{\text{in}}(c_1)) \geq 0, \quad B(\widetilde{r}_{\text{out}}(c_1)) = B(r_{\text{out}}(c_1)) \geq 0,$$

which implies c_1 is also a point of $\mathfrak{P}(\widetilde{\gamma})$. If c_2 is a point of $K \setminus \mathfrak{P}(\gamma)$, then at least one of $B(\widetilde{r}_{\text{in}}(c_2)) = B(r_{\text{in}}(c_2)) < 0$ and $B(\widetilde{r}_{\text{out}}(c_2)) = B(r_{\text{out}}(c_2)) < 0$ holds, which deduces that c_2 is not a point of $\mathfrak{P}(\widetilde{\gamma})$. From the convexity of $\mathfrak{P}(\widetilde{\gamma})$ (see Theorem 2.1), all points of the domain $\widetilde{K} \setminus K$ are not positive centers of $\widetilde{\gamma}$. Therefore $\mathfrak{P}(\gamma) = \mathfrak{P}(\widetilde{\gamma})$. □

Now, we study the positive center set of a constant width curve γ with width equal to w. For such curves, there is a famous theorem (cf. [15, p. 75]): The largest circle inscribed in γ and the smallest circle circumscribed about γ must be concentric, and the sum of their radii equals the width w of γ. In other words, $r_{\text{in}}(o) + r_{\text{out}}(o) = w$, where o is the center of the minimal annulus of γ. It becomes an interesting question whether the sum of $r_{\text{in}}(c)$ and $r_{\text{out}}(c)$ still equals w for each point c of the domain K enclosed by γ. The following two lemmata answer it positively. We should point out that Jin and Guo in [9] have obtained this result by using the Minkowski asymmetry of convex bodies; our method here is different from theirs.

Lemma 2.13 *Let γ be a constant width curve with width w and composed of circular arcs with radius w. Denote by M the set of all corners of γ and N the set $\gamma \setminus M$. If y_1, y_2 are two points of γ such that $r_{\text{in}}(c) = |\overline{cy}_1|$ and $r_{\text{out}}(c) = |\overline{cy}_2|$, where $c \in K^{\circ}$, then $y_1 \in N$ and $y_2 \in M$.*

Proof Because both the width of γ and the radii of the circular arcs are equal to w, each point of M must be the center of a circular arc. From the definition of $r_{\text{in}}(c)$ and $r_{\text{out}}(c)$, it is clear that $|\overline{cy}_1|$ and $|\overline{cy}_2|$ are the minimal distance and maximal distance from c to γ, respectively.

If $y_1 \notin N$, one can draw a circle with center c and radius $r_{\text{in}}(c) = |\overline{cy}_1|$; then there exists a point $y_1'' \in \gamma$ such that $|\overline{cy''}_1| < |\overline{cy}_1|$, which contradicts the minimality of $|\overline{cy}_1|$.

Let y_1', y_2' be points of γ such that y_1, c, y_1' and y_2, c, y_2' are collinear, respectively. From the fact that $|\overline{cy}_1|$ is the minimal distance from c to γ, it yields that $y_1' \in M$ and $|\overline{cy}_1|+|\overline{cy'}_1| = w$. If $y_2 \notin M$, then $y_2 \in N$; furthermore $y_2' \in M$ or $y_2' \in N$. If $y_2' \in M$, it deduces $|\overline{cy}_2| + |\overline{cy'}_2| \leq w$. From $|\overline{cy'}_2| > |\overline{cy}_1|$, it follows that $|\overline{cy'}_1| > |\overline{cy}_2|$, which contradicts the maximality of $|\overline{cy}_2|$. If $y_2' \in N$, then $|\overline{cy}_2| + |\overline{cy'}_2| < w$, since the distance between two points of circular arcs is smaller than width w. In this case, $|\overline{cy'}_2| \geq |\overline{cy}_1|$, and thus $|\overline{cy'}_1| > |\overline{cy}_2|$, which is impossible. □

Lemma 2.14 *If γ is a curve of constant width w and K is the domain enclosed by γ, then*

$$r_{\text{in}}(c) + r_{\text{out}}(c) = w, \quad c \in K.$$

Proof Observe that a curve of constant width w can be approximated by constant width curves consisting of circular arcs of radius w (cf. [15, pp. 76–80]). It is sufficient to show that $r_{\text{in}}(c) + r_{\text{out}}(c) = w$ for the curve constructed by circular arcs with radius w. In the following, we use the same notations as in Lemma 2.13.

