APPENDIX: HASH FUNCTIONS
(From Kleinberg &Tardos Book: Algorithm Design

Dictionary data type

Dictionary. Given a universe U of possible elements, maintain a subset $S \subseteq U$ so that inserting, deleting, and searching in S is efficient.

Dictionary interface.

- *create*(): initialize a dictionary with $S = \emptyset$.
- *insert*(u): add element $u \in U$ to S.
- *delete*(u): delete u from S (if u is currently in S).
- *lookup*(u): is u in S ?

Challenge. Universe U can be extremely large so defining an array of size $|U|$ is infeasible.

Applications. File systems, databases, Google, compilers, checksums, P2P networks, associative arrays, cryptography, web caching, etc.

HASH FUNCTIONS

Hashing

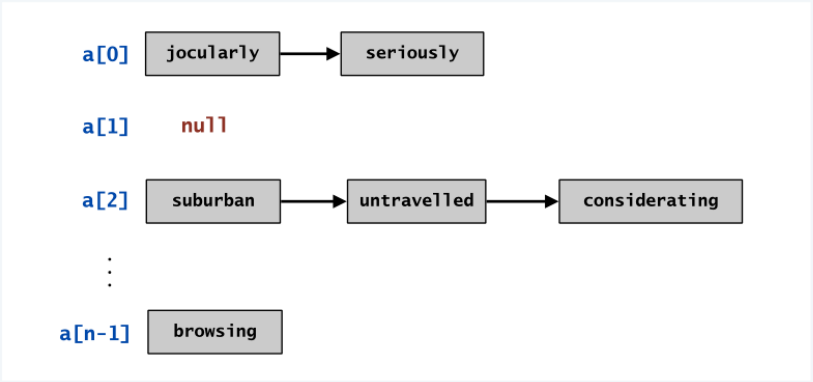
Hash function. $h : U \rightarrow \{ 0, 1, \ldots, n-1 \}$.

Hashing. Create an array a of length n. When processing element u, access array element $a[h(u)]$.

Collision. When $h(u) = h(v)$ but $u \neq v$.

birthday paradox

- A collision is expected after $\Theta(\sqrt{n})$ random insertions.
- Separate chaining: $a[i]$ stores linked list of elements u with $h(u) = i$.



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HASH FUNCTIONS

Hashing performance $h: U \to [n]$; $|U| \gg n$; $\forall S \subseteq U$ s.t. $|S| = m$

Ideal hash function. Maps m elements uniformly at random to n hash slots.

- Running time depends on length of chains.
- Average length of chain = $\alpha = m / n$.
- Choose $n \approx m \Rightarrow$ expect $O(1)$ per insert, lookup, or delete.

Challenge. Hash function h that achieves $O(1)$ per operation.

Approach. Use randomization for the choice of h.

adversary knows the randomized algorithm you're using, but doesn't know random choice that the algorithm makes

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Universal Family of Hash Functions

Universal hashing (Carter–Wegman 1980s)

A universal family of hash functions is a set of hash functions H mapping a universe U to the set $\{ 0, 1, \ldots, n-1 \}$ such that

- For any pair of elements $u \neq v$: $\Pr_{h \in H}\left[h(u) = h(v) \right] \leq 1/n$
- Can select random h efficiently.
- Can compute $h(u)$ efficiently.

chosen uniformly at random

Ex. $U = \{ a, b, c, d, e, f \}$, $n = 2$.

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
$h_2(x)$	0	0	0	1	1	1

$H = \{h_1, h_2\}$

$\Pr_{h \in H}[h(a) = h(b)] = 1/2$

$\Pr_{h \in H}[h(a) = h(c)] = 1$

$\Pr_{h \in H}[h(a) = h(d)] = 0$

. . .

not universal

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
$h_2(x)$	0	0	0	1	1	1
$h_3(x)$	0	0	1	0	1	1
$h_4(x)$	1	0	0	1	1	0

$H = \{h_1, h_2, h_3, h_4\}$

$\Pr_{h \in H}[h(a) = h(b)] = 1/2$

$\Pr_{h \in H}[h(a) = h(c)] = 1/2$

$\Pr_{h \in H}[h(a) = h(d)] = 1/2$

$\Pr_{h \in H}[h(a) = h(e)] = 1/2$

$\Pr_{h \in H}[h(a) = h(f)] = 0$

. . .

universal

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Universal Family of Hash Functions

Universal hashing: analysis

Proposition. Let H be a universal family of hash functions mapping a universe U to the set $\{ 0, 1, \ldots, n-1 \}$; let $h \in H$ be chosen uniformly at random from H; let $S \subseteq U$ be a subset of size at most n; and let $u \notin S$. Then, the expected number of items in S that collide with u is at most 1.

Pf. For any $s \in S$, define random variable $X_s = 1$ if $h(s) = h(u)$, and 0 otherwise. Let X be a random variable counting the total number of collisions with u.

$$E_{h \in H}[X] = E[\textstyle\sum_{s \in S} X_s] = \sum_{s \in S} E[X_s] = \sum_{s \in S} \Pr[X_s = 1] \leq \sum_{s \in S} \frac{1}{n} = |S| \frac{1}{n} \leq 1$$

linearity of expectation — X_s is a 0–1 random variable — universal

Q. OK, but how do we design a universal class of hash functions?

Universal Family of Hash Functions

Designing a universal family of hash functions

Modulus. We will use a prime number p for the size of the hash table.

Integer encoding. Uniquely identify each element $u \in U$ with a base-p integer of r digits: $x = (x_1, x_2, \ldots, x_r)$.

distinct elements have different encodings

Hash function. Let A = set of all r-digit, base-p integers. For each $a = (a_1, a_2, \ldots, a_r)$ where $0 \le a_i < p$, define

$$h_a(x) = \left(\sum_{i=1}^{r} a_i x_i \right) \bmod p$$

maps universe U to set $\{ 0, 1, \ldots, p-1 \}$

Hash function family. $H = \{ h_a : a \in A \}$.

Universal Family of Hash Functions

$U \subseteq [p]^r$

Designing a universal family of hash functions

$h_a(x) = \sum_{i=1}^{r} a_i x_i \mod p$

Theorem. $H = \{ h_a : a \in A \}$ is a universal family of hash functions.

Pf. Let $x = (x_1, x_2, \ldots, x_r)$ and $y = (y_1, y_2, \ldots, y_r)$ encode two distinct elements of U.

We need to show that $\Pr[h_a(x) = h_a(y)] \le 1/p$.

- Since $x \neq y$, there exists an integer j such that $x_j \neq y_j$.
- We have $h_a(x) = h_a(y)$ iff

$$(1) \qquad a_j \underbrace{(y_j - x_j)}_{z} \equiv \underbrace{\sum_{i \neq j} a_i (x_i - y_i)}_{m} \mod p$$

Principle of Deferred D
This is fixed before
a_j is chosen i.u.a.r.

- Can assume a was chosen uniformly at random by first selecting all coordinates a_i where $i \neq j$, then selecting a_j at random. Thus, we can assume a_i is fixed for all coordinates $i \neq j$.
- Since p is prime, $a_j z \equiv m \mod p$ has at most one solution among p possibilities. ← see lemma on next slide
- Thus $\Pr[h_a(x) = h_a(y)] \le 1/p$. ▪ → it is the a_j value s.t. (1) is satisfied

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Universal Family of Hash Functions

Number theory fact

Fact. Let p be prime, and let $z \not\equiv 0 \bmod p$. Then $\alpha z \equiv m \bmod p$ has at most one solution $0 \le \alpha < p$.

Pf.

- Suppose $0 \le \alpha_1 < p$ and $0 \le \alpha_2 < p$ are two different solutions.
- Then $(\alpha_1 - \alpha_2)\, z \equiv 0 \bmod p$; hence $(\alpha_1 - \alpha_2)\, z$ is divisible by p.
- Since $z \not\equiv 0 \bmod p$, we know that z is not divisible by p.
- It follows that $(\alpha_1 - \alpha_2)$ is divisible by p.
- This implies $\alpha_1 = \alpha_2$. ▪

here's where we use that p is prime

Bonus fact. Can replace "at most one" with "exactly one" in above fact.

Pf idea. Euclid's algorithm.

Universal Family of Hash Functions

Universal hashing: summary

Goal. Given a universe U, maintain a subset $S \subseteq U$ so that insert, delete, and lookup are efficient.

Universal hash function family. $H = \{ h_a : a \in A \}$.

$$h_a(x) = \left(\sum_{i=1}^{r} a_i x_i \right) \bmod p$$

- Choose p prime so that $m \le p \le 2m$, where $m = |S|$.
- Fact: there exists a prime between m and $2m$. ← can find such a prime using another randomized algorithm (!)

Consequence.

- Space used = $\Theta(m)$.
- Expected number of collisions per operation is ≤ 1

  $\Rightarrow$ $O(1)$ time per insert, delete, or lookup.

