

A Length-Based Stock Assessment Method Utilizing a Generalized Delay-Difference Model

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We present a length-based stock assessment method based on a generalization of the delay-difference methods of Deriso. Data inputs to the model include the estimated weight of the catch, the estimated fishing effort, the estimated mean weight of the catch, and the estimated central moments of the length distribution of the catch. The model's performance is demonstrated by applying it to data from a simulated exploited fish population exhibiting biomass-dependent catchability which would cause conventional catch-effort models to seriously underestimate the extent of stock depletion. The model was generally able to detect the presence of the biomass-dependent catchability and to correctly estimate the optimal level of fishing effort.

Les auteurs présentent une méthode d'évaluation du stock basée sur la longueur d'après une généralisation des méthodes de différence du décalage de Deriso. Les données utilisées pour l'élaboration du modèle comprennent les estimations du poids des prises, de l'effort de pêche, du poids moyen des prises et des moments centrés de la distribution des longueurs dans les prises. On obtient un aperçu du rendement du modèle en l'appliquant à des données obtenues d'une population hypothétique exploitée de poissons qui montre un potentiel de capture dépendant de la biomasse; dans le cas des modèles conventionnels sur les prises et l'effort, ce dernier facteur sous-estimerait fortement l'importance de l'appauvrissement du stock. En général, le modèle pouvait déceler la présence du potentiel de capture dépendant de la biomasse et estimer précisément le niveau optimum d'effort de pêche.

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Deriso (1980) introduced a population dynamics model, based on concepts of age-structured populations, which had dynamical equations describing the evolution of the population over time that were independent of the age structure. This enabled the population dynamics to be described in terms of biologically meaningful parameters determining such things as growth, mortality, and the relationship between stock and recruitment while avoiding the necessity for specifying exactly what the numbers at age in the population were at any time.

While Deriso's approach represents a great conceptual advance over stock-production models such as Schaefer's model (Schaefer 1954; Schnute 1977), or generalizations thereof (Pella and Tomlinson 1969), its usefulness in practice has been hampered by the fact that while his model has considerably more parameters to estimate than stock-production models, it uses about the same amount of data, e.g. a time series of catch and effort data, to estimate these parameters. To overcome this difficulty, we shall demonstrate how Deriso's approach can be extended so that it will predict the values of other statistics easily collected with catch data, such as the mean length and weight of the catch, and the higher sampling moments of the length distribution of the catch. This approach increases the number of observations to be fitted by the model at the cost of introducing only a small number of additional parameters. It

also produces a model which is a natural tool for length-based stock assessment.

To estimate the parameters in the model, we have adopted a Bayesian approach. This approach enables the user of the model to include prior knowledge or opinion into the analysis and to incorporate at the outset an optimistic or conservative view of the ability of the stock to withstand exploitation.

In keeping with this Bayesian viewpoint, parameterizations of relationships have been incorporated into the model even though they may not be well determined by the data alone. For example, the model utilizes a general form of stock-recruitment relationship which includes both the Ricker and Beverton-Holt stock-recruitment relationships as special cases.

To introduce the methods used in the paper in the simplest fashion, we have begun with the one group case considered by Deriso. The difference equations satisfied by the biomass of the population and the total number of fish in the population are derived.

The main part of the paper deals with the extra complexity brought about by the incorporation of length data, multiple fisheries, and age-dependent effects into the model.

A good criterion justifying the introduction of a complicated new theory and a model resulting from it would be that one must demonstrate that situations might reasonably exist where the complicated model can produce a better analysis than simple alternative models. Ludwig and Walters (1985) showed

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that a very simple model could outperform Deriso's model for simulated data where the data were generated according to Deriso's model's hypotheses. It would seem therefore that Deriso's model itself has already failed this criterion and one might ask why an even more complicated model should be expected to perform better.

Both Deriso's model and Ludwig and Walter's model (which appeared originally in Ludwig and Hilborn 1983) assume that there is constant catchability, i.e. that a simple relationship such as

$$C_i = B_i(1 - \exp(-q\hat{E}_i))$$

holds in the i th year where C_i is the catch in year i , B_i is the biomass in year i , \hat{E}_i is the estimated fishing effort in year i , and q is the constant catchability.

The use of constant catchability has been criticized on the grounds that the behavior of some species, especially schooling behavior (Clark and Mangel 1979), may result in catchability increasing as the biomass is reduced by exploitation. Such an effect can have important management implications.

The reason that the constant catchability assumption is made in models which utilize only catch and effort data is that there is not enough information in such data to determine whether the catchability is biomass dependent or not. To date, the only methods which can investigate the relationship between biomass and catchability involve the use of data derived by aging the fish, or by using age estimates produced by using an age-length relationship. Aging data are difficult and expensive to obtain. Age-length relationships only produce useful age estimates when the relationship is known with sufficient accuracy and when the length distributions of neighboring age classes do not overlap too much. An alternative method which used length data from a fishery directly, without the need to impute ages to the fish and which could detect biomass-dependent changes in catchability, could be very useful.

To demonstrate the model's ability to analyze fisheries data without the need for age data and without the necessity for making the constant catchability hypothesis, an approach similar to that of Ludwig and Walters (1985) was used. A series of simulated data sets were generated and analyzed and the results used to estimate the optimal level of effort. The main extension to the assumptions adopted by Ludwig and Walters was the inclusion of a biomass-dependent effect in the catchability. This effect would lead to a severe underestimation of the extent of stock depletion if it were not accounted for. The model was generally able to detect the biomass-dependent component of the catchability and correctly determine the optimal level of exploitation.

Simplest Case

The general model contains a large number of concepts and this may obscure the main ideas. For this reason we first present a less formal discussion of the necessary ideas in the simplest case.

To avoid the introduction of an excessive amount of notation we shall introduce notation in this section which has a slightly different interpretation from that used in the rest of the paper.

Notation

i indexes the fishing year
 j indexes the ages

N_{ij} total number of fish of age j at the beginning of year i
 $N_{i\cdot}$ total number of fish in year i
 R_i recruitment in year i (in numbers of fish)
 l survival rate (from natural mortality)
 s_i survival rate from fishing in year i
 C_i total weight of the catch in year i
 \hat{C}_i estimated total weight of the catch in year i
 \bar{M}_i mean weight of the fish in the population in year i
 \hat{M}_i estimated mean weight of the fish in the population in year i
 w_j mean weight of an age j fish
 p Brody growth coefficient
 B_i total weight of the fish in the population in year i
 k age at which the fish recruit to the population
 q catchability in the effort - fishing mortality relationship
 \hat{E}_i effect fishing effort in year i
 \hat{E}_i estimated fishing effort in year i

We assume that there is a lag of k years before the fish recruit. After recruitment all fish are assumed to be equally vulnerable to the fishery regardless of their ages. For simplicity assume that the fishery takes place at the beginning of the year over a time period so short that natural mortality can be ignored. Let s_i denote the survival rate from fishing and l the survival rate from natural mortality. Then $(1 - s_i)$ is the proportion of fish caught in the fishery.

Assume that the stock dynamics of the population are given by the equations

$$(1.1) \quad N_{ik} = R_i$$

and for $j > k$

$$(1.2) \quad N_{ij} = l s_{i-1} N_{i-1,j-1}.$$

The simplest difference equation is satisfied by the total number of fish in the population denoted by $N_{i\cdot}$:

$$(1.3) \quad N_{i\cdot} = \sum_{j \geq k} N_{ij}.$$

In expression 1.3, split off the first term to obtain the equivalent expression

$$(1.4) \quad N_{i\cdot} = \sum_{j > k} N_{ij} + R_i$$

or by expression 1.2

$$N_{i\cdot} = l s_{i-1} \sum_{j \geq k} N_{i-1,j} + R_i$$

or

$$(1.5) \quad N_{i\cdot} = l s_{i-1} N_{i-1\cdot} + R_i.$$

Expression 1.5 is a first-order difference equation satisfied by the $N_{i\cdot}$. By employing expression 1.5 iteratively it is possible to express any $N_{i\cdot}$ in terms of the recruitments R_p for $2 \leq p \leq i$, the survival rates from fishing s_p for $1 \leq p \leq i-1$, the survival rate from natural mortality l , and the total number of fish in the population in year 1, $N_{1\cdot}$. For example, consider $N_{3\cdot}$:

$$N_{3\cdot} = l s_2 N_{2\cdot} + R_3$$

or

$$(1.6) \quad N_{3\cdot} = l s_2 (l s_1 N_{1\cdot} + R_2) + R_3.$$

Assume that w_j , the mean weight at age j of the fish, satisfies the relationship

$$(1.7) \quad w_j = a \left(\frac{1 - \rho^j}{1 - \rho} \right) \quad 0 < \rho < 1$$

where a is the mean weight of an age 1 fish.

It is easily verified that the w_j satisfy the first-order difference equation

$$(1.8) \quad w_j - \rho w_{j-1} = a.$$

The biomass in year i , B_i , is given by

$$(1.9) \quad B_i = \sum_{j \geq k} w_j N_{ij}.$$

The relationship 1.8 implies that

$$\sum_{j \geq k} (w_j - \rho w_{j-1}) N_{ij} = a N_i.$$

or

$$(1.10) \quad B_i = \rho \sum_{j \geq k} w_{j-1} N_{ij} + a N_i.$$

Using expression 1.2, one obtains

$$(1.11) \quad \sum_{j \geq k} w_{j-1} N_{ij} = l s_{i-1} \sum_{j \geq k} w_j N_{i-1,j} + w_{k-1} R_i.$$

Substituting the relationship 1.11 into expression 1.10 yields

$$(1.12) \quad B_i = \rho l s_{i-1} B_{i-1} + a N_i + \rho w_{k-1} R_i.$$

Expression 1.12 shows that the B_i also satisfy a first-order difference equation independent of the age structure. Since in expression 1.12 the B_i also depend on the N_i , the two expressions 1.5 and 1.12 together form a first-order system of difference equations.

Note that the derivation of the difference equation 1.12 satisfied by the B_i depended on the fact that the coefficients themselves satisfied the difference equation 1.8. This enabled the B_i to be expressed in terms of the N_i and the "lagged" term

$$(1.13) \quad \sum_{j \geq k} w_{j-1} N_{ij}.$$

Expression 1.13 is not of the same type as B_i but it would be if the index j in N_{ij} matched the index $j-1$ in w_{j-1} . The relationships 1.1 and 1.2 enable this to be done. This is the general approach which is used to derive the desired difference equations in this paper.

