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CPT205 Computer Graphics

# Transformation Pipeline and Geometric Transformations

Lecture 04
2025-26

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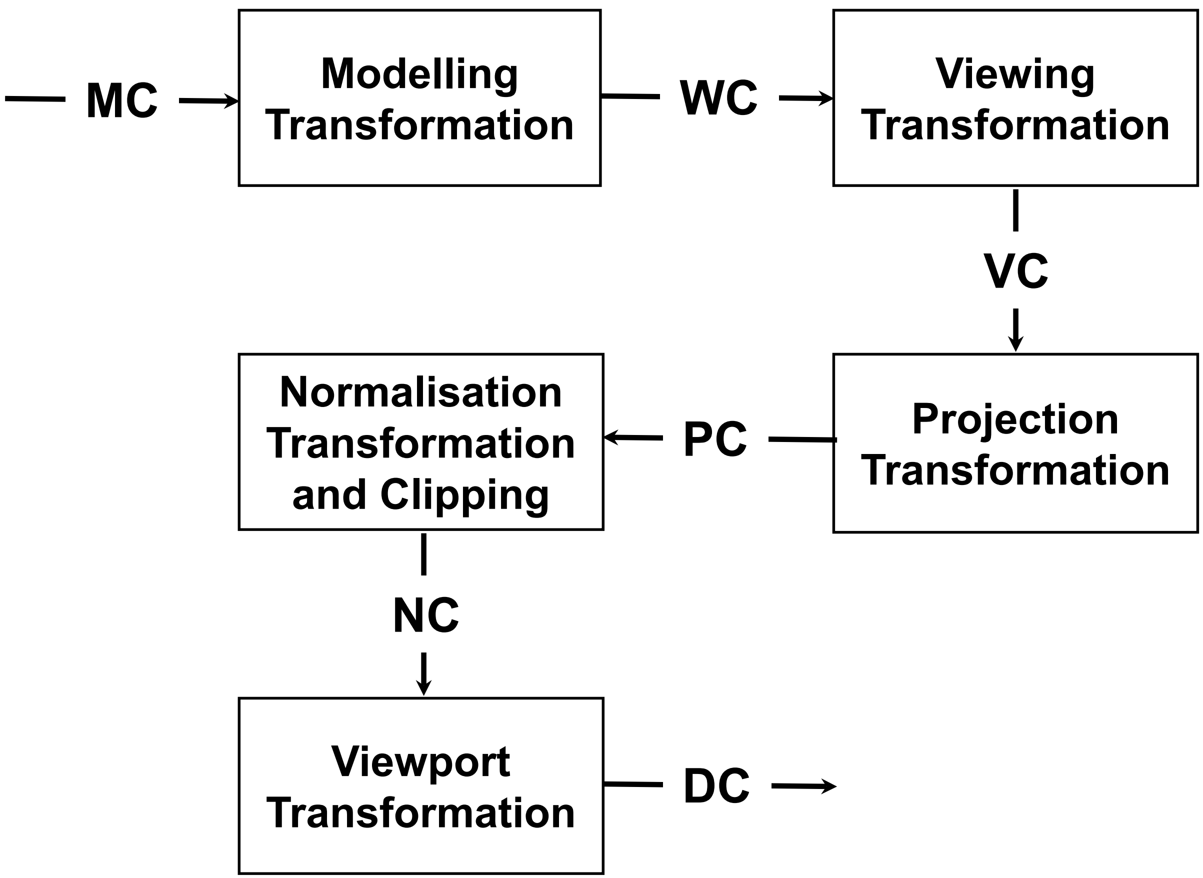
# Topics for today

- Transformation pipeline
- Standard transformations
  - Translation
  - Rotation
  - Scaling
  - Reflection
  - Shearing
- Homogeneous co-ordinate transformation matrices
- Composite (arbitrary) transformation matrices from simple transformations
- OpenGL functions for transformations

# Transformation pipeline (1)

- The Transformation Pipeline is the series of transformations (alterations) that must be applied to an object before it can be properly displayed on the screen.
- The transformations can be thought of as a set of processing stages.
  - If a stage is omitted, very often the object will not look correct.
  - For example if the *projection* stage is skipped then the object will not appear to have any depth to it.
- Once an object has passed through the pipeline it is ready to be displayed as either a wire-frame item or as a solid item.

# Transformation pipeline (2)



# Transformation pipeline (3)

- Modelling Transformation - to place an object into the Virtual World.
- Viewing Transformation - to view the object from a different vantage point in the virtual world.
- Projection Transformation - to see depth in the object.
- Viewport Transformation - to temporarily map the volume defined by the "window of interest" plus the front and rear clipping planes into a unit cube. When this is the case, certain other operations are easier to perform.
- Device Transformation - to map the user defined "window of interest" (in the virtual world) to the dimensions of the display area.