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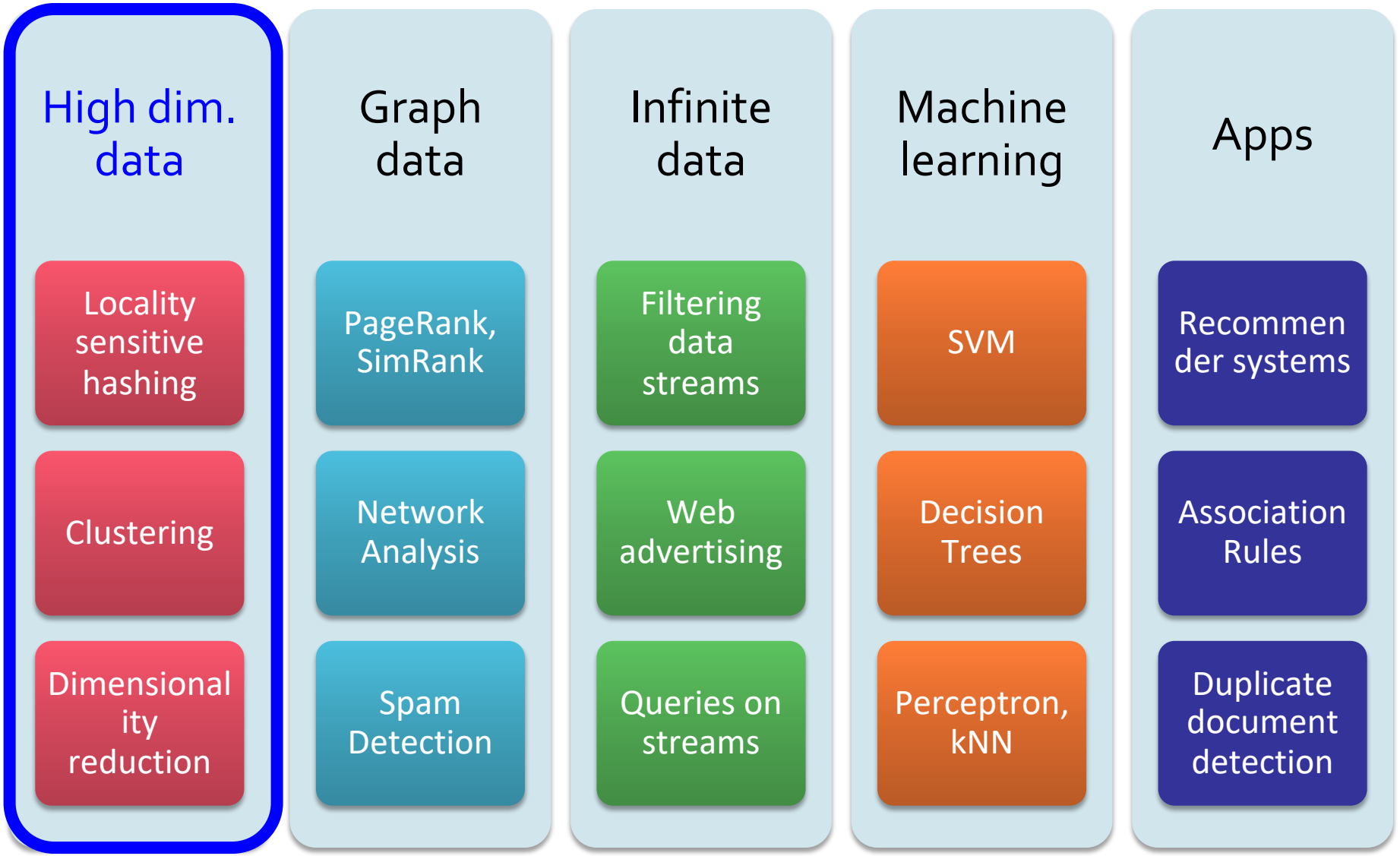
Finding Similar Items in Large Data Sets

Lecturer: **Andrea Clementi**
University of Rome *Tor Vergata*

Note: These slides are an adaptation, with more details, of the slides by Jure Leskovec, Anand Rajaraman, Jeff Ullman available at http://www.mmds.org

New thread: High dim. data

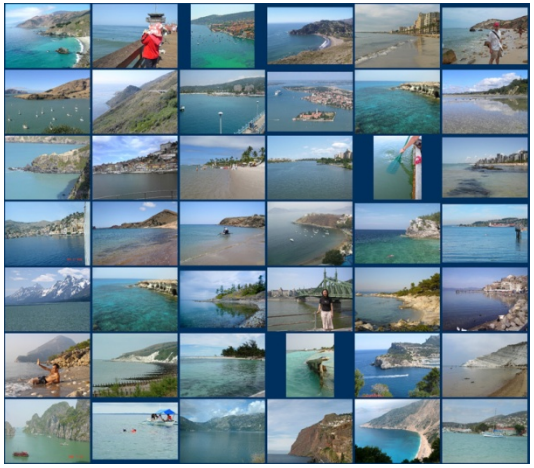


[Hays and Efros, SIGGRAPH 2007]

Application: Scene Completion Problem

target item

many items



selection of *similar* items

closest item

A Common Metaphor

Many problems can be expressed as:

finding "similar" sets

Examples:

- **Pages with a large fraction of similar words**
  - For duplicate detection, classification by topic
- **Customers who purchased a large fraction of similar products**
  - Products with similar customer sets
- **Images with similar features**
  - Users who visited similar websites

High-Dimensional Data and Distance

Input: *High-Dimensional* data points $\boldsymbol{x_1, x_2, \ldots, x_N}$

- Image x is a *long* vector of pixel colors

$$x = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow [1\ 4\ 1\ 0\ 2\ 1\ 0\ 1\ 0] \in \{0,1,\ldots,c\}^h \text{ , } (h = \text{data dimension})$$

Introduce a distance function $\boldsymbol{d(x_1, x_2)}$

- Which quantifies the "distance" between any pair of data points $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$

Goal: Find ***all* pairs of data points** $\boldsymbol{(x_i, x_j)}$ that are within some *distance threshold* $\boldsymbol{d(x_i, x_j) \leq s}$

High-Dimensional Data and Distance

Goal: Find **all pairs of data points** (x_i, x_j) that are within some distance threshold $d(x_i, x_j) \leq s$

Naïve solution would take $O(N^2 * h)$ ☹

where N is the number of data points and h is the data dimension.

MAGIC: **This can be done in $O(N * h')$ with $h' << h$, *with high confidence* !!**

Why? How?

Our Application: Finding Similar Documents

Goal: Given a large number (N in the millions or billions) of Docs, find "near duplicate" pairs

Applications:

- Mirror websites, or approximate mirrors
  - Don't want to show both in search results
- Similar news articles at many news sites
  - Cluster articles by "same story"

Technical Problems:

- Many small *pieces* of one Doc can appear out of *order* in another
- Too many Docs to compare *all* pairs
- Docs are so large or so many that they cannot fit in main memory

Choosing the *right* Distance Measures

For each application, we first need to define what "**distance**" means

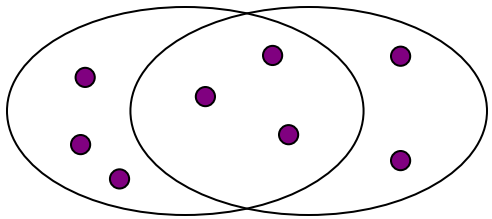
Our Choice: *Jaccard* distance/similarity

over a family of subsets

- The ***Jaccard* similarity** of two **sets** is:

$$sim(C_1, C_2) = |C_1 \cap C_2| / |C_1 \cup C_2|$$

- **Jaccard distance: $d(C_1, C_2) = 1 - sim(C_1, C_2)$**



3 in intersection
8 in union
Jaccard similarity= 3/8
Jaccard distance = 5/8

Application: Documents

A First Important Question: How apply ***Jaccard*** *distance/similarity* to Docs (i.e. to finite-size *strings*)?

Doc ∈ U* where U is the Alphabet

J.Similarity $sim(C_1, C_2) = |C_1 \cap C_2| / |C_1 \cup C_2|$

Docs ⇔ **Sets** **?** 🤔

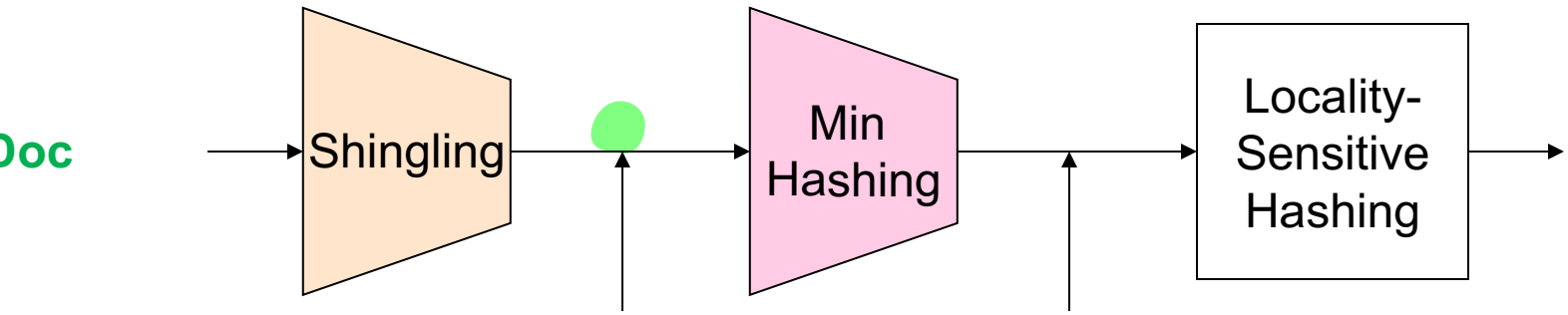
3 Essential Steps for Similar Docs

1. ***Input:*** A Huge *Universe* of *Docs*

2. ***Shingling:*** *Convert* Docs to (large) Sets
3. ***Min-Hashing:*** *Convert* large Sets to short Signatures, while preserving ***J.*** *Similarity*
4. ***Locality-Sensitive Hashing:*** Focus on pairs of Signatures likely to be from J.similar Docs (overcome the Θ(N^2) barrier!!)