To see how the above results could be used in practice, assume that for some fishery one has estimates of the total catch, the total fishing effort, and the mean weight of the catch for each year of the fishery. Let \hat{C}_i , \hat{E}_i , and \hat{M}_i denote the values of these estimates for year i . Assume also that the survival rate s_i satisfies the relationship

$$(1.14) \quad s_i = \exp(-qE_i)$$

where q is an unknown parameter, the catchability. Since all fish are assumed to be equally vulnerable to the fishery, we shall assume that the total catches C_i satisfy the relationship

$$(1.15) \quad C_i = (1 - s_i) B_i.$$

For the recruitment R_i , we shall assume that a Ricker stock-recruitment relationship holds so that

$$(1.16) \quad R_{i+k} = \alpha B_i \exp(-\beta B_i).$$

Note that the recruitment cannot be calculated for the years 1 through k because corresponding biomass is not known. We shall deal with this problem by using the biomass B_i in expression 1.16 to compute the R_i for $2 \leq i \leq k+1$.

For a given value of the parameters $B_1, N_1, l, q, \alpha, \beta, a, \rho$, and the $s_p, 1 \leq p \leq i-1$, it is possible to use equations 1.5 and 1.12 to calculate the quantities B_i and N_i for all the years of the fishery. The mean weight M_i of the fish in the population can then be calculated from the relationship

$$(1.17) \quad M_i = B_i / N_i.$$

The method used for calculating the s_i, C_i , and E_i depends on the assumptions made about the form of the errors present in the data. If it is assumed that the estimated effort \hat{E}_i is equal to the effective effort E_i , while the errors are in the observed catches, then the s_i can be calculated from 1.14 with E_i set equal to \hat{E}_i and then the catches C_i calculated from 1.15 so that

$$s_i = \exp(-q\hat{E}_i)$$

and

$$C_i = (1 - \exp(-q\hat{E}_i)) B_i.$$

If it is assumed that the true catches are equal to the estimated catches and the errors are in the estimated fishing effort, then the s_i can be calculated from 1.15 with C_i set equal to \hat{C}_i giving

$$s_i = 1 - \hat{C}_i / B_i$$

and from the relationship 1.14

$$E_i = -\log_e(s_i)/q.$$

Finally, if it is assumed that there are errors both in the estimated catches and the estimated effort, then the effective efforts E_i become independent model parameters and the C_i and s_i are calculated from expressions 1.14 and 1.15.

Since all fish in the population are assumed to be equally vulnerable (after recruitment), the mean weight of the fish in the population is also the mean weight of the fish in the catch. To estimate the correct values of the model parameters, we pick those values of the model parameters for which the predicted values of the C_i, M_i , and E_i agree best with their observed values \hat{C}_i, \hat{M}_i , and \hat{E}_i .

Of course, the notion of best agreement depends on which criterion for measuring closeness is chosen. This criterion will depend on the type of probabilistic mechanisms which are assumed to generate the observations. One could assume, for example, that there is no error in the estimated effort so that $E_i = \hat{E}_i$ and that the observed catches are lognormally distributed and that the quantities $\log_e(\hat{C}_i)$ are normally distributed with means $\log_e(C_i)$, while the \hat{M}_i are normally distributed with means M_i . Then the maximum likelihood criterion for closeness would be to pick that value of the parameters $B_1, N_1, l, q, \alpha, \beta, a$, and ρ for which the sum of squares

$$(1.18) \quad \sum_i (\log_e(C_i) - \log_e(\hat{C}_i))^2 + p \sum_i (M_i - \hat{M}_i)^2$$

is minimized. The quantity p in expression 1.18 is a penalty weight which determines the extent to which the model attempts to fit the observed catches or the observed mean weights. In general, there is not enough information in the data to determine the correct value of p . Such information must be supplied from external sources or the desired value of p deter-

mined by external considerations. This sort of ambiguity is common in this type of model where different types of data, in this case total catch and mean weight data, are being combined. However, a general statement can be made.

If we regard the catch data \hat{C}_i as the primary data and the mean weight data \hat{M}_i as data supplying auxiliary information, then p would be set small enough so that the requirement of the model to fit the observed mean weight data would not prevent it from fitting the observed catch data.

Alternatively, one could assume that there is no error in the catch estimates so that $C_i = \hat{C}_i$ and that the estimated effort is lognormally distributed with the mean value of $\log_e(\hat{E}_i)$ equal to $\log_e(E_i)$. Then the maximum likelihood criterion for closeness would be to minimize the sum of squares

$$(1.19) \sum_i (\log_e(E_i) - \log_e(\hat{E}_i))^2 + p \sum_i (M_i - \hat{M}_i)^2.$$

The simple model may be adequate for many purposes but it has several limitations. Because all the fish in the population are equally vulnerable to fishing, the model is not suitable for fisheries in which recruitment occurs gradually over time. Also, many fish populations are exploited by several fisheries where each fishery has its own age-dependent pattern of selectivity. It is desirable to develop a theory which allows for this level of complexity. Finally, with the assumed mean weight at age relationship 1.7, it is impossible to incorporate length data into the model while at the same time retaining a realistic weight-length relationship. The general model is intended to overcome these limitations.

General Model

To describe the general model, we introduce the following notation which will be used throughout the rest of the paper.

Notation

i	indexes the fishing year
I	total number of years of fishing
j	indexes the age classes
f	indexes the fisheries
N_f	total number of fisheries
g	indexes the groups the fish belong to
N_{ijg}	= number of age j fish in group g at the beginning of year i
w_j	= mean weight of an age j fish
R_i	= recruitment in year i
B_{ig}	= biomass of fish in group g at the beginning of year i
t_{fig}	= proportion of the fish removed by fishing from group g caught by fishery f in year i
s_{ig}	= survival rate (from fishing) for fish in group g in year i
l_g	= survival rate (from natural mortality) in group g
τ_j	= mean length of an age j fish
m_g	= probability that a fish in group g remains in group g for another year
q_{fig}	= catchability for group g in fishery f in year i
a	= mean weight of the age 1 fish
ρ	= Brody growth coefficient
b	= coefficients in the weight-length relationship
RB_i	= reproductive biomass in year i
Q_{f1g}	= constant component of the catchability
Q_{f2g}	= biomass-dependent component of the catchability
C_{fi}	= weight of the catch in fishery f in year i

\hat{C}_{fi}	= estimated weight of the catch in fishery f in year i
n_{fi}	= number of fish sampled for weight and length data from fishery f in year i
\hat{M}_{fi}	= mean weight of the catch in fishery f in year i
\hat{M}_{fi}	= mean weight of the fish in the random sample of the catch in fishery f in year i
M_{fi}^2	= second central moment of the distribution of the weight of the catch for fishery f in year i
\hat{M}_{fi}	= mean weight of the catch in fishery f in year i
\hat{M}_{fi}	= mean weight of the fish in the random sample of the catch in fishery f in year i
M_{fi}^2	= second central moment of the distribution of the weight of the catch for fishery f in year i
μ_{fi}^r	= r th central moment of the lengths of the fish in the catch from fishery f in year i
$\hat{\mu}_{fi}^r$	= r th central moment of the lengths of the fish in the random sample of the catch from fishery f in year i
E_{fi}	= (effective) fishing effort in fishery f in year i
\hat{E}_{fi}	= estimated fishing effort in fishery f in year i
σ_E^2	= variance in the stochastic component of the relationship between estimated fishing effort and effective fishing effort
σ_R^2	= variance in the stochastic component of the stock recruitment relationship
σ_L^2	= variance in the distribution of the lengths of the fish in each age class around their mean

Data

Assume that there is a fish population which has been exploited by a number of fisheries for several years. Each fishery has its own age-dependent selectivity pattern. For simplicity we assume that all the fisheries occur simultaneously for a short period at the beginning of each year. (Short in this context means that mortality during the fishing period from causes other than fishing can be ignored.) We assume that for each year and fishery there are estimates of the total weight of the catch \hat{C}_{fi} and the total fishing effort \hat{E}_{fi} . In addition a random sample of n_{fi} fish has been taken from each catch and the weight and length of each fish noted. (It is not necessary that the weight and length data come from the same sample. This assumption is only made for simplicity.) Let \hat{M}_{fi} and $\hat{\mu}_{fi}^1$ denote the mean weights and mean lengths of the samples and μ_{fi}^2 , μ_{fi}^3 , and μ_{fi}^4 denote the second, third, and fourth central moments of the length distributions of the samples. To calculate the \hat{M}_{fi} and $\hat{\mu}_{fi}^1$, let x_{fiy} and z_{fiy} denote the length and weight of the y th fish taken from fishery f in year i . Then

$$\hat{M}_{fi} = \frac{1}{n_{fi}} \sum_{y=1}^{n_{fi}} z_{fiy}$$

and

$$\hat{\mu}_{fi}^1 = \frac{1}{n_{fi}} \sum_{y=1}^{n_{fi}} x_{fiy}$$

while for $r > 1$

$$\hat{\mu}_{fi}^r = \frac{1}{n_{fi}} \sum_{y=1}^{n_{fi}} (x_{fiy} - \hat{\mu}_{fi}^1)^r.$$

The observations \hat{C}_{fi} , \hat{E}_{fi} , \hat{M}_{fi} , and $\hat{\mu}_{fi}^r$, $r = 1, \dots, 4$, are the data which will be used in the model. The \hat{M}_{fi} and $\hat{\mu}_{fi}^r$ are random variables. Let $M_{fi} = E\{\hat{M}_{fi}\}$ and $\mu_{fi}^r = E\{\hat{\mu}_{fi}^r\}$ denote

the expected value of these random variables. Let C_{fi} denote the actual weight of the catch.

Fitting the Model

Assume for the moment that the system of difference equations has been derived so that the quantities C_{fi} , E_{fi} , M_{fi} , and the μ_{fi}^r can be expressed in terms of a (relatively) small number of model parameters. We shall denote the set of all model parameters by Θ . (The exact number and form of the model parameters will vary in different applications, so we wish to refer to them in a generic fashion at this point.)

The proper method to fit the model depends on the form of the probabilistic assumptions which are made. It will be assumed that the \hat{M}_{fi} are normally distributed with mean M_{fi} and variance v_1^2 and that the $\hat{\mu}_{fi}^r$, $r = 1, \dots, 4$, are normally distributed with means μ_{fi}^r and variances v_{r+1}^2 .

