INT201 Decision, Computation and Language

Lecture 7 – Context-Free Languages (2)

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Assistant Professor-Dr Chunchuan Lyu

- Graduated from The University of Edinburgh and XJTLU

Studied computational semantics but moved to unsupervised reinforcement learning (what an agent should do if no moral gold standard is given?)

- Office hour: 15:00-17:00 Wednesday at SD543 (or by appointment)
- Contact: chunchuan.lyu@xjtlu.edu.cn

Overall Study Tips

- Theory of Computation in 12 Hours by Easy Theory Youtuber

  Really clear explanation
- Theory of Computation 2020 by Michael Sipser MIT OCW

  We are following closely
- The Nature of Computation

  Good complementary book
- ND-CSE-30151 by Professor Chiang, University of Notre Dame

  python notebook that demonstrates automata

Please come to office hour, if you are having difficulty or question about anything.

Overview for the second half

- We will be climbing the ladder of what is computationally possible. What kind of languages/problems can be recognized by what kind of machine?

- After the end of this module, you will know that there are problems can not be solved by any reasonable computing machine.

- You will be writing proofs.

Recap

- Regular languages are context-free
- Every context-free grammar has a Chomsky Normal Form

Today

- Closure property of context-free grammar
- Syntactic parsing (*optional)
- Pushdown Automata

Noam Chomsky 1928-now

An American professor, father of modern linguistics

- Transformational Analysis (1955)
- Syntactic Structures (1957)
- Minimalist program (1995)

What is language?

Why does it have the properties it has?

Formal Basis of a Language Universal (2021 Miloš Stanojević, Mark Steedman)

Noam Chomsky 1928-now

A public intellectual

- Manufacturing Consent (1988 with Edward S. Herman)
- On Palestine (2015 with Ilan Pappé)
- Consequences of Capitalism (2021 with Marv Waterstone)

"one of the most notable contemporary champions of the people“

"pathological hatred of his own country"

Noam Chomsky 1928-now

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- Manufacturing Consent (1988 with Edward S. Herman)
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Israel responsible for four genocidal acts in Gaza, inquiry chair tells General Assembly

© UNRWA | Destruction in northern Gaza

Gaza, a city a **quarter** the size of **SIP**, was subjected to a minimum of **five** times more tons of explosives than fell on **London** throughout the second world war.

Closure Properties of Context Free Language

Theorem: If $L_1$and L_2 are context-free languages, their union $L_1 \cup L_2$ is also context free.

Example:

$$L_1 = \{a^n b^n c^m | m \geq 0, n \geq 0\}$$

$$L_2 = \{a^n b^m c^m | m \geq 0, n \geq 0\}$$

$$L_3 = L_1 \cup L_2 = \{a^i b^j c^k | i \geq 0, j \geq 0, k \geq 0, i = j \; or \; j = k\}$$

Proof idea:

For $L_1$and L_2 , there exists corresponding context free grammars $G_1 = (V_1, \Sigma_1, R_1, S_1)$ and $G_2 = (V_2, \Sigma_2, R_2, S_2)$. Let $G_3 = (S \cup V_1 \cup V_2, \Sigma_1 \cup \Sigma_2, , R_1 \cup R_2 \; \cup \{S \rightarrow S_1 | S_2\}, S)$, clearly L(G_3) = $L_1 \cup L_2$.

Closure Properties of Context Free Language

Theorem: If $L_1$and L_2 are context-free languages, their concatenation $L_1 \circ L_2$ is also context free.

Example:

$$L_1 = \{a^n | n \geq 0\}$$

$$L_2 = \{b^n | n \geq 0\}$$

$$L_3 = L_1 \circ L_2 = \{a^i b^j | i \geq 0, j \geq 0\}$$

Proof idea:

For $L_1$and L_2 , there exists corresponding context free grammars $G_1 = (V_1, \Sigma_1, R_1, S_1)$ and $G_2 = (V_2, \Sigma_2, R_2, S_2)$. Let $G_3 = (S \cup V_1 \cup V_2, \Sigma_1 \cup \Sigma_2, , R_1 \cup R_2 \ \cup \{S \rightarrow S_1 S_2\}, S)$, clearly L(G_3) = $L_1 \circ L_2$.

Closure Properties of Context Free Language

Theorem: If $L_1$is context-free languages, their Kleene closure L_1^* is also context free.

Example:

$$L_1 = \{a^n b^n | n \geq 0\}$$

$$L_2 = L_1^* = \{(a^{n_k} b^{n_k})^k \, | n_k \geq 0, k \geq 0\}$$

Proof idea:
For L_1 , there exists corresponding context free grammars $G_1 = (V_1, \Sigma_1, R_1, S_1)$. Let $G_2 = (S \cup V_1, \Sigma_1, , R_1 \cup \{S \to S_1 S | \epsilon\}, S)$, clearly L(G_2) =L_1^*.

Parsing Natural Language with Context-Free Grammar

Given CFG $G = (V, \Sigma, R, S)$

Variables $V = \{S, NP, VP, Det, Nominal, Noun, PP, Preposition, Verb\}$

Terminals $\Sigma = \{$The, spy, saw, cop, with, a , telescope$\}$

Rules:

Grammar

- S → NP VP
- NP → Det Nominal
- Nominal → Noun | Nominal PP
- VP → VP PP | Verb NP
- PP → Preposition NP