5. **Output: Candidate *Doc* pairs!**

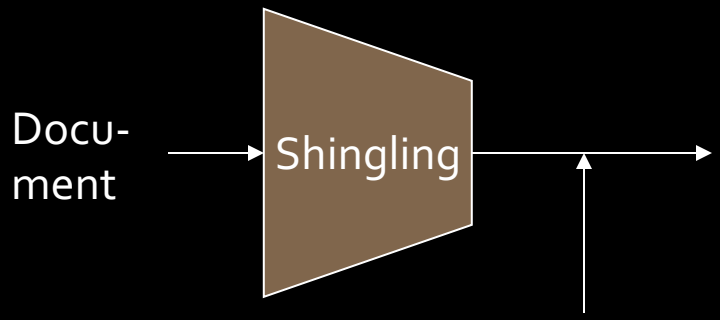
The Big Picture



The **Set** of strings of length ***k*** that appear in the Doc

Signatures: short integer vectors that represent the **Sets**, and reflect their J.similarity

Candidate pairs: those pairs of signatures that we need to test for similarity



Shingling

Step 1: ***Shingling:*** Convert documents to sets

Documents as High-Dim Data

Step 1: *Shingling:* Convert Documents to Sets

Simple approaches:

- Document = set of *words* appearing in document
- Document = set of "important" *words*

Don't work well for this application.....😟

Why?

Need to account for ordering of words!

A different approach: Shingles!

Definition of Shingles

A *k*-shingle (or *k*-gram) for a Doc is a sequence of ***k*** **tokens** that appears in the Doc

- **Tokens** can be ***characters***, *words* or something else, depending on the application:
  - **U** ≡ Universe of ***all*** possible Tokens
- **Example**: Assume Tokens = **characters and k=2**
  - Then for Doc $\mathbf{D_1}$ = abcab → Set of 2-shingles is $\mathbf{S(D_1)}$ = {ab, bc, ca}

- **Option:** Shingles as a bag (multiset). Count ab twice: $\mathbf{S'(D_1)}$ = {ab, bc, ca, ab}

Doc Representation by Shingles

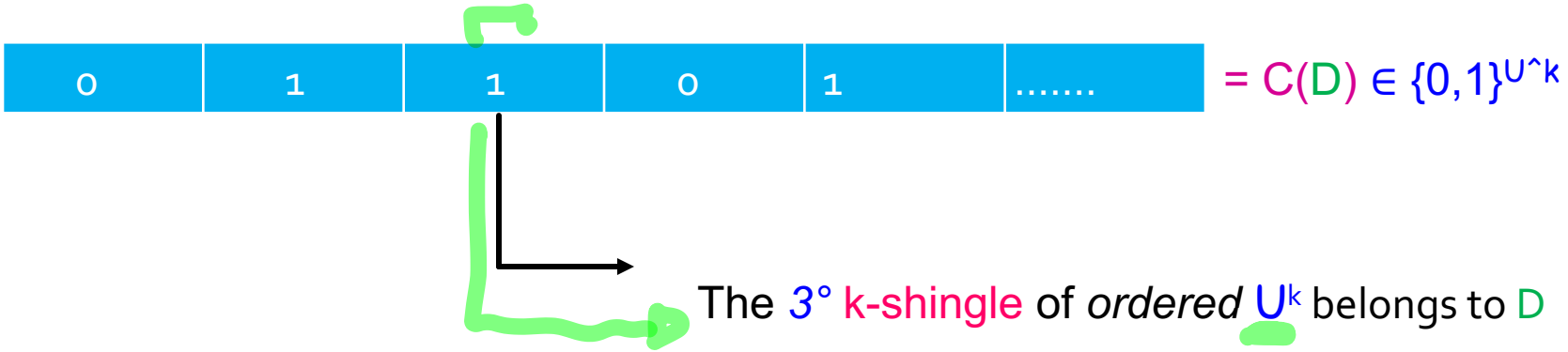
Each Doc D ∈ U^* is represented as the **set** of its **k-shingles C=S(D)**

Equivalently:

each Doc is represented as a *long binary vector* **C=S(D)** having one component for each element of the set U^k of *all* possible *k*-shingles

- Each unique shingle is a *dimension* (*component*) of **C=S(D)**
- ***Note:*** *Vectors* have several dimensions but are very sparse

Example:



Similarity Metric for Shingles

Our strategy:

(a) Doc D is represented as the **subset** of its **k-shingles C_j=S(D)**

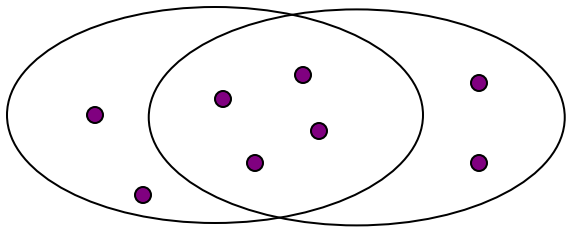
Equivalently, each **Doc** is a **binary vector** in the space of all possible *k*-shingles $\cup^k$

(b) A natural ***similarity*** measure **(=1- distance)** is the ***Jaccard similarity*** (and, thus, ***distance***):

$J.sim(D_1, D_2) \equiv |C_1 \cap C_2| / |C_1 \cup C_2| = Pr(C_1 \cap C_2 \mid C_1 \cup C_2)$

$dist(D_1, D_2) \equiv 1 - sim(D_1, D_2)$

PROBABILISTIC INTERPRETATION



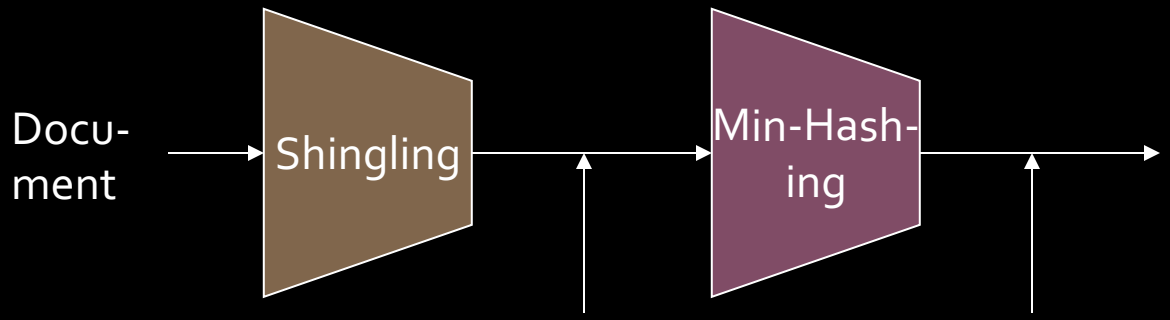
Working Hypothesis

Docs that have lots of k-shingles in common have *similar* text, *even* if the text appears in different order

Practice: You must pick ***k*** large enough, or most Docs will have most shingles → large intersection → large (not relevant) ***J. Sim.***

Parameter tuning from Real World:

- ***k*** = 5 is **OK** for *short* Docs
- ***k*** = 10 is better for *long* Docs



The set of strings of length *k* that appear in the doc-ument

Signatures: short integer vectors that represent the sets, and reflect their similarity

MinHashing

Step 2: ***Min-hashing:*** Convert **large sets** (i.e. long binary sparse *vectors*) to **short signatures**, while **preserving similarity**

Recall: Motivation for Min-Hashing

Suppose we need to find near-duplicate **Docs** among $N = 10^6$ **Docs** and, hence, among 10^6 *k−shingles C=S(D)*

Then....

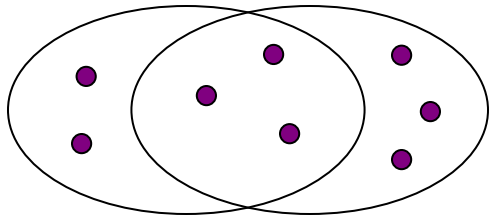
Min-Hashing does not avoid $\Theta(N^2) \simeq 5*10^{11}$ pair *comparisons!*

However ...

It makes *each* pair comparison (***J.*** *similarity* computations) much more **efficient**

Note: if **C** $\in \{0,1\}^{U\hat{}k}$ → cost(Naïve-Comparison) = $\Theta(|U|^k)$. For alphabet size $|U| = 27$ and $k = 10$ → .....😢

Encoding Sets as Binary Vectors



Many ***similarity*** problems can be formalized as **finding subsets** that have significant **intersection (recall *J. Sim.*):**

(a) Encode sets using binary vectors

- One dimension (component) per *element* (shingle) in the *Universe* U^k

(b) Interpret: (i) *set intersection* as bitwise ***AND***

(ii) *set union* as bitwise ***OR***

Example: C_1 = 10111; C_2 = 10011

- Size of intersection **= 3**; size of union **= 4**,
- **Jaccard similarity** (not distance) **= 3/4**
- **Distance: $d(C_1,C_2) = 1 – (J.Sim(C_1,C_2)) = 1/4$**

From Sets to Large, Sparse Boolean Matrices

Rows = elements (shingles) = vector components (**n.** of raws = **m** = $|U|^k$)

Columns = Sets = Docs (N col)

- **1** in row ***s*** and column ***e*** *iff* ***s*** $\in$ ***e***
- Column similarity is the **J.sim** of the corresponding sets
- **Remark:** Typical matrix **M** is sparse!