Since the sample moments \hat{M}_{fi} and the $\hat{\mu}_{fi}^r$ are asymptotically normally distributed, the assumption that they are normally distributed is fine as long as the sample sizes n_{fi} are large enough, and this assumption did not appear to cause any problems with the results from the simulations. However, if the length distribution of the catch is highly skewed, or the sample sizes are small, the distribution of the third and fourth sample moments may also be highly skewed. The higher sampling moments are also extremely sensitive to outliers in the data. For these reasons it may be advantageous to employ statistics with better statistical properties than the higher sampling moments and to fit them by procedures which are more robust than least squares.

A simple alternative to using the third and fourth sample moments of the catch would be to use the sample skewness $\hat{\mu}_{fi}^3/(\hat{\mu}_{fi}^2)^{1.5}$ and the sample kurtosis $\hat{\mu}_{fi}^4/(\hat{\mu}_{fi}^2)^2$.

With respect to the observed catches \hat{C}_{fi} and estimated efforts \hat{E}_{fi} , two approaches are commonly used. The first is to assume that the effective fishing effort E_{fi} is equal to the estimated fishing effort \hat{E}_{fi} and that all the error is between the observed catches \hat{C}_{fi} and the true catches C_{fi} . The \hat{C}_{fi} are assumed to be lognormally distributed and $\log_e(\hat{C}_{fi})$ is assumed to be normally distributed with mean $\log_e(C_{fi})$ and variance v_0^2 . The second approach is to assume that the estimated efforts \hat{E}_{fi} are lognormally distributed and that $\log_e(\hat{E}_{fi})$ is normally distributed with mean $\log_e(E_{fi})$ and variance v_6^2 .

The second approach is probably more realistic in that there is likely to be a large component of variability in the relationship between estimated fishing effort and the resulting fishing mortality, while the catch can be well estimated at least in principle by using good fleet sampling schemes. The implementation of the second approach in the general model involves considerable technical difficulties, however, so we shall forego further discussion of it until later.

Let $V = (v_0^2, v_1^2, \dots, v_5^2)$ denote the vector of variance parameters.

With the first approach the log-likelihood function $F(\Theta, V)$ for the parameters is given by

$$(2.1) \quad F(\Theta, V) = - \sum_{fi} \{ \log_e(C_{fi}(\Theta)) - \log_e(\hat{C}_{fi}) \}^2 / v_0^2 \\ - \sum_{fi} \{ M_{fi}(\Theta) - \hat{M}_{fi} \}^2 / v_1^2 \\ - \sum_{fi} \{ \mu_{fi}^1(\Theta) - \hat{\mu}_{fi}^1 \}^2 / v_2^2$$

$$- \sum_{fi} \{ \mu_{fi}^2(\Theta) - \hat{\mu}_{fi}^2 \}^2 / v_3^2 \\ - \sum_{fi} \{ \mu_{fi}^3(\Theta) - \hat{\mu}_{fi}^3 \}^2 / v_4^2 \\ - \sum_{fi} \{ \mu_{fi}^4(\Theta) - \hat{\mu}_{fi}^4 \}^2 / v_5^2 \\ - \sum_{fi} \sum_{u=0}^5 \log_e(v_u).$$

In expression 2.1 the parameter v_0^2 is the variance of the random variables $\log_e(\hat{C}_{fi})$. The terms v_1^2 through v_5^2 are the variances of the corresponding sample moments. There are many different structural assumptions that can be made about the variance parameters V . The form of the assumption made depends on how much is assumed to be known, a priori, about these parameters. We shall describe three such assumptions here. They include those assumptions which assume the most a priori knowledge and the least a priori knowledge about these parameters.

The terms v_0^2 through v_5^2 can either be assumed to be known, assumed to be known up to a multiple of each other, or be estimated. The maximum likelihood estimates for the parameters would be found by maximizing expression 2.1 with respect to Θ and V . The estimates obtained depend on the form of the structural assumptions made about V .

Rather than adopting this approach, we have chosen a Bayesian approach which involves the addition of an extra term to the log-likelihood function.

Because some of the parameters in the model such as the growth determining coefficient ρ , the natural survival rates l_g , and the mean length of the fish at the age of recruitment have an interpretation independent of the model, there will often be information about them besides that contained in the data. This additional information is included in the model by assuming the existence of a probability distribution on these parameters whose probability density function is denoted by $G(\Theta)$. The logarithm of the posterior probability distribution for the parameters is given by

$$(2.2) \quad \Phi(\Theta, V) = F(\Theta, V) + \log_e(G(\Theta)).$$

The mode of the posterior distribution is given by those values of the parameters Θ and V which maximize 2.2. These parameter estimates have been used in the estimation and formulation of the optimal policy in the simulation examples.

The values of the parameters obtained when maximizing 2.2 will depend on which of the three assumptions about the parameters V is used. Each assumption has its advantages and disadvantages.

Since $G(\Theta)$ does not depend on V , to maximize expression 2.2 with respect to V , it suffices to first maximize expression 2.1 with respect to V and to substitute the resulting expression $\tilde{F}(\Theta)$ which does not depend on V into 2.2.

Case 1. The v 's are assumed known and fixed

In this case the term

$$\sum_{fi} \sum_{u=0}^5 \log_e(v_u)$$

is constant and can be ignored. Then

$$(2.3) \quad \tilde{F}(\Theta) = - \sum_{fi} \{ \log_e(C_{fi}(\Theta)) - \log_e(\hat{C}_{fi}) \}^2 / v_0^2$$

$$\begin{aligned}
& - \sum_{fi} \{M_{fi}(\Theta) - \hat{M}_{fi}\}^2 / v_1^2 \\
& - \sum_{fi} \{\mu_{fi}^1(\Theta) - \hat{\mu}_{fi}^1\}^2 / v_2^2 \\
& - \sum_{fi} \{\mu_{fi}^2(\Theta) - \hat{\mu}_{fi}^2\}^2 / v_3^2 \\
& - \sum_{fi} \{\mu_{fi}^3(\Theta) - \hat{\mu}_{fi}^3\}^2 / v_4^2 \\
& - \sum_{fi} \{\mu_{fi}^4(\Theta) - \hat{\mu}_{fi}^4\}^2 / v_5^2
\end{aligned}$$

where in expression 2.3 the v 's are known constants.

Case 2. The v 's are known multiples of each other

Assume that

$$(2.4) \quad p_i = v_0^2 / v_i^2$$

where the p_i are assumed to be known. Then expression 2.1 can be rewritten as

$$\begin{aligned}
(2.5) \quad F(\Theta, V) = & - p_0 \sum_{fi} \{\log_e (C_{fi}(\Theta)) - \log_e (\hat{C}_{fi})\}^2 / v_0^2 \\
& - p_1 \sum_{fi} \{M_{fi}(\Theta) - \hat{M}_{fi}\}^2 / v_0^2 \\
& - p_2 \sum_{fi} \{\mu_{fi}^1(\Theta) - \hat{\mu}_{fi}^1\}^2 / v_0^2 \\
& - p_3 \sum_{fi} \{\mu_{fi}^2(\Theta) - \hat{\mu}_{fi}^2\}^2 / v_0^2 \\
& - p_4 \sum_{fi} \{\mu_{fi}^3(\Theta) - \hat{\mu}_{fi}^3\}^2 / v_0^2 \\
& - p_5 \sum_{fi} \{\mu_{fi}^4(\Theta) - \hat{\mu}_{fi}^4\}^2 / v_0^2 \\
& - 6 \sum_{fi} \log_e (v_0) \\
& + \sum_{fi} \sum_{u=0}^5 \log_e (p_u) / 2.
\end{aligned}$$

In expression 2.5 a term p_0 has been included. This term would normally have the value of 1. It has been included so that the catch data can be given a greater weight when fitting the model if this is desired by giving p_0 a value larger than 1.

The value of v_0 maximizing expression 2.5 satisfies the condition

$$(2.6) \quad \partial F(\Theta, V) / \partial v_0 = 0.$$

Differentiating expression 2.5 with respect to v_0 and solving the equation 2.6 for v_0 yields

$$\begin{aligned}
(2.7) \quad v_0^2 = & [p_0 \sum_{fi} \{\log_e (C_{fi}(\Theta)) - \log_e (\hat{C}_{fi})\}^2 \\
& + p_1 \sum_{fi} \{M_{fi}(\Theta) - \hat{M}_{fi}\}^2 \\
& + p_2 \sum_{fi} \{\mu_{fi}^1(\Theta) - \hat{\mu}_{fi}^1\}^2 \\
& + p_3 \sum_{fi} \{\mu_{fi}^2(\Theta) - \hat{\mu}_{fi}^2\}^2
\end{aligned}$$

$$\begin{aligned}
& + p_4 \sum_{fi} \{\mu_{fi}^3(\Theta) - \hat{\mu}_{fi}^3\}^2 \\
& + p_5 \sum_{fi} \{\mu_{fi}^4(\Theta) - \hat{\mu}_{fi}^4\}^2] / IN_t.
\end{aligned}$$

Substituting this expression for v_0^2 into 2.5 and discarding the constant terms (which are irrelevant for the maximization) yields

$$\begin{aligned}
(2.8) \quad \tilde{F}(\Theta) = & - \log_e [p_0 \sum_{fi} \{\log_e (C_{fi}(\Theta)) - \log_e (\hat{C}_{fi})\}^2 \\
& + p_1 \sum_{fi} \{M_{fi}(\Theta) - \hat{M}_{fi}\}^2 \\
& + p_2 \sum_{fi} \{\mu_{fi}^1(\Theta) - \hat{\mu}_{fi}^1\}^2 \\
& + p_3 \sum_{fi} \{\mu_{fi}^2(\Theta) - \hat{\mu}_{fi}^2\}^2 \\
& + p_4 \sum_{fi} \{\mu_{fi}^3(\Theta) - \hat{\mu}_{fi}^3\}^2 \\
& + p_5 \sum_{fi} \{\mu_{fi}^4(\Theta) - \hat{\mu}_{fi}^4\}^2] / 2.
\end{aligned}$$

Case 3. All the v 's estimated

In this case, solving the equation

$$\partial F(\Theta, V) / \partial v_i = 0$$

for each v_i and substituting the resulting expression for the v 's into expression 2.1 and discarding the constant terms yields

$$\begin{aligned}
(2.9) \quad \tilde{F}(\Theta) = & - \log_e [\sum_{fi} \{\log_e (C_{fi}(\Theta)) - \log_e (\hat{C}_{fi})\}^2 / 2 \\
& - \log_e [\sum_{fi} \{M_{fi}(\Theta) - \hat{M}_{fi}\}^2 / 2 \\
& - \log_e [\sum_{fi} \{\mu_{fi}^1(\Theta) - \hat{\mu}_{fi}^1\}^2 / 2 \\
& - \log_e [\sum_{fi} \{\mu_{fi}^2(\Theta) - \hat{\mu}_{fi}^2\}^2 / 2 \\
& - \log_e [\sum_{fi} \{\mu_{fi}^3(\Theta) - \hat{\mu}_{fi}^3\}^2 / 2 \\
& - \log_e [\sum_{fi} \{\mu_{fi}^4(\Theta) - \hat{\mu}_{fi}^4\}^2 / 2.
\end{aligned}$$

The advantage of the first method is that one can specify the v 's exactly. This allows one to completely control their effect on the solution obtained. The disadvantage of course is that one must know what their values are.

