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Magnetoelastic modelling of elastomers

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Abstract

In this paper we first summarize the equations governing the deformation of magneto-sensitive (MS) elastic solids with particular reference to elastomers whose mechanical properties may be changed rapidly by the application of a magnetic field. These 'smart materials' typically consist of micron-sized ferrous particles dispersed within an elastomeric matrix. Constitutive relations for isotropic MS-elastic solids are examined. The equations are then applied to a representative geometry appropriate for applications, that in which the material is confined to a circular cylindrical tube in the presence of a radial magnetic field. The material is then subject to an axial shear deformation. Results are illustrated for two specific material models, for each of which the shear stiffness of the material increases with the magnetic field strength, as observed in experiments on MS elastomers.

Keywords: Elastomers; Magnetoelasticity

1. Introduction

Magneto-sensitive (MS) elastomers, also referred to as magnetorheological (MR) elastomers in the literature, are materials that change their mechanical behaviour in response to the application of magnetic fields. They have attracted considerable interest, especially in recent years, because of their potential for providing relatively simple and quiet variable-stiffness devices as rapid-response interfaces between electronic controls and mechanical systems. See, for example (Kordonsky, 1993; Carlson and Jolly, 2000; Jolly et al., 1996). MS elastomers have been used, for example, in adaptive tuned vibration absorbers, stiffness tunable mounts and suspensions and automotive bushing.

Typically, the magnetic response is achieved and optimized by distributing within an elastomeric matrix particles with a high magnetic saturation, such as an alloy of iron, and volume fractions between 0.1 and 0.5. The choice of the matrix material is based on its thermomechanical properties and, for example, silicone and other elastomers are found to be suitable materials.

In a recent paper (Dorfmann and Brigadnov, 2003) MS elastomers were considered as homogenized single non-polar isotropic continua. The relevant equations were based on those in the classic work of (Pao, 1978) in which the equations of motion for an isotropic non-polar continuum in an electromagnetic field are described by Maxwell's equations and the mechanical and thermodynamical balance laws. The basic system of constitutive equations for MS Cauchy-elastic solids was then developed using a phenomenological approach based on experimental data of (Carlson and Jolly, 2000). The resulting equations, coupled with suitable boundary and initial conditions, were then used to illustrate the application of the constitutive model and the theory by considering the simple shearing of a MS elastomer between two parallel plates with a magnetic field normal to the plates.

In this paper we first summarize, in Section 2, the relevant general theory and then, in Section 3, specialize the theory by making simplifying assumptions on the electromagnetic properties appropriate for MS elastomers. The corresponding

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mechanical constitutive properties are discussed in Section 4. The reduced theory is then applied, in Section 5, to a realistic problem in which the material is confined within a rigid circular cylindrical tube and subjected to an axial shearing deformation in the presence of a radial magnetic field. The influence of the magnetic field strength on the axial shear stress-strain response is illustrated for two particular constitutive models. For these example the shear stress required to achieve a given axial displacement increases with the strength of the magnetic field.

2. Governing equations

We consider a continuous deformable solid occupying a region $\mathcal{B}_r \subset \mathbb{R}^3$ in a stress-free (natural) configuration, with material particles labelled by their position vectors $\mathbf{X} \in \mathcal{B}_r \cup \partial\mathcal{B}_r$, where $\partial\mathcal{B}_r$ is the boundary of $\mathcal{B}_r$. In the deformed configuration the point $\mathbf{X}$ occupies the position $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \in \mathbb{R}^3$, where the vector field $\boldsymbol{\chi}$ describes the deformation (or motion) of the continuum and t is time. We consider $\boldsymbol{\chi}$ to be a one-to-one, orientation-preserving mapping with appropriate regularity properties and defined on $\mathcal{B}_r \cup \partial\mathcal{B}_r$ for every $t > 0$.

As required, vectors and tensors are referred to the (Cartesian) orthonormal right-handed set of basis vectors $\mathbf{e}_i,\ i \in \{1, 2, 3\}$. The permutation symbol is defined by $\varepsilon_{ijk} = (\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k$, so that $\varepsilon_{ijk} = 1\ (-1)$ if (ijk) is a cyclic (anticyclic) permutation of (123) and zero otherwise.

The deformation gradient $\mathbf{F}$, the velocity $\mathbf{v}$, its spatial gradient $\boldsymbol{\Gamma}$ and the deformation rate tensor $\boldsymbol{\Sigma}$ are defined, respectively, by

$$\mathbf{F} = \operatorname{Grad}\boldsymbol{\chi}, \qquad \mathbf{v} = \frac{\partial\boldsymbol{\chi}}{\partial t}, \qquad \boldsymbol{\Gamma} = \operatorname{grad}\mathbf{v}, \qquad \boldsymbol{\Sigma} = \frac{1}{2}\left(\boldsymbol{\Gamma} + \boldsymbol{\Gamma}^{\mathrm{T}}\right), \tag{2.1}$$

where the superscript $^{\mathrm{T}}$ denotes the transpose of a tensor and Grad and grad denote the gradient operators with respect to $\mathbf{X}$ and $\mathbf{x}$ respectively. For the gradient of a vector we use the convention such that, in component form, $(\operatorname{grad}\mathbf{v})_{ij} = \partial v_i/\partial x_j$, for example.

The other relevant mechanical variables are the mass density ρ, the Cauchy stress tensor $\boldsymbol{\sigma}$ and the external body force $\mathbf{f}$ per unit mass (see, for example, (Holzapfel, 2001; Ogden, 1997)). Additionally, we use the notations T for absolute temperature and $\mathbf{Q}$ for the (Eulerian) heat flux vector (Holzapfel, 2001; Müller and Ruggeri, 1993).

The material time derivatives of a scalar-valued function $\varphi = \varphi(\mathbf{x}, t)$ and a vector-valued function $\mathbf{a} = \mathbf{a}(\mathbf{x}, t)$ are expressed using a superposed dot and defined by

$$\dot{\varphi} = \frac{\partial\varphi}{\partial t} + \mathbf{v}\cdot\operatorname{grad}\varphi, \qquad \dot{\mathbf{a}} = \frac{\partial\mathbf{a}}{\partial t} + \mathbf{v}\cdot\operatorname{grad}\mathbf{a}. \tag{2.2}$$

For the electromagnetic variables we use the following notations: the electric field intensity $\mathbf{E}$, magnetic field intensity $\mathbf{H}$, electric induction or displacement $\mathbf{D}$, electric polarization density $\mathbf{P}$, magnetic induction $\mathbf{B}$, magnetic polarization density $\mathbf{M}$, free electric current density $\mathbf{J}$; all of these are vectors. Additionally, we have the (scalar) free electric charge density q. For full details we refer to, for example, (Eringen and Maugin, 1989; Jackson, 1983; Pao, 1978).

