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Int 201: Decision Computation and Language
Tutorial 10 Solution

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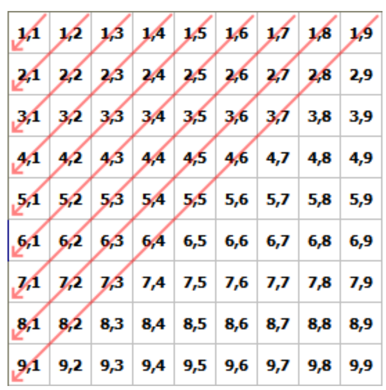
Question 1. Show the set $\{(i, j)|i, j \in \mathcal{N}\}$ is countable.

Solution 1. We use the same zigzag strategy as in the mapping between natural numbers and rational numbers.



1,1	1,2	1,3	1,4	1,5	1,6	1,7	1,8	1,9
2,1	2,2	2,3	2,4	2,5	2,6	2,7	2,8	2,9
3,1	3,2	3,3	3,4	3,5	3,6	3,7	3,8	3,9
4,1	4,2	4,3	4,4	4,5	4,6	4,7	4,8	4,9
5,1	5,2	5,3	5,4	5,5	5,6	5,7	5,8	5,9
6,1	6,2	6,3	6,4	6,5	6,6	6,7	6,8	6,9
7,1	7,2	7,3	7,4	7,5	7,6	7,7	7,8	7,9
8,1	8,2	8,3	8,4	8,5	8,6	8,7	8,8	8,9
9,1	9,2	9,3	9,4	9,5	9,6	9,7	9,8	9,9

Question 2. Show the Kleene star of any countable set of strings of finite length is countable.

Solution 2. Let's denote the countable set S, and since it is countable, we index its' strings as s_i. We further index the subset of S that includes all strings until s_i by $S_i = \{s_j | j \leq i\}$. Each S_i has exactly i strings. Now, we construct k finite unions of S_i. $C_{ik} = \cup_{1 \leq j \leq k} S_i^j$, where $S_i^j = \{wv | w \in S_i, v \in S_i^{j-1}\}$ and $S_i^1 = S_i$. Clearly, each C_{ik} has finite number of strings. Therefore, we still have a table of countable rows and countable columns. We perform the same zigzag strategy across the cells but inside each i, k cell, we need to count finite number of strings and skip the ones we have encountered earlier. Of course, we count the ϵ as our first element.

The injectivity of the mapping is guaranteed by the skipping of encountered strings. The surjective part is also easy to see, as any string $w \in S^*$, there exists $k \in \mathbf{N}$ and s.t. $w = w(1) \cdots w(k)$ where each $w(i) \in S$. Let $m = \text{argmin}_n \forall i \leq k, w(i) \in S_n$. Clearly, $w \in C_{mk}$.Therefore, it must have been countered before we move to the next cell (maybe earlier in the k index).

Question 3. Show for any finite set S, $|\mathcal{P}(S)| > |S|$, where $\mathcal{P}(S)$ is the power set, by using the diagonalization method.

Solution 3. Clearly, the mapping $\forall s, f(s) = \{s\} \in \mathcal{P}(S)$ is an injective mapping. Therefore $|\mathcal{P}(S)| \geq |S|$.

To apply the diagonal method, for any given mapping, we will show it is not surjective, hence the equality cannot hold. Notice that we can index the finite set, $f(s_i)$ can be represented by the indicator vector of $f_i[j] = [s_j \in f(s_i)]$. For example, consider the set $\{s_1, s_2, s_3\}$. $f(s_1), f(s_2), f(s_3) = \{s_1, s_2\}, \{s_1, s_3\}, \{s_3\}$ can be represented by $[1, 1, 0], [1, 0, 1], [0, 0, 1]$. Since any indicator vector represents a subset, we only need to find a vector that differs from any of the given one by f. We pick a vector $X \in \{0, 1\}^{|S|}$ where $X_i = 1 - f_i[i]$. Clearly, X differs from any $f(s_i)$ at ith element. Therefore, no function cannot be surjective and $|\mathcal{P}(S)| > |S|$.

