Introduction to Biological Imaging
Homework

Winter term 2018/19

Contents

1 Homework 1

Assignment 1

In the following, the code to answer the questions of the first assignment is plotted.

Task 1

```
%% 1) Create a phantom for simulating XCT measurements.
% The phantom should contain % ellipses and polygons of varying intensity
% (use phantom() and augment the resulting
% image). Show an image of your phantom.
% Size of quadratic Phantom in pixels
img_width = 256;
img_length = img_width;
% Determine Ellipe count (min = 3)
num_ellipses = 6;
% Initialize array
E = zeros(num_ellipses,6);
% Get basic skull structure from phantom
[~,base] = phantom('Modified Shepp-Logan', img_width);
E(1:2,:) = base(1:2,:);
% Get remaining array size
[M,N] = size(E(3:end,:));
% Create random ellipse along boundaries
E(3:end,1) = rand(M,1);
E(3:end,2) = -0.2 + (0.3+0.3)*rand(M,1);
E(3:end,3) = -0.4 + (0.4+0.4)*rand(M,1);
E(3:end,4) = -0.4 + (0.4+0.4)*rand(M,1);
E(3:end,5) = -0.45 + (0.45+0.45)*rand(M,1);
E(3:end,6) =  180*rand(M,1);
% Build phantom only of ellipses
phntm = phantom(E,img_width);
% Increase inensity of pixels in a square of the matrix
phntm(img_width/2-16:img_width/2+16,img_width/2-16:img_width/2+16) =...
    phntm(img_width/2-16:img_width/2+16,img_width/2-16:img_width/2+16) +
        0.5;
% visualize the phantom
figure; imagesc(phntm);
axis equal tight; colormap gray; colorbar; xlabel('x'); ylabel('y');
title('Random Shepp-Logan Phantom');
```

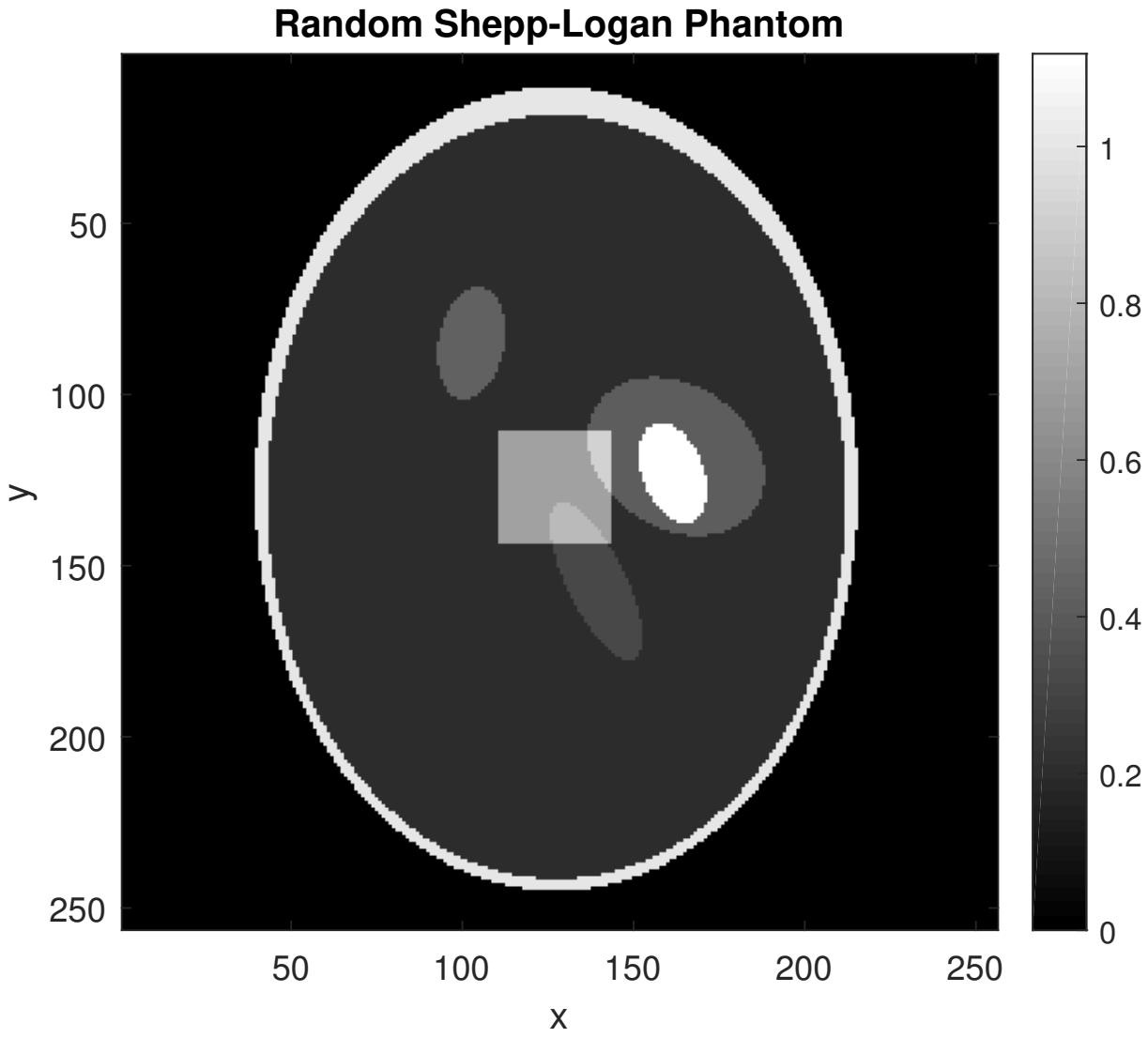


Figure 1: Randomly generated skull

Task 2

```
%% 2) Compute the views (projections) for the range of angles from 0 to 179 degrees with spacing
% of 1, 5 and 10 degrees. Show the projections at 0, 30, 45 and 90 degrees (in one axes). Show the
% sinogram with the most angles/projections.
% specify projection angles
theta = {0:1:179;
    0:5:179;
    0:10:179};
% pad the image with zeros so nothing gets lost during rotation
img_diag = sqrt(img_length^2 + img_width^2);
padding = ceil(img_diag - img_width) + 4;
pad_img = zeros(img_length + padding, img_width + padding);
pad_img(ceil(padding/2):(ceil(padding/2) + img_length - 1),...
    ceil(padding/2):(ceil(padding/2) + img_width - 1)) = phntm;
% loop over the number of angles and summarize
th = theta{1};
n = length(th);
img_pr = zeros(size(pad_img,2), n);
for i = 1:n
    tmp_img = imrotate(pad_img,180+th(i), 'bilinear', 'crop');
    img_pr(:,i) = (sum(tmp_img))';
end
% create a sinograms for the specified angles
for i = 1:length(theta)
sinogram{i}(:,:) = radon(phntm, theta{i});
end
% Plot sinogram data at specific points in the same axes
figure;
plot(sinogram{1}(:,1),'DisplayName',[num2str(theta{1}(:,1))  ' degrees']);
hold on
plot(sinogram{1}(:,31),'DisplayName',[num2str(theta{1}(:,31)) ' degrees']);
plot(sinogram{1}(:,46),'DisplayName',[num2str(theta{1}(:,46)) ' degrees']);
plot(sinogram{1}(:,91),'DisplayName',[num2str(theta{1}(:,91)) ' degrees']);
title('Radon Transform at specific angles'); legend
% visualize the sinogram
figure;
imagesc(sinogram{1});
title(['Sinogram @ ' num2str(length(theta{1})) 'angles (' num2str(theta{1}(:,1)) '\textdegree -' num2str(theta{1}(:,end)) '\textdegree)']);
xlabel('Angle');colormap gray;colorbar
```

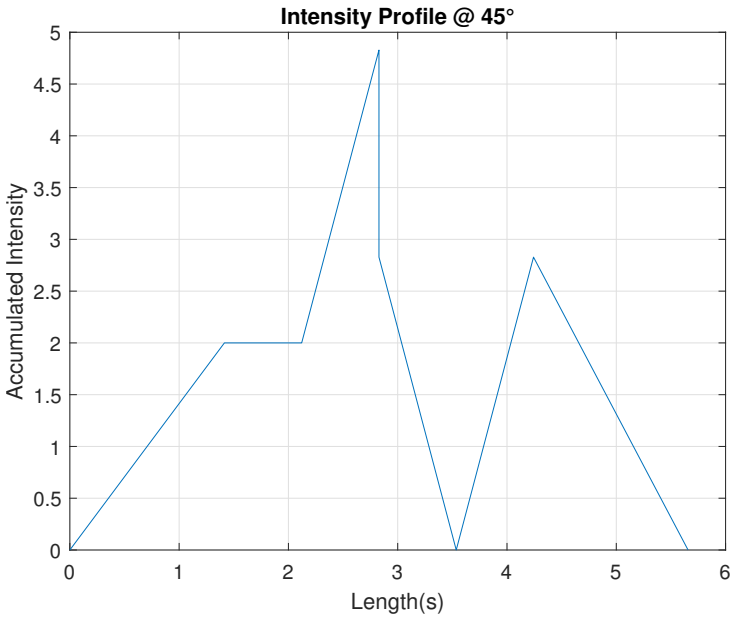


Figure 2: Sinogram with the most angles (180)

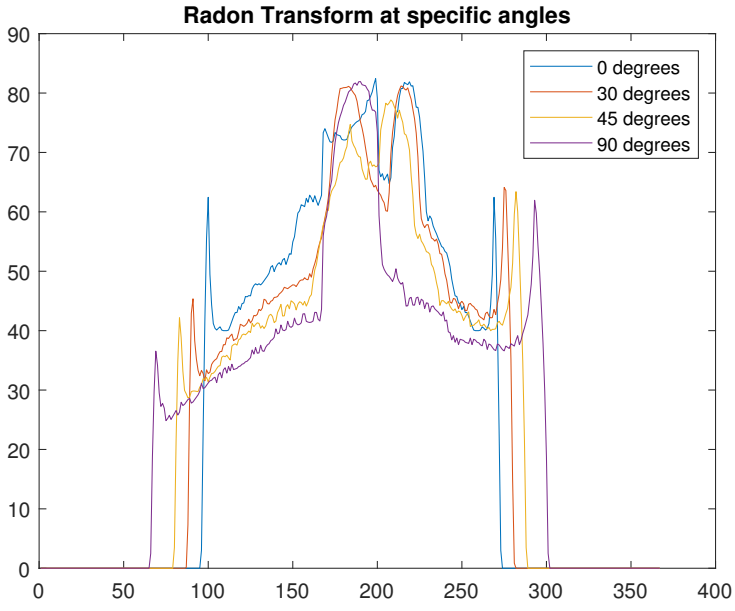


Figure 3: Projections at the specific angles

Task 3

The following code was implemented as a seperate MATLAB function, in order to reduce the amount and simplify the code.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Backprojection Algorithm %%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Parameters:
% sinogram: Input 2D - sinogram matrix
% theta: corresponding angles of sinogram
% filter_shape('none','Ramp','Cos','Hamming'): Selection of applied filter
% method
% Output:
% rec: Reconstructed image
function rec = backprojection(sinogram,theta,filter_shape)
if nargin <3
    filter_shape = 'none';
end
% figure out how big the picture is going to be.
sideSize = size(sinogram,1);
%filter setups
x = linspace(-1,1,sideSize);
ramp = abs(x);
switch filter_shape
    case 'Ramp'
        filter = ramp;
    case 'Hamming'
        filter = ramp .* hamming(sideSize)';
    case 'Cos'
        filter = ramp .* cos((x./2).*pi).^2;
    otherwise
        filter = 1;
end
% set up the image
rec = zeros(sideSize,sideSize);
for i = 1:length(theta)
    line = sinogram(:,i);
    % perform fft
    line_fft = fftshift(fft(ifftshift(line)));
    % filter in frquency space
    line_fft_filtered = line_fft .* filter';
    % transform back into time domain
    line_filtered = ifftshift(ifft(fftshift(line_fft_filtered)));
    % backproject
    image = repmat(line_filtered,1,sideSize);
    image = imrotate(image,theta(i),'crop');
    % sum up final picture
    rec = rec + image;
end
% rotate image as the original
rec = real(imrotate(rec,90))./length(theta);
end
```

Task 4

```
%% 4) Reconstruct the phantom data with the specified angular spacings
    using your
% backprojection algorithm without filtering. Show the obtained
    reconstructions.
recon = backprojection(sinogram{1},theta{1});
% visualize reconstruction results
figure;
subplot(1,2,1);
imagesc(phntm);
axis equal tight;
colormap gray;
title('Original'); xlabel('x'); ylabel('y');
subplot(1,2,2);
imagesc(recon);
axis equal tight;
colormap gray;
title('Unfiltered BP'); xlabel('x'); ylabel('y');
```

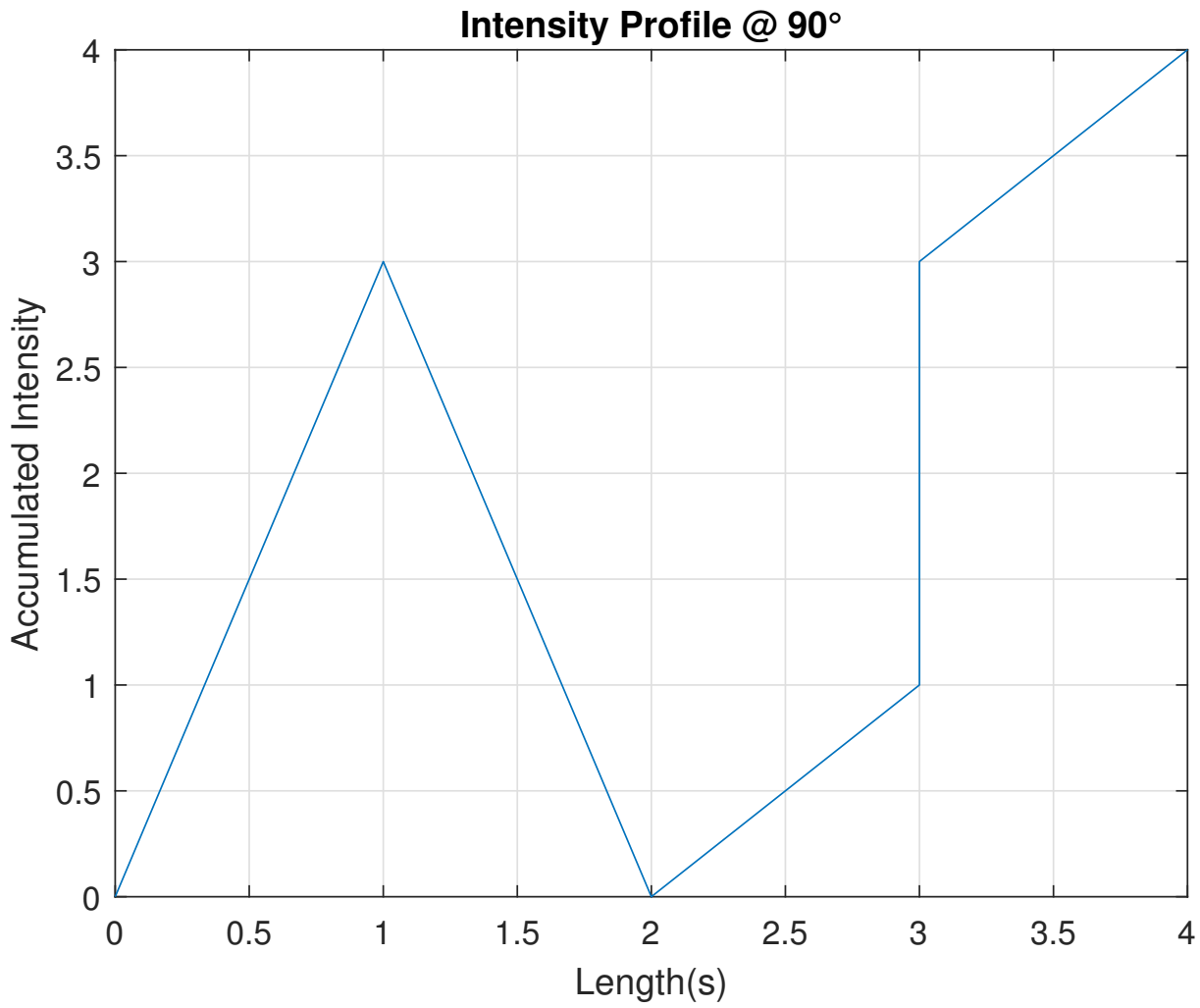


Figure 4: Reconstruction from 1°resolution without any filtering

Task 5