2.1. Maxwell's equations for a continuum

In this section we summarize Maxwell's equations in standard SI units (Eringen and Maugin, 1989; Jackson, 1983).

In vacuo the electric field $\mathbf{E}$ and the magnetic flux $\mathbf{B}$ are regarded as the basic variables. For condensed matter additional variables, such as the electric polarization density $\mathbf{P}$ and the magnetic polarization density $\mathbf{M}$ are introduced. These two variables are used in the relations

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}, \qquad \mathbf{B} = \mu_0\mathbf{H} + \mathbf{M}, \tag{2.3}$$

where the constants ε_0 and μ_0 are, respectively, the vacuum (electric) permittivity and the vacuum (magnetic) permeability. They satisfy the identity $\varepsilon_0\mu_0 = c^{-2}$, where c is the vacuum speed of light. The notation $\mathbf{M}$ used here differs from that in (Pao, 1978), which is recovered by replacing $\mathbf{M}$ by $\mu_0\mathbf{M}$.

The Gauss, Faraday and Ampère laws are, respectively,

$$\varepsilon_0 \operatorname{div}\mathbf{E} = q - \operatorname{div}\mathbf{P}, \tag{2.4}$$

$$\operatorname{curl}\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = \mathbf{0}, \tag{2.5}$$

$$\mu_0^{-1}\operatorname{curl}\mathbf{B} = \varepsilon_0\frac{\partial\mathbf{E}}{\partial t} + \frac{\partial\mathbf{P}}{\partial t} + \mu_0^{-1}\operatorname{curl}\mathbf{M} + \mathbf{J}, \tag{2.6}$$

where $-\operatorname{div}\mathbf{P}$ is the polarization charge, $\partial\mathbf{P}/\partial t$ is the polarization current, and $\mu_0^{-1}\operatorname{curl}\mathbf{M}$ is the magnetization current.

From (2.5), on the assumption that $\operatorname{div}\mathbf{B}=0$ at some initial time, we obtain the conservation equation for the magnetic flux, namely

$$\operatorname{div}\mathbf{B}=0. \tag{2.7}$$

The conservation equation for electric charge follows from (2.4) and (2.6) in the form

$$\frac{\partial q}{\partial t}+\operatorname{div}\mathbf{J}=0. \tag{2.8}$$

2.2. Mechanical balance laws

The conservation of mass equation for a continuum may be written in either of the equivalent forms

$$\dot{\rho}+\rho\operatorname{div}\mathbf{v}=0, \qquad J\rho=\rho_0, \tag{2.9}$$

where ρ_0 is the mass density in the reference configuration and $J=\det\mathbf{F}>0$ is the volume ratio.

The equation of linear momentum balance is

$$\rho\dot{\mathbf{v}}=\operatorname{div}\boldsymbol{\sigma}+\rho\mathbf{f}+\mathbf{f}_e, \tag{2.10}$$

where $\mathbf{f}_e$ is the electromagnetic force (per unit volume). For full details of the mechanical equations we refer to, for example, (Holzapfel, 2001; Ogden, 1997).

For the dipole-current circuit model for a moving continuum (Pao, 1978) $\mathbf{f}_e$ is given by

$$\mathbf{f}_e=q\mathbf{E}+\mathbf{J}\times\mathbf{B}+\mu_0^{-1}(\operatorname{grad}\mathbf{B})^{\mathrm{T}}\mathbf{M}+(\operatorname{grad}\mathbf{E})^{\mathrm{T}}\mathbf{P}+\frac{\partial}{\partial t}(\mathbf{P}\times\mathbf{B})+\operatorname{div}(\mathbf{vP}\times\mathbf{B}), \tag{2.11}$$

and the balance of angular momentum takes the local form

$$\boldsymbol{\varepsilon\sigma}+\left(\mu_0^{-1}\mathbf{M}+\mathbf{v}\times\mathbf{P}\right)\times\mathbf{B}+\mathbf{P}\times\mathbf{E}=\mathbf{0}, \tag{2.12}$$

where $\boldsymbol{\varepsilon}$ denotes the third-order permutation tensor with components ε_{ijk} and $(\boldsymbol{\varepsilon\sigma})_i=\varepsilon_{ijk}\sigma_{jk}$.

2.3. Thermodynamic equations

The relevant thermodynamic equations have the following forms (see, for example (Holzapfel, 2001; Müller and Ruggeri, 1993; Pao, 1978)). The first equation is the balance of energy in the local form

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}\left(U+\frac{1}{2}|\mathbf{v}|^2\right)+\operatorname{div}\mathbf{Q}=\operatorname{div}(\boldsymbol{\sigma}\mathbf{v})+\rho\mathbf{f}\cdot\mathbf{v}+\rho R+w_e, \tag{2.13}$$

where U and R denote the specific (per unit mass) internal energy and the radiant heating, respectively. The electromagnetic power, denoted w_e, is defined by

$$w_e=\mathbf{f}_e\cdot\mathbf{v}+\mathbf{J}_e\cdot\mathbf{E}_e-\mathbf{M}_e\cdot\dot{\mathbf{B}}+\rho\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{P}}{\rho}\right)\cdot\mathbf{E}_e, \tag{2.14}$$

where $\mathbf{J}_e$ is the effective conduction current, $\mathbf{E}_e$ is the effective electric field intensity and $\mathbf{M}_e$ is the effective magnetization in the rest frame. According to (Minkowski, 1908) these are related to the laboratory frame variables by

$$\mathbf{J}_e=\mathbf{J}-q\mathbf{v}, \qquad \mathbf{E}_e=\mathbf{E}+\mathbf{v}\times\mathbf{B}, \qquad \mathbf{M}_e=\mu_0^{-1}\mathbf{M}+\mathbf{v}\times\mathbf{P}. \tag{2.15}$$

Using (2.9) with the standard relation $\dot{J}=J\operatorname{div}\mathbf{v}$ and (2.10) we obtain the reduced form of (2.13), namely

$$\rho\dot{U}+\operatorname{div}\mathbf{Q}=\operatorname{tr}\left(\boldsymbol{\sigma}\boldsymbol{\Gamma}^{\mathrm{T}}\right)-\mathbf{M}_e\cdot\dot{\mathbf{B}}+\mathbf{J}_e\cdot\mathbf{E}_e+\rho R+\dot{\mathbf{P}}\cdot\mathbf{E}_e+(\mathbf{P}\cdot\mathbf{E}_e)\operatorname{div}\mathbf{v}, \tag{2.16}$$

where tr denotes the trace of a second-order tensor. Next we note the Clausius–Duhem inequality in the local form

$$\rho\dot{S}+\operatorname{div}\left(\frac{\mathbf{Q}}{T}\right)-\rho\frac{R}{T}\geqslant 0, \tag{2.17}$$

where S is the specific entropy.

