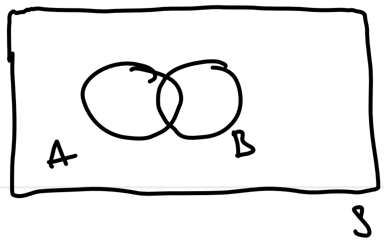
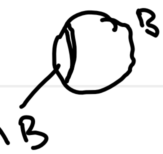
Statistics 213 – 1.2: Conditional Probability



out of B? if B has occured?

↳



if B has occurred then prob of A?

$$↳ P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B) \cdot P(A|B) = P(A \cap B)$$

Textbook:

3.5, 3.6, (3.8)

Objectives:

- Be familiar with the concept of ***conditional probability*** and be able to apply and manipulate the conditional probability formula
- Create and use a ***contingency table*** to find specific probabilities
- Create and use a ***tree diagram*** to find specific probabilities
- Prove ***independence***/dependence using probability

Motivation:

In the 1.1 notes, we looked at some basic probability problems where no conditions were assumed apart from those conditions that defined the experiment.

Examples (from Unit 1)

- If I flip a fair coin three times in a row, what is the chance that I will get three heads?
- How likely are the Astros to win the World Series this year?

While problems such as these might be interesting to us in many situations, there are other situations where we may be interested in calculating the probability of something occurring given some certain conditions.

Examples

- If I flip a fair coin three times in a row and the first coin comes up heads, what is the chance that I will get three heads in total?
- Assuming that the Astros win the American League Championship Series (ALCS) this year, how likely are they to win the World Series?

These probabilities are a bit different from those presented in the 1.1 notes in the sense that they rely in part on additional information or conditions (the first coin being a "heads," the Astros winning the ALCS).

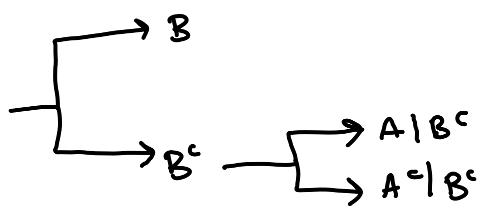
These probabilities are what we call *conditional probabilities*. In this set of notes, we'll discuss the rules for calculating conditional probabilities as well as a few techniques that can help make these calculations easier and more intuitive!

Introducing Conditional Probability

A ***conditional probability*** is a probability that reflects additional knowledge that affects the outcome of an experiment. This knowledge can be thought of as information that we're "given."

Suppose we had two events, A and B, in a sample space S.

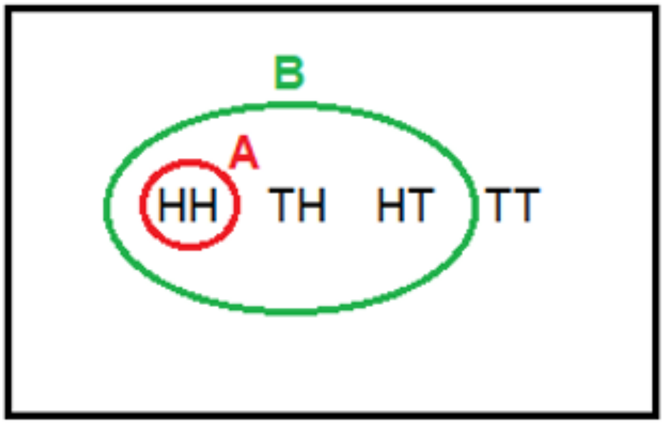
$P(A|B)$ is read as "the probability of A given B" and is calculated as:



With a conditional probability, the probability of event A is conditional upon another event occurring (event B). Because of this, we actually "restrict" our sample space…it is no longer all of S, but now just B, the "given" information we have. So now, $P(A|B)$ is the ratio of $n(A \cap B)$ over $n(B)$.

