Xi'an Jiaotong-Liverpool University
西交利物浦大学

CPT205 Computer Graphics

Mathematics for Computer Graphics

Lecture 02
2025-26

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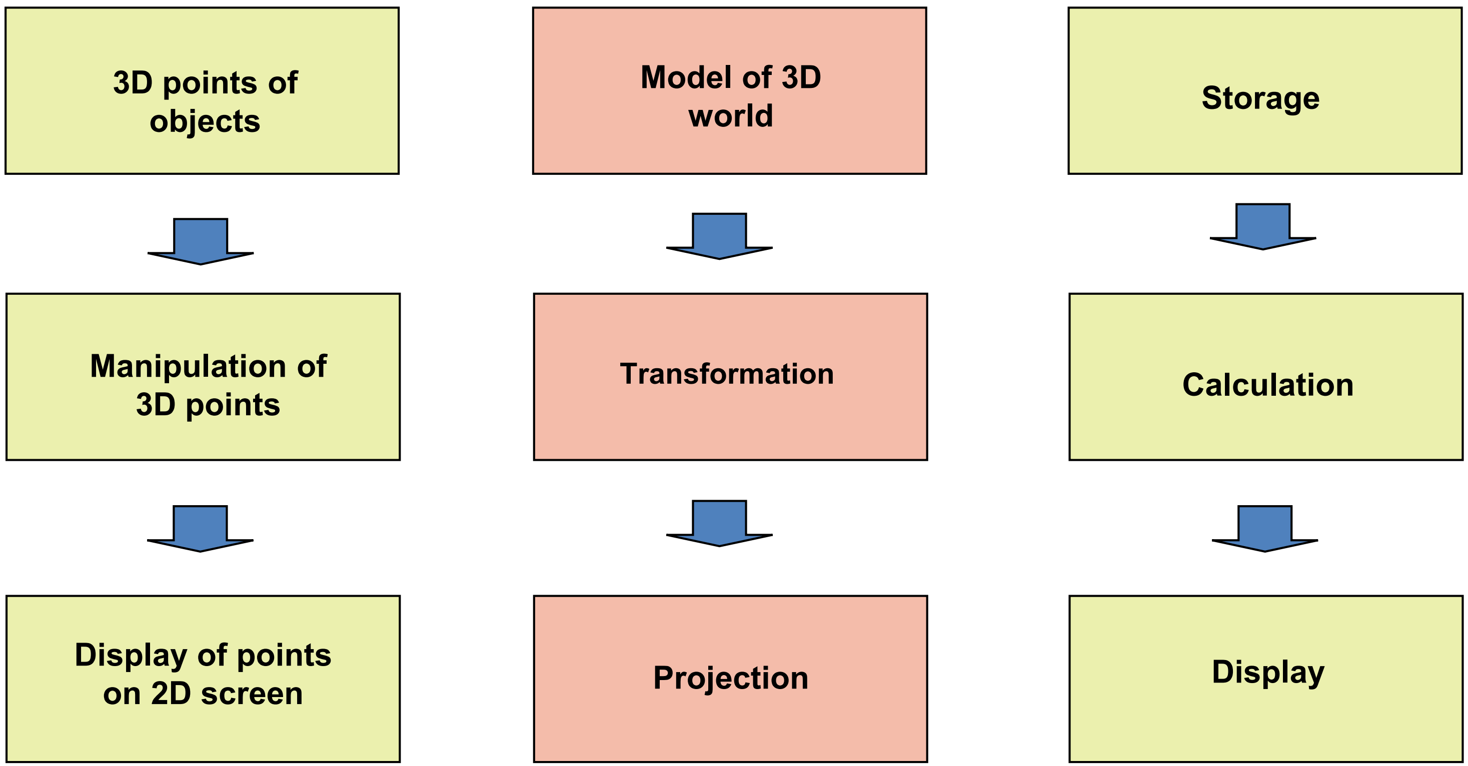
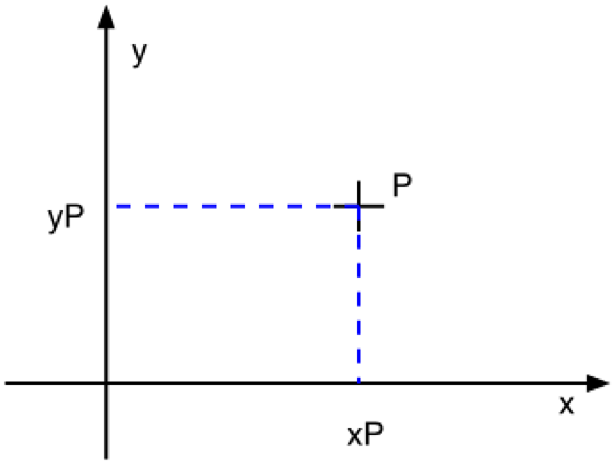
Topics for today

- Computer representation of objects
- Cartesian co-ordinate system
- Points, lines and angles
- Trigonometry
- Vectors (unit vector) and vector calculations (addition, subtraction, scaling, dot product and cross product)
- Matrices (dimension, transpose, square/symmetric/identity and inverse) and matrix calculations (addition, subtraction and multiplication)

