University of Cape Town

APG4012S Geodesy

Determination of N using Spherical Harmonics

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2.1 Overview

The objective of this paper is the determination of orthometric heights for five Trignet stations, given a GGMO2S Grace geopotential model to degree and order 160. The GRS80 ellipsoid characteristics will be used in the calculations.

In order to calculate the orthometric heights of the stations, the Geoidal height above the Ellipsoidal must be calculated as a function of gravity. The following steps breakdown this process.

1. Find an expression for N as a function of the disturbing potential T
2. Relate the disturbing potential T on the Geoid to the gravity anomoly Δg This will allow an expression for N as a function of $\Delta g$to be derived.

2.2 Find an expression for N as a function of the disturbing potential T

This can be accomplished using Bruns formula.

$$N = \frac{T_p}{\gamma_0} \tag{2.1}$$

2.3 Relate the disturbing potential T on the Geoid to the gravity anomoly

$\Delta g = (g_P - \gamma_O)$ and simplifying the Fundamental Gravimetric Equation to derive

$$\Delta g = -\frac{\delta T}{\delta n} - \frac{2}{a}.T \tag{2.2}$$

using Laplace's Equation $\Delta T \equiv 0$ and the boundary condition above, solving for T.

3. *Geoid Determination*

3.1 Laplace Equation

In order to use spherical harmonics to represent a surface, the gravitational potential at a point inside the body must satisfy Poisson's Equation:

$$\Delta V = -4\pi G\sigma \tag{3.1}$$

Inside the body has a density greater than zero. For a point outside the body, Laplace's equation (below) must be satisfied (here there is no density outside the body):

$$\Delta V \equiv 0 \tag{3.2}$$

3.2 Spherical Harmonics

Functions that satisfy Laplace's Equation are known as harmonic functions. Using a spherical coordinate system, spherical harmonic functions can then be used to represent the Earth's gravity field. A Legendre function $P_{nm}(cos\theta)$ is often used to calculate the geoid. The sign changes in any $Y_{nm}(\theta, \lambda)$ in both directions divide the Earth into a chequer board pattern of $(n - m + 1) \times 2m$ tiles as seen in the figure below [Sneeuw, 2006]:

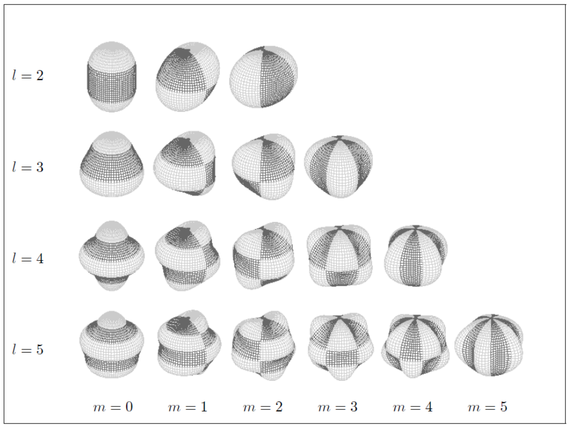

Figure 3.1: Spherical Harmonics up to Degree (l) and order (m) 5
Sneeuw [2006]

4. *Method*

4.1 Preliminaries

4.1.1 Ellipsoid Radius

The local ellipsoid radius is calculated using the Cartesian coordinates or the geographic coordinates as follows:

$$r(\varphi) = \sqrt{x^2 + y^2 + z^2} = a\sqrt{1 - \frac{e^2(1-e^2)\sin^2\varphi}{1 - e^2 sin^2\varphi}} \tag{4.1}$$

e^2 is the ellipsoidal first eccentricity. $e = \dfrac{E}{a}$ where$E = \sqrt{a^2 - b^2}$. a and b are the semi-major and semi-minor axes respectively.