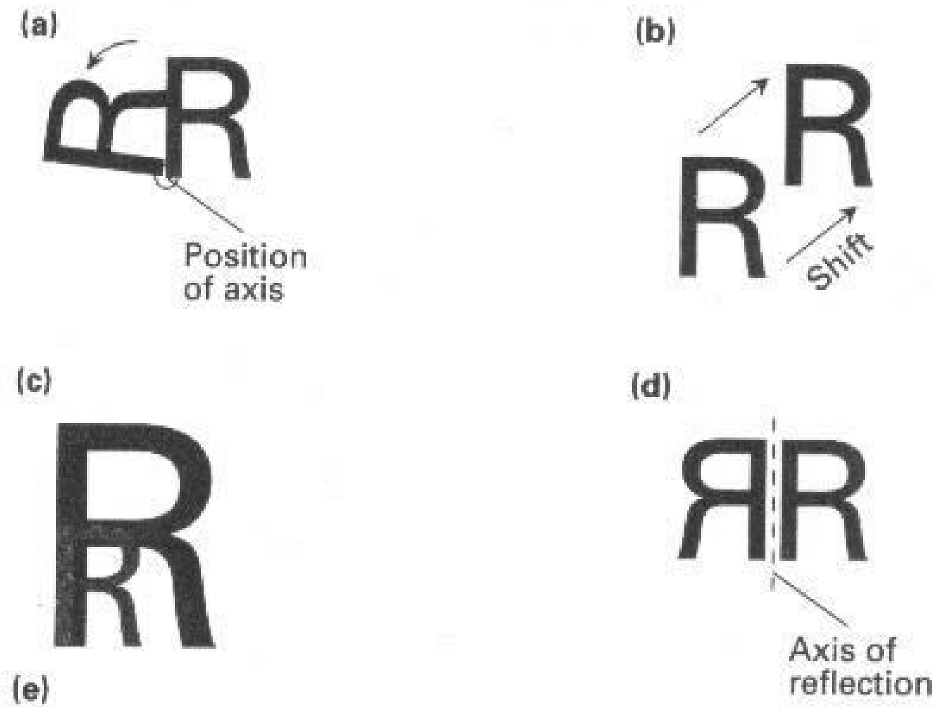
# Transformation pipeline (4)

- We start when the object is loaded from a file and is ready to be processed.
- We finish when the object is ready to be displayed on the computer screen.
- We should be able to draw a picture of a simple object, say a cuboid, and show visually what happens to it as it passes through each pipeline stage.

# Types of geometric transformation

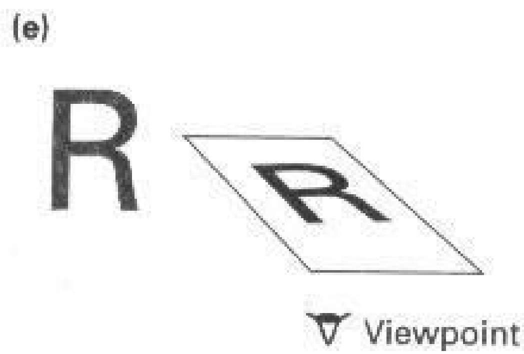
Rotation

Translation




Scaling

Reflection

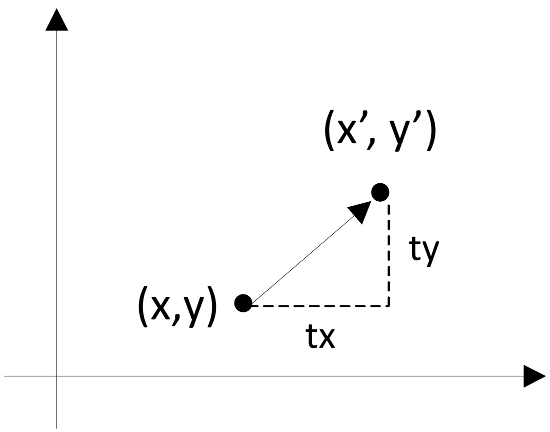



Shearing

# 2D translation (1)

- Translating a point from P(*x*, *y*) to P’(*x*’, *y*’) along vector T
- Importance in computer graphics – we need to only transform the two endpoints of a line segment and let the implementation draw the line segment between the transformed endpoints

$$\begin{cases} x' = x + t_x \\ y' = y + t_y \end{cases}$$



# 2D translation (2)

**P** = $\begin{bmatrix} x \\ y \end{bmatrix}$, **P'** = $\begin{bmatrix} x' \\ y' \end{bmatrix}$, **T** = $\begin{bmatrix} t_x \\ t_y \end{bmatrix}$

