European Journal of Physics

PAPER

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To cite this article: Tarek Echekki 2020 *Eur. J. Phys.* **41** 015002

IOP Publishing European Journal of Physics

Eur. J. Phys. **41** (2020) 015002 (16pp) https://doi.org/10.1088/1361-6404/ab4b62

In the rain with and without an umbrella? The Reynolds transport theorem to the rescue

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Received 12 June 2019, revised 2 October 2019
Accepted for publication 7 October 2019
Published 27 November 2019

Abstract

The age-old question of whether to run or walk in the rain still raises interest because of its practical relevance. A number of published studies have been carried out to address this question. These studies showed that there is no trivial answer and that different factors, including the rain conditions (rate and angle), the walking/running speeds and even the geometry used for analysis and the shape of the person matter. In the present paper, we present an analysis of the rate and amount of rain accumulation on a person walking/running through the rain using the Reynolds transport theorem (RTT) and its implementation for mass conservation. A new twist to earlier analyses is added, which involves the usage of a simplified model for an umbrella. The present work illustrates the use of RTT to establish the observations made using different analyses as well as demonstrates the RTT problem with this new twist.

Keywords: Reynolds transport theorem, running in the rain, fluid mechanics

(Some figures may appear in colour only in the online journal)

1. Introduction

Addressing the question of whether it is worth running or walking in the rain (RWiR) has been the subject of a number of studies [1–18]. One of the principal objectives of these studies was to determine the amount of rain that accumulates on a person during travel over a prescribed path. From the different analyses, a number of observations can be made.

- With vertical rain, it is better to run than to walk to safety, although the rate of rain accumulation is higher in the front by virtue of running into the rain [1, 3, 9, 11].

- With an angle prescribed along the direction of travel, potentially different recommendations may be made. With a component of the wind at a person's back (or motion upwind), there may be an optimum speed to run, which corresponds to the wind speed (see, for example, [4, 16]). However, Bailey [11] also showed that the shape can be a factor with this consideration and that an optimum speed may or may not correspond to the wind speed based on the body shape. Other authors [12, 15, 16] also developed analyses based on alternative shape models, including cylindrical and ellipsoidal shapes.

While most analyses involve the assumption of a parallelepiped shape for the person, a few studies also considered cylindrical and ellipsoidal shapes (see for example, [12–16]). Shape considerations seem to expand the range of conclusions as well. In a more recent study, Kurusingal [18] addresses for the first time the use of an umbrella while running in the rain. This study addressed primarily the viewpoints of interpreting the invariance of distances and angles under a Galilean transformation from the standpoints of a moving or stationary observers. The present study (see section 4) will address the use of an umbrella within the context of the problem of determining the rain accumulation on a person under different rain angles and other optimization parameters.

Much of the divergence in the results and conclusions associated with studies of rain accumulation in the absence of an umbrella may be attributed to (1) the choice of the model and its inherent assumptions, and (2) the choice of the shape. A principal scope of the present study is to introduce a different and potentially more transparent strategy for the analysis of rain accumulation on a person using the Reynolds transport theorem (RTT). RTT is an integral representation of the conversion of conservation laws (e.g. mass) from a control mass of a finite-volume to a control volume when coupled with conservation laws. With a moving person, the rate of accumulation is related to the fluxes entering all surfaces on this person. Elements of this analysis already can be identified in the analysis presented by Bocci [16] who explicitly used flux considerations, which are inherent in RTT.

In many respects, this study is about the introduction of RTT as a unique representation of conservation laws with many applications in physics as much as it is the presentation of a solution to the RWiR problem.

Within the context of the present study, RTT is coupled with mass conservation where the control volume corresponds to a person running towards safety. We will first revisit the problem of RWiR without an umbrella. We will explore different scenarios of rain conditions (vertical rain with no wind conditions to wind conditions in the directions of walking/running as well as the sides). Then, we investigate a new twist to the problem by allowing the person to carry an umbrella. In the following sections, the RTT is introduced in section 2. The RWiR problem for a parallapiped shape is presented in section 3, a geometry that has been widely used in previous analyses. In section 4, we investigate how the use of an umbrella can contribute to the shielding of a person from the rain while walking or running towards a target. Finally, conclusions and final remarks are made in section 5.

2. The RTT

Engineering students taking their first introductory fluid mechanics course are generally familiar with the RTT. RTT provides a compact integral form to transform conservation laws from a control mass framework (where the mass is allowed to move with the flow) to a control volume framework (where the volume is set, yet it can deform in shape, remain stationary, move at constant velocity or accelerate). Named after Osborne Reynolds (1842–1912), the theorem is often associated with applications in continuum mechanics, and

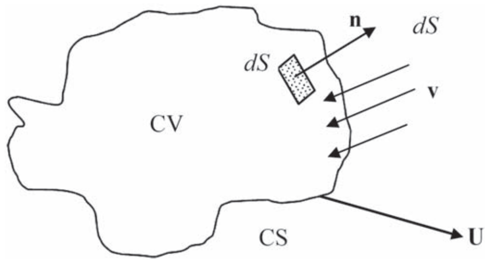

Figure 1. Schematics of control volume and control surfaces for RTT.

in particular fluid mechanics. However, its potential relevance extends beyond these applications in science and engineering. Besides principles of conservation laws, RTT has been used in applications related to shape optimization [19] and free boundary problems (see for example [20]).

In its 'modern' form, it may be interpreted mathematically as an extension of the Leibniz integral rule [21]; and in engineering, the theorem represents a generalization of bookkeeping rules, which are related to the original work by Osborne Reynolds [22]. Vincenti [23] provides additional details on how these original concepts based on these rules have evolved to the present form of the RTT.

