Research Hypothesis: A Computational Construction of Algebraic Cycles via the Abel-Jacobi Map**

Core Hypothesis: It is possible to computationally construct and verify the image of a non-trivial algebraic cycle of high codimension on an abelian variety under the Abel-Jacobi map, using a refined methodology that combines symbolic computation (SageMath, Magma) and high-precision numerical analysis (Arb). This process will provide an explicit, verifiable case study that advances the constructive approach to the Hodge Conjecture.

Primary Research Question: Can we algorithmically compute the Abel-Jacobi image of the cycle \(Z = \{P_1\} \times E_2 \times \{P_3\} \) on the abelian threefold \(X = E_1 \times E_2 \times E_3 \) with sufficient numerical precision to rigorously verify its properties, thereby creating a blueprint for computational investigations in higher codimension?

Theoretical Framework and Justification

This research is situated within the framework of **refined Abel-Jacobi maps and normal functions**. The Abel-Jacobi map provides a bridge between algebraic cycles (geometric objects) and complex tori (analytic objects). For a homologically trivial algebraic cycle \(Z \) of codimension \(p \) on a smooth projective variety \(X \), the Abel-Jacobi map \(\text{AJ}^p \) sends \(Z \) to a point in the intermediate Jacobian \(J^p(X) \), a complex torus.

The hypothesis is justified by:

1. **Historical Precedent:** The map is a generalization of the Abel-Jacobi theorem for algebraic curves, a cornerstone of algebraic geometry.

2. **Abelian Variety Structure:** The choice of \(X \) as a product of elliptic curves simplifies the homology and cohomology computations, making the problem computationally tractable while remaining non-trivial (codimension 2 on a threefold).

3. **Constructive Pathway:** Successfully implementing this map computationally is a critical step towards the larger goal of constructing algebraic cycles from Hodge classes.

Methodological Plan: Phases and Subtasks

The hypothesis will be tested through a two-phase computational strategy:

Phase 0: Pilot Study on an Elliptic Curve (Codimension 1)

*   **Objective:** To de-risk the project by validating the entire software toolchain and developing numerical integration templates on a well-understood problem.

*   **Method:** Implement the Abel-Jacobi map for a degree-zero divisor \(D = (P) - (O) \) on a single elliptic curve \(E \). This involves:

    1. Computing the period lattice \(\Lambda \) by integrating the holomorphic 1-form over a basis of \(H_1(E, \mathbb{Z}) \).

    2. Computing \(\text{AJ}(D) \) by numerically integrating the 1-form along a path from \(O \) to \(P \).

    3. Verifying that \(\text{AJ}(D) \in \Lambda \) to high precision.

*   **Success Criteria:** Reproduction of the classical result with >100 digits of precision.

Phase 1: Core Computation on the Abelian Threefold (Codimension 2)

*   **Subtask 1: Define \(X \) and \(Z \).** Define non-isomorphic elliptic curves \(E_1, E_2, E_3 \) in SageMath/Magma. Construct \(X \) as their product and define the algebraic cycle \(Z = \{P_1\} \times E_2 \times \{P_3\} \).

*   **Subtask 2: Compute \(J^2(X) \).**

    1. Compute a basis for \(H^{2,0}(X) \) (exterior products of 1-forms from \(E_i \)).

    2. Compute a basis for the integer homology group \(H_2(X, \mathbb{Z}) \).

3. Numerically compute the period integrals of the 2-forms over the 2-cycles to form the period matrix Ω, defining the intermediate Jacobian $J^2(X) = \mathbb{C}^g / \Lambda$.

* **Subtask 3: Compute $\text{AJ}(Z)$.** This is the core technical challenge.

1. **Construct a 3-chain Σ** such that $\partial \Sigma = Z - Z_0$, leveraging the group law of X. The chosen construction is $\Sigma = \gamma_{P_1 \to 0} \times E_2 \times \gamma_{P_3 \to 0}$, where γ are paths on the respective elliptic curves.

2. **Perform high-precision numerical integration** of the holomorphic 2-forms over Σ. This will employ a multi-faceted strategy to ensure stability:

    * **Adaptive Cubature:** Using Arb's arbitrary-precision adaptive integration routines.

    * **Iterated Integrals:** Exploiting the product structure to break the triple integral into manageable parts.

    * **Singularity Handling:** Monitoring the Jacobian of the parameterization and preemptively subdividing the domain or applying coordinate transformations to avoid numerical instability.

    * **Precision & Validation:** Running computations at multiple precision levels and using alternative parameterizations to cross-verify results.

* **Subtask 4: Verification.**

1. **Lattice Agreement:** Reduce the computed image vector $\text{AJ}(Z)$ modulo the period lattice Λ. The result must be numerically zero within a tolerance of 10^{-90}.

2. **Torsion Verification:** If Z is constructed to be homologically trivial, test if $N \cdot \text{AJ}(Z)$ is a lattice point for some integer N, confirming it is a torsion point.

3. **Reproducibility:** All code and parameters will be published to allow independent verification.

Expected Outcomes and Validation

A successful verification of the hypothesis will be determined by:

1. **Numerical Proof:** The final, reduced vector of \(\text{AJ}(Z) \) modulo \(\Lambda \) is numerically indistinguishable from zero (\(|\text{vector}| < 10^{-90} \)).

2. **Torsion Property:** The image is confirmed to be a torsion point in \(J^2(X) \).

3. **Stability and Reproducibility:** Results are stable under perturbation and can be reproduced independently using the provided code and documentation.

4. **Expert Endorsement:** The methodology and interpretation withstand peer review by experts in computational algebraic geometry.

Significance: Success would provide a concrete, reproducible computational protocol for studying algebraic cycles in high codimension, directly addressing the "construction problem" and providing a powerful tool for exploring cases related to the Hodge Conjecture. Failure would still yield valuable insights, highlighting specific computational or theoretical obstacles in higher codimension, and would result in a detailed public post-mortem to guide future research.

This hypothesis represents a rigorous, step-by-step plan to move from theoretical discussion to practical, verifiable computation at the frontier of algebraic geometry.