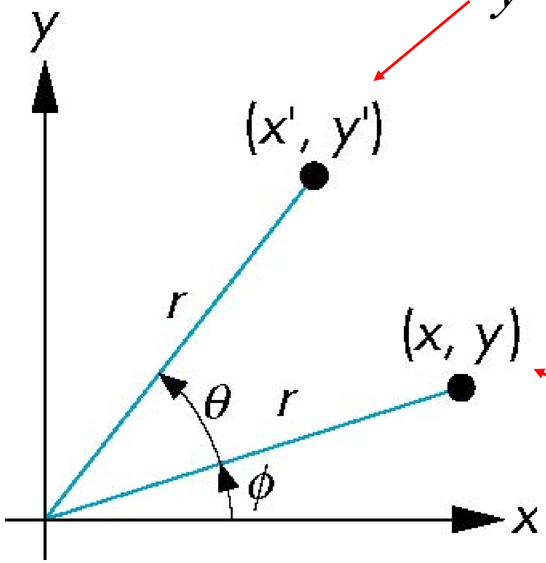
where **P**($x$, $y$) and **P'**($x$', $y$') are the original and new positions, and **T** is the distance translated.

**P' = P+T** or $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$

# 2D rotation (1)

Rotating a point from P($x$, $y$) to P'($x'$, $y'$) about the origin by angle $\theta$ - radius stays the same, and angle increases by $\theta$.

$$x' = r\cos(\phi + \theta) = x\cos\theta - y\sin\theta$$
$$y' = r\sin(\phi + \theta) = x\sin\theta + y\cos\theta$$




$$x = r\cos\phi$$
$$y = r\sin\phi$$

# 2D rotation (2)

$$\begin{cases} x' = r\cos(\phi+\theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ y' = r\sin(\phi+\theta) = r\cos\phi\sin\theta + r\sin\phi\cos\theta \end{cases}$$

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$$

$$\begin{cases} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{cases}$$

# 2D rotation (3)

**P' = R · P**

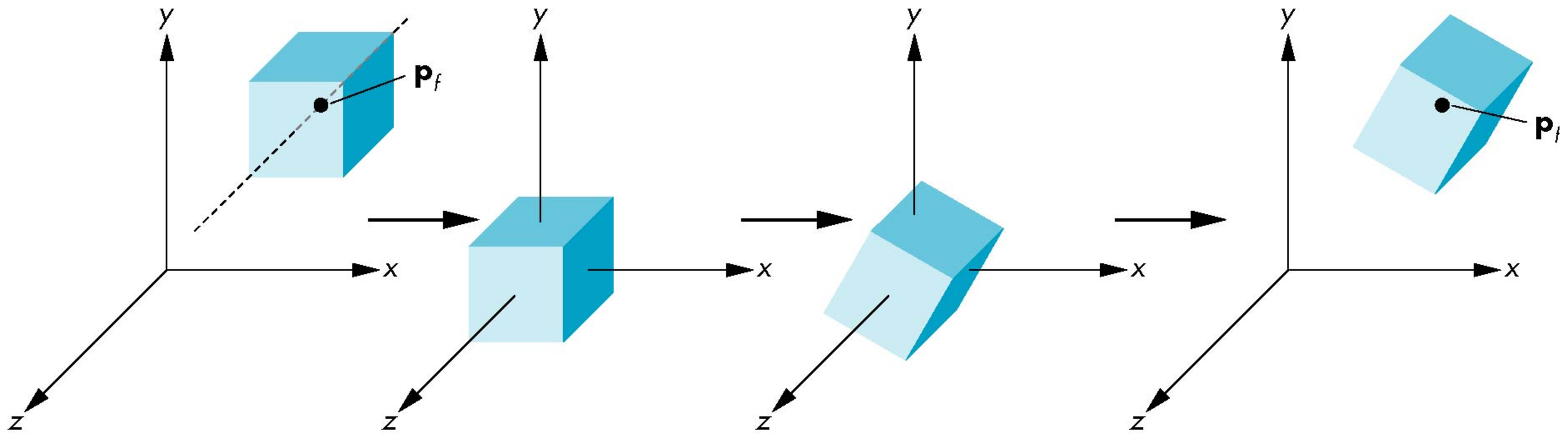
**R =** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ so $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

where $\theta$ is the rotation angle and $\phi$ is the angle between the *x*-axis and the line from the origin to $(x, y)$.