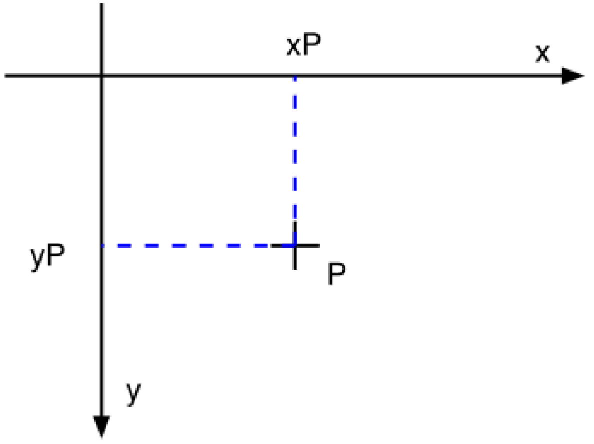
Computer representation of objects



Cartesian co-ordinate system

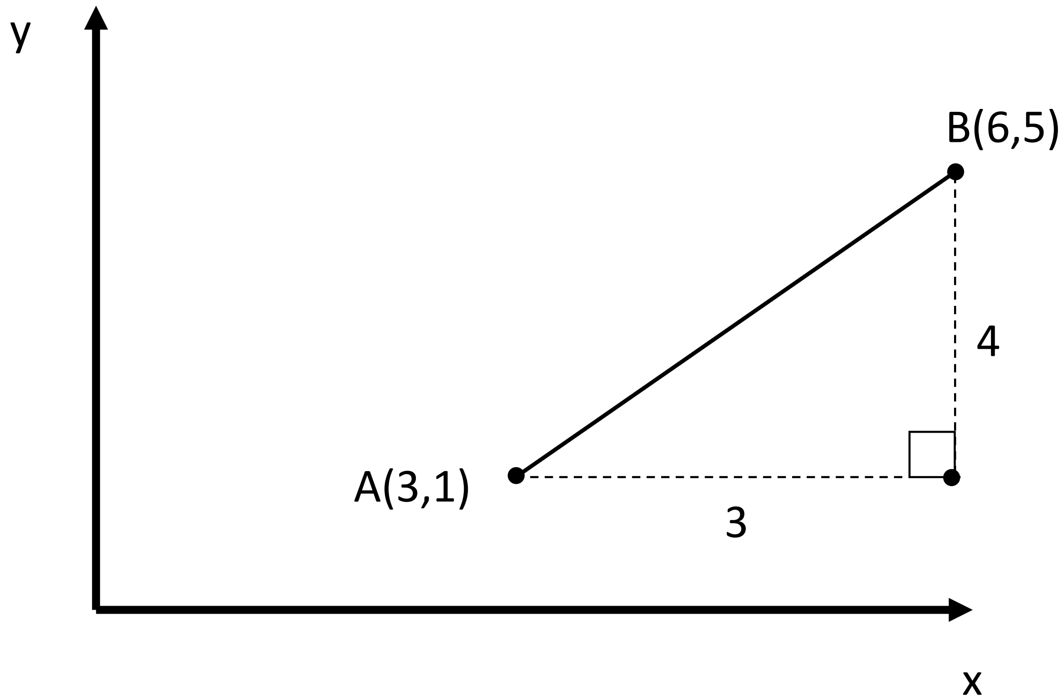


Representation of point
$P(x_p, y_p)$ in Cartesian co-ordinates



Representation of point
$P(x_p, y_p)$ on computer screen

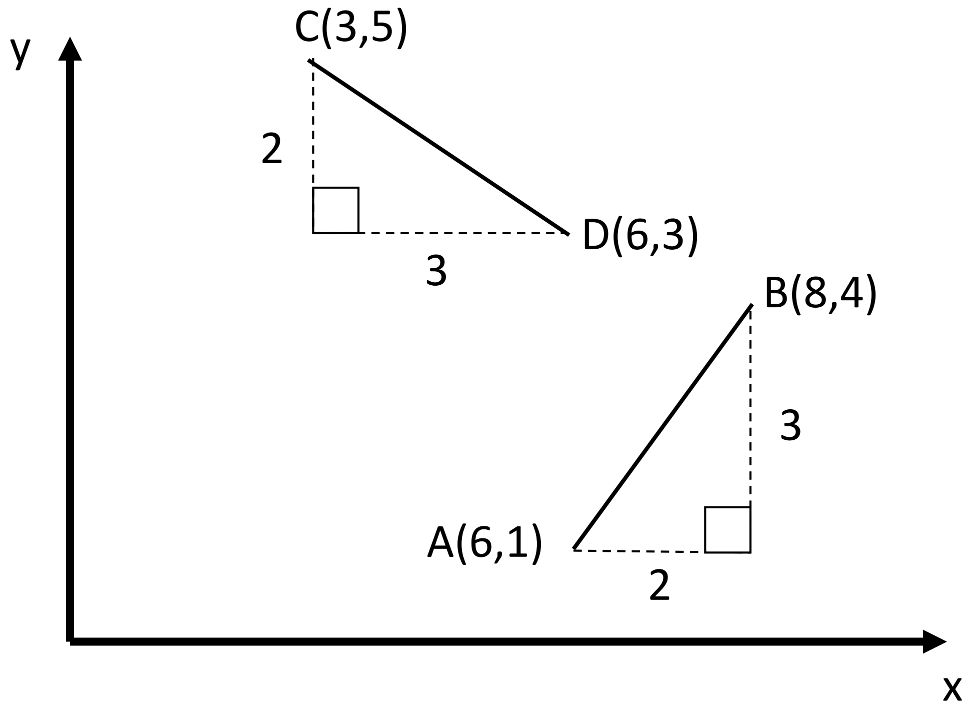
Straight line



Using Pythagoras theorem:

AB = sqrt $(4^2 + 3^2)$ = 5

Gradient of a line



Gradient AB = $\Delta y/\Delta x$ = (4-1) / (8-6) = 3/2
Gradient CD = $\Delta y/\Delta x$ = (3-5) / (6-3) = -(2/3)

An uphill line (direction is ‘bottom left to top right’) has a **positive** gradient.
A downhill line (direction is ‘top left to bottom right’) has a **negative** gradient.