The third method is the most elegant. One need know nothing about the v 's. In fact they no longer appear in the estimation scheme at all. The disadvantage is that one has no control over the effect of the various types of data on the parameter estimates obtained.

The second method is a compromise between the other two methods with some of the advantages and disadvantages of both. It is not necessary to specify the absolute size of the variance terms v_u , but only their relative sizes. This enables one to regard the catch and effort data as the primary data with the length and weight moments providing auxiliary information by controlling the influence of the length and weight moments on the parameter estimates obtained.

We consider the second method to be preferable to the other two methods. It enables the user to examine the residuals to the model fit and to adjust the p 's until they seem correct to the user. The user who finds this ad hoc approach unsettling can use the third approach. The second approach has been used in the simulations.

Determining the p 's

When setting values for the p 's, we have adopted the point of view that the catch and effort data \hat{C}_{fi} and \hat{E}_{fi} are the main data while the weight and length moments are used to provide auxiliary information. For this reason we advocate setting the term p_0 large enough relative to p_1 through p_5 so that the fit between the $\log_e(C_{fi})$ and $\log_e(\hat{C}_{fi})$ is acceptably good. It remains to pick relative values for the terms p_1 through p_5 .

It is not necessary to compute the exact variances for the sampling moments, but only to use estimates which are roughly correct. To get an idea of these variances, we have used the variances for the sampling moments for a normal distribution taken from Kendall and Stuart (1958):

Sampling moment	Variance
\hat{M}_{fi}	M_{fi}^2 / n_{fi}
$\hat{\mu}_{fi}^1$	μ_{fi}^2 / n_{fi}
$\hat{\mu}_{fi}^2$	$2(\mu_{fi}^2)^2 / n_{fi}$
$\hat{\mu}_{fi}^3$	$6(\mu_{fi}^2)^3 / n_{fi}$
$\hat{\mu}_{fi}^4$	$96(\mu_{fi}^2)^4 / n_{fi}$

where n_{fi} is the sample size from fishery f in year i .

These variances are only exact for the sampling moments of a normal distribution, but they should reflect the approximate magnitude of the variances of these random variables relative to each other. The quantities M_{fi} and μ_{fi}^2 are not known but they can be approximated by the corresponding values of the sampling statistics \hat{M}_{fi}^2 and $\hat{\mu}_{fi}^2$. Thus, for example, the variance of $\hat{\mu}_{fi}^4$ can be approximated by $96(\hat{\mu}_{fi}^2)^4 / n_{fi}$ and p_5 set equal to $v_0 n_{fi} / [96(\hat{\mu}_{fi}^2)^4]$.

These values for the p 's should be considered to be approximations. The higher sampling moments are much more sensitive to deviations from the model's hypotheses than the observed catches. Besides, the results of the analysis lack credibility if they fit the fourth moment of the length distribution perfectly but fail to predict the observed catches. For these reasons if the observed-catch predicted-catch residuals are too large, then the value of p_0 should be increased until an acceptable agreement is obtained.

Availability and Selectivity

Two important notions in fisheries management are availability and selectivity. Availability is a concept associated with an entire age class and a fishery. It is defined to be the proportion of individuals in the age class which have a positive probability of being caught in the fishery, i.e. are vulnerable to the fishery.

Selectivity is a between age class concept. For an age class j with positive availability, the selectivity in fishery f is defined

to be the probability that a vulnerable age class j fish will be caught by one unit of fishing effort in fishery f . For unavailable age classes the selectivity is defined to be 0.

If a fish is picked at random from an age class, the probability that it is caught by one unit of effort in a fishery is the product of its availability for that fishery times its selectivity for that fishery.

Availability and selectivity will be termed age dependent if they differ among age classes.

The difference between availability and selectivity becomes clear when the fishing effort is very large, for in this case essentially all vulnerable fish are caught. Consider an age class which has an availability of 0.50 and a selectivity of 0.05 versus one with an availability of 1.0 and a selectivity of 0.025. A fish picked at random from either cohort will have a probability of 0.025 of being caught by one unit of fishing effort. For very large fishing effort, however, only 50% of the former cohort will be caught while all of the latter cohort will be caught.

Most age-structured models use age-dependent selectivity while the availability for all age classes is assumed to be 1 (Doubleday 1976; Fournier and Archibald 1982; Deriso et al. 1985). In addition, in these models the selectivity is independent of the history of exploitation of the stock by the fishery.

The theory we shall develop can include both age-dependent availability and selectivity, and the age dependence of both can depend on the history of exploitation of the fishery.

Groups

Age-dependent availability and selectivity are introduced into the model by using the concept of groups first introduced by Deriso.

The general idea is to consider the members of a fish population as belonging to a set of distinct groups. The members of any group are all assumed to experience the same natural and fishing mortality in any year. New recruits are assumed to enter the first group after a lag of k years. During their lifetimes, individual fish proceed sequentially through the groups so that in any year a fish presently in group g has a probability $1 - m_g$ of proceeding on to group $g + 1$. Otherwise it remains in group g .

The movement of fish from one group to the next is assumed to take place at the end of each year. The fisheries are assumed to take place at the beginning of each year over a period of time so short that mortality from natural causes can be ignored. Mortality from natural causes operates throughout the year. (This particular form of the assumed relationship between fishing mortality and natural mortality is adopted for convenience. It could easily be modified to accommodate more complex situations.)

There are two different schemes by which the fish can move through the groups. In the first scheme, which seems to be a generalization of the methods employed by Deriso for his partial recruitment model, the fish that move from group g to group $g + 1$ at the end of a given year are considered to be group $g + 1$ fish and then have a probability $1 - m_{g+1}$ of moving to group $g + 2$ in the same year. Under this scheme it is possible for a group 1 fish to move through all the groups in one year. Under the second scheme, group g fish move to group $g + 1$ with probability $1 - m_g$ and remain in that group until the next year.

The two sets of schemes yield the same stock dynamics for the one group case and become increasingly different as the

number of groups is increased.

Since an understanding of the concept of the groups is essential for an appreciation of the general theory, some examples are presented. In these examples we shall assume that there is only one fishery and that the second scheme is employed for the movement of the fish through the groups, i.e. the fish can move at most one group in each year.

To see how the group concept can produce age-dependent availability and selectivity, it is necessary to be able to calculate the probability that a fish of age j belongs to group g . This probability will depend on the survival rates from natural mortality l_u , $1 \leq u \leq g$, the fishing survival rates s_{iu} , $1 \leq u \leq g$, and those years i after the fish recruited, and the transition probabilities m_u , $1 \leq u \leq g$. Note that the probability that fish belongs to a group g is independent of l_u , s_{iu} , and m_u for $u > g$ so that it is independent of the number of groups G as long as $G > g + 1$.

Assume that there is no fishing mortality for the first two groups so that $s_{ig} = 1$ for $g = 1, 2$ and all years i .

At the beginning of a year consider a fish which recruited r years before. It is of age $r + k$ if it is still alive. If the fish is in group 1, then it has remained there for r years. For each year the probability that the fish lives and remains in group 1 is $l_1 m_1$. Thus, the probability that it remains in group 1 for r years is $l_1^r m_1^r$. (The events are assumed to be independent.)

To compute the probability that an age $r + k$ fish is in group 2, note that it can move to group 2 at the end of any one of the r years beginning with the year it recruited. Suppose it moves at the end of the q th year. That means that it experienced the event "remain in group 1" $q-1$ times and experienced the event "move to group 2" one time. Also, since it was in group 1 for q years, it experienced the event "survive in group 1" q times. Since it has spent $r-q$ years in group 2, it has experienced the event "survive in group 2" $r-q$ times and the event "remain in group 2" $r-q$ times. The collection of these events has a probability of $l_1^q m_1^{q-1} (1 - m_1) l_2^{r-q} m_2^{r-q}$ of occurring, which on summing over q yields

$$(3.1) \quad \sum_{q=1}^r l_1^q m_1^{q-1} (1 - m_1) l_2^{r-q} m_2^{r-q}$$

as the probability that an age $r + k$ fish lies in group 2.

If there are only two groups, then $m_2 = 1$. If we also assume that $l_1 = l_2 = 1$, expression 3.1 simplifies to

$$(3.2) \quad l' (1 - m'_1).$$

Expression 3.2 is the probability that a fish of age $r + k$ lies in group 2 for the 2 group model when there has not been any fishing for the last r years.

For an age $r + k$ fish which is still alive, it follows that the probability that it is in group 1 is m'_1 while the probability that it is in group 2 is $1 - m'_1$. If group 1 fish have no probability of being caught in the fishery while group 2 fish do, then when fishing begins in the next year there is an age-dependent fishing availability curve which is equal to 0 for all age classes consisting of fish of age less than $k + 1$ and $1 - m'_1$ for all age classes consisting of fish of age $r + k$. Since all group 2 fish have an equal probability of being caught, this model has no age-dependent selectivity.

The form of the age-dependent availability depends on the history of exploitation. To see this, assume that for the previous year the fishing has been so heavy that the second group is essentially empty. At the beginning of the year the probability that a (living) fish of age $r + k$ is in group 1 is m_1 and $1 - m_1$

that it is in group 2. Thus, in this case the availability of the fish is equal to 0 for all age classes consisting of fish of age less than $k + 1$ and equal to $1 - m_1$ for all age classes consisting of fish of age greater than k .