Introducing the specific Helmholtz free energy Ψ through

$$\Psi=U-TS-\frac{1}{\rho}\mathbf{E}_e\cdot\mathbf{P}, \tag{2.18}$$

and substituting (2.18) and (2.13) into (2.17) we obtain the main dissipation inequality

$$-\rho(\dot{\Psi}+\dot{T}S)+\mathrm{tr}\big(\boldsymbol{\sigma}\boldsymbol{\Gamma}^{\mathrm{T}}\big)-\mathbf{M}_e\cdot\dot{\mathbf{B}}-\frac{1}{T}\,\mathbf{Q}\cdot\mathrm{grad}\,T+\mathbf{J}_e\cdot\mathbf{E}_e-\mathbf{P}\cdot\dot{\mathbf{E}}_e\geqslant 0. \tag{2.19}$$

3. Reduced system of equations for MS elastomers

To the equations discussed above have to be appended appropriate material constitutive laws. From a thermodynamical point of view, the variables $(\mathbf{F},\mathbf{v},\mathbf{E}_e,\mathbf{B},T,\mathrm{grad}\,T)$ are independent quantities for non-potential materials (Truesdell, 1991; Truesdell and Noll, 1992) and we therefore need to define constitutive relations for $(U,S,\boldsymbol{\sigma},\mathbf{P},\mathbf{J}_e,\mathbf{M}_e,\mathbf{Q})$.

The heat flux vector $\mathbf{Q}$ is taken here to be given by the Fourier law of heat conduction

$$\mathbf{Q}=-k\,\mathrm{grad}\,T, \tag{3.1}$$

where k, the thermal conductivity, is a positive constant. In the remaining constitutive relations we omit the dependence on the temperature gradient $\mathrm{grad}\,T$ (Holzapfel, 2001).

From the principle of material objectivity (Truesdell, 1991) it follows that all constitutive relations are independent of $\mathbf{v}$. As a result, we have

$$\boldsymbol{\Phi}=\hat{\boldsymbol{\Phi}}(T,\mathbf{F},\mathbf{E}_e,\mathbf{B}), \tag{3.2}$$

where $\boldsymbol{\Phi}=(U,S,\boldsymbol{\sigma},\mathbf{P},\mathbf{J}_e,\mathbf{M}_e)$ is the generalized vector of the considered variables.

Using (2.9), (3.1), (3.2) and computing $\dot{\Psi}$ in (2.19) we obtain the following form of the Clausius–Duhem inequality:

$$\begin{aligned}&-\rho\left(\frac{\partial\Psi}{\partial T}+S\right)\dot{T}-\mathrm{tr}\left[\left(\rho\frac{\partial\Psi}{\partial\mathbf{F}}-\big(\mathbf{F}^{-1}\boldsymbol{\sigma}\big)^{\mathrm{T}}\right)\dot{\mathbf{F}}\right]-\left(\rho\frac{\partial\Psi}{\partial\mathbf{E}_e}+\mathbf{P}\right)\cdot\dot{\mathbf{E}}_e\\&\quad-\left(\rho\frac{\partial\Psi}{\partial\mathbf{B}}+\mathbf{M}_e\right)\cdot\dot{\mathbf{B}}+k\frac{|\,\mathrm{grad}\,T|^2}{T}+\mathbf{J}_e\cdot\mathbf{E}_e\geqslant 0.\end{aligned} \tag{3.3}$$

This inequality must be satisfied at all times and at every fixed point in space for all *admissible thermodynamic processes*, i.e. processes compatible with the balance laws and the constitutive response functions. Since the quantities $\dot{\mathbf{F}}$, $\dot{T}$, $\dot{\mathbf{E}}_e$ and $\dot{\mathbf{B}}$ are independent (Truesdell and Noll, 1992) and since the inequality (3.3) is linear in these rates we obtain

$$S=-\frac{\partial\Psi}{\partial T},\qquad \mathbf{F}^{-1}\boldsymbol{\sigma}=\rho\frac{\partial\Psi}{\partial\mathbf{F}},\qquad \mathbf{P}=-\rho\frac{\partial\Psi}{\partial\mathbf{E}_e},\qquad \mathbf{M}_e=-\rho\frac{\partial\Psi}{\partial\mathbf{B}}. \tag{3.4}$$

Then, using (2.9), the reduced dissipation inequality becomes

$$k\,\frac{|\,\mathrm{grad}\,T|^2}{T}+\mathbf{J}_e\cdot\mathbf{E}_e\geqslant 0. \tag{3.5}$$

Linear isotropic continuous media are described by the following well-known experimental laws for electromagnetic variables (Jackson, 1983; Pao, 1978):

$$\mathbf{J}=\eta\mathbf{E},\qquad \mathbf{D}=\varepsilon\varepsilon_0\mathbf{E},\qquad \mathbf{B}=\mu\mu_0\,\mathbf{H}. \tag{3.6}$$

In (3.6) $\eta\geqslant 0$ is the electric conductivity, while $\varepsilon\geqslant 1$ and $\mu\geqslant 1$ are the dielectric permittivity and magnetic permeability, respectively. In vacuo $\eta=0$, $\varepsilon=1$ and $\mu=1$, while for dielectrics $\mu=1$ and for conductors $\varepsilon=1$.

From (2.3), (2.15) and (3.6) we obtain the constitutive relations for the effective conduction current $\mathbf{J}_e$ and the effective magnetization $\mathbf{M}_e$ in the forms

$$\mathbf{J}_e=\eta\mathbf{E}-q\mathbf{v},\qquad \mathbf{M}_e=\gamma\mathbf{B}+(\varepsilon-1)\varepsilon_0\,\mathbf{v}\times\mathbf{E}, \tag{3.7}$$

where $\gamma=(\mu-1)(\mu_0\mu)^{-1}\geqslant 0$ is a function of magnetic saturation that may be determined easily from the standard $\mu_0 H$ against B experimental curves (Jolly et al., 1996).