RTT can be implemented for any general bulk quantity B (e.g. mass, energy, momentum, angular momentum) defined in a given volume whose specific quantity, b, can be defined as: $b = \lim_{\mathrm{d}m \to 0} \frac{\mathrm{d}B}{\mathrm{d}m}$. Here, $\mathrm{d}m$ is an infinitesimal mass within the volume, which also can be expressed in terms of the product of density, ρ, and the corresponding infinitesimal volume, $\mathrm{d}\vartheta$. B can, then, be expressed as an integral quantity of b as follows:

$$B = \int_V \rho \, b \, \mathrm{d}\vartheta. \tag{1}$$

It is important to note that the volume can be deformable and, as such, the limits of the volume integral can vary in time to reflect an evolving shape. A schematic of the control volume is shown in figure 1. It is characterized by a control surface, a shell around the volume, and a normal vector, $\mathbf{n}$, at each incremental surface, $\mathrm{d}S$, that is pointing away from the volume and has a magnitude of unity. RTT for a quantity B can be expressed as

$$\frac{DB_{\mathrm{CM}}}{Dt} = \frac{\partial B_{\mathrm{CV}}}{\partial t} + \int_{\mathrm{CS}} \rho \, b \, (\mathbf{v}_r \cdot \mathbf{n}) \; \mathrm{d}S = \frac{\partial}{\partial t}\left(\int_V \rho \, b \, \mathrm{d}\vartheta\right) + \int_{\mathrm{CS}} \rho \, b \, (\mathbf{v}_r \cdot \mathbf{n}) \; \mathrm{d}S. \tag{2}$$

Here, the subscripts CM and CV refer to the control mass and control volume perspectives. Note here that there are two time derivatives in the expression for RTT. The first derivative, D/Dt, the material derivative, tracks the evolution of the control mass in time as it moves in space subject to a velocity, interpreted here as a fluid velocity, $\mathbf{v}$. The second time derivative, $\partial/\partial t$, which is implemented on the control volume, accounts for the rate of change while keeping the other spatial independent variables fixed. As a generalization of the RTT, we also allow the control surface to move at a prescribed speed, $\mathbf{U}$. Of course, $\mathbf{U}$ can vary along the control surface for the case of a moving and deforming control volume. Similarly, $\mathbf{v}$ can vary along the control surface because of flow non-uniformity. Neither case is applicable for the present study under the stated assumptions. The relative velocity in equation (2) corresponds to the difference: $\mathbf{v}_r = \mathbf{v} - \mathbf{U}$. The second terms on the right-hand side of equation (2) represents the flow rates of B across the volume boundary, the control surface. Only the normal component to the surface of the relative velocity can transfer B in or out of the control volume.

The RTT can be combined with conservation laws to relate control volume integrals and the conservation relations. For mass conservation, RTT may be written as

$$\frac{Dm_{\mathrm{CM}}}{Dt} = \frac{\partial m_{\mathrm{CV}}}{\partial t} + \int_{\mathrm{CS}} \rho(\mathbf{v}_r \cdot \mathbf{n})\ \mathrm{d}S = \frac{\partial}{\partial t}\left(\int_V \rho\ \mathrm{d}\vartheta\right) + \int_{\mathrm{CS}} \rho(\mathbf{v}_r \cdot \mathbf{n})\ \mathrm{d}S, \ \text{where } b = 1 \tag{3}$$

while mass conservation is expressed as

$$\frac{Dm_{\mathrm{CM}}}{Dt} = 0. \tag{4}$$

Perhaps, one of the most attractive attributes of the use of RTT for conservation laws is the manner with which the fluxes are implemented. The contributions to the flux term can be broken down into identifying the fluid velocity, $\mathbf{v}$, the control surface velocity, $\mathbf{U}$, both making up the relative velocity, $\mathbf{v}_r$, the control surface normal, $\mathbf{n}$, and the surface area, which involves identifying the form of the infinitesimal surface, dS, and whether it can be simplified into a line integral instead of a surface integral and the corresponding limits. In analysis problems, it is often easier to break down a complex problem and integral into potentially easier components as illustrated below.

It is important to emphasize that the present analysis is framed within the context of RTT and conservation principles. In Bocci [16], mass conservation is treated by tracking the surface fluxes (as given Bocci's equation (6) [16] or equation (4) in our analysis) and carrying out a bookkeeping analysis based on the alignment of the fluxes with the surfaces. This alignment in our present analysis is enforced by the dot product of the relative velocity vector at the surface and the surface normal. Therefore, while conservation laws are framed differently between this analysis and that of Bocci's [16], the results of both analyses should be identical for any given problem.

In the following analysis, we attempt to use RTT to address the RWiR problem. Section 3 is based on the classical problem of running without an umbrella. The goal is to illustrate RTT for this problem and illustrate the approach's predictions. In section 4, we illustrate the use of RTT in the presence of an umbrella. Of course, this problem introduces additional optimization parameters, which will be discussed in this section.

3. RWiR problem formulation: parallelepiped geometry

We illustrate our analysis first using a parallelepiped geometry for the walking/running person. The configuration under study is shown in figure 2. A person's volume represented by a parallelepiped of height, h, a width, w, and a depth, d traveling at constant speed, U, towards safety, which is located at a distance, L, at the start of the walk/run. Rain is modeled as a continuous rain falling with a speed, $\mathbf{v}$, which is redirected by the wind. The rain vector $\mathbf{v}$ makes an angle φ with the vertical y-axis along fixed x–y planes as shown in figure 2(b). The angle φ can range from $-\pi/2$ to $\pi/2$. Therefore, it can take positive and negative values. The rain angle also has a component in the z- or transverse direction relative to the x–y planes, which is represented by angle θ as shown in figure 2(b). Because the motion of the person does not impact the rate at which the rain will fall on his/her side, we consider only the configuration of rain landing to the left of the person. Moreover, we primarily are concerned with the angle relative to the z-axis; therefore, we will consider the sines and cosines of θ to