The removal of the age dependence in the availability curve at high exploitation rates is a consequence of the simplicity of the model. Since all fish in the first group are considered to be identical, they all have the same probability of moving to group 2 in any year, independent of what their age happens to be. If this assumption is considered to be too simplistic it is possible to remedy it by employing more complex models with more groups.

Consider a three-group model where all the groups have the same survival rate from natural mortality which is denoted simply by l and that $m_1 = m_2 = m$. We assume that only the third group is vulnerable to the fishery.

At the beginning of a year, it follows that the probability that a fish of age $r + k$ has a probability $l^r m^r$ of being alive and in group 1, while from expression 3.1 it follows that it has a probability of $r l^r m^{r-1} (1 - m)$ of being alive and in group 2.

Let

$$(3.3) \quad u = r (1 - m) / m.$$

It follows that given that a fish of age $r + k$ is alive and lies in group 1 or 2, then the probability is

$$(3.4) \quad 1 / (1 + u)$$

that it lies in group 1 and

$$(3.5) \quad u / (1 + u)$$

that it lies in group 2.

Now assume that in the year in question the fishing has been very intensive so that group 3, the only group vulnerable to the fishery, has been fished out. This means that in the following year all group 3 fish must come from group 2. From expressions 3.4 and 3.5 it follows that the probability that an age $k + r + 1$ fish is alive and lies in group 1 or group 2 and is therefore not vulnerable to the fishery is

$$(3.6) \quad (1 + mu) / (1 + u)$$

while the probability that it is alive and in group 3 and vulnerable to the fishery is

$$(3.7) \quad (1 - m)u / (1 + u).$$

Using the relationship 3.3, we obtain the age-dependent availability for an age $k + r + 1$ fish:

$$(3.8) \quad (1 - m) / \{m / [(1 - m) (r - 1)] + 1\}.$$

Thus, this model will produce an age-dependent availability curve even when the exploitation rate is very high. The model "remembers" some aspects of the age structure of the fish which are not vulnerable to the fishery.

This model still has no age-dependent selectivity. If both the second and third groups are vulnerable to the fishery, but with fish in these two groups having different probabilities of being captured for a unit of fishing effort, then a model with both age-dependent availability and age-dependent selectivity is obtained.

In general, as more groups are employed the model becomes more nearly age structured. Unfortunately, the model also becomes more complicated, having more parameters which must be estimated. In a given case the answer to the question of how complicated a model to employ depends on both the quality of

the data available and the complexity of the fishery being analyzed.

To illustrate the connection between the group concept and age-structured models, suppose that $m_g = 0$, $1 \leq g \leq G - 1$, while $m_G = 1$. In this case all the fish in the first $G - 1$ groups move to the next group in each year so that these groups can be identified with age classes and the model becomes age structured. The last group consists of all the older fish.

Stock Dynamics

The movement of individual fish through the groups has been given in probabilistic terms. For entire age classes we shall assume that the number of fish is so large that the proportion of fish moving through the groups can be described in a deterministic manner. Then the stock dynamics for the two schemes of moving the fish through the groups are given by equations 4.1 through 4.4 and 5.1 through 5.4, respectively.

Scheme 1

For $g = 1$:

$$(4.1) \quad N_{ik1} = m_1 R_i$$

$$(4.2) \quad N_{ij1} = m_1 l_{1s_{i-1,1}} N_{i-1,j-1,1} \quad j > k.$$

For $g > 1$:

$$(4.3) \quad N_{ikg} = m_g ((1 - m_{g-1})/m_{g-1}) N_{ikg-1}$$

$$(4.4) \quad N_{ijg} = m_g (l_{gs_{i-1,g}} N_{i-1,j-1,g} + (1 - m_{g-1})/m_{g-1} \times N_{ijg-1} \quad j > k.$$

To understand the rationale behind equation 4.3, consider the case when $g = 2$. Of the recruitment R_i the $m_1 R_i$ remains in group 1 and $(1 - m_1) R_i$ moves to group 2. Because of the dynamics, only $m_2(1 - m_1) R_i$ of this remains in group 2. Thus, $N_{ik2} = m_2(1 - m_1) R_i$. Expression 4.3 is obtained by noticing that $R_i = N_{ik1}/m_1$. The derivation of 4.4 is similar.

Scheme 2

For $g = 1$:

$$(5.1) \quad N_{ik1} = R_i$$

$$(5.2) \quad N_{ij1} = m_1 l_{1s_{i-1,1}} N_{i-1,j-1,1} \quad j > k.$$

For $g > 1$:

$$(5.3) \quad N_{i,k+g-1,g} = (1 - m_{g-1}) l_{g-1s_{i-1,g-1}} N_{i-1,k+g-2,g-1}$$

$$(5.4) \quad N_{ijg} = m_g l_{gs_{i-1,g}} N_{i-1,j-1,g} + (1 - m_{g-1}) l_{g-1s_{i-1,g-1}} \times N_{i-1,j-1,g-1} \quad j > k + g - 1.$$

Relationship between Stock and Recruitment

To investigate the relationship between stock and recruitment, the three-parameter stock-recruitment relationship of Deriso (1980) and Schnute (1985) with the option of including stochastic variation or "noise" has been employed. It has the form

$$(6.1) \quad R_{i+k} = \alpha RB_i / (1 + \gamma \beta [RB_i / RB_1 - 1])^{1/\gamma} e^{\epsilon_i}$$

where RB_i is the reproductive biomass in year i and the ϵ_i are independent normally distributed random variables with mean

0 and equal variances σ_R^2 .

Deriso's parameterization has been modified so that the parameters are more stable. As γ tends to 0, the relationship tends to a Ricker curve:

$$(6.2) \quad R_{i+k} = \alpha RB_i \exp(-\beta [RB_i / RB_1 - 1]) e^{\epsilon_i}$$

while for $\gamma = 1$ the Beverton-Holt stock-recruitment relationship is obtained. For a deterministic stock-recruitment relationship, the ϵ_i are set equal to 0.

Incorporating Fishing Effort Data

For simplicity we have assumed that all fisheries occur simultaneously at the beginning of the fishing period and that the fishing period is so short that mortality from causes other than fishing can be ignored. The relationship between effective effort and the survival rate from fishing is assumed to have the form

$$(7.1) \quad s_{ig} = \exp\left(-\sum_f q_{fig} E_{fi}\right)$$

while the proportion of the fish in group g which are caught by fishery f denoted by t_{fig} is assumed to satisfy the relationship

$$(7.2) \quad t_{fig} = q_{fig} E_{fi} (1 - s_{ig}) / \sum_{f'} q_{f'ig} E_{f'i}.$$

Equations 7.1 and 7.2 are simply the Beverton-Holt catch equations for simultaneous fisheries during a period where there is no natural mortality and where the instantaneous fishing mortality rate in each fishery is assumed to be proportional to $q_{fig} E_{fi}$. These equations can easily be modified to include fisheries which occur at different times of the year.

The relationships 7.1 and 7.2 serve to determine the effective effort E_{fi} .

The number of parameters q_{fig} in expressions 7.1 and 7.2 must be reduced to obtain a useful theory. The simplest method of accomplishing this is to assume that the catchability is constant so that

$$(7.3) \quad q_{fig} = Q_{f1g}$$

where each group has its own constant catchability coefficient for each fishery independent of time.

A more complicated situation arises when the catchability depends on the biomass. There are many ways to attempt to parameterize a biomass-dependent effect in the catchability. Let \bar{B}_g denote the average biomass, i.e.

$$\bar{B}_g = \frac{1}{I} \sum_i B_{ig}.$$

The simplest method is to assume the relationship

$$(7.4) \quad q_{fig} = Q_{f1g} + Q_{f2g} (B_{ig} - \bar{B}_g).$$

Expression 7.4 will produce increasing catchability with decreasing biomass if $Q_{f2g} < 0$. There are two difficulties with this parameterization, however. It is bad dimensionally because the term Q_{f2g} must contain the units of (biomass)⁻¹. This can lead to numerical instability in the model. For some values of Q_{f2g} it can produce negative values for the catchability. Finally, since the catchability will be needed to calculate the survival rates which are then used in the difference equations which determine the B_{ig} , it is impossible to calculate the average biomass \bar{B}_g in advance. A parameterization that overcomes all these difficulties is given by

$$(7.5) \quad q_{fi g} = Q_{f1 g} \exp [Q_{f2 g} (1 - B_{ig}/B_{1g})].$$

In expression 7.5 we have employed the dimensionless expression $1 - B_{ig}/B_{1g}$ to express the relative change in biomass from year 1. In year 1 the relationship 7.5 gives a catchability $q_{f1 g} = Q_{f1 g}$, while if the biomass B_{ig} tends to 0 the catchability will tend to the limiting value $Q_{f1 g} \exp[Q_{f2 g}]$. If the $Q_{f2 g} = 0$, then the catchability is constant. If the coefficient $Q_{f2 g} > 0$, then the catchability increases as the biomass decreases.

Modelling Errors in the Estimated Effort \hat{E}_{fi}

We shall assume that the estimated effort \hat{E}_{fi} is lognormally distributed and that $\log_e(\hat{E}_{fi})$ has mean $\log_e(E_{fi})$. If it is assumed that the observed catches C_{fi} are equal to the estimated catches \hat{C}_{fi} , the log-likelihood function for the observations is simply expression 2.5 with the first sum

$$- p_0 \sum_i \{\log_e(C_{fi}) - \log_e(\hat{C}_{fi})\}^2 / v_0^2$$

replaced by

$$- p_0 \sum_i \{\log_e(E_{fi}) - \log_e(\hat{E}_{fi})\}^2 / v_0^2.$$

Unfortunately, in the general situation where more than one group contributes to a fishery, it is not possible to proceed so directly. In equation 7.1 the effective effort \hat{E}_{fi} determines the survival rates for each group. The catches, however, do not determine the survival rates except in the case where there is only one group in the fishery. Instead the relationships 7.1 become a system of constraints which the E_{fi} must satisfy. The easiest way to overcome this problem is to assume that there are errors in both the observed catches and the estimated effort, but that the errors in the observed catches are very small compared with the errors in the estimated efforts. Then the objective function 2.5 is modified by adding a term

$$p_6 \sum_{fi} \{\log_e(E_{fi}) - \log_e(\hat{E}_{fi})\}^2 / v_0^2$$

to it. The penalty weight p_0 in the term

$$p_0 \sum \{\log_e(C_{fi}) - \log_e(\hat{C}_{fi})\}^2 / v_0^2$$

is then set large relative to p_6 .