In the following some of our assumptions are based on experimental data of (Carlson and Jolly, 2000; Jolly et al., 1996).

The main assumption is that for MS materials the electric polarization is negligible, and we therefore set

$$\mathbf{P}=\mathbf{0}, \tag{3.8}$$

so that $\varepsilon=1$ in the above relations. From $(3.7)_2$ we then have $\mathbf{M}_e=\mu_0^{-1}\mathbf{M}=\gamma\mathbf{B}$. As a result, from the law of the balance of angular momentum in the form (2.12) it follows that the stress tensor $\boldsymbol{\sigma}$ is symmetric, i.e. it is the *Cauchy stress tensor*, and the equality $\mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{\Gamma}^{\mathrm{T}})=\mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{\Sigma})$ therefore holds.

We can now write our system of equations as

$$\operatorname{div}\mathbf{E}=\frac{q}{\varepsilon_0},\qquad \operatorname{curl}\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=\mathbf{0},\tag{3.9}$$

$$\operatorname{curl}\big(\mu^{-1}\mathbf{B}\big)=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}+\mu_0\eta\,\mathbf{E},\quad \operatorname{div}\mathbf{B}=0,\tag{3.10}$$

$$\rho\dot{\mathbf{v}}=\operatorname{div}\boldsymbol{\sigma}+\rho\mathbf{f}+\mathbf{f}_e,\quad \rho=\rho_0\,J^{-1},\tag{3.11}$$

$$\rho\dot{U}-k\Delta T=\operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{\Sigma})-\gamma\,\mathbf{B}\cdot\dot{\mathbf{B}}+\rho\,R+\mathbf{E}\cdot\mathbf{G},\tag{3.12}$$

together with the dissipation inequality

$$k\,\frac{|\operatorname{grad}T|^2}{T}+\mathbf{E}\cdot\mathbf{G}\geqslant 0,\tag{3.13}$$

where $\mathbf{f}_e$ and $\mathbf{G}$ are given, respectively, by

$$\mathbf{f}_e=q\mathbf{E}+\eta\mathbf{E}\times\mathbf{B}+\gamma(\operatorname{grad}\mathbf{B})^{\mathrm{T}}\mathbf{B},\tag{3.14}$$

$$\mathbf{G}=\eta(\mathbf{E}+\mathbf{v}\times\mathbf{B})-q\mathbf{v}.\tag{3.15}$$

In the system of Eqs. (3.9)–(3.15) some variables and expressions differ by several orders of magnitude in problems of practical interest. In order to identify the relevant magnitudes a non-dimensional analysis of constitutive relations was carried out by Rajagopal and Růžička (2001), and a similar analysis can be conducted here. Rather than doing this, however, we base our approximations here on experimental data for commercial MS elastomers (Carlson and Jolly, 2000; Jolly et al., 1996).

We find that for commercially available MS elastomers, the electric charge and the electric conductivity are very small, and, therefore, we shall assume that $q\equiv 0$ and $\eta\equiv 0$. From $(3.7)_1$ and (3.15) it then follows that $\mathbf{J}_e=\mathbf{0}$ and $\mathbf{G}=\mathbf{0}$.

Next we note that experimental data for MS elastomers show that the influence of the electric field is non-essential, and we therefore omit it from (3.2) and write

$$\boldsymbol{\Phi}=\hat{\boldsymbol{\Phi}}(T,\mathbf{F},\mathbf{B}),\tag{3.16}$$

where the generalized vector of unknown values for MS elastomers now becomes $\boldsymbol{\Phi}=(U,\boldsymbol{\sigma})$.

From $(3.4)_2$, $(3.4)_3$ and (3.8) it follows that the Helmholtz free energy function Ψ is independent of $\mathbf{E}_e$. Moreover, it is assumed that $\Psi=\Psi(T,\mathbf{F},\mathbf{B})$ is a smooth function (Holzapfel, 2001; Truesdell, 1991). Then, using (2.9), (2.18) with $\mathbf{P}=\mathbf{0}$, and (3.4) and (3.7) with $\mathbf{M}_e=\gamma\mathbf{B}$, we obtain the expression

$$\begin{aligned}\rho\dot{U}&=\rho\frac{\mathrm{d}}{\mathrm{d}t}\left(\Psi-T\,\frac{\partial\Psi}{\partial T}\right)=\rho\operatorname{tr}\left(\frac{\partial\Psi}{\partial\mathbf{F}}\dot{\mathbf{F}}\right)+\rho\frac{\partial\Psi}{\partial\mathbf{B}}\cdot\dot{\mathbf{B}}-\rho T\frac{\partial}{\partial T}\left(\frac{\mathrm{d}\Psi}{\mathrm{d}t}\right)\\&=\operatorname{tr}\left[\left(\boldsymbol{\sigma}-T\,\frac{\partial\boldsymbol{\sigma}}{\partial T}\right)\boldsymbol{\Sigma}\right]\operatorname{div}\mathbf{v}-\gamma\mathbf{B}\cdot\dot{\mathbf{B}}+\rho\left(-T\,\frac{\partial^2\Psi}{\partial T^2}\right)\dot{T}-\rho T\,\frac{\partial^2\Psi}{\partial T\,\partial\mathbf{B}}\cdot\dot{\mathbf{B}}\end{aligned}\tag{3.17}$$

for the rate of the internal energy.

The *specific heat capacity* is defined as

$$c_v:=-T\,\frac{\partial^2\Psi}{\partial T^2}>0,\tag{3.18}$$

and using (3.17) we re-write (3.12) in the form

$$c_v\rho\dot{T}-k\Delta T-\rho R=\operatorname{tr}\left[\left(T\,\frac{\partial\boldsymbol{\sigma}}{\partial T}\right)\boldsymbol{\Sigma}\right]+\rho T\,\frac{\partial^2\Psi}{\partial T\,\partial\mathbf{B}}\cdot\dot{\mathbf{B}}.\tag{3.19}$$

It follows from (3.11) and from the independence of $\mathbf{F}$ and T that the density ρ and magnetic field intensity $\mathbf{H}$ do not depend on the temperature T. Therefore, from (3.4), (3.7), (3.8) and the definition of γ following (3.7), we have

$$\rho T\frac{\partial^2\Psi}{\partial T\,\partial\mathbf{B}}=-T\frac{\partial}{\partial T}(\gamma\mathbf{B})=-\frac{T}{\mu_0\mu}\frac{\partial\mu}{\partial T}\mathbf{B}.\tag{3.20}$$