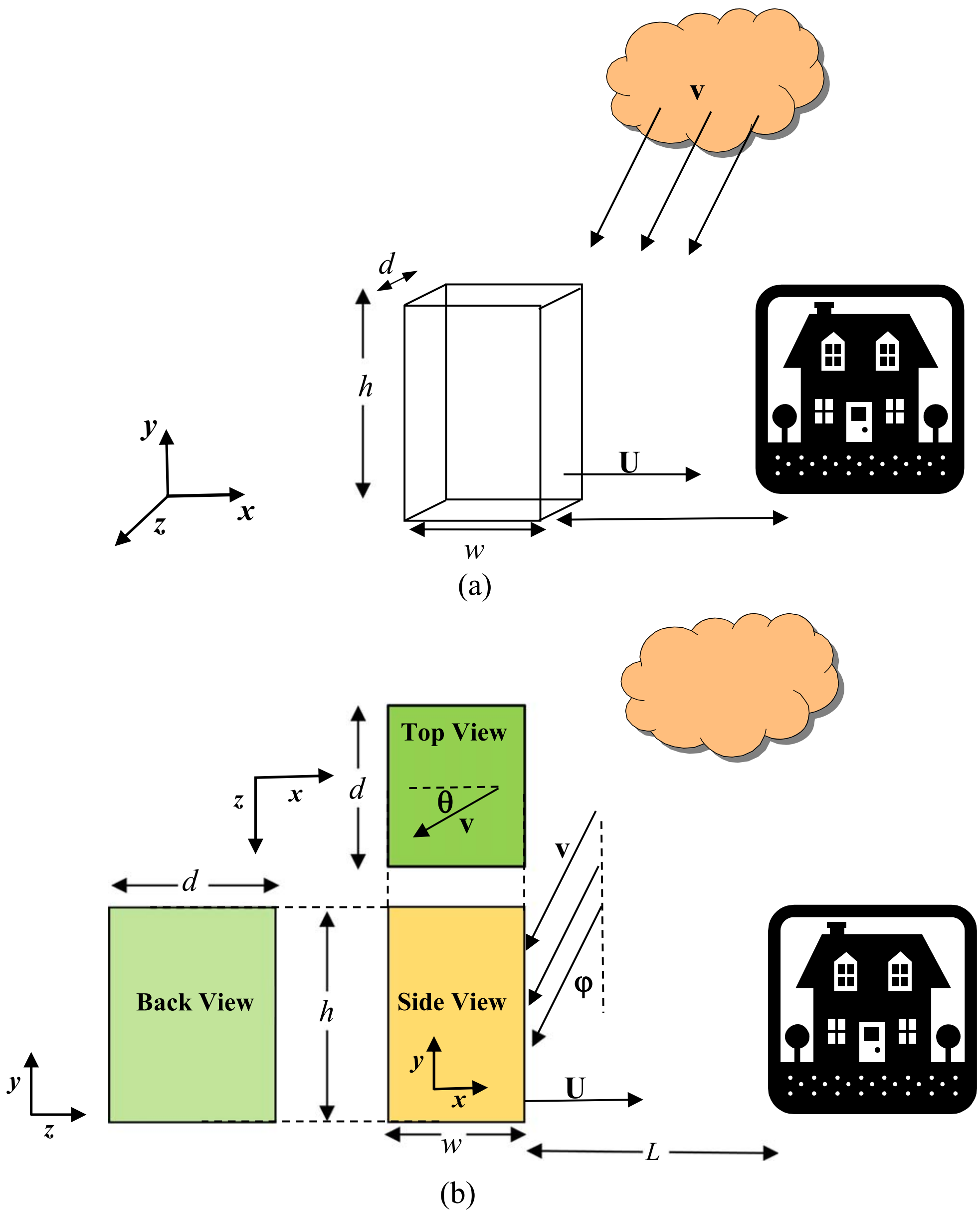

Figure 2. Schematic of problem configuration (a) 3D view, (b) side, top and back views.

be always positive. If the rain lands to the left, then, it will be an outlet on the left and the two conditions are mutually exclusive.

The person is modeled as a rectangular control volume of height h, width, w, and depth, d and is running to safety at a speed, U, in the x-direction over a distance L. Therefore, the velocity vector $\mathbf{U} = (U, 0, 0)$. Rain falls with a speed of $\mathbf{v}$ with a y-component pointing downwards; while its x-component can point either in the direction of safety or away from it based on the angle

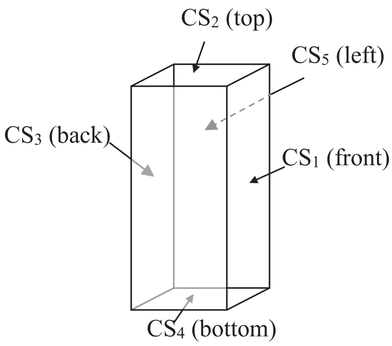

Figure 3. Description of the control volume and control surfaces.

φ. The rain velocity is then prescribed as: $\mathbf{v} = -(V\sin\varphi\ \cos\theta\ , V\cos\varphi, V\sin\varphi\ \sin\theta)$. Note that $\cos\varphi$ is always positive, $\sin\varphi$ can be positive or negative, while $\cos\theta$ and $\sin\theta$ are treated as always positive. We may assume that water is absorbed entirely by the person; although, models can be incorporated to allow for rain water removal.

3.1. Derivation of the rate of accumulation

In this section, we use the RTT relation presented in equation (2) applied to mass conservation to derive an expression for the accumulation of rain water on the person. A control volume will be drawn to include the person. We will proceed to write the RTT mass balance as usual; the key distinction in the analysis here is that fluxes leaving the control volume are set to zero while evaluating the storage term. Stated differently, the contributions to the surface integrals are set to 0 when this contribution corresponds to a net flow out.

