4.2: Consistency of Estimators

Textbook: 9.3

Objectives

- Know the definition of a <u>consistent estimator</u>
- Be able to determine if a given estimator is consistent

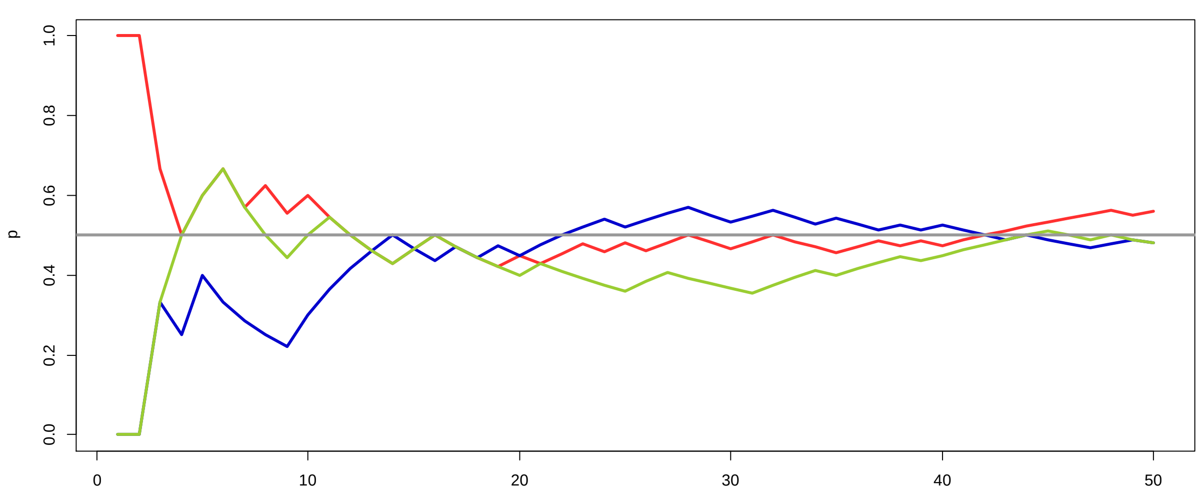
Motivation

To help us understand the topic discussed in these notes, let's start with a demonstration. Suppose we had a coin and we wanted to determine if the coin was fair. Let's use n to denote the number of times we flip the coin and X denote the number of times the coin comes up "heads." The ratio $\frac{X}{n}$ is a proportion; specifically, it is the proportion of flips that result in a "heads."

$$\hat{p} = \frac{X}{n}$$

We know that a fair coin would land "heads" 50% of the time, or $p = \frac{1}{2}$. One way we could go about determining if a coin was fair would be to observe what happens to $\hat{p}$ as n gets large. If the coin is in fact fair, we would suspect that $\hat{p}$ would approach p (the proportion of observed heads should approach $\frac{1}{2}$) as we increase the number of times we flip the coin.

The following plot shows three iterations of this experiment, each involving a total of 50 flips. The x-axis shows the number of flips that have occurred, and the y-axis shows the proportion of heads observed out of the number of flips.



You can see that while the sequences are not identical, all three of them do seem to show a "convergence" of sorts to $p = \frac{1}{2}$, or the "true" proportion of heads we would observe if the coin was fair. This convergence is visible by comparing the wider spread of the values of the estimates for smaller values of n to the smaller spread of the values of the estimates for larger values of n.

In this set of notes, we will formally define this idea of convergence with the property of consistency and see how we can determine whether or not a given estimator is consistent.

Consistency

Let's continue using the above example to define the property we're observing. Because the estimator $\hat{p}$ is a random variable, we can express its "closeness" to p, the parameter we're trying to estimate, in terms of probability. Consider the difference between the estimator and the target parameter as $|\hat{p} - p|$. We want to determine the probability that this difference is less than some arbitrary positive real number ε.

If n is large and our intuition about the estimator is correct, then

$$P(|\hat{p} - p| \leq \varepsilon)$$

should tend to 1, or

$$P(|\hat{p} - p| \leq \varepsilon) \approx 1$$

If this probability does in fact tend to 1 as $n \to \infty$, then we say that $\hat{p}$ is a consistent estimator of p. Another way to express this is to say that $\hat{p}$ converges in probability to p.

To generalize this for any estimator $\hat{\theta}_n$, we say that $\hat{\theta}_n$ is a <u>consistent estimator</u> of θ ($\hat{\theta}_n$ <u>converges in probability</u> to θ) if and only if, for any positive number ε,

$$\lim_{n\to\infty} P\left(\left|\hat{\theta}_n - \theta\right| \leq \varepsilon\right) = 1$$

or, equivalently,

$$\lim_{n\to\infty} P\left(\left|\hat{\theta}_n - \theta\right| > \varepsilon\right) = 0$$

where $\hat{\theta}_n$ is an estimator based off of a sample of size n.

*Example 4.2.1**
Recall Example 3.2.3 from the 3.2 notes: let $X_1, X_2, \dots, X_n$ represent a random sample of student UCID numbers. Assume $X_i \sim Uniform(0, N)$ where N is the total number of U of C students. We are interested in estimating $\theta = N$. We determined the following results:

$$f_{X_{(n)}}(x) = \frac{n}{N^n} x^{n-1} \qquad E\left[X_{(n)}\right] = \frac{Nn}{n+1}, 0 < X_{(n)} < N$$

and showed that $X_{(n)}$ was a biased estimator for N. Determine if $X_{(n)}$ is a consistent estimator for N.

Consistency of Unbiased Estimators

Theorem 4.2.1

If $\hat{\theta}_n$ is an unbiased estimator for θ, then it is also a consistent estimator of θ if

$$\lim_{n\to\infty} VAR[\hat{\theta}_n] = 0$$

*Example 4.2.2**

Consider $\hat{p}$, $\bar{x}$, and s^2, which are all unbiased estimators for their respective parameters p, μ, and σ^2. Determine if they are also consistent estimators for their respective parameters.

*Example 4.2.3**
A random sample of n values is taken from a population that follows a normal distribution with a mean of μ and a known variance $\sigma = 1$. Consider the first observation X_1 as an estimator for μ.

1. Show that X_1 is an unbiased estimator for μ.

2. Find $P(|X_1 - \mu| \leq 1)$.

R function: `pnorm(x)`

`pnorm(  )  - pnorm(  )`

3. Based on the above result, is X_1 a consistent estimator for μ?

Additional Results
The following are a few additional results that may help in determining the consistency of estimators.

Suppose $\hat{\theta}_n \to \theta$ and $\hat{\theta}_n^* \to \theta^*$.

i. $\hat{\theta}_n + \hat{\theta}_n^* \to \theta + \theta^*$

ii. $\hat{\theta}_n \hat{\theta}_n^* \to \theta\theta^*$

iii. $\hat{\theta}_n / \hat{\theta}_n^* \to \theta / \theta^*$

iv. If $g(\cdot)$ is a function that is continuous at θ, then $g(\hat{\theta}_n) \to g(\theta)$