Statistics 213 – 1.3: Counting

UNIVERSITY OF CALGARY

Textbook:

3.7

Objectives:

- Be familiar with counting rules/formulas (***probability of the intersection of independent events***, ***multiplicative rule***) used in this course
- Be familiar with the equations for ***permutations*** and ***combinations*** and know when to use them
- Be able to apply the counting rules/formulas in different probability calculations
- Be able to combine the counting rules with previously learned probability concepts (unions, intersections, etc.)

Motivation:

In many of the probability problems we've looked at so far, we've been able to easily list out all the elements in our sample space S and in some event of interest (let's call it A).

> Example
>
> When rolling a fair six-sided die, let A be the event that we observe an even number.
>
> $S = \{1, 2, 3, 4, 5, 6\};\ n(S) = 6$
>
> $A = \{2, 4, 6\};;\ n(A) = 3$
>
> $P(A) = \frac{3}{6} = \frac{1}{2} = 0.5$

In other situations, listing all the elements in S may be too tedious. However, we've learned how to compute the number of elements in S, $n(S)$, and have seen scenarios where we can still list the number of elements in a particular event of interest A. We've then used this information to calculate probabilities.

> *Example*
>
> When rolling five six-sided dice, let A be the event that all five dice show the same number.
>
> $$n(S) = r^m = 6^5 = 7776$$
>
> A = {11111, 22222, 33333, 44444, 55555, 66666}; n(A) = 6
>
> $$P(A) = \frac{6}{7776} = 0.0008$$

However, we may run into situations where the above techniques are not going to be very helpful to us. Maybe S and A are *both* too large for us to list all the relevant elements.

> *Example*
>
> When rolling five six-sided dice, let A be the event that at least two of the dice show the same number.
>
> $$n(S) = r^m = 6^5 = 7776$$
>
> $$A = \{11234, 11123, 11112, 11111, 11345, 11134, 11113, 11456, 11156, 11116, \ldots\}$$

How do we go about finding $n(A)$ and $P(A)$ in the above example? In this set of notes, we'll learn some rules and techniques that will help us in situations where the number of elements in our sample space S and the number of elements in an event of interest are both very large.

Some Useful Counting Rules and Formulas:

As we will see, there are a lot of scenarios like the one described above, where there are a large number of elements in both the sample space S and in a particular event of interest. The methods we've used thus far (drawing a picture, listing the whole sample space) are usually not very practical in these types of situations.

Counting in probability refers to techniques that allow us to calculate probabilities without having to list out the elements. It also refers to a few additional rules and formulas that allow us to calculate probabilities for independent events and for figuring out the number of outcomes that can occur in an experiment. For us, we will focus on two main rules and two additional calculations/formulas that will be useful when solving more complicated probability questions.

Rule: The Probability of the Intersection of Independent Events [from the 1.2 notes]

when independent

$\to P(A) \to P(B) \to P(A)\cdot P(B) = P(A\cap B)$

when not independent

$\to P(A) \to P(B|A) \to P(A)\cdot P(B|A) = P(A\cap B)$

As we learned in the 1.2 notes, if events A and B are independent, then

$$P(A \cap B) = P(A) \cdot P(B)$$

This rule can be extended to more than two events. Suppose events $A, B, C, \ldots, Z$ are all independent events. Then:

$$P(A \cap B \cap C \cap \ldots \cap Z) = P(A) \cdot P(B) \cdot P(C) \cdots P(Z)$$

This rule is useful when: we need to calculate the probability of more than two independent events occurring at a time. That is, when we want the probability of the intersection of more than two independent events.

Example

A recent city poll found that 57% of Calgarians have a favorable view of Mayor Nenshi. Suppose two Calgarians are randomly chosen and are asked if they have a favorable view of Nenshi.