Figure 3 shows the control volume based on an assumed shape of a parallelepiped adopted for the person. We can break the overall control surface around the control volume into 5 *control surfaces*, labeled as CS_1 (front), CS_2 (top), CS_3 (back), CS_4 (bottom) and CS_5 (left); and the rate of accumulation according to RTT is expressed as:

$$\begin{aligned}\frac{\partial m_{\mathrm{CV}}}{\partial t} = -\Bigg(&\int_{\mathrm{CS}_1}\rho(\mathbf{v}_r\cdot\mathbf{n})\ \mathrm{d}S + \int_{\mathrm{CS}_2}\rho(\mathbf{v}_r\cdot\mathbf{n})\ \mathrm{d}S + \int_{\mathrm{CS}_3}\rho(\mathbf{v}_r\cdot\mathbf{n})\ \mathrm{d}S\\ &+\int_{\mathrm{CS}_4}\rho(\mathbf{v}_r\cdot\mathbf{n})\ \mathrm{d}S + \int_{\mathrm{CS}_5}\rho(\mathbf{v}_r\cdot\mathbf{n})\ \mathrm{d}S\Bigg).\end{aligned} \tag{5}$$

The total accumulation of rain water corresponds to the amount of rain water that enters the volume through these control surfaces over the duration of travel, $t = L/U$, $\Delta m_{\mathrm{CV}} = (\partial m_{\mathrm{CV}}/\partial t)\times(L/U)$:

$$\begin{aligned}\Delta m_{\mathrm{CV}} = -\frac{L}{U}\Bigg(&\int_{\mathrm{CS}_1}\rho(\mathbf{v}_r\cdot\mathbf{n})\ \mathrm{d}S + \int_{\mathrm{CS}_2}\rho(\mathbf{v}_r\cdot\mathbf{n})\ \mathrm{d}S + \int_{\mathrm{CS}_3}\rho(\mathbf{v}_r\cdot\mathbf{n})\ \mathrm{d}S\\ &+\int_{\mathrm{CS}_4}\rho(\mathbf{v}_r\cdot\mathbf{n})\ \mathrm{d}S + \int_{\mathrm{CS}_5}\rho(\mathbf{v}_r\cdot\mathbf{n})\ \mathrm{d}S\Bigg).\end{aligned} \tag{6}$$

The table below lists the values of various terms inside the control surfaces' integrals given that: $\mathbf{v} = -(V\sin\varphi\ \cos\theta\ , V\cos\varphi, V\sin\varphi\ \sin\theta)$ and $\mathbf{U} = (U, 0, 0)$, which results in

Table 1. Summary of normal components of the fluid velocities.

CS	$\mathbf{n}$	$(\mathbf{v}_r \cdot \mathbf{n})$	Area	Side
CS_1	e_x	$-(U + V\sin\varphi\cos\theta)$	$h\ d$	Front
CS_2	e_y	$-V\cos\varphi < 0$	$w\ d$	Top
CS_3	$-e_x$	$+(U + V\sin\varphi\cos\theta)$	$h\ d$	Back
CS_4	$-e_y$	$V\cos\varphi > 0$	$w\ d$	Bottom
CS_5	e_z	$-V\lvert\sin\varphi\rvert\ \sin\theta$	$h\ w$	Left

$\mathbf{v}_r = -(U + V\sin\varphi\ \cos\theta\ , V\cos\varphi, V\sin\varphi\ \sin\theta)$. Note that e_x, e_y and e_y are the three components of the orthonormal basis of vectors in the Cartesian coordinates.

Note that the flux term across CS_2 is always negative; and therefore, it must be included regardless of the value of the angle of the rain. Similarly, the flux term across surface CS_4 is always positive; and therefore, it will be excluded from the sum. The sign of the remaining terms depends on the angle φ and the relative magnitudes of the running speed and the rain speed. The integrals across surfaces CS_1 and CS_3 are mutually exclusive. If one is considered, the second should be set to 0. Therefore, their combined effect is a term of the following form: $|U + V\sin\varphi\cos\theta|$ to represent the contributions to $(\mathbf{v}_r \cdot \mathbf{n})$ in the x-direction.

Based on the signs of the dot products of the relative velocity to the normal vectors, an expression for the rate of accumulation is obtained:

$$\frac{\partial m_{\mathrm{CV}}}{\partial t} = \underbrace{\rho(V\cos\varphi)\ (w\ d)}_{\text{top}} + \underbrace{\rho(|U + V\sin\varphi\cos\theta|)\ (h\ d)}_{\text{front or back}} + \underbrace{\rho(V|\sin\varphi|\sin\theta)\ (h\ \ w)}_{\text{side}}, \tag{7}$$

which can be written as follows:

$$\frac{\partial m_{\mathrm{CV}}}{\partial t} = (\rho\ V\ h\ d)\left(\frac{w}{h}\cos\varphi + \left|\frac{U}{V} +\ \sin\varphi\cos\theta\right| + \frac{w}{d}|\sin\varphi|\sin\theta\right). \tag{8}$$

The total accumulation is obtained by multiplying this rate by the time of travel, L/U:

$$\Delta m_{\mathrm{CV}} = \rho\left(\frac{V}{U}h\ d\ L\right)\left(\frac{w}{h}\cos\varphi + \left|\frac{U}{V} +\ \sin\varphi\cos\theta\right| + \frac{w}{d}|\sin\varphi|\sin\theta\right). \tag{9}$$

It is convenient to normalize this expression by dividing the total accumulation by $\rho(h\ d\ L)$:

$$\Delta m_{\mathrm{CV}}^{*} \equiv \frac{\Delta m_{\mathrm{CV}}}{\rho(h\ d\ L)} = \frac{V}{U}\left(\frac{w}{h}\cos\varphi + \left|\frac{U}{V} +\ \sin\varphi\cos\theta\right| + \frac{w}{d}|\sin\varphi|\sin\theta\right). \tag{10}$$

The volume $(h\ d\ L)$ represents the volume swept by the running person on his/her way to safety.

