

Int 201: Decision Computation and Language

Tutorial 11 Solution

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Question 1. Show if $A \leq_m B$ and B is co-Turing-recognizable, then A is co-Turing-recognizable.

Solution 1. Proof: Let $TM_{\overline{B}}$ be the TM that recognizes \overline{B} , and the map be f . $TM_{\overline{B}} =$ “On input w :

1. Compute $f(w)$.
2. Run $TM_{\overline{B}}$ on $f(w)$ and output whatever $TM_{\overline{B}}$ outputs.”

Clearly, if $w \notin A$, then $f(w) \notin B$ by the mapping reduction definition. Thus $TM_{\overline{A}}$ accepts w , as $TM_{\overline{B}}$ accepts $f(w)$.

If $w \in A$, $f(w) \in B$ and $TM_{\overline{B}}$ either rejects or loops, and consequently $TM_{\overline{A}}$ either rejects or loops.

Question 2. Show L_{tm} is not mapping reducible to $E_{\text{TM}} = \{\langle M \rangle \mid M \text{ is TM and } L(M) = \emptyset\}$ Hint: Is E_{TM} co-Turing recognizable?

Solution 2. Proof: E_{TM} is co-recognizable as we can find an acceptance by search. If L_{tm} is mapping reducible to E_{TM} , by q1, we have L_{tm} co-recognizable. As L_{tm} is recognizable, it will be decidable. However, L_{tm} is not decidable. So, it is not mapping reducible to E_{TM} .

Question 3. Show $EQ_{\text{TM}} = \{\langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$ is undecidable. Hint: reduction from the emptiness problem E_{TM} .

Solution 3. Proof by contradiction: Suppose we have a decider EQ for EQ_{TM} , we reduce E_{TM} to EQ_{TM} . Let TM_{\emptyset} be a Turing machine that accepts no string.

$E =$ “On input $\langle M \rangle$: Run EQ on $(\langle M \rangle, \langle TM_\emptyset \rangle)$ and output whatever TM_B outputs.”

We claim E is a decider for E_{TM} , as EQ_{TM} decides $L(M) = \emptyset$. As we know E_{TM} is undecidable, we have a contradiction. So, EQ_{TM} cannot be decidable.

Question 4. Show the two property conditions for the Rice’s theorem are equivalent.

1. for any two TMs M_1 and M_2 with $L(M_1) = L(M_2)$, $\langle M_1 \rangle \in P \iff \langle M_2 \rangle \in P$.
2. $\exists p : \{L(M) | M \text{ is a TM}\} \rightarrow \{0, 1\}$ such that $\langle M \rangle \in P \iff p(L(M)) = 1$

Solution 4. To show $1 \implies 2$

We construct the p function as:

$$p(L) = \begin{cases} 1, & \text{if } \exists M \text{ such that } L(M) = L \text{ and } \langle M \rangle \in P \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

Now we need to show $\langle M \rangle \in P \iff p(L(M)) = 1$.

If $\langle M \rangle \in P$, clearly there exists M itself such that $L(M) = L(M)$ and $\langle M \rangle \in P$. So, we have $p(L(M)) = 1$.

If $p(L(M)) = 1$, then there exists M' such that $L(M') = L(M)$ and $\langle M' \rangle \in P$. By the condition 1, we have $\langle M \rangle \in P$.

To show $2 \implies 1$

We have the existence of p function and we consider M_1 and M_2 with $L(M_1) = L(M_2)$.

First, if $\langle M_1 \rangle \in P$, then by the condition 2, we have $p(L(M_1)) = 1$. However, $p(L(M_2)) = p(L(M_1)) = 1$ as $L(M_1) = L(M_2)$. Apply the condition 2 again, we have $\langle M_2 \rangle \in P$.

By symmetry, we also have $\langle M_2 \rangle \in P \implies \langle M_1 \rangle \in P$.

Question 5. Show $FINITE_{TM} = \{\langle M \rangle | M \text{ is a TM and } \exists n \in \mathcal{N}, |L(M)| = n\}$ is undecidable by using the Rice’s theorem.

Solution 5. We need to check the three conditions of the Rice's theorem.

1. A Turing machine that only accepts empty tape, has $L(M)=1$, therefore it is in $\text{FINITE}_{\text{TM}}$ and this checks the non-emptiness.
2. A Turing machine that accepts all strings have $L(M) = \infty$, therefore it is not in $\text{FINITE}_{\text{TM}}$ and this checks the proper subset condition.
3. For any two TMs M_1 and M_2 with $L(M_1) = L(M_2)$, if $\langle M_1 \rangle \in P$, then $\exists n \in \mathcal{N}, |L(M_1)| = n$. Therefore, $|L(M_2)| = |L(M_1)| = n$, and $\langle M_2 \rangle \in P$. By symmetry, we have the only if part too. This checks the language property condition.

In all, we have shown that $\text{FINITE}_{\text{TM}}$ is a non-trivial property of TM descriptions. Therefore, by the Rice's theorem it is undecidable.