Example 1.2.1

Suppose we were to toss two fair coins. Define A as “observing exactly two heads” and B as “observing at least one head.” The following Venn diagram shows the samples space and the events A and B.



1. What is $P(A)$?

$$P(H_1 \cap H_2) = P(H_1)P(H_1 \mid H_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P(B) = P(\text{@ least one head}) = 3/4$$

	A	A^c	
B	1/4	2/4	3/4
B^c	0	1/4	1/4
	1/4	3/4	1

2. What is $P(A|B)$?

$$P(A \cap B) = P(\text{2 heads "And" @ least one head}) = 1/4$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Notice that the condition B "restricts our sample space" from S to B.

Example 1.2.2

In a survey of 100 people, 57 people said that they owned a cat. Of those 57 people, 40 were female (that is, 40 people owned a cat and were female). We select one person at random from this survey. Given that the selected person owns a cat, what is the probability that they are female?

$P(\text{owning cat}) = \frac{57}{100}$ $\qquad$ $P(\text{female} \mid \text{cat}) = \frac{40}{57}$

$P(\text{female} \mid \text{cat}) = \frac{40}{57}$

$P(f \cap \text{cat}) = \left(\frac{57}{100}\right)\left(\frac{40}{57}\right) = \frac{40}{100}$

	C	C^c	
F	0.4		
F^c	0.17		
	$\frac{57}{100}$	$\frac{43}{100}$	1

Try using

RStudio

```
0.40/0.57
```

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## [1] 0.7017544
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"Given" vs. "And" (Conditional vs. Intersection)

When first learning about conditional probability, it might be tough to distinguish between a conditional probability (such as $P(A|B)$) versus an intersection (such as $P(A \cap B)$). It is important to note that they have different meanings!

- $P(A \cap B)$ is the probability of an element belonging to both A and B without restricting our overall sample space to either A or B.
- $P(A|B)$ is the probability of an element belonging to A after restricting our sample space to B.
- $P(B|A)$ is the probability of an element belonging to B after restricting our sample space to A.

current $\qquad$ historical

Examples

- "The probability of a college student owning both a laptop computer and a desktop computer is estimated to be 0.24." This is telling us: $P(laptop \cap desktop) = 0.24$
- "If a college student owns a laptop computer, the probability that they also own a desktop computer is 0.32." This is telling us: $P(desktop|laptop) = 0.32$
- "Knowing that a college student owns a desktop computer, the probability that they also own a laptop computer is 0.20." This is telling us: $P(laptop|desktop) = 0.20$

Some common conditional probability keywords include *if, given, assuming, knowing…*

Strategies for Conditional Probability Problems

Before we do some more examples with conditional probability, let's first discuss two different but related techniques that can help us organize our thoughts when solving a conditional probability-related problem. Depending on what probabilities you know in a given problem, constructing a contingency table or drawing a tree diagram (or both!) can help you understand the problem and make it easier to make the correct calculations.

Contingency Tables

A ***contingency table*** is useful if you have two events, A and B, and want to organize the probability relationships between the two events. A common way to construct a contingency table for events A and B is as follows:

	$\mathbf{A}$	$\mathbf{A^C}$	
$\mathbf{B}$	$n(\mathbf{A} \cap \mathbf{B})$	$n(\mathbf{A^C} \cap \mathbf{B})$	$n(\mathbf{B})$
$\mathbf{B^C}$	$n(\mathbf{A} \cap \mathbf{B^C})$	$n(\mathbf{A^C} \cap \mathbf{B^C})$	$n(\mathbf{B^C})$
	$n(\mathbf{A})$	$n(\mathbf{A^C})$	$n(\mathbf{S})$

You can also construct the table so that it contains probabilities rather than numbers of elements in each cell/margin.