3.2. Vertical rain

At this point, it is useful to consider the trivial solution of vertical rain, which corresponds to $\varphi = \theta = 0$. Table 2 summarizes the values of the relative velocity for the different surfaces corresponding to this condition.

Table 2. Summary of normal components of the fluid velocities for vertical rain.

CS	$(\mathbf{v}_r \cdot \mathbf{n})$	Comment
CS_1	$-U < 0$	Keep, inlet
CS_2	$-V < 0$	Keep, inlet
CS_3	$U > 0$	Remove, outlet
CS_4	$V > 0$	Remove, outlet
CS_5	0	Remove, no flux

Considering the fact that we only keep negative contributions to the dot product of relative velocity vector and the normal vector, only fluxes from surfaces CS_1 and CS_2 are maintained, and the rate of accumulation can be written as follows:

$$\frac{\partial m_{\mathrm{CV}}}{\partial t} = \rho(w\ V + h\ U)\, d, \tag{11}$$

resulting in a total normalized accumulation of

$$\Delta m^*_{\mathrm{CV}} = 1 + \frac{w}{h}\frac{V}{U}. \tag{12}$$

This expression carries two contributions. The first term, $\frac{w}{h}\frac{V}{U}$, is the rain accumulation on the person from the top. This term is clearly reduced by running faster, since the person's speed component is in the denominator. The second contribution, 1, corresponds to the accumulation of the rain from the front. This term does not change with the running speed. The faster the person runs, the higher is the rate of running into the vertical rain, thus neutralizing the effects of a shorter travel duration.

Note that the rate expression (equation (11)) indicates that this rate is increased with an increased speed of travel, U; while the opposite trend is obtained for the total accumulation (equation (12)). Another observation can be made in relation to the amount of accumulation under vertical run. Given the same volume, $h \times w \times d$, a taller person can accumulate more rain than a shorter person. In practical terms, bending forward while running, can reduce the amount of rain accumulation.

3.3. No component to the side of the person in the rain

Next, we consider the more complex case of rain that falls in the person's travel direction; therefore, there is no component to the side of the person or $\theta = 0$. In this case, the rate and the total accumulation are expressed as

$$\frac{\partial m_{\mathrm{CV}}}{\partial t} = (\rho\ V\ h\ d)\left(\frac{w}{h}\cos\varphi + \left|\frac{U}{V} + \sin\varphi\right|\right) \tag{13}$$

and

$$\Delta m^*_{\mathrm{CV}} = \frac{V}{U}\left(\frac{w}{h}\cos\varphi + \left|\frac{U}{V} + \sin\varphi\right|\right). \tag{14}$$

From these relations, a number of observations can be made. First, as far as the rate of accumulation is concerned (equation (13)), an optimum speed of travel can be achieved with a tailwind (or negative angle, φ):

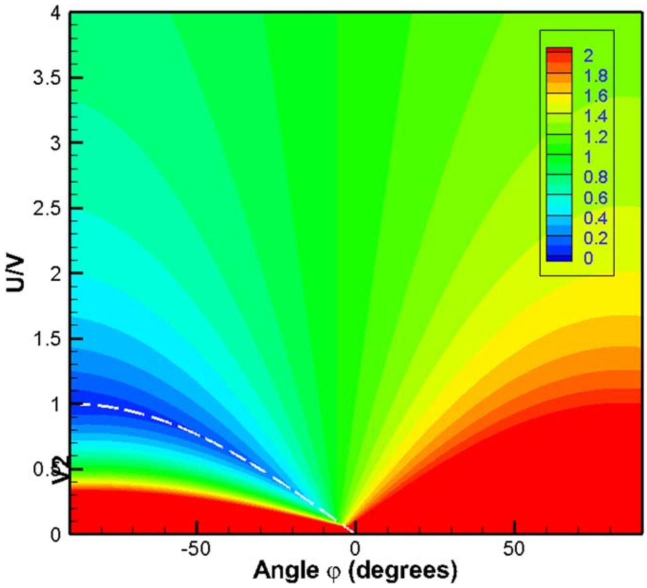

Figure 4. Contours of the indicator of the total accumulation $\frac{V}{U}\left(\frac{w}{h}\cos\varphi + \left|\frac{U}{V} + \sin\varphi\right|\right)$. Color contour values range from low (blue) to high (red). Dashed white line: optimum speed line.

$$U_{\text{optimum}} = -V \sin\varphi. \tag{15}$$

In this case, traveling with a tailwind will only accumulate rain from the top and not the front or the sides. When the person is running at a different speed from this optimum value, he or she will accumulate rain water from the front if running faster than the optimum speed or from the back if running slower than this speed.

Figure 4 shows contours of Δm^{*}_{CV} for an angle $\theta = 0$ and $w/h = 0.15$ and a range of U/V values from 0 to 10 and angles φ from -90 to 90 degrees. Also shown on the figure (in dashed lines) is the contour that corresponds to a minimum rate of accumulation as prescribed by equation (16). The figure shows the following.

- At negative angles, φ, there is, again, an optimum speed at which the total accumulation is minimized. Below this speed, the ratio U/V quickly yields significant accumulation of rain water. Therefore, if runners are not sure about the optimum speed, they should run until no rain is felt on their backs.
- At positive angles, φ, there is no optimum speed. Here, running faster tends to reduce the amount of rain accumulation.

An extension of the optimum speed for the arbitrary angle includes the following more general expression:

$$U_{\text{optimum}} = -V \sin\varphi \cos\theta, \tag{16}$$

which takes into consideration the presence of a wind component in the transverse direction.