1. What is the probability that both people have a favorable view of Nenshi?

if independent then $(0.57)(0.57) = 0.57^2 = 0.3249$

2. What is the probability that neither person has a favorable view of Nenshi?

$(1-0.57)(1-0.57) = (1-0.57)^2 \to$ if independent

$= 0.1849$

3. One out of the two prefer Nenshi?

YN or NY $\to$ $(0.57)(1-0.57) + (1-0.57)(0.57)$

$= 2\cdot(0.57)(1-0.57)$

$\downarrow$

$+$

\# of ways to order

$\binom{2}{2,0} = \frac{2!}{2!0!} = 1 \to$ combo of ways for N + N

$\binom{2}{1,1} = \frac{2!}{1!1!} = 2 \to$ combos of ways for N + Y

$\binom{2}{0,2} = \frac{2!}{0!2!} = 1 \to$ combo of ways for Y + Y

Example

A quiz consists of three multiple choice questions. Each question has four possible answers, only one of which is right. Suppose you complete the quiz by randomly selecting an answer for each question.

1. What is the probability that all of your answers are correct?

order C C C

$$\underbrace{1}_{\text{order}} \cdot \underbrace{\left(\tfrac{1}{4}\right)^3}_{P(\text{success})} \underbrace{\left(\tfrac{3}{4}\right)^0}_{P(\text{failure})} = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4}$$

one correct of three

$$\binom{\overset{\text{options}}{3}}{\underset{\text{correct}}{1},\ \underset{\text{incorrect}}{2}} \left(\tfrac{1}{4}\right)^1 \left(\tfrac{3}{4}\right)^2$$

$$\text{ways} = \left(\frac{3!}{1!\,2!}\right)\left(\tfrac{1}{4}\right)^1\left(\tfrac{3}{4}\right)^2 = 3 \cdot \frac{1}{4} \cdot \frac{9}{16} = 3 \cdot \frac{9}{64} = \frac{27}{64}$$

2. What is the probability that at least two of your answers are correct?

let x be # of correct out of 4 $\quad x \geq 0, 1, 2, 3 \quad P(x \geq 2) = 1 - \left(P(x=0) + P(x=1)\right)$

Rule: The Multiplicative Rule [generalized from the 1.1 notes]

In the 1.1 notes, we learned the simplified multiplicative rule. For a sample space S:

$$n(S) = r^m$$

where r is the number of outcomes/results and m is the number of instances/trials for a given scenario.

The *general* multiplicative rule is useful when r is not the same for every m. Suppose you have independent instances $m_1, m_2, \ldots, m_k$, where m_1 has r_1 outcomes, m2 has r2 outcomes, …, and mk has rk outcomes. The multiplicative rule tells us that the number of elements in $(m_1 \cap m_2 \cap \ldots \cap m_k)$ is

$$r_1^{m_1} \cdot r_2^{m_2} \cdot \ldots \cdot r_k^{m_k}$$

This rule is useful when: you have two or more independent instances and need to find the number of elements in the intersection of those instances.

Example

In an experiment, you roll four six-sided dice and toss three fair coins. How many elements are in the sample space?

$6^4 \quad 2^3 \quad = \underline{6} \cdot \underline{6} \cdot \underline{6} \cdot \underline{6} \cdot \underline{2} \quad \underline{2} \cdot \underline{2}$

$= 10\,368$

with replacement
independent

Permutations

A ***permutation*** is an arrangement of objects in which order matters. If there are a total of n objects and we wish to find the total number of ways to order r of them, we can calculate this as:

$$P_r^n = \frac{n!}{(n-r)!}$$

where n is the total number of objects and r is the number of objects we wish to select.

Side note: $n!$ is read as "n factorial" and is equal to $n(n-1)(n-2) \ldots (3)(2)(1)$.

$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

$12! = 12 \cdot 11 \cdot 10 \cdot 9 \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = 479001600$

Note also that
$0! = 1$

${}_7P_3 = \underline{7} \cdot \underline{6} \cdot \underline{5}$

Sleeplessness

$\underbrace{\frac{13!}{5!4!2!1!1!}}_{\#\text{ of ways}}$

$\binom{13}{5!2!4!1!1!}$

$\underbrace{\frac{5!4!2!1!1!}{13!}}_{P(\text{correct spelling})}$

Example

A total of 30 athletes compete in a track meet. At the end of the meet, three medals—gold, silver, and bronze—are awarded to the top three competitors. How many different ways could the medals be given out to the athletes? In other words, how many different ways can three of the 30 athletes be ordered in 1st, 2nd, and 3rd place?