Lexicon

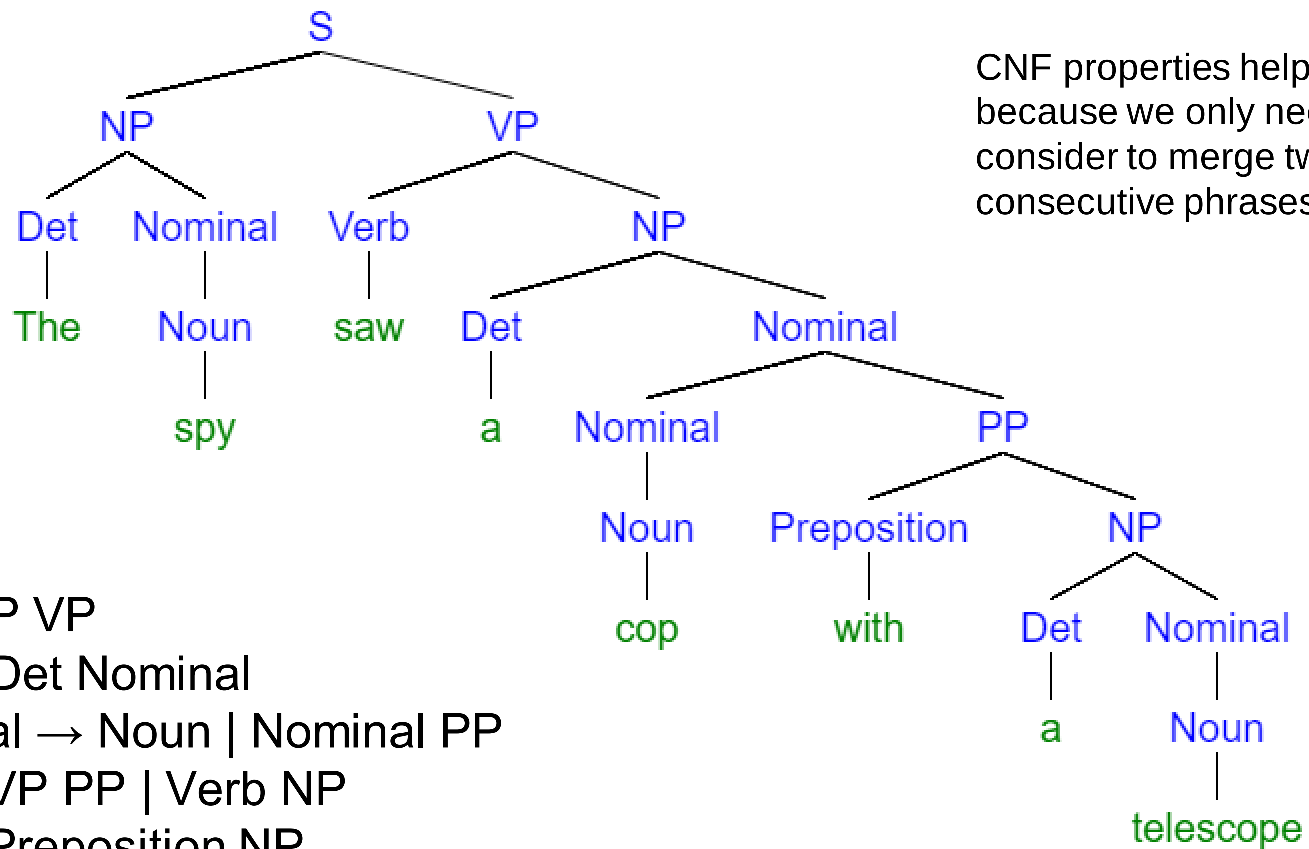
- Det → The|a
- Noun → spy|cop|telescope
- Verb → saw
- Preposition → with

Is this CNF?

How to generate

The spy saw a cop with a telescope

Parsing Natural Language with Context-Free Grammar

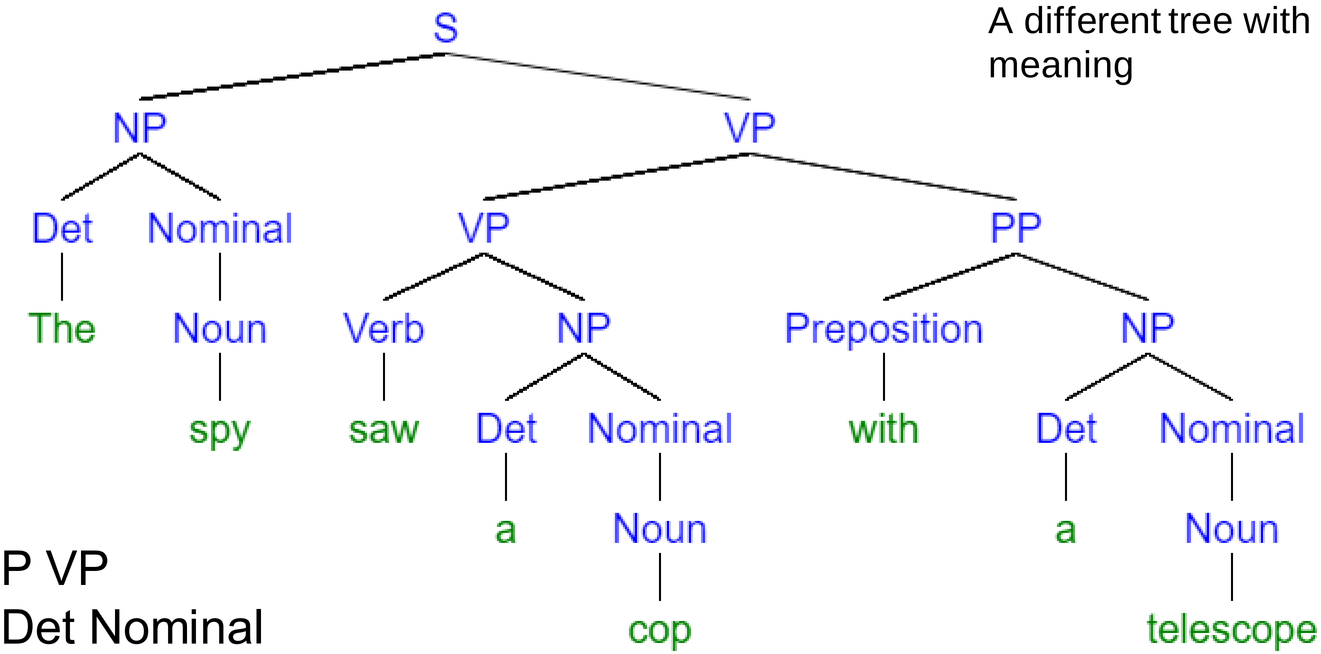


CNF properties help, because we only need to consider to merge two consecutive phrases

Rules:

S → NP VP
NP → Det Nominal
Nominal → Noun | Nominal PP
VP → VP PP | Verb NP
PP → Preposition NP

Parsing Natural Language with Context-Free Grammar



A different tree with different meaning

Rules:

S → NP VP

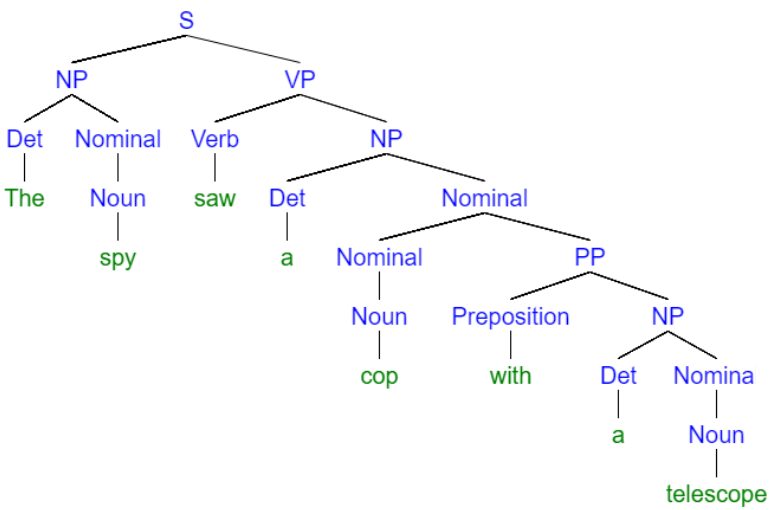
NP → Det Nominal

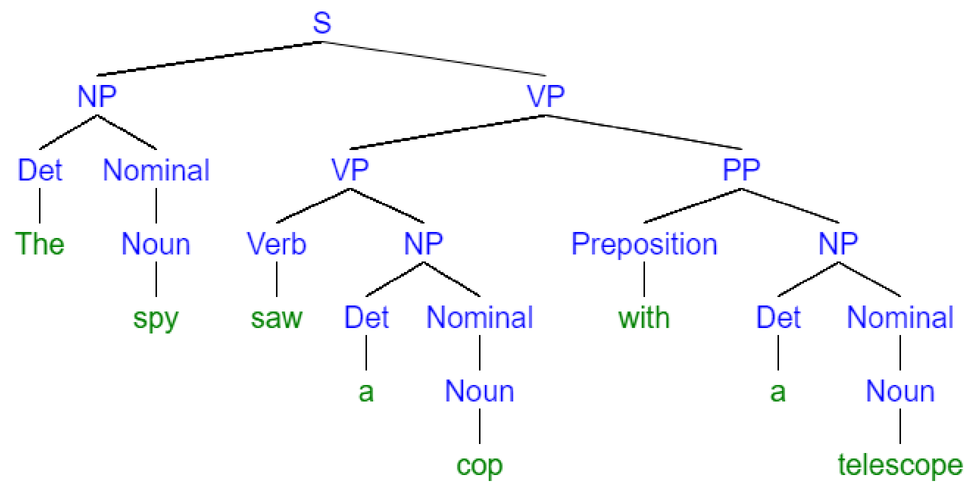
Nominal → Noun | Nominal PP

VP → VP PP | Verb NP

PP → Preposition NP

Parsing Natural Language with Context-Free Grammar





Different derivation corresponds to different meaning.

Liu, Alisa et al. "We're Afraid Language Models Aren't Modeling Ambiguity." *ArXiv* abs/2304.14399 (2023): n. pag.

Pushdown Automata (PDAs)

The class of languages that can be accepted by pushdown automata is exactly the class of context-free languages (finite automata are for regular languages).

- The input for a pushdown automaton is a string w in Σ^*.
- PDA accepts or doesn't accept w.
- Different from finite automata, PDAs have a stack.
- Stack have 2 different operations:

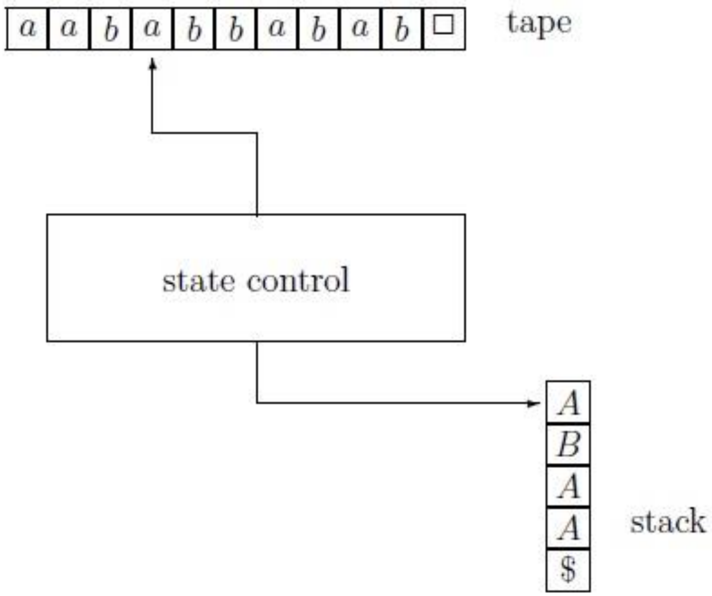
(1) push – adds item to top of stack

(2) pop – removes item from top of stack

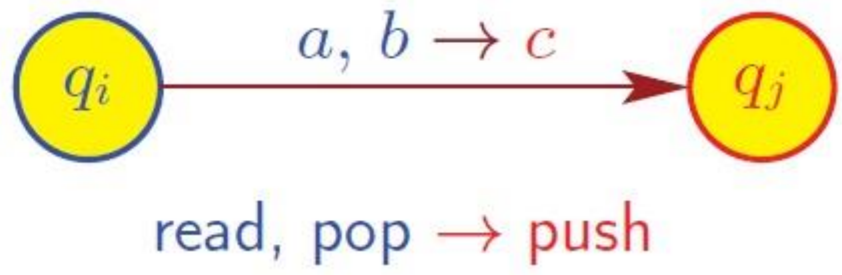
Pushdown Automata (PDAs)

A PDA consists of: a tape, a stack and a state control

- **Tape**: divided into cells that store symbols belonging to $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$.
- **Tape head**: move along the tape, one cell to the right per move.
- **Stack**: containing symbols from a finite set Γ, called the stack alphabet. This set contains a special symbol $\$$ (often mark bottom of stack).
- **Stack head**: reads the top symbol of the stack. This head can also pop the top symbol, and it can push symbols of Γ onto the stack.
- **State control**: can be in any one of a finite number of states. The set of states is denoted by Q. The set Q contains one special state q, called the start state.