Draw tree

	A	**A**^{C}	
B	P(**A** ∩ **B**)	P(**A**^{C} ∩ **B**)	P(**B**)
B^C	P(**A** ∩ **B**^{C})	P(**A**^{C} ∩ **B**^{C})	P(**B**^{C})
	P(**A**)	P(**A**^{C})	P(**S**)

Contingency tables are a good technique to use if you are told one or more "and" (intersection) probabilities and need to find conditional probabilities.

Example 1.2.3

A study was done to examine the relationship between peanut (PN) allergies and tree nut (TN) allergies. In a sample of 200 people, the researchers found that 150 people had PN allergies, 26 had TN allergies but not PN allergies, and 78 people had both.

1. Construct a contingency table for PN and TN allergies.

$P(PN) = \frac{150}{200}$

$P(TN \cap PN^c) = \frac{26}{200}$

$P(PN \cap TN) = \frac{78}{200}$

	PN	PN^c	
TN	78/200	26/200	104/200
TN^c	72/200	24/200	96/100
	$\frac{150}{200}$	50/200	1

2. Find
$P(TN \cap PN)$. $= \frac{78}{200}$

3. Find
$P(PN^C)$. $= 50/200$

4. Find
$P(TN^C | PN^C)$. $\frac{P(TN^c \cap PN^c)}{P(PN^c)} = \frac{\frac{24}{200}}{\frac{50}{200}} = \frac{50}{24} = \frac{25}{12}$.

Draw tree

Example 1.2.4

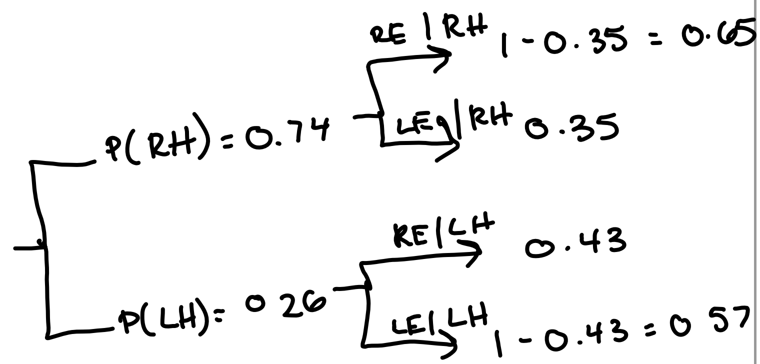
In a study examining the relationship handedness and eye-dominance, 35% of right-handers were left-eye dominant and 43% of left-handers were right-eye dominant. 74% of people in the study were right-handed.

1. Construct a contingency table for handedness and eye-dominance.

$P(LE \mid RH) = 0.35$

$P(RE \mid LH) = 0.43$

$P(RH) = 0.74$



	LE	RE	
LH	0.1482	0.1118	0.26
RH	0.256	0.481	0.74
	0.4072	0.5929	1

Tree about margins +

Table about ands + intersections

2. What is the probability that a randomly selected person from this study is left-handed and left-eye dominant?

0.1482

Bay's Theorem

3. Suppose a randomly selected person from this study is right-eye dominant. What is the probability they are left-handed?

$$P(LH \mid RE) = \frac{0.1118}{0.5928} = \frac{43}{228} \simeq 0.188596491228$$

We know $P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A|B) \cdot P(B) = P(A \cap B)$

Tree Diagrams

A ***tree diagram*** is another way of organizing the probability relationships between two events. Tree diagrams are a good technique to use if you know one or more “given” (conditional) probabilities and need to find “and” (intersection) probabilities. If you have two events, A and B, a common method for constructing a tree diagram is as follows:

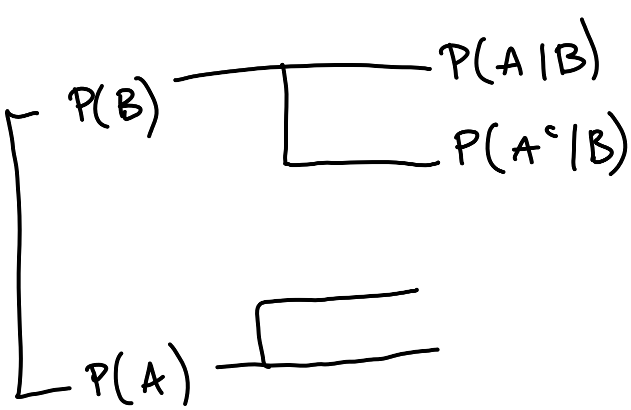
Step 1: If A is a condition for B, begin by constructing a branch for A and a branch for A^C.