Question 4. Show for any set S, $|\mathcal{P}(S)| > |S|$, where $\mathcal{P}(S)$ is the power set, by using the abstract diagonalization method. Hint: now we do not have a table, but for any given $f : S \to \mathcal{P}(S)$, try define an $X(f) \subset S$ such that $X(f) \notin \{f(s) | \forall s \in S\}$. [1]

Solution 4. Clearly, the mapping $\forall s, f(s) = \{s\} \in \mathcal{P}(S)$ is an injective mapping. Therefore $|\mathcal{P}(S)| \geq |S|$.

We apply the diagonalization method for showing no surjective mapping can exist. We construct the $X(f)$ directly without the indicator representation. Consider any mapping f from S to $\mathcal{P}(S)$, we construct the element $X(f) = \{s \in S | s \notin f(s)\}$. [2] We claim $X(f) \notin \{f(s) | s \in S\}$.

This is equivalent to say $\forall s, f(s) \neq X(f)$. Let's prove this by contradiction, assume there exists s s.t. $f(s) = X(f)$. Does $s \in f(s)$? If $s \in f(s)$, then by the definition of $X(f)$, $s \notin X(f)$, therefore $s \notin f(s)$. If $s \notin f(s)$, then we

[1] Corollary: power set operations allow construction of set that is larger than the real numbers.

[2] This exactly corresponds to the diagonal method if we are working with the finite sets. We are flipping the $\in$ relationship at s and $f(s)$ (the diagonal).

have $s \in X(f)$, therefore $s \in f(s)$. In both cases, we have a contradiction. Therefore, no function cannot be surjective and $|\mathcal{P}(S)| > |S|$.

Question 5. Show $\mathrm{E_{DFA}} = \{< A > |$A is a DFA and L(A)=$\emptyset\}$ is a decidable language.

Solution 5. If DFA reads anything at all then accept the string, the DFA language won't be the empty language. T = "On input $< A >$, where A is a DFA:

1. Mark the start state of A.
2. Repeat until no new states get marked:
3. Mark any state that has a transition coming into it from any marked state
4. If no accept state is marked, accept ; otherwise, reject ."

We claim that this machine terminates and accepts iff L(A)=$\emptyset$.

As each DFA has finite transition rules and finite states, the marking procedure will terminate as we do BFS/DFS starting from the start state.

If the machine rejects, then clearly there is a path from starting state to one of the acceptance state. The strings that consumed alone the path will be accepted by the DFA, therefore, the language is not empty.

If the language is not empty, there must exist a path from starting state to one of the acceptance state. The marking procedural will traverse such path to reach the acceptance state, therefore, the machine will reject it.

Question 6. Show $\mathrm{EQ_{DFA}} = \{< A, B > |$A and B are DFAs and L(A)=L(B)$\}$ is a decidable language.

Solution 6. We construct the language $L(C) = (L(A) \cap L(\bar{B})) \cup (L(\bar{A}) \cap L(B))$. Since regular language is closed under complement, union and intersection, we can construct a corresponding DFA for C. Now we claim $L(C) = \emptyset \iff L(A) = L(B)$. If $L(C) = \emptyset$, we have $L(A) \cap L(\bar{B}) = \emptyset$, which means no element in $L(A)$ is not in $L(B)$. In other words, $L(A) \subset L(B)$. Similarly, we have $L(B) \subset L(A)$. Therefore, $L(C) = \emptyset \implies L(A) = L(B)$. On the other hand, if $L(A) = L(B)$, clearly, $L(C) = (L(A) \cap L(\bar{B})) \cup (L(\bar{A}) \cap L(B)) = L(C) = (L(A) \cap L(\bar{A})) \cup (L(\bar{A}) \cap L(A)) = \emptyset \cup \emptyset$, and we have $L(C) = \emptyset$. Now, as the previous question have shown $\mathrm{E}_{\mathrm{DFA}}$ is decidable, we ask whether $L(C)$ is empty, and accept if it is, and reject if it is not.