Each Doc is a column:

- **Example: sim(C_1 ,C_2) = ?**
  - Size of intersection = 3; size of union = 6, *Jaccard similarity* (not distance) = 1/2
  - **d(C_1,C_2) = 1 – (J.Sim (C_1,C_2)) = 1/2**

Matrix **M:**

|Docs| = N

Shingles				
	1	1	1	0
	1	1	0	1
	0	1	0	1
	0	0	0	1
	1	0	0	1
	1	1	1	0
	1	0	1	0

$m = |U|^k$

Outline: Finding Similar Columns

So far:

- Docs → ***Subsets*** of Shingles
- Represent *Subsets* as *binary columns* of a *matrix* **M**

Next goal: Find similar columns by computing *small* column ***signatures*** (**Min-Hashing**)

Technical goal: define a ***good*** signature *algorithm* so that

J.Sim of Columns ≈ **"Similarity" of Signatures**

Outline: Finding Similar Columns

Next Goal: Find similar columns using small signatures

Our Approach:

- **1) Signatures of columns =** small *summaries* of columns
- **2) Examine pairs of signatures** to find similar columns
  - **Essential:** Similarities of signatures and columns must be related
- **3) Optional:** Check that columns with similar signatures are really *similar*

Warnings:

- Comparing *all pairs* may take too much time: **Task for LSH**
- These methods can produce *false negatives*, and even *false positives* (if the optional check Step (3) is not made)

Hashing Columns (Signatures)

Key idea: "hash" each m-size column ***C*** to a small ***signature h(C)***, such that: $|h(C)| << |C| = m$

- **(1)** ***h(C)*** is small enough that the signature fits in RAM (Fast Memory)
- **(2)** ***sim(C_1, C_2)*** is "*close*" to the *equivalence* of signatures ***h(C_1)*** and ***h(C_2):***

Goal: Find a hash function *h(·)* such that:

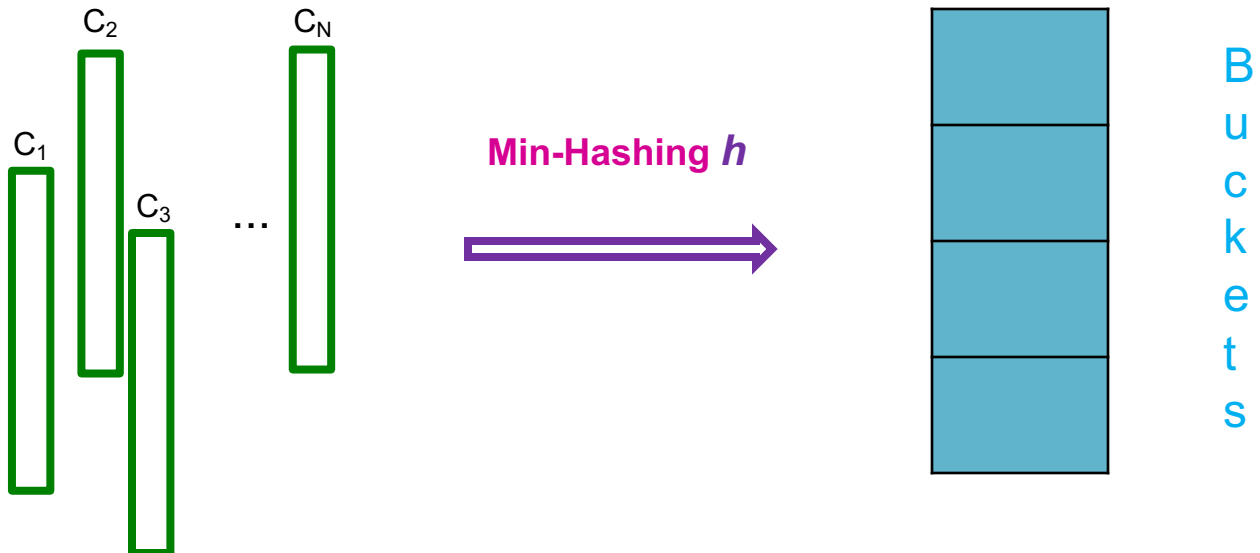
- If ***sim(C_1,C_2)*** is high, then **with high prob.** ***h(C_1) = h(C_2)***
- If ***sim(C_1,C_2)*** is low, then **with high prob.** ***h(C_1) ≠ h(C_2)***

Hashing Columns (Signatures)

Goal: Find a hash function *h(·)* such that:

- If ***sim(C_1,C_2)*** is high, then **with high prob.** ***h(C_1) = h(C_2)***
- If ***sim(C_1,C_2)*** is low, then **with high prob.** ***h(C_1) ≠ h(C_2)***

Intuition: Hash Docs into ***Buckets***. Expect that "most" pairs of near duplicate Docs hash into the same **Bucket**!



Min-Hashing

Goal: Find a hash function $h(\cdot)$ such that:

- If $sim(C_1,C_2)$ is high, then **with high prob.** $h(C_1) = h(C_2)$
- If $sim(C_1,C_2)$ is low, then **with high prob**. $h(C_1) \neq h(C_2)$

Clearly, the hash function depends on the *similarity* metric:

- Not all similarity metrics have a suitable hash function

Good News: There is a suitable hash function for the Jaccard similarity: It is called **Min-Hashing**

Min-Hashing

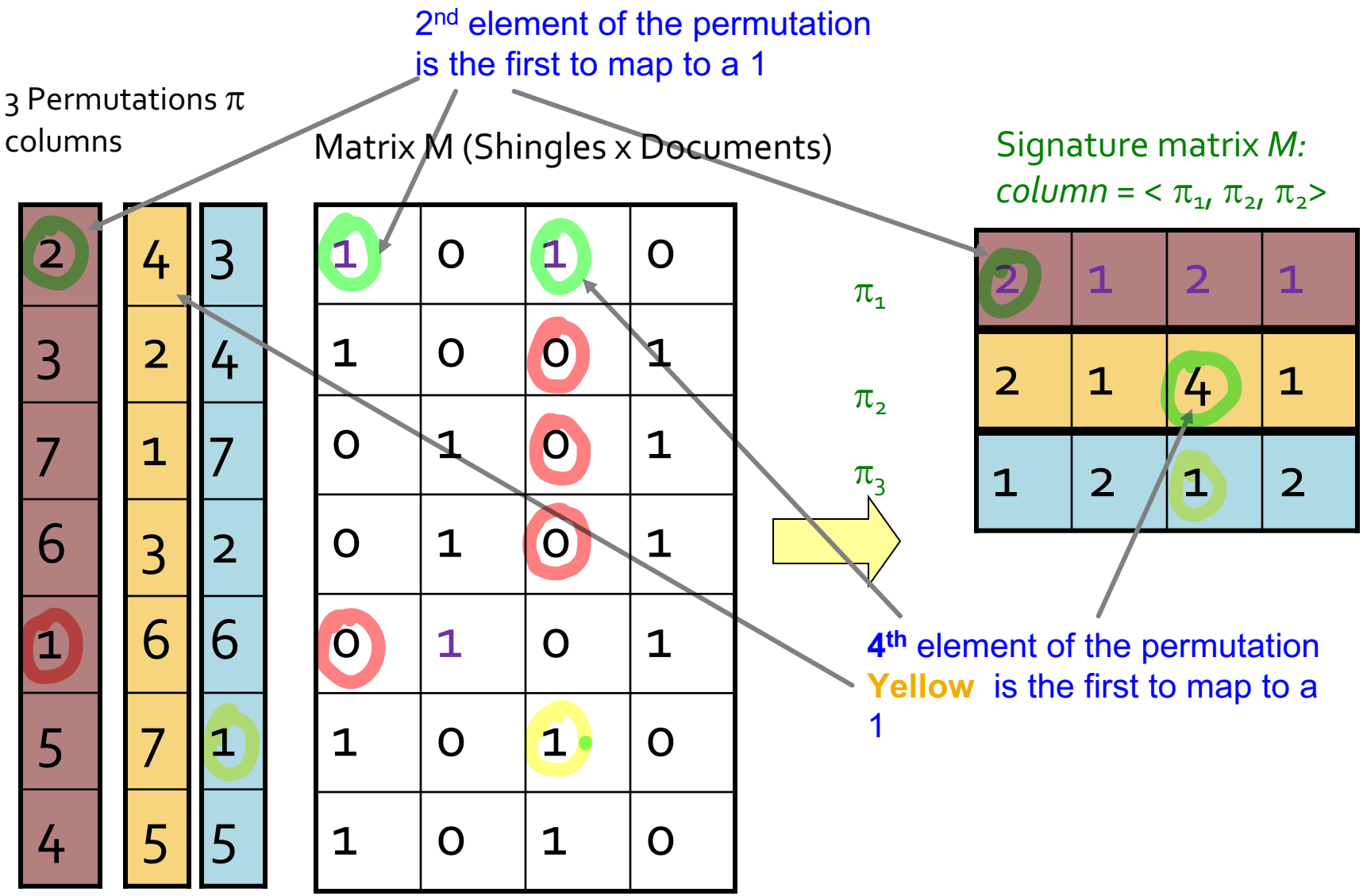
Imagine the *rows* of matrix **M** permuted under ***random permutation*** π

Def. For each column C, define: **Min-Hash function $h_\pi(C)$** ≡ the **index** of the **first** (in the permuted order π) row in which column **C** has value **1** (we named it as **min(π(C))**)

Practical Strategy: Use several (e.g., 100) independent rnd hash functions (that is, rnd permutations) to create a *signature* of **C**

see later....

Min-Hashing Example with N=4, m= 7



MEMORY - COST (MIN-HASH)

$h_\pi(C) \longrightarrow [m] \equiv \{1, 2, \ldots, m\}$

where $|C| \equiv |U|^k \equiv m$

and SPACE $(h_\pi(C)) \equiv \Theta(\lg m)$

$m \xrightarrow{h_\pi} \Theta(\lg m)$

$\sqcup$ m

it is just a row-index of M

The Min-Hash Property

0	0
0	0
1	1
0	0
0	1
1	0

J-SIMILARITY PRESERVING

THM : Fix C_1, C_2. Choose *u.a.r.* a permutation π. Then,

- $\Pr_\pi [h_\pi(C_1) = h_\pi(C_2)] = J.sim(C_1, C_2)$

1/3

Proof.

- Let **X** be a Doc (column of shingles), $y \in X$ is one of its shingle. Then: $\Pr[\pi(y) = \min(\pi(X))] = 1/|X|$ since:
  - It is equally likely that any row $y \in X$ is mapped to the ***min*** index

bit = 1

- Let y be s.t. $\pi(y) = \min(\pi(C_1 \cup C_2))$ π is rnd uniq
- **Then either:** $\pi(y) = \min(\pi(C_1))$ if $y \in C_1$, **or** → UNION
  $\pi(y) = \min(\pi(C_2))$ if $y \in C_2$

One of the two cols had to have 1 at position ***y***

- So the prob. that **both** are **true** is the prob. of $y \in C_1 \cap C_2$
- $\Pr[\min(\pi(C_1)) = \min(\pi(C_2))] = |C_1 \cap C_2| / |C_1 \cup C_2| = sim(C_1, C_2)$

$h(C_1)$ $h(C_2)$

$$= \frac{|C_1 \cap C_2| \cdot 1/|X|}{|C_1 \cup C_2| \cdot 1/|X|}$$

Similarity for Signatures

Given any two columns C_1 and C_2 , **Main THM** → the Prob. that C_1 and C_2 get the same Min-Hash signature, i.e. $h_\pi(C_1) = h_\pi(C_2)$, **equals** the value of **J.*sim*(C_1, C_2)**

Increase the "confidence"!? How?