[Note that order matters here! Alice, Bob, Charles winning gold, silver, and bronze, respectively, is different from Charles, Alice, Bob winning gold, silver, and bronze, respectively] total order $30! = 30 \times 29 \times 28 \times \cdots$

$$\text{winners} = \underset{1}{\underline{30}} \times \underset{2}{\underline{29}} \times \underset{3}{\underline{28}} = {}_{30}P_3 = \binom{30}{\underset{G}{1},\underset{S}{1},\underset{B}{1},27}$$

30 athletes, 3 winners. You have 2 friends in RACE

$$P(\text{1 friend gold + 2 friend gets bronze}) = \frac{\underset{G}{(2)}\,\underset{S}{(\cancel{28})}\,\underset{B}{(1)}}{30 \cdot 29 \cdot \cancel{28}} = \frac{2}{30 \cdot 29} = \frac{1}{15 \cdot 29}$$

sample space $\rightarrow 30 \cdot 29 \cdot 28$

$$= \frac{\binom{2}{1,0,1,0}\binom{28}{0,1,0,27}}{30 \cdot 29 \cdot 28}$$

G S B L

Combinations

A ***combination*** is an arrangement of objects in which order does not matter (think of a combination just as a "group" or "selection" rather than an "order"). If there are a total of n objects and we wish to find the total number of ways to select r of them, we can calculate this as:

$$C_r^n = \binom{n}{r} = \binom{n}{r, n-r} = \frac{n!}{r!(n-r)!}$$

where n is the total number of objects and r is the number of objects we wish to select.

Example

A total of 30 athletes compete in a track meet. At the end of the meet, three athletes (regardless of performance) are selected to compete in a larger track meet the following month. How many ways could these three athletes be selected? In other words, how many different ways can three of these 30 athletes be grouped?

[Note that order does not matter here! “Alice, Bob, Charles” is the same group of three as “Charles, Alice, Bob”]

$$\binom{30}{3,27} = \binom{30}{3} = \binom{30}{27} = {}_{30}C_3 = \frac{30!}{3!\,27!} = \frac{30!}{3!(30-3)!}$$

order doesn't matter

Using Counting to Compute Probabilities

The above rules and formulas can be used together to help you solve for more complicated probabilities. Notice that the above rules/formulas give you counts, or the number of possible outcomes. Remember that probability is all about ratios of counts—usually the ratio of $n(A)$ to $n(S)$, if A is an event of interest in some sample space S.

One way to approach a lot of more difficult-looking questions is to try and figure out the probability calculation's denominator and numerator separately, and then combine them to get your answer.

The denominator will represent the total number of possible outcomes in a sample space.

The numerator will represent the subset(s) of the outcomes that fit particular criteria.

If finding the probability of some event A seems quite difficult, try figuring out what AC is. In some cases, the probability of the complement is actually a lot easier to find—then you can just use the complement rule to find the probability you're asked for.

At first, problems involving counting may seem quite difficult. However, he more you practice with counting, the more “intuitive” it will seem! So let's do some more examples.

Example like Q 12 + 13

Assignment 2 has a total of 15 questions. Suppose you haven't started the assignment yet, but decide to randomly select four problems to complete tonight. Of the 15 questions, you know how to solve eight of them without using your notes.

1. How many different ways could you select the four problems to complete

tonight?

total # of possibilities

$_{15}C_4 = 1365$

2. What is the probability that you will be able to solve all four selected problems without consulting your notes?

know how to do

don't know how to do

$$\frac{\binom{8}{4,\,4}\binom{7}{0,\,7}}{\binom{15}{4}} = \frac{70}{1365}$$

total

3. What is the probability that you will be able to solve at most one of the four selected problems without consulting your notes?

let x be # of q's I get right $\qquad x = \{0, 1, 2, 3, 4\}$

$$P(x \le 1) = P(x=0) \quad P(x=1)$$

$$= \frac{\binom{8}{0,\,8}\cdot\binom{7}{4,\,3}}{\binom{15}{4,\,11}} + \frac{\binom{8}{1,\,7}\cdot\binom{7}{3,\,4}}{\binom{15}{4,\,11}} = \frac{\binom{8}{0}\binom{7}{4} + \binom{8}{1}\binom{7}{3}}{\binom{15}{4}}$$

Example

 like 12 + 13

A first grade class consists of 20 children. Of these 20 children, 13 are boys and seven are girls. Their teacher wants to select a random set of five children from the class to attend an academic competition.