PDA Transition



If PDA

- in state q_i
- reads $a \in \Sigma_\varepsilon$
- pops $b \in \Gamma_\varepsilon$ off the stack

If $a = \varepsilon$, then no input symbol is read.

If $b = \varepsilon$, then nothing is popped off stack.

then PDA

- moves to state q_j
- push $c \in \Gamma_\varepsilon$ onto top of stack

If $c = \varepsilon$, then b is popped from stack.

PDA Definition

Definition

A **pushdown automaton** is a 6-tuple $M = (Q, \Sigma, \Gamma, \delta, q, F)$:

- Q is finite set of states
- Σ is (finite) input (tape) alphabet
- Γ is (finite) stack alphabet
- δ is the transition function: $Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow P(Q \times \Gamma_\varepsilon)$
- q is start state, $q \in Q$
- F is set of accept states, $F \subseteq Q$, **PDA accepts as long as it is in F regardless of the stack.**

Let $r, r' \in Q$, $a \in \Sigma^*$ and $b, c \in \Gamma^*$

$$\delta(r, a, b) = \{ (r', c) \}$$

In state r, PDA reads a on the tape and pop b from the stack, move to state r' and push c to the stack. The tape head moves to the right.

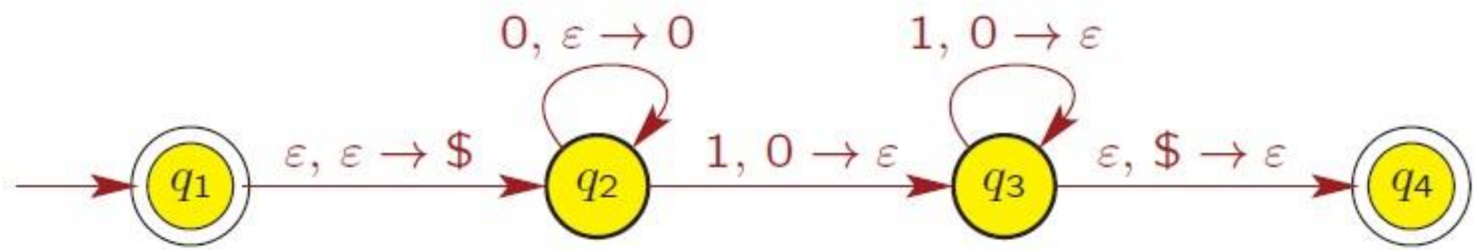
Example

Given a PDA $M = (Q, \Sigma, \Gamma, \delta, q_1, F)$

- $Q = \{q_1, q_2, q_3, q_4\}$
- $\Sigma = \{0, 1\}$
- $\Gamma = \{0, \$\}$
- q_1 is start state
- $F = \{q_1 , q_4\}$
- $\delta: Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow P(Q \times \Gamma_\varepsilon)$

Input:	0			1			ε		
Stack:	0	\$	ε	0	\$	ε	0	\$	ε
q_1									$\{(q_2, \$)\}$
q_2			$\{(q_2, 0)\}$	$\{(q_3, \varepsilon)\}$					
q_3				$\{(q_3, \varepsilon)\}$				$\{(q_4, \varepsilon)\}$	
q_4									

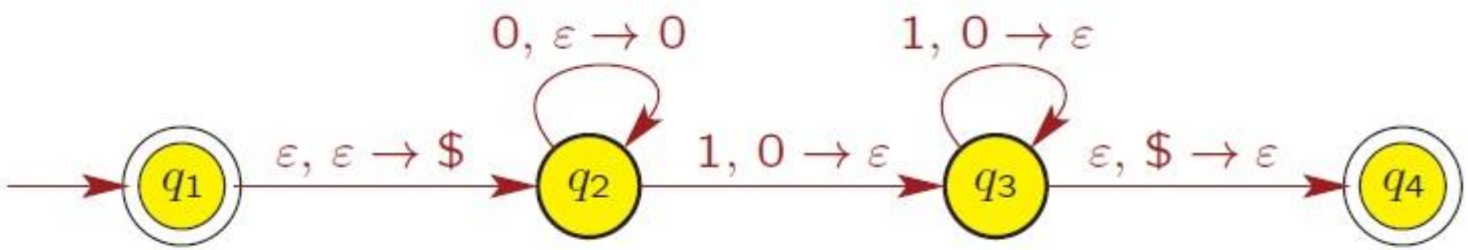
Example

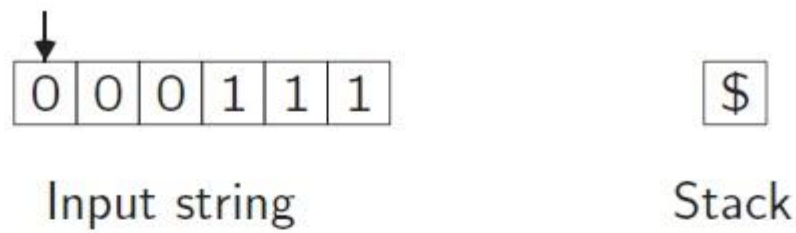


Process string 000111

- Start in start state q_1 with stack empty.
- No input symbols read so far.
- Next go to state q_2
  - reading nothing, popping nothing, and pushing \$ on stack.

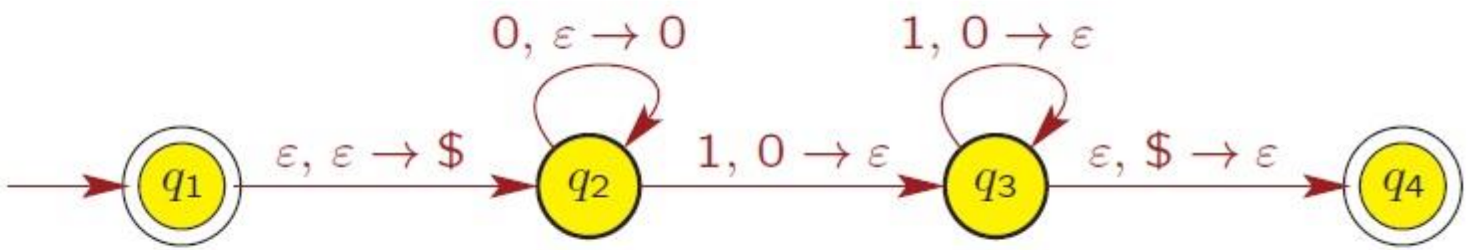
Example





- Next return to state q_2
  - reading input symbol 0
  - popping nothing from stack
  - pushing 0 on stack.