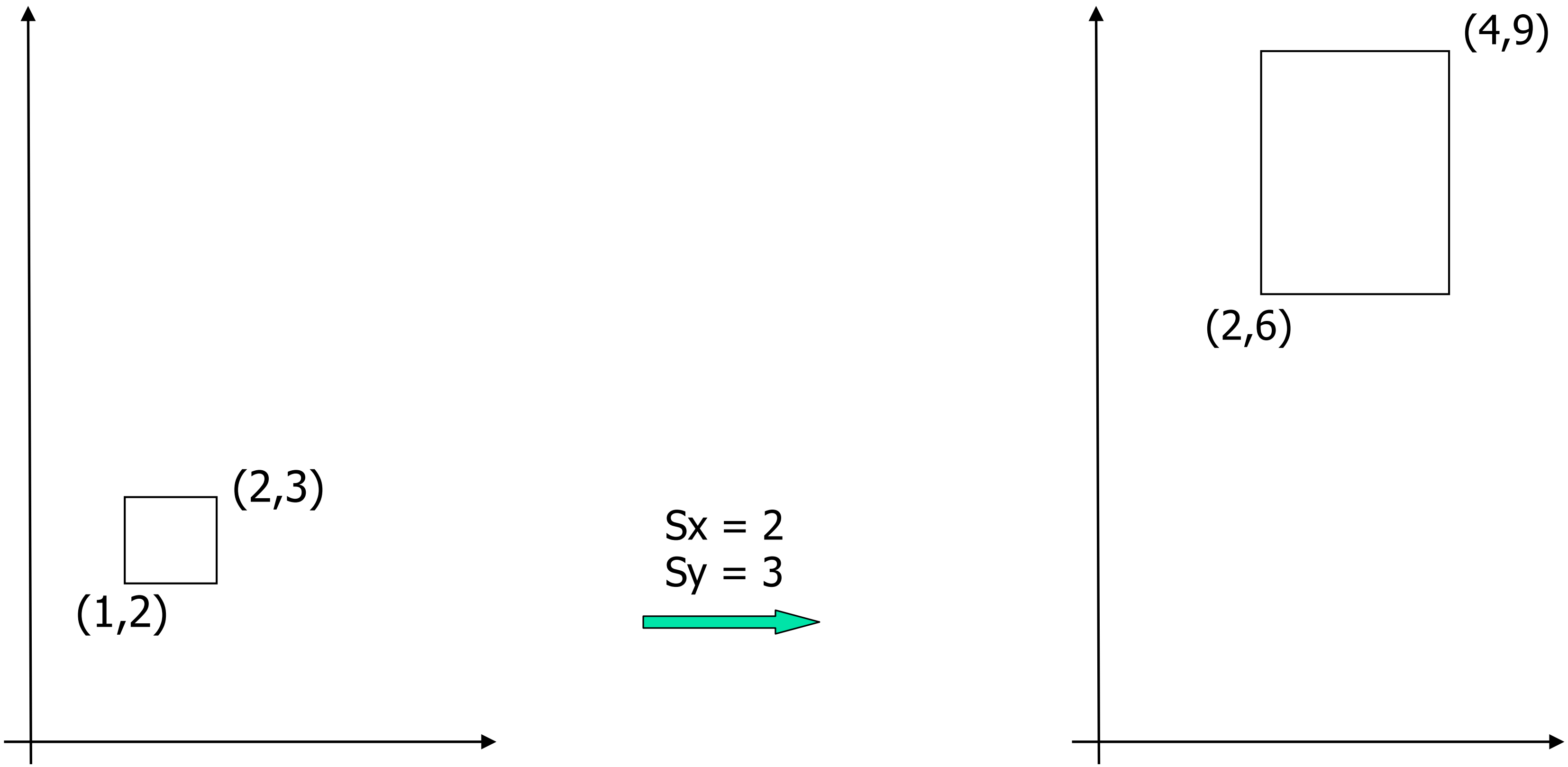
Notice that the rotation point (or pivot point) is <u>the co-ordinate origin</u>.

# Rotation about a fixed point rather than the origin

- Move the fixed point to the origin
- Rotate the object
- Move the fixed point back to its initial position
- $\mathbf{M} = \mathbf{T}(p_f)\, \mathbf{R}(\theta)\, \mathbf{T}(-p_f)$



# 2D scaling (1)




- When an object is scaled, both the size and location change.
- To scale at a fixed point rather than the origin, the same process applies as for rotating about a fixed point.