```
%% 5) Incorporate filtering in your backprojection. Implement 3 filters (ramp, cosine and
% hamming) and test their influence on the reconstruction using your phantom data (pick
% a single angle spacing). Show the reconstruction results
for i = 1:3
    % Compute filtered backprojections
    recon_cos = backprojection(sinogram{i},theta{i},'Cos');
    recon_ramp = backprojection(sinogram{i},theta{i},'Ramp');
    recon_hamming = backprojection(sinogram{i},theta{i},'Hamming');
    % visualize reconstruction results
    figure;
    subplot(2,2,1);
    imagesc(phntm);
    title(['w/o Filtering,' num2str(length(theta{i})) ' angles']);
    colormap gray; colorbar; xlabel('x'); ylabel('y');
    subplot(2,2,2);
    imagesc(recon_ramp);
    title(['w/ Ramp Filtering, ' num2str(length(theta{i})) ' angles']);
    colormap gray; colorbar; xlabel('x'); ylabel('y');
    subplot(2,2,3);
    imagesc(recon_hamming);
    title(['w/ Hamming Filtering, '  num2str(length(theta{i})) ' angles']);
    colormap gray; colorbar; xlabel('x'); ylabel('y');
    subplot(2,2,4);
    imagesc(recon_cos);
    title(['w/ Cosine Filtering, ' num2str(length(theta{i})) ' angles']);
    colormap gray; colorbar; xlabel('x'); ylabel('y');
end
```

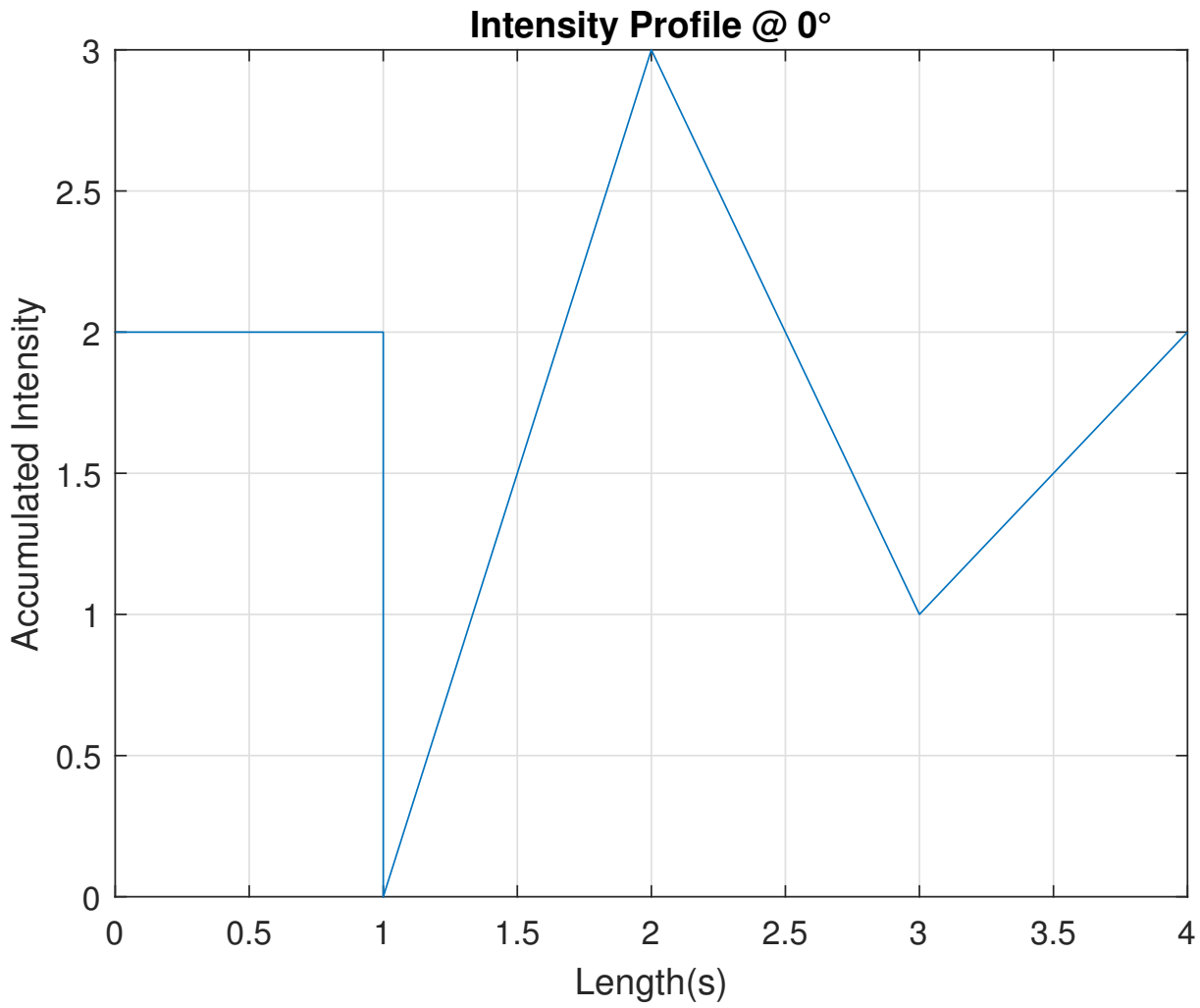


Figure 5: 1°resolution

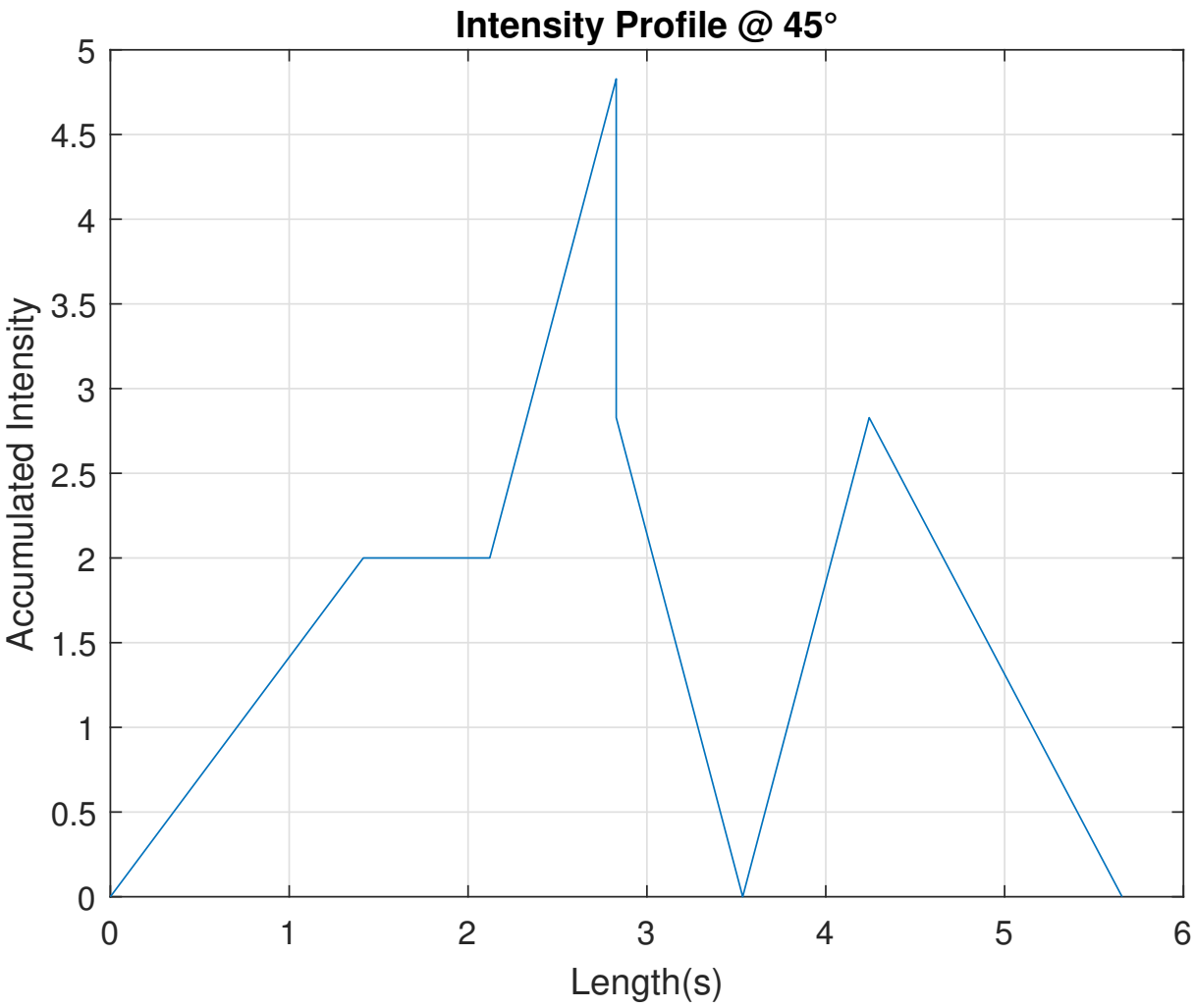


Figure 6: 5°resolution

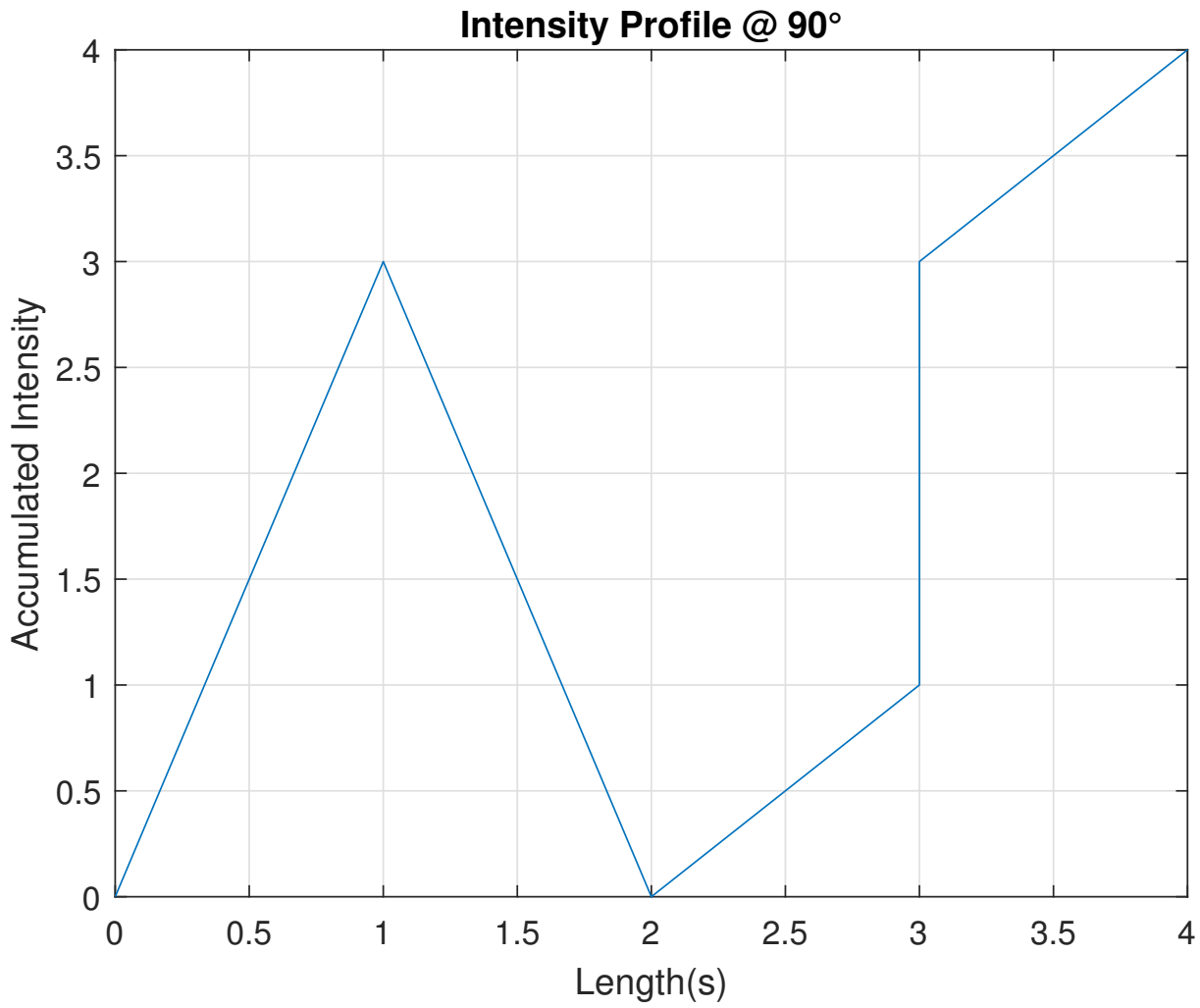


Figure 7: 10°resolution

Task 6

```
%% 6) Reconstruct the provided datasets (CT_2018.mat) with your
    backprojection algorithm
% without filtering and with each of the implemented filters, respectively.
    Show the
% reconstructed images.

% load sample data
S = load('CT_2018.mat');
% loop over sample data, compute backprojections and plot them each in a
% single figure
for i = 1:3
    % change variable name dynamically
    sino_name = S.(['sino' num2str(i-1)]);
    angle = S.(['theta' num2str(i-1)]);
    % visualize
    fig_name = sprintf('%s with %d angles from %d  to %d ', ['sino'
        num2str(i-1)],length(angle), angle(1),angle(end));
    figure('Name',fig_name);
    subplot(2,2,1);
    recon = backprojection(sino_name,angle);
    imagesc(recon);
    title('w/o Filtering');
    colormap gray; colorbar; xlabel('x'); ylabel('y');
    subplot(2,2,2);
    recon = backprojection(sino_name,angle,'Ramp');
    imagesc(recon);
    title('w/ Ramp Filtering');
    colormap gray; colorbar; xlabel('x'); ylabel('y');
    subplot(2,2,3);
    recon = backprojection(sino_name,angle,'Hamming');
    imagesc(recon);
    title('w/ Hamming Filtering');
    colormap gray; colorbar; xlabel('x'); ylabel('y');
    subplot(2,2,4);
    recon = backprojection(sino_name,angle,'Cos');
    imagesc(recon);
    title('w/ Cosine Filtering');
    colormap gray; colorbar; xlabel('x'); ylabel('y');
end
```

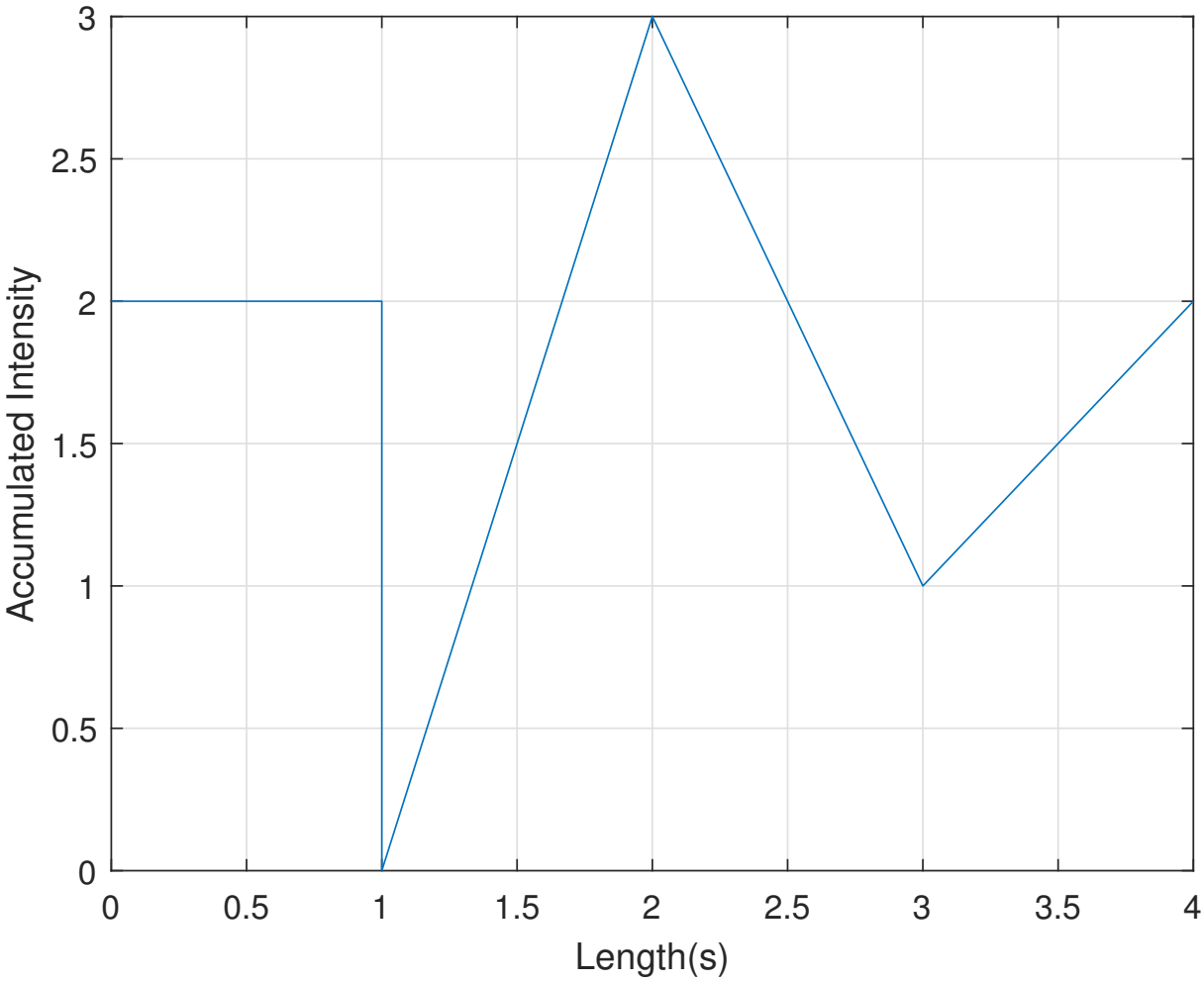


Figure 8: 1°resolution

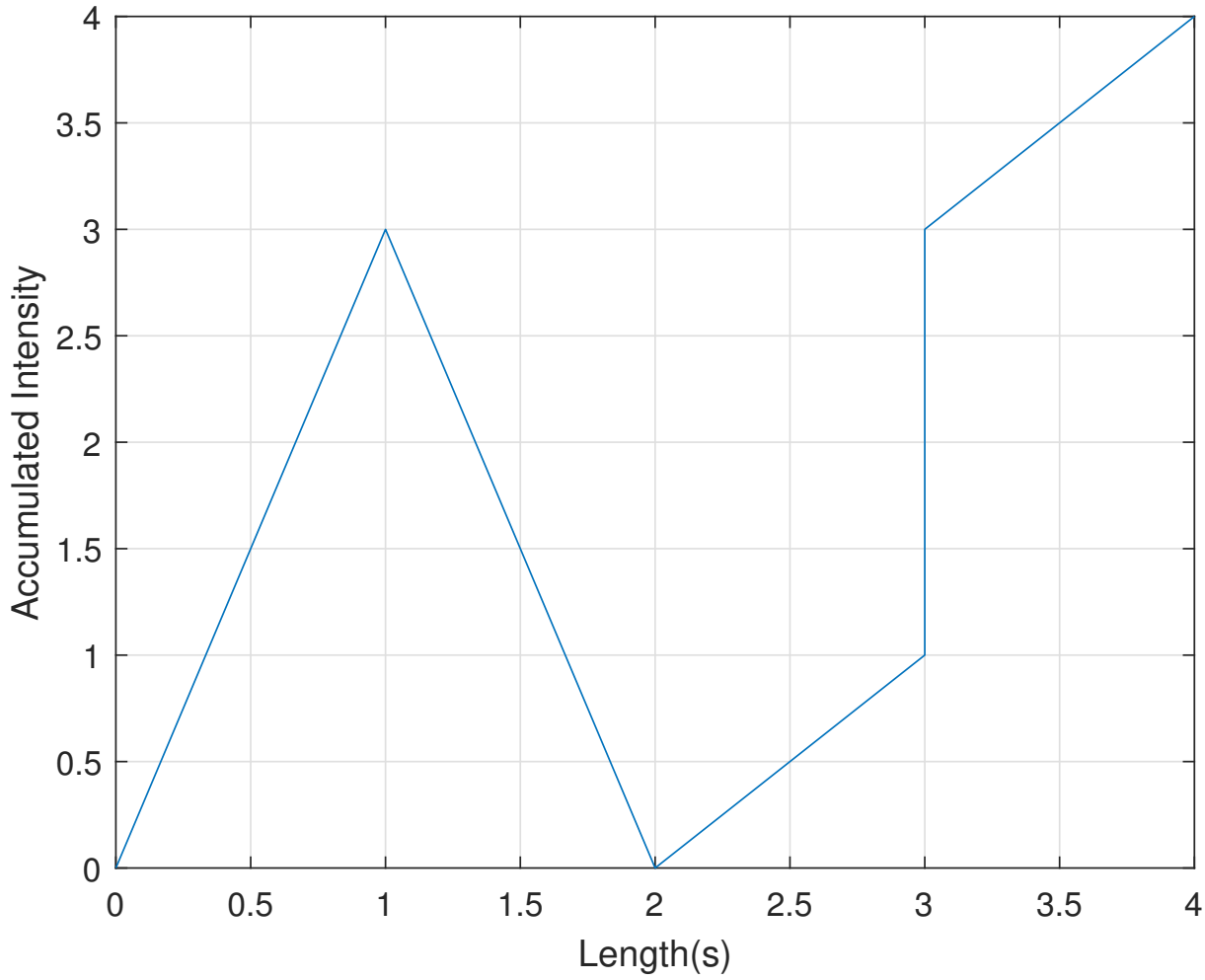


Figure 9: 2°resolution

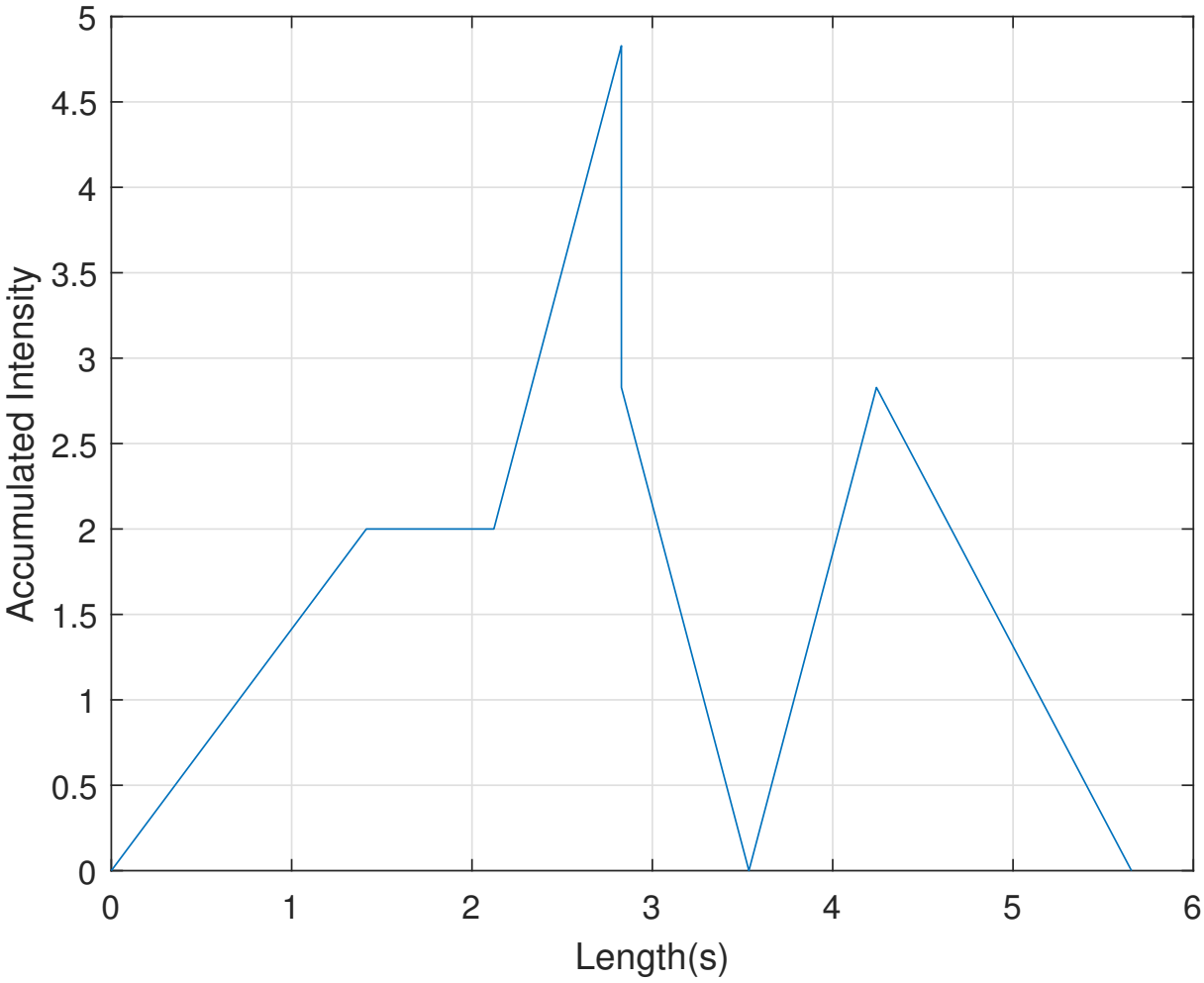


Figure 10: 0.5°resolution

Task 7

Shortly interpret your results:

(a) What is the effect of different angular spacings on the reconstruction?
Finer Spacings <==> More Information, therefore better reconstruction, more details visible.

(b) How do the different filters change the reconstruction results?
Filters affect the edge sharpness, the residual noise and the overall brightness of an image

(c) Which filter performs best? Why? Under which circumstances?
The Hamming filter performs best overall, because he does not damp the low frequencies to zero,as cosine and ramp filter do.

(a) What is the effect of different angular spacings on the reconstruction?
Finer spacing provide more information, therefore a better reconstruction is possible.

(b) How do the different filters change the reconstruction results?
Different filters affect different radial resolutions. They either provide more or less noise.

(c) Which filter performs best? Why? Under which circumstances?
In the case of the self-made phantom, the Hamming filter performs best due to its elaborate filtering function. In comparison of the three filters the hamming flter is a good compromise between high contrast and noise. The cos filter looses a lot of information for low frequencies. Filtering is always a trade off between reducing the noise and getting a high resolution.

Assignment 2

Assume the 2x2 discretization of the spatial distribution of absorption coefficient shown in Fig. 1 and 2 point detectors with spacing L.

Task 1

Design a model matrix A that relates X-ray measurements $p = (p_1, p_2, p_3, p_4, p_5, p_6)^T$ at 3 shown angles to the (unknown) absorption $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)^T$ as in: $\mathbf{A} = \mathbf{p}$. Show A.