Simple Idea from *Statistics*: Exploit *concentration* on the *Expected Value*: use multiple, mutually-independent Min-Hash **signatures *SIG*(C) = <** $h_{\pi 1}(C)$, $h_{\pi 2}(C)$,..., $h_{\pi t}(C)$> for some fixed **t >>1** and set:

DEF. *Similarity* Sign-Sim(C_1, C_2) of two (vector) signatures *SIG*(C_1) and ***SIG*(C_2) = the fraction of the chosen** Min-Hash signatures **in which they *agree***

AVERAGE $t \to t\infty$ EXPECTATION

Formally,

Def. binary r.v. **Z = 1 iff "min($\pi(C_1)$) = min($\pi(C_2)$)"** .

Goal. Estimate **Pr[Z=1]**

THM 1

we KNOW Pr[Z=1] III S.SIM(C_1,C_2)

Min-Hashing Example

Permutation π

2	4	3
3	2	4
7	1	7
6	3	2
1	6	6
5	7	1
4	5	5

Input matrix (Shingles x Documents)

1	0	1	0
1	0	0	1
0	1	0	1
0	1	0	1
0	1	0	1
1	0	1	0
1	0	1	0

Signature Matrix *SIG(π,C)*

2	1	2	1
2	1	4	1
1	2	1	2

Similarities: $|C_i \cap C_j| / |C_i \cup C_j|$

	1-3	2-4	1-2	3-4
Col/Col	6/7	0.75	0	0
Sig/Sig	2/3	1.00	0	0

Min-Hash: The Algorithm

Rnd Algorithm ***Doc-Pair Check***

Input: Columns $C_1, C_2 \in \{0,1\}^h$; Confidence Parameter **t**

For j=1 to t

- Choose u.a.r. a raw permutation $\pi_j \in \Pi_m$
- Compute hash $h_{\pi j}(C_1)$ and $h_{\pi j}(C_2)$

Return **Sign-Sim(C_1, C_2) = $|\{j : h_{\pi j}(C_1) = h_{\pi j}(C_2)\}| / t$**

Note: J.*Sim*(C_1, C_2) ≠ Sign-Sim(C_1, C_2) but… Because of the Min-Hash property (Main THM): J.*Sim*(C_1,C_2) ≡ E_π(Sign-Sim(C_1, C_2)), so

Coroll. 1 (of Main THM). Sign-Sim(C_1,C_2) → J.*Sim*(C_1,C_2) as t → ∞

Important Note: since the **t** hash functions are *mut. indep.* → we have *concentration results* on | Sign-Sim(C_1,C_2) - J.*Sim*(C_1,C_2) |

Min-Hash: Space Complexity

Pick t=100 random permutations of the **m rows**

Think of ***SIG*(C)** as a *short* column vector of a signature Matrix

- ***SIG*(i,C) =** the index of the first row that has a 1 in column **C** according to the *i*-th rnd permutation,

$$SIG(i,C) \text{ is an index!} \quad \Theta(\log |C|) = \Theta(\log m))$$

Note: The sketch (signature) ***SIG*(*,C)** of **C** is **small: ~100 bytes (in general: $\Theta(t*\log m)$)!**

Recall: $m = |U|^k$ is the number of *all* possible **k-shingles**

We achieved our goal: **We "compressed" long-bit vectors C into short signatures *SIG*(*,C) (m → t log m)**

Implementation Trick for Min-Hash Signatures

Min-Hash function $h_\pi(C)$ requires to generate several independent rnd permutations π of the raw set U^k ($|U|^k = m$)

Permuting m rows even once is too expensive !

Idea: Replace **t Row Permutations** with **t Row Hash Functions**

Ordering under an ***Hash Function* f** gives an (*almost*) random π

Pick **t = 100 hash** functions ***f_i : [m]→[m]*** ***(there might be few collisions: neglect them!)***

raw index of the first 1 "after" hashing f_i

$\Theta(\lg m)$ bits

Original Binary Matrix M

1	0	1	1	0	0
0	1	1	1	1	0
0	1	0	0	0	0
0	0	0	1	0	0
1	0	0	0	1	1
0	0	0	1	0	1

m raws = k-shingles

N docs = columns

Min-Hash using f_i

Signature Matrix *SIG*(i,C)

f_1						
f_2						
…						
f_t						

N docs = columns

Algorithm for *Signature* Matrix *SIG(i,C)*

FOR each column ***C=1…N*** and (hash-function) **i=1…t** DO
- Initialize the signature matrix ***SIG(i,C) = ∞***

FOR each row **j=1…m** of the original matrix **M** DO
- FOR each column ***C=1…N*** DO
  - IF ***M(j,C) = 1*** THEN (**)
    - FOR each (hash-function) **i=1…t** DO
    - Compute the i-hash of raw **j**: $f_i(j)$
    - IF $f_i(j) < SIG(i,C)$ THEN update $SIG(i,C) \leftarrow f_i(j)$

MIN - Computation

SEE PREVIOUS LESSONS!

Performance Analysis: Θ(mN) iterations

Each iteration requires **t** hash computations $f_i(j)$ *iff* **Condition** (**) holds.

Good News: i) No full computations of rnd permutations π_m ;
ii) Original Matrix M is sparse → (**) is unlike !

Bad News: Hash may generate collisions (no permutation)

How to pick a random hash function h(x)?

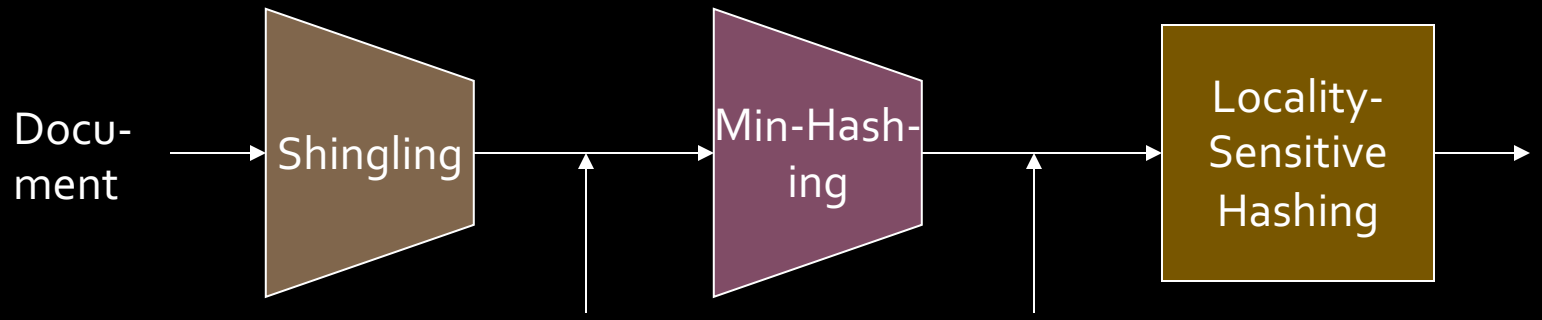
Universal hashing:

$h_{a,b}(x)=((a \cdot x+b) \bmod p) \bmod m$

where:

a,b … random integers

p … prime number (p > m)



Locality Sensitive Hashing

Step 3: ***Locality-Sensitive Hashing:***
Focus on pairs of signatures likely to be from similar documents

Motivation for LSH

Suppose we need to find *near*-duplicates among $N = 10^6$ Docs

Naïvely, we would have to compute pairwise

J. similarities for **every pair of Docs**

$N(N-1)/2 \approx 5*10^{11}$ comparisons !!!

- At 10^5 *secs/day* and 10^6 comparisons/sec, it would take **5 days**
- For $N = 10^7$, it takes more than a year…!

Remark: **Min-Hashing** does not avoid $\Theta(N^2)$ pair *comparisons*. It makes *each* pair comparison (*J. similarity* computations) much more **efficient:**

Original Matrix M(k,C) → **Signature Matrix *SIG*(i,C)**

m raws → **t log m raws**

index size

hash functions

Local Sensitive Hashing (LSH)

Goal: Find Doc pairs with similarity at least ***s*** (for some ***similarity threshold s***, e.g., ***s*** = 0.8)

└ INPUT PARAMETER

LSH – General idea: Use a function ***f(x,y)*** that tells whether ***x*** and ***y*** is a ***candidate*** *Doc Pair:* a pair of Docs whose *similarity* must be evaluated in a deeper way.

General Idea of LSH: Hashing Docs Signature

Define **Candidate Function *f(x,y):***

- **Consider (Min-Hash) Signature Matrix *SIG*(i,C) (i=1…t; C=1…N)**

SIG(i,C)*

2	1	4	1
1	2	1	2
2	1	2	1

NOTE: From now on, we "forget" matrix M, and we only work with matrix SIG(i,C)*

- Hash columns (Docs) of *SIG*(i,C) to *many* Buckets
- Each pair of Docs that hashes into the same Bucket is a candidate pair: check it!