Perpendicular lines

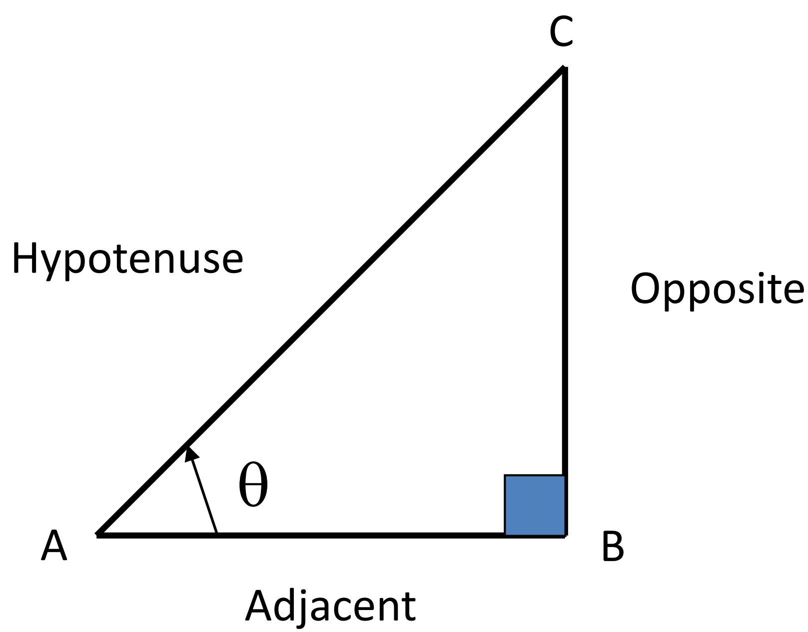
- Given that the gradient of AB = 3/2 and gradient of CD = -2/3, when the two gradients are multiplied together we have: (3/2) * (-2/3) = -1.

- Thus we, conclude that lines AB and CD are perpendicular.

- Prove this using graph paper.

- What can you say about lines with same gradient?

Angles and trigonometry



sine (θ) = Opposite / Hypotenuse = BC / AC

cosine (θ) = Adjacent / Hypotenuse = AB / AC

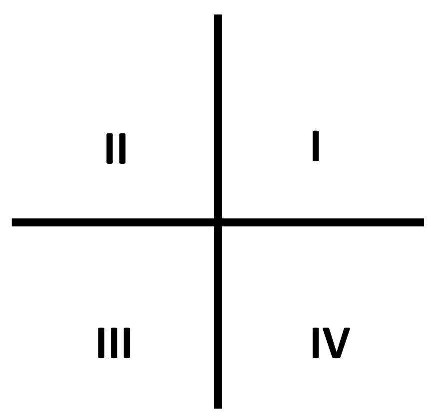
tangent (θ) = Opposite / Adjacent = BC / AB

Angles and trigonometry

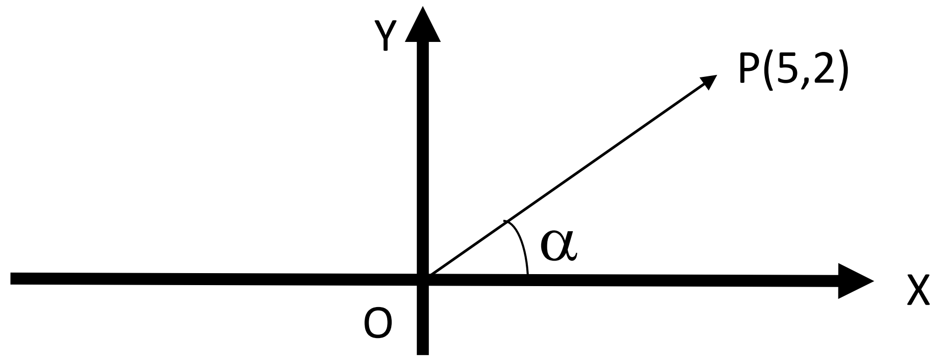
A complete revolution gives 360° or 2π (rad).

The following diagram is used to find the values of the trigonometric ratios:

- All trigonometric ratios of angles in quadrant 1 have positive ratios.
- Only sine of angles in quadrant 2 have positive ratios.
- Only tangent of angles in quadrant 3 have positive ratios.
- Only cosine of angles in quadrant 4 have positive ratios.



Vectors



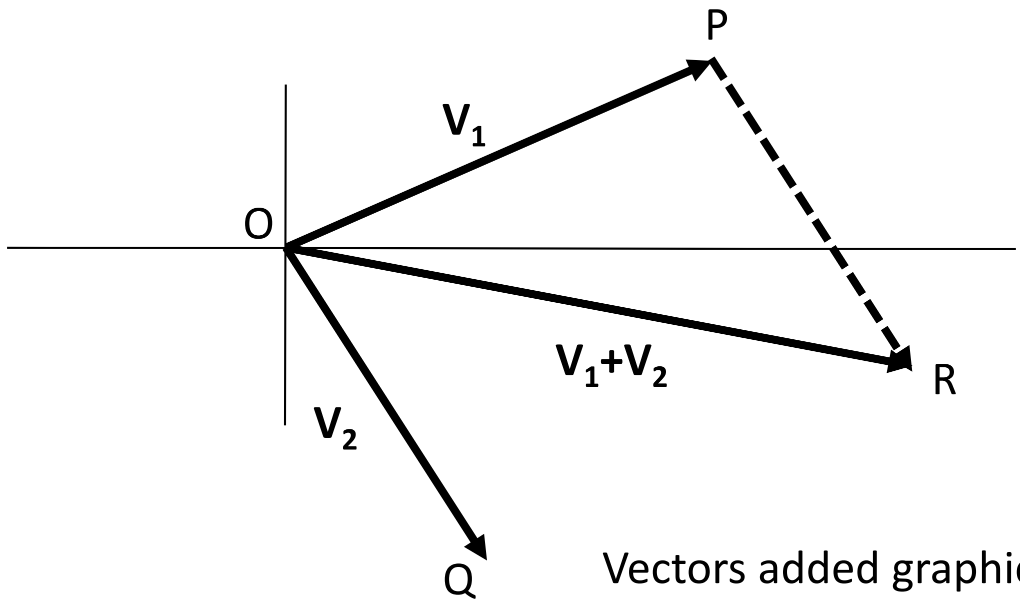
- **OP** = x**i** + y**j**
  where **i** and **j** are **unit vectors** along the x- and y-axes, respectively.