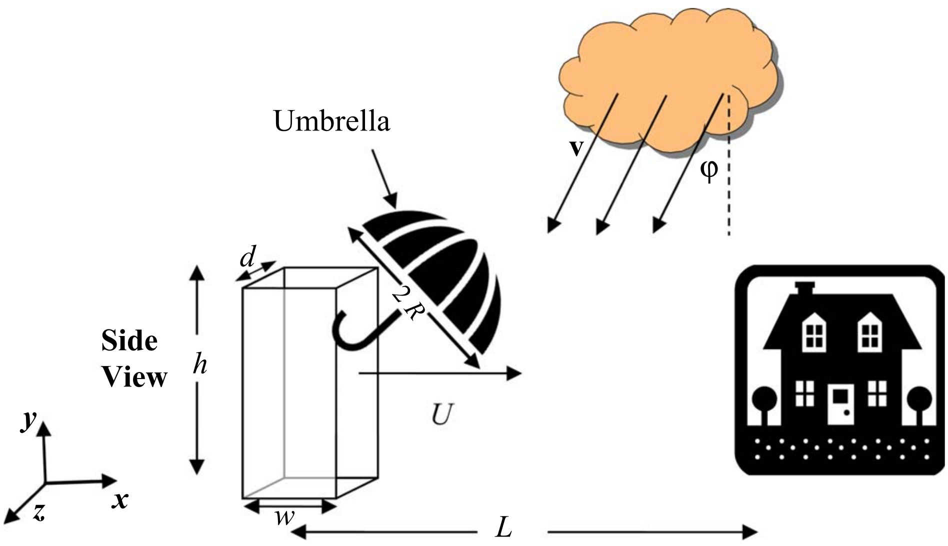

Figure 5. Schematic of problem configuration.

4. Problem formulation: walking/running with an umbrella

In the previous discussion, we have focused on the familiar problem of running in the rain without an umbrella. In this section, we explore a new twist to this problem by allowing the person to hold an umbrella with a radius *R*. The schematic of the person and the umbrella arrangements is shown in figure 5.

Figure 6 shows a close-up of the person and the umbrella indicating the angle, β, made by the umbrella's handle with the vertical direction as well as the distances of the umbrella from the person in the horizontal direction, *a*, and vertical distance from the ground, *b*. The shaded volume in figure 7 indicates the volume swept by the umbrella coverage. Therefore, in this volume, the rain velocity can be set to zero. *From the figure, it is clear that setting the angle β to φ will make the optimum coverage of the umbrella.*

Although, the analysis can be generalized to any arbitrary rain angle from the front and the sides, we consider only the case of rain falling in the direction of travel (i.e. angle $\theta = 0$) and with a positive angle φ as shown. When the angle φ is negative, the trivial solution of an umbrella overhead and an optimum speed $U_{\text{optimum}} = -V \sin\varphi$ are sufficient to keep the person dry. Moreover, for simplicity, we carry out the analysis by adopting a simpler shape for the umbrella's cross-section as a $2R \times 2R$ square instead of a circular shape with a radius *R*.

In figure 6, we label 2 positions M and N on the front of the person whose vertical positions from the ground can be expressed:

$$\begin{cases} y_{\mathrm{M}} = y_{\min} = (b - a\cot\varphi) - R\cos\beta \cdot \left(1 + \dfrac{\tan\beta}{\tan\varphi}\right) \\ y_{\mathrm{N}} = y_{\max} = (b - a\cot\varphi) + R\cos\beta \cdot \left(1 + \dfrac{\tan\beta}{\tan\varphi}\right). \end{cases} \tag{17}$$

These expressions are derived by evaluating the heights of M and N in relation to the top of the umbrella's handle with coordinates (a, b).

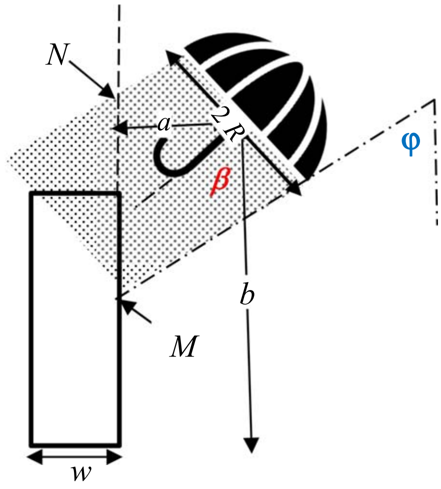

Figure 6. Definitions of angles and distances for the RWiR problem.

These points indicate the lower and upper tips of the umbrella coverage at the front of the person. If the upper tip is greater than the person's height, h, then, at least some of person's top or all of it is covered by the umbrella. Therefore, practically speaking, M should be above the ground and N can be below h to cover only the front or above h to cover both the front and the top of the person.

The choices of a and b serve to shift the coverage from front to top or vice versa. For these parameters, it is clear that for optimum coverage, there is a range of values for these two parameters that can be adopted; stated differently, if either parameter is set, then, an optimum value for the remaining parameter can be determined. There are, of course, limitations to the choice of the optimum values. First, the position of the umbrella is limited by the height and arm length and the umbrella's handle. Second, to allow for some coverage from the top, the length b must be higher than the person's height, h.

In addition to the choice of the rain angle, we make additional assumptions to simplify our analysis.

- The umbrella radius is wider than the person, i.e. $R \geqslant d$.
- The projection of the coverage of the umbrella onto the front and the top of the person can generate relatively complex expressions for the resulting ellipsoidal shapes for these projections. With a reasonably large umbrella, we estimate the projection of the umbrella on the person to be a rectangle of width d and a length that depends on the projection of the umbrella on the vertical and horizontal, front and top, surfaces. Corrections can be made to express an equivalent radius to account for this conversion. However, this is not attempted here. We assume that the observations and conclusions can be made based on the simplified analysis of a rectangular umbrella.

