Lab 8 Hint

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Let's start with the form:

$$u(x_i) = f(x_i) + \int_a^b K(x_i, t)u(t)dt \tag{1}$$

for each $i \in \{0, \dots, m\}$

1 Step 1

Expand the integral $\int_a^b K(x_i, t)f(t)dt$ using the Composite Trapezoidal rule:

$$\int_a^b K(x_i, t)u(t)dt \approx \frac{h}{2}\big[K(x_i, x_0)u(x_0) + 2K(x_i, x_1)u(x_1) + \dots$$
$$+2K(x_i, x_{m-1})u(x_{m-1}) + K(x_i, x_m)u(x_m)\big]$$

Plugging this into (1), you get the equation:

$$u(x_i) = f(x_i) + \frac{h}{2}\big[K(x_i, x_0)u(x_0) + 2K(x_i, x_1)u(x_1) + \dots$$
$$+2K(x_i, x_{m-1})u(x_{m-1}) + K(x_i, x_m)u(x_m)\big]$$

for each $i \in \{0, \dots, m\}$.

2 Step 2

We want to solve for $u(x_i)$ terms. Hence, we will want to write each equation with the $u(x_i)$ terms on one side and all other terms $(f(x_i))$ on the other:

$$\begin{aligned}
-f(x_i) &= -u(x_i) + \frac{h}{2}\big[K(x_i, x_0)u(x_0) + 2K(x_i, x_1)u(x_1) + \dots \\
&\qquad +2K(x_i, x_{m-1})u(x_{m-1}) + K(x_i, x_m)u(x_m)\big] \\
&= \frac{h}{2}K(x_i, x_0)u(x_0) + hK(x_i, x_1)u(x_1) + \dots + (hK(x_i, x_i) - 1)u(x_i) + \dots \\
&\qquad + hK(x_i, x_{m-1})u(x_{m-1}) + \frac{h}{2}K(x_i, x_m)u(x_m)
\end{aligned}$$

for each $i \in \{1, \ldots, m-1\}$. Similarly, for $i = 0, m$

$$-f(x_0) = (\frac{h}{2}K(x_0, x_0) - 1)u(x_0) + hK(x_0, x_1)u(x_1) + \ldots$$
$$+ hK(x_0, x_{m-1})u(x_{m-1}) + \frac{h}{2}K(x_0, x_m)u(x_m)$$

and

$$-f(x_m) = \frac{h}{2}K(x_m, x_0)u(x_0) + hK(x_m, x_1)u(x_1) + \ldots$$
$$+ hK(x_m, x_{m-1})u(x_{m-1}) + (\frac{h}{2}K(x_m, x_m) - 1)u(x_m)$$

3 Step 3

The goal is to solve for $u(x_i)$. We will do so by writing

$$\vec{u} = \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_{m-1}) \\ u(x_m) \end{pmatrix}$$

and

$$\vec{f} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) \end{pmatrix}$$

See if you can write the equations developed in Step 2 by using the form:

$$A\vec{u} = -\vec{f}$$