4.1: Relative Efficiency of Estimators

Textbook: 9.2

Objectives

- Be able to compute and interpret the relative efficiency of estimates

Motivation

Back in the 3.1 notes, we talked about how a point estimator, $\hat{\theta}$, can be used to estimate an unknown population parameter θ. The "goodness" of an estimator $\hat{\theta}$ can be assessed by calculating either the bias $B(\hat{\theta})$ or the mean squared error (MSE) $MSE[\hat{\theta}]$.

These values, in addition to being used to evaluate the "goodness" of a single estimator, can also be used to compare the "goodness" of two competing estimators. In this set of notes, we will discuss the idea of relative efficiency as a way to compare two estimators, $\hat{\theta}_1$ and $\hat{\theta}_2$, of some parameter θ.

Relative Efficiency

The efficiency of an estimator is a measure of its quality in terms of how many observations the estimator needs in order to achieve a given performance. A more efficient estimator will require fewer observations than a less efficient estimator in order to achieve the same level of performance. Put more plainly, a more efficient estimator requires a smaller sample size to achieve the same level of performance of a less efficient estimator.

If we have two estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ that are both used to estimate the unknown target parameter θ, then assessing the relative efficiency of these estimators can help us determine which estimator to choose.

The relative efficiency of $\hat{\theta}_1$ with respect to $\hat{\theta}_2$ is the ratio of the estimators' MSE values.

$$RE(\hat{\theta}_1, \hat{\theta}_2) = \frac{MSE[\hat{\theta}_1]}{MSE[\hat{\theta}_2]} = \frac{VAR[\hat{\theta}_1] + B(\hat{\theta}_1)^2}{VAR[\hat{\theta}_2] + B(\hat{\theta}_2)^2}$$

When $RE(\hat{\theta}_1, \hat{\theta}_2) < 1$, then $\hat{\theta}_1$ is considered to be the more efficient estimator.

Similarly, when $RE(\hat{\theta}_1, \hat{\theta}_2) > 1$, then $\hat{\theta}_2$ is considered to be the more efficient estimator.

Note that if both $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased estimators of θ, then we prefer the estimator with the smallest variance.

$$RE\left(\hat{\theta}_1, \hat{\theta}_2\right) = \frac{VAR\left[\hat{\theta}_1\right]}{VAR\left[\hat{\theta}_2\right]}$$

Example 4.1.1
Let $X_1, X_2, \ldots, X_n$ represent a random sample of student UCID numbers. Assume $X_i \sim Uniform(0, N)$ where N is the total number of U of C students. We are interested in estimating N. Consider the following related quantities to help with calculations:

$$E\left[X_{(n)}\right] = \frac{Nn}{n+1}, 0 < X_{(n)} < N \qquad VAR\left[\left(\frac{n+1}{n}\right)X_{(n)}\right] = \frac{N^2}{n(n+2)}$$

1. Find the relative efficiency of the estimators $\hat{\theta}_1 = 2\bar{x}$ and $\hat{\theta}_2 = \left(\frac{n+1}{n}\right)X_{(n)}$. Then, assuming $n = 43$, compute the relative efficiency and interpret its meaning.

Example 4.1.2
Let $X_1, X_2, \ldots, X_n$ represent a random sample taken from a population of values that is modeled by the following probability cumulative distribution function and probability distribution functions for $x > 0$:

$$F_X(x) = 1 - e^{-\frac{X}{\theta}} \qquad\qquad f_X(x) = \frac{e^{-\frac{X}{\theta}}}{\theta}$$

Find the relative efficiency of the estimators $\hat{\theta}_1 = nX_{(1)}$ and $\hat{\theta}_2 = \bar{x}$ and interpret the meaning.