Example



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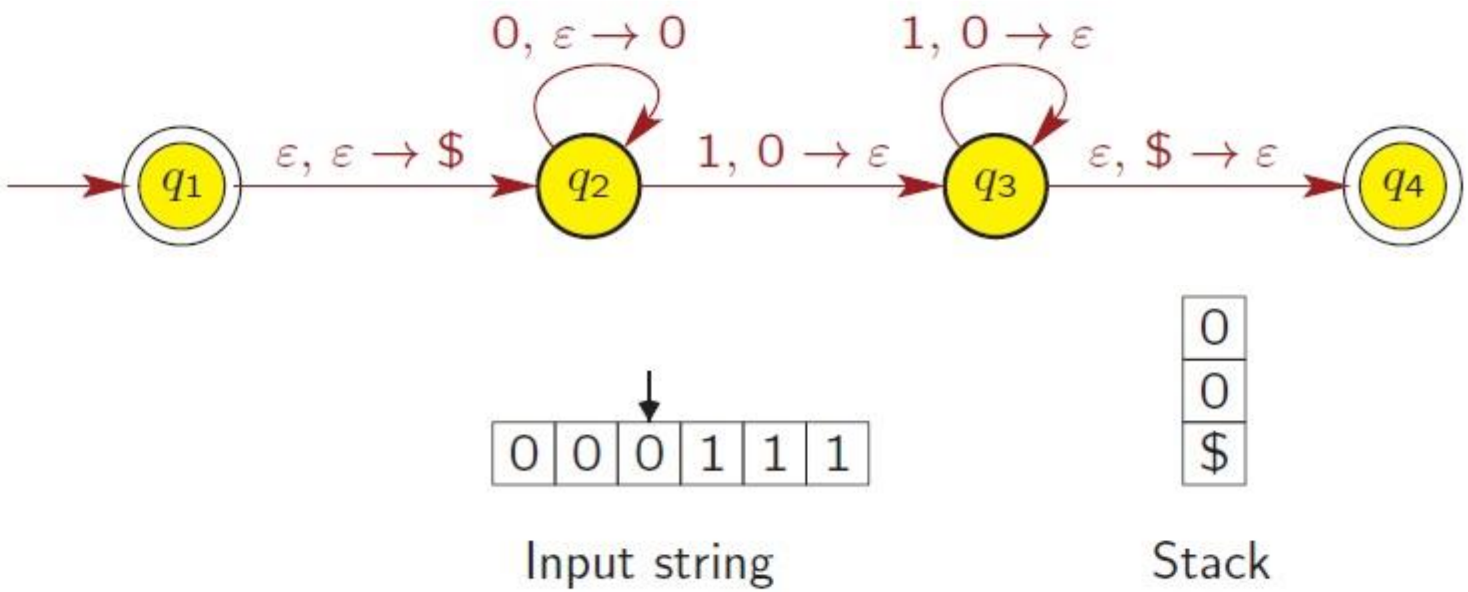
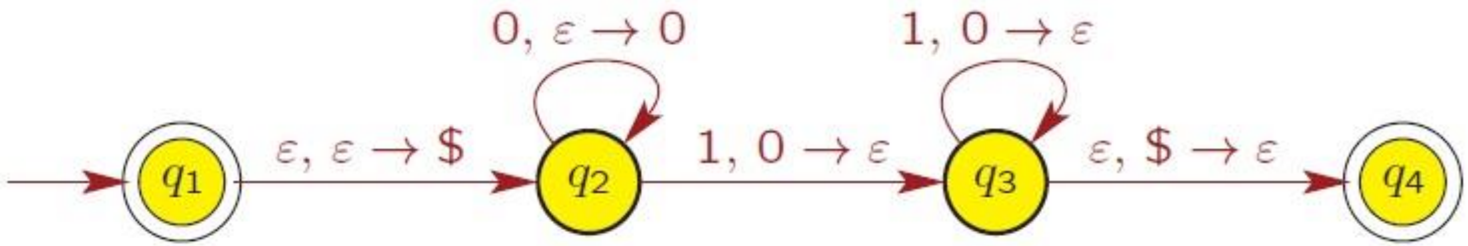
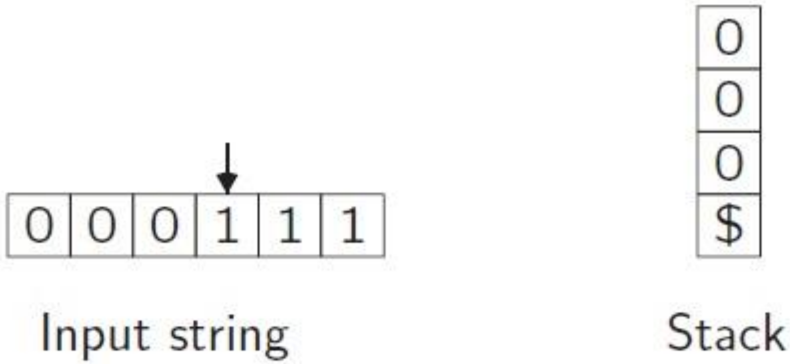


- Next return to state q_2
  - reading input symbol 0
  - popping nothing from stack
  - pushing 0 on stack.