The final form of the system of Eqs. (3.9)–(3.12) now gives

$$\operatorname{div}\mathbf{E} = 0, \qquad \operatorname{curl}\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = \mathbf{0}, \tag{3.21}$$

$$\operatorname{div}\mathbf{B} = 0, \qquad \operatorname{curl}\big(\mu^{-1}\mathbf{B}\big) = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}, \tag{3.22}$$

$$\rho\dot{\mathbf{v}} = \operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{f} + \frac{1}{2}\gamma\operatorname{grad}(\mathbf{B}\cdot\mathbf{B}), \quad \rho = \rho_0 J^{-1}, \tag{3.23}$$

$$c_v\rho\dot{T} - k\Delta T - \rho R = \operatorname{tr}\left[\left(\frac{\partial\boldsymbol{\sigma}}{\partial T}\right)\boldsymbol{\Sigma}\right] - \varphi\mathbf{B}\cdot\dot{\mathbf{B}}, \tag{3.24}$$

while inequality (3.13) is automatically satisfied since $k > 0$. The functions γ and φ used in the above are given by

$$\gamma = \frac{\mu - 1}{\mu_0\mu}, \qquad \varphi = \frac{T}{\mu_0\mu}\frac{\partial\mu}{\partial T} \tag{3.25}$$

and may be determined from experimental data in which $\mu_0 H$ is plotted against B for different temperatures.

Each set of fields $(\mathbf{E}, \mathbf{B}, \mathbf{v}, \boldsymbol{\sigma}, \mathbf{F}, T)$ satisfying the system of Eqs. (3.21)–(3.24) is said to be *admissible*. Assuming that the internal and surface electric charge and the electric current are absent, the *solution* of the system (3.21)–(3.24) is the set of admissible fields satisfying appropriate boundary conditions. Referring to (Pao, 1978), we list the boundary conditions as follows, where $\mathbf{n}$ denotes the unit normal vector to a material surface in the current configuration. The square brackets indicate a discontinuity across a boundary surface of the considered material.

(i) For the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$:

$$\mathbf{n}\cdot[\mathbf{E}] = 0, \qquad \mathbf{n}\times[\mathbf{E} + \mathbf{v}\times\mathbf{B}] = \mathbf{0}, \tag{3.26}$$

$$\mathbf{n}\cdot[\mathbf{B}] = 0, \qquad \mathbf{n}\times\big[\mu^{-1}\mathbf{B} - c^{-2}\mathbf{v}\times\mathbf{E}\big] = \mathbf{0}. \tag{3.27}$$

(ii) For the velocity $\mathbf{v}$, the Cauchy stress tensor $\boldsymbol{\sigma}$ and heat flux $\mathbf{Q} = -k\operatorname{grad}T$:

$$[\mathbf{v}] = \mathbf{0}, \qquad [\boldsymbol{\sigma} + \boldsymbol{\tau}]\mathbf{n} = \mathbf{0}, \tag{3.28}$$

$$\mathbf{n}\cdot\left[(\boldsymbol{\sigma} + \gamma\mathbf{B}\otimes\mathbf{B})\mathbf{v} + \frac{1}{2}\big(\varepsilon_0|\mathbf{E}|^2 + \mu_0^{-1}|\mathbf{B}|^2\big)\mathbf{v} - \varepsilon_0\mathbf{E}\times\mathbf{B} + k\operatorname{grad}T\right] = 0, \tag{3.29}$$

where $\boldsymbol{\tau}$ is the electro-dynamic stress tensor defined by

$$\boldsymbol{\tau} = \frac{1}{2}\big(\varepsilon_0|\mathbf{E}|^2 + \big(\mu_0^{-1} - 2\gamma\big)|\mathbf{B}|^2\big)\mathbf{I} - \big(\varepsilon_0\mathbf{E}\otimes\mathbf{E} + (\mu_0\mu)^{-1}\mathbf{B}\otimes\mathbf{B}\big), \tag{3.30}$$

$\mathbf{I}$ denotes the identity tensor and $\otimes$ the tensor product.

The initial conditions are

$$\mathbf{E}\big|_{t=0} = \mathbf{E}^0, \qquad \mathbf{B}\big|_{t=0} = \mathbf{B}^0, \qquad \mathbf{v}\big|_{t=0} = \mathbf{v}^0, \tag{3.31}$$

$$\boldsymbol{\sigma}\big|_{t=0} = \boldsymbol{\sigma}^0, \qquad -k\operatorname{grad}T\big|_{t=0} = \mathbf{Q}^0, \qquad T\big|_{t=0} = T^0, \tag{3.32}$$

where the zero superscript indicates a prescribed function.

In order to complete the system we require a constitutive law which gives the Cauchy stress tensor in terms of the independent variables.

4. Constitutive relations for MS-elastic materials

We begin by considering a purely elastic material with no electromagnetic interaction. For a Cauchy elastic material there is no strain-energy function (see, for example (Truesdell and Noll, 1992; Ogden, 1997). According to the Rivlin–Ericksen representation theorem the Cauchy stress tensor for an isotropic material states that

$$\boldsymbol{\sigma} = \alpha_0\mathbf{I} + \alpha_1\mathbf{b} + \alpha_2\mathbf{b}^2, \tag{4.1}$$

where $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ is the left Cauchy–Green deformation tensor, and the coefficients $\alpha_0, \alpha_1, \alpha_2$ are functions of its principal invariants, which are defined by

$$I_1(\mathbf{b}) = \operatorname{tr}\mathbf{b}, \qquad I_2(\mathbf{b}) = \frac{1}{2}\big[(\operatorname{tr}\mathbf{b})^2 - \operatorname{tr}\big(\mathbf{b}^2\big)\big], \qquad I_3(\mathbf{b}) = \det\mathbf{b} = J^2. \tag{4.2}$$

The undeformed reference configuration is stress free provided that, for $\mathbf{b} = \mathbf{I}$, $\boldsymbol{\sigma} = \mathbf{O}$, the zero second-order tensor. When $\mathbf{b} = \mathbf{I}$ we have $I_1 = 3$, $I_2 = 3$ and $I_3 = 1$ and

$$\alpha_0 + \alpha_1 + \alpha_2 = 0 \tag{4.3}$$

must hold.

For an incompressible material Eq. (4.1) is replaced by

$$\boldsymbol{\sigma} = -p\mathbf{I} + \alpha_1 \mathbf{b} + \alpha_2 \mathbf{b}^2, \tag{4.4}$$

where p is a Lagrange multiplier associated with the incompressibility constraint

$$J = \det \mathbf{F} = 1 \tag{4.5}$$

and is interpreted as a hydrostatic pressure (see, for example (Ogden, 1997)). It is not determined by a constitutive law but by solution of the governing equations coupled with appropriate boundary conditions.

