Statistics 213 – 1.1: Probability Basics

UNIVERSITY OF CALGARY

Probability is a measure between 0 - 1 for likelihood (Based on history, feel etc.)

2 things → · options possible
· sense of how likely those options are

Textbook:

3.1, 3.2, 3.3, 3.4

Objectives:

- Be familiar with commonly-used probability terms such as ***random experiment***, ***sample space***, ***element***, and ***event***
- Compute probabilities for events of interest
- Understand the meanings of probability relationships (***complementary events***, ***unions***, ***intersections***) and be able to express these relationships using set notation
- Apply probability laws to compute more complex probabilities
- Show if events are ***mutually exclusive*** using probability

Motivation:

The concept of probability is something that we're all familiar with to some degree. It is likely that you could come up with a few basic examples of situations that would involve calculating or estimating a probability.

Examples

- If I flip a fair coin three times in a row, what is the chance that I will get three heads?
- If I randomly select a single M&M from a pack, what is the probability that the M&M is red?
- How likely are the Astros to win the World Series this year?

While all of these examples involve this idea of "chance" or "probability," the interpretation of probability differs from example to example.

The first example involves a situation where we could exactly calculate a probability based on what we know about the situation (we know that there is a 50% chance for "heads" and 50% chance for "tails" for any given flip).

The second example involves a situation where we might base our probability calculation on previous or additional observations of similar situations (suppose the M&Ms website states that the percentage of red M&Ms in any given package is approximately 13%).

The third example is a bit more complicated. At this point in the season, there are still a good number of things that could happen to affect the Astros' chance of winning the World Series, so calculating the probability would take a lot of work.

For this course, we will be focusing on probability as the likelihood of a particular outcome occurring in a given situation (rather than as an estimation or an opinion) and will mainly be dealing with situations like those in the first two examples above—situations where we either know a specific probability of something happening or can base our probability calculations on additional information about the situation of interest.

In this first set of notes, we will discuss some of the basic definitions and rules used in probability and then look at a few "basic" probability problems. Let's start with a few definitions that we'll use throughout the course.

Key Definitions:

There are several terms that are commonly used when discussing probability.

A ***random experiment*** is a process that leads to a single outcome. This outcome cannot be predicted with certainty. An experiment can have any number of possible outcomes, but will only result in one of these outcomes at a time.

An ***element*** is a specific outcome of an experiment.

The ****sample space*** (denoted S) is the collection of all possible outcomes for a given experiment.

An ***event*** is a collection of elements that are of interest to us or share a common characteristic in a given experiment. An event can contain just one element or several elements and is a subset of the sample space S.

The ***probability*** of event X, denoted $P(X)$, is the measurement of the likelihood that X will occur when an experiment is performed.

$$P(X) = \frac{\text{\# number of ways X can occur}}{\text{total \# of outcomes}} = \frac{n(X)}{n(S)}$$

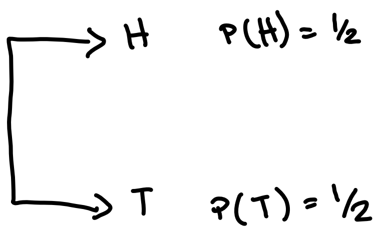
Example

When tossing one fair coin, we are interested in the probability of observing "heads."

Experiment: tossing one fair coin

Sample space: $\{H, T\}$ ⟶ options are mutually exclusive so can't happen @ same time