Based on the extent of the projections of the umbrella on the front and top surfaces, the area of exposure from the front and the top can be expressed as follows:

$$\begin{cases} A_{\text{exposure,top}} = \left\{1 - \min\left[1,\ \max\left(0,\ \dfrac{y_{\max}-h}{w}\tan\varphi\right)\right]\right\} \times (w\ d) \\ A_{\text{exposure,front}} = \min\left[1,\ \max\left(0,\ \dfrac{y_{\min}}{h}\right)\right] \times (h\ d). \end{cases} \tag{18}$$

The min's and the max's are used to derive compact expressions for these areas and address conditions where (1) coverage of the umbrella is limited to either the front or the top, or (2) where portion of the coverage does not shield the person. Based on the flux breakdown information in table 1, the rate of accumulation from the front and top is

$$\frac{\partial m_{\text{CV}}}{\partial t} = \underbrace{\rho\ (V\cos\varphi)A_{\text{exposure, top}}}_{\text{top}} + \underbrace{\rho\ (U + V\sin\varphi)\ A_{\text{exposure, front}}}_{\text{front}} \tag{19}$$

or by using the expressions for the exposure areas, we get

$$\begin{aligned} \frac{\partial m_{\text{CV}}}{\partial t} &= \underbrace{\rho\ (V\cos\varphi)(wd)\cdot\left\{1 - \min\left[1,\ \max\left(0,\ \frac{y_{\max}-h}{w}\tan\varphi\right)\right]\right\}}_{\text{top}} \\ &+ \underbrace{\rho\ (U + V\sin\varphi)\ (hd)\cdot\left\{\min\left[1,\ \max\left(0,\ \frac{y_{\min}}{h}\right)\right]\right\}}_{\text{front}}. \end{aligned} \tag{20}$$

Note that the terms in the curly brackets $\{\cdot\}$ in the above expression correspond to the attenuations from unity of the rate of accumulation due to the presence of the umbrella. The corresponding total accumulation based on the travel time is

$$\begin{aligned} \Delta m^*_{\text{CV}} &= \frac{V}{U} \\ &\times \left(\begin{array}{c} \underbrace{\dfrac{w}{h}\cos\varphi\cdot\left\{1 - \min\left[1,\ \max\left(0,\ \dfrac{y_{\max}-h}{w}\tan\varphi\right)\right]\right\}}_{\text{top}} \\ + \underbrace{\left(\dfrac{U}{V} + \sin\varphi\right)\cdot\left\{\min\left[1,\ \max\left(0,\ \dfrac{y_{\min}}{h}\right)\right]\right\}}_{\text{front}} \end{array}\right), \end{aligned} \tag{21}$$

where $y_{\min}$ and $y_{\max}$ with the additional restriction of $\varphi = \beta$, can be expressed as

$$y_{\min} = (b - a\cot\varphi) - R(1 + \cot^2\varphi)\sin\varphi,\quad y_{\max} = (b - a\cot\varphi) + R(1 + \cot^2\varphi)\sin\varphi. \tag{22}$$

The expressions for the total accumulation with an umbrella can be contrasted with the expressions without umbrella when the angle θ is set to 0 and the direction of the rain is prescribed by figure 5 from equation (10) above:

$$\Delta m^*_{\text{CV}} = \frac{V}{U}\left[\underbrace{\frac{w}{h}\cos\varphi}_{\text{top}} + \underbrace{\left(\frac{U}{V} + \sin\varphi\right)}_{\text{front}}\right] = 1 + \frac{V}{U}\left(\frac{w}{h}\cos\varphi + \sin\varphi\right). \tag{23}$$

Therefore, the terms in the curly brackets, $\{\cdot\}$, in equation (21) are replaced with 1's in the absence of an umbrella (equation (23)). Invariably, a higher reduction of the total accumulation can be achieved through a reduction in the exposure surfaces due to the use of the umbrella.

An alternative form of the total accumulation is as follows:

$$\Delta m_{CV}^{*} = \frac{V}{U}\left(\left\{\frac{w}{h}\cos\varphi \cdot \quad 1 - \min 1,\ \max\left(0,\ \frac{y_{\max}-h}{w}\tan\varphi\right)\right.\right.$$
$$\left.\left. + \sin\varphi \cdot \left\{\min\left[1,\ \max\left(0,\ \frac{y_{\min}}{h}\right)\right]\right\}\right\}\right)$$
$$+ \left\{\min\left[1,\ \max\left(0,\ \frac{y_{\min}}{h}\right)\right]\right\}. \tag{24}$$

In this expression, the first term on the RHS represents the contribution that is partially dependent on the running/walking speed; while, the second contribution is not. More importantly, in the analysis without an umbrella, this latter contribution is constant and corresponds to 1; while, this contribution can be reduced by ensuring maximum coverage on the front side of the person. Regardless, equation (24) clearly states that increasing the running speed will result in a reduced total accumulation. However, in addition to running faster, the person can also optimize the placement of the umbrella to minimize the total rain accumulation. To achieve this goal, it is important to identify the different parameters of the problem.

- *The model constraints*. These include the person's dimensions, h, w and d, the rain speed and angle, V and φ, the umbrella size, R, and the distance of travel, L. The person should run as fast as she or he can; but, every person has a maximum speed that can be sustained over the travel distance.
- *The choice parameters*. These include geometry parameters that can be chosen, which do not restrict the optimization process or are consistent with the optimization process. These include the choice of the umbrella angle, β, which we have stated should be set to φ. Moreover, the person can prioritize which parts of his/her body to shield from the rain.
- *The model optimized parameters*. They include the parameters that are optimized to minimize the total accumulation, which for the specific problem correspond to determining the parameters a and b.