- The magnitude or modulus of **OP** = 5**i** + 2**j** is
  |**OP**| = sqrt(5^2 + 2^2) = 5.39

- Unit vector of OP is
  (OP) = **OP** / |**OP**| = (5**i** + 2**j**) / 5.39 = 0.93**i** + 0.37**j**

- sin(α) = 2 / |**OP**| = 2/5.39 = 0.37
  cos(α) = 5 / |**OP**| = 5/5.39 = 0.93

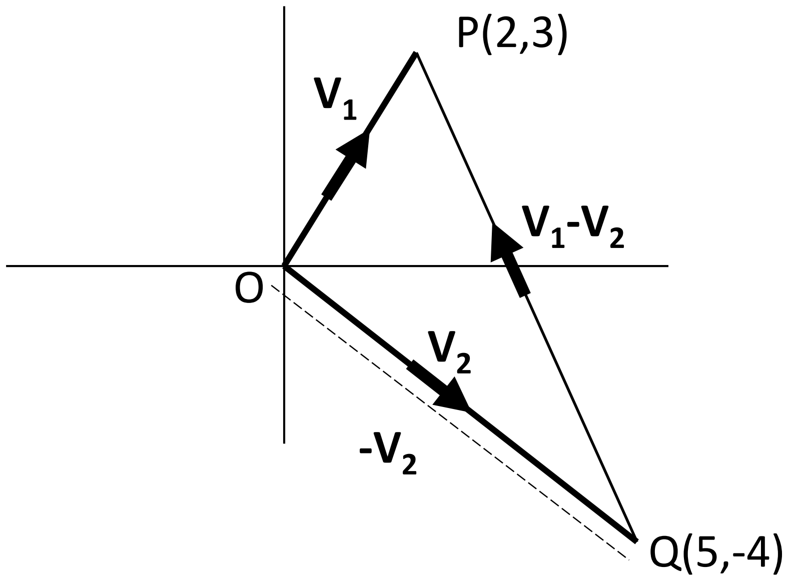
Vector addition



Vectors added graphically

- For two vectors **OP** and **OQ** such as
  OP = $\mathbf{V_1}$ = 5**i** + 2**j** and **OQ** = $\mathbf{V_2}$ = 2**i** - 4**j**
- The vector addition is the sum of vectors **OP** and **OQ**
  $\mathbf{V_1} + \mathbf{V_2} = (5\mathbf{i} + 2\mathbf{j}) + (2\mathbf{i} - 4\mathbf{j}) = 7\mathbf{i} - 2\mathbf{j}$
- The direction of $\mathbf{V_1} + \mathbf{V_2}$ with respect to the x-axis is
  $\cos(\alpha) = 7 / |\mathbf{V_1} + \mathbf{V_2}| = 7 / \mathrm{sqrt}(7^2 + (-2)^2) = 0.962$