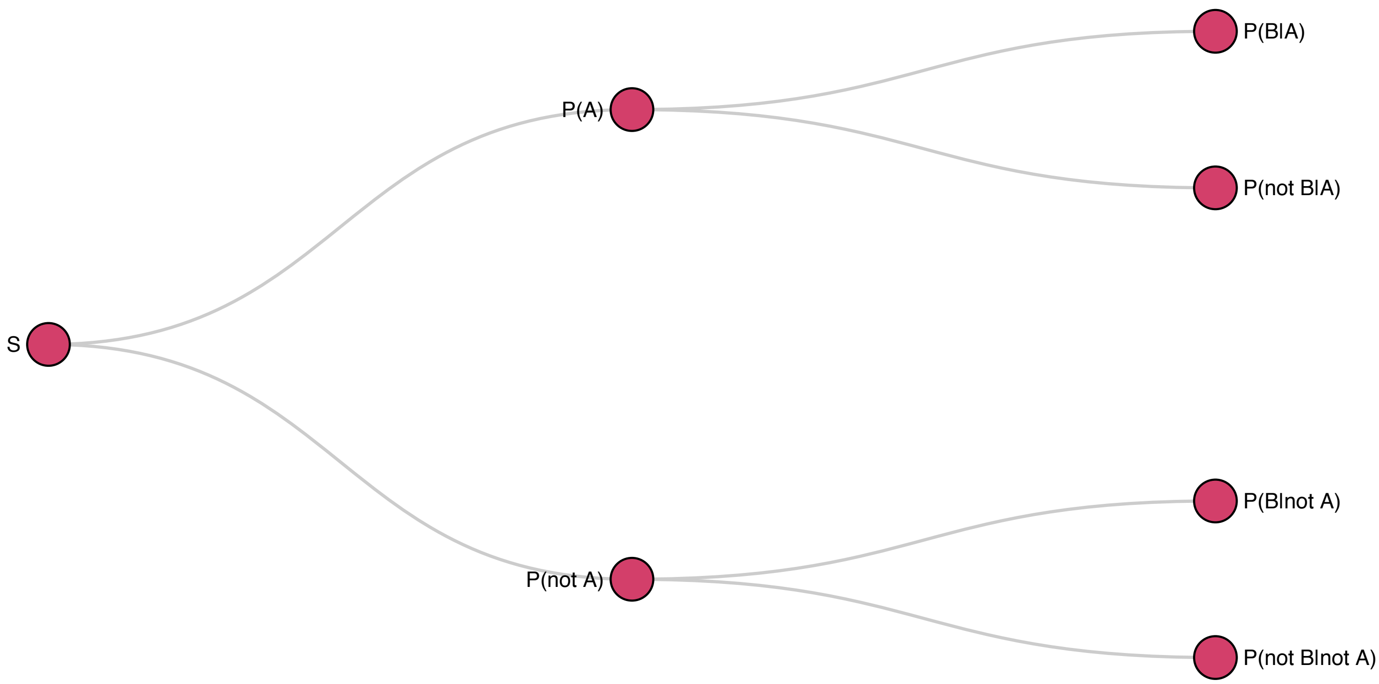
Step 2: For each branch constructed in step 1, construct branches for the conditional probabilities of B and B^C.