Therefore, in contrast with the problem without an umbrella, considering an umbrella introduces additional parameters to optimize the amount of accumulation. To illustrate the model results, we consider the following model constraints: $w/h = 0.15$, $R/h = 0.25$ and $V/U = 0.1$. The values of a and b are constrained to be less than $0.5h$ and $1.5h$, respectively. We also impose the choice parameter that the person's head and shoulders must be entirely shielded from the rain. This condition results in: $\frac{y_{\max}-h}{w}\tan\varphi = 1$. By substituting for the expression of $y_{\max}$ and solving for b in terms of a, we get

$$b = h + (w + a)\cot\varphi - R\,(1 + \cot^2\varphi)\sin\varphi. \tag{25}$$

Figure 7 shows the optimized values for a, b and the normalized accumulation Δm_{CV}^{*} as functions of the angle φ. Also shown on the figure is the normalized accumulation without an umbrella (in dashed lines) based on equation (23). As suggested by this equation, the minimum value for the normalized accumulation is 1 already illustrating the reduction by more than a factor of 2 in this quantity by using the umbrella.

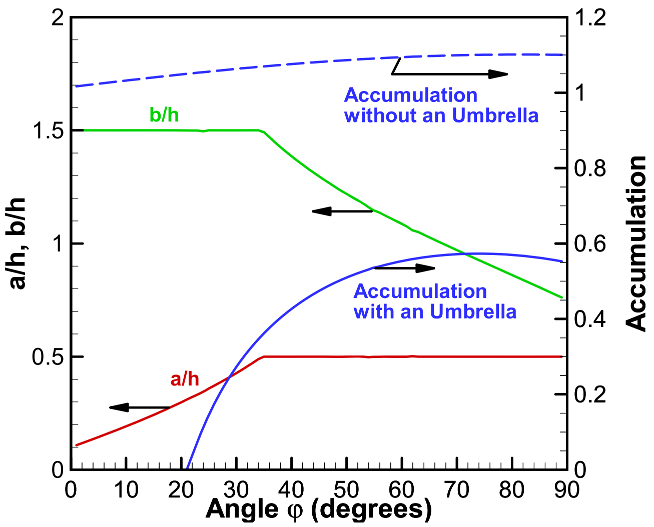

Figure 7. Optimized values of a, b and the total accumulation versus rain angle.

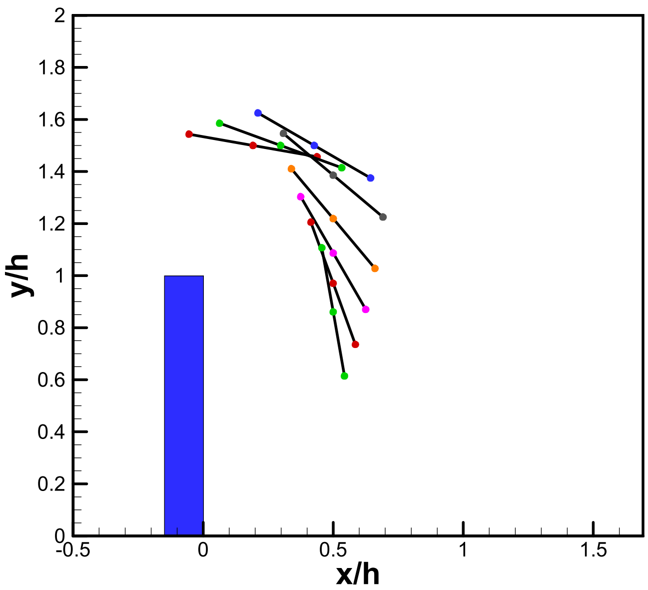

Figure 8. Optimum umbrella positon and angle at different rain angles from 10 to 90 degrees and 10 degree increments.

Figure 8 further illustrates the umbrella placement relative to the person (shown as a blue box) at angles from 10 to 80 degrees and at 10 degree increments (with the steepest slopes corresponding to the higher angles). Both figures 7 and 8 show that the initial placement of the umbrella is closer to the person's body and maintained high enough (the low angles all require the maximum value of 1.5 for b). However, as the rain angle shifts to point more towards the person's front, the placement of the umbrella is shifted lower and its horizontal position, indicated by the parameter a, is advanced for larger angles to its maximum value.

The normalized accumulation, $\Delta m^{*}_{\mathrm{CV}}$, is zero for low angles since an overhead umbrella can shield both the top and front sides of the person. There an intermediate peak for this accumulation around 70 degrees, which eventually decays at larger angles. Adopting other constraints than the ones chosen can result in variations in the outcome for the placement of the umbrella.

5. Conclusions

Analysis of the rate and total accumulation of rain on a person traveling to safety with and without an umbrella is carried out with the aid of the RTT combined with mass conservation. Most of the contributions presented with the analysis in the absence of an umbrella are not new. However, the key advantage of the analysis is its ease of implementation, which can enable the consideration of more complex shapes for the traveling person. The analysis offers a quite elegant teaching tool for students who can be motivated by a practical implementation of RTT.

This study uses RTT to incorporate the effects of using an umbrella to address the RWiR problem. This case is analyzed in section 4 and demonstrated that with new parameters involving the size and angle of the umbrella and the position of the umbrella relative to the person, an optimization process can be implemented to minimize the amount of rain accumulation.

The present analysis can be easily extended to allow for a combination of ellipsoid shapes and parallelepiped to closely approximate a moving person's shape. Regardless of any further refinements, the basic qualitative results are expected to be maintained.

In this paper, we have sought to reproduce the results of previous observations and develop new analysis related to the running in the rain problem using RTT. However, the ultimate goal is to illustrate some of the key attributes of RTT. The analysis is reduced to the evaluation of integrals that are in turn sub-divided into different sub-tasks related to identifying the rain velocity and the person velocity from which a relative velocity is determined and identifying the surface area and the surface normal to determine the next flux in or out of the control volume. Breaking down the problem into more trivial sub-tasks, in principle, can have an inherent pedagogical value and can help generalize a problem if only one or two of these sub-tasks are modified (e.g. changing the person's model for geometry or implementing the problem to optimize rain catching).

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