Event of interest: H

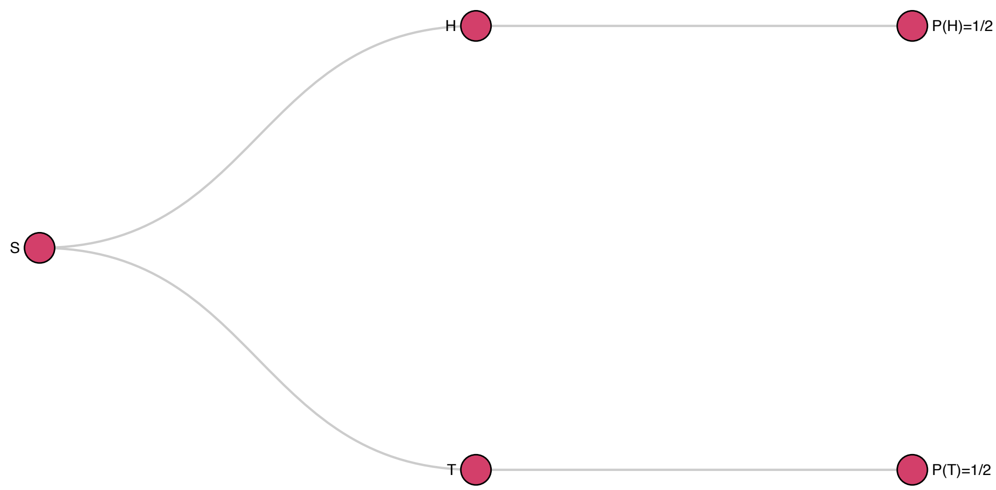


"one out of two"

Probability:
$P(H) = \frac{1}{2} = 0.5$ Try using RStudio as a calculator: type(or copy paste) into the "Console" window.

```
1/2
```

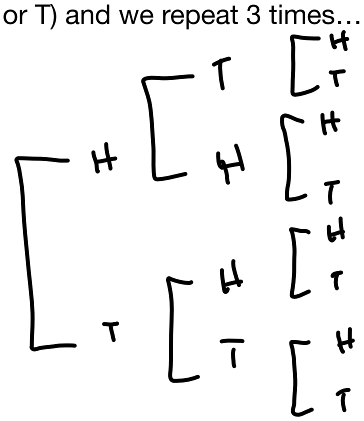
```
## [1] 0.5
```



Example

When tossing three fair coins, we are interested in the probability of observing **at least** one "head."

Experiment: tossing three fair coins How many outcomes are there to consider? Each toss has two options (H or T) and we repeat 3 times…



$2^3 = 8$

$$n(S) = 2 \cdot 2 \cdot 2 = 2^3 = 8$$

Try using

RStudio

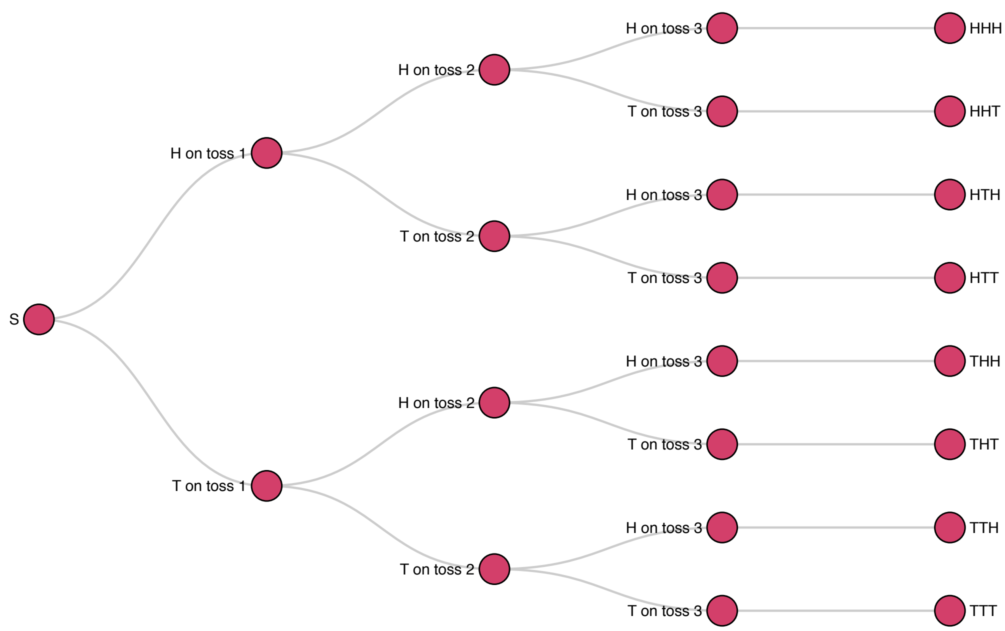
```
2*2*2
```

```
## [1] 8
```

```
2^3
```

```
## [1] 8
```