# 2D scaling (2)

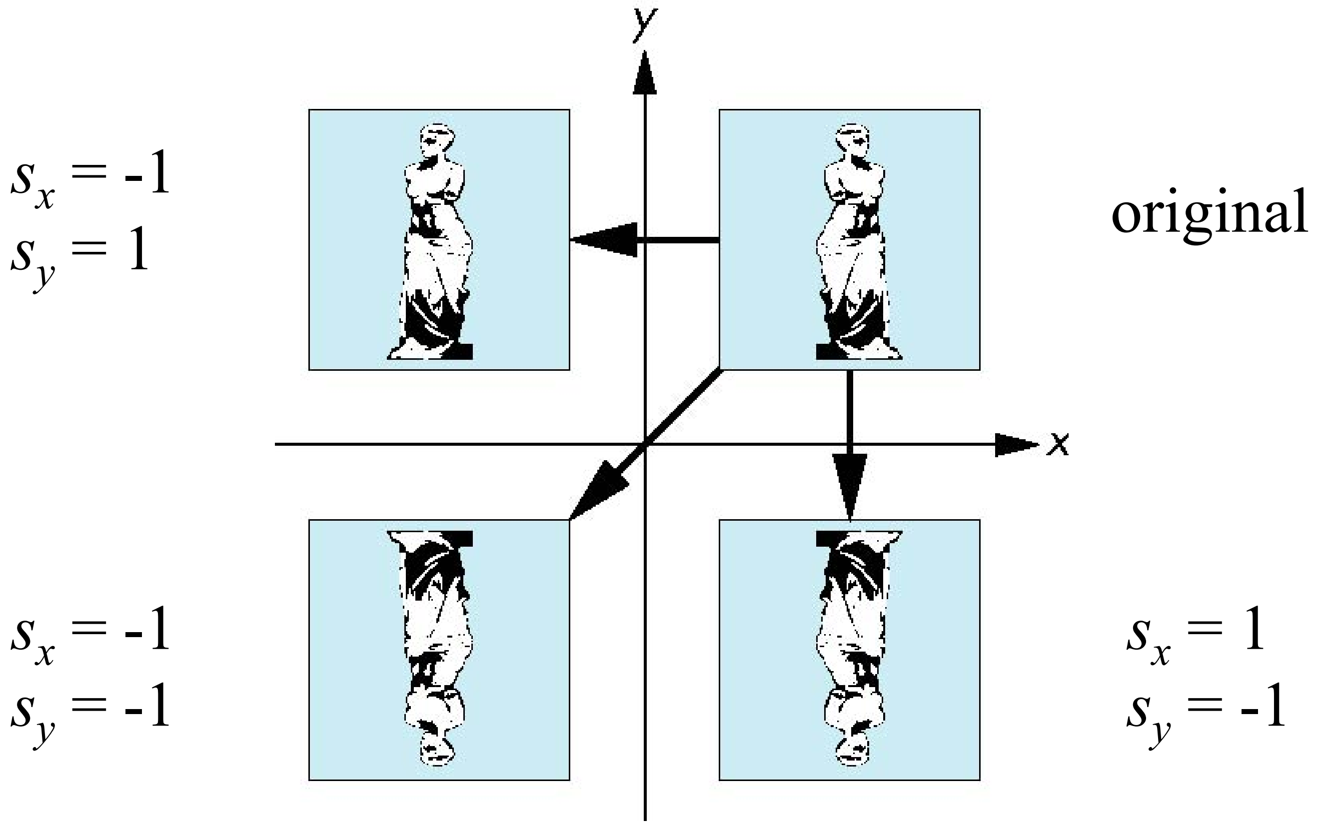
$$\begin{cases} x' = x \cdot s_x \\ y' = y \cdot s_y \end{cases}$$

**P’ = S · P** so $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

where **P**, and **P’** are the original and new positions, and $s_x$ and $s_y$ are the scaling factors along the $x$- and $y$-axes.