1. How many possible ways could the five children be selected from the class?

$$\binom{20}{5,15} = \binom{20}{5} = {}_{20}C_3 = 15504$$

2. What is the probability that exactly three of the selected children are boys?

$$\frac{\binom{13}{3,10}\binom{7}{2,5}}{\binom{20}{5,15}} = \frac{\binom{13}{3}\binom{14}{2}}{\binom{20}{5}}$$

3. How likely is it that all five selected children are girls?

$$\frac{\binom{13}{0,13}\binom{7}{5,2}}{\binom{20}{5,15}} = \frac{{}_{13}C_0 + {}_7C_5}{{}_{20}C_5}$$

4. What is the probability that there is at least one boy selected?

let x be # of boys selected $x = \{0, \underbrace{1, 2, 3, 4, 5}\}$

1 - all girls (means 0 boys chosen which is what we don't want)

$$1 - \frac{\binom{13}{0}\binom{7}{5}}{\binom{20}{5}} =$$

Example like q11 + 14

You roll five six-sided die in a game of Yahtzee.

5 dice 6 options per die (1, 2, 3, 4, 5, 6)

1. How many possible outcomes are there?

$\underline{6}\ \underline{6}\ \underline{6}\ \underline{6}\ \underline{6} = 6^5$

$= 7776$

2. What is the probability that at least two of the dice show the same number?

let x be # of dice that are same

can't have 1 dice that's the same

$x = \{0, \cancel{1}, \underbrace{2, 3, 4, 5}\}$

$$1 - \frac{\binom{6}{5,1}\binom{5}{1,1,1,1,1}}{6^5}$$

extras

singles

1 - all of them are different

$P(x \geq 2) = 1 - P(x=0)$ =

3. What is the probability that you get a Yahtzee (that is, all the dice show the same number)?

$x=0$ ABCDE: $\frac{\binom{6}{5,1}\binom{5}{1,1,1,1,1}}{6^5}$

$x=2$ AABCD AABBC: $\frac{\binom{6}{1,3,2}}{6^5} + \frac{\binom{6}{2,1,3}\binom{5}{2,2,1}}{6^5}$

$x=3$ AAABC AAABB: $\frac{\binom{6}{1,2,3}\binom{5}{3,1,1}}{6^5} + \frac{\binom{6}{1,1,4}\binom{5}{3,2}}{6^5}$

$x=4$ AAAAB: $\frac{\binom{6}{1,1,4}\binom{5}{4,1}}{6^5}$

$x=5$ AAAAA: $\frac{\binom{6}{1,5}\binom{5}{5,0}}{6^5} = \frac{(6)(1)}{6^5} = \frac{6}{6^5} = \left(\frac{6!}{1!\,5!}\right)\left(\frac{5!}{5!\,0!}\right)$

Example 11, 14

On a Sunday night, there are four people on a bus. The bus has five stops left before it ends its route. Suppose each person will get off the bus at one of the stops, and will do so randomly.