Sample space: $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$



Event of interest: "At least one" let x be # of H's $P(x \geq 1)$

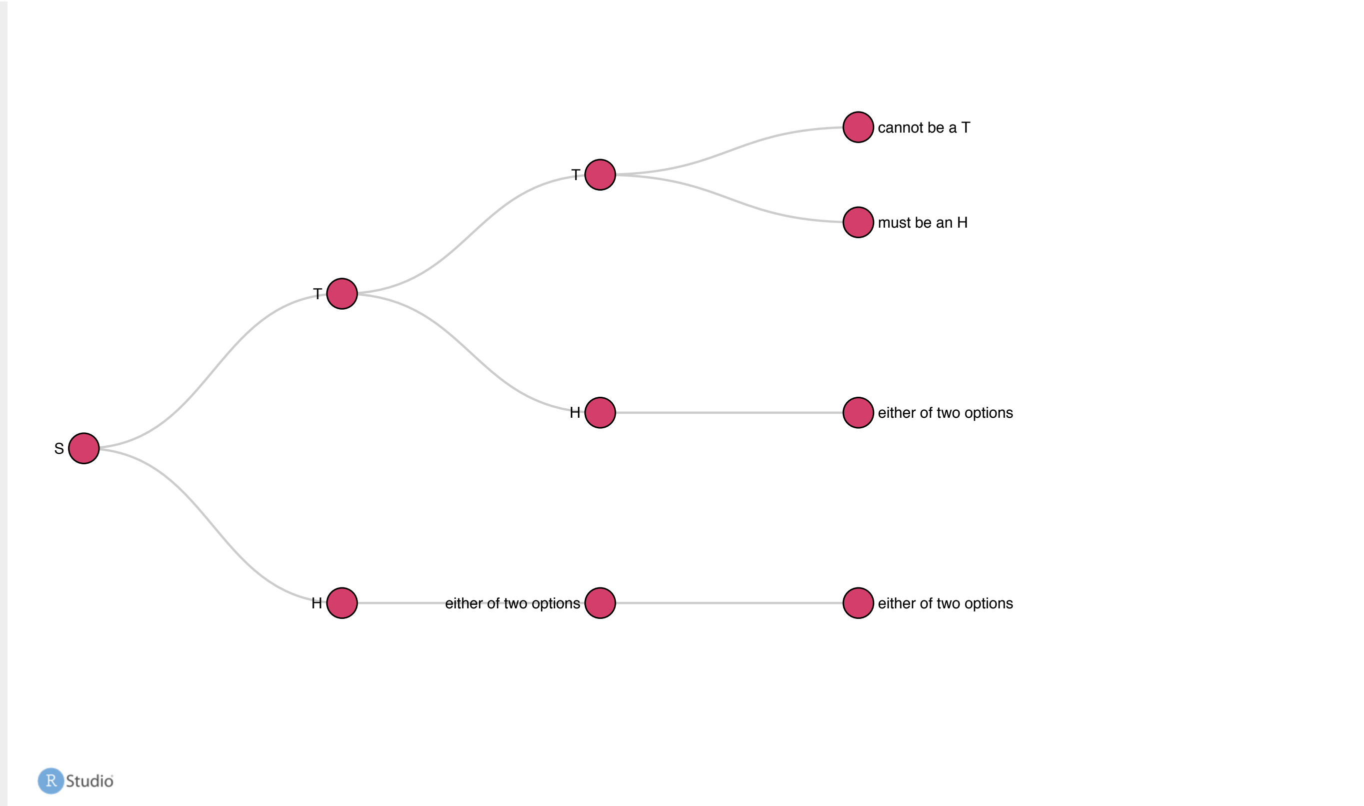
x could be $0, \boxed{1, 2, 3}$

Probability:

$P(x \geq 1) = 1 - \frac{1}{8} = \frac{7}{8}$ (take 1 - what we don't want)

called "Law of Complement"

Alternatively…




```
8-1
```

```
## [1] 7
```

```
1*1*1+1*1*2+1*2*2
```

```
## [1] 7
```

What if outcomes are not equally likely?