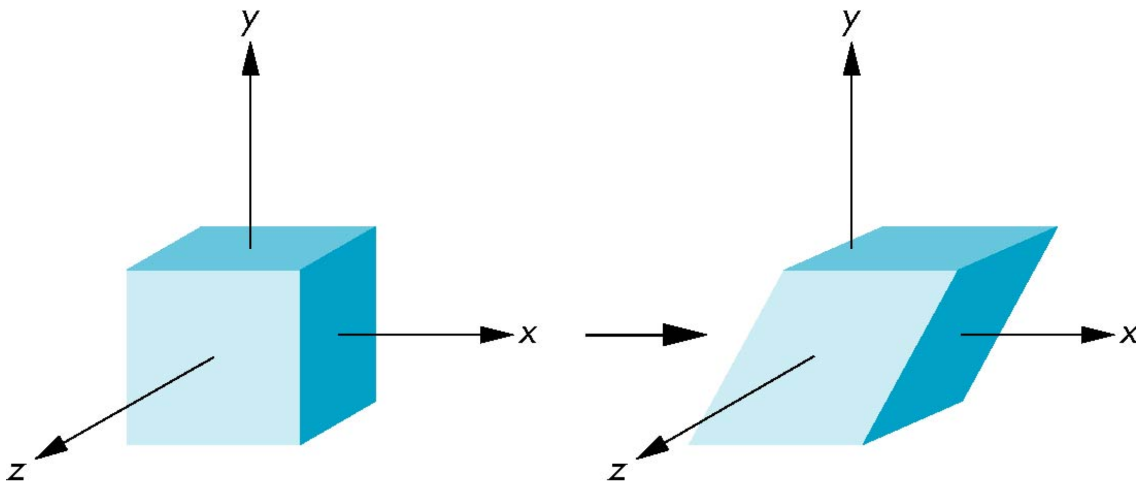
# 2D reflection

Special case of scaling - corresponding to negative scale factors.



# 2D shearing (1)

Equivalent to pulling faces in opposite directions.

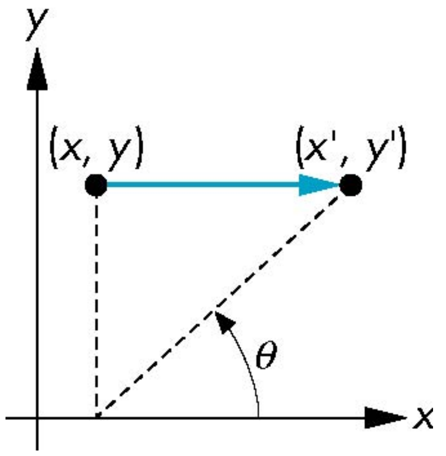


# 2D shearing (2)

Consider simple shearing along the *x* axis.

$$x' = x + y\cot\theta$$
$$y' = y$$
$$z' = z$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & \cot\theta \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$



# 2D homogeneous co-ordinates

- By expanding the 2x2 matrices to 3x3 matrices, homogeneous co-ordinates are used.
- A homogeneous parameter is applied so that each 2D position is represented with homogeneous co-ordinates ($h \cdot x$, $h \cdot y$, $h$).
- The homogeneous parameter has a non-zero value, and can be set to 1 for convenient use.
- This makes all the transformation matrices into the same format, i.e. 3x3 matrices, including the translation matrix.
- The homogenous transformation matrices can then be combined into a composite transformation matrix.

# 2D homogeneous matrices (1)

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$ (2D translation)

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$ (2D rotation)

# 2D homogeneous matrices (2)

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(2D scaling)

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \cot\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(2D shearing)

# 2D composite transformation (1)

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} rs_{xx} & rs_{xy} & trs_x \\ rs_{yx} & rs_{yy} & trs_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- where elements *rs* are the multiplicative rotation-scaling terms in the transformation (which involve only rotation angles and scaling factors);
- elements *trs* are the translation terms, containing combination of translation distances, pivot-point and fixed-point co-ordinates, rotation angles and scaling parameters.

# 2D composite transformation (2)

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

The composite transformation becomes an identity matrix (**I**) when there is no transformation.

# 3D translation

3D translation and scaling can be simply extended from the corresponding 2D methods.

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

# 3D co-ordinate axis rotations (1)

- The extension from 2D rotation methods to 3D rotation is less straightforward (because this is about an arbitrary axis instead of an arbitrary point).
- Equivalent to rotation in two dimensions in planes of constant $z$ (i.e. about the origin).

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

(About $z$-axis)

# 3D co-ordinate axis rotations (2)

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

(About $y$-axis)

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

(About $x$-axis)

# General rotation about the origin

A rotation by $q$ about an arbitrary axis can be decomposed into the concatenation of rotations about the $x$, $y$, and $z$ axes.