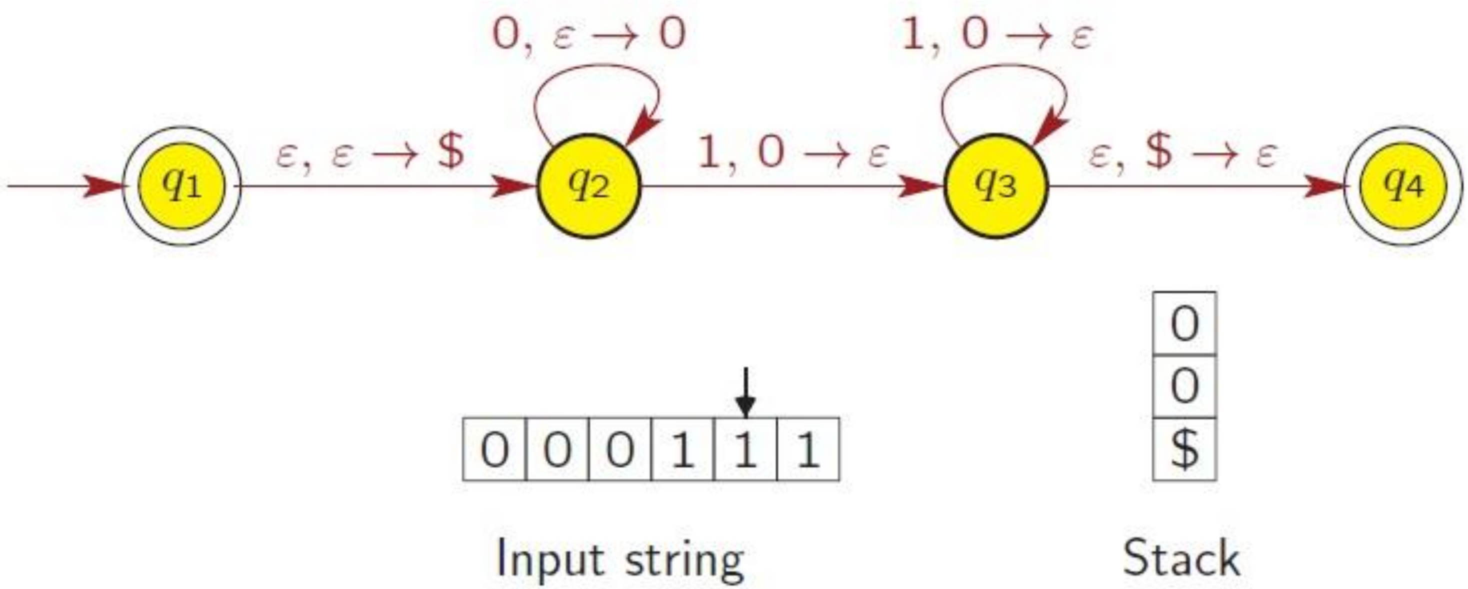
Example





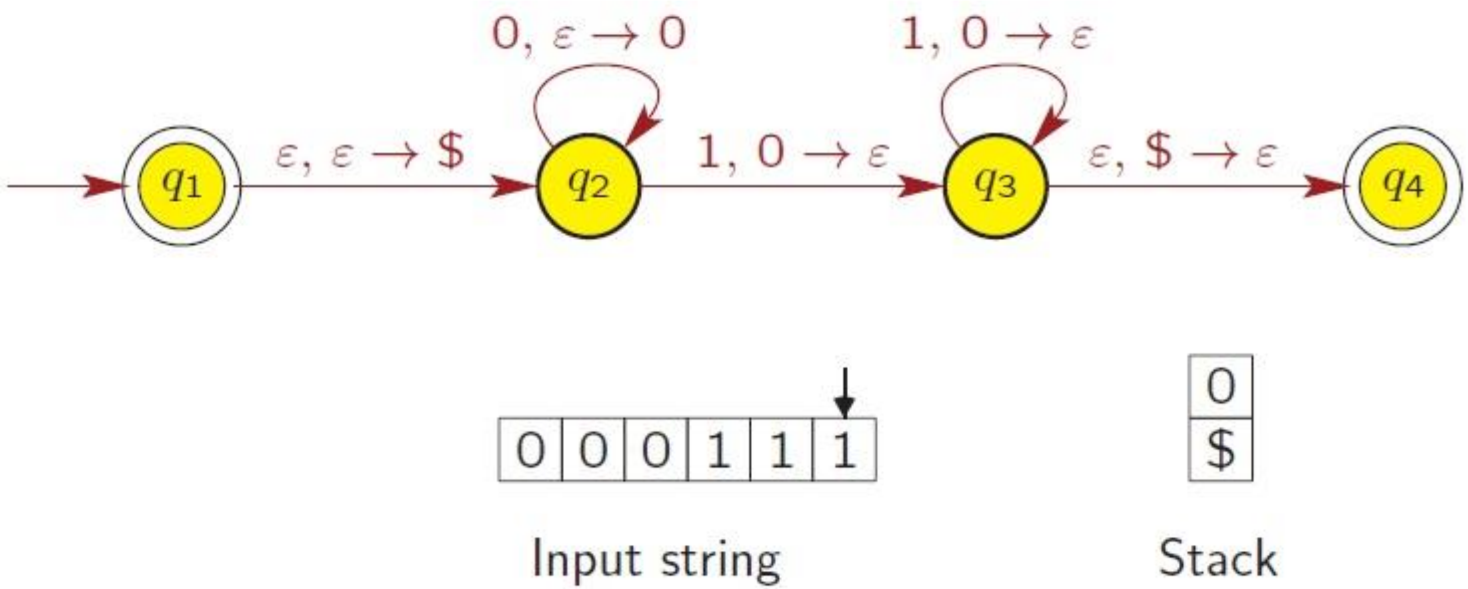
- Next go to state q_3
  - reading input symbol 1
  - popping 0 from stack
  - pushing nothing on stack.

Example



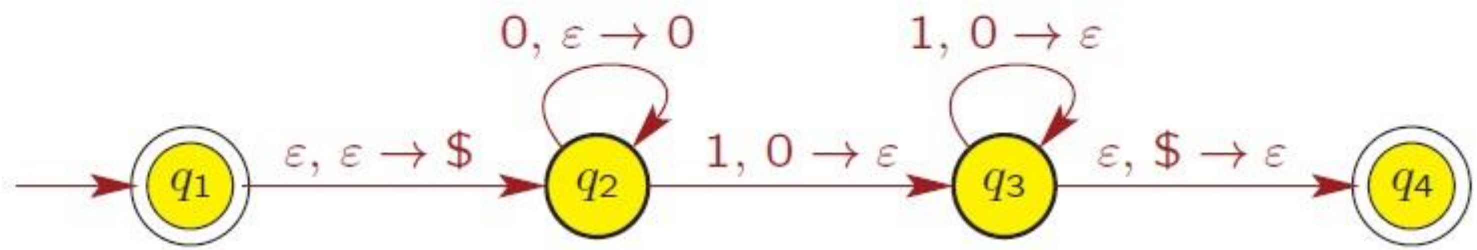
- Next return to state q_3
  - reading input symbol 1
  - popping 0 from stack
  - pushing nothing on stack.