```
%% 1) Design a model matrix A that relates X-ray measurements
% p=(P1,P2,P3,P4,P5,P6)T at 3 shown angles to
% the (unknown) absorption mu=(mu1,mu2,mu3,mu4)T as in: A*mu = p. Show A.

% scaling for 45   beams
a = sqrt(2) - 1;

syms L;

% setup model matrix
A = [L, 0, L, 0;
0, L, 0, L;
a*L, 0, L, a*L;
a*L, L, 0, a*L;
0, 0, L, L;
L, L, 0, 0];

pretty(A);
```

$$\mathbf{A} = \begin{pmatrix} L & 0 & L & 0 \\ 0 & L & 0 & L \\ (\sqrt{2}-1)L & 0 & L & (\sqrt{2}-1)L \\ (\sqrt{2}-1)L & L & 0 & (\sqrt{2}-1)L \\ 0 & 0 & L & L \\ L & L & 0 & 0 \end{pmatrix} \tag{1}$$

Task 2

```
%% 2) Assume a specific distribution (values) of mu_test and a specific
    value of L
% Simulate the corresponding measurements mu_test for this distribution
% using the model matrix A. Show p_test.

L_spec = 4;
A_subs = double(subs(A,L,L_spec));
mu_test = rand([4,1]); % assume (random) values of absorption mu

p_test = A_subs*mu_test % simulate corresponding measurements b
```

$$P = \begin{pmatrix} 5,13361736274707 \\ 6,49676346881041 \\ 5,18217061046964 \\ 5,35509078478611 \\ 6,32384329449394 \\ 5,30653753706355 \end{pmatrix} \tag{2}$$

Task 3

```
%% 3) Using the simulated measurements p_test, reconstruct absorption
    mu_rec,
% i.e. solve A*mu_rec=p_test. Show both the assumed (mu_test) and
% the reconstructed (mu_rec) absorption distributions.

% now we try to recover mu back from our measurements:
% A*mu = p => mu = inv(A)*p - we try to solve for mu that is assumed
    unknown
% inv(A)*p % doesn't work, rank of A is 3!

mu_rec = lsqr(A_subs, p_test); % min||A*mu-p||.^2 - we try to find a
    solution using minmization procedure
rel_error_perc = abs(mu_test - mu_rec)./mu_test*100 % looking at the
    relative error in percent, we're pretty far off from the real values
```

$$4\mu_{rec} = \begin{pmatrix} 0,647617630172602 \\ 0,679016754093115 \\ 0,635786710513997 \\ 0,945174113109320 \end{pmatrix}$$

$$\mu_{test} = \begin{pmatrix} 0,647617630172602 \\ 0,679016754093115 \\ 0,635786710513997 \\ 0,945174113109320 \end{pmatrix}$$

$$rel - error - perc = \begin{pmatrix} 1.2703 \cdot 10^{-11} \\ 1.2796 \cdot 10^{-11} \\ 1.3672 \cdot 10^{-11} \\ 8.6099 \cdot 10^{-12} \end{pmatrix}$$

Task 4

(a) Does the reconstruction μ_{rec} correspond to the assumed distribution mutest of the absorption coefficient?
According to variable rel-error-perc, μ_{rec} matches the array μ_{test} at a relative error smaller than 1E-10 percent. This means, the arrays can considered to be the same.

(b) Did you use inv() for inverting the model? Why?
No, because model matrix a A is no square matrix and has a rank of 4.

2 Homework 2

2.1 Finite Difference Method

2.1.1 Come up with a simple one dimensional differential equation $\mathbf{f'(x) = g(x)}$ on $\mathbf{\Omega = [0, 1]}$ by specifying g(x) and setting appropriate boundary condition (e. g. $\mathbf{f(0) = 0}$).

Initial assumption:

$$g(x) = \cos^2(\pi x) \tag{3}$$

Dirichlet boundary condition:

$$f(0) = 0 \tag{4}$$

2.1.2 Solve the obtained boundary value problem analytically. Show your solution as well as its plot.

$$\int f'(x)\ dx = \int \cos^2(\pi x)\ dx \tag{5}$$

$$f(x) = \frac{x}{2} + \frac{\sin(2\,\pi\,x)}{4\,\pi} + c_1 \tag{6}$$

$$f(0) = \frac{0}{2} + \frac{\sin(2\,\pi\,0)}{4\,\pi} + c_1 \overset{!}{=} 0 \tag{7}$$

$$\Rightarrow c_1 = 0 \tag{8}$$

The corresponding MATLAB code was imeplemented as follwing:

```
x_cont = linspace(0,1,1000);
x_interval = [0,1];
%% Plot ODE f'(x) = cos(pi x)^2 and its solution
syms f(x)
ODE = diff(f,x) == (cos(pi.*x)).^2;
bc = f(0) == 0;
sol = dsolve(ODE,bc);

figure;
yyaxis right
plot(x_cont,(cos(pi.*x_cont)).^2,'DisplayName',['ODE (f''(x) =' ...
    'cos(x \pi)^2)']); hold on
xlabel('x');
ylabel('f''(x)');
yyaxis left
fplot(sol,x_interval,'DisplayName',['Analytical Solution ' ...
    '(f(0)=0, ' newline 'f(x)=^{x}/_{2}+sin(2\pix)/4\pi )']);
legend;
xlabel('x');
ylabel('f(x)');
```

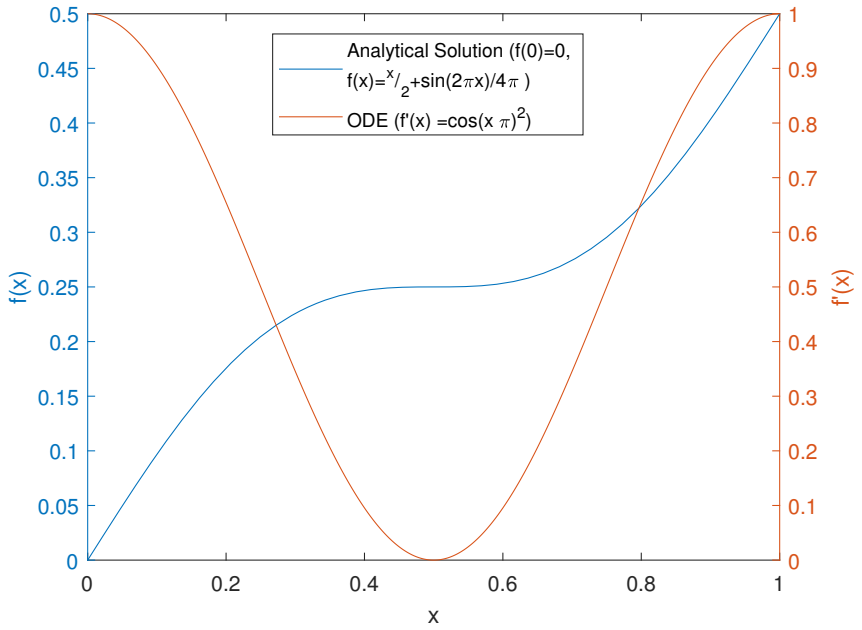


Figure 11: Original ODE with analytical solution

2.1.3 Solve the boundary value problem using the Finite Difference Method for step h = 0.2, 0.1 and 0.01. Pick the discretization scheme yourself (forward/backward/central difference), discretize the equation and put it into matrix form. Use linsolve() to solve the resulting system of linear equations. Hint: unless you know what you are doing, avoid central difference scheme.

Choosing the *forward Method* for the Finite Differences Method, where $x_i = i \cdot h$ with $i = 1, 2, ..., N + 1$ and $N + 1 = \frac{1}{h}$. For given h values of 0.2, 0.1 and 0.01 the dimension M is 5, 10 and 100.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{9}$$

$$\frac{f(x+h) - f(x)}{h} = \cos^2(\pi x) \tag{10}$$

$$\text{Discretizing}: \frac{f(x_{i+1}) - f(x_i)}{h} = \cos^2(\pi x_i) \tag{11}$$

$$\text{Rewriting}: f(x_{i+1}) - f(x_i) = h \cos^2(\pi x_i) \tag{12}$$

$$\text{Using Eq. 4}: \; f_1 = f(0) = 0 \tag{13}$$

$$\underbrace{\begin{bmatrix} 1 & & & & 0 \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -1 & 1 & \\ 0 & & & -1 & 1 \end{bmatrix}}_{M \times M} \underbrace{\begin{bmatrix} f_2 \\ f_3 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix}}_{M \times 1} = h \underbrace{\begin{bmatrix} \cos^2(\pi x_1) \\ \cos^2(\pi x_2) \\ \vdots \\ \cos^2(\pi x_{N-2}) \\ \cos^2(\pi x_{N-1}) \end{bmatrix}}_{M \times 1} \tag{14}$$

The Finite Differences were computed by the following MATLAB script:

```
%% Finite Difference Method
% Initialize Matrices
h = [0.2 0.1 0.01];
mat_size = 1./h;
A = cell(length(mat_size),1);
b = cell(length(mat_size),1);
f = cell(length(mat_size),1);
x = cell(length(mat_size),1);
fin_diff = cell(length(mat_size),1);

figure;
fplot(sol,x_interval,'DisplayName','Analytical Solution');hold on
% Populate A and b
for i = 1:length(mat_size)
    x{i} = linspace(0,1,mat_size(i))';
    A{i} = eye(mat_size(i));
    for j = 2:(mat_size(i))
        A{i}(j,j-1) = -1;
    end

    b{i} = double(h(i) .* ones(mat_size(i),1) .* (cos(pi .* x{i})).^2);

    % Evaluate Af = b
    f{i} = linsolve(double(A{i}),b{i});

    % Plot solution
    plot(x{i},[0;f{i}(1:end-1)],'DisplayName',['Finite Difference h = '
        num2str(h(i))]);
end
legend({},'Location','northwest');
xlabel('x_i');
ylabel('f(x_i)');
```

2.1.4 Plot the resulting solutions along with the exact solution you have computed in step 2

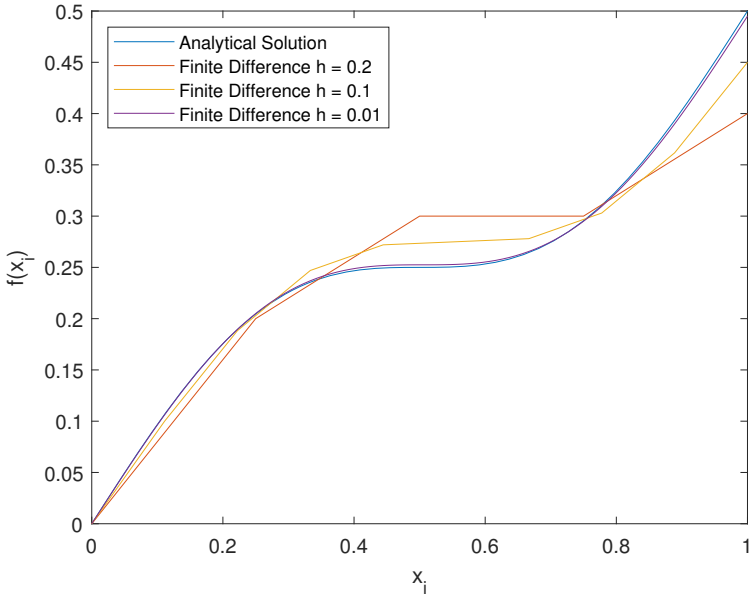


Figure 12: Analytical and Finite Difference solutions of the ODE

2.2 Assignment 2. Finite Element Method

2.2.1 Derive quadratic interpolation functions for the finite element method. Plot them in local coordinates. Show your derivation.

As shown in the lecture, for the square shape functions, we can write our trial solution as:

$$\widehat{T} = a_0 + a_1 x + a_2 x^2 \tag{15}$$

$$= \underbrace{\begin{bmatrix}1 & x & x^2\end{bmatrix}}_{X} \underbrace{\begin{bmatrix}a_0 & a_1 & a_2\end{bmatrix}^T}_{A} \tag{16}$$

$$T_{i_1} = T(0) = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 \tag{17}$$

$$T_{i_2} = T(1/2) = a_0 + \frac{a_1}{2} + \frac{a_2}{2^2} \tag{18}$$

$$T_{i_3} = T(1) = a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 \tag{19}$$

$$\underbrace{\begin{bmatrix}T_{i_1}\\ T_{i_2}\\ T_{i_3}\end{bmatrix}}_{T} = \underbrace{\begin{bmatrix}1 & 0 & 0\\ 1 & \frac{1}{2} & \frac{1}{2^2}\\ 1 & 1 & 1\end{bmatrix}}_{L} \underbrace{\begin{bmatrix}a_0\\ a_1\\ a_2\end{bmatrix}}_{A} \tag{20}$$

$$XA = NLA \tag{21}$$

$$N = XL^{-1} \tag{22}$$

$$= \begin{bmatrix}2x^2 - 3x + 1 & 4x - 4x^2 & 2x^2 - x\end{bmatrix} \tag{23}$$

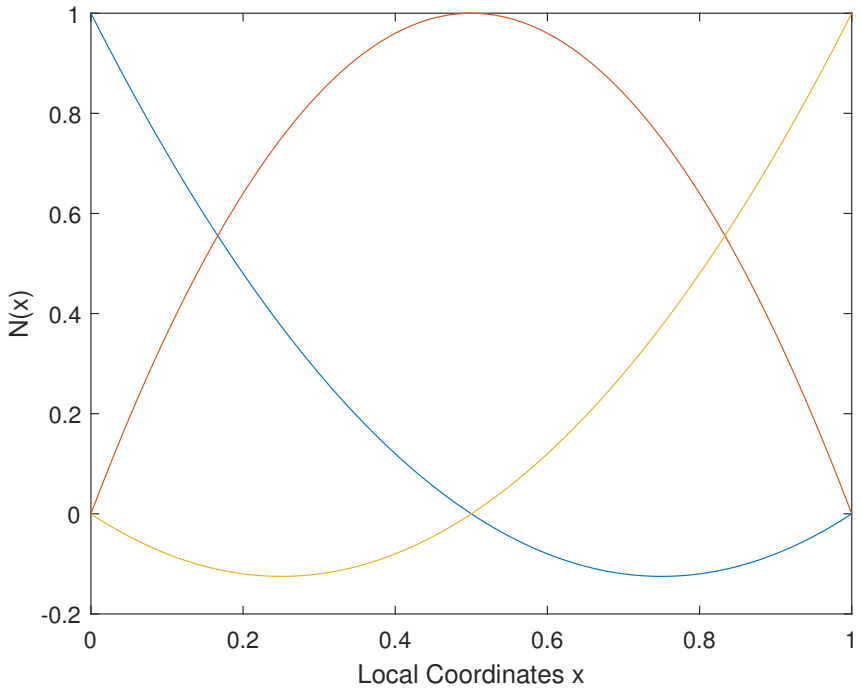


Figure 13: Quadratic FEM Interpolation Functions

2.2.2 Consider a 1D steady state heat conduction problem, where k and S are constant. How would the stiffness matrix for one element look like for the shape functions derived in step 1?

Problem:

$$kT'' + S = 0 \text{ on } \Omega \in [0,1] \tag{24}$$

$$T(0) = 0,\ T(1) = 0 \tag{25}$$

$$k_{stiff}(i,j) = \int_0^1 N'(i) \cdot N'(j) dx \tag{26}$$

$$k_{stiff} = \begin{bmatrix} 2.\bar{3} & -2.\bar{6} & 0.\bar{3} \\ -2.\bar{6} & 5.\bar{3} & -2.\bar{6} \\ 0.\bar{3} & -2.\bar{6} & 2.\bar{3} \end{bmatrix} \tag{27}$$

```
%% 3D Finite Element Method

% Local x
x_local = 0:0.01:1;
L = [1,0,0; 1,0.5,0.25;1,1,1];
L_inv = inv(sym(L));

syms x
X = [1,x,x^2];
N = X*L_inv;

figure; %visualize cubic shape functions
for i = 1:length(N)
    plot(x_local, subs(N(i),x_local),'DisplayName',['Square Shape Function
        ' num2str(i)]); hold on;
end
xlabel('Local Coordinates x');
ylabel('N(x)');
%% Stiffness Matrix
k_stiff = zeros(length(N));

for i = 1:length(N)
    for j = 1:length(N)
        k_stiff(i,j) = double(int( diff(N(i), x).*diff(N(j), x), x, 0, 1 ))
            ;
    end
end

nodes = 3; % number of elements
Ke = double(k_stiff);
RHSe = int(N, x, 0, 1)';


L = 1; % domain length
k = 1; %for k = [100,10,1,0.1] %; % thermal conductivity,1
S = 10; %    for S = [100,10,1,0.1]% % source,100
l = L/nodes; % node spacing assuming equidistant nodes
dim = nodes*(nodes - 1) +1; %dimension of stiffness matrix
K = zeros(dim); %initialize stiffness matrix with zeros
```

```
RHS = zeros([dim, 1]); %initialize load vector with zeros

for i = 1:nodes %cycle through elements

    %compute indices of entries for a given element
    ind_start = 1+(i-1)*(nodes-1);
    ind_end = ind_start + (nodes-1);

    % update stiffness matrix with the stiffness matrix of a single element
    K(ind_start:ind_end, ind_start:ind_end) = K(ind_start:ind_end, ...
        ind_start:ind_end) + Ke;