Further, if the material is hyperelastic, i.e. possesses a strain-energy function $W(\mathbf{F})$ per unit reference volume then, for a compressible isotropic material, the dependence of W reduces to dependence on (I_1, I_2, I_3). Then,

$$\alpha_0 = 2JW_3, \qquad \alpha_1 = 2J^{-1}(W_1 + I_1 W_2), \qquad \alpha_2 = -2J^{-1}W_2 \tag{4.6}$$

where the subscripts on W indicate partial differentiation with respect to the invariants. The corresponding expressions for an incompressible material are

$$\alpha_1 = 2(W_1 + I_1 W_2), \qquad \alpha_2 = -2W_2, \tag{4.7}$$

in (4.4) with $I_3 \equiv 1$.

We recall that the Cauchy stress tensor $\boldsymbol{\sigma}$ is an *objective* tensor. In Section 3 it was shown that for an isotropic MS elastomer the stress tensor is *symmetric*. General results from invariant theory (Spencer, 1971) allow us to write the constitutive relation for a compressible isotropic MS Cauchy elastic solid in its most general form as

$$\boldsymbol{\sigma} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{b} + \alpha_2 \mathbf{b}^2 + \alpha_3 \mathbf{B} \otimes \mathbf{B} + \alpha_4 (\mathbf{bB} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{bB}) + \alpha_5 \big(\mathbf{b}^2\mathbf{B} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{b}^2\mathbf{B}\big), \tag{4.8}$$

where α_i, $i = 0, 1, \ldots, 5$, are functions of

$$T, \quad I_1(\mathbf{b}), \quad I_2(\mathbf{b}), \quad I_3(\mathbf{b}), \quad |\mathbf{B}|^2, \quad (\mathbf{bB}) \cdot \mathbf{B}, \quad \big(\mathbf{b}^2\mathbf{B}\big) \cdot \mathbf{B}. \tag{4.9}$$

The corresponding expression for an incompressible material is obtained from the above by replacing α_0 by $-p$ and setting $I_3 = 1$.

For the hyperelastic counterpart of the above equations we require the energy function W to depend on (4.9), with $I_3 = 1$ in the incompressible case. If we now restrict attention to incompressible materials then, on setting $W = \rho_0 \Psi$, we note from $(3.4)_2$, suitably modified to account for the incompressibility constraint, that the Cauchy stress is given by the standard formula

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mathbf{F}\frac{\partial W}{\partial \mathbf{F}}. \tag{4.10}$$

In order to calculate $\boldsymbol{\sigma}$ it is convenient to introduce the notations

$$K_1 = (\mathbf{bB}) \cdot \mathbf{B}, \qquad K_2 = \big(\mathbf{b}^2\mathbf{B}\big) \cdot \mathbf{B}, \tag{4.11}$$

from which it follows that

$$\mathbf{F}\frac{\partial K_1}{\partial \mathbf{F}} = 2\mathbf{bB} \otimes \mathbf{B}, \qquad \mathbf{F}\frac{\partial K_2}{\partial \mathbf{F}} = 2\mathbf{bB} \otimes \mathbf{bB} + 2\mathbf{b}^2\mathbf{B} \otimes \mathbf{B}. \tag{4.12}$$

The contributions of the derivatives of W with respect to K_1 and K_2 to the Cauchy stress are therefore non-symmetric. Thus, for $\boldsymbol{\sigma}$ to be symmetric W must be *independent of* K_1 *and* K_2. For an incompressible material it therefore suffices to consider W as a function of T, $I_1(\mathbf{b})$, $I_2(\mathbf{b})$ and $|\mathbf{B}|^2$.

The Cauchy stress now simplifies to

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2(W_1 + I_1 W_2)\mathbf{b} - 2W_2\mathbf{b}^2, \tag{4.13}$$

with $W(T, I_1, I_2, |\mathbf{B}|^2)$. We use this model in the following section for the purely isothermal situation so that explicit dependence on T is omitted.

We emphasize here that the fact that W is independent of K_1 and K_2 is a consequence of the symmetry of $\boldsymbol{\sigma}$, which itself is a consequence of the assumed special constitutive relation $(3.6)_3$. If $(3.6)_3$ is not adopted then symmetry of $\boldsymbol{\sigma}$ is replaced by symmetry of $\boldsymbol{\sigma} + \mathbf{M}_e \otimes \mathbf{B}$, or equivalently, symmetry of

$$\mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - \frac{\partial W}{\partial \mathbf{B}} \otimes \mathbf{B},$$

with W dependent on I_1, I_2, K_1, K_2 and $|\mathbf{B}|^2$. This generalization does not have a significant effect on the subsequent analysis so in the following section we retain the specialization considered above.

5. Application to circular cylindrical geometry

The equations remaining to be solved are

$$\operatorname{div}\mathbf{B} = 0 \tag{5.1}$$

and, in the absence of mechanical body forces,

$$\operatorname{div}\boldsymbol{\sigma} + \frac{1}{2}\gamma\,\operatorname{grad}(\mathbf{B}\cdot\mathbf{B}) = \mathbf{0}. \tag{5.2}$$

The geometry to which this is applied is a right circular cylindrical tube with internal and external radii A and B respectively in the reference configuration. The material is contained within the annular region $A \leqslant R \leqslant B$, where we use cylindrical polar coordinates R, Θ, Z associated with the reference configuration. We suppose that the deformation is defined as *pure axial shear* by the equations

$$r = R, \qquad \theta = \Theta, \qquad z = Z + u(R), \tag{5.3}$$

where r, θ, z are cylindrical polar coordinates in the deformed configuration. For a recent discussion of axial shear in the elastic context we refer to (Jiang and Ogden, 2000), which also contains relevant references to analysis of this problem.

The components of the deformation gradient $\mathbf{F}$, referred to the two sets of cylindrical polar coordinate axes, are represented by the matrix F, which is given by

$$\mathsf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u'(r) & 0 & 1 \end{pmatrix}, \tag{5.4}$$

where we are now regarding u as a function of $r(= R)$; correspondingly, we write $a = A, b = B$. We also use the notation $\kappa = u'(r)$ and note that the deformation is locally a simple shear with amount of shear κ. Then, the corresponding matrix for $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, written b, is given by

$$\mathsf{b} = \begin{pmatrix} 1 & 0 & \kappa \\ 0 & 1 & 0 \\ \kappa & 0 & 1+\kappa^2 \end{pmatrix}, \tag{5.5}$$

and it follows that $I_1 = I_2 = 3 + \kappa^2$.