LSH STRATEGY

LSH: Step I (Candidates from Min-Hash Signatures)

Choose a *suitable* **similarity threshold** s ($0 < s < 1$)

CRITERION

DEF. Columns x and y of $SIG(*,*)$ are a **candidate pair** if their Min-Hash signatures $SIG(*,x)$ and $SIG(*,y)$ agree on at least fraction s of their rows:
$|\{i \in [t] : SIG(i,x) = SIG(i,y)\}| \; / \; t \geq s$

Working Assumption: Docs x and y have the same ***similarity*** as their **Signatures** (thanks to **Corollary 1** of **Main THM**)

Note: Similarity of two column **signatures** does not come from *small* differences in some/several of their row values: there must be several/most **identical** values!

index gap $|i-j|$ has NO meaning !!

LSH: Step II (Hash Signatures to Buckets)

Main idea: Hash Columns of ***SIG*(*,*)** to a set **B** of Buckets for several times

Key Fact: (only) **candidate** Column sets are likely to hash to the same Bucket, *with high probability. (Key Property of Hash Functions)*

Hence:

Candidate pairs are those that hash to the same Bucket

Excercise: ***LSH*** overcomes the $\Theta(N^2)$ *barrier*: Why? When?

min-hash
DOCS with SAME INDEX will be hashed to the same BUCKET

Is this a 2nd level of HASHING in LSH

LSH. Step II (Details: Partition *SIG*(*,*) into Bands)

Technical Problem: Hashing the full signature of a Doc into a one Bucket may generate several false positive and false negative.

A Good Solution.

Partition each doc signature, i.e. each column of ***SIG*(*,*)** into a set of ***Bands*** (i.e. subsets of *adjacent* raws):

Divide matrix ***SIG*(*,*)** into ***b*** Bands, each of ***r*** rows

For each **Band**, hash its raw-portion of each column to a hash table with ***c*** Buckets

LSH

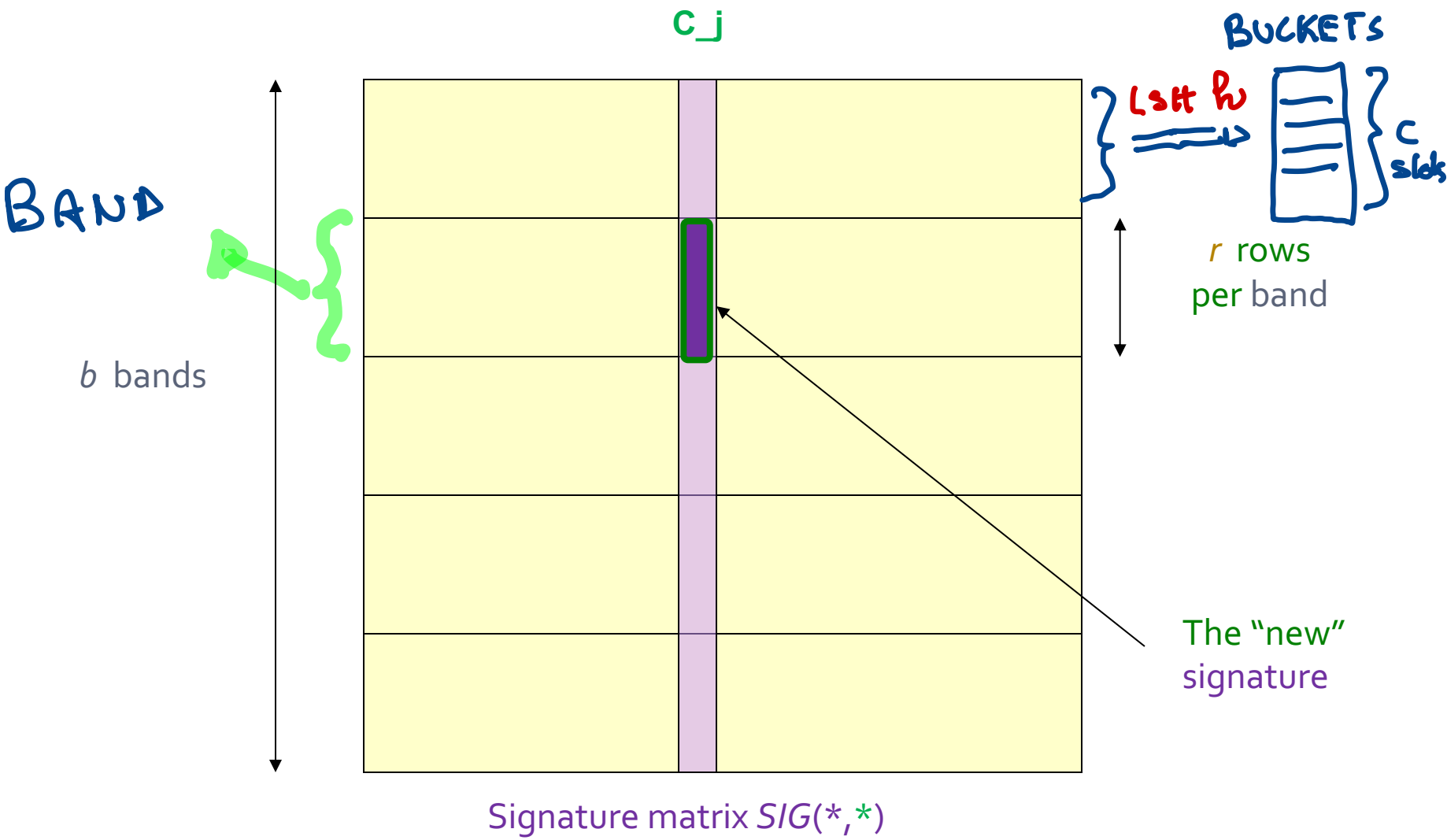
- Make ***c*** large enough (*w.r.t.* ***r***) to *whp* avoid **collisions** of different signatures

Candidate Criterium: Let's **candidate** column *pairs* those that hash to the same Bucket for **at least one Band**

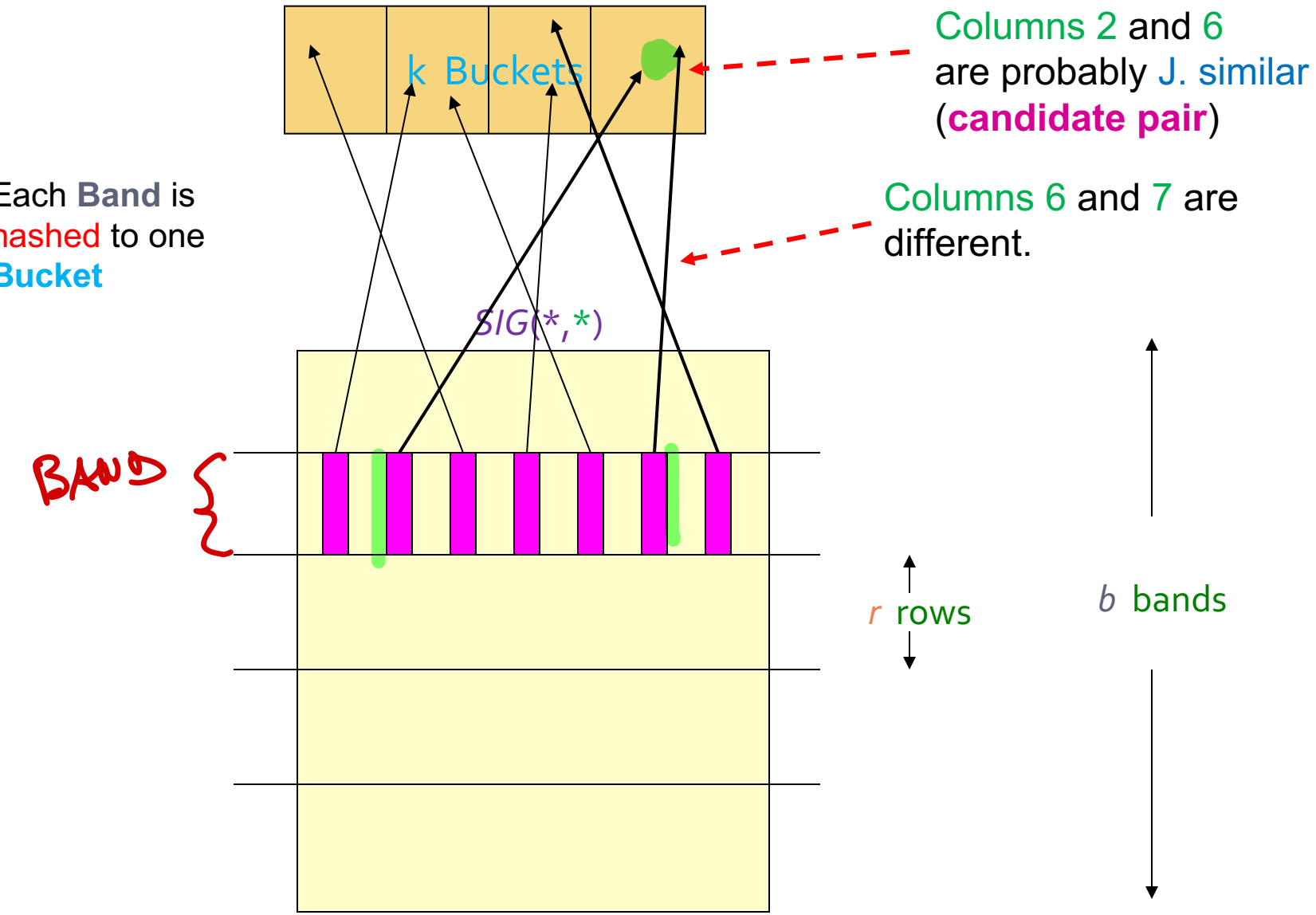
Practical Step: Tune ***b, r (and thus t= total n. of raws)*** to catch most similar *pairs*, but few non-similar *pairs*

Partition *SIG*(*,*) into *b* Bands

2	1	4	1
1	2	1	2
2	1	2	1



Hashing Bands



LSH: Simplifying Assumptions

There are **enough Buckets** (**c** >> **r**) that Columns are unlikely to hash to the same Bucket *unless* they are **identical** in a particular **Band**

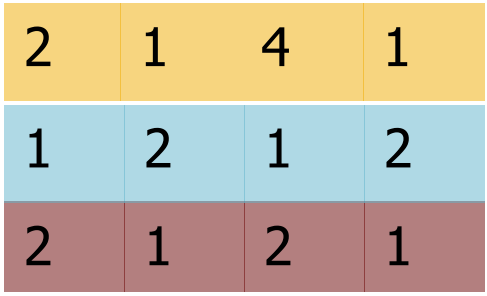
We assume that: "**same Bucket**" means "**identical in that Band**" (Perfect Hashing)

(real app = universal hashing)

Assumptions above needed only to simplify analysis, not for correctness of algorithm

Recall: **candidate** column pairs = those that hash to the same Bucket for **at least one Band**

Example of Bands



Assume the following case:

Suppose 100,000 columns of ***SIG*(*,*)** (100k Docs)

Signatures of 100 integers (rows)

Therefore, signatures take 40 Mb

Choose ***b*** = 20 bands, each of ***r*** = 5 integers.