    %update load vector
    RHS(ind_start:ind_end) = RHS(ind_start:ind_end)+RHSe;

end

K = (k/l)*K; %multiply by constants
F = S*l*RHS;
large_number = 1e6; %a trick not to exclude T(0)
K(1,1) = K(1,1)+large_number;
T_FEM = linsolve(K, F)% solve for T
%
x_coord_FEM = 0:l/(nodes-1):1; %coordinates of nodes
figure;
plot(x_coord_FEM, T_FEM,'k--','DisplayName',['FEM, S=',num2str(S),' k=',...
    num2str(k)]); hold on %plot FEM solution
x_coord_solution = 0:0.01:1; %a grid to compute direct solution on
solution = -S/(2*k)*x_coord_solution.^2+S/k*x_coord_solution; %values of
    the direct
plot(x_coord_solution, solution,'DisplayName', ['Solution, S=',num2str(S),'
    k=',...
    num2str(k)]); hold on %visualize the direct solution
xlabel('x');
ylabel('T(x)');
legend({},'Location','northwest');
```

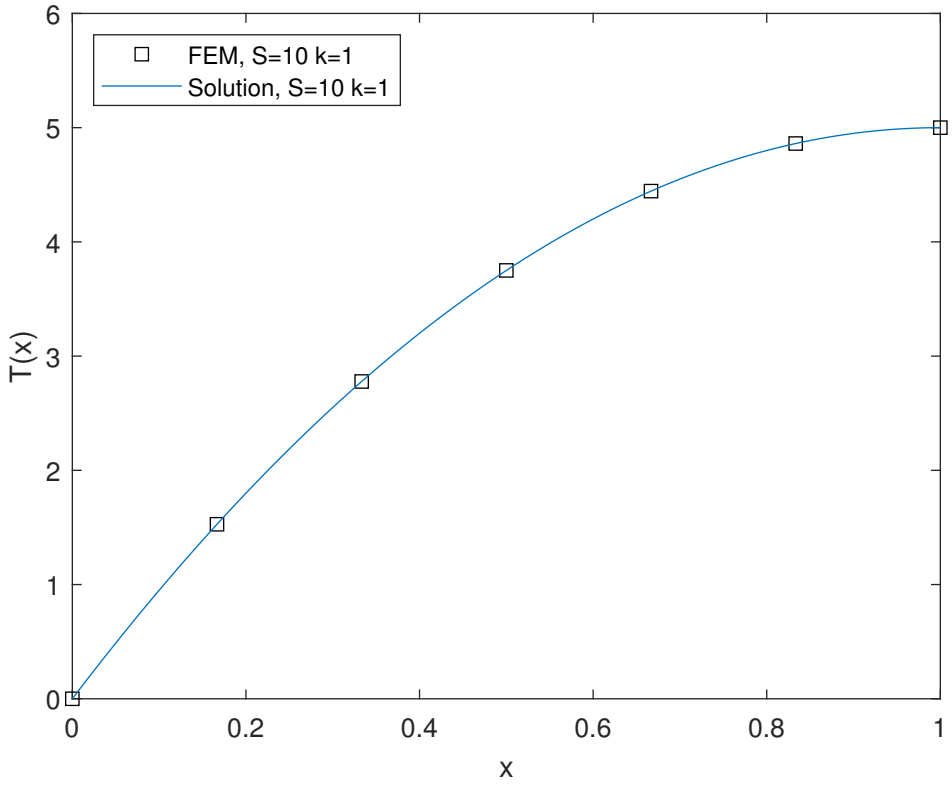


Figure 14: Analytical Solution and FEM Approximation

Midterm Exam

Q1

a In CT images it is easy to differentiate different types of soft tissues.

False, soft tissues have a similar effective attenuation coefficient. Therefore, it is not easy to distinguish between soft tissues, but soft and hard tissue.

b There are no biological hazards due to x-ray exposure.

False, due to the high energy of x-ray beams, which causes ionization of tissue, the biological risk is of a mutagenic resp. cancerogenic manner.

c The 2D Fourier transform of the image can be obtained from the acquired sinogram.

True, the Fourier slice theorem specifies exactly this issue.

d No representative images are rendered with back-projection when the views are not filtered.

True, in order to enhance the contrast of features in the picture you have to filter it, though.

e The total attenuation of a non-monochromatic x-ray beam increases linearly with depth for a uniform medium.

True,

$$\mathbf{I} = I_0 \; e^{-\mu_1 L_1} e^{-\mu_2 L_2} e^{-\mu_3 L_3} = I_0 e^{\underbrace{-\mu_1 L_1 - \mu_2 L_2 - \mu_3 L_3}_{\text{linear terms}}} \tag{28}$$

Q2

a

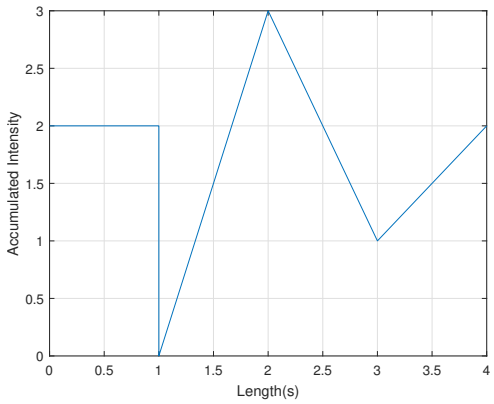


Figure 15: Signal shape at 0°

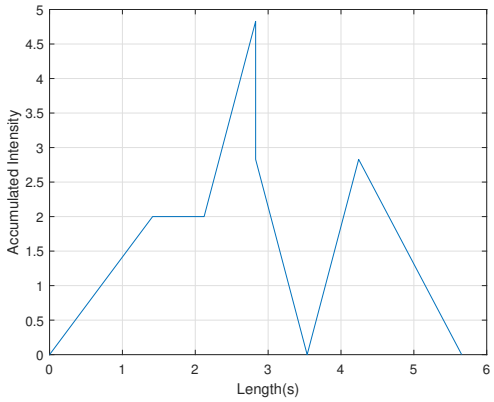


Figure 16: Signal shape at 45°

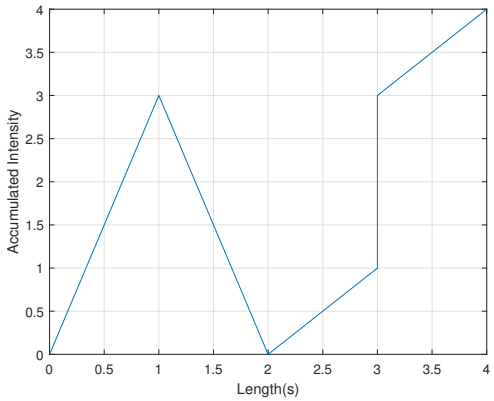


Figure 17: Signal shape at 90°

b

As seen in the figures 15,16 and 17, the highest peak of the accumulated intensity is at the signal from 45°. The position of $2\sqrt{2}$ on the x-axis of plot 16 corresponds thereby to the complete diagonal ray from the lower left to the upper right corner in the lower scheme (Fig. 18).

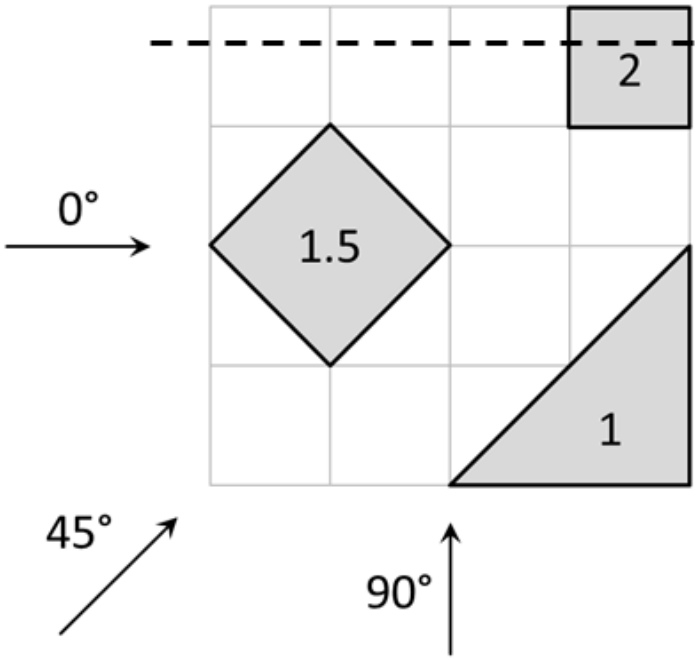


Figure 18: Perfect Ray

Q3

Calculate resolution of an optical microscope given the following characteristics:

- Magnification $\mathbf{M} = 60\text{x}$
- Lens Diameter $\mathbf{D} = 9.4\,\text{mm}$
- Focal Length $\mathbf{f} = 2.4\,\text{mm}$
- Immersion Medium: oil, $\mathbf{n} = 1.33$
- Illumination $\boldsymbol{\lambda} = 560\,\text{nm}$

$$
\begin{aligned}
d &= \frac{\lambda}{2\text{NA}} && (29)\\
&= \frac{\lambda}{2n\sin\theta} && (30)\\
&= \frac{\lambda}{2n\left(\frac{0.5D}{\sqrt{(0.5D)^2+f^2}}\right)} && (31)\\
&= \frac{560\,\text{nm}}{2\ \cdot\ 1.33\ \cdot\ \left(\frac{0.5\cdot 9.4\,\text{mm}}{\sqrt{(0.59.4\,\text{mm})^2+(2.9\,\text{mm}^2}}\right)} && (32)\\
&= 0.25\,\mu\text{m} && (33)
\end{aligned}
$$

Q4

Which parameters influence the depth (slice thickness) that can be efficiently visualized by an optical microscope? How could high resolution images of thicker slices be obtained by a conventional microscope?

The main parameter to influence the penetration into the sample is the scattering inside. Furthermore, the wavelength and light intensity are also crucial physical parameters. At the optical microscope Numerical Aperture (resp. the diffraction), the optical density and reflectance of the sample influence imaging.
To overcome these hurdles an increase in NA resp. light intensity and/or a decrease in wavelength could yield more resolution. Generally speaking, the the overall diffraction limit (Abbe) has to be maxed out.

Q5

a The width of a signal in the time domain is proportional to the width in the frequency domain.

True, the width in time domain is indirect proportional to a signal in the frequency domain.

b The central pixel in the 2D k-space (k_x=0,k_y=0) corresponds to a uniform image in the space domain.

True, the center of the 2D k-space is a uniform image. One pixel corresponds to a constant picture frequency which means a uniform image in the space domain.

c A one-dimensional wave characterized by $u = u_0 \cos(5x + 3t2.5)$ propagates towards the negative x direction.

False, it propagates in the positive x-Direction (+3t Term)

d Deconvolution is particularly suited for systems with a narrow frequency response.

True, a narrow frequency response implies that the image possesses only few strong gradients and therefore the image has poor feature edge quality. A deconvolution then sharpens the image.

e For a given aperture, diffraction is more prominent in the ultraviolet than in the infrared.

False, diffraction increases with increasing wavelength. The focal size of a microscope is ultimately diffraction limited and proportional to the wavelength.

$$d = \frac{\lambda}{2NA} \tag{34}$$

Q6

For a diffuse medium with prominent Rayleigh scattering, provide the wavelength dependence of the effective attenuation coefficient in a spectral range where the absorption coefficient increases linearly with wavelength.
Effective Attenuation Coefficient:

$$\mu_{eff} = \sqrt{\frac{\mu_a}{D}}, \text{ with } D = \frac{1}{3(\mu_a + \mu_s(1-g)}, \text{ with } g \in [0.65, 0.95] \tag{35}$$

$$= \sqrt{\mu_a \ \cdot \ 3(\mu_a + \mu_s(1-g))} \tag{36}$$

According the diffusion approximation that can be applied to solve the RTE scattering events are dominant over the absorption $\mu_s \gg \mu_a$. We therefore neglect the additive μ_a.
Furthermore, Rayleigh scattering effects are dominant over others, which simplifies the equation according to $\mu_s \propto \lambda^{-4}$. With the wavelength dependence of the absorption coefficient $\mu_a \propto \lambda$, this leads to:

$$\mu_{eff} = \sqrt{3\mu_a\mu_s(1-g)} \tag{37}$$

$$= \sqrt{3\lambda\lambda^{-4}(1-g)} \tag{38}$$

$$\Rightarrow \mu_{eff} \propto \lambda^{-3/2} \tag{39}$$

Q7

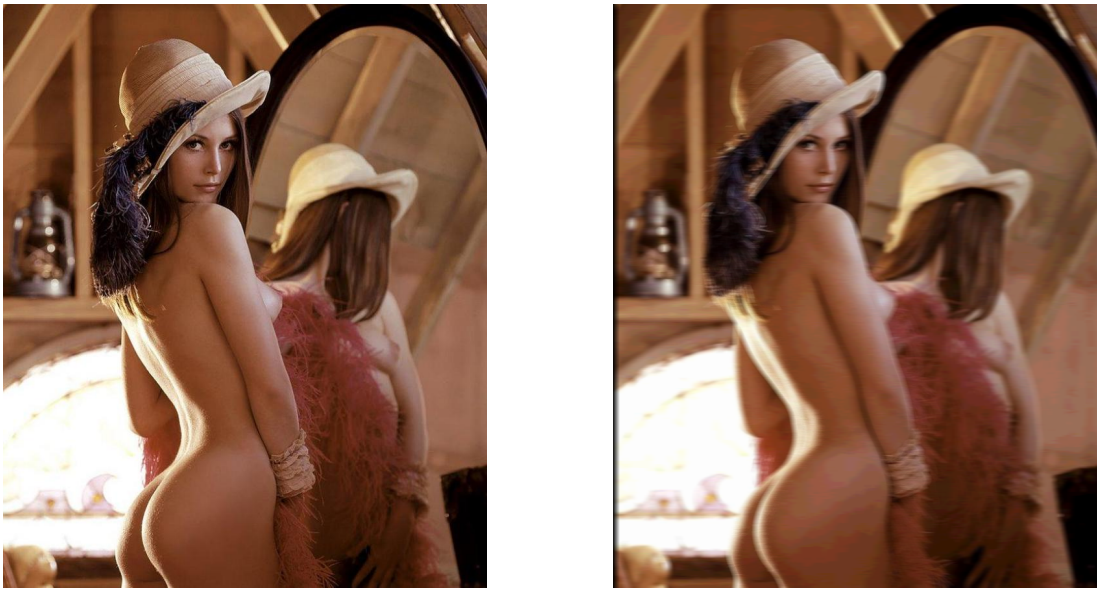
An ideal camera moves in the x direction with constant velocity ν during exposure. If the exposure time is ν derive an expression for the point spread function and the transfer function of the imaging system. Demonstrate the effect of such motion on an image of your choice, assuming that $\nu T = 10$ pixels. Provide Matlab code.

The PSF can be computed by:

$$f(x,y) = \begin{cases} \frac{1}{v\ T_{end}}, & for\ 0 \leq x \leq vT_{end} \\ 0, & otherwise \end{cases} \tag{40}$$

Then, the images are overlapped along the interval by tranlation of the original picture.

```
vT = 10;
img_orig = importdata('lena_full.png');
img = img_orig/(vT+1);
for i = 1:vT
    img = img + imtranslate(img_orig/(vT+1),[i 0]);
end
figure;
imshow(img_orig);
figure;
imshow(img);
imwrite(img,'lena_blurry.png');
```



(a) Original Figure (b) Blurry Picture

Figure 19: Pictures before and after processing

Q8

An infinitely large non absorbing medium with a high concentration of small spherical scatters is irradiated with blue, yellow and red light. Which beam penetrates deeper? Why? Give an analytical explanation.

As shown in the Planck-Einstein-Relation $E = h\frac{c}{\lambda}$, the light with low wavelength contains more energy than light with higher wavelengths. Therefore, photons with high energy scatter more often in tissue than photons with lower energy. Hence, even with anisotropic scattering high-energy photons pass more times the mean free path between to scattering events than low-energy photons.

Q9

Derive central, forward and backward finite difference approximations for the second order derivative $f''(x)$ of a function of one variable $f(x)$? What is the truncation error of the derived approximations?
Forward Difference:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \tag{41}$$

$$Then,\ f''(x) \approx \frac{f'(x+h) - f'(x)}{h} \tag{42}$$

$$= \frac{\frac{f(x+2h)-f(x+h)}{h} - \frac{f(x+h)-f(x)}{h}}{h} \tag{43}$$

$$= \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} \tag{44}$$

Truncation Error: $O(h)$
Backward Difference:

$$f'(x) \approx \frac{f(x) - f(x+h)}{h} \tag{45}$$

$$Then,\ f''(x) \approx \frac{f'(x) - f'(x-h)}{h} \tag{46}$$

$$= \frac{\frac{f(x)-f(x-h)}{h} - \frac{f(x-h)-f(x-2h)}{h}}{h} \tag{47}$$

$$= \frac{f(x) - 2f(x-h) + f(x-2h)}{h^2} \tag{48}$$

Truncation Error: $O(h)$

Central Difference:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \tag{49}$$

$$Then,\ f''(x) \approx \frac{f'(x+2h) - f'(x-h)}{2h} \tag{50}$$

$$= \frac{\frac{f(x+2h)-f(x)}{2h} - \frac{f(x)-f(x-2h)}{h}}{2h} \tag{51}$$

$$= \frac{f(x+2h) - 2f(x) + f(x-2h)}{4h^2} \tag{52}$$

$$\overset{h'=\frac{h}{2}}{=} \frac{f(x+h') - 2f(x) + f(x-h')}{h'^2} \tag{53}$$

Truncation Error: $O(h^2)$

Q10

Using the central difference approximation, derive a coefficient matrix **A** corresponding to the following boundary problem:

$$f''(x) = (1 - x\sin x)f(x),\ x \in [0, 2] \tag{54}$$
$$f(0) = \alpha \tag{55}$$
$$f(2) = \beta \tag{56}$$

with the equation 53 and 54

$$(1 - x\sin x)f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \tag{57}$$
$$(1 - x_i\sin(x_i))f(x_i) \underset{x-h=x_{i-1}}{\overset{x+h=x_{i+1}}{=}} \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} \tag{58}$$
$$(2 + h^2 - h^2x_i\sin(x_i))f(x_i) - f(x_{i+1}) - f(x_{i-1}) = 0 \tag{59}$$
$$\text{Substitute } \xi\ = (2 + h^2 - h^2x_i\sin(x_i)) \tag{60}$$

$$\begin{bmatrix} \xi & -1 & 0 & \dots & \dots & \dots & 0 \\ -1 & \xi & -1 & 0 & & & \vdots \\ 0 & -1 & \xi & -1 & 0 & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & & & 0 & -1 & \xi & -1 \\ 0 & \dots & \dots & \dots & 0 & -1 & \xi \end{bmatrix} \begin{bmatrix} f_2 \\ f_3 \\ f_4 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \\ \beta \end{bmatrix} \tag{61}$$

3 Homework 3

3.1 Assignment: Newton's Method

This method is a standard way in mathematics to find zeros of a function numerically. Graphically speaking, one computes the zero-crossing point of a tangent to a point on the function itself in each iteration. The projection of the zero onto the function serves as the new starting point in the following iteration. In each step, the algorithm approximates the zero more precisely until the method is aborted below a certain error. For the exercise, the starting point (62) for a function (63) as well as the precision (64) and its abortion criterion (67) are given. In this case, we do not want to find the zeros of a provided function, but of its derivative in order to find the extremum. Therefore, substituting equation 65 with $g(x) = f'(x)$ simplifies to eq. 66.

$$x_0 = 0.5 \tag{62}$$

$$f(x) = \frac{x}{2} - \sin(x) \tag{63}$$

$$\epsilon = 1 \times 10^{-5} \tag{64}$$

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)} \tag{65}$$

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)} \tag{66}$$

$$\epsilon > |x_{k+1} - x_k| \tag{67}$$

To the specified maximal error, the algorithm converges in five steps. Figure 20 shows the function, the starting points (grey) and the approximated zeros (red). The last iterations cannot be distinguished in the plot, because the approximating lies already close to the zero of the function. By inspection of the plot, we can validate that the found extremum is a minimum.

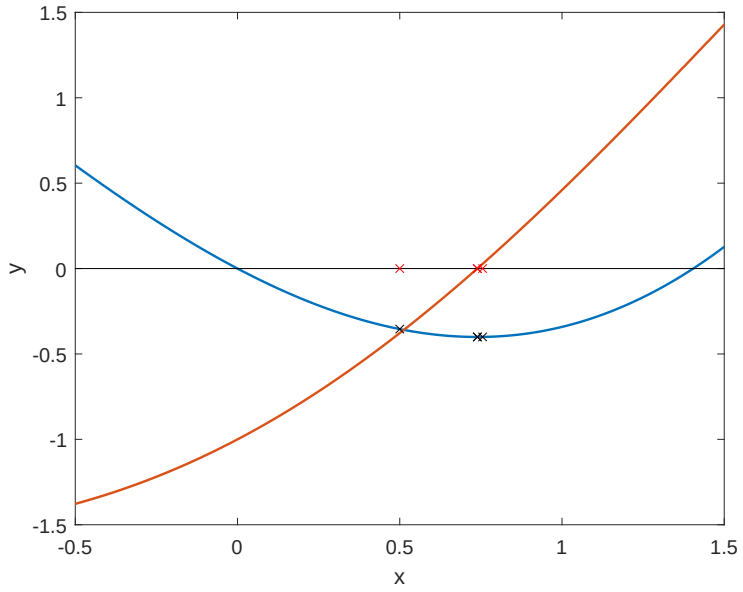


Figure 20: Approximated zeros of a function by the Newton Method. The orange Graph depicts the derivative of the blue function (63).