If $c \in \partial K = \gamma$, then $r_{\text{in}}(c) = 0$ and $r_{\text{out}}(c) = w$; the conclusion is evident. If $c \in K^\circ$, from Lemma 2.13, it follows that $y_1 \in N$ and $y_2 \in M$. It is obvious that $r_{\text{in}}(c)+r_{\text{out}}(c) = w$ when y_1, c, and y_2 are collinear. If y_1, c, and y_2 are non-collinear, we can claim that

$$|\overline{cy}_1| + |\overline{cy'}_1| = |\overline{cy}_2| + |\overline{cy'}_2|. \tag{2.10}$$

Otherwise $w = |\overline{cy}_1| + |\overline{cy'}_1| > |\overline{cy}_2| + |\overline{cy'}_2|$. From $|\overline{cy}_1| \leq |\overline{cy'}_2|$, one can see that $|\overline{cy'}_1| > |\overline{cy}_2|$, which contradicts the maximality of $|\overline{cy}_2|$. Meanwhile $|\overline{cy}_1| = |\overline{cy'}_2|$, otherwise $|\overline{cy}_1| < |\overline{cy'}_2|$. By (2.10), it implies that $|\overline{cy'}_1| > |\overline{cy}_2|$, which is a contradiction. Thus $|\overline{cy}_2| = |\overline{cy'}_1|$ and

$$r_{\text{in}}(c) + r_{\text{out}}(c) = |\overline{cy}_1| + |\overline{cy}_2| = |\overline{cy}_1| + |\overline{cy'}_1| = w,$$

which completes the proof. □

According to the above lemmata, we can describe the positive center set of a constant width curve.

Theorem 2.15 *If γ is a constant width curve with width w and enclosed area A, then*

(i) $\mathfrak{P}(\gamma)$ *is its inner parallel body K_{-r_1}, where r_1 is the smaller zero point of the Bonnesen function $B(r)$.*
(ii) $B(r_{\text{in}}(c)) = B(r_{\text{out}}(c)) = 0$ *holds for each boundary point c of $\mathfrak{P}(\gamma)$.*

(iii) *The area of* $\mathfrak{P}(\gamma)$, $A(\mathfrak{P}(\gamma))$, *satisfies*

$$\pi(r_{\text{in}} - r_1)^2 \le A(\mathfrak{P}(\gamma)) \le A - \pi r_1^2,$$

where r_{in} *is the inradius of* γ. *And furthermore, if* γ *is non-circular, then* $A(\mathfrak{P}(\gamma)) > 0$.

Proof (i) Let K be the domain bounded by γ. Since γ is a curve of constant width w, from Lemma 2.14, it follows that

$$r_{\text{in}}(c) + r_{\text{out}}(c) = w, \quad c \in K. \tag{2.11}$$

The two roots r_1, r_2 of the quadratic equation $B(r) = 0$ satisfy

$$r_1 + r_2 = \frac{L}{\pi} = w. \tag{2.12}$$

(2.11) and (2.12) show that $r_{\text{in}}(c)$ and $r_{\text{out}}(c)$ are symmetric with respect to $\frac{w}{2}$ and so are r_1 and r_2. Thus, if $r_{\text{in}}(c) \ge r_1$, then $r_{\text{out}}(c) \le r_2$. From the definition of $\mathfrak{P}(\gamma)$ and that of the inner parallel body, $\mathfrak{P}(\gamma)$ is its inner parallel body K_{-r_1}.

(ii) By the symmetry of $r_{\text{in}}(c)$ and $r_{\text{out}}(c)$, $B(r_{\text{in}}(c)) = B(r_{\text{out}}(c))$. Thus, from Theorem 2.1, it follows that $B(r_{\text{in}}(c)) = B(r_{\text{out}}(c)) = 0$ holds for any point c on the boundary of $\mathfrak{P}(\gamma)$.