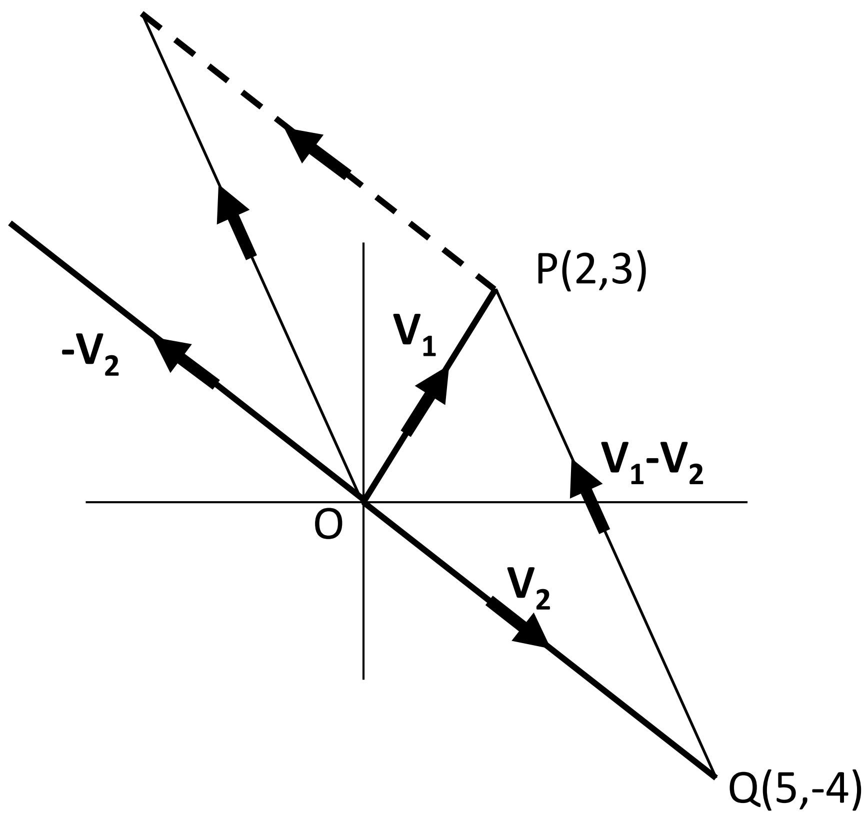
Vector subtraction



Vectors subtracted graphically

- For two vectors **OP** and **OQ** such as **OP** = $\mathbf{V_1}$ = 2**i** + 3**j** and **OQ** = $\mathbf{V_2}$ = 5**i** - 4**j**
- $\mathbf{V_1} - \mathbf{V_2} = (2\mathbf{i} + 3\mathbf{j}) - (5\mathbf{i} - 4\mathbf{j}) = -3\mathbf{i} + 7\mathbf{j}$
- The direction of $\mathbf{V_1}$ - $\mathbf{V_2}$ with respect to the x-axis is $\cos(\alpha) = -3 / |\mathbf{V_1} + \mathbf{V_2}| = -3 / \mathrm{sqrt}((-3)^2 + 7^2) = -0.394$

Vector subtraction



Vectors subtracted graphically

Vector scaling

- A vector may be scaled up or down by multiplying it with a scalar number. Assume the following vector

$$\mathbf{V} = 4\mathbf{i} + 3\mathbf{j}$$

multiplying by 3, we have

$$3*\mathbf{V} = 3*(4\mathbf{i} + 3\mathbf{j}) = 12\mathbf{i}+9\mathbf{j}$$

multiplying by 1/2, we have

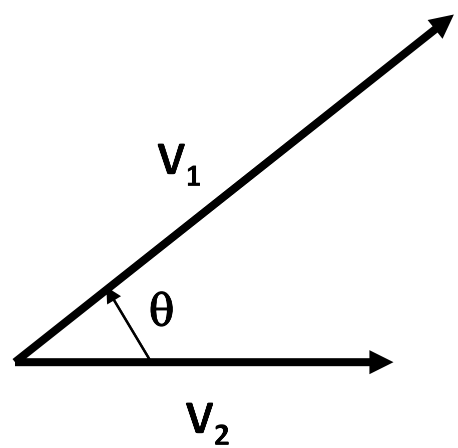
$$(1/2)*\mathbf{V} = (1/2)*(4\mathbf{i} + 3\mathbf{j}) = 2\mathbf{i} + 1.5\mathbf{j}$$

Dot product of two vectors

- Given vectors $\mathbf{V_1}$ and $\mathbf{V_2}$, their dot product is a scalar.

  $\mathbf{V_1} \bullet \mathbf{V_2} = |\mathbf{V_1}|\,|\mathbf{V_2}|\cos(\alpha)$ where $0 \le \alpha \le 180^o$

  $\cos(\alpha) = \mathbf{V_1} \bullet \mathbf{V_2} / (|\mathbf{V_1}|\,|\mathbf{V_2}|)$



Dot product of two vectors

- The product $\mathbf{V_1 \bullet V_2}$ for $\mathbf{V_1} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{V_2} = x_2\mathbf{i} + y_2\mathbf{j}$ is

$$\begin{aligned}\mathbf{V_1 \bullet V_2} &= (x_1\mathbf{i})*(x_2\mathbf{i} + y_2\mathbf{j}) + (y_1\mathbf{j})*(x_2\mathbf{i} + y_2\mathbf{j}) \\ &= (x_1*x_2)*\mathbf{i}*\mathbf{i} + (y_1*y_2)*\mathbf{j}*\mathbf{j} + (x_1*y_2)*\mathbf{i}*\mathbf{j} + (y_1*x_2)*\mathbf{j}*\mathbf{i}\end{aligned}$$

- Because $\mathbf{i*i} = \mathbf{j*j} = 1$ and $\mathbf{i*j} = \mathbf{j*i} = 0$, therefore

$$\mathbf{V_1 \bullet V_2} = x_1*x_2 + y_1*y_2$$

- The dot product is also expressed as

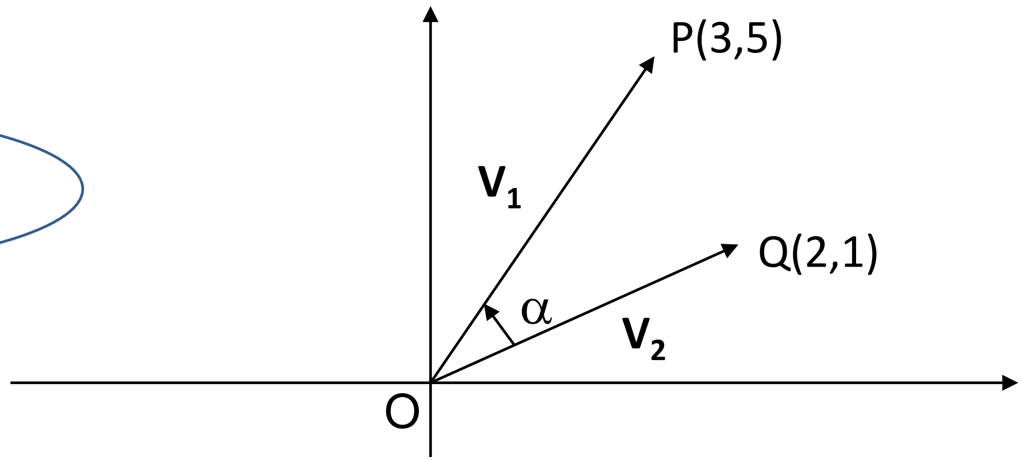
$$\mathbf{V_1 \bullet V_2} = |\mathbf{V_1}||\mathbf{V_2}|\cos(\alpha)$$

therefore

$$\begin{aligned}\cos(\alpha) &= \mathbf{V_1 \bullet V_2} / (|\mathbf{V_1}||\mathbf{V_2}|) \\ &= (x_1*x_2 + y_1*y_2) / (|\mathbf{V_1}||\mathbf{V_2}|)\end{aligned}$$

Example use of dot product

Find angle α?