```
syms f(x)
f(x) =  ((x.^2)./2) - sin(x);
df = diff(f,x);
ddf = diff(df,x);
x_prev = 0.5;
e = 1;
iterations = 1;
x = -0.5:0.001:1.5;
figure;
plot(x,double(f(x)),'LineWidth',1.2,'DisplayName','f(x)');
hold on
plot(x,double(df(x)),'LineWidth',1.2,'DisplayName','f''(x)');
l = refline([0 0]);
l.Color = [0,0,0];
l.LineStyle = '-';
l.LineWidth = 0.1;
while e >= 1e-5
    x_next = x_prev - (double(df(x_prev))/double(ddf(x_prev)));
    e = abs(x_next - x_prev);
    plot(x_prev,double(f(x_prev)),'kx');
    plot(x_prev,0,'rx');
    x_prev = x_next;
    iterations = iterations + 1;
end
xlabel('x');
ylabel('y');
print('fig/NewtonMethod.pdf','-dpdf');
```

3.2 Assignment: An imaging problem

Consider the following imaging problem (see Fig. 1): 3 light emitters ($E_j,\ j \in [1,3]$) are positioned in known locations in a room. In order to find the power emanating from the emitters, 4 detectors ($D_i,\ i \in [1,4]$) are positioned at certain locations. The power I_d measured by the i-th detector is given by:

$$I_{d,i} = \sum_{j=0}^{N} \frac{I_{e,j}}{|r_{ij}|^2}, \tag{68}$$

$$r_{ij} = \sqrt{(D_i - E_j)^2} \tag{69}$$

$$\text{Imaging Forward Problem} : \mathbf{AI_e} = \mathbf{I_d} \tag{70}$$

$$\text{Imaging Inverse Problem} : \mathbf{I_e} = \mathbf{A^{-1}I_d} \tag{71}$$

$$\tag{72}$$

where $\mathrm{I}_{e,j}$ is the power of the j-th intensity source and r_{ij} is the Distance between the detector and emitter.

3.2.1 Build the matrix for the imaging problem defined above

```
%% Assignment 2 An imaging problem
% a
s.orig.D = [-1,1;1,1;1,-1;-1,-1];
s.orig.E = [-0.5,0.5;0,0;0.5,0.5];
s.orig.I_e = [4,3,1];
s.orig.A = zeros(length(s.orig.D),length(s.orig.E));

for i = 1:length(s.orig.D)
    for j = 1:length(s.orig.E)
        s.orig.A(i,j) = 1./sqrt(sum((s.orig.D(i,:)-s.orig.E(j,:)).^2));
    end
end
```

3.2.2 What are the detected values for the source values $[\mathbf{I_{e,1}}, \mathbf{I_{e,2}}, \mathbf{I_{e,3}}] = [4, 3, 1]$? Consider these detected values as the measured data.

```
%% b
s.orig.I_d = s.orig.A * s.orig.I_e';
```

$$\mathbf{A} = \begin{pmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}\sqrt{5}}{5} \\ \frac{\sqrt{2}\sqrt{5}}{5} & \frac{\sqrt{2}}{2} & \sqrt{2} \\ \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}\sqrt{5}}{5} \\ \frac{\sqrt{2}\sqrt{5}}{5} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{3} \end{pmatrix} \tag{73}$$

$$\mathbf{I_d} = \begin{pmatrix} 8.411 \\ 6.065 \\ 4.639 \\ 5.123 \end{pmatrix} \tag{74}$$

3.2.3 Calculate the solution of the inverse problem based on the pseudo-inverse of the matrix

```
%% c
tol = 1e-5;
[s.orig.moore.I_e, s.orig.moore.A_inv] = SolPseudoInvMoore(s.orig.A,s.orig.
    I_d,tol);
```

```
function [I_e_moore,A_inv_moore] = SolPseudoInvMoore(A,b,tol)
A_inv_moore = pinv(A,tol);
I_e_moore = A_inv_moore * b;
end
```

$$\mathbf{A_{inv,moore}} = \begin{pmatrix} 1.144 & -0.08321 & -0.6567 & -0.4039 \\ -0.8279 & -0.8279 & 1.535 & 1.535 \\ -0.08321 & 1.144 & -0.4039 & -0.6567 \end{pmatrix} \tag{75}$$

$$\mathbf{I_{e,moore}} = \begin{pmatrix} 4.0 \\ 3.0 \\ 1.0 \end{pmatrix} \tag{76}$$

3.2.4 Calculate the solution of the inverse problem based on the singular value decomposition (SVD) of the matrix

$$\mathbf{A_{inv,SVD}} = \begin{pmatrix} 1.144 & -0.08321 & -0.6567 & -0.4039 \\ -0.8279 & -0.8279 & 1.535 & 1.535 \\ -0.08321 & 1.144 & -0.4039 & -0.6567 \end{pmatrix} \tag{77}$$

$$\mathbf{I_{e,SVD}} = \begin{pmatrix} 4.0 \\ 3.0 \\ 1.0 \end{pmatrix} \tag{78}$$

```
%% d
[s.orig.SVD.I_e, s.orig.SVD.A_inv] = SolPseudoInvSVD(s.orig.A,s.orig.I_d,0)
    ;
```

```
function [I_e_SVD,A_inv_SVD] = SolPseudoInvSVD(A,b,truncate)
[U,S,V] = svd(A);

[m,n] = size(U);
U = U(1:m,1:n-truncate);

[m,n] = size(V);
V = V(1:m,1:n-truncate);

[m,n] = size(S);
S = S(1:m-truncate,1:n-truncate);

S_degger = S;
S_degger(S_degger > 0) = 1./S_degger(S_degger > 0);
A_inv_SVD = V * S_degger' * U';
I_e_SVD = A_inv_SVD * b;
end
```

3.2.5 Calculate the solution of the inverse problem based on an iterative inversion algorithm (lsqr)

$$\mathbf{I_{e,lsqr}} = \begin{pmatrix} 4.0 \\ 3.0 \\ 1.0 \end{pmatrix} \tag{79}$$

```
%% e
s.orig.LSQR.I_e = lsqr(s.orig.A,s.orig.I_d);
```

3.2.6 Repeat steps c-d when noise is added to the measurements. Compare the results obtained by changing the locations of the detectors. Comment on the conditioning of the problem.

Noise with an amplitude of 1% around actual value was added to the detection.
After moving the detectors three time the distance away from the sources, the error does not influence the third digit, but also the second and first after the point.
The posed problem is **well-posed**, because the three necessary conditions are satisfied:

1. Existence $rank(A) = rank(A|I_{d,noise}) = 3$
2. Uniqueness $rank(A) = size(A, 2) = 3$
3. Stability $pinv(A) * A = \mathbf{I}$

The conditions number of MATLAB's intrinsic *cond()* then computes for the original A to 7.7272 and for the moved detectors to 25.01. These numbers are both larger than 1, which shows, that the matrix inversion is sensitive to small changes before the inversion. Terefore we call both matrices **ill-conditioned**.

$$\mathbf{A_{inv,moore,noise}} = \begin{pmatrix} 1.144 & -0.08321 & -0.6567 & -0.4039 \\ -0.8279 & -0.8279 & 1.535 & 1.535 \\ -0.08321 & 1.144 & -0.4039 & -0.6567 \end{pmatrix} \tag{80}$$

$$\mathbf{I_{e,moore,noise}} = \begin{pmatrix} 3.998 \\ 3.0 \\ 0.9992 \end{pmatrix} \tag{81}$$

$$\mathbf{A_{inv,SVD,noise}} = \begin{pmatrix} 1.144 & -0.08321 & -0.6567 & -0.4039 \\ -0.8279 & -0.8279 & 1.535 & 1.535 \\ -0.08321 & 1.144 & -0.4039 & -0.6567 \end{pmatrix} \tag{82}$$

$$\mathbf{I_{e,SVD,noise}} = \begin{pmatrix} 3.998 \\ 3.0 \\ 0.9992 \end{pmatrix} \tag{83}$$

Detectors moved 3-times farer away:

$$\mathbf{I_{e,moore,noise}} = \begin{pmatrix} 4.041 \\ 2.836 \\ 1.122 \end{pmatrix} \tag{84}$$

$$\mathbf{I_{e,SVD,noise}} = \begin{pmatrix} 3.901 \\ 2.976 \\ 1.121 \end{pmatrix} \tag{85}$$

```
%% f
interval = [-0.1,0.1];
noise = interval(1) + (interval(2) - interval(1)) * rand(length(s.orig.I_d)
    ,1);
s.noise.I_d = s.orig.I_d + noise;
[s.noise.moore.I_e,s.noise.moore.A_inv] = SolPseudoInvMoore(s.orig.A,s.
    noise.I_d,tol);
[s.noise.SVD.I_e, s.noise.SVD.A_inv] = SolPseudoInvSVD(s.orig.A,s.noise.I_d
    ,0);
% Compare Results with detectors manifold times farer away
factor = 3;
s.orig.moved.D = s.orig.D * factor;
for i = 1:length(s.orig.D)
```

```
    for j = 1:length(s.orig.E)
        s.orig.moved.A(i,j) = 1./sqrt(sum((s.orig.moved.D(i,:)-s.orig.E(j
            ,:)).^2));
    end
end
s.noise.moved.I_d = (s.orig.moved.A * s.orig.I_e') + noise;
[s.noise.moved.moore.I_e,s.noise.moved.moore.A_inv] = ...
    SolPseudoInvMoore(s.orig.moved.A,s.noise.moved.I_d,tol);
[s.noise.moved.SVD.I_e, s.noise.moved.SVD.A_inv] = ...
    SolPseudoInvSVD(s.orig.moved.A,s.noise.moved.I_d,0);
% Conditioning
cond_num = cond(s.orig.A); %7.7273 --> ill-conditioned
cond_num_hat = cond(s.orig.moved.A); % 25.01 --> ill-conditioned
```

3.2.7 Add 2 other sources (E_4 and E_5) close to E_3 in the problem above.

$$\mathbf{A} = \begin{pmatrix} 1.414 & 0.7071 & 0.6325 & 0.6063 & 0.6565 \\ 0.6325 & 0.7071 & 1.414 & 1.768 & 1.179 \\ 0.4714 & 0.7071 & 0.6325 & 0.6063 & 0.6565 \\ 0.6325 & 0.7071 & 0.4714 & 0.4419 & 0.5051 \end{pmatrix} \tag{86}$$

```
%% g
distance = [0.1, 0.1];
s.orig.added.E = [s.orig.E;s.orig.E(3,:) + distance;s.orig.E(3,:) -
    distance];
for i = 1:length(s.orig.D)
    for j = 1:length(s.orig.added.E)
        s.orig.added.A(i,j) = 1./sqrt(sum((s.orig.D(i,:)-s.orig.added.E(j
            ,:)).^2));
    end
end
```

3.2.8 Calculate the solution of the inverse problem with and without adding noise. Assume source values of $[\mathbf{I_{e,1}}, \mathbf{I_{e,2}}, \mathbf{I_{e,3}}, \mathbf{I_{e,4}}, \mathbf{I_{e,5}}] = [\mathbf{4}, \mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}]$

$$\mathbf{I_{e,noise}} = \begin{pmatrix} 4.003 \\ 2.992 \\ 0.9956 \\ 1.001 \\ 1.009 \end{pmatrix} \tag{87}$$

$$\mathbf{I_d} = \begin{pmatrix} 9.674 \\ 9.012 \\ 5.902 \\ 6.07 \end{pmatrix} \tag{88}$$

$$\mathbf{I_{d,noise}} = \begin{pmatrix} 9.675 \\ 9.014 \\ 5.902 \\ 6.068 \end{pmatrix} \tag{89}$$

```
%% h
s.orig.added.I_e = [s.orig.I_e'; 1; 1];
s.orig.added.I_d = s.orig.added.A * s.orig.added.I_e;
noise = interval(1) + (interval(2) - interval(1)) * rand(length(s.orig.
    added.I_e),1);
s.noise.added.I_e = s.orig.added.I_e + noise;
s.noise.added.I_d = s.orig.added.A * s.noise.added.I_e;
```

3.2.9 For the problem in g, perform the inversion with standard SVD and truncated SVD for different levels of noise. Comment on the results.

Truncation: 0		
Noise:0%	Noise:100%	Noise: 1000%
$I_e = \begin{pmatrix} 4.0 \\ 2.998 \\ 0.9516 \\ 1.018 \\ 1.031 \end{pmatrix}$	$I_e = \begin{pmatrix} 4.026 \\ 2.92 \\ 0.9775 \\ 1.001 \\ 1.077 \end{pmatrix}$	$I_e = \begin{pmatrix} 4.265 \\ 2.218 \\ 1.211 \\ 0.8445 \\ 1.49 \end{pmatrix}$
Truncation: 1		
Noise:0%	Noise:100%	Noise: 1000%
$I_e = \begin{pmatrix} 3.979 \\ 2.692 \\ 1.169 \\ 0.5941 \\ 1.598 \end{pmatrix}$	$I_e = \begin{pmatrix} 4.008 \\ 2.643 \\ 1.174 \\ 0.6178 \\ 1.589 \end{pmatrix}$	$I_e = \begin{pmatrix} 4.264 \\ 2.209 \\ 1.218 \\ 0.8317 \\ 1.507 \end{pmatrix}$
Truncation: 2		
Noise:0%	Noise:100%	Noise: 1000%
$I_e = \begin{pmatrix} 4.149 \\ 2.403 \\ 1.179 \\ 0.7079 \\ 1.53 \end{pmatrix}$	$I_e = \begin{pmatrix} 4.149 \\ 2.404 \\ 1.182 \\ 0.7123 \\ 1.533 \end{pmatrix}$	$I_e = \begin{pmatrix} 4.144 \\ 2.413 \\ 1.211 \\ 0.7512 \\ 1.555 \end{pmatrix}$

Table 1: Different solutions of the inverse problem, with different margins of truncation for the SVD matrices and different levels of noise. Noise levels are computed relative to the noise level used in the exercises before.

```
%% i
s.orig.added.SVD.truncation = 0:1:2;
s.noise.added.SVD.I_e = cell(1,length(s.orig.added.SVD.truncation));
s.noise.added.SVD.A = cell(1,length(s.orig.added.SVD.truncation));

Level = noise*[0,1,10];

for i = s.orig.added.SVD.truncation
    for j = 1:size(Level,2)
     s.noise.added.SVD.I_d = s.orig.added.A * (s.orig.added.I_e + Level(:,j
         ));
  [s.noise.added.SVD.I_e{i+1},s.noise.added.SVD.A{i+1}] = SolPseudoInvSVD(s
      .orig.added.A,s.noise.added.SVD.I_d,i);
    digits(4);
    sprintf('Truncation:%d, Noise Level:%d\n',i,Level(1,j)./noise(1))
    latex(vpa(sym(s.noise.added.SVD.I_e{i+1})))
    end
end
```

3.2.10 For the problem in g, compute the solution using Tikhonov regularization. Using Lcurve, determine an optimal regularization parameter in range $[10^{-4}; 10^{-4} \cdot 2^{14}]$ (simply double your regularization parameter every iteration). Show your L-curve, explain its meaning and how the optimal regularization parameter was selected.

An L-curve shows the behavior of the approximation error to the norm of the augmented regularization matrix L (Numerical stability of method). By choosing the regularization parameter λ in the corner of this graph, we can obtain a suffictient small numerical error and simultaneously minimize the numerical instabilities.

$$\mathbf{L} = \begin{pmatrix} 0.9595 & -0.9144 & -0.5103 & 0.2028 & -0.6406 \\ 0.2704 & 1.692 & -1.536 & -0.7097 & -1.102 \\ -0.6313 & -0.03136 & 1.313 & -0.3647 & -0.2856 \\ 0.5552 & 0.2245 & -0.1433 & 1.574 & 0.107 \\ -0.4948 & -0.1816 & -0.09867 & 0.2989 & 1.471 \end{pmatrix} \tag{90}$$

$$\lambda = 0.071194016876375 \tag{91}$$

$$\mathbf{q}(\lambda) = \begin{pmatrix} 4.004 \\ 2.293 \\ 2.556 \\ -0.6331 \\ 1.895 \end{pmatrix} \tag{92}$$

Figure 21: Typical L-curve scheme with the smallest possible lambda with the most stable value.

```
%% Tikhonov Regularization
M = s.orig.added.A;
s_tik = s.noise.added.I_d;
range = 1e-4 * 2.^(0:1:14);
x = zeros(length(range));
y = zeros(length(range));
L = eye(5) + 0.5*randn(5);
q_lam = @(lam) (M'*M+lam*(L'*L))\M'*s_tik;
mq_norm = @(lam) norm(M*q_lam(lam)-s_tik,2);
lq_norm = @(lam) norm(L*q_lam(lam),2);
q_norm = @(lam) norm(q_lam(lam),2);
x = arrayfun(mq_norm,range);
y = arrayfun(lq_norm,range);
dy = [0,diff(y)];
ddy = [0, diff(dy)];
dddy = [0, diff(ddy)];
max_dist = max(dddy);
fig =figure;
plot(x,y,'x-','LineWidth',1.2,'DisplayName','L-curve');
hold on
legend;
xlabel('||Mq(\lambda) - s||');
ylabel('||Lq(\lambda)||');
pos_opt = find(max(dddy)== dddy)+2;
plot(x(pos_opt),y(pos_opt),'ro','MarkerSize',10,'DisplayName',...
'Optimal regularization parameter');
```

```
27 q_lam(x(pos_opt))
28 print(fig,'fig/Tikhonov.eps','-dpdf');
```

4 Homework 4

4.1 Bloch's Equation

The pulse sequence diagram of an inversion-recovery spin echo sequence is given below. The sequence is composed of a π_y pulse, a $-\frac{\pi}{2}_y$ pulse and a π_x pulse and it uses the following parameters $TE = 40\,\text{ms}$, $TI = 300\,\text{ms}$ and $TR = 1500\,\text{ms}$. The sequence is employed to image a tissue with $T_1 = 1000\,\text{ms}$ and $T_2 = 80\,\text{ms}$.

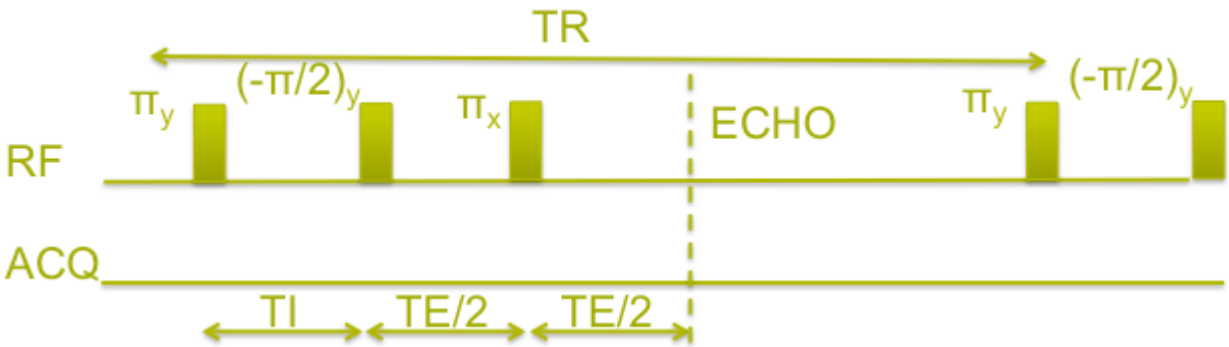


Figure 22

4.1.1 Consider a spin that precesses with off-resonance frequency equal to 1 Hz. Plot the evolution of the angle of the transverse magnetization for the above spin for the 2nd TR. Show that the spin echo is formed at time TE from the $-\frac{\pi}{2}_y$ RF pulse.