$$\mathbf{R}(q) = \mathbf{R}_z(q_z)\, \mathbf{R}_y(q_y)\, \mathbf{R}_x(q_x)$$

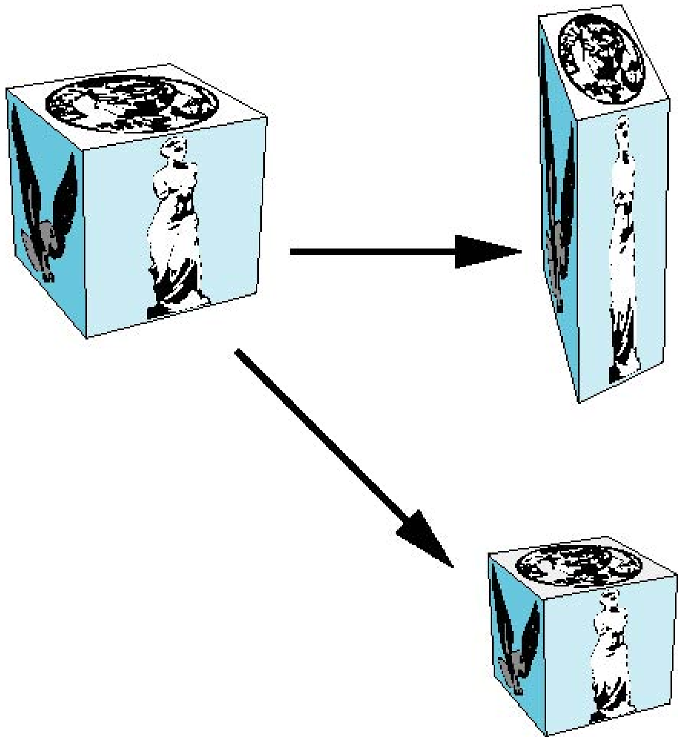
where $q_x$, $q_y$ and $q_z$ are called the Euler angles.

Note that rotations do not commute though we can use rotations in another order but with different angles.

# 3D scaling

$$x' = s_x x$$
$$y' = s_y y$$
$$z' = s_z z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



# 3D composite transformation (1)

- As with 2D transformation, a composite 3D transformation can be formed by multiplying the matrix representations for the individual operations in the transformation sequence.
- There are further forms of transformation, namely reflection and shearing which can be implemented with the other three transformations.
- Translation, scaling, rotation, reflection and shearing are all affine transformations in that transformed point P'($x'$,$y'$,$z'$) is a linear combination of the original point P($x$,$y$,$z$).

# 3D composite transformation (2)

- Matrix multiplication is associative.

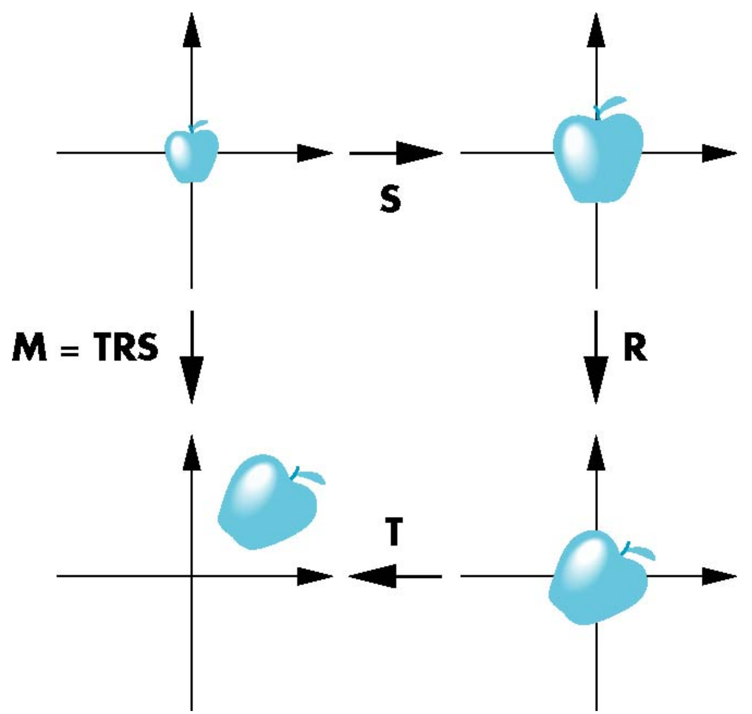
  $M3 \cdot M2 \cdot M1 = (M3 \cdot M2) \cdot M1 = M3 \cdot (M2 \cdot M1)$

- Transformation products are not always commutative.

  $A \cdot B \neq B \cdot A$

# Instancing

- In modelling, we often start with a simple object centred at the origin, oriented with an axis, and of a standard size.
- We apply an *instance transformation* to its vertices to
  - Scale
  - Orient
  - Locate



# OpenGL matrices (1)

- OpenGL is (the OpenGL API is defined as) a <u>state machine</u>.
- Modes and attributes in OpenGL remain in effect until they are changed.
- Most state variables can be enabled or disabled with **`glEnable()`** or **`glDisable()`**.
- In OpenGL, matrices are <u>part of the state</u>.

