4.3: Method of Moments Estimators

Textbook: 9.6

Objectives

- Know the steps to compute a method of moments estimator

Motivation

Thus far in Unit 4, we have discussed two main properties of estimators: bias and consistency, and we know how to compute these two properties for any estimator we may wish to consider. However, we haven't yet given much thought as to how estimators are determined. In other words, how and where do we "start" if we wished to come up with an estimator (for which we can later determine bias and consistency)?

In this set of notes, we'll discuss one of the oldest and simplest methods for finding an estimator for one or more population parameters: the method of moments.

The Method of Moments

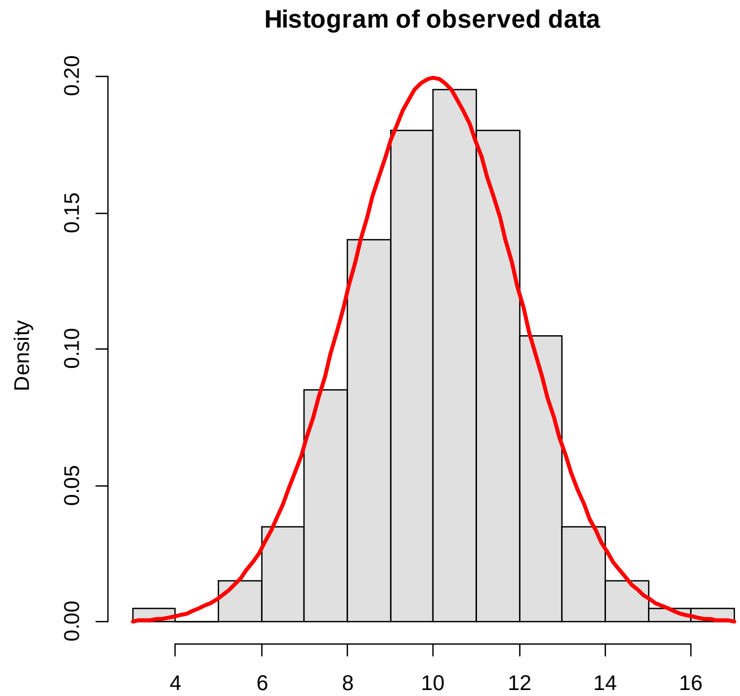
Before we discuss the method itself, let's recall the definitions of theoretical moments as well as introduce the definitions of sample moments.

Definitions

1. $\mu' = E[X^K]$ is the k^th theoretical moment of the distribution (about the origin) for $k = 1, 2, \ldots$.
2. $M_k' = \frac{1}{n}\sum_{i=1}^{n} X_i^k$ is the k^th sample moment (about the origin) for $k = 1, 2, \ldots$.
3. $\mu^* = E[(X-\mu)^K]$ is the k^th theoretical moment of the distribution (about the mean) for $k = 1, 2, \ldots$.
4. $M_k^* = \frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X})^2$ is the k^th sample moment (about the mean) for $k = 1, 2, \ldots$.

The method of moments is based on the idea that the sample moments should be good estimates for the corresponding population moments since the empirical (observed) distribution converges in some sense to the probability distribution. Thus, the corresponding moments should be approximately equal.

Example
The following is a histogram of a sample of $n = 200$ observations from a normal distribution with $\mu = 10$ and $\sigma = 2$. The red curve represents the corresponding probability distribution for a normal$(10, 2)$ distribution. If we were to increase n, the shape of the histogram would converge to the shape of the red distribution.



Note also that for this sample, $\bar{x} = 10.17957$ and $s = 1.956454$…both decently good estimates of μ and σ, respectively.

Method of Moments: One Version
The method of moments involves equating sample moments with theoretical moments. The following series of steps can be used to carry out the method of moments.

1. Equate the first sample moment about the origin, $M_1' = \frac{1}{n}\sum_{i=1}^{n} X_i^1 = \bar{x}$, to the first theoretical moment, $E[X]$.
2. Equate the second sample moment about the origin, $M_2' = \frac{1}{n}\sum_{i=1}^{n} X_i^2$, to the second theoretical moment $E[X^2]$.
3. Continue equating sample moments about the origin, M_k', with the corresponding theoretical moments $E[X^k]$, $k = 3, 4, \ldots$ until there are as many equations as there are parameters.
4. Solve for the parameters. The resulting values are <u>method of moments estimators</u>.

*Example 4.3.1**
Let $X_1, X_2, \dots, X_n$ be Bernoulli random variables with parameter p. What is the method of moments estimator of p?

*Example 4.3.2**
Let $X_1, X_2, \dots, X_n$ be normal random variables with mean μ and variance σ^2.

1. What is the method of moments estimator for μ?

2. What is the method of moments estimator for σ^2?

Method of Moments: Another Version
Another version of the method of moments can be carried out using the following steps.

1. Equate the first sample moment about the origin, $M_1' = \frac{1}{n}\sum_{i=1}^{n} X_i^1 = \bar{x}$, to the first theoretical moment $E[X]$.

2. Equate the second sample moment about the mean, $M_2^* = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$, to the second theoretical moment $E[(X-\mu)^2]$.

3. Continue equating sample moments about the origin, M_k^*, with the corresponding theoretical moments $E[(X-\mu)^k]$, $k = 3, 4, \ldots$ until there are as many equations as there are parameters.

4. Solve for the parameters. The resulting values are method of moments estimators.

*Example 4.3.3**
Let $X_1, X_2, \ldots, X_n$ be gamma random variables with mean $\alpha\beta$. What are the method of moments estimators for α and β?