```
%% Assignment 1:
close all;
clear all;
df = 1; % Hz off-resonance.
T1 = 1000; % ms.
T2 = 80; % ms.
TE = 40; % ms.
TR = 1500; % ms.
TI = 300; % ms.
flip1 = pi; % pi y pulse
flip2 = -pi/2;% -pi/2 y pulse
flip3 = pi; % pi x pulse
dT = 1;
N_tr = round(TR/dT);
N_ti = round(TI/dT);
N_te = round(TE/dT);
N_tehalf = round(TE/(2*dT));
N_ex = 5; % number of RF excitations.
% magnetization vector
M=zeros(3,N_ex*N_tr);
% initial magnetization
M(:,1) = [0;0;1];
% Bloch equation matrices
R_flip1 = yrot(flip1);
```

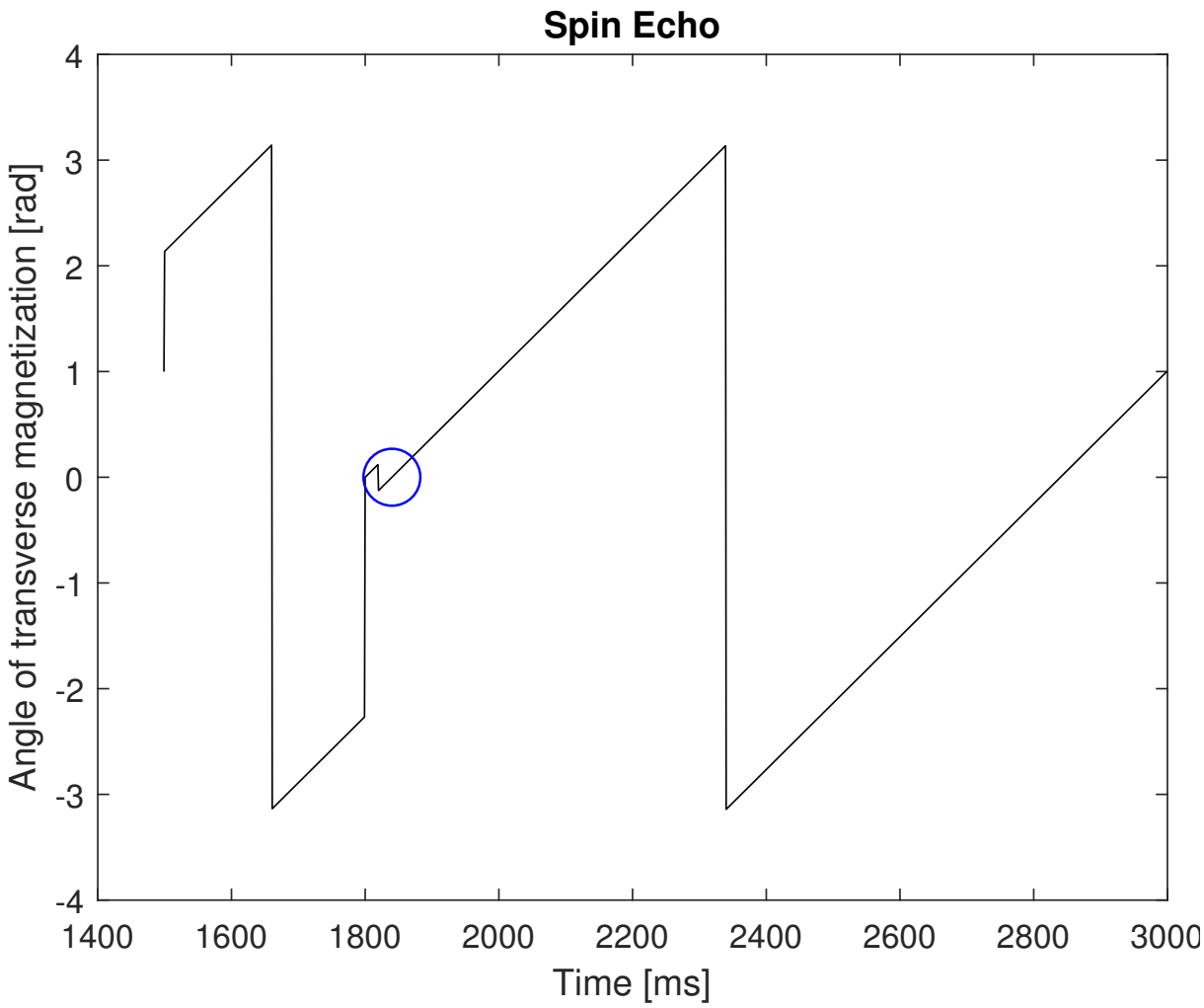


Figure 23: A spin echo is formed at $T_{echo} = TR + TI + TE = 1840\ ms$ (red circle).

```
R_flip2 = yrot(flip2);
R_flip3 = xrot(flip3);
[A1,B1] = freeprecess(dT,T1,T2,df);
% simulate Bloch equations
M_count=1;
% simulate Nex TRs
for n=1:N_ex
% pi RF excitation
M(:,M_count) = R_flip1*M(:,M_count);
% free-precession and relaxation over period TI
for k=1:N_ti
M_count=M_count+1;
M(:,M_count)=A1*M(:,M_count-1)+B1;
end
% pi/2 RF excitation
M(:,M_count) = R_flip2*M(:,M_count);
% free-precession and relaxation over period TR-TI
```

```
for k=1:N_tehalf
M_count=M_count+1;
M(:,M_count)=A1*M(:,M_count-1)+B1;
end
% pi RF excitation
M(:,M_count) = R_flip3*M(:,M_count);
% free-precession and relaxation over period TR-TI-TE/2
for k=1:(N_tr-N_ti-N_tehalf)

M_count=M_count+1;
M(:,M_count)=A1*M(:,M_count-1)+B1;
end
end
time = [0:M_count-1]*dT;
%% verify formation of echo by plotting angle of transverse magnetization
% for second TR
M_trans=M(1,:)+j*M(2,:);
figure;
plot(time(N_tr:2*N_tr),angle(M_trans(N_tr:2*N_tr)),'k-');
hold on;
scatter(N_tr+TI+TE,angle(M_trans(N_tr+TI+TE+1)), 300, 'b');
xlabel('Time [ms]');
ylabel('Angle of transverse magnetization [rad]');
title('Spin Echo');
print('img/Spin_Echo.eps','-depsc');
```

4.1.2 Plot the evolution of longitudinal and transverse magnetization for 5 TRs. How many TRs are required in order to establish a steady-state in the magnetization evolution?

The stead state in the magnetization evolution is reached after 2 TR. (Fig. 24)

```
%% plot magnetization evolution
figure;
time = [0:M_count-1]*dT;
plot(time,M(1,:),'b-',time,M(2,:),'g-',time,M(3,:),'r-');
legend('M_x','M_y','M_z');
xlabel('Time [ms]');
ylabel('Magnetization');
axis([min(time) max(time) -1 1]);
title('Inversion Recovery');
print('img/Inversion_Recovery.eps','-depsc');
```

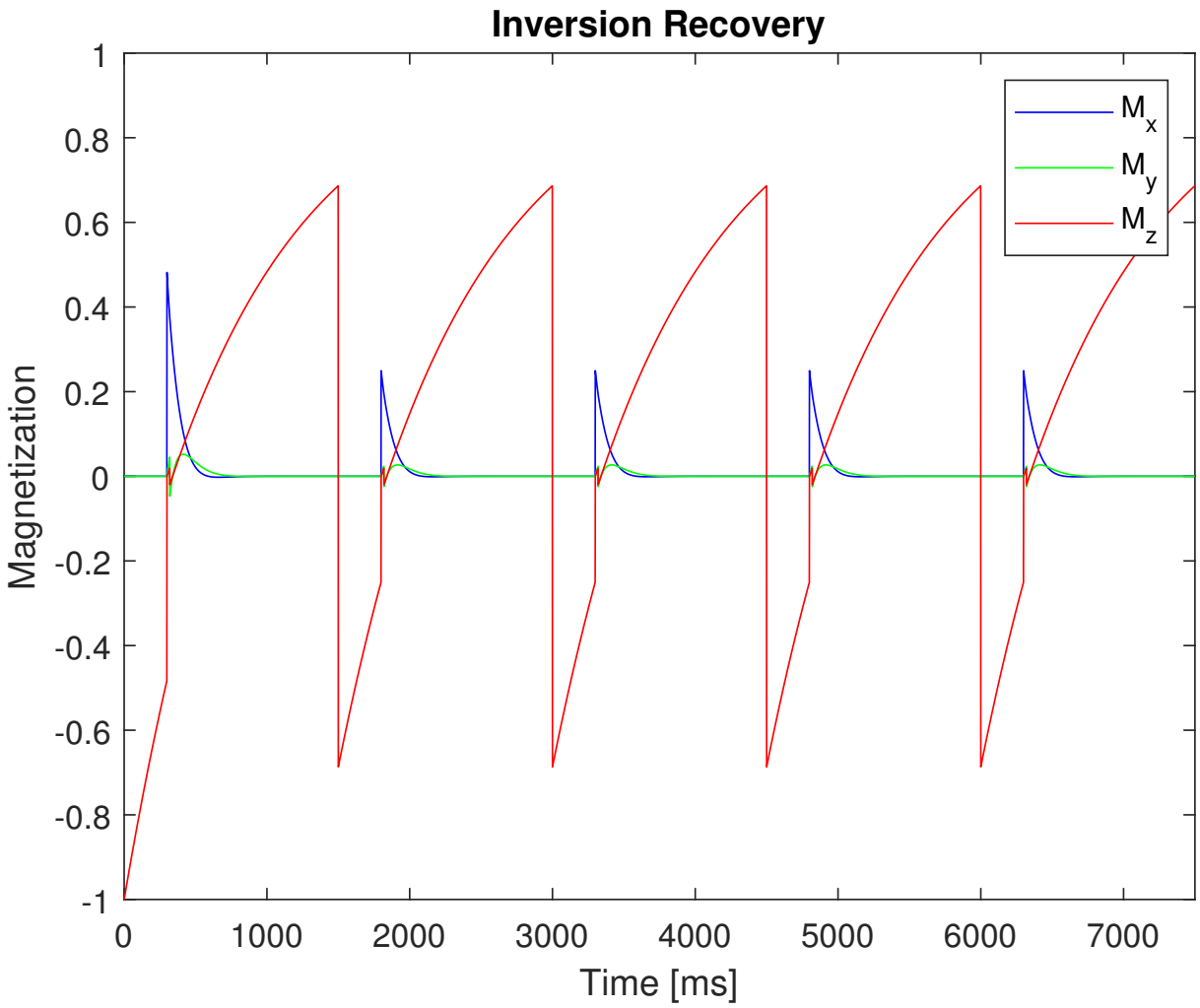


Figure 24: Inversion Recovery splitted into trajectories of the three cartesian field components.

4.1.3 Based on the numerical simulation of the Bloch equation, determine the signal of the spin echo at the 5th TR (assume $M_0 = 1$). Compare the signal of the formed spin-echo at the 5th TR to the following analytical expression (depending on T_1 and T_2 relaxation times):

The error between simulated and analytical solution is in the 3^{rd} digit. Hence, the simulation shows an accurate approximation.

$$S = M_0 \left(1 - 2e^{-\frac{TI}{T_1}} + 2e^{-\frac{TR-\frac{TE}{2}}{T_1}} - e^{-\frac{TR}{T_1}}\right) \cdot e^{-\frac{TE}{T_2}} \tag{93}$$

$$M_{echo,\ 5^{th}TR_x} = 0.1532 \tag{94}$$

$$M_{analytical,\ x} = 0.1568 \tag{95}$$

```
%% signal at TE of the 5th TR based on numerical Bloch equation simulation
M_echo_5thTR = abs(M(:,4*N_tr+ N_ti + N_te));
M_echo_5thTR_x = M_echo_5thTR(1, :)
% signal based on analytical formula
M_analytical_x = abs((exp(-TE/T2))*(1-2*exp(-TI/T1)+exp(-TR/T1)))
```

4.1.4 The above equation can be simplified for TE << TR: Based on the analytic above expression, it is possible for a given TR to select the inversion time TI to null the signal of a tissue with a given T_1. Assume that the inversion recovery spin echo sequence will be used to null the signal from fat (T_1=360 ms). Find the inversion time required to null the fat signal.

$$S = M_0 \left(1 - 2e^{-\frac{TI}{T_1}} + 2e^{-\frac{TR-\frac{TE}{2}}{T_1}} - e^{-\frac{TR}{T_1}}\right) \cdot e^{-\frac{TE}{T_2}} \tag{96}$$

$$S \overset{TE<<TR}{=} M_0 \left(1 - 2e^{-\frac{TI}{T_1}} + e^{-\frac{TR}{T_1}}\right) e^{-\frac{TE}{T_2}} \tag{97}$$

$$S \overset{!}{=} 0,\ T_1(fat) = 360\,\text{ms} \wedge TR = 150\,\text{ms} \tag{98}$$

$$TI = -T_1 \ln\left(0.5 + 0.5e^{-\frac{TR}{T_1}}\right) \tag{99}$$

$$TI = -360\,\text{ms} \cdot \ln\left(0.5 + 0.5e^{-\frac{150\,\text{ms}}{360\,\text{ms}}}\right) = 243.94\,\text{ms} \tag{100}$$

$$TI_{sim} = 243.94\,\text{ms} \tag{101}$$

```
T1fat=360; %ms
TR=1500; %ms
TI=-T1fat*log(.5+.5*exp(-TR/T1fat))
S=(1-2*exp(-TI/T1fat) + exp(-TR/T1fat))
```

4.2 Reconstruction from k-space

The k-space data of a T_1-weighted sagittal brain image can be loaded from the workspace braint1data.mat.

4.2.1 Reconstruct the brain image using the given complex k-space data. Is it possible to reconstruct the MR image using only the magnitude or the phase of the measured k-space data? Show the corresponding images.

A reconstruction with just the phase or magnitude information is not possible. Reducing one whole dimension causes too much information loss!

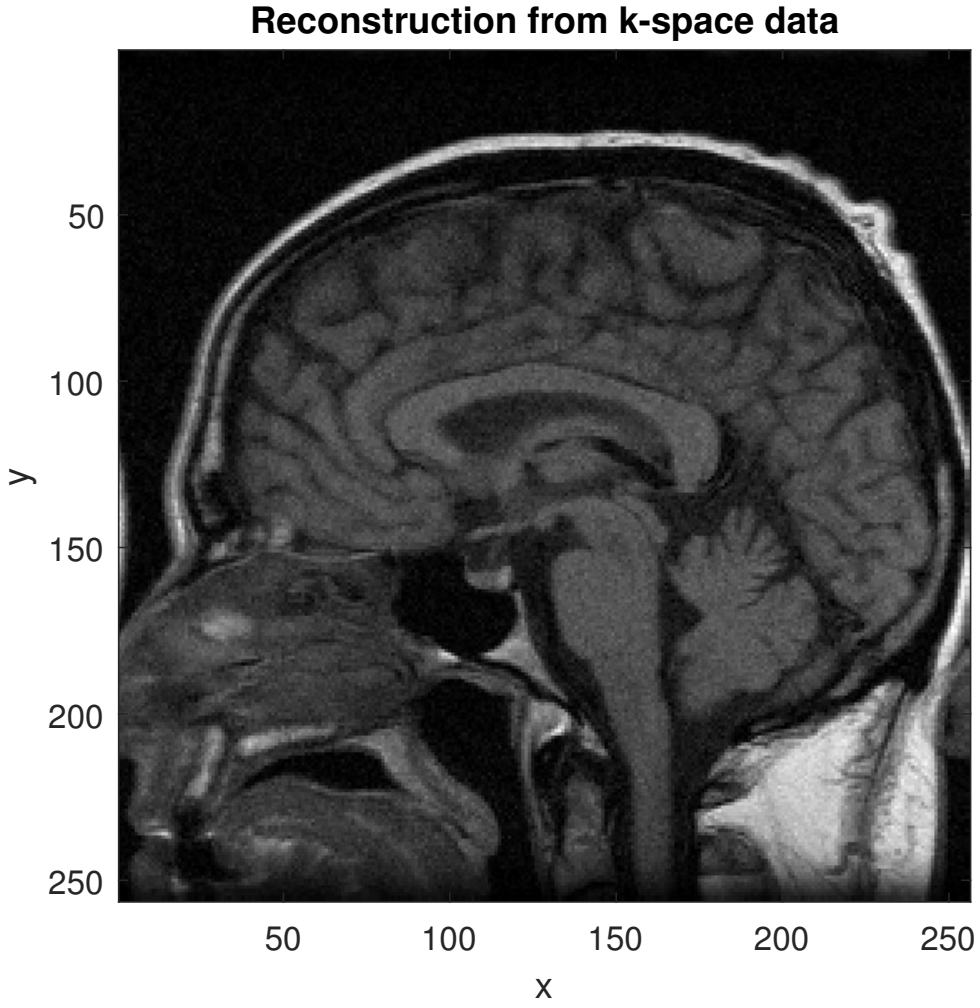


Figure 25: Fully reconstructed image

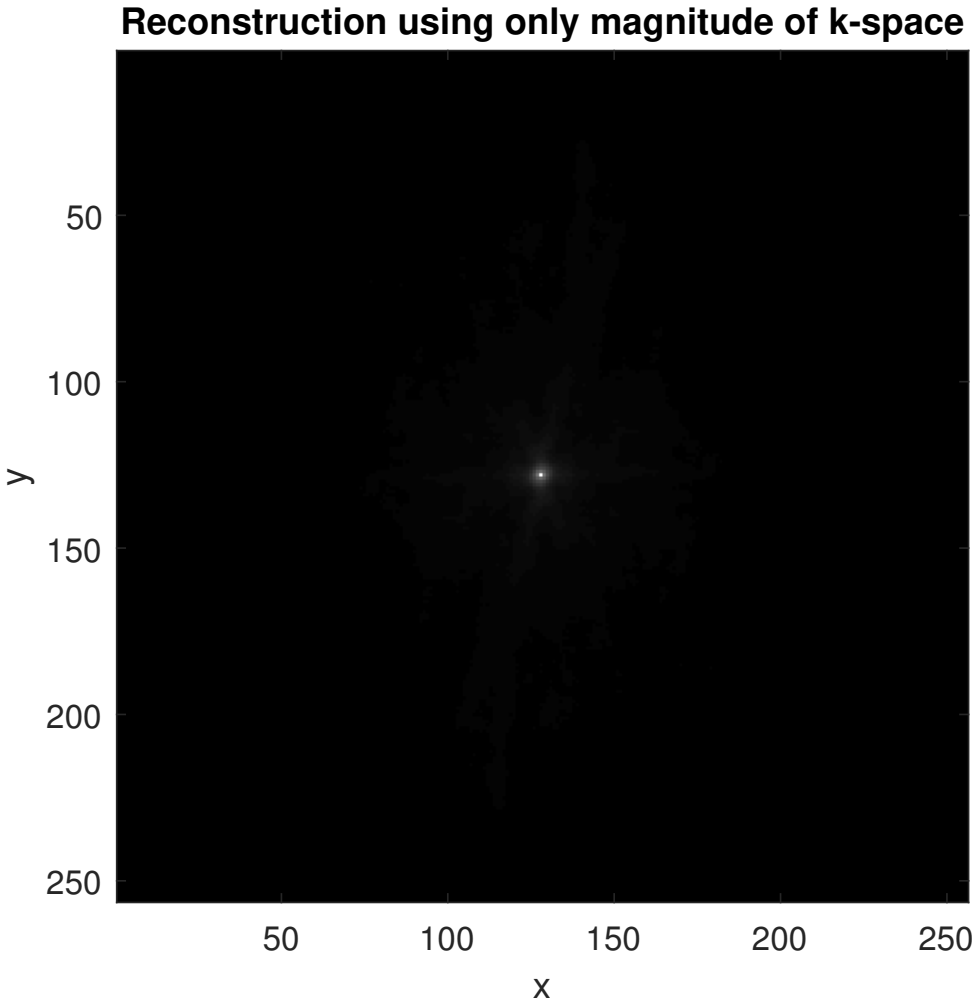


Figure 26: Reconstructed image without phase information

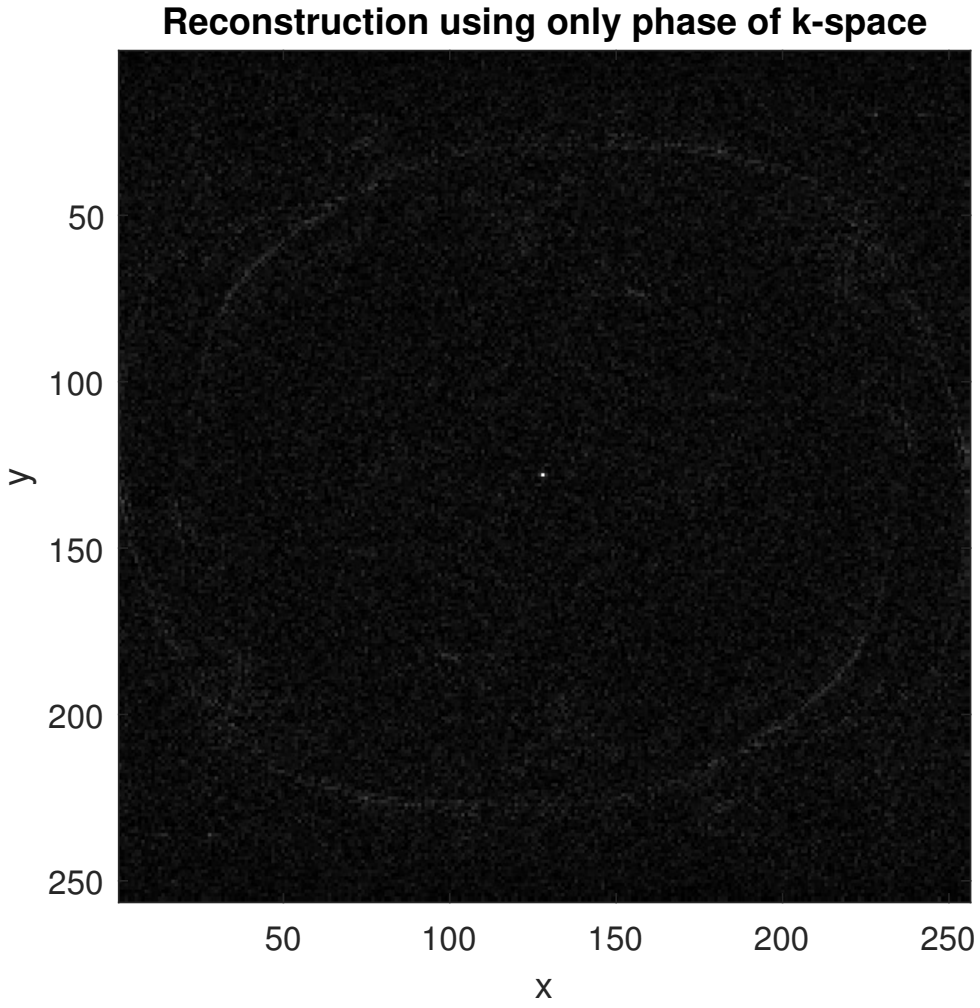


Figure 27: Reconstructed image without magnitude information