Because of the cylindrical symmetry Eq. (5.1) reduces to

$$\frac{\mathrm{d}(rB_r)}{\mathrm{d}r} = 0, \tag{5.6}$$

where B_r is the radial component of $\mathbf{B}$, while (5.2) gives

$$\frac{\mathrm{d}}{\mathrm{d}r}\sigma_{rr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{1}{2}\gamma\frac{\mathrm{d}}{\mathrm{d}r}\big(B_r^2 + B_z^2\big) = 0 \tag{5.7}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r}(r\sigma_{rz}) = 0. \tag{5.8}$$

Note that in general (for the considered symmetry) the axial component B_z of $\mathbf{B}$ is non-zero and depends only on r.

Since $I_1 = I_2$ we may write the strain-energy function $W(I_1, I_2, |\mathbf{B}|^2)$ as a function of $I_1 = I_2$ through the definition $\hat{W}(I_1, |\mathbf{B}|^2) = W(I_1, I_1, |\mathbf{B}|^2)$. The Cauchy stress given by (4.13) may then be written as

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\hat{W}_1\mathbf{b} + 2(I_1 - 3)W_2\mathbf{e}_\theta \otimes \mathbf{e}_\theta, \tag{5.9}$$

where $\mathbf{e}_\theta$ is the unit basis vector in the θ direction, p is adjusted to absorb hydrostatic terms arising from the transformation to $\hat{W}$, and W_2 is evaluated for $I_2 = I_1$.

For the considered cylindrical problem the non-zero components of $\boldsymbol{\sigma}$ are given by

$$\sigma_{rr} = -p + 2\hat{W}_1, \qquad \sigma_{\theta\theta} = \sigma_{rr} + 2W_2\kappa^2, \tag{5.10}$$

$$\sigma_{zz} = -p + 2\hat{W}_1\big(1+\kappa^2\big), \qquad \sigma_{rz} = 2\hat{W}_1\kappa, \tag{5.11}$$

from which we deduce the universal relation $\sigma_{zz} - \sigma_{rr} = \kappa\sigma_{rz}$.

Since $I_1 = 3 + \kappa^2$ we may also regard the strain energy as a function of κ instead of I_1. For this purpose we use the notation w and write

$$w(\kappa, |\mathbf{B}|^2) = \hat{W}(I_1, |\mathbf{B}|^2). \tag{5.12}$$

On using $(5.11)_2$ and integrating (5.8), we then have simply

$$\sigma_{rz} = w_\kappa = \frac{\tau b}{r}, \tag{5.13}$$

where τ is the value of σ_{rz} on the boundary $r = b$ and the subscript κ indicates partial differentiation with respect to κ. For given $\mathbf{B}$ and w Eq. (5.13) can in principle be solved for κ and hence, by integration, for the displacement function $u(r)$. Eq. (5.13) is the same equation as arises in the corresponding elastic problem, and for discussion of requirements on the function w for this equation to yield a solution we refer to (Jiang and Ogden, 2000). Here, of course, w depends additionally on $|\mathbf{B}|^2$.

The corresponding integral of (5.6) is

$$B_r = \frac{B_b b}{r}, \tag{5.14}$$

where B_b is the value of B_r on $r = b$.

Eq. (5.7) reduces to

$$r\sigma'_{rr} + \gamma r(B_r B'_r + B_z B'_z) - 2W_2\kappa^2 = 0, \tag{5.15}$$

where the prime signifies differentiation with respect to r. This equation serves to determine p when κ and γ are known and appropriate boundary conditions are specified. The quantity κ itself is determined from Eq. (5.13), as indicated above, while γ needs to be specified from information about the material properties. Here, the choice of boundary conditions is constrained by the nature of the considered deformation (5.3). A suitable set of boundary conditions is

$$u(a) = 0, \qquad u(b) = d, \tag{5.16}$$

which corresponds to the inner boundary of the tube being held fixed and the outer boundary being displaced by a prescribed amount d. The axial stress σ_{zz} does not enter the equations but can be used to calculate the resulting axial load on the tube once the equations have been solved. We do not include such a calculation here.

Note that, in general, B_z is not known. However, it may be determined as follows. Let $\mathbf{B}^{(r)}$ be the referential magnetic flux (i.e. the Lagrangian counterpart or pull-back of $\mathbf{B}$) so that

$$0 = \int_{\mathcal{B}} \operatorname{div}\mathbf{B}\,\mathrm{d}v = \int_{\partial\mathcal{B}} \mathbf{B}\cdot\mathbf{n}\,\mathrm{d}a = \int_{\partial\mathcal{B}_r} \mathbf{B}^{(r)}\cdot\mathbf{N}\,\mathrm{d}A = \int_{\mathcal{B}_r} \operatorname{Div}\mathbf{B}^{(r)}\,\mathrm{d}V,$$

where Div is the divergence operator in $\mathcal{B}_r$, $\mathcal{B}$ and $\partial\mathcal{B}$ are the images of $\mathcal{B}_r$ and $\partial\mathcal{B}_r$ respectively under the deformation, $\mathbf{N}$ is the unit outward normal to $\partial\mathcal{B}_r$ and Nanson's formula connecting the unit normals $\mathbf{N}$ and $\mathbf{n}$ has been used in the form $\mathbf{n}\,\mathrm{d}a = J\mathbf{F}^{-\mathrm{T}}\mathbf{N}\,\mathrm{d}A$. For the considered incompressible material this yields the connection $\mathbf{B} = \mathbf{F}\mathbf{B}^{(r)}$. If we assume that $\mathbf{B}^{(r)}$ is purely radial, with radial component $B_r^{(r)}$ then, for the deformation with gradient components given by (5.4), this yields $B_r = B_r^{(r)}$, $B_z = \kappa B_r^{(r)}$. In the following examples, for simplicity of calculation and illustration, we take $\mathbf{B}$ itself to be purely radial, so that $B_z = 0$.