# OpenGL matrices (2)

- Multiple types
  - Model-View (**`GL_MODELVIEW`**)
  - Projection (**`GL_PROJECTION`**)
  - Texture (**`GL_TEXTURE`**)
  - Color (**`GL_COLOR`**)
- <u>Single</u> set of functions for manipulation
- Select which to be manipulated by
  - **`glMatrixMode(GL_MODELVIEW);`**
  - **`glMatrixMode(GL_PROJECTION);`**

# Current transformation matrix

- Conceptually there is a 4x4 homogeneous coordinate matrix, the *current transformation matrix* (CTM) that is part of the state and is applied to all vertices that pass down the pipeline.
- The CTM is defined in the user program and loaded into a transformation unit.
- The CTM can be altered either by loading a new CTM (e.g., Identity matrix) or by postmutiplication.
- OpenGL has a model-view and a projection matrix in the pipeline which are concatenated together to form the CTM.
- The CTM can manipulate each by first setting the correct matrix mode.

# CTM operations

The CTM can be altered either by loading a new CTM or by postmutiplication.

- Load an identity matrix: $\mathbf{C} \leftarrow \mathbf{I}$
- Load an arbitrary matrix: $\mathbf{C} \leftarrow \mathbf{M}$

- Load a translation matrix: $\mathbf{C} \leftarrow \mathbf{T}$
- Load a rotation matrix: $\mathbf{C} \leftarrow \mathbf{R}$
- Load a scaling matrix: $\mathbf{C} \leftarrow \mathbf{S}$

- Postmultiply by an arbitrary matrix: $\mathbf{C} \leftarrow \mathbf{CM}$
- Postmultiply by a translation matrix: $\mathbf{C} \leftarrow \mathbf{CT}$
- Postmultiply by a rotation matrix: $\mathbf{C} \leftarrow \mathbf{CR}$
- Postmultiply by a scaling matrix: $\mathbf{C} \leftarrow \mathbf{CS}$

# Arbitrary Matrices

- We can load and multiply by matrices defined in the application program
  - `glLoadMatrixf(m)`
  - `glMultMatrixf(m)`
- Matrix **`m`** is a <u>one-dimension array of 16 elements</u> which are the components of the desired 4 x 4 matrix stored by <u>columns</u>.
- In **`glMultMatrixf`**, **`m`** multiplies the existing matrix on the right.

# Matrix stacks (1)

- CTM is not just one matrix but a <u>matrix stack</u> with the “current” at top.
- In many situations we want to <u>save transformation matrices</u> for use later.
  - Traversing hierarchical data structures
  - Avoiding state changes when executing display lists
- Pre 3.1 OpenGL maintains stacks for each type of matrix.
  - Access present type (as set by **`glMatrixMode)`** by

    **`glPushMatrix()`**

    **`glPopMatrix()`**
- Right now just 1-level CTM.

# Matrix stacks (2)

- We can also access matrices (and other parts of the state) with *query* functions
  - **`glGetIntegerv()`**
  - **`glGetFloatv()`**
  - **`glGetBooleanv()`**
  - **`glGetDoublev()`**
  - **`glIsEnabled()`**

- For matrices, we use as
  - **`double m[16];`**
  - **`glGetFloatv(GL_MODELVIEW, m);`**

# OpenGL transformation functions (1)

- **`glTranslate*()`**
  Specify translation parameters
- **`glRotate*()`**
  Specify rotation parameters for rotation about any axis through the origin
- **`glScale*()`**
  Specify scaling parameters with respect to the co-ordinate origin
- **`glMatrixMode()`**
  Specify current matrix for geometric-viewing, projection, texture or colour transformations
- **`glLoadIdentity()`**
  Set current matrix to identity
- **`glLoadMatrix*(elems)`**
  Set elements of current matrix

# OpenGL transformation functions (2)

- **`glMultMatrix*(elems)`**
  Post-multiply the current matrix by the specified matrix
- **`glGetIntegerv()`**
  Get max stack depth or current number of matrices in the stack for selected matrix mode
- **`glPushMatrix()`**
  Copy the top matrix in the stack and store copy in the second stack position
- **`glPopMatrix()`**
  Erase top matrix in stack and move second matrix to top stack
- **`glPixelZoom()`**
  Specify 2D scaling parameters for raster operations

# Example of rotation

- Rotation about the $z$ axis by 30 degrees with a fixed point of (1.0, 2.0, 3.0) – equivalent to rotation about a point at (1.0, 2.0) in a 2D space

```
glMatrixMode(GL_MODELVIEW);
glLoadIdentity();
glTranslatef(1.0, 2.0, 3.0);
glRotatef(30.0, 0.0, 0.0, 1.0);
glTranslatef(-1.0, -2.0, -3.0);
```

- Note that the last matrix specified in the program is the first applied.

# Summary

- Transformation pipeline
- Standard transformations
  - Rotation
  - Translation
  - Scaling
  - Reflection
  - Shearing
- Homogeneous co-ordinate transformation matrices
- Composite (arbitrary) transformation matrices from simple transformations
- OpenGL functions for transformations