```
%% reconstruct image using complex k-space data
img = ifft2c(rawkspace);
figure;
imagesc(abs(img));
colormap(gray);
axis equal tight;
title('Reconstruction from k-space data');
xlabel('x');
ylabel('y');
print('img/recon.eps','-depsc');
% using only magnitude of k-space
img = ifft2c(abs(rawkspace));
figure;
imagesc(abs(img));
colormap(gray);
axis equal tight;
title('Reconstruction using only magnitude of k-space');
xlabel('x');
ylabel('y');
print('img/recon_mag.eps','-depsc');
% using only phase of k-space
img = ifft2c(angle(rawkspace));
figure;
imagesc(abs(img));
colormap(gray);
axis equal tight;
title('Reconstruction using only phase of k-space');
xlabel('x');
ylabel('y');
print('img/recon_phase.eps','-depsc');
```

4.2.2 Assume that only every other k-space point along the horizontal direction has been sampled. Reconstruct the MR image. What type of artifacts do you observe? Why?

Sampling every second column causes a insufficient spacing in the k-space frquencies and therefore "orders" of the image. With large space between these, the higher orders will not be separated and overlap the central image. The overlap of the +1 and -1 "diffracted" order (in the Fourier space) with the central image (zeroth order) is cause of the "wrap". This effect is called aliasing in image processing.

```
%% remove every other data point from k-space along x-axis
reduced_rawkspace = rawkspace;
reduced_rawkspace(:,1:2:end)=[];
%reconstruct image from reduced k-space data
reduced_ima = ifft2c(reduced_rawkspace);
figure;
imagesc(abs(reduced_ima));
colormap(gray);axis equal tight;
title(['Recon with only every other point from k-space' ...
' in horizontal direction']);
xlabel('x');
ylabel('y');
print('img/recon_every2nd.eps','-depsc');
```

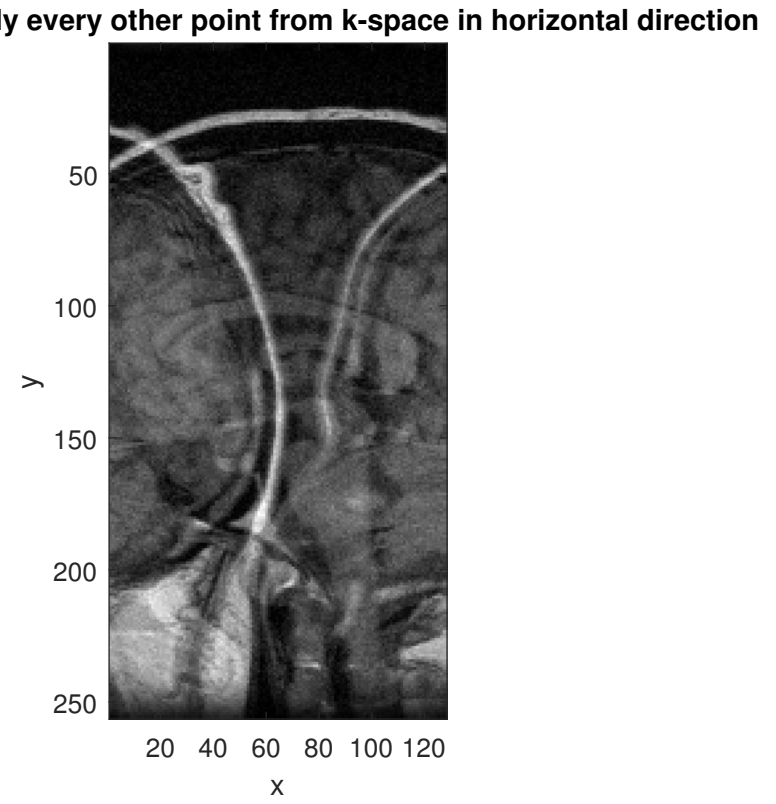


Figure 28: Reconstructed image with every other column in the k-space matrix.

4.2.3 Raw k-space can be corrupted with noisy spikes of various origins. Introduce a spike artifact in k-space, by setting the complex signal at the pixel location (160,160) equal to 10^4. Reconstruct the MR image. How do k-space spike artifacts appear in image space?

$\frac{160}{256} \cdot 360 = 180 + 45$ -> information from a 225°angle in the image get superimposed. Normalization of the image with the highest magnitude squeezes the intensities of the k-space pixels togeher. Thus, it unitizes the reconstructed image's brightness. (What can be oberserved in increasing the pixels intensity to 10^5 or higher)

```
%% disturb pixel (160,160)
disturbed_kspace = rawkspace;
disturbed_kspace(160,160) = 10^4;
disturbed_img = ifft2c(disturbed_kspace);
figure;
imagesc(abs(disturbed_img));
colormap(gray);
axis equal tight;
title('Reconstruction with disturbed pixel in k-space [160,160]');
xlabel('x');
ylabel('y');
print('img/recon_pixelanomaly.eps','-depsc');
```

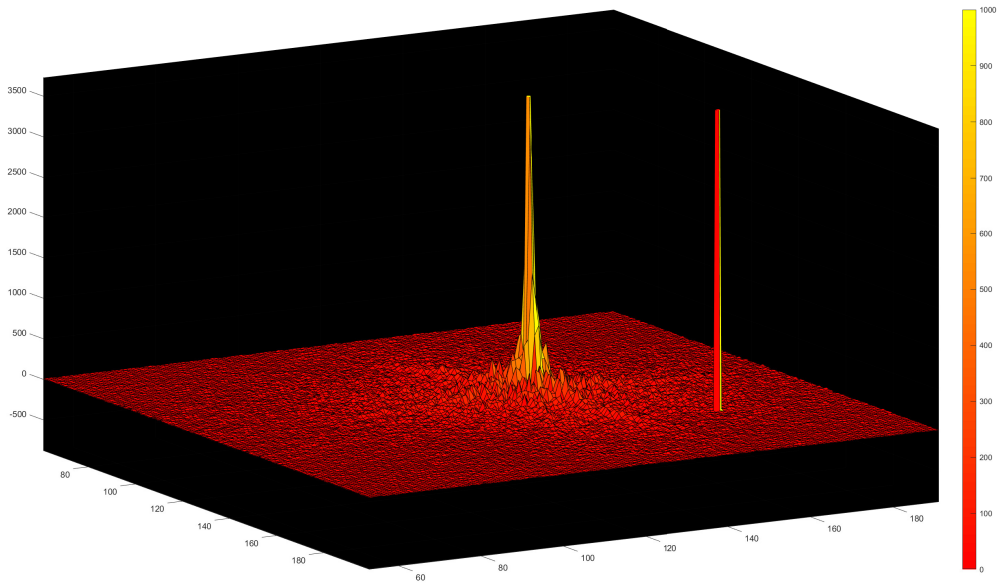


Figure 29: Magnitude plot of a changed pixel in k-space at (160,160)

4.2.4 Introduce a stripe artifact in k-space, by setting the complex signal for all pixels at column 160 equal to 10^2. Reconstruct the MR image. How do k-space stripe artifacts appear in image space?

Because frequency encodes for one spatial dimension, we get a line along a certain position along that spatial dimenstion. This is mostly caused by RF contamination. A vertical stripe artifact in k-space appears as a stripe in image space:

```
%% disturb column 160
disturbed_kspace2 = rawkspace;
disturbed_kspace2(:,160) = 10^2;
disturbed_img2 = ifft2c(disturbed_kspace2);

figure;
imagesc(abs(disturbed_img2));
colormap(gray);
axis equal tight;
title('Reconstruction with disturbed column in k-space 160');
xlabel('x');
ylabel('y');
print('img/recon_columnanomaly.eps','-depsc');
```

4.2.5 Measure the noise by computing the standard deviation in a signal-less region in the image. Display the image in SNR units, and add a colorbar to show the SNR scale.

```
%% SNR in signal
img = ifft2c(rawkspace);
% calculate SNR from small window with no signal
```

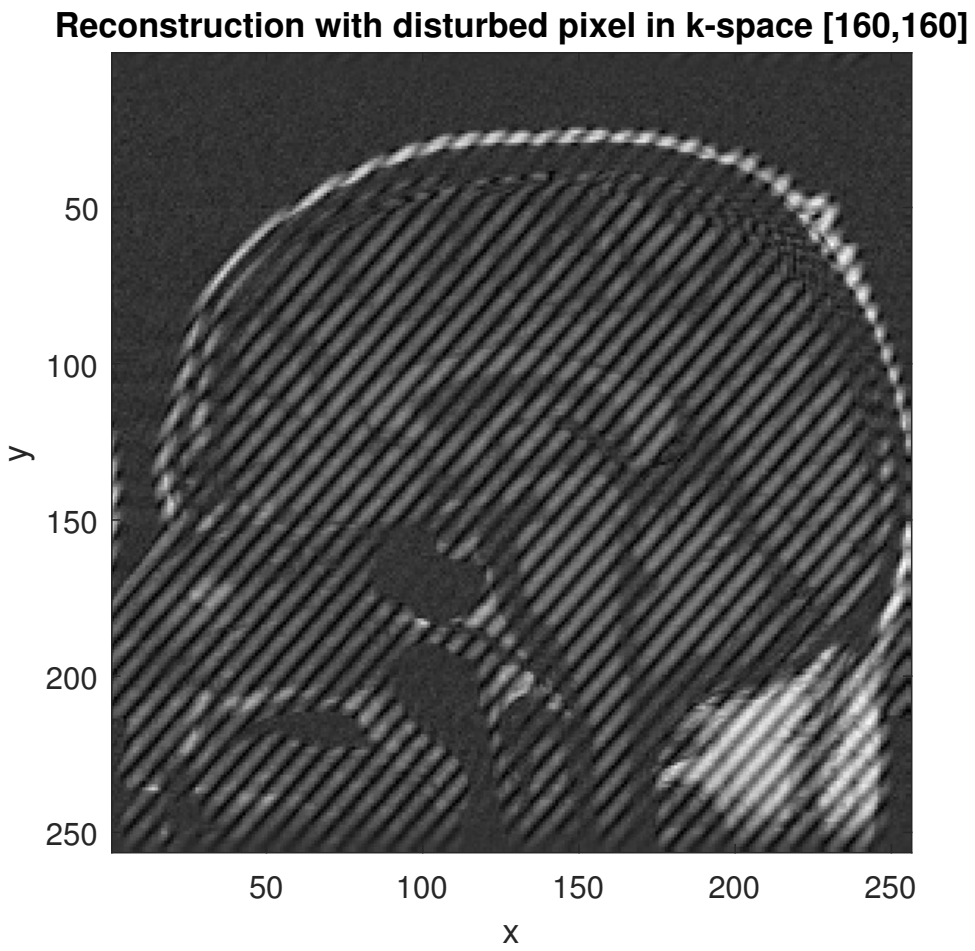


Figure 30: Reconstructed image with a changed pixel in k-space at (160,160)

```
y_inn = 1;
y_outn = 50;
x_inn = 1;
x_outn = 50;
small_window = abs(img(y_inn:y_outn,x_inn:x_outn));
standard_deviation = std(std(small_window));
overall_SNR = mean(mean(abs(img)))/standard_deviation
%% SNR measurements
%% original data
y_ins=110;
y_outs=115;
x_ins=90;
x_outs=95;
y_inn=1;
y_outn=50;
x_inn=1;
x_outn=50;
SNRorig = mean(mean(abs(img(y_ins:y_outs,x_ins:x_outs))))/...
    std(std(abs(img(y_inn:y_outn,x_inn:x_outn))))
% SNR has been defined as the ratio of the average signal value to the
```

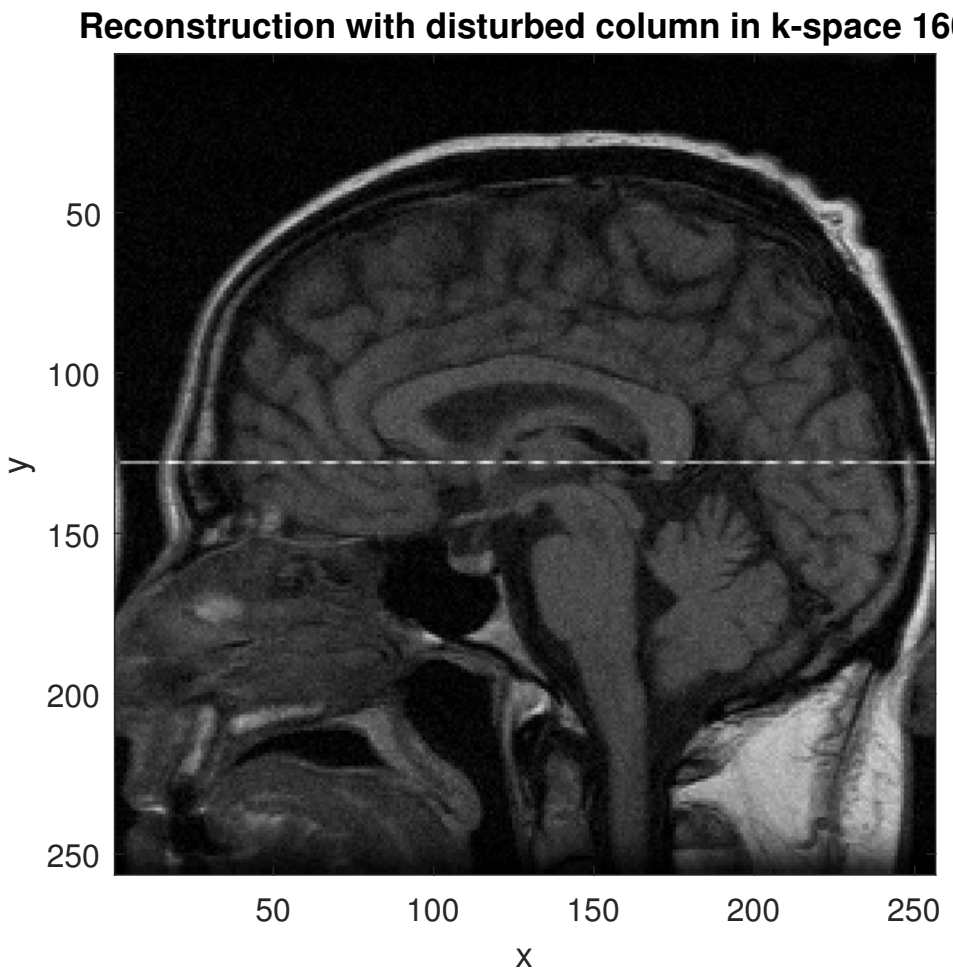


Figure 31: Reconstructed image with every entry in column 160 set to 100

```
% standard deviation
% average signal over 5x5 window
average_mask = fspecial('average', 5);
averaged_image = imfilter(abs(img), average_mask);
% devide by standard_deviation
SNRorig = averaged_image/standard_deviation;
figure;
imagesc(SNRorig);
colormap(jet);
axis equal tight;
colorbar
title('Image SNR over 5x5 window average');
xlabel('x');
ylabel('y');
textColor = 'black';
textBackground = 'white';
text(200, 15, ...
['Overall SNR: ', num2str(overall_SNR) ], ...
'Color', textColor, ...
```

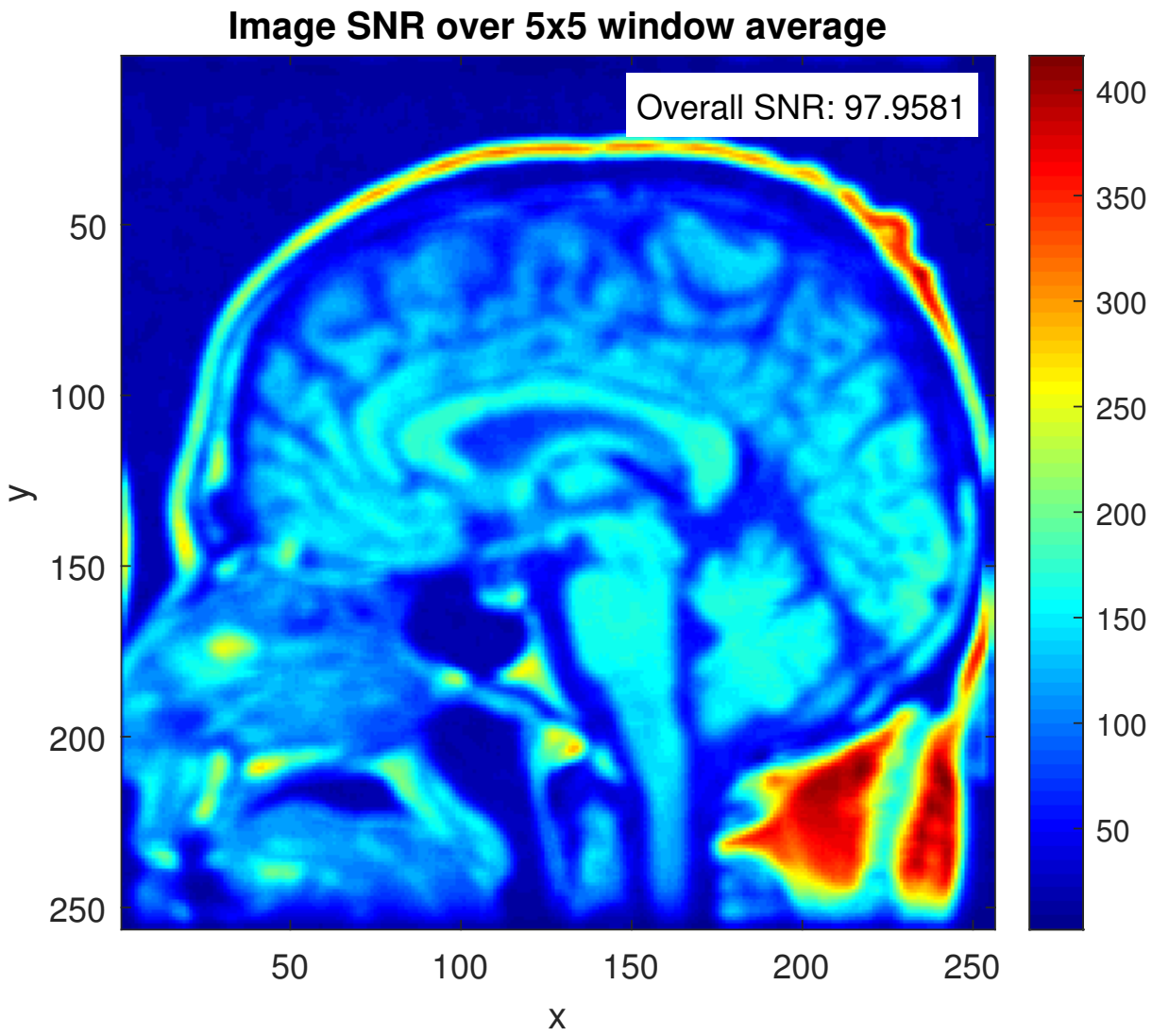


Figure 32: SNR of the reconstructed image