Goal: Find **pairs** of Docs that are at least ***s*** = *0.8* **similar**

Case I: C_1, C_2 are 80% similar

(1) **Find pairs of $\geq s$=0.8 similarity**, set b=20, r=5

Assume: J.sim(C_1, C_2) = 0.8

- Since sim(C_1, C_2) $\geq$ **s**, we want C_1, C_2 to be a **candidate pair**: We want them to hash to **at least one common Bucket** (i.e. **at least one Band** is identical)

Fact 1. Probability C_1, C_2 get identical in one particular **Band is**

(i) $(0.8)^5 = 0.328$ {each of the 5 row pair must match}

Fact 2. Probability C_1, C_2 get ***not*** identical in *all* of the 20 **Bands** is

(ii) $(1-0.328)^{20} = 0.00035$ {each of the 20 Bands does not satisfy **(i)**} (false negative)

Confidence Rate: about 1/3000th of the 80%-**similar** column pairs are **false negatives** (we miss them) → **We would find 99.965% pairs of truly similar Docs**

Case II: C_1, C_2 are 30% similar

(ii)

Find pairs of $\geq$ *s*=0.8 similarity, set **b**=20, **r**=5

Assume: J.sim(C_1, C_2) = 0.3

- Since J.sim(C_1, C_2) << **s** we want C_1, C_2 to hash to **NO common** Buckets (*all* **Bands** should be different)

Fact 1. Probability C_1, C_2 get identical in one particular **Band** is $(0.3)^5 = 0.00243$ r = 5

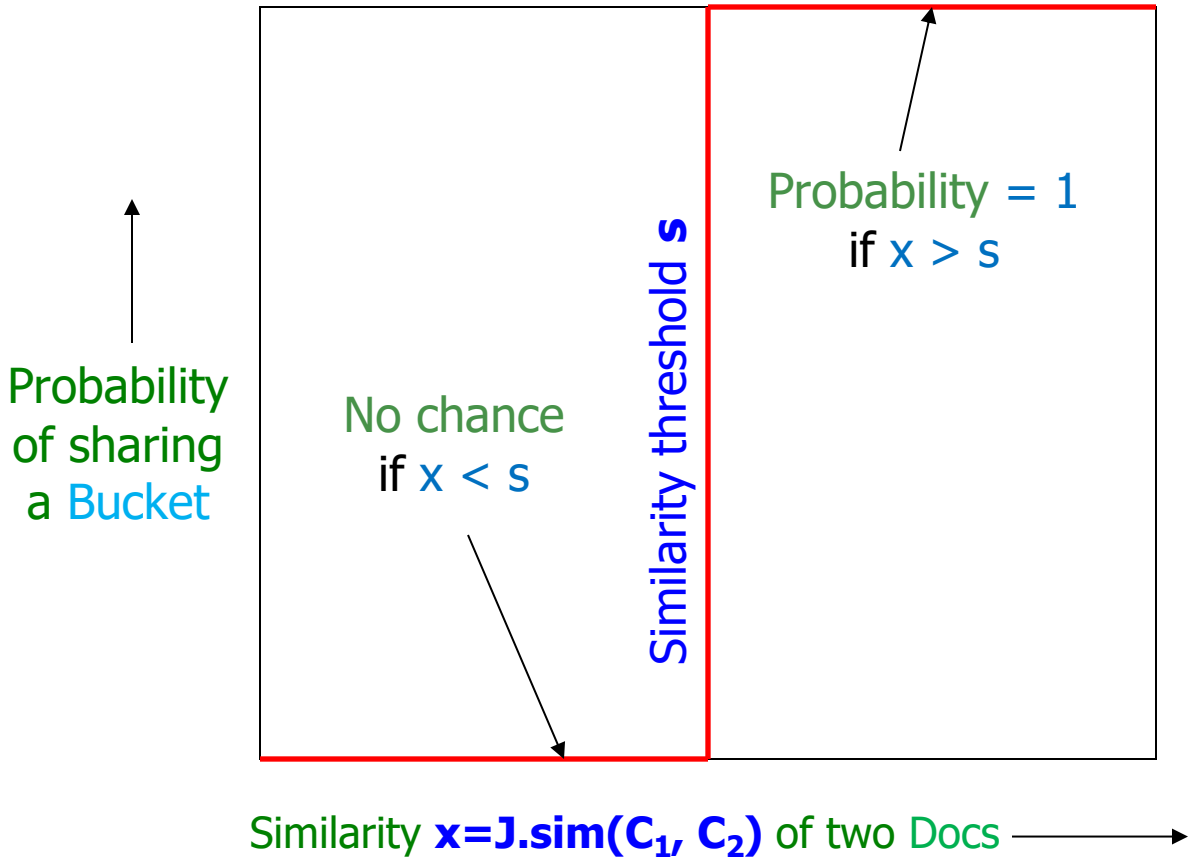
Fact 2. Probability C_1, C_2 get ***not***-identical in *all* of the 20 **Bands** is $1 - (1 - 0.00243)^{20} = 0.0474$ (false positive) → BAD EVENT

Confidence Rate: about 4.74% pairs of Docs with J.sim 0.3% end up becoming **candidate pairs**: They are **false positives.**

Since we will have to examine them (they are candidate pairs) but then it will turn out their similarity is below threshold **s**

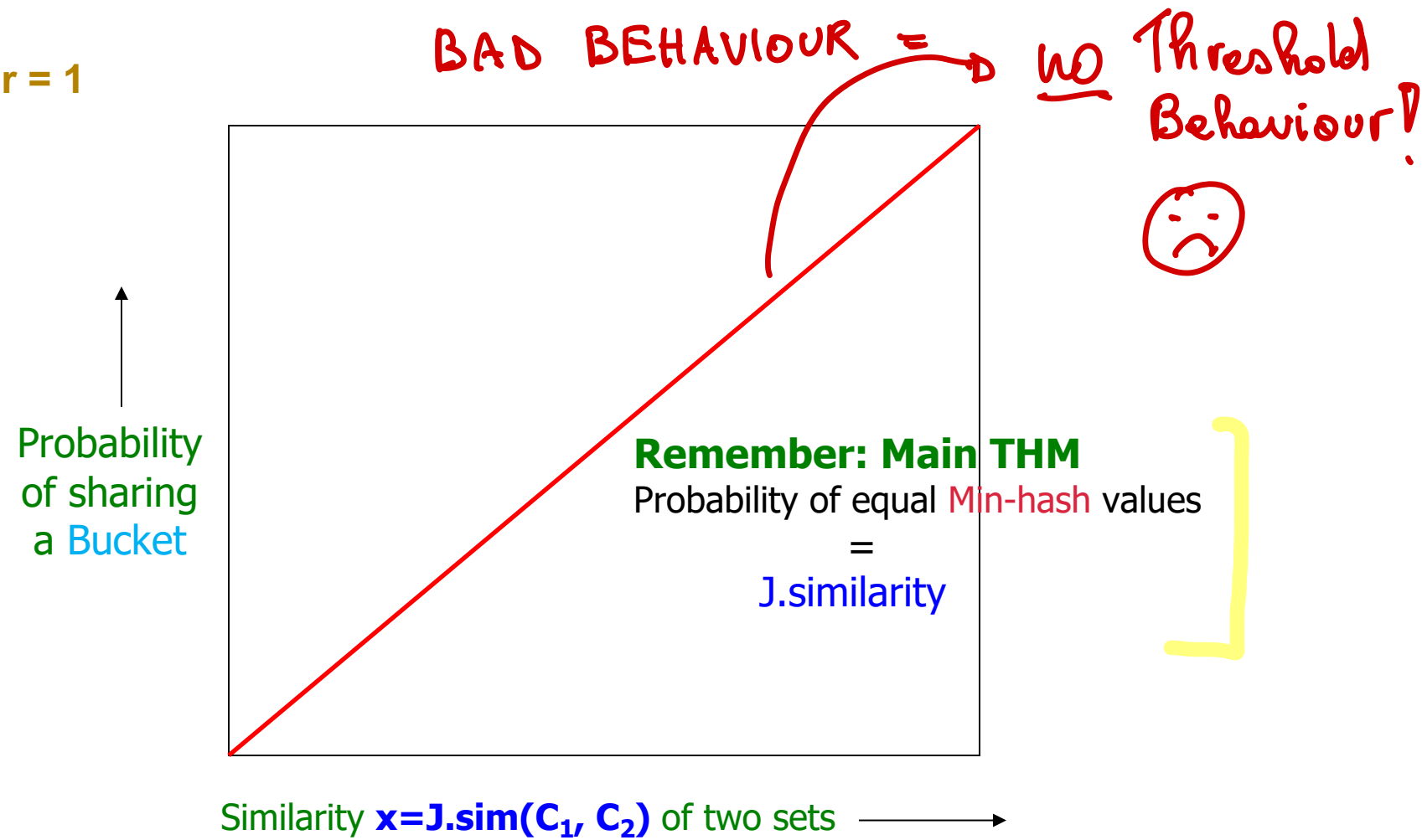
Analysis of LSH – Ideal Performance

x = J.sim(C_1, C_2); s = target similarity threshold



What 1 Band of 1 Row Gives You

Case **r = 1**



Using b Bands, *r* rows/band

suppose Docs C_1 and C_2 have J.similarity ***x.***

Pick any Band (with *r* rows):