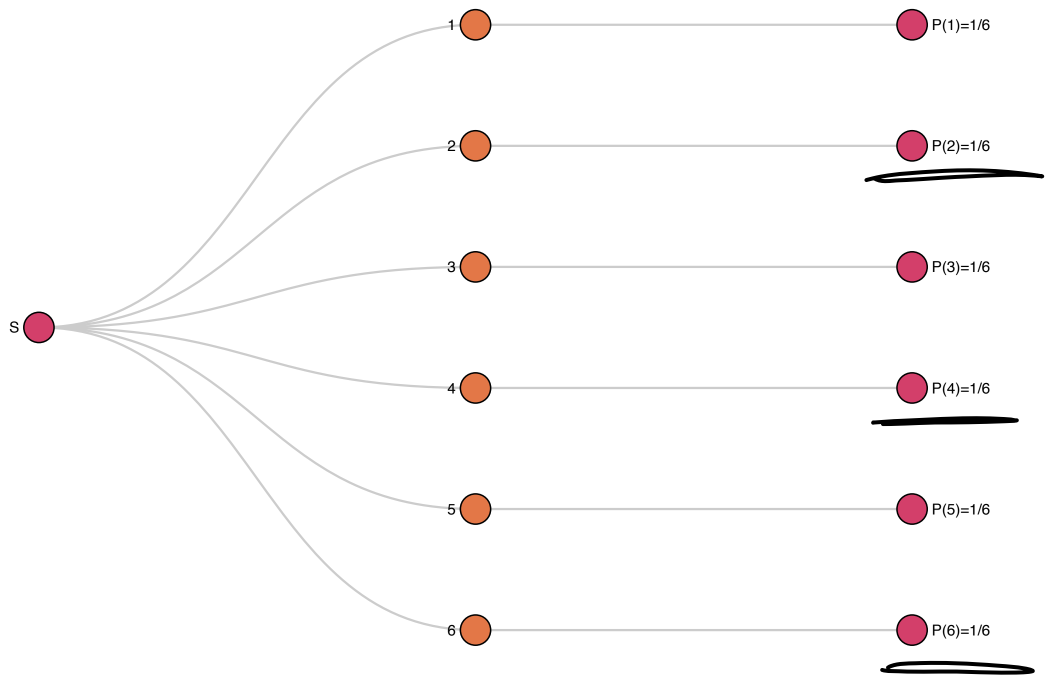
H(1/4)
- H(1/4)
  - H(1/4) → $P(H\,H\,H) = (1/4)(1/4)(1/4) = 1/64$
  - T(3/4) → $P(H\,H\,T) = (1/4)(1/4)(3/4) = 3/64$
- T(3/4)
  - H(1/4)
  - T(3/4)

T(3/4)
- H(1/4)
  - H(1/4)
  - T(3/4)
- T(3/4)
  - H(1/4)
  - T(3/4) → $P(T\,T\,T) = (3/4)^3$

P(@ least one head) $= 1 - (3/4)^3 = 37/64$

Example When rolling a fair six-sided die, we are interested in the probability of observing an even number.

Experiment: rolling a six-sided die



Sample space:

$\{1, 2, 3, 4, 5, 6\}$

Event of interest: an even number

$\{2, 4, 6\}$

Probability:

$P(\text{even}) = 3/6 = 1/2$

Determining the Number of Elements in the Sample Space

In the above examples, it is fairly quick and easy for us to figure out and list all of the elements in the sample space. But what if it is more difficult to do so? For example, suppose we wanted to roll five six-sided dice at once (like in Yahtzee). How many elements would be in the sample space —that is, how many different combinations of the five dice are there? If you tried to list them, you'd quickly find out that there would be way too many to list!