$\mathbf{V_1} = 3\mathbf{i} + 5\mathbf{j}$

$\mathbf{V_2} = 2\mathbf{i} + \mathbf{j}$

$\mathbf{V_1} \bullet \mathbf{V_2} = 3*2 + 5*1 = 11$

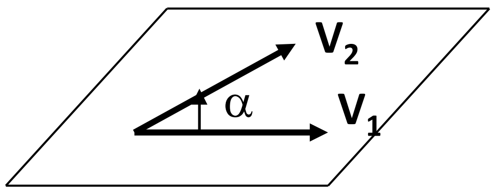
$|\mathbf{V_1}| = \text{sqrt}(3^2 + 5^2) = \text{sqrt}(34) = 5.831$

$|\mathbf{V_2}| = \text{sqrt}(2^2 + 1) = \text{sqrt}(5) = 2.236$

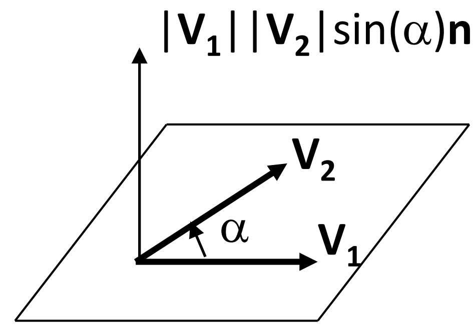
$\cos(\alpha) = \mathbf{V_1} \bullet \mathbf{V_2} / (|\mathbf{V_1}||\mathbf{V_2}|) = 11 / (5.831*2.236) = 0.8437$

$\alpha = 32.47^o$

Cross product of two vectors



(i)



(ii)

- For two vectors $\mathbf{V_1}$ and $\mathbf{V_2}$ lying on a plane (Figure i), their cross product is another vector, which is perpendicular to the plane (Figure ii).

- The cross product is defined as

$$\mathbf{V_1} \times \mathbf{V_2} = |\mathbf{V_1}||\mathbf{V_2}|\sin(\alpha)\mathbf{n}$$

where $0 <= \alpha <= 180$ and **n** is a unit vector along the direction of the plane normal obeying the right-hand rule.

Cross product of two vectors

- $\mathbf{V_1} \times \mathbf{V_2} = -\mathbf{V_2} \times \mathbf{V_1}$
- $\mathbf{V_1} \times \mathbf{V_2} = |\mathbf{V_1}||\mathbf{V_2}|\sin(\alpha)\mathbf{n}$, thus $|\mathbf{V_1} \times \mathbf{V_2}| = |\mathbf{V_1}||\mathbf{V_2}|\sin(\alpha)$
- When $\alpha = 0$, $\sin(\alpha) = 0$. Hence $\mathbf{i}\mathrm{x}\mathbf{i} = \mathbf{j}\mathrm{x}\mathbf{j} = \mathbf{k}\mathrm{x}\mathbf{k} = 0$
  where **i**, **j** and **k** are unit vectors along the x, y and z axes, respectively.
- When $\alpha = 90$, $\sin(\alpha) = 1$. Hence $\mathbf{i}\mathrm{x}\mathbf{j} = \mathbf{k}$, $\mathbf{j}\mathrm{x}\mathbf{k} = \mathbf{i}$ and $\mathbf{k}\mathrm{x}\mathbf{i} = \mathbf{j}$
- From the identity above, the reverse is true,
  i.e. $\mathbf{j}\mathrm{x}\mathbf{i} = -\mathbf{k}$, $\mathbf{k}\mathrm{x}\mathbf{j} = -\mathbf{i}$ and $\mathbf{i}\mathrm{x}\mathbf{k} = -\mathbf{j}$

Matrices

- Matrices are techniques for applying transformations.
- A matrix is simply a set of numbers arranged in a rectangular format.
- Each number is known as an element.
- Capital letters are used to represent matrices, bold letters when printed **M**, or underlined when written $\underline{M}$.
- A matrix has dimensions that refer to the number of rows and the number of columns it has.

Dimensions of matrices

$$\mathbf{A} = \begin{array}{c} \begin{array}{ccc} \text{Col 1} & \text{Col 2} & \text{Col 3} \end{array} \\ \begin{bmatrix} 1 & 6 & 3 \\ -1 & 2 & 4 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Row 1} \\ \text{Row 2} \end{array}$$

The dimensions of A are (2×3)

$$\mathbf{B} = \begin{array}{c} \begin{array}{ccc} \text{Col 1} & \text{Col 2} & \text{Col 3} \end{array} \\ \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ 1 & -4 & 6 \\ 0 & 2 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \text{Row 4} \end{array}$$

The dimensions of B are (4×3)

$$\mathbf{C} = \begin{array}{c} \begin{array}{cc} \text{Col 1} & \text{Col 2} \end{array} \\ \begin{bmatrix} 1 & 6 \\ 2 & 9 \\ -3 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \text{Row 4} \\ \text{Row 5} \end{array}$$

The dimensions of C are (5×2)

$$\mathbf{D} = \begin{array}{c} \begin{array}{ccc} \text{Col 1} & \text{Col 2} & \text{Col 3} \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0.5 \\ 0.3 & 0 & 0.2 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{array}$$

The dimensions of D are (3×3)