To find an “and” (intersection) probability for two events (A and B, A^C and B, etc.), multiply the probabilities along the branch of interest. This is actually just a manipulation of the formula for conditional probability.

$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$...rearrange to get...$P(A \cap B) = P(A \mid B)P(B)$

Here is a general picture of the setup for a basic tree diagram:




S
P(A)
P(not A)
P(B|A)
P(not B|A)
P(B|not A)
P(not B|not A)

Example 1.2.5

Suppose the probability of carrying a certain disease is 0.012. A particular procedure has been developed to test for the presence of this disease. If an individual has the disease, the test shows "positive" for the disease with a probability of 0.98. However, if the individual does not have the disease, the test shows "positive" for the disease with a probability of 0.05.

1. Construct a tree diagram for this scenario.

$P(D) = 0.012$

$P(+|D) = 0.98$

$P(+|D^c) = 0.05$

$0.012 \cdot 0.98$

$0.012 \cdot 0.02$

	+	−	
D	0.01176	0.00024	0.012
D^c	0.0494	0.9386	0.988
	0.06116	0.93884	1

- $P(D) = 0.012$
  - $P(+|D) = 0.98$
  - $P(-|D) = 0.02$
- $P(D^c) = 0.988$
  - $P(+|D^c) = 0.05$
  - $P(-|D^c) = 0.95$

2. What is the probability that a randomly selected person is carrying the disease and has a test that shows "negative" for the disease?

$P(D \cap -) = 0.00024$

3. What is the probability that a randomly selected person will test "negative" for the disease? $P(-) = 0.93884$

$$P(D \mid -) = \frac{P(D \cap -)}{P(-)} = \frac{0.00024}{0.93884} \simeq 0.0002556346$$

$$P(D^c \mid +) = P(D^c \cap +) = \frac{0.0494}{0.06116} \simeq 0.8077175$$

when $P(A) = P(A \mid B) \rightarrow$ this means that B doesn't affect A

independence

Independence

In probability, two events A and B are independent if the occurrence of A does not affect the probability of B occurring, and vice-versa. There are two different ways we can prove independence using probability. If you ever need to do so, either one of these methods is fine (one may be easier than the other, depending on what you know in the situation of interest).

only true when independent | always true | only true if independent

A and B are independent if: $P(A) = P(A \mid B)$ | $P(B) \cdot P(A \mid B) = P(A \cap B)$ | $P(B)\ P(A) = P(A \cap B)$

or…

A and B are independent if: $P(B) = P(B \mid A)$ | $P(A)\ P(B \mid A) = P(A \cap B)$ | $P(A)\ P(B) = P(A \cap B)$

Note that independence cannot necessarily be determined using a picture (such as a Venn diagram). You must use one of the above calculations in order to prove independence!

Note: if A and B are independent, then so are A and B^C; A^C and B; A^C and B^C.

X

Example 1.2.6

The following contingency table shows the probabilities for the peanut (PN) and tree nut (TN) allergy study described in Example 1.2.3. Is having peanut allergies independent of having tree nut allergies? Show why/why not.

if true then

if ind $P(A)\cdot P(B) \stackrel{?}{=} P(A\cap B)$

	PN	not PN	Totals
TN	0.39	0.13	0.52
not TN	0.36	0.12	0.48
Totals	0.75	0.25	1.00

if 1 of the margins margin = intersection then all combos will = their intersection

Are TN + PN independent?