One rule that we can use to help us quickly determine $n(S)$, or the number of elements in the sample space S, is a simplified version of what's known as the multiplicative rule.

The ***multiplicative rule*** (simplified) states that if you have m instances (or trials), each with r outcomes (or results), the number of elements in the sample space S is given as:

recall 3 tosses of a coin $\Rightarrow 2^3 = 8$

$$n(S) = r^m$$

Example You roll five six-sided dice. How many elements are in S?

$6^5 = n(S) = 7776$

with replacement $\to$ everytime I do the experiment the same options available (each time 6 sides available)

#'s in hat w/o replacement select 3

$5 \cdot 4 \quad 3 = \dfrac{5!}{(5-3)!}$ } permutation (incomplete factorial)

Example You ask three friends to choose their favorite color out of blue, green, red, and yellow. What is n(S)?

$4^3 = n(S) = 64$

flip a coin then roll a die (not the same # of events in both)

$2 \times 6 = 12$ outcomes

Axioms of Probability

Finally, before going on to discuss probability relationships and some set theory notation, let's make note of a few axioms of probability that help further define the idea of probability. Suppose event A is a subset of sample space S.

Axiom 1: $0 \leq P(A) \leq 1$ → ratio: % or decimal? "out of" 1

Axiom 2: $P(S) = 1$ → all events must be included

These axioms can be helpful when calculating probabilities, as we will see later with a few examples.

→ sample space needs all outcomes

Set Theory Notation and Probability Relationships

Probability expressions can and often do involve more than one event of interest. For example, maybe we were tossing a coin and rolling a die at the same time and wanted to find the probability of getting a "head" and a "4." How would we write this?

Using ***set theory notation*** to write probability expressions allows us to have a convenient and consistent way of expressing expressions that are more complicated. They also allow us to express specific relationships amongst events in which we are interested.

→ shape with "space" representing liklihood or probability

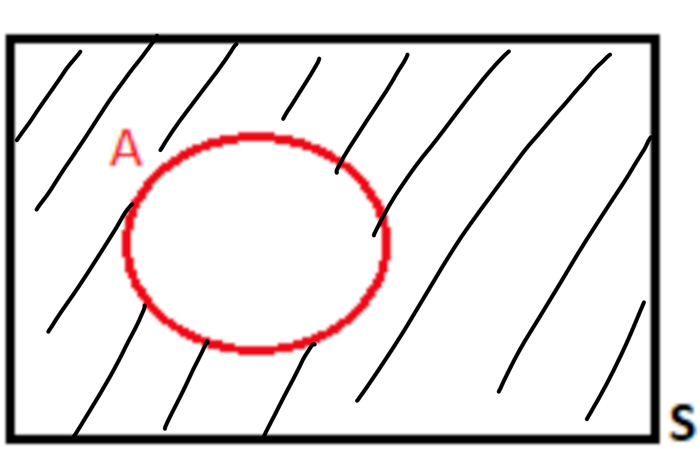
Probability Relationships and Formulas using Set Theory Notation

Suppose we have an event A. "NOT"

The ***complement*** of A is the set of all elements in the sample space S that do not belong to A. In other words, it is the event that A does not occur. The complement of A is denoted as A^C or $\overline{A}$. What is the complement of white? every color that is not white

Note that A and A^C cover the entire sample space and do not contain any of the same elements.

A related probability formula is: $P(A) + P(A^C) = 1$



$A^c \rightarrow$ not A

Always true law of complement

$\rightarrow P(A) + P(A^c) = 1$

$1 - P(A) = P(A^c)$

$1 - P(A^c) = P(A)$

Now, in addition to event A, define another event B.