Transpose matrix

- When a matrix is rewritten so that its rows and columns are interchanged, then the resulting matrix is called the transpose of the original.

$$\mathbf{A} = \begin{bmatrix} 1 & 6 & 3 \\ -1 & 2 & 4 \end{bmatrix}$$

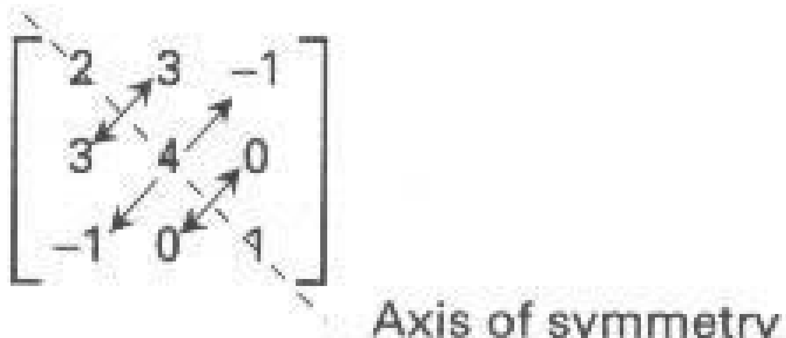
The dimensions of A are (2×3)

$$\mathbf{A}' = \begin{bmatrix} 1 & -1 \\ 6 & 2 \\ 3 & 4 \end{bmatrix}$$

The dimensions of A' are (3×2)

Square and symmetric matrices

- A **square matrix** is a matrix where the number of rows equals the number of columns (e.g., Matrix D in slide 21).
- A **symmetric matrix** is a **square matrix** where the rows and columns are such that its transpose is the same as the original matrix, i.e., elements $a_{ij} = a_{ji}$ where $i \neq j$.



Identity matrices

- An **identity matrix, I** is a square and symmetric matrix with zeros everywhere except its diagonal elements which have a value of 1.
- Examples of 2x2, 3x3, and 4x4 matrices are

$$\mathbf{I}_{(2\times 2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{I}_{(3\times 3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \mathbf{I}_{(4\times 4)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Adding matrices

- Matrices **A** and **B** may be added if they have the same dimensions.
- That is, the corresponding elements may be added to yield a resulting matrix.
- The sum is **commutative**, i.e., **A** + **B** = **B** + **A**

$$\begin{bmatrix} 2 & \boxed{1} & 3 \\ 4 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & \boxed{2} & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & \boxed{3} & 3 \\ 4 & 2 & -1 \end{bmatrix}$$

Subtracting matrices

- Matrix **B** may be subtracted from matrix **A** if they have the same dimensions, i.e., the corresponding elements of **B** may be subtracted from those of **A** to yield a resulting matrix.

$$\begin{bmatrix} 2 & 1 \\ 6 & \boxed{5} \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & \boxed{2} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & \boxed{3} \\ -1 & 1 \end{bmatrix}$$

- The result is **not commutative**. Reversing the order of the matrices yields different results, i.e. **A** - **B** $\neq$ **B** -**A**

$$\begin{bmatrix} 6 & 2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & -1 \end{bmatrix}$$

Reversing the operation

$$\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -2 & 1 \end{bmatrix}$$

Different result

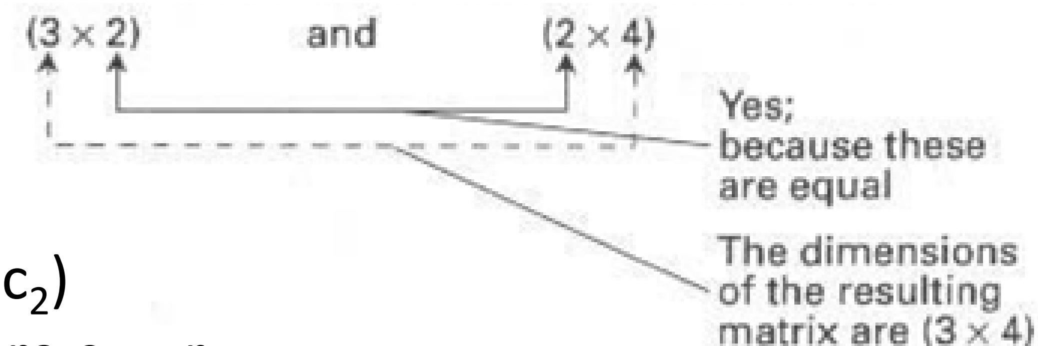
Multiplying matrices

- By a constant

$$3\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix} = \begin{bmatrix}3 & 6\\6 & 3\end{bmatrix}$$

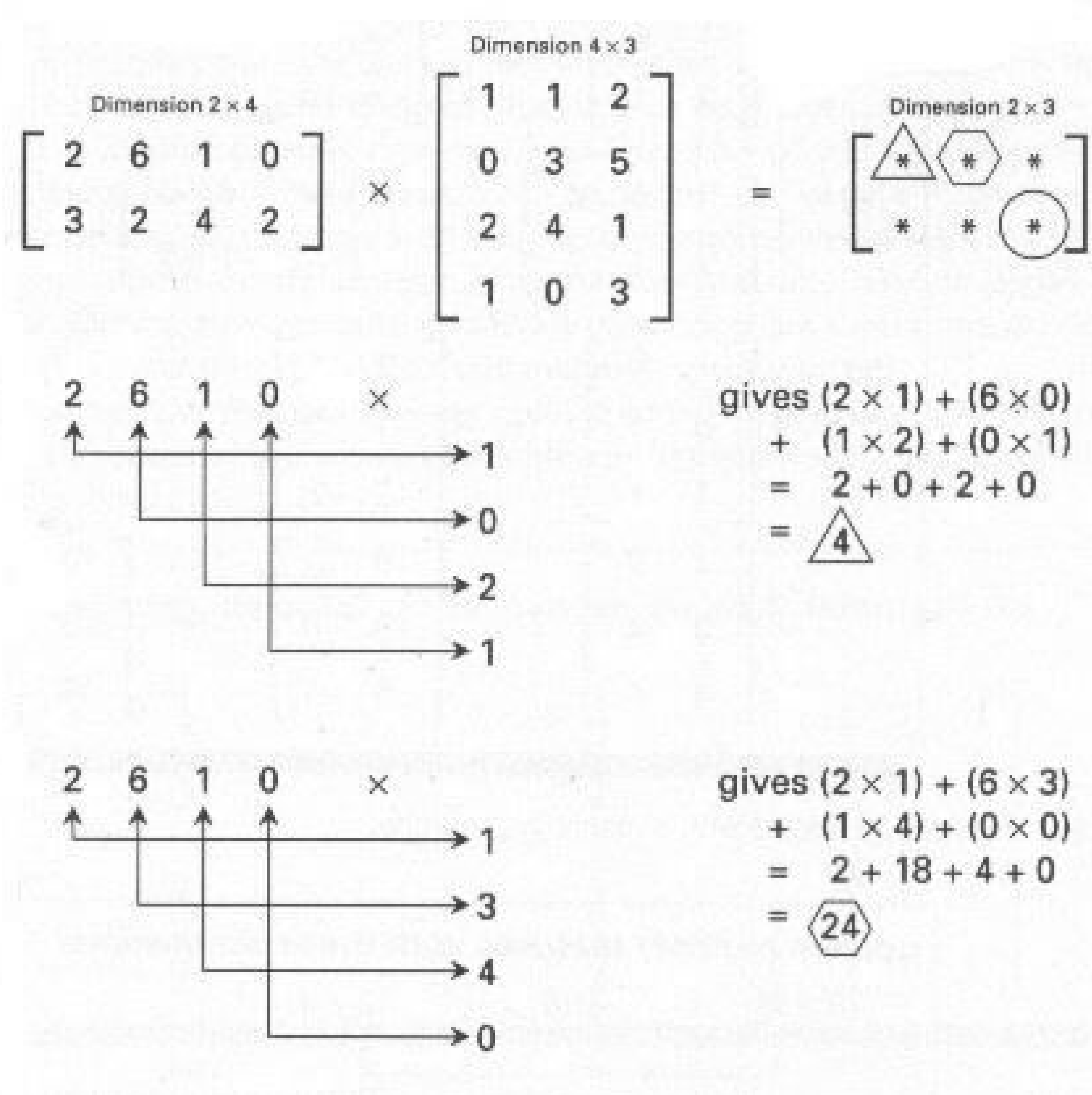
$$-1\begin{bmatrix}1 & 0 & 1\\2 & 1 & 2\end{bmatrix} = \begin{bmatrix}-1 & 0 & -1\\-2 & -1 & -2\end{bmatrix}$$

- By a matrix - The rule for multiplying one matrix to another is simple: if the **number of columns** in the first matrix is the **same as the number of rows** in the second matrix, the multiplication can be done.

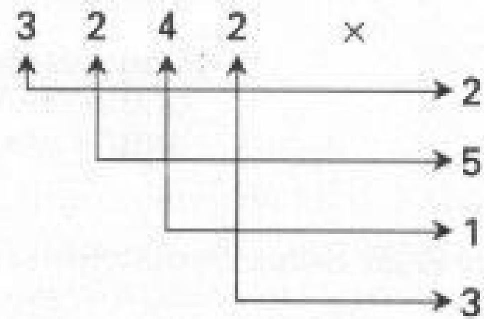


So $\mathbf{M_1}(r_1 * c_1) * \mathbf{M_2}(r_2 * c_2)$
$= \mathbf{M_3}(r_1 * c_2)$ where $c_1 = r_2$

Multiplying matrices



Multiplying matrices – example



gives $(3 \times 2) + (2 \times 5)$
$+ \quad (4 \times 1) + (2 \times 3)$
$= \quad 6 + 10 + 4 + 6$
$= \quad 26$

$$\begin{bmatrix} 2 & 6 & 1 & 0 \\ 3 & 2 & 4 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 5 \\ 2 & 4 & 1 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 24 & 35 \\ 13 & 25 & 26 \end{bmatrix}$$

The overall result

Non-commutative property of matrix multiplication

- Matrix multiplication is not **commutative.**

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 14 \\ 7 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 9 \\ 22 & 14 \end{bmatrix}$$

- Reversing the order of the matrices yields different results.

Non-commutative property of matrix multiplication

- Reversing the order of the matrices yields different results (e.g., Slide 30) or the condition for matrix multiplication will not be satisfied (e.g., Slide 28).

Further example:

$$\mathbf{A} = [1 \;\; 2 \;\; 3], \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Will the following multiplications be possible?

A*B
B*A

Inverse matrices

- If two matrices **A** and **B**, when multiplied together, results in an identity matrix **I**, then matrix **A** is the inverse of matrix **B** and vice versa, i.e.,

$$\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A} = \mathbf{I}$$

$$\mathbf{A} = \mathbf{B}^{-1} \text{ and } \mathbf{B} = \mathbf{A}^{-1}$$

e.g.,

$$\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$$

Topics covered today

- Computer representation of objects
- Cartesian co-ordinate system
- Points, lines and angles
- Trigonometry
- Vectors (unit vector) and vector calculations (addition, subtraction, scaling, dot product and cross product)
- Matrices (dimension, transpose, square/symmetric/identity and inverse) and matrix calculations (addition, subtraction and multiplication)