$P(TN)\ P(PN) = (0.52)\cdot(0.75) = 0.39 = P(TN \cap PN)$

R Studio

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0.75*0.52
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## [1] 0.39
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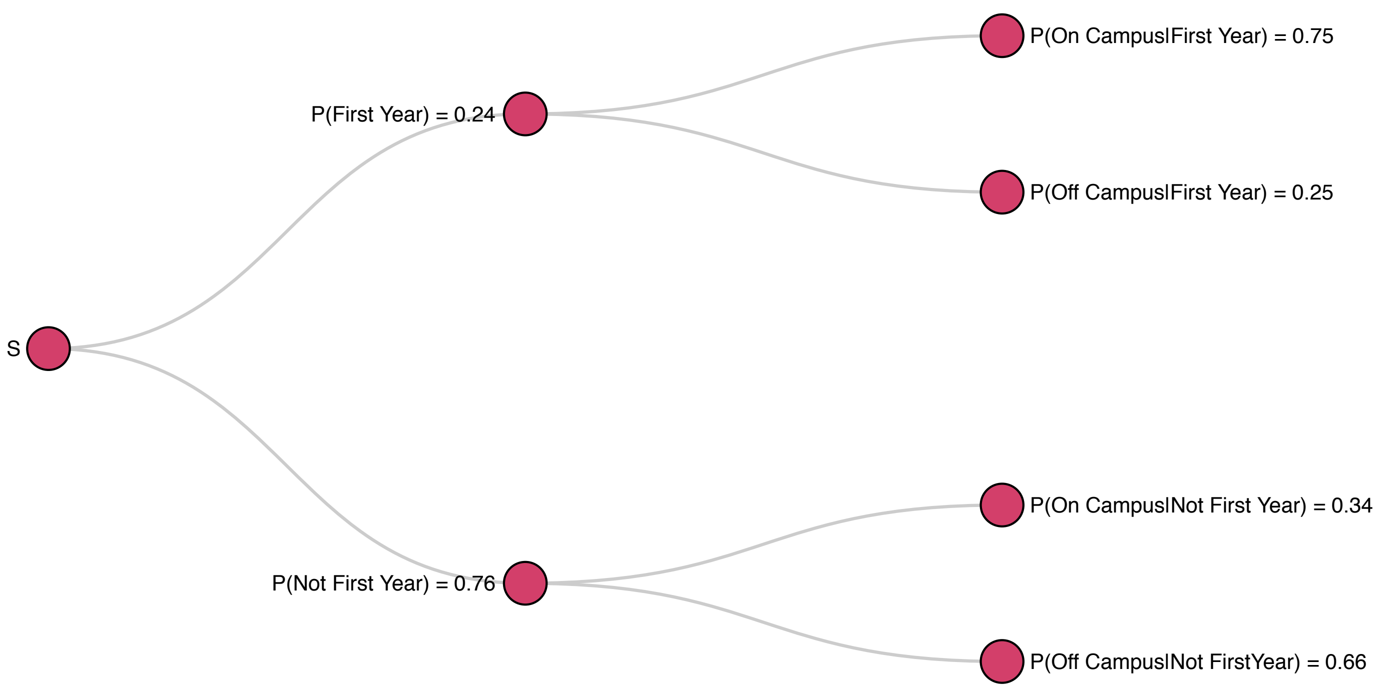
what is ME? $\longrightarrow$ $P(A \cap B) = 0$ intersection $= 0$

what is independence? $\rightarrow$ $P(A)\cdot P(B) = P(A\cap B)$

Example 1.2.7

In a certain university, 24% of the students are first-year students (the rest are, of course, non-first-year students). If a student is in their first year, there is a 75% chance that they live on campus. If a student is not in their first year, there is only a 34% chance that they live on campus. The following tree diagram displays this information.



1. Use the tree diagram to construct a contingency table for this scenario.

	C	C^c	
F	0.18	0.06	0.24
F^c	0.		0.76
			1

2. If a randomly selected student lives on campus, what is the probability that they are not in their first year?

$$P(F^c|C) = \frac{P(F^c \cap C)}{P(C)} = \frac{0.2584}{0.4384} = \frac{323}{548} \simeq 0.589416058 4$$

3. Is being a first-year student independent of living on campus? Show why/why not.

$(0.4384)(0.24) = 0.105216 \neq 0.18$

$P(C)P(F) \neq P(F \cap C) \rightarrow$ not independent, dependent