```
'BackgroundColor', textBackground, ...
'HorizontalAlignment', 'Center');
print('img/recon_snr.eps','-depsc');
```

4.2.6 Zero-fill the k-space data to a 512x512 matrix by symmetrically adding zeros around the data. This will interpolate the image to a larger matrix size. Reconstruct the image, compute the SNR, and display in SNR units.

Adding zeros around the image increases the field of view

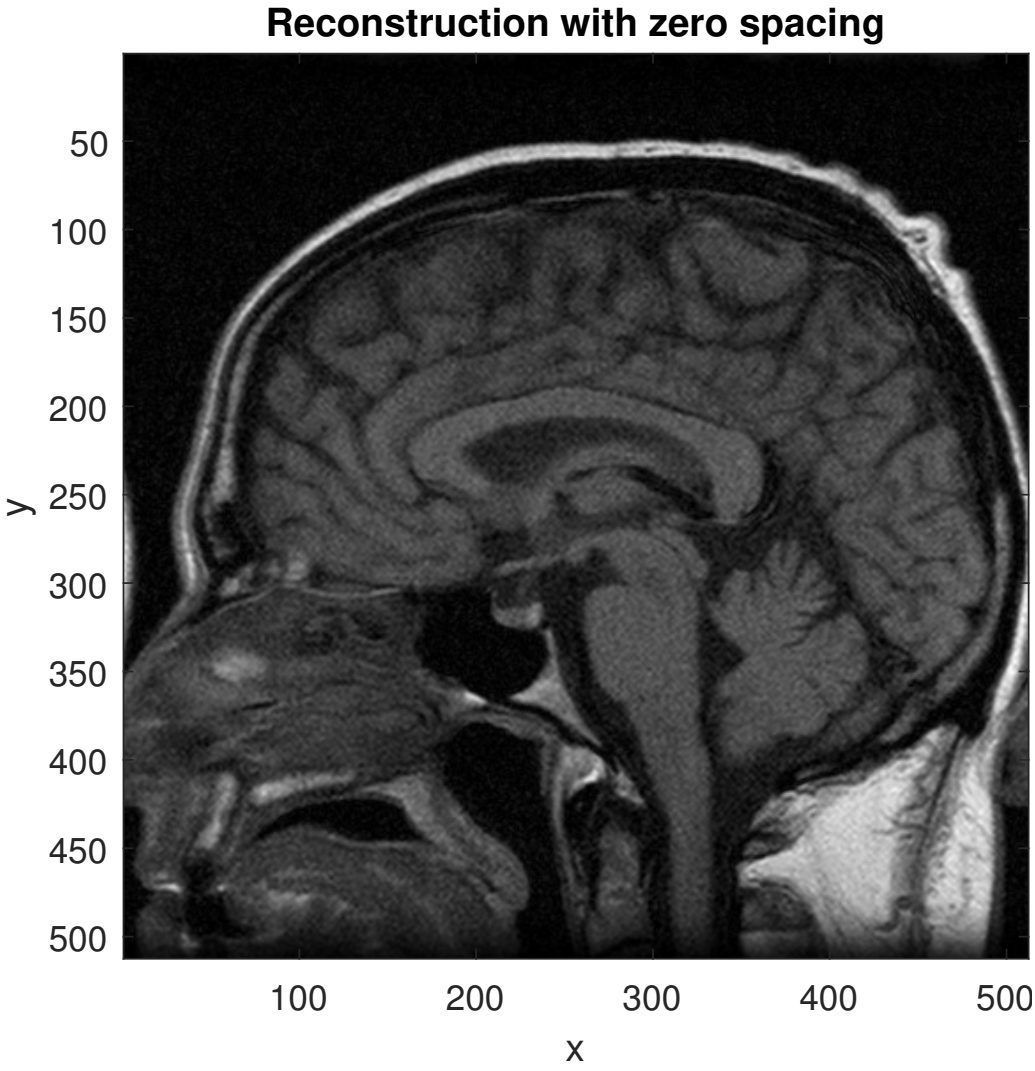


Figure 33: Reconstructed image from an enlarged, zero-filled k-space

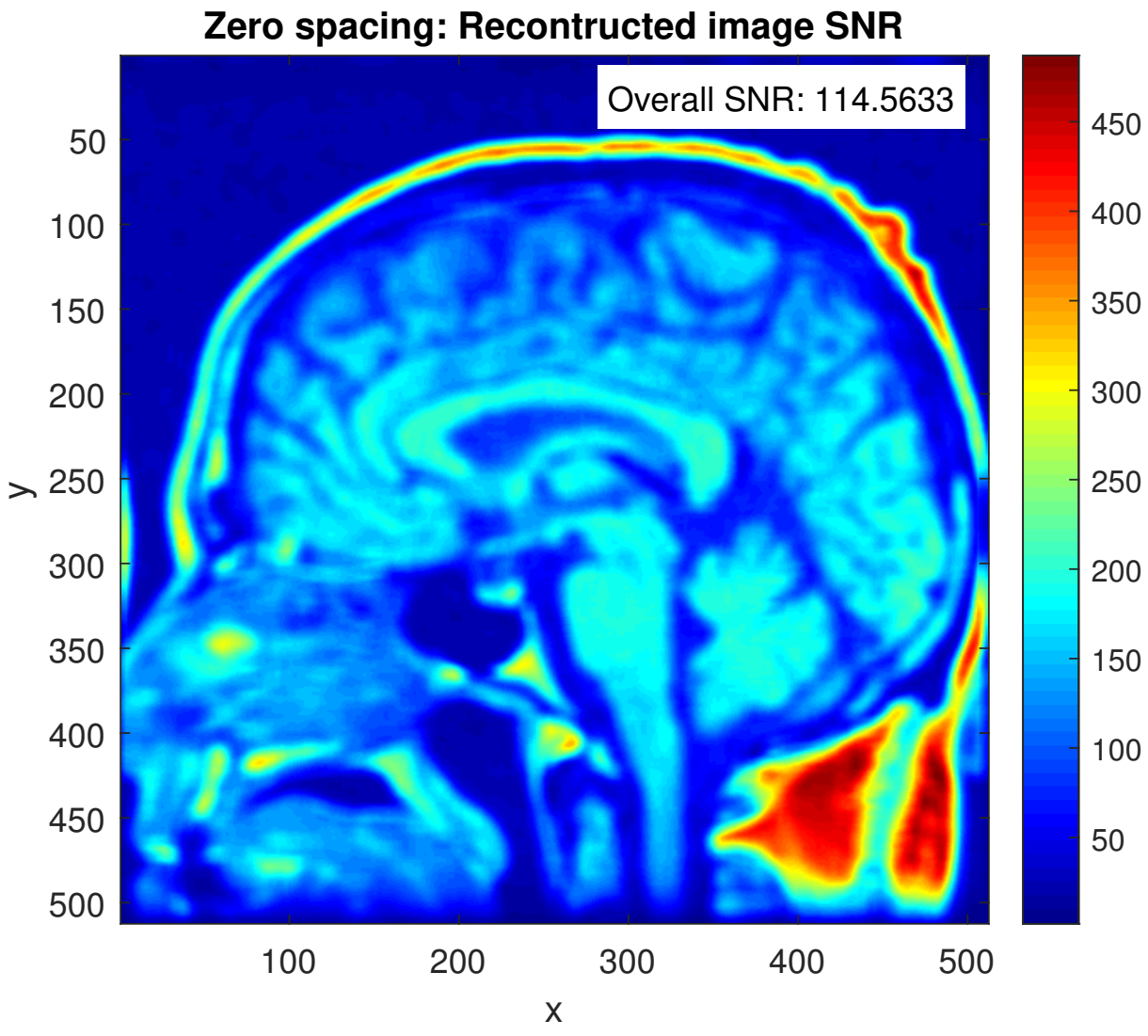


Figure 34: SNR from figure 33

```
%% zero-filling to 512x512
% image 512
rawkspacefill = zeros(512,512);
rawkspacefill((1:256) + 128,(1:256) + 128) = rawkspace;
imfill = ifft2c(rawkspacefill);
% reconstruct the imagefigure;
imagesc(abs(imfill));
colormap(gray);axis equal tight;
title('Reconstruction with zero spacing');
xlabel('x');
ylabel('y');
print('img/recon_zerofilled.eps','-depsc');
% now SNR of image --> double size of window and filter
y_inn = 1;
y_outn = 100;
x_inn = 1;
x_outn = 100;
small_window = abs(imfill(y_inn:y_outn,x_inn:x_outn));
standard_deviation_2 = std(std(small_window));
%overall SNR
overall_SNR_imfill = mean(mean(abs(imfill)))/standard_deviation_2;
average_mask = fspecial('average', 10);
averaged_imfill = imfilter(abs(imfill), average_mask);
% devide by standard_deviation
SNRimfill = averaged_imfill/standard_deviation_2;
figure;
imagesc(SNRimfill);
colormap(jet);
axis equal tight;
colorbar
title('Zero spacing: Recontructed image SNR');
xlabel('x');
ylabel('y');
textColor = 'black';
textBackground = 'white';
text(390, 25, ...
['Overall SNR: ', num2str(overall_SNR_imfill) ], ...
'Color', textColor, ...
'BackgroundColor', textBackground, ...
'HorizontalAlignment', 'Center');
print('img/recon_zerofilledSNR.eps','-depsc');
```

4.2.7 Windowing: Multiply the data by a 2D Hanning window. Reconstruct the image, compute the SNR, and display in SNR units. How has the SNR changed? How has the image changed? How are these changes related?

The resolution becomes worse, the image is more blurry. By Hanning-filtering, we amplify low frequencies in k-space and squeeze the high frequencies together. Hence, we are removing fine details, but also the noise. It improves the SNR and makes medium-sized features better visible, but lowers the resolution.

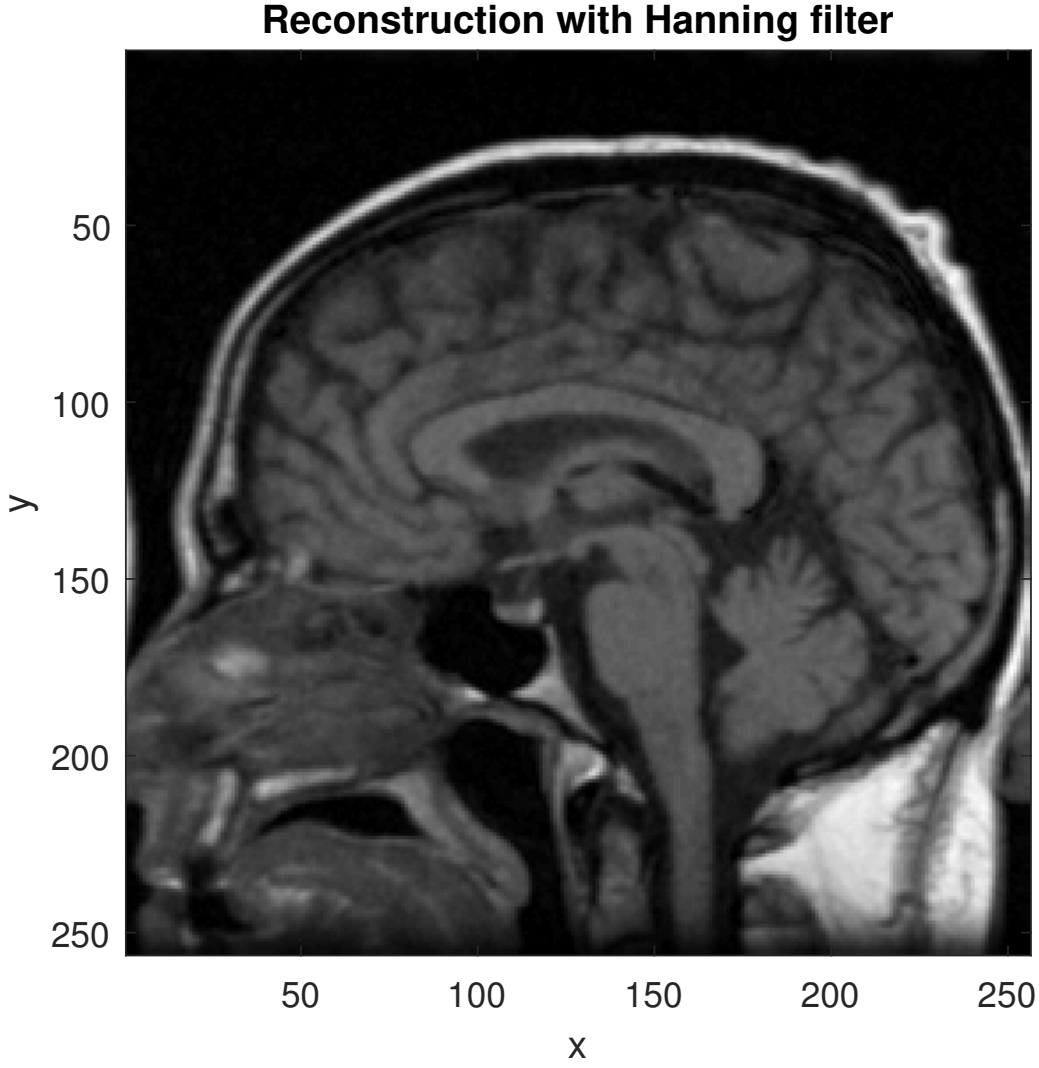


Figure 35: Reconstructed image with a 2D hanning filtering.

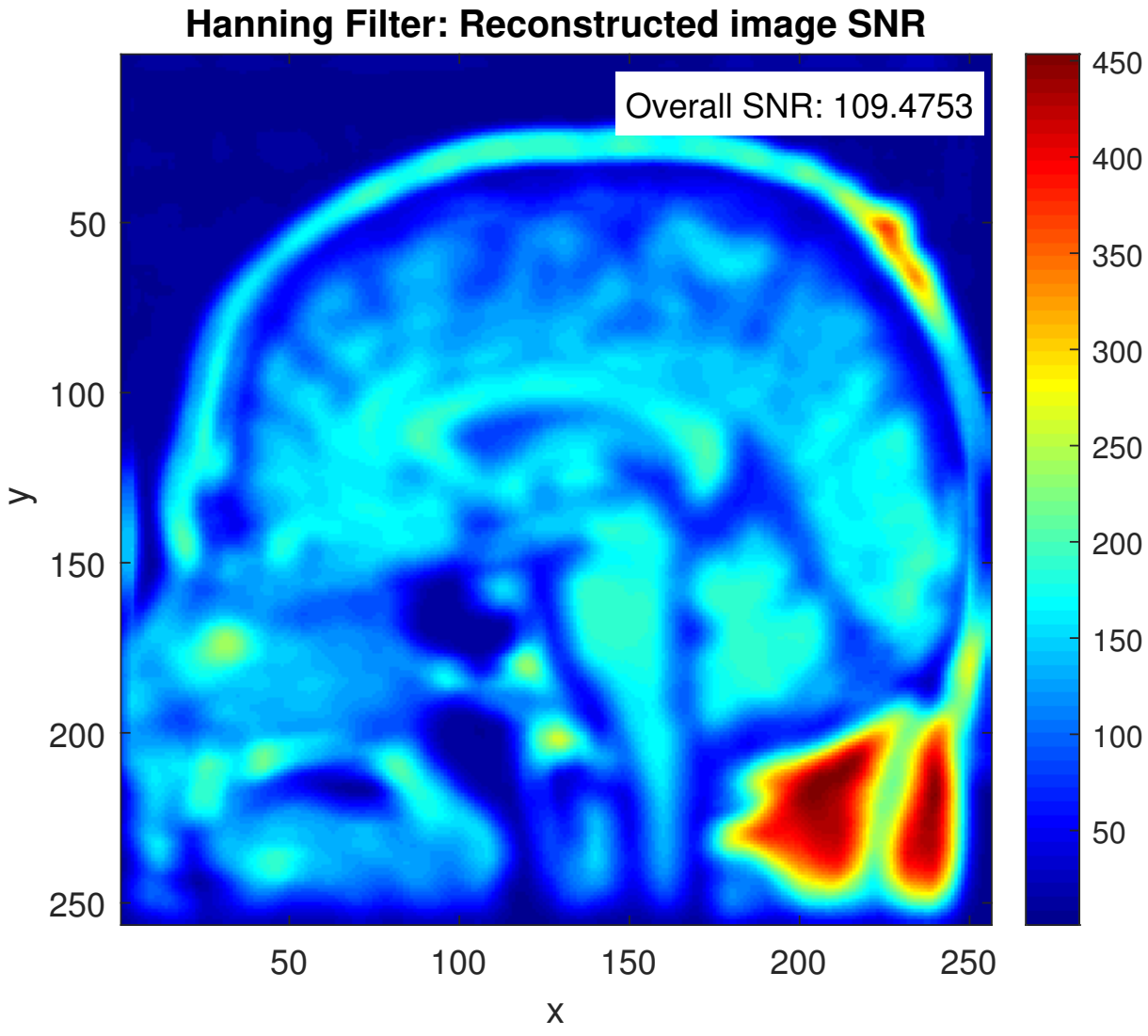


Figure 36: SNR of figure 35

```
%% Hanning window
% hanning window in k-space
rawkspace_hann = rawkspace.* (hann(256)* hann(256)');
ima_hann = ifft2c(rawkspace_hann);
%reconstruct image
figure;
imagesc(abs(ima_hann));
colormap(gray);axis equal tight;
title('Reconstruction with Hanning filter');
xlabel('x');
ylabel('y');
print('img/recon_hanning.eps','-depsc');
% calculate SNR from small window with no signal
y_inn=1; y_outn=50;
x_inn=1; x_outn=50;
small_window_hann = abs(ima_hann(y_inn:y_outn,x_inn:x_outn));
standard_deviation_hanning = std(std(small_window_hann));
overall_SNR_hanning = mean(mean(abs(ima_hann)))/standard_deviation_hanning;
% SNR has been defined as the ratio of the average signal value to the
% standard deviation
% average signal over 5x5 window
average_mask_hann = fspecial('average', 5);
averaged_image = imfilter(abs(ima_hann), average_mask);
% devide by standard_deviation
SNRhann = averaged_image/standard_deviation_hanning;
figure;
imagesc(SNRhann);
colormap(jet);
axis equal tight;
colorbar
title('Hanning Filter: Reconstructed image SNR');
xlabel('x');
ylabel('y ');
textColor = 'black';
textBackground = 'white';
text(200, 15, ...
['Overall SNR: ', num2str(overall_SNR_hanning) ], ...
'Color', textColor, ...
'BackgroundColor', textBackground, ...
'HorizontalAlignment', 'Center');
print('img/recon_hanningSNR.eps','-depsc');
```