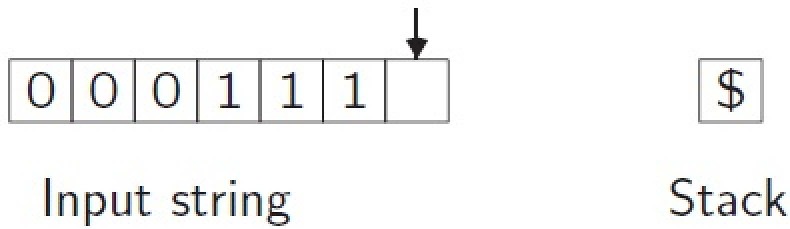
Example



- Next return to state q_3
  - reading input symbol 1
  - popping 0 from stack
  - pushing nothing on stack.

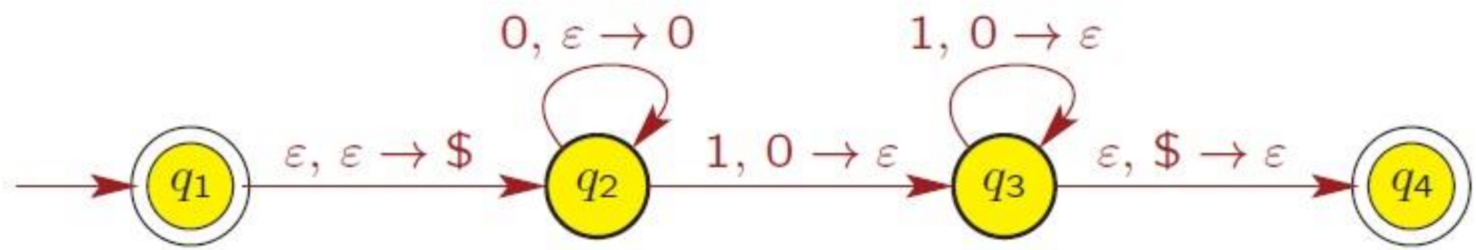
Example

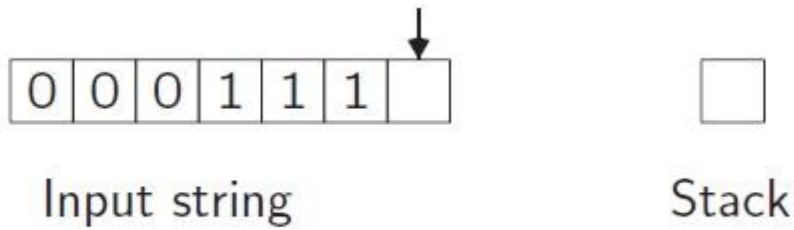




- Next go to state q_4
  - reading nothing
  - popping \$ from stack
  - pushing nothing on stack.

Example



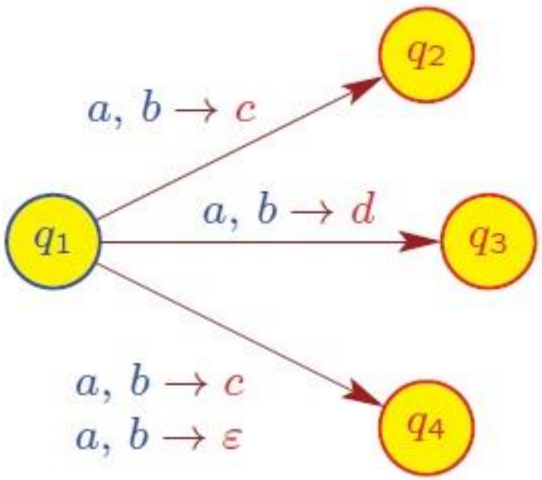


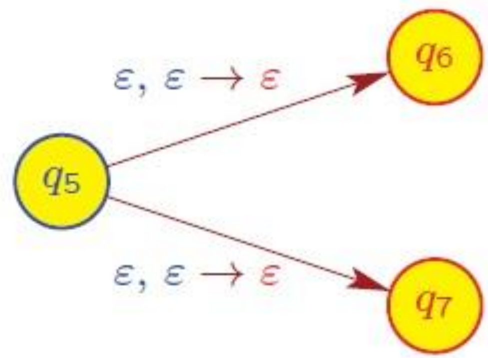
- String 000111 is **accepted** by PDA because
  - q_4 is an accept state and
  - PDA read the entire input string without crashing.

PDA is Nondeterministic

PDA transition function allows for nondeterminism

$$\delta: Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow P\,(Q \times \Gamma_\varepsilon)$$





ε-transitions

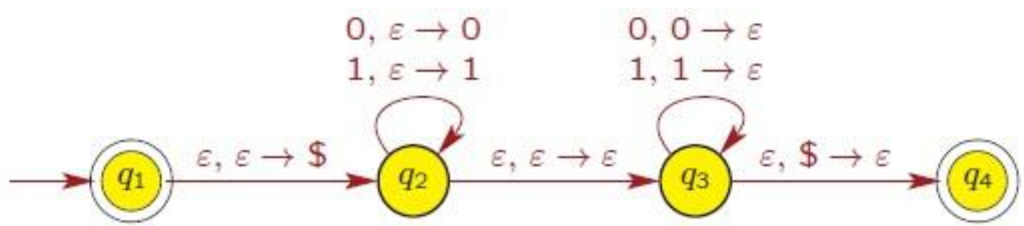
Language accepted by PDA

Definition

The set of all input strings that are accepted by PDA M is the language recognized by M and is denoted by $\mathrm{L(M)}$.