$\Pr\{$ *All* rows in Band are *equal* $\} = x^r$

$\Pr\{\exists$ row in Band *unequal* $\} = 1 - x^r$

$\Pr\{\nexists$ Band identical $\} = (1 - x^r)^b$

$\Pr\{\exists$ Band identical $\} = 1 - (1 - x^r)^b$

Increases with **b**

Decreases with **r**

LSH Involves a Tradeoff

For a fixed threshold s, tune:

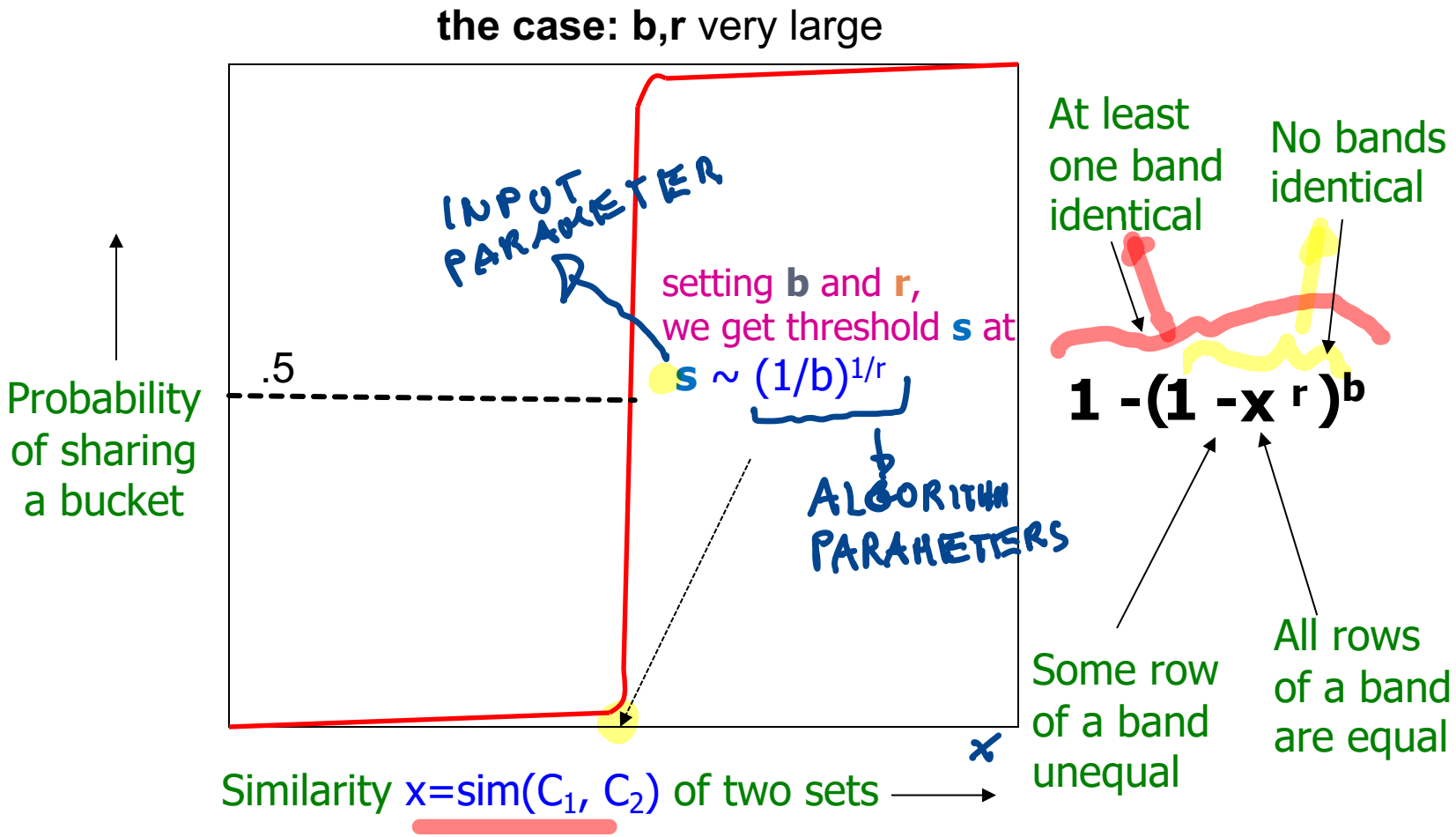
- The number **t** of Min-Hashes (rows of ***SIG*(*,*)**)
- The number of **Bands *b***, and, hence:
- The number of rows ***r*** per band

to balance false positives/negatives

(a) and (b)

Example: If we decreased to **15 Bands** of **5** rows, the number of false positives (Case II before) would *go down*, but the number of false negatives (Case I before) would *go up*

What *b* Bands of *r* Rows Gives You



LSH: The Algorithmic Goal

Task: Given the **t** Signature columns for each Doc (i.e. Matrix ***SIG*(*,*)**) and the *sim threshold* **s** (with $0 < s \leq 1$) , **find** all the Doc pairs having: Signature Sim ≥ **s**

***LSH* Solution**: Apply the **LSH** **Band** technique to ***SIG*(*,*)**, by setting parameters **b** and **r** such that: $b \cdot r \leq t$ and $(1/b)^{1/r} \approx s$

Output: *All* (candidate) pairs of Docs having the same signature in *all* the **r** raws of *at least one* **Band**

Possible Parameter Tuning:

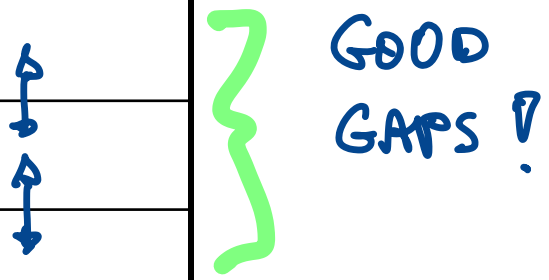
- increasing **b** → decreases the number of false negatives
- increasing **r** → decreases the number of false positives

Example: $b = 20$; $r = 5$

J.Sim(C_1,C_2) = x ; b = 20; r = 5

Prob. that at least 1 Band is identical:

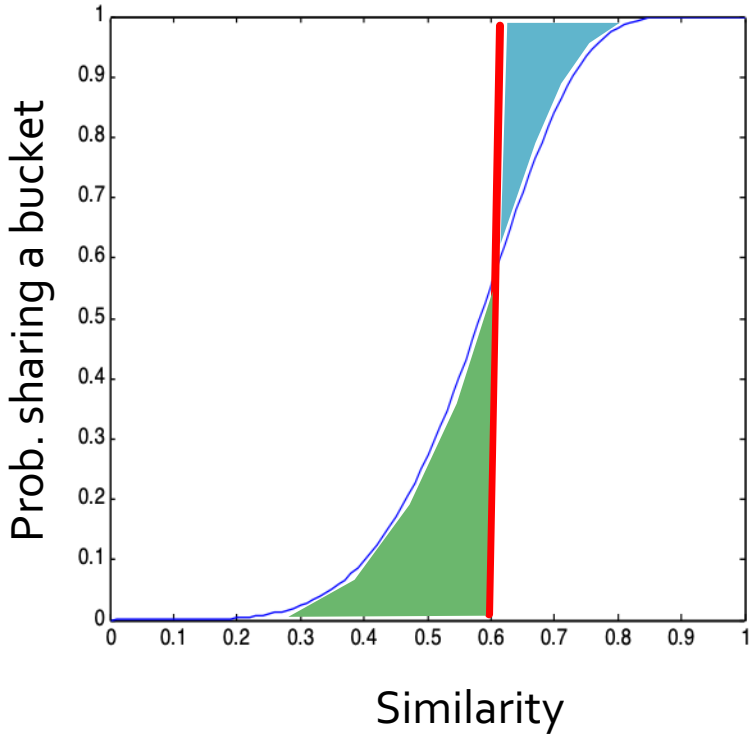
x	$1-(1-x^r)^b$
.2	.006
.3	.047
.4	.186
.5	.470
.6	.802
.7	.975
.8	.9996



Picking *r* and *b*: The S-curve

Picking *r* and *b* to get the *best S-curve*

- **t= 50** hash-functions (**r**=5, **b**=10)



Blue area: False Negative rate
Green area: False Positive rate

LSH Summary

Tune ***b*** ***and*** ***r*** to get almost all pairs with similar signatures, but eliminate most pairs that do not have similar signatures: *most-balanced* choice is: **s** $\sim (1/b)^{1/r}$

Check in main memory that **candidate pairs** really do have **similar signatures** (i.e. same values in several raws)

Optional: In another pass through data, check that the selected candidate pairs really represent similar Docs

↳ avoid false Pos

Summary: 3 Steps

Shingling: Convert Docs to **Sets**

- We used **hashing** to assign each **k-shingle** an ***ID (one raw of the Big Matrix M)***

- J. SIM. for SUBSETS

Min-Hashing: Convert large sets to short signatures, while preserving *similarity*

- We used **similarity-preserving hashing** to generate signatures with property $\mathbf{Pr[}h_\pi(C_1) = h_\pi(C_2)\mathbf{]} = sim(C_1, C_2)$
- We used **hashing** to get around generating random ***permutations***

Locality-Sensitive Hashing: Focus on *pairs* of signatures likely to be from *similar* docs

- We used **hashing** to find **candidate pairs** of similarity $\geq$ **s**

[the BAND Technique]

Summary: 3 Steps

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