If only one group is present in the fishery, the relationship 7.1 becomes

$$(7.6) \quad s_{ig} = \exp(-q_{fi g} E_{fi})$$

where the subscript g takes on the value appropriate value for that fishery. Assuming that $C_{fi} = \hat{C}_{fi}$, then

$$(7.7) \quad s_{ig} = (1 - \hat{C}_{fi}/B_{ig}).$$

By using the relationships 7.6 and 7.7 it is possible in this case to express the effective effort E_{fi} in terms of the observed catch \hat{C}_{fi} giving

$$(7.8) \quad E_{fi} = -\log_e(1 - \hat{C}_{fi}/B_{ig})/q_{fi g}.$$

Calculating the M_{fi} and μ_{fi}^r

Recall that in the simple one-group model the mean weight of the fish in the catch was assumed to be equal to the mean weight of the fish in the population (i.e. the group). This assumption was made because all the fish in the group were assumed to be equally likely to be caught in the fishery. In the

present multigroup case, fish in different groups have different probabilities of being present in each fishery. This leads to the moments of the weight and length distributions of the catch being mixtures of the moments of the weight and length distributions of the groups in each fishery.

Let $t_{fi g}$ denote the proportion of the fish in group g which are caught in fishery f in year i . Let $Y_{fi g}$ be the number of age j fish from group g which are caught in fishery f in year i . Then since all fish in a group are assumed to be equally likely to be caught regardless of age:

$$Y_{fi g} = t_{fi g} N_{ij g}.$$

Let w_j and τ_j denote the mean weight and mean length of an age j fish. Let $z_{fi g y}$ be the weight of the y th age j fish from group g caught in fishery f in year i . The $z_{fi g y}$ are random variables. Let $E\{z\}$ denote the expected value of the random variable z . Then $E\{z_{fi g y}\} = w_j$. The mean weight of the catch is a random variable whose expected value M_{fi} is given by

$$(8.1) \quad M_{fi} = \sum_g \sum_{j \geq k} \sum_{y=1}^{Y_{fi g}} E\{z_{fi g y}\} / \sum_g t_{fi g} N_{i \cdot g}$$

which simplifies to

$$(8.2) \quad M_{fi} = \sum_g t_{fi g} \sum_{j \geq k} w_j N_{ij g} / \sum_g t_{fi g} N_{i \cdot g}.$$

The expected value for the moments of the length distribution of the catches can be calculated in a similar fashion. For the higher moments there are two extra difficulties, however. It is easier to work with the expected values of the noncentral moments of the length distribution of the catches because these depend in a simple way on the noncentral moments of the length distribution of the groups present in the fishery, while the central moments depend in a more complicated fashion on the central moments of the component groups.

In addition, the higher moments of the length distribution of the catch depend on the distribution of the lengths-at-age of the fish in each age class around their means, so that this within age class variation in length must be taken into account.

Assume that for each age the lengths of the fish are normally distributed around their mean length τ_j with variance σ_L^2 where it is assumed that the variance is independent of the age j . Let $x_{fi g y}$ denote the length of the y th age j fish from group g caught in fishery f in year i . Then $x_{fi g y}$ is a random variable which can be written as $x_{fi g y} = \tau_j + \epsilon_{fi g y}$ where the $\epsilon_{fi g y}$ are normally distributed random variables with mean 0 and variance σ_L^2 . Let λ_{fi}^r denote the r th noncentral moment of the length distribution of the catch of fishery f in year i . The λ_{fi}^r are given by

$$(8.3) \quad \lambda_{fi}^r = \sum_g \sum_{j \geq k} \sum_{y=1}^{Y_{fi g}} E\{(\tau_j + \epsilon_{fi g y})^r\} / \sum_g t_{fi g} N_{i \cdot g}.$$

For a normally distributed random variable ϵ with mean 0 and variance σ_L^2 , $E\{\epsilon\} = 0$, $E\{\epsilon^2\} = \sigma_L^2$, $E\{\epsilon^3\} = 0$, and $E\{\epsilon^4\} = 3(\sigma_L^2)^2$. This gives the following formulas for the λ_{fi}^r :

$$(8.4) \quad \lambda_{fi}^1 = \sum_g t_{fi g} \sum_{j \geq k} \tau_j N_{ij g} / \sum_g t_{fi g} N_{i \cdot g}$$

$$(8.5) \quad \lambda_{fi}^2 = \left(\sigma_L^2 \sum_g t_{fi g} N_{i \cdot g} + \sum_g t_{fi g} \sum_{j \geq k} \tau_j^2 N_{ij g} \right) / \sum_g t_{fi g} N_{i \cdot g}$$

$$(8.6) \quad \lambda_{fi}^3 = \left(3\sigma_L^2 \sum_g t_{fig} \sum_{j \geq k} \tau_j N_{ijg} + \sum_g t_{fig} \sum_{j \geq k} \tau_j^3 N_{ijg} \right) / \sum_g t_{fig} N_{i \cdot g}$$

$$(8.7) \quad \lambda_{fi}^4 = \left(3(\sigma_L^2)^2 \sum_g t_{fig} N_{i \cdot g} + 6\sigma_L^2 \sum_g t_{fig} \sum_{j \geq k} \tau_j^2 N_{ijg} + \sum_g t_{fig} \sum_{j \geq k} \tau_j^4 N_{ijg} \right) / \sum_g t_{fig} N_{i \cdot g}$$

The relationship between the central and the noncentral moments of the length distribution of the catch is given by

$$(8.8) \quad \mu_{fi}^1 = \lambda_{fi}^1$$

$$(8.9) \quad \mu_{fi}^2 = \lambda_{fi}^2 - (\lambda_{fi}^1)^2$$

$$(8.10) \quad \mu_{fi}^3 = \lambda_{fi}^3 - 3\lambda_{fi}^2 \lambda_{fi}^1 + 2(\lambda_{fi}^1)^3$$

$$(8.11) \quad \mu_{fi}^4 = \lambda_{fi}^4 - 4\lambda_{fi}^3 \lambda_{fi}^1 + 6\lambda_{fi}^2 (\lambda_{fi}^1)^2 - 3(\lambda_{fi}^1)^4$$

To include the biomass and mean weight of the fish in the catch in the model, it is necessary to assume that there is a weight-length relationship. In fisheries literature it is often assumed that the mean weight at age j of the fish w_j satisfies the relationship

$$(9.1) \quad w_j = b\tau_j^\nu$$

where the exponent ν is estimated by regressing the logarithm of weight on the logarithm of length. The exponent ν often has a value close to 3. In the present context, only integer exponents can be considered because the procedure for deriving the difference equations only works for integral exponents. We shall therefore assume that the value of ν is equal to 3. If this relationship is not sufficient, then a polynomial weight-length relationship of the form

$$w_j = b_1\tau_j + b_2\tau_j^2 + b_3\tau_j^3$$

could be considered. The model can easily be modified to deal with this assumption.

Now assuming the weight-length relationship 9.1 with $\nu = 3$ holds, and simply denoting b_3 by b , the biomass B_{ig} of the fish in group g is given by

$$(9.2) \quad B_{ig} = b \sum_{j \geq k} \tau_j^3 N_{ijg}$$

Similarly, the biomass of the catch C_{fi} taken in fishery f is given by

$$(9.3) \quad C_{fi} = b \sum_g t_{fig} \sum_{j \geq k} \tau_j^3 N_{ijg}$$

while the mean weight M of the fish in the catch is given by

$$(9.4) \quad M_{fi} = \left(b \sum_g t_{fig} \sum_{j \geq k} \tau_j^3 N_{ijg} \right) / \left(\sum_g t_{fig} N_{i \cdot g} \right)$$

Notice that in the above expressions we have managed to express everything in terms of some coefficients multiplied by terms of the form

$$\sum_{j \geq k} \tau_j^r N_{ijg}$$

where r is an integer between 0 and 4.

Difference Equations

To derive the desired difference equations, it is necessary to make an assumption about the form of the mean lengths τ_j . We shall assume that they lie on a von Bertalanffy curve so that

$$(10.1) \quad \tau_j = c + (a - c)(1 - \rho^j)/(1 - \rho)$$

where $0 < \rho < 1$. (This form for parameterizing a von Bertalanffy curve is discussed in Schnute and Fournier 1980.) Then

$$(10.2) \quad \tau_j^r = \sum_{q=0}^r C_q^r c^q (a - c)^{r-q} \left(\frac{1 - \rho^j}{1 - \rho} \right)^{r-q}$$

where $C_q^r = r!/[q!(r - q)!]$ so that

$$(10.3) \quad \sum_{j \geq k} \tau_j^r N_{ijg} = \sum_{q=0}^r C_q^r c^q (a - c)^{r-q} \times \sum_{j \geq k} \left(\frac{1 - \rho^j}{1 - \rho} \right)^{r-q} N_{ijg}$$

For each positive integer r , let T_{ig}^r be defined by

$$(10.4) \quad T_{ig}^r = \sum_{j \geq k} \left(\frac{1 - \rho^j}{1 - \rho} \right)^r N_{ijg}$$

In terms of the T_{ig}^r , expression 10.3 can be rewritten as

$$(10.5) \quad \sum_{j \geq k} \tau_j^r N_{ijg} = \sum_{q=0}^r C_q^r c^q (a - c)^{r-q} T_{ig}^{r-q}$$

Expression 10.1 gives the general von Bertalanffy growth curve. It is possible and may be desirable to require that the length at age 0 is 0. This is accomplished by setting c equal to 0. In this case, expression 10.5 simplifies to

$$(10.6) \quad \sum_{j \geq k} \tau_j^r N_{ijg} = a^r T_{ig}^r$$

Since all expressions of the form

$$\sum_{j \geq k} \tau_j^r N_{ijg}$$

can be expressed in terms of the T_{ig}^r , it follows that B_{ig} , C_{fi} , M_{fi} , and the μ_{fi}^r can be as well. Therefore, it only remains to show that T_{ig}^r satisfy the desired system of difference equations.