Example

PDA for language $\{ww^R \mid w \in \{0, 1\}^*\}$



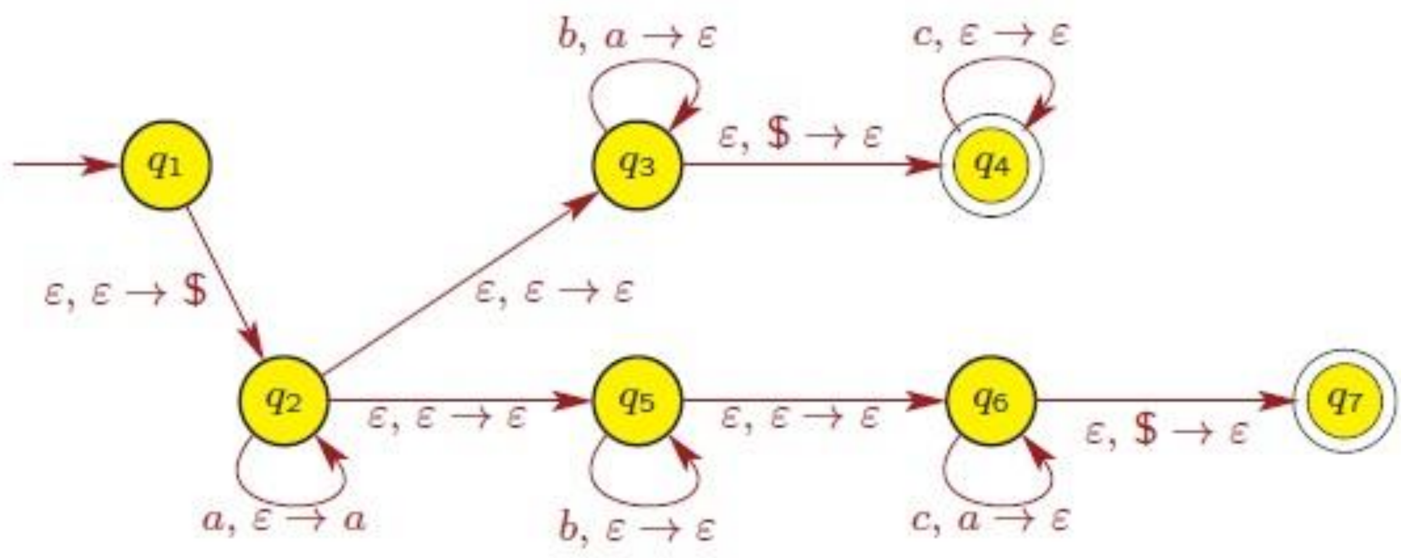
What happens if we replace $ at the last transition with epsilon ?

- $q_1 \to q_2$: First pushes $ on stack to mark bottom
- $q_2 \to q_2$: Reads in first half w of string, pushing it onto stack
- $q_2 \to q_3$: Guesses that it has reached middle of string
- $q_3 \to q_3$: Reads second half $w^{\mathcal{R}}$ of string, matching symbols from first half in reverse order (recall: stack LIFO)
- $q_3 \to q_4$: Makes sure that no more input symbols on stack

Language accepted by PDA

Example

PDA for language $\{ a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k \}$



Quick review

- CFLs are closed under concatenation, union and Kleene closure
- CFLs/Natural language exhibits ambiguities (* optional)
- Pushdown automata has an additional stack to store information

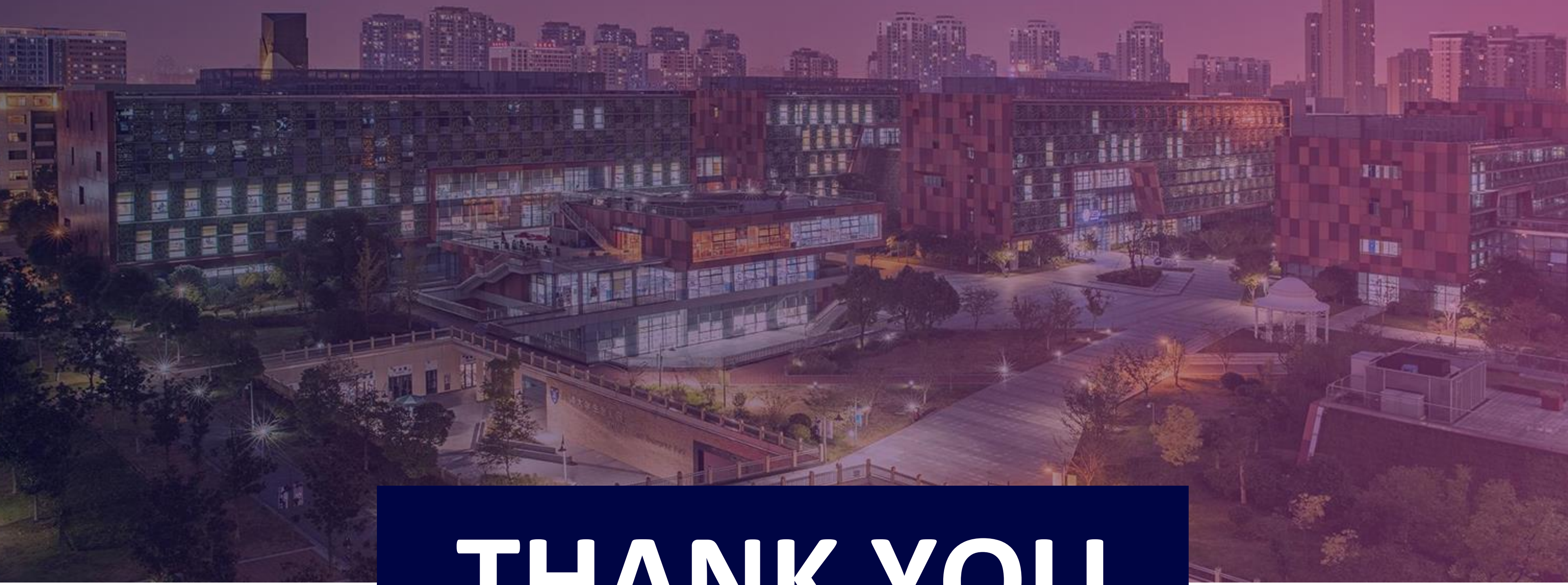
Q&A

- Does the stack elements have any influence on the accepting condition of PDA?

  No, the acceptance is solely decidec by the state.

- Why we put the $ at the beginning for some PDA?

  Yes, we do this for some, but it is not necessary. Putting $ and combining this with poping $ before accepting, we make sure that all things being added later will be processed.



THANK YOU

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