4.1.2 Geocentric Coordinates

Spherical harmonic models are formulated in geocentric coordinates (the ellipsoidal coordinates). The longitude remains the same for both but the latitude for a given geographic coordinate location is converted to a geocentric location using the formula :

$$(\varphi^*) = \left[(\frac{b}{a})^2 tan\varphi \right] \tag{4.2}$$

4.1.3 Normal Gravity

The normal gravity γ_0 is a function of latitude is calculated using:

$$\gamma(\phi) = \gamma_e \frac{1 + ksin^2\phi}{\sqrt{1 - e^2 sin^2\phi}} \tag{4.3}$$

where ϕ is the latitude of the point of interest. the values of $\gamma_e, k\ and\ e^2$ are inserted as geometric constants gathered from the ellipsoid of choice, where γ_e is the normal gravity at the equator and e^2 is the first eccentricity. k is given as a constant but can also be calculated $k = \dfrac{b_{y_b} - a_{y_a}}{a_{y_a}}$.

4.2 Legendre Polynomials and Functions

Zonal Legendre functions of order 0 are called Legendre polynomials Sneeuw [2006]. They are polynomials in $t = cos\theta$. The Legendre polynomials were calculated as follows...

Step 1 using Rodrigues Formula:

$$P_l(t) = \frac{1}{2^l l!} \frac{d^l (t^2 - 1)^l}{dt^l} \tag{4.4}$$

Step 2 using Ferrers Formula:

$$P_{lm}(t) = (1 - t^2)^{m/2} \frac{d^m P_l(t)}{dt^m} \tag{4.5}$$

4.2.1 Fully Normalised associated Legendre Functions

The abbreviations $t = sin\varphi^*\ and\ u = cos\varphi^*$ are used below. The normalisation process applied to an associated Legendre function is as follows:

$$\bar{P}_{n,m}(t) = \sqrt{(k(2n+1)) \frac{(n-m)!}{(n+m)!}} P_{n,m}(t) \tag{4.6}$$

4.2.2 Recursive Formula

$P_{n,m}(t)$ can be calculated using recursive formulas as well:

$$\begin{aligned} P_{n+1,0}(t) &= (2n+1)tP_{n,0}(t) - nP_{n-1,0}(t) \\ P_{n,n}(t) &= (2n-1)uP_{n-1,n-1}(t) \\ P_{n,m}(t) &= P_{n-2,m}(t) + (2n-1)uP_{n-1,m-1}(t) \end{aligned} \tag{4.7}$$

The following starting values are required:

$$\begin{aligned}
P_{0,0}(t) &= 1 \\
P_{1,0}(t) &= t \\
P_{1,1}(t) &= u \\
P_{2,0}(t) &= \frac{3}{2}t^2 - \frac{1}{2} \\
P_{2,1}(t) &= 3ut \\
P_{2,2}(t) &= 3u^2
\end{aligned} \tag{4.8}$$

4.3 (Calculate Geoid undulation N as a function of Longitude and Latitude)

$$N(\lambda, \varphi) = \frac{GM_g}{\gamma(\varphi)r(\varphi)} \sum_{n=2}^{\infty} (\frac{a_g}{r(\varphi)}) \sum_{m=0}^{n} \left[\bar{C}_{n,m} cos(m\lambda) + \bar{S}_{n,m} sin(m\lambda)\right] \bar{P}_{nm}(cos\varphi^*) \tag{4.9}$$

$\bar{C}_{n,m}$ and $\bar{S}_{n,m}$ are the spherical harmonic coefficients of degree and order n , m respectively. The mass gravity constant GM_{g} and the scale factor a_{g} are from the geopotential model. $\bar{P}_{n,m}(cos\bar{\varphi})$ fully normalized associated Legendre functions. The harmonics $\bar{P}_{n,m}$ are evaluated at the geocentric latitude φ^*, (not at the geographical latitude) φ.

5. *Results*

References

Nico Sneeuw. Physical geodesy. Technical report, University of Stuttgart, 2006.