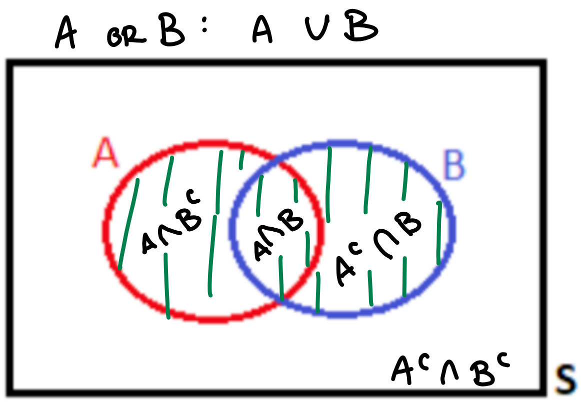
"OR"

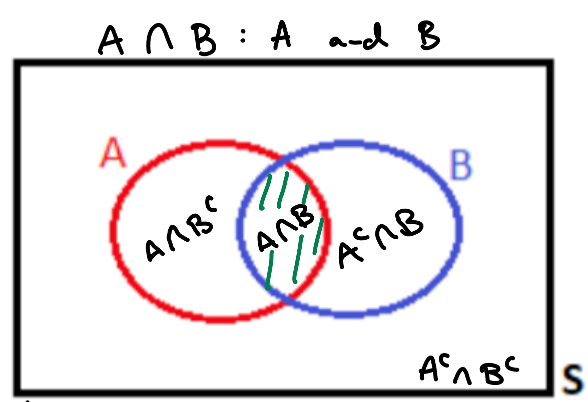
The ***union*** of A and B is the set of all elements that belong to A or B or both. The union of A and B is denoted as $A \cup B$.

The related probability formula is: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $= P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$

The ***intersection*** of A and B is the set of all elements that belong to A and B. The intersection of A and B is denoted as $A \cap B$.

The related probability formula is: $P(A \cap B) = P(A) + P(B) - P(A \cup B)$





	A	A^c	
B	$P(A\cap B)$	$P(A^c\cap B)$	$P(B)$
B^c	$P(A\cap B^c)$	$P(A^c\cap B^c)$	$P(B^c)$
	$P(A)$	$P(A^c)$	1

Other Useful Probability Relationships and Rules

In addition to the formulas above, a few other formulas and rules can be useful when computing probabilities.

DeMorgan's laws can help simplify a probability calculation. The rules are:

to take complement outside of bracket change ∩ to ∪

$$P(A^{\cancel{C}} \cap B^{\cancel{C}}) = P(A \cup B)^{\cancel{C}} = 1 - P(A \cup B)$$

	A	A^c
B		
B^c		✓

=

	A	A^c
B	x	x
B^c	x	

= 1 −

	A	A^c
B	✓	✓
B^c	✓	

$$P(A^C \cup B^C) = P(A \cap B)^C = 1 - P(A \cap B)$$

$$P(A^C \cup B) = P(A \cap B^C)^C = 1 - P(A \cap B^C)$$

	A	A^c
B	✓	✓
B^c		✓

=

	A	A^c
B		
B^c	x	

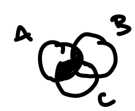
= 1 −

	A	A^c
B		
B^c	✓	

The ***law of total probability*** states that for any events A and B:

$$P(A) = P(A \cap B) + P(A \cap B^C)$$

The ***distributive laws*** work kind of like distributive laws in math. If we define a third event C:



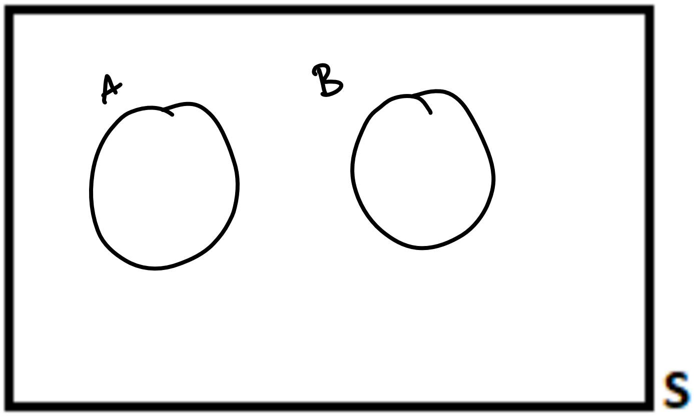
$$P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C))$$

$$P(A \cup (B \cap C)) = P((A \cup B) \cap (A \cup C))$$

Note that these relationships and laws can be extended to more than two or three events!