1. How many different ways could the people get off the bus?

$\underline{5} \cdot \underline{5} \cdot \underline{5} \cdot \underline{5} = 5^4$

2. What is the probability that all four people get off the bus on the first stop?

$$\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}\right) = \left(\frac{1}{5}\right)^4 = \frac{1}{5^4}$$

3. What is the probability that all four people get off the bus on the same stop?

$$\frac{\binom{5}{1,4}\binom{4}{4,0}}{5^4} = \frac{5(1)}{5^4} = \frac{1}{5^3}$$

let x be the # of ppl off on same stop

$x = \{0, 2, 3, 4\}$

ABCD

$$P(x=0) = \frac{\binom{5}{4,1}\binom{4}{1,1,1,1}}{5^4}$$ } different stops for everyone

4. What is the probability that exactly three of the four people get off the bus on the same stop?

AAAB

stop, order

$$\frac{\binom{5}{1,0,1,3}\binom{4}{3,1}}{5^4} = \frac{\frac{5!}{1!1!3!} \cdot \frac{4!}{3!1!}}{5^4}$$

triple, double, single

Straight or Run

1, 2, 3, 4 or 2, 3, 4, 5

$$2\left(\frac{1}{5}\right)^4\binom{4}{1,1,1,1}$$

$$= \frac{2(4!)}{5^4}$$

Example

A baseball team consists of nine players. Suppose the batting order of the players in a particular game is chosen at random.

1. How many different batting orders are possible?

$$9! = \binom{9}{1,1,1,1,1,1,1,1,1}$$

2. What is the probability that the pitcher bats last if the batting order is chosen at random?

$$\frac{\binom{8}{1,1,1,1,1,1,1,1}\binom{1}{0,1}}{9!} = \frac{8!}{9!}$$

→ only 1 option bc pitcher has to be last option

Example 16

A hat contains cards numbered 1 through 9. A card is randomly selected, its number recorded, and the card is replaced in the hat. This process repeats three more times (for a total of four cards).

1. How many different selections are possible?

$9 \cdot 9 \cdot 9 \cdot 9 = 9^4$ with replacement

$\binom{9}{4,5} = \frac{9!}{4!5!} = 9 \times 8 \times 7 \times 6$ without replacement

2. What is the probability that exactly two of the four cards are even-numbered?

with R $P(odd) = \frac{5}{9}$ $P(even) = \frac{4}{9}$

$\binom{4}{2,2} = \left(\frac{4}{9}\right)^2 \left(\frac{5}{9}\right)^2$

without R $\frac{\binom{5}{2}\binom{4}{2}}{\binom{9}{4}}$

3. What is the probability that at least one of the four cards are odd-numbered?

1 - all even

Poker 5 cards from 52 4 suits

(H, D, C, S)

13 denominations

(A, 2 ....., Q, K)

AABBC

$$P(\text{two pairs}) = \frac{\binom{13}{2,1,10}\underbrace{\binom{4}{2}}_{A}\underbrace{\binom{4}{2}}_{B}\underbrace{\binom{4}{1}}_{C}}{\binom{52}{5}}$$

AAABB

$$P(\text{full house}) = \frac{\binom{13}{1,1,11}\underbrace{\binom{4}{3}}_{A}\underbrace{\binom{4}{2}}_{B}}{\binom{52}{5}}$$

any card combo in a row

P(straight) .. A→5, 2→6, 3→7, 4→8, 5→9, 6→10, 7→J, 8→Q, 9→K, 10→A

(10 straights)

$$\frac{10\binom{4}{1}^5}{\binom{52}{5}}$$

all same suit

$$P(\text{flush}) = \frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}$$

There's an Ace that is also spade so you have to consider this case

$$P(2\text{ Aces} \cap 3\text{ spades}) = \frac{\underset{A\cap S}{\binom{1}{0}}\underset{A\cap S^c}{\binom{3}{2}}\underset{A^c\cap S}{\binom{12}{3}}\underset{A^c\cap S^c}{\binom{36}{0}} + \underset{A\cap S}{\binom{1}{1}}\underset{A\cap S^c}{\binom{3}{1}}\underset{A^c\cap S}{\binom{12}{2}}\underset{A^c\cap S^c}{\binom{36}{1}}}{\binom{52}{5}}$$

$$\underbrace{P(2\text{ Aces} \cap 3\text{ 10's})}_{\text{mutually exclusive}} = \frac{\underset{A}{\binom{4}{2}}\underset{10}{\binom{4}{3}}\underset{A\cap 10}{\binom{0}{0}}\underset{A^c\cap 10^c}{\binom{44}{0}}}{\binom{52}{5}}$$