(iii) From the definition of inner parallel body, it yields

$$K_{-r_1} + r_1 D \subseteq K, \quad K_{-r_{\text{in}}} + (r_{\text{in}} - r_1)D \subseteq K_{-r_1},$$

where D is the unit disk. Combining the Steiner polynomial (see the second formula of (2.8)) of γ can give us

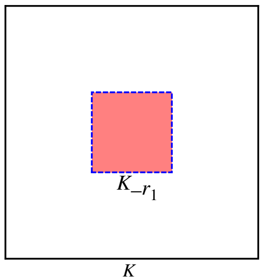
$$A(K_{-r_1}) + r_1 L(K_{-r_1}) + \pi r_1^2 = A(K_{-r_1} + r_1 D) \le A(K)$$

and

$$\begin{aligned} &A(K_{-r_{\text{in}}}) + (r_{\text{in}} - r_1)L(K_{-r_{\text{in}}}) + \pi(r_{\text{in}} - r_1)^2 \\ &\quad = A(K_{-r_{\text{in}}} + (r_{\text{in}} - r_1)D) \le A(K_{-r_1}). \end{aligned}$$

From (i) it deduces that $\pi(r_{\text{in}} - r_1)^2 \le A(\mathfrak{P}(\gamma)) \le A - \pi r_1^2$.

Fig. 6 The positive center set of a square



Since $r_1 = r_{in}$ if and only if γ is a circle or a sausage curve, from the fact that γ is a constant width curve, it follows that $A(\mathfrak{P}(\gamma)) \geq \pi(r_{in} - r_1)^2 > 0$ when γ is not a circle. $\square$

The next example shows that if γ is not a constant width curve, its inner parallel body K_{-r_1} may also be its positive center set.

Example 2.16 Let K be a square with side equal to 2; a direct calculation shows that its positive center set is the inner parallel body K_{-r_1} of K, where $r_1 = \frac{4-2\sqrt{4-\pi}}{\pi}$ (see Fig. 6).

Acknowledgments This work was supported by the National Science Foundation of China (No. 11171254) and a Grant of "The First-class Discipline of Universities in Shanghai." The authors would like to thank the referees for their careful reading of the original manuscript of this paper and giving them some helpful suggestions and invaluable comments.

References

1. Bonnesen, T.: Les Problèmes des Isopérimètres et des Isépiphanes. Gauthier-Villars, Paris (1929)
2. Böröczky, K.J., Lutwak, E., Yang, D., Zhang, G.: The log-Brunn–Minkowski inequality. Adv. Math. **231**(3–4), 1974–1997 (2012)
3. Burago, Y.D., Zalgaller, V.A.: Geometric Inequalities. Springer, Berlin (1988)
4. Gage, M.E.: An isoperimetric inequality with applications to curve shortening. Duke Math. J. **50**(4), 1225–1229 (1983)
5. Gage, M.E.: Curve shortening makes convex curves circular. Invent. Math. **76**(2), 357–364 (1984)
6. Gage, M.E.: Positive centers and the Bonnesen inequality. Proc. Am. Math. Soc. **110**(4), 1041–1048 (1990)
7. Green, M., Osher, S.: Steiner polynomials, Wulff flows, and some new isoperimetric inequalities for convex plane curves. Asian J. Math. **3**(3), 659–676 (1999)
8. Henk, M., Cifre, M.A.H.: On the location of roots of Steiner polynomials. Bull. Braz. Math. Soc. (N.S.) **42**(1), 153–170 (2011)
9. Jin, H.L., Guo, Q.: Asymmetry of convex bodies of constant width. Discrete Comput. Geom. **47**(2), 415–423 (2012)
10. Osserman, R.: Bonnesen-style isoperimetric inequalities. Am. Math. Mon. **86**(1), 1–29 (1979)
11. Peri, C., Zucco, A.: On the minimal convex annulus of a planar convex body. Monatsh. Math. **114**(2), 125–133 (1992)
12. Santaló, L.A.: Integral Geometry and Geometric Probability, 2nd edn. Cambridge University Press, Cambridge (2004)
13. Schneider, R.: Convex Bodies: The Brunn–Minkowski Theory. Encyclopedia of Mathematics and Its Applications, vol. 44. Cambridge University Press, Cambridge (1993)
14. Treibergs, A.: The Strong isoperimetric inequality of Bonnesen. University of Utah (2006). http://www.math.utah.edu/~treiberg/isoperim/Bonn.pdf. Accessed 30 June 2015
15. Yaglom, I.M., Boltyanskiĭ, V.G.: Convex Figures. Translated by Kelly, P.J. and Walton, L.F., Holt, Rinehart and Winston, New York (1960)