Mutual Exclusivity

The last concept we'll discuss in this first set of notes is the concept of mutual exclusivity. Suppose we have two events A and B. These events A and B are ***mutually exclusive*** (or ***disjoint***) if: $P(A \cap B) = 0$



zero chance of A and B
happening @ the same time

	A	A^c
B	0	
B^c		

if A + B are mutually exclusive
then $P(A \cap B) = 0$

That is, A and B are mutually exclusive if there are no elements that are in both A and B. Note that when events A and B are mutually exclusive, our additive rule simplifies: $P(A \cap B) = P(A) + P(B)$

Let's put all of these new notations and rules into practice with a few examples!

Example

You have the following two spinners. In an experiment, you spin each spinner once, adding the two resulting numbers.

1. How many elements are in S? List them.

1
- 2 – 1,2
- 3 – 1,3
- 4 – 1,4

3
- 2 – 3,2
- 3 – 3,3
- 4 – 3,4

5
- 2 – 5,2
- 3 – 5,3
- 4 – 5,4

$3 \times 3 = 9 = n(S)$

	1	3	5
2			
3			
4			

2. Define event A as "the sum of the numbers is odd." What is $n(A)$? What is $P(A)$?

Spinner 1

Spinner 2

	1	3	5	
2	3	5	7	odd
3	4	6	8	even
4	5	7	9	odd

P(Sum odd)

$= P(A) = \frac{6}{9}$

3. Define event B as "the sum of the two numbers is less than 5." What is $\boxed{P(A \cap B)}$?

5 is not <5

what is probability of the sum being odd and <5

	1	3	5
2	3^{A}_{B}	5^{A}	7^{A}
3	4_{B}	6	8
4	5^{A}	7^{A}	9^{A}

$P(B) = \frac{2}{9}$

$P(A \cap B) = \frac{1}{9}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= P(A \cap B) + P(A^c \cap B) + P(A \cap B^c)$$
$$= \frac{7}{9}$$

	A	A^c	
B	1/9	1/9	2/9
B^c	5/9	2/9	7/9
	6/9	3/9	

Example

Suppose $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.2$.

	A	A^c	
B	0.2	0.3	0.5
B^c	0.2	0.3	0.5
	0.4	0.6	1

1. What is $P(A \cup B)$?

$P(A) + P(B) - P(A \cap B)$

$0.4 + 0.5 - 0.2$

$= 0.7 = 1 - 0.3$

2. What is $P(A \cap B)^C$?

$1 - P(A \cap B) = P(A \cap B)^c$

$1 - 0.2 = 0.8$

3. What is $P(A \cap B^C)$? where A exists but B doesn't

$= 0.2$

4. Are A and B mutually exclusive? Why/why not?

Definition: if $P(A \cap B) = 0$ then they're mutually exclusive.

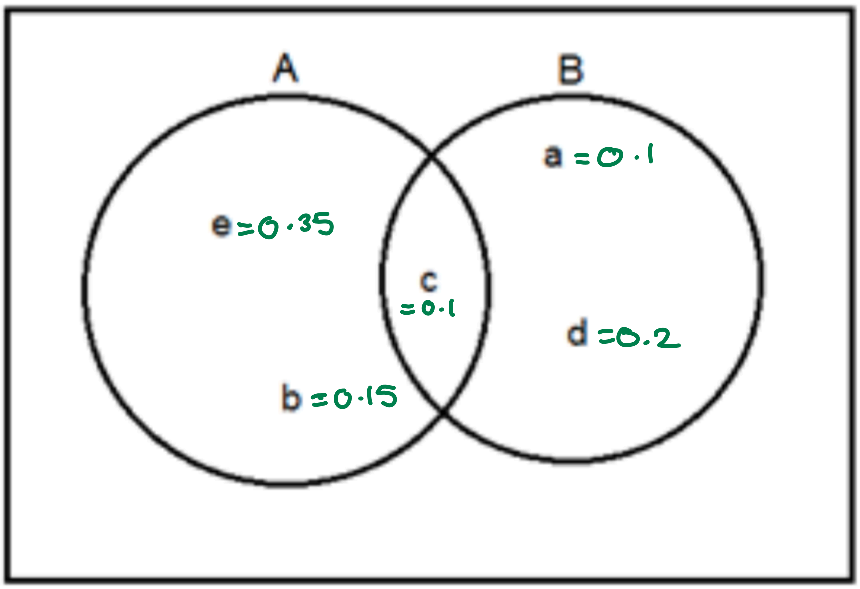
look @ table where A and B intersect $A \cap B = 0.2 \neq 0$

