MATH 3043, Numerical Analysis I
Fall 2020

Lab 11

This lab will have you implementing the Jacobi and Gauss-Seidel methods to approximate solutions of linear systems.

Solutions must be submitted on Canvas by **December 6** at **11:59 PM**. Please submit a single script file `Lab11Lastname.m` and the corresponding published file `Lab11Lastname.pdf` (for example, my submitted files would be `Lab11Zumbrum.m` and `Lab11Zumbrum.pdf`). Each solution should

- be contained in a separate cell which includes the problem number and short problem description,
- run independent of other cells,
- be adequately commented.

1. Write MATLAB functions `Jacobi` and `GaussSeidel` to implement the Jacobi and Gauss-Seidel methods respectively. The functions should accept five inputs (the matrix and right-hand side vector for the linear system, the maximum number of iterations, the stopping criteria tolerance, and the column vector with the initial approximation) and return two outputs (the number of iterations required and the approximation of the solution). Output the number of iterations required and the approximation for the solution (displaying at least k digits for solutions accurate to within 10^{-k}). Use the stopping criteria

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < \text{tolerance}.$$

   Hint: $\|\mathbf{x}\|_\infty$ can be computed using the built-in function `norm` using `norm(x, inf)`.

2. Consider the linear system

$$\begin{array}{rcrcrcrcr} 10x_1 & - & x_2 & + & 2x_3 & & & = & 6 \\ -x_1 & + & 11x_2 & - & x_3 & + & 3x_4 & = & 25 \\ 2x_1 & - & x_2 & + & 10x_3 & - & x_4 & = & -11 \\ & & 3x_2 & - & x_3 & + & 8x_4 & = & 15. \end{array}$$

   (a) Use the Jacobi method to solve the linear system with initial approximation $\mathbf{x}^{(0)} = \mathbf{0}$ and tolerance 10^{-3}.
   Hint: This problem is Section 7.3 Example 1.

   (b) Use the Gauss-Seidel method to solve the linear system with initial approximation $\mathbf{x}^{(0)} = \mathbf{0}$ and tolerance 10^{-3}.

3. Consider the linear system

$$\begin{array}{rcrcrcrcrcrcr} 4x_1 & - & x_2 & & & & & & & & & = & 0 \\ -x_1 & + & 4x_2 & - & x_3 & & & & & & & = & 5 \\ & - & x_2 & + & 4x_3 & & & & & & & = & 0 \\ & & & & & + & 4x_4 & - & x_5 & & & = & 6 \\ & & & & & - & x_4 & + & 4x_5 & - & x_6 & = & -2 \\ & & & & & & & - & x_5 & + & 4x_6 & = & 6. \end{array}$$

(a) Use the Jacobi method to solve the linear system with initial approximation $\mathbf{x}^{(0)} = \mathbf{0}$ and tolerance 10^{-4}.

(b) Use the Gauss-Seidel method to solve the linear system with initial approximation $\mathbf{x}^{(0)} = \mathbf{0}$ and tolerance 10^{-4}.

4. Consider the linear system

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & x_3 & = & -1 \\ 2x_1 & + & 2x_2 & + & 2x_3 & = & 4 \\ -x_1 & - & x_2 & + & 2x_3 & = & -5 \end{array}$$

with exact solution $\mathbf{x} = (1, 2, -1)^T$.

(a) Compute $\rho(T_j)$ and $\rho(T_g)$.
Hint: The eigenvalues of a matrix can be easily computed using the built-in function `eig`.

(b) Use your answers from part (a) to choose the method that you would use to approximate the solution.

5. Consider the linear system

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & - & 2x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 2x_2 & + & x_3 & = & 5 \end{array}$$

with exact solution $\mathbf{x} = (1, 2, -1)^T$.

(a) Compute $\rho(T_j)$ and $\rho(T_g)$.

(b) Use your answers from part (a) to choose the method that you would use to approximate the solution.

6. Suppose that an object can be at any one of $n+1$ equally spaced points $x_0, x_1, \ldots, x_n$. When an object is at location x_i, it is equally likely to move to either x_{i-1} or x_{i+1} and cannot directly move to any other location. Consider the probability $\{P_i\}_{i=0}^n$ that an object starting at location x_i will reach the left endpoint x_0 before reaching the right endpoint x_n (so that $P_0 = 1$ and $P_n = 0$). Since the object can move to x_i only from x_{i-1} or x_{i+1} and does so with probability $\frac{1}{2}$ for each of these locations,

$$P_i = \frac{1}{2}P_{i-1} + \frac{1}{2}P_{i+1}, \text{ for each } i = 1, 2, \ldots, n-1,$$

yielding the linear system

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & \cdots & \cdots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \ddots & & \vdots \\ 0 & -\frac{1}{2} & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \cdots & \cdots & 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

(a) Solve this system using the Gauss-Seidel method with initial approximation $\mathbf{x}^{(0)} = \mathbf{0}$ and tolerance 10^{-10} for $n = 10, 50,$ and 100.
Hint: The coefficient matrix can be easily constructed using the built-in function `diag`.

(b) Change the probabilities to α and $1 - \alpha$ for movement to the left and right respectively, derive the linear system equivalent to that above, and repeat part (a) with $\alpha = \frac{1}{3}$.