Define A_j by

$$A_j = \left(\frac{1 - \rho^j}{1 - \rho} \right) \quad 0 < \rho < 1.$$

Then for all positive integers r , one has

$$(11.1) \quad \rho^r A_{j-1}^r = (A_j - 1)^r$$

Applying the binomial theorem to 11.1 yields the relationship

$$(11.2) \quad \rho^r A_{j-1}^r = \sum_{q=0}^r (-1)^q C_q^r A_j^{r-q}$$

or

$$(11.3) \quad A_j^r - \rho^r A_{j-1}^r = \sum_{q=1}^r (-1)^{q+1} C_q^r A_j^{r-q}$$

From expression 11.3 it follows that

$$\sum_{j \geq k} (A_j^r - \rho^r A_{j-1}^r) N_{ijg} = \sum_{q=1}^r (-1)^{q+1} C_q^r \sum_{j \geq k} A_j^{r-q} N_{ijg}$$

or

$$(11.4) \quad T_{ig}^r = \sum_{q=1}^r (-1)^{q+1} C_q^r T_{ig}^{r-q} + \rho^r \sum_{j \geq k} A_{j-1}^r N_{ijg}.$$

To obtain the desired system of difference equations, it is necessary to deal with the lagged term

$$(11.5) \quad \sum_{j \geq k} A_{j-1}^r N_{ijg}.$$

Lemma 1

Assume that the population dynamics are described by equations 4.1 through 4.4. Let U_j be any set of coefficients. Then for $g = 1$:

$$(11.6) \quad \sum_{j \geq k} U_{j-1} N_{ij1} = m_1 l_1 s_{i-1,1} \sum_{j \geq k} U_j N_{i-1,j,1} + m_1 U_{k-1} R_i$$

and for $g > 1$:

$$(11.7) \quad \sum_{j \geq k} U_{j-1} N_{ijg} = m_g l_g s_{i-1,g} \sum_{j \geq k} U_j N_{i-1,j,g} + m_g (1 - m_{g-1})/m_{g-1} \sum_{j \geq k} U_{j-1} N_{i,j,g-1}.$$

Applying this result to the lagged term 11.5 and substituting the result into expression 11.4 yields, for $r = 0$, theorem 1.

Theorem 1

Suppose that the populations' dynamics are described by equations 4.1 through 4.4. Then the quantities $N_{i \cdot g}$ satisfy the relationships for $g = 1$:

$$(11.8) \quad N_{i \cdot 1} = m_1 l_1 s_{i-1,1} N_{i-1,1} + m_1 R_i$$

and for $g > 1$:

$$(11.9) \quad N_{i \cdot g} = m_g l_g s_{i-1,g} N_{i-1,g-1} + m_g (1 - m_{g-1})/m_{g-1} N_{i \cdot g-1}.$$

For convenience let T_{ig}^0 denote $N_{i \cdot g}$. For $r > 0$ the result is theorem 2.

Theorem 2

Assume that the population dynamics are determined by equations 4.1 through 4.4. Then setting $U_j = A_j^r$ and applying lemma 1 for all integers $r > 0$ and for $g = 1$ and $i > 1$:

$$(11.10) \quad T_{i1}^r = \rho^r l_1 m_1 s_{i-1,1} T_{i-1,1}^r + \sum_{q=1}^r (-1)^{q+1} C_q^r T_{i1}^{r-q} + \rho^r m_1 A_{k-1}^r R_i$$

and for $g > 1$ and $i > 1$:

$$(11.11) \quad T_{ig}^r = \rho^r l_g m_g s_{i-1,g} T_{i-1,g}^r + \sum_{q=1}^r (-1)^{q+1} C_q^r T_{ig}^{r-q} + m_g (1 - m_{g-1})/m_{g-1} \sum_{q=0}^r (-1)^q C_q^r T_{ig}^{r-q}.$$

The corresponding result for the scheme of moving through the groups given by equations 5.1 through 5.4 is lemma 2.

Lemma 2

Assume that the population dynamics are described by equations 5.1 through 5.4. Let U_j be any set of coefficients. Then

for $g = 1$:

$$(11.12) \quad \sum_{j \geq k} U_{j-1} N_{ij1} = m_1 l_1 s_{i-1,1} \sum_{j \geq k} U_j N_{i-1,j,1} + U_{k-1} R_i$$

and for $g > 1$:

$$(11.13) \quad \sum_{j \geq k+g-1} U_{j-1} N_{ijg} = m_g l_g s_{i-1,g} \times \sum_{j \geq k+g-1} U_j N_{i-1,j,g} + (1 - m_{g-1}) l_{g-1} s_{i-1,g-1} \times \sum_{j \geq k+g-2} U_j N_{i-1,j,g-1}.$$

Theorem 3

Suppose that the population dynamics satisfy equations 5.1 through 5.4. Then the $N_{i \cdot g}$ satisfy the relationships for $g = 1$ and $i > 1$:

$$(11.14) \quad N_{i \cdot 1} = m_1 l_1 s_{i-1,1} N_{i-1,1} + R_i$$

and for $g > 1$ and $i > 1$:

$$(11.15) \quad N_{i \cdot g} = m_g l_g s_{i-1,g} N_{i-1,g} + (1 - m_{g-1}) l_{g-1} s_{i-1,g-1} N_{i-1,g-1}.$$

For $r > 0$, the result is theorem 4.

Theorem 4

Assume that the populations dynamics are determined by equations 5.1 through 5.4. Then setting $U_j = A_j^r$ and applying lemma 2 yields for all integers $r > 0$ and for $g = 1$ and $i > 1$:

$$(11.16) \quad T_{i1}^r = \rho^r l_1 m_1 s_{i-1,1} T_{i-1,1}^r + \sum_{q=1}^r (-1)^{q+1} C_q^r T_{i1}^{r-q} + \rho^r A_{k-1}^r R_i$$

and for $g > 1$ and $i > 1$:

$$(11.17) \quad T_{ig}^r = \rho^r l_g m_g s_{i-1,g} T_{i-1,g}^r + \sum_{q=1}^r (-1)^{q+1} C_q^r T_{ig}^{r-q} + \rho^r (1 - m_{g-1}) l_{g-1} s_{i-1,g-1} T_{i-1,g-1}^r.$$

This completes the derivation of the desired systems of difference equations.

Calculating the Model Parameters

These results can be used to calculate the values of the model parameters as follows.

Step 1

Pick values for all free parameters, free parameters being those parameters which are not calculated in terms of other parameters. Whether a parameter is free or not may depend on the particular form of the model hypotheses. For example, if the form of the model where it is assumed that there is no error in the estimated efforts \hat{E}_{fi} is used, then the E_{fi} are set equal to the \hat{E}_{fi} so they are not free parameters. In general, the free parameters may include Q_{f1g} , Q_{f2g} , E_{fi} , T_{ig}^r , a , b , ρ , α , β , λ , a , l_g , and σ_L^2 .

Any of these parameters may be given a fixed value if their true value is assumed to be known. For example, b could be determined by performing a regression of the logarithm of the fish weights on the logarithm of the fish lengths. The variance term σ_L^2 could be estimated by aging fish, inference from tag recovery, or by examination of length frequency data. If a

group is not considered available to the fishery, then the coefficients Q_{f1g} and Q_{f2g} would be set equal to 0 for that group and fishery. Let $i = 1$.

Step 2

Calculate the biomasses B_{ig} from equations 10.6 and 9.2.

Step 3

Calculate the catchabilities q_{fig} from equation 7.5.

Step 4

Calculate the survival rates s_{ig} from equation 7.1.

Step 5

Calculate the proportions caught t_{fig} from equation 7.2.

Step 6

Calculate the catches C_{fi} from equations 10.6 and 9.3.

Step 7

Calculate the mean weights of the catches M_{fi} from equations 10.6 and 9.4.

Step 8

Calculate the noncentral moments of length distribution λ_{fi}^r of the catches from equations 10.6 and 8.4 through 8.7.

Step 9

Calculate the central moments of the length distribution μ_{fi}^r of the catches from equations 8.8 through 8.11.

Step 10

Calculate the reproductive biomass RB_i . This calculation depends on the particular model assumptions. For example, in the two-group case used in the simulations, group 1 fish were assumed to be sexually immature and RB_i was set equal to B_{i2} , the biomass of the sexually mature group 2 fish.

Step 11

Calculate R_{i+1} from the stock–recruitment relationship, equation 6.1. For $2 \leq i+1 \leq k$ the reproductive biomass for the recruitment R_{i+1} is not known. We have used RB_i in equation 6.1 for $2 \leq i+1 \leq k$.

Step 12

Calculate the $T_{i+1,g}^r$ by using equations 11.8 through 11.11 if the population dynamics are given by equations 4.1 through 4.4 and by using equations 11.14 through 11.17 if the population dynamics are determined by equations 5.1 through 5.4.

Step 13

Set $i = i + 1$. If $i \leq I$ (the number of years of fishing data), then go to step 2; otherwise, stop.

Simulations

When choosing the hypotheses used for the simulations, several objectives were considered. We wanted to simulate data which had enough stochastic irregularity to represent a real fishery. Also, we wanted to be able to relate our results to the previous work of Ludwig and Walters. To this end our generalization of Deriso's partial recruitment model was used in the simulations. To make the simulations realistic, considerable stochastic error was included in both the stock–recruitment relationship and the relationship between estimated effort and effective effort. Finally, a relatively short time period of 20 yr (Ludwig and Walters used 50 yr) of fishing data were employed.

The simple von Bertalanffy growth curve with a length of 0

at age 0 was used.

The model has two groups of fish. The first group is assumed to be completely unavailable to the fishery, and the catchability for fish in this group is fixed at 0. The second group consists of the fishable stock.

The parameters determining the population dynamics of the stock (Table 1) have been chosen so that the stock consists of slow-growing fish with low productivity. The catchability has been given a large biomass-dependent component which leads to approximately a 90% increase in catchability when the stock has been fished to 50% of its initial biomass. The stochastic noise in the relationship between the observed effort and the effective effort was assumed to be multiplicative lognormal with a variance of 0.09.

A Ricker stock–recruitment relationship was employed with a multiplicative lognormal random variation with variance 0.09. The form of estimation with the stochastic error in the effort – fishing mortality relationship was employed.

The particular combination of growth, mortality, and stock–recruitment parameters produces a virgin biomass at a level which is 1.5 times the biomass which produces the maximum recruitment. The reproductive biomass RB_i was set equal to the biomass of the fishable stock B_{i2} .

To produce length data which could not be used for age estimation through the use of an age–length relationship or by length frequency analysis using a method such as that described in Schnute and Fournier (1980), the variance of the distribution of the lengths-at-age of the fish was chosen so that the length distributions of neighboring age classes have a large overlap. The mean lengths of age 3 and age 4 fish are 5.1 and 6.4 so that the mean length increment is 1.3. The variance σ_L^2 was set equal to 1.0:

		Mean length											
Age		1	2	3	4	5	6	7	8	9	10	11	12
Mean													
length		2.0	3.7	5.1	6.4	7.4	8.3	9.1	9.7	10.2	10.7	11.1	11.4
Length													
change		1.7	1.4	1.3	1.0	0.9	0.8	0.6	0.5	0.5	0.4	0.3	

so that the difference between the mean lengths divided by the sum of the standard deviations for the first two age classes in the fishery is equal to 0.65. Hasselblad (1966) demonstrated that for mixtures of normal distributions there is almost no information about the means or proportions of the components of the mixture when the difference between the means divided by the sum of the standard deviations is less than 1.0. In the present situation the separation is much less than that, so that no analysis of the length frequency data by itself could be expected to produce accurate estimates of the age structure of the catch.

The catch samples were simulated by generating random samples of 500 fish from the catch.

Table 2 gives catch data for one realization of the simulations. All the data here have their true values. Thus, for example, the effort shown is the true effective effort. The estimated effort used in the estimations process had a 30% random error imposed on it. Of course the manager does not know the biomass of the fishable populations or the survival rates. They are included here for comparison. Because of the effect of density-dependent catchability, the catch per unit effort (CPUE) is still about 51% of its initial value, while the

TABLE 1. Parameters values for example 1.

	Remarks
$l = 0.90$	l has the same value for both groups
$p = 0.85$	Slow growing. Requires about 14 yr to reach 90% of maximum length
$a = 2.0$	Mean length at age 1
$b = 2.5$	Parameter determining length-weight relationship
$Q_{112} = 0.0018$	Gives a fishing mortality rate of 0.11 for the first year
$Q_{122} = 1.3$	Produces a strong density-dependent component to the catchability
$m_1 = 0.50$	Annual rate of entry to the fishable stock
$V_6 = 0.30$	Standard deviations in the effort - fishing mortality relationship
$\alpha = 0.36 \times 10^{-4}$	Coefficients of the Ricker stock-recruitment relationship
$\beta = 1.5$	
$\sigma_R^2 = 0.30$	Standard deviation in the stock-recruitment relationship
$\sigma_L^2 = 1.0$	Standard deviations of the distribution of the lengths of the fish around their mean
Age at recruitment to group 1	3

biomass of the fishable population has actually decreased to about 20% of its initial value.

Over the 20-yr period the mean weight of the fish in the catch has decreased from 31 to 18. This observation might cause the manager some concern, but with a model that uses only catch and effort data there is no way to incorporate the mean weight data into the analysis.

The Bayesian prior distributions (Table 3) were chosen to somewhat overestimate the ability of the stock to withstand exploitation in that the growth rate is higher than the true growth rate and the natural survival rate is underestimated, while the degree of biomass dependence is underestimated by over 50%. The reason for picking these values for the Bayesian prior distributions is that we are interested in determining whether the model is capable of detecting the fact that the stock is much more vulnerable to fishing than the parameter values favored by the Bayesian assumptions would indicate.

Note that it was necessary to impose an upper limit on the mean length of the fish in group 1. Otherwise the model tended to allow the fish to remain in group 1 for too long a period so that the mean lengths of group 1 and group 2 fish became almost the same. In the simulations, the actual mean length of fish in group 1 was never greater than 6.4, so that the assumption that the manager knows that it should be less than 7.0 is not unreasonable.

Choosing an Optimal Policy

Many criteria could be used to define an optimal policy. The criterion chosen was to attempt to maximize long-term average yield by choosing the optimal level of fishing effort. To determine the long-term average yield the simulator was run using the true parameter values for 60 yr at a fixed level of effort and the average yield for the years 50–60 calculated. This process was repeated 100 times so that the distribution of the average yields and the average biomass between years 50 and 60 could be determined. This Monte Carlo approach can be easily adapt-

ed to investigate other policies. The true expected mean yields at each effort level together with their coefficient of variation and the corresponding mean biomass levels are as follows:

Effort	0	10	15	20	25	30	35	40	45	50
Catch	0.0	100	140	180	200	200	170	140	90	60
Coeff.										
of var.	0.00	0.02	0.03	0.06	0.09	0.11	0.15	0.16	0.17	0.18
Biomass	5600	4600	4000	3300	2500	1800	1107	701	308	201

The maximum expected average yield is at an effort of 25. The coefficient of variation of the catch shows an increasing trend which is caused by the decrease in stability of the population as the harvesting rate is increased.

This procedure allows one to estimate the optimal level of effort provided that the correct values of the parameters are known. For investigating the performance of the model the values of the parameters are not known but must be estimated by analyzing simulated data. The procedure used is the following.

The simulator is run with the true parameter values to produce 20 yr of simulated fisheries data. These simulated data are then analyzed by the model and estimates of the parameters obtained. These parameter estimates are then used in the simulator with the 60-yr fishery to estimate long-term yields at various effort levels. The simulations were repeated 25 times at each effort level so that an estimate of the expected mean yield together with an estimate of the standard deviation in expected mean yield was produced. The rule used to pick the optimal level of effort was to choose the minimum effort level for which the estimated expected mean yield was within one standard deviation of the maximum expected mean yield. This produces a slightly conservative policy compared with just picking the effort level for which the estimated expected mean yield is the largest, and also penalizes the expected mean yield slightly if it has a large estimated variance.

This entire set of analyses and simulations was repeated 50 times to investigate how well the method was able to detect the biomass-dependent component of catchability and how well it was able to estimate the optimal effort level:

Effort level	10	15	20	25	30	35	40	45	50	55
Number of times picked	3	3	12	12	12	5	0	1	0	2

The mean of the estimates for Q_{122} was 1.17, while their standard deviation was 0.42.

To insure that the apparent ability of the model to detect biomass-dependent catchability was not just an artifact of the estimation process, the simulations were carried out again with the effort corrected so that the biomass-dependent effect was removed. The resulting estimates for Q_{122} had a mean of 0.34 and a standard deviation of 0.38.

Determining the Age at Recruitment

The true age at recruitment of 3 may not be known. The sensitivity of the model to changes in this parameter were investigated by using estimated ages of recruitment of 2 and 4 and redoing the 50 sets of simulations. For assumed age of recruitment of 2:

TABLE 2. Annual statistics for the simulated catch.

Year	Biomass	Catch (weight)	Effort	Mean weight	Mean length	Survival rate	CPUE
1	5606	573	60	31	10.2	0.898	9.5
2	5043	583	60	30	10.2	0.884	9.7
3	4487	586	60	30	10.1	0.869	9.8
4	3947	579	60	29	10.0	0.853	9.6
5	3437	562	60	28	9.8	0.836	9.4
6	2968	535	60	26	9.6	0.819	8.9
7	2549	502	60	25	9.3	0.803	8.3
8	2186	464	60	23	9.1	0.788	7.7
9	1878	153	20	21	8.9	0.918	7.6
10	1900	154	20	21	8.8	0.919	7.7
11	1934	156	20	20	8.8	0.919	7.8
12	1972	158	20	21	8.9	0.920	7.9
13	2007	159	20	21	8.9	0.920	8.0
14	2040	161	20	21	9.0	0.921	8.1
15	2069	162	20	21	9.0	0.922	8.1
16	2096	453	60	21	9.0	0.784	7.5
17	1818	416	60	21	8.9	0.771	6.9
18	1571	378	60	20	8.8	0.759	6.3
19	1360	341	60	19	8.6	0.749	5.6
20	1184	307	60	18	8.5	0.740	5.1

TABLE 3. Bayesian prior distributions.

Parameter	Mean	SD	Remarks
ρ	0.80	0.1	Normal distribution
l	0.80	0.1	Normal distribution
m_1	0.40	0.1	Normal distribution
Mean length at recruitment	5.0	0.7	Normal distribution true value = 5.2
Q_{112}	0.0036	0.7	Multiplicative lognormal distribution
Q_{122}	0.75	0.7	Normal distribution
Mean length of fish in group 1	<7.0		Necessary to stabilize the estimation procedure

Effort											
level	10	15	20	25	30	35	40	45	50	55	
Number of times picked	5	1	10	12	5	5	3	5	2	0	

The results are degraded considerably by underestimating the age of recruitment. If no other data are available, a natural way of determining which value to use for the age at recruitment is to pick the one for which the mode of the posterior distribution achieves its higher value. Using this criterion, the correct value of the age of recruitment of 3 would have been picked over that of 2 in 46 of 48 cases considered (two runs aborted with numerical errors).

The estimates for an age of recruitment of 4 are as follows:

Effort											
level	10	15	20	25	30	35	40	45	50	55	
Number of times picked	2	3	11	15	9	4	4	0	0	2	

The estimates are evidently not so sensitive to over-estimation of the age of recruitment as they are to its under-estimation. This is fortunate because the model was not as capable of distinguishing between the age of recruitment estimates of 3 and 4 as it was between 2 and 3. The correct value of the age of recruitment of 3 would have been picked over that of 4 in 37 of the 50 cases.

Conclusions

The results of the simulations illustrate that through the use of the total catch, the mean weight of the catch, and effort and length data, it is possible to estimate optimal harvesting rates in cases where catchability is not constant so that there is no simple index of abundance provided by CPUE data, provided that the fishery is managed in such a way as to provide sufficient contrast in the data.

To a person accustomed to certainty and precision, the number of times that the method picked a suboptimal policy might appear disconcerting. It is, however, a fact that in fisheries management one must deal with noisy (i.e. to some extent unpredictable) systems which themselves must be analyzed with data of varying accuracy. The purpose of mathematical modelling in fisheries management is to provide the best analysis possible in a given circumstance.

It should also be remembered that for a real fishery the analysis would be ongoing. For the scenarios where the estimated optimal level of fishing is too high the model would obtain valuable information from subsequent years of fisheries data as the stock declined further. Eventually it would be realized that the optimal level of effort is less than it had previously been estimated to be.

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