so they are not mutually exclusive

Example

The following Venn diagram shows a sample space with five elements. Events A and B are defined as in the diagram. Let $P(a) = P(c) = 0.1, P(b) = 0.15, P(d) = 0.2$, and $P(e) = 0.35$.

	A	A^c	
B	0.1	0.3	0.4
B^c	0.5	0.1	0.6
	0.6	0.4	1



1. Find $P(A^C \cup B^C)$.

$P(A^c) + P(B^c) - P(A^c \cap B^c)$

$= 0.4 + 0.6 - 0.1$

$= 0.9$

OR

$P(A \cap B)^c$

$= 1 - P(A \cap B)$

$= 1 - 0.1$

$= 0.9$

2. Find $P(A \cap B^C)$.

$= 0.5$

3. Find
   $P(B \cap B^C)$. $= 0$

B and B^c can't exist @ the same time

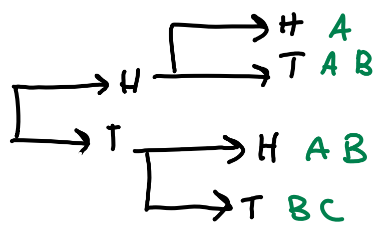
B and B^c are mutually exclusive bc $P(B \cap B^C) = 0$

Example

Suppose we were to toss two fair coins. Define A as "observing at least one head," B as "observing at least one tail," and C as "observing exactly two tails."

1. What are the probabilities of these three events occurring?

$P(A) = 3/4$ $P(B) = 3/4$ $P(C) = 1/4$



	Coin 1 H	Coin 1 T
Coin 2 H	A	AB
Coin 2 T	AB	BC

2. What is the probability of observing $\underbrace{\text{at least one head}}_{A}$ $\underset{\cap}{\text{and}}$ $\underbrace{\text{exactly two tails}}_{C}$?

$P(A \cap C) = 0 \rightarrow$ Mutually Exclusive

3. What is
   $P(C \cup (A \cap B))$? with if i have a C then great but if I dont then I i should have an A and B

4. What is $P(A \cup B \cup C)$?

H → H A
H → T AB
T → H AB
T → T B

All $\frac{4}{4} = 1$

5. Are A and C mutually exclusive? Why/why not?

Yes they are mutually exclusive bc they can't occur simultaneously

$P(A \cap C) = 0$

ex. roll two 7 sided die (1 - 7)

Events

A : sum is 7
B : product is 12
C : exactly one die shows a 5
D : difference between dice is 3

	1	2	3	4	5	6	7
1				D	C	A	
2					AC	B	
3				AB	C	D	
4	D		AB		C		D
5	C	ACD	C	C		C	C
6	A	B	D		C		
7				D	C		

A) neither A nor B → NOT A and NOT B

$P(A^c \cap B^c)$ $P(A \cup B)^c = 1 - P(A \cap B) = 1 - \frac{8}{49} = \frac{41}{49}$

B) $P(A \cup (B \cap C)) = 6/49$

+ A or B+C = A and C

C) Are B and C mutually exclusive

$P(B \cap C) = 0 \rightarrow$ <u>ME</u>

d) $P(A \cap B \cap C \cap D) = 0$

e) $P(A \cup B \cup C \cup D) = \frac{24}{49}$