Example (i). For illustration we first consider w to have the simple form

$$w = \frac{1}{2}\mu(B_r^2)\kappa^2, \tag{5.17}$$

where μ is a constitutive function given by

$$\mu = \mu_0 + \mu_1 B_r^2, \tag{5.18}$$

μ_0 and μ_1 being material constants. (Note that this μ differs from that in the electromagnetic equations.) This is a very simple choice of function but it serves to illustrate the desired effect. When $B_r = 0$ this reduces to the quadratic dependence $\mu_0\kappa^2/2$ on κ which is associated with the classical neo-Hookean solid commonly used as a prototype model for the description of the stress-strain behaviour of rubberlike solids, $\mu_0 > 0$ being the shear modulus. Eq. (5.13) leads to

$$\kappa = u'(r) = \frac{\tau b r}{\mu_0 r^2 + \mu_1 b^2 B_b^2}, \tag{5.19}$$

which integrates and gives, after use of the boundary condition $(5.16)_1$, the closed-form solution

$$u(r) = \frac{\tau b}{2\mu_0} \log\left(\frac{\mu_0 r^2 + \mu_1 b^2 B_b^2}{\mu_0 a^2 + \mu_1 b^2 B_b^2}\right). \tag{5.20}$$

Hence, on use of $(5.16)_2$, the displacement d on the outer boundary $r = b$ is given by

$$d = u(b) = \frac{\tau b}{2\mu_0} \log\left(\frac{\mu_0 b^2 + \mu_1 b^2 B_b^2}{\mu_0 a^2 + \mu_1 b^2 B_b^2}\right), \tag{5.21}$$

which shows that the relationship between τ and d is linear in this case. The logarithmic function on the right-hand side of the above equation is a decreasing function of B_b, which is a measure of the magnetic field strength. Thus, as B_b increases a given displacement d requires a larger value of the shear stress τ to maintain mechanical equilibrium. In other words, the response of the geometrical device becomes stiffer.

Example (ii). In this second example we use a strain-energy function that leads to a nonlinear relationship between τ and d. For the purely elastic situation (i.e. without any electromagnetic coupling) the problem of axial shear has been examined recently by Jiang and Ogden (2000) for a number of different forms of strain-energy function for compressible materials. Their solutions also apply in the incompressible case. One class of strain-energy function used in (Jiang and Ogden, 2000), and also in (Jiang and Ogden, 1998) for the corresponding azimuthal shear problem, generates a class of incompressible materials with strain-energy functions of the form

$$W = \frac{\mu}{2^k k}(I_1 - 1)^k - \frac{\mu}{k}, \tag{5.22}$$

where k is a dimensionless material parameter. Note that for $k = 1$ this reduces to the neo-Hookean strain energy with shear modulus μ.

In respect of (5.22) Eq. (5.13) gives

$$w_\kappa = \frac{\mu\kappa}{2^{k-1}}\left(2 + \kappa^2\right)^{k-1} = \frac{\tau b}{r}, \tag{5.23}$$

which (allowing for different notations) is exactly Eq. (5.22) in (Jiang and Ogden, 2000). The solutions of (5.23) for κ, and hence the displacement, were given by Jiang and Ogden (2000) for several different values of k. Here, for illustration, we examine just one value, namely $k = 3/2$. Because the expression of the solution is quite lengthy we omit it here and refer to (Jiang and Ogden, 2000) for the details. Again, we consider μ to be given by (5.18). We use the dimensionless notations $\bar{d} = d/b$, $\bar{\tau} = \tau/\mu_0$ and $\bar{B}_b = B_b\sqrt{\mu_1/\mu_0}$ and take $\mu_1 > 0$ (given that $\mu_0 > 0$, as in the purely elastic case). Sample results, with $\bar{\tau}$ plotted against $\bar{d}$, are shown in Fig. 1 for a representative value $b/a = \sqrt{2}$ of the ratio of exterior to interior radius of the tube. Results are compared for three values of $\bar{B}_b$, namely $0, 1, \sqrt{2}$. The figure shows that the stiffness of the material increases with the magnetic field strength; in other words, for a given axial displacement d, the required axial stress τ increases with $\bar{B}_b$.

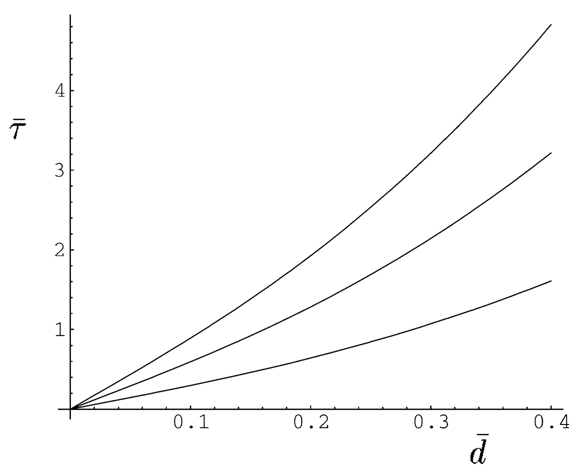

Fig. 1. Plot of the dimensionless shear stress $\bar{\tau}$ against the dimensionless displacement $\bar{d}$ for magnetic field strengths given by $\bar{B}_b = 0, 1, \sqrt{2}$. The stiffness increases with the magnitude of $\bar{B}_b$.

For the model in Example (i) these curves are straight lines, while a choice of k in (5.23) such that $1/2 < k < 1$ would give curves that are concave, as distinct from the convex curves in Fig. 1. Thus, while, for a given value of $\bar{B}_b$, the curves in Fig. 1 reflect a material that stiffens with shear, values of k in the interval $(1/2, 1)$ correspond to materials that soften with shear.

Eq. (5.15) has not been used since it merely enables the radial stress (equivalently p) to be determined and hence the resulting radial traction on the boundaries (on which the displacements are prescribed).

6. Conclusions

In this paper we have summarized the system of constitutive equations for an isotropic magneto-sensitive Cauchy-elastic solid within the framework of the electromechanical and thermomechanical theories. The simplifications appropriate for the constitutive law of such a material have been discussed and incorporated into the material model. The theory has been applied to the model problem of axial shear of a circular cylindrical tube in the presence of a radial magnetic field. Two very specialized forms of the model have been used to illustrate the results. These show that the effect of the magnetic field is to stiffen the shear response of the material. Variations on the results can be produced by adopting different models for the elastic part of the constitutive response and different dependence of the shear modulus (and other elastic parameters in more general models) on the magnetic field strength. In particular, there is an indication in the available data that the shear modulus, while at first increasing with the magnetic field strength, reaches a maximum (Jolly et al., 1996). The model can be modified to accommodate such behaviour if required. However, the choice of realistic models must be determined by detailed experiments and at present the available data are not sufficiently extensive to justify a more refined analytical treatment.

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