MATH 3043, Numerical Analysis I
Fall 2020

Lab 10

This lab will have you implementing matrix factorizations.

Solutions must be submitted on Canvas by **November 29** at **11:59 PM**. Please submit a single script file `Lab10Lastname.m` and the corresponding published file `Lab10Lastname.pdf` (for example, my submitted files would be `Lab10Zumbrum.m` and `Lab10Zumbrum.pdf`). Each solution should

- be contained in a separate cell which includes the problem number and short problem description,
- run independent of other cells,
- be adequately commented.

1. Find the Doolittle factorization for A (or PA if necessary) and use it to solve the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 3 & -6 & 9 & 3 \\ 2 & 1 & 4 & 1 \\ 1 & -2 & 2 & -1 \\ 1 & -2 & 3 & 0 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

   Hints:
   - The Doolittle factorization (and permutation matrix P) for the matrix PA can be found in MATLAB using `[L, U, P] = lu(A)`
   - What do you notice about P in this problem?

2. Repeat Problem 1 for the following coefficient matrix:

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 4 & 3 \\ 2 & -1 & 2 & 4 \\ 2 & -1 & 2 & 3 \end{bmatrix}$$

   Hint: What do you notice about P in this problem?

3. [**Section 12.1 Example 1**] Determine the steady-state heat distribution in a thin square metal plate with dimensions 0.5 m by 0.5 m using $n = 4$ subintervals. Two adjacent boundaries are held at 0°C, and the heat on the other boundaries increases linearly from 0°C at one corner to 100°C where the sides meet.

To find a numerical solution for this problem, solve the linear system

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{bmatrix} = \begin{bmatrix} 25 \\ 50 \\ 150 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \\ 25 \end{bmatrix}$$

for temperatures w_i at interior grid points (at the corresponding points labeled in Figure 12.7). The code below constructs the coefficient matrix and right-hand side vector for a system with a general number of subintervals n.

```
%specifies the number of subintervals in each direction
n = 4;

%creates the coefficient matrix
A = toeplitz([4 -1 zeros(1, n-3) -1 zeros(1, (n-2)*(n-1)-1)]);

%updates specific coefficient matrix entries in the identity blocks
for i = 1:n-2
    A(i*(n-1), i*(n-1)+1) = 0;
    A(i*(n-1)+1, i*(n-1)) = 0;
end

%creates the right-hand side vector
b = zeros((n-1)*(n-1), 1);

%updates entries based on the boundary conditions
b(1:n-1) = b(1:n-1) + 100/n*[1:n-1]';
b(n-1:n-1:end) = b(n-1:n-1:end) + 100/n*[n-1:-1:1]';
```

(a) What do you notice about the coefficient matrix? Use the LU decomposition of the coefficient matrix to solve the system and save the temperatures in a vector w.

(b) With the numerical approximations for the temperature at the interior grid points stored in the column vector `w`, the code below plots the resulting numerical solution for the temperature distribution on the entire metal plate.

```
%creates a matrix for temperatures at all grid points
W = zeros(n+1, n+1);

%inserts temperatures at interior grid points
W(2:end-1, 2:end-1) = reshape(w, n-1, n-1)';

%inserts temperatures on the boundaries
W(1, :) = 100/n*[0:n];
W(:, end) = 100/n*[n:-1:0];

%creates vectors for plotting
x = 0.5/n*[0:n];
y = 0.5/n*[n:-1:0];

[xx, yy] = meshgrid(x,y);

surf(xx,yy,W)
```

Produce the plot for the numerical solution of the temperature distribution with $n = 4$ intervals.

(c) Redo parts (a) and (b) for $n = 32$ intervals.

(d) Update the code to use the temperature distributions $f(x) = 100x^4$ and $f(y) = 100y^4$ at the non-zero boundaries and use $n = 32$ intervals. **Hint:** Only update the right-hand side vector and solve the new system for the approximations of the temperature at the interior grid points; use the LU decomposition you found in part (c)!

Note: If this problem is interesting to you, register